Mathematisches Forschungsinstitut Oberwolfach

Report No. 42/2019

DOI: 10.4171/OWR/2019/42

Large Scale Stochastic Dynamics

Organized by
Thierry Bodineau, Palaiseau
Fabio Toninelli, Lyon
Bálint Tóth, Bristol/Budapest

15 September – 21 September 2019

Abstract. The goal of this workshop was to explore the recent advances in the mathematical understanding of the macroscopic properties which emerge on large space-time scales from interacting microscopic particle systems. There were 55 participants, including postdocs and graduate students, working in diverse intertwining areas of probability and statistical mechanics. During the meeting, 29 talks of 45 minutes were scheduled and an evening session was organised with 10 more short talks of 10 minutes, mostly by younger participants. These talks addressed the following topics : randomness emerging from deterministic dynamics, hydrodynamic limits, interface growth models and slow convergence to equilibrium in kinetically constrained dynamics.

Mathematics Subject Classification (2010): 60K35, 74Q0, 82C22, 76P05, 82B40.

Introduction by the Organizers

The workshop *Large scale stochastic dynamics* organised by Thierry Bodineau (Palaiseau), Fabio Toninelli (Lyon) and Bálinth Toth (Bristol/Budapest), was well attended with over 50 participants with broad geographic representation.

The workshop was devoted to the wide mathematical problem of understanding emergent structures on large space-time scales in the evolution of physical systems. These are modelled by particle systems, namely high-dimensional Markov processes and/or by systems of particles with deterministic (Hamiltonian) dynamics where randomness comes only with the initial conditions. In our choice of 29 talks, we tried to illuminate major recent advances in the field and to expose and address at least some aspects of the works for each of the participants. An evening session with short talks was the occasion to learn about the recent results of 10

early career participants and to trigger further discussions afterwards. A more detailed account of the long presentations is given below.

From deterministic dynamics to kinetic equations and stochastic processes.

Several talks were devoted to the derivation of kinetic equations and more generally of macroscopic limits from microscopic systems following the deterministic laws of classical physics starting with randomized initial conditions.

Herbert Spohn explained the challenges for deriving hydrodynamics of integrable many-particle systems and used the classical Toda chain to illustrate recent advances.

Domokos Szasz presented a simplified – but still sufficiently complex – version of the Gaspard-Gilbert model for microscopic mechanism of Fourier's law of heat conduction, the so-called *piston model*. In this one-dimensional fully deterministic (Hamiltonian) model of interacting particles the authors are able to implement the first part of the Gaspard-Gilbert program. Namely, deriving with full mathematical rigour an effective stochastic approximation in a particular low-density limit.

Giada Basile presented a gradient flow approach to the linear Boltzmann equation and she explained how it is related to the large deviations. Recent progress on the large deviations for a Kac-like walk were also reviewed.

François Huveneers discussed the slow thermalization in a chain of classical anharmonic oscillators and how the thermalization is related to the predictions from the Boltzmann-Peierls equation.

Christopher Lutsko presented recent results on the diffusive scaling limit of the Lorentz gas with randomly placed spherical scatterers, in three dimensions. It was shown that in the low-density (a.k.a. Boltzmann-Grad) limit up to divergent time scales (given explicitly in terms of the kinetic time scale of the problem) a diffusive approximation formulated as the invariance principle for the tagged particle trajectory holds.

Alessia Nota surveyed the different kinetic limits arising from a Lorentz Gas with long-range interactions. Depending on the power law decay of the interaction potential, the linear Boltzmann equation or the linear Landau equation can be recovered.

Carlangelo Liverani defined a family of walks on the d-dimensional integer lattice, where an internal state of the local mechanism determines the next step of the walker. The family of models is motivated by the urge of understanding the large scale dynamics of the Lorentz gas with randomized positions of the fixed scatterers. It was shown that under some particular conditions on the model family the walk exhibits a loss of memory quantified by a uniform Gibbsian approximation. This may open the way to a diffusive scaling limit.

Makiko Sasada surveyed recent results on discrete time versions of the KdV equation and the Toda lattice starting from random initial conditions. She has shown how a generalization of Pitman's transform allows to study these models.

Isabelle Gallagher presented some recent results on the correlation in hard sphere gas dynamics and she showed how non-equilibrium fluctuations around Boltzmann equation can be derived from these estimates.

Sergio Simonella explained how correlations can be controlled in a wide range of collisional dynamics including the Kac model for the Boltzmann equation, and large quantum particle systems in the mean field regime. In this case optimal bounds wrt the system size can be obtained.

Mario Pulvirenti has shown how the number of collisions in the backward clusters of hard-sphere systems can be controlled in a low-density regime.

Tomasz Komorowski considered the limit of a linear phonon Boltzmann equation, with reflection-transmission-absorption at an interface, with a degenerate scattering kernel. He described the diffusive and superdiffusive asymptotics which can be obtained in different regimes.

Jani Lukkarinen discussed the existence of stationary non-equilibrium solutions for coagulation systems associated with interaction kernels relevant to atmospheric coagulation phenomena.

Approach to equilibrium and interface dynamics in reversible systems

Riddhipratim Basu showed that, for Last Passage Percolation in the infinite square grid $\mathbb{Z}^2$ with i.i.d. exponential weights, almost surely no infinite geodesics exist, except for the trivial ones (horizontal and vertical geodesics).

In a closely related talk, Márton Balázs presented an alternative, more probabilistic, proof of non-existence of infinite geodesics for Last Passage Percolation on $\mathbb{Z}^2$.

Chiranjib Mukherjee discussed the Kardar-Parisi-Zhang (KPZ) equation in space dimension $d \geq 3$. When the noise is suitably rescaled and regularized, the space-time fluctuations of the process converge to a limit Gaussian process as the regularization parameter is sent to zero.

Giuseppe Cannizzaro discussed stochastic growth models in (1+1)-dimensions: he showed that beyond the well-known Edwards-Wilkinson and KPZ universality classes, another class of models exists, with new growth exponents. The conjectural scaling limit of these models, named "Brownian Castle", was discussed.

Tadahisa Funaki described an ongoing research project about dynamical large deviations for a Markov reversible dynamics of 3-dimensional Young diagrams or lozenge tilings

Roland Bauerschmidt described a multiscale method which generalizes the Bakry Emery criterion and is designed to prove Log-Sobolev inequality for non-convex interactions. An application was given for the continuum Sine-Gordon model.

Hubert Lacoin gave a panorama of new results on the Gaussian Free Field (GFF) in dimension $d \geq 3$, subject to a disordered pinning potential. Results included sharp behavior of the free energy and of the typical interface height, when the pinning strength is close to the critical point.

Patrik Ferrari discussed time-time correlations, under various initial conditions, for models in the $(1+1)$-dimensional KPZ universality class, and notably for Last Passage Percolation.

Sunil Chhita presented results on fluctuations of level lines of random domino tilings of the Aztec diamonds, with two-periodic weights. At the boundary between liquid and gas regions, a suitably coarse-grained point process associated to the level lines converges to the Airy kernel point process.

Alexandre Stauffer discussed multi-particle versions of the celebrated Diffusion Limited Aggregation process. He showed that in any dimension $d \geq 2$, the model has a regime of positive speed of growth, where the aggregate includes a bulky set.

Approach to equilibrium in reversible systems.

Glauber dynamics are Markov chains that are reversible w.r.t. the Gibbs distribution of a statistical mechanics system. Understanding how quickly the process approaches the equilibrium distribution, and the occurrence of slowdown phenomena, gives insight on phase transitions, glassy or metastable behavior.

Ivailo Hartarsky discussed Kinetically Constrained Models (KCM), especially on the two-dimensional lattice $\mathbb{Z}^2$. He focused especially on the deep relation between KCM and bootstrap percolation and he presented results and conjectures on universality classes for both models.

Assaf Shapira presented results about KCM in random environments. One striking result is that, at large particle density $1 - q \approx 1$, the first time where the origin is emptied scales like $q^{-\alpha}$, with α a *random* exponent.

Finally, we had a few talks that cannot be classified under the above themes, and that broadened the workshop topic to other aspects of large scale stochastic dynamis:

Balazs Rath presented a recent result where it is shown that although the so-called frozen percolation process on the binary tree is a well defined process – as constructed by Aldous (2000) – it is *not determined* by the local random variables which intuitively dictate its local dynamics. Extra randomness sweeps in from the boundary (at infinity). The result answers a question of Aldous (2000) formulated explicitly in his cited work left open since then.

Dmitry Ioffe showed that although the stationary random interchange process on large discrete tori reflect the d-dimensional geometry of the underlying graph, in the thermodynamic limit the dynamics of the large cycles of these random permutations, properly scaled, converge to the canonical split-and-merge process which is independent of geometry. The question is motivated by stochastic representations of quantum spin systems.

Alessandra Faggionato presented stochastic homogeneization results for random walks in amorphous random media with unbounded jump range. Examples include simple random walks on Delaunay triangulations or Mott variable range hopping in doped semiconductors.

Tyler Helmuth showed how Pirogov-Sinai theory can be used to construct algorithms that approximately sample the Gibbs measure of the Potts model with large q in any dimension, with running time that is polynomial wrt the system size and wrt the desired approximation precision.

Summary. The workshop helped to update the participants on the state of the art and on the important pending open problems in the fields related to their domain of research. The workshop triggered interactions between people working in different fields: probability, PDE analysis and dynamical systems. It was also the occasion to initiate and pursue collaborations. The scientific presentations proved that this research field is still very active and is absorbing new ideas from other branches of mathematics.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Riddhipratim Basu in the "Simons Visiting Professors" program at the MFO.

Workshop: Large Scale Stochastic Dynamics

Table of Contents

Abstracts

Fourier law from Hamiltonian dynamics

Domokos Szász
(joint work with P. Bálint, IP. Tóth, Th. Gilbert, P. Nándori)

The derivation of Fourier's celebrated heat equation from Hamiltonian principles has long attracted particular attention of both mathematicians and physicists. All this without any hope until in 2008 the Belgian physicists Gaspard and Gilbert, [2] came up with a billiard type model and a two-step strategy applied to it. (The model is an adaptation of a 1992 system suggested by Bunimovich-Liverani-Pellegrinotti-Sukhov, [1] making life simpler by ensuring that there is no mass transport.)

In the first part of the Gaspard-Gilbert strategy, by taking the rare (but strong) interaction limit of the energies of the particles, they provide heuristics showing that the limit is a Markov jump process. Rigorous arguments for this step require methods of dynamical systems. Then - in the second, probabilistic part of their program - by executing the hydrodynamic limit of this Markov jump process one is, indeed, expected to derive the heat equation.

Since the dynamical part of their scenario was still unreachable for a mathematical treatment, we modified it and initiated the study of the so-called piston model instead (see [3]). In my talk I will explain the dynamical approach to the rare interaction limit of the piston model resulting, really, in the Markov jump process conjectured by [3]. With this result in hand, the next big - and at the same time quite realistic challenge - is to take the hydrodynamic limit of this Markov jump process.

This requires answers to two stimulating questions. The first one seems to be simpler. This question is, roughly saying, the following. For the execution of the hydrodynamic limit it is necessary to have an appropriate lower bound for the spectral gap of the N particle piston model (concretely $\frac{\text{const.}}{N^2}$ where const. > 0). For N particle GG-models this has been proved in a beautiful paper by Sasada, [4] in 2015. Similar bound is expected for the piston model though one should keep in mind that the piston model has less symmetry than the GG models. The second question is then the hydrodynamic limit itself. Here it is reasonable challenge to apply the method of Varadhan's 1993 fundamental work [5] but its conditions of the coefficients should be weakened.

References

[1] L.A. Bunimovich, C. Liverani, A. Pellegrinotti, Y.M. Suhov, *Ergodic system of n balls in a billiard table*, Communications in Mathematical Physics **146** (1992), 357–396.

[2] P. Gaspard, T. Gilbert, *Heat onduction and Fourier's law by consecutive local mixing and thermalization* Physical Review Letters **101**, (2008) 20601.

[3] P. Bálint, Th. Gilbert, P. Nándori, D. Szász, IP. Tóth, *On the limiting Markov process of energy exchanges in a rarely interacting ball-piston gas*, Journal of Statistical Physics, **166**, (2017), 903–925.

[4] M. Sasada: *Spectral gap for stochastic energy exchange model with nonuniformly positive rate function*, Ann. of Probability **43**, (2015) 1663-1711.
[5] S. R. S. Varadhan: *Nonlinear diffusion limit for a system with nearest neighbor interactions-II*, in Asymptotic Problems in Probability Theory: Stochastic Models and Diffusions on Fractals. Longman. (1993), 75-128.

A gradient flow formulation of kinetic equations and connection with large deviations

Giada Basile
(joint work with Dario Benedetto, Lorenzo Bertini and Carlo Orrieri)

We consider linear Boltzmann equations of the following form

$$(\partial_t + b(v)\cdot\nabla_x) f(t,x,v) = \int_{\mathcal{V}} \pi(\mathrm{d}v')\sigma(v,v')\big[f(t,x,v') - f(t,x,v)\big], \tag{1}$$

where $\pi(\mathrm{d}v)$ is a probability measure on the velocity space $\mathcal{V}$, b is the drift, $\sigma(v,v') = \sigma(v',v) \geq 0$ is the scattering kernel and f is the density of the one-particle distribution. The condition on σ follows from a reversibility assumption. The velocity space $\mathcal{V}$ can be either $\mathbb{R}^d$, $\mathbb{T}^d$, $\mathbb{S}^{d-1}$, or a discrete state space and the position space is the d dimensional torus.

We consider the current as a dynamical variable. Setting $\eta^f = \sigma(v,v')\big[f(t,x,v) - f(t,x,v')\big]$, LBE is equivalent to

$$\begin{cases} \big(\partial_t + b(v)\cdot\nabla_x\big) f(t,x,v) + \int \pi(dv')\,\eta(t,x,v,v') = 0, \\ \eta = \eta^f. \end{cases}$$

The first equation is a balance equation. Together with Dario Benedetto and Lorenzo Bertini, we provided a formulation of the LBE in terms of an entropy dissipation inequality. Given a couple (f,η) of density and current satisfying the balance equation, we say that (f,η) satisfies the LBE if and only if the following inequality holds

$$H(f_T) + \int_0^T dt \int dx \mathcal{D}(f_t) + \mathcal{R}(f,\eta) \leq H(f_0), \tag{2}$$

where $H(f_t)$ is the entropy of f at time t, $\mathcal{D}(f)$ is the Dirichlet form of the square root of f, namely, denoting by L the collision operator, $\mathcal{D}(f) := (\sqrt{f}, -L\sqrt{f})$, and $R(f,\eta) = \int_0^T dt \Psi(f,\eta)$, $\Psi \geq 0$ and convex (see [1] for the explicit form of Ψ). For any couple (f,η) satisfying the balance equation, the other way inequality holds. We proved existence and uniqueness of the solution to LBE in [1].

From a probabilistic point of view, (1) is the Fokker-Planck equation of the process $(V(t), X(t))_{t\geq 0}$ where $V(\cdot)$ is an autonomous jump process, taking value in $\mathcal{V}$, with jump rate $\pi(\mathrm{d}v')\sigma(v,v')$ and $X(t) = \int_0^t b(V_s)ds$. Given N independent copies of the process $(V(t), X(t))_{t\geq 0}$, one can construct the empirical measure

$\alpha_t^N(dx,dv) = \frac{1}{N}\sum_{i=1}^N \delta_{X^i(t)}(dx)\delta_{V^i(t)}(dv)$ and and the associated empirical current, satisfying a balance equation. Then (1) is the law of large numbers of the empirical measure in the limit $N \to \infty$.

The gradient flow formulation of the LBE is related to the large deviations.We look at the asymptotic probability that the couple empirical measure-current differ from the typical one (f, η^f), solution to the LBE, and it is close to (g, ζ), satisfying the balance equation. This probability is exponentially small for diverging N, and the action functional has the form

$$I(g,\zeta) = H(g_0) + \int_0^T dt \int dx \iint_{\mathcal{V}\times\mathcal{V}} \mathrm{d}\pi \otimes \mathrm{d}\pi\, \Phi_\sigma(g, g', \zeta),$$

where $g' = g(t, x, v')$. The first term on the r.h.s. is entropy of g at time 0 and Φ_σ is a convex, positive function which is equal to zero iff the couple g, ζ satisfies the LBE. Inequality (2) can be directly derived by the condition $\Phi_\sigma(f, f', \zeta) \geq 0$, which characterize the zero level set of the functional.

In a work in progress together with D. Benedetto, L. Bertini and C. Orrieri we investigate the large deviation asymptotics for a Kac-like walk, which is described at the kinetic level by a non linear homogeneous Boltzmann equation. At the microscopic level, at random times two particles with velocities v and v_* "collide" and change their velocities with rate σ in such a way the total momentum is preserved, i.e., denoting by v', v'_* their velocities after collision, $v + v_* = v' + v'_*$. We prove a large deviation principle for the empirical measure and current. The action functional is given by

$$I(g,\theta) = H(f_0) + \int_0^T dt \int_{\mathbb{R}^{4d}} d\pi \otimes \mathrm{d}\pi \otimes \mathrm{d}\pi \otimes \mathrm{d}\pi\, \Phi_\sigma(gg_*, g'g'_*, \zeta)$$

where, using the usual kinetic notations, $g_* = g(t, v_*)$, $g'_* = g(t, v'_*)$. The solution to the homogeneous Boltzmann equation is then the zero level set of the rate function. The condition $\Phi_\sigma \leq 0$, which characterizes the zero level set of the functional, provides an inequality of the same type of (2).

References

[1] B. Basile, D. Benedetto, L. Bertini *A gradient flow approach to linear Boltzmann equation*, to appear on the Annali della Scuola Normale di Pisa - Classe di Scienze arxiv.org:1707.09204v2 (2018)

Slow thermalization in a chain of classical oscillators

François Huveneers

(joint work with Jani Lukkarinen)

Chains of oscillators in the thermodynamic limit are good models of crystal-like extended systems. In this talk, I consider a one-dimensional, classical, chain of harmonic oscillators, coupled by nearest neighbor harmonic couplings. In addition, an anharmonic pinning potential acts on each atom. In the absence of the latter, i.e. for a purely harmonic chain, the energy of each mode is conserved and the

system is integrable. Instead, as soon as the anharmonic potential is added, there is evidence that the energy is the only conserved quantity in the system. In the regime of weak anharmonic interactions, the process of thermalization is expected to take place on the so called kinetic time-scales.

In some cases, however, recent numerical results point towards thermalization on a much slower rate [1]. These observations do confirm the predictions from the Boltzmann-Peierls equation, but no hint is given on the actual time scales for thermalization. This phenomenon is due to the presence of a pseudo-conserved quantity that can be identified as the *number of phonons* in the chain, and that do not relax on the usual kinetic scale.

It is the aim of the talk to explain a possible mechanism behind this, and to provide predictions for the actual time scales of thermalization. Our theory is in line with the recent framework proposed in [2], though the case at hand does not properly falls in the class of systems analyzed there. Our predictions are backed up by numerical results.

References

[1] C. B. Mendl, J. Lu and J. Lukkarinen *Thermalization of oscillator chains with onsite anharmonicity and comparison with kinetic theory*, Physical Review E **94** (2016), 062104.

[2] K. Mallayya, M. Rigol and W. De Roeck, *Prethermalization and Thermalization in Isolated Quantum Systems*, Physical Review X **9** (2019), 021027.

Invariance Principle for the Random Lorentz Gas – Beyond the Boltzmann-Grad Limit

Christopher Lutsko

(joint work with Bálint Tóth)

In this report we discuss results concerning two classical models of diffusion. Foremost we consider the *random Lorentz gas*. That is, given parameters $r > 0$ and $\rho > 0$ let $\mathcal{P}$ be a Poisson point process of intensity ρ and consider the set of scatterers $\mathcal{B}_r^d + \mathcal{P}$, where $\mathcal{B}_r^d$ is the d-dimensional ball of radius r. The Lorentz process is then a process $t \mapsto X^{r,\rho}(t)$ which starts at the origin ($X^{(r,\rho}(0) = 0$) with a randomly chosen initial velocity and moves in straight lines with unit speed until there is a collision with a scatterer at which point the point particle collides elastically.

The randomness is introduced entirely in the initial condition: in the scatterer configuration and the initial velocity. Hence there are two relevant notions of convergence: The *Quenched Limit* - For a typical realization of the Poisson point process averaging over the initial velocity. The *Averaged-Quenched Limit* - Averaging over the initial velocity *and* the Poisson configuration. In both of these settings, a major open problem for this model is the validity of the invariance principle in the *diffusive limit*:

$$t \mapsto \frac{X^{r,\rho}(Tt)}{\sqrt{T}} \qquad \text{as } T \to \infty. \tag{1}$$

That is, to show that the process in (1) converges weakly (in either the quenched or averaged-quenched limit) to a Wiener process. See either [7] or [8] for a detailed survey. This question remains entirely open.

In the periodic setting (when scatterers are arranged on a lattice) this problem is relatively well-understood (see the surveys by Golse [3] or Marklof [5]). This owes to the fact that, one can use the machinary of hyperbolic dynamics, and reduce the problem to a problem about additive functionals of the billiard flow.

In the random setting, what progress there has been towards the invariance principle has been in the *Boltzmann-Grad limit*, that is

$$\rho \to \infty, \qquad r \to 0 \qquad \rho r^{d-1} \to C_d \tag{2}$$

where C_d is a constant depending on dimension chosen such that the mean free path length is 1 (for the Lorentz gas C_d is the volume of the $d-1$ dimensional unit ball). In this limit there are three results relevant to our work: Gallavotti [2] and Spohn [6] showed that the random Lorentz gas converges to a Markovian flight process (defined below) in the averaged-quenched limit (Spohn's result considers very general scatterer configurations and potentials) and Boldrighini, Bunimovich, and Sinai [1] who proved the same convergence in the quenched setting.

In a recent work [4], Bálint Tóth and the author proved the invariance principle in an intermediate scaling limit where the diffusive limit (1) is taken *simultaneously* with the Boltzmann-Grad scaling (2) in dimension 3. Specifically

Theorem 1 ([4, Theorem 2])**.** *In dimension $d = 3$. Let $T = T(r)$ be such that $\lim_{r\to 0} T(r) = \infty$ and $\lim_{r\to 0} r^2 |\log r|^2 T(r) = 0$. Then,*

$$\left\{ t \mapsto T^{-1/2} X^{r,\rho}(Tt) \right\} \Rightarrow \{ t \mapsto W(t) \}, \tag{3}$$

as $r \to 0$, in the averaged-quenched sense. On the right hand side of (3) *W is a standard Wiener process of variance* 1 *in $\mathbb{R}^3$.*

The proof is based on a novel coupling method: Given a sequence $\{\xi_n\}_{n=1}^{\infty}$ $EXP(1)$ of i.i.d exponential flight times and velocities $\{u_n\}_{n=1}^{\infty} \sim UNI(S^{d-1})$ define the Markovian flight process to be a process $\{t \mapsto Y(t)\}$ which begins at the origin ($Y(0) = 0$) and flies for time ξ_1 in the direction u_1 before adopting the velocity u_2 for time ξ_2 and so forth. Classically, the invariance principle holds for the Markovian flight process. Thus our proof will follow by coupling (jointly realizing on the same probability space) the Markovian flight process and the Lorentz process. Then we show that with high probability, on our time scales the distance $\sup_{t<T} |Y(t) - X^{r,\rho}(t)|$ is negligable compared to the $T^{1/2}$ coming from the diffusive scaling in (3).

The coupling is as follows: Given the random variables $\{\xi_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$, we construct the process $\{t \mapsto Y(t)\}$ as described. The Lorentz process then

begins at the origin and flies for time ξ_1 in direction u_1, then it adopts the velocity u_2. At this point we place a scatterer at the position corresponding to the velocity change u_1 and u_2. We then continue evolving the Lorentz process parallel to the Markovian process, placing the appropriate scatterers along the path. There are two occurances when this coupling is broken: if the Lorentz process encounters a previously placed scatterer (*recollision*) then it respects the collision, or if a given jump would correspond to a scatterer being placed over the old path (*shadowed collision*) this scattering event is ignored. In both these *mismatches* the Markovian flight process and the Lorentz process are no longer parallel, then at the next collision time (i.e the next time the Poisson clock rings) the two processes attempt to make the next jump in the sequence. Because of the Poisson configuration of scatterers, we recover the Lorentz process from this construction.

With that coupling, we implement the following argument: First we show that up till times of order $o(r^{-1})$, with high probability, mismatches will not occur. Then we show that on our time scales only geometrically 'simple' mismatches occur, and after a mismatch, recoupling is possible after an order $\mathcal{O}(1)$ time. These mismatches are isolated in time and altogether create a separation between the Markovian and Lorentz processes which is of order $o(T^{1/2})$. The log corrections in Theorem 1 do not appear in this simplified argument, however arise in the fine-tuning. Formalizing this argument requires estimates on Green's functions, a third auxilliary processes used to interpolate between X and Y, a decomposition of this auxilliary process, and precise estimates controlling the geometry of mismatches.

In a work currently in preparation, Bálint Tóth and the author showed that the invariance principle in Theorem 1 also holds for the random *wind-tree* process. That is, a similar model with the spheres replaced by cubes of side length r, all oriented along the axes. More precisely

Theorem 2 ((In preparation))**.** *In dimension $d = 3$. Let $T = T(r)$ be such that $\lim_{r\to 0} T(r) = \infty$ and $\lim_{r\to 0} r^2 T(r) = 0$. For any initial velocity $v = (v_1, v_2, v_3)$ such that $v_1, v_2, v_3 > 0$, let $\{t \mapsto \mathcal{X}^{r,\rho}(t)\}$ denote the random wind-tree process with initial velocity v. Then,*

$$\left\{t \mapsto T^{-1/2}\mathcal{X}^{r,\rho}(Tt)\right\} \Rightarrow \{t \mapsto W_v(t)\}, \tag{4}$$

as $r \to 0$, in the averaged-quenched sense. On the right hand side of (3) *W_v is a standard Wiener process with covariance matrix* $\mathrm{diag}(v_1^2, v_2^2, v_3^2)$ *in $\mathbb{R}^3$.*

Theorem 2 follows a similar proof with a few notable differences: firstly note that the log correction in Theorem 1 is not present in Theorem 2. This is because the log correction comes from those collisions where the angle between the entry velocity and the exit velocity are very small. For the wind-tree model the velocities are restricted to a finite set. Therefore this angle is bounded. Similarly the geometry of mismatches in the wind-tree model are far simpler which aids the calculations. Lastly, the velocity chain for the Markovian flight process associated to the wind-tree model is not i.i.d but a genuine Markov process. Therefore the coupling argument is slightly more complicated. In particular recoupling is trickier. That said, a similar strategy of proof holds.

References

[1] C. Boldrighini, L.A. Bunimovich, Y.G. Sinai, *On the Boltzmann equation for the Lorentz gas*, J. Stat. Phys. **32:** (1983) 477-501

[2] G. Gallavotti, Rigorous theory of the Boltzmann equation in the Lorentz gas, Nota Interna Univ di Roma **358** (1970)

[3] F. Golse: *The periodic Lorentz gas in the Boltzmann-Grad limit.* In: Proceedings of the International Congress of Mathematicians – 2006 Madrid **3** (2006), 183-201, European Mathematical Society, Zürich, 2006.

[4] C. Lutsko, B. Tóth, *Invariance principle for the random Lorentz gas - beyond the Boltzmann-Grad limit* arXiv:1812.11325 [math.PR] (2019).

[5] J. Marklof, *The low-density limit of the Lorentz gas: periodic, aperiodic and random.* In: Proceedings of the International Congress of Mathematicians – 2014 Seoul **3** (2014), 623-646, Kyung Moon Sa, Seoul.

[6] H. Spohn, *The Lorentz process converges to a random flight process*, Comm. Math. Phys. **60:** (1978) 277-290.

[7] H. Spohn, *Kinetic equations from Hamiltonian dynamics: the Markovian approximations*, Rev. Mod. Phys. **52:** (1980) 569-611

[8] H. Spohn, *Large Scale Dynamics of Interacting Particles*, Springer, Berlin-Heidelberg, 1991

On the derivation of linear kinetic equations from a Lorentz Gas with long-range interactions

Alessia Nota

(joint work with Sergio Simonella, Juan J.L. Velázquez)

During the last couple of decades, several important results have been achieved in the derivation of kinetic equations from particle systems. Most of the rigorous results available in this direction have been obtained for dilute gases and for systems with short-range interactions. The validity (or non-validity) of the Boltzmann equation in the case of long-range interaction potentials still represents an open problem. A first contribution in this direction has been proposed in [4]. The aim of this talk is to report on the recent results obtained in [4] as well as on ongoing progress on the problem of the derivation of kinetic equations from particle systems with long-range interactions. More precisely, in [4] we consider a simpler particle model where a single tagged particle moves in force fields produced by a random distribution of fixed scatterers, which is known in the literature as Lorentz Gas. We assume that the scatterers are distributed according to a Poisson measure with density of order one. Each scatterer yields a potential $Q_i\Phi(x-x_i)$ $x_1, x_2, \cdots$ in $\mathbb{R}^3$ with possibly different charges $Q_i \in \mathbb{R}$. We denote as $\omega = \left\{(x_n, Q_n)\right\}_{n\in\mathbb{N}} \in \Omega \otimes I$ a charged scatterer configuration.

We first study the properties of the random force field F associated to a potential Φ that decays at infinity as a power law $\Phi(x) \sim |x|^{-s}$ with $s > \frac{1}{2}$. Two physically relevant examples are the case of Newtonian or Coulombian potentials, corresponding to the case $s = 1$. More precisely, we assume that Φ belongs to the

class of interaction potentials

$$\mathcal{C}_s := \Big\{ \Phi \in C^2\left(\mathbb{R}^3 \setminus \{0\}; \mathbb{R}\right) \text{ s.t. } \Phi(x) = \Phi(|x|) \text{ and } \exists\, A \neq 0,\ r > \max(s,2)$$
$$\text{s.t. for } |x| \geq 1,\ \left|\Phi(x) - \frac{A}{|x|^s}\right| + |x| \left|\nabla\Big(\Phi(x) - \frac{A}{|x|^s}\Big)\right| \leq \frac{C}{|x|^r} \Big\}\,.$$

We first prove the existence of the limit stochastic force field

$$F(x;\omega) = \lim_{R\to\infty} F_U^{(R)}(x;\omega) := -\sum_{x_n \in RU} Q_{j_n} \nabla\Phi(x - x_n)\,,\ x \in \mathbb{R}^3,\ U \subset \mathbb{R}^3 \text{ open},$$

and identify the conditions under which this field is invariant under translations. We denote the limit field as "generalized Holtsmark field" (cf. [1, 2]) associated to $\Phi \in \mathcal{C}_s$. For instance, in the case of potentials decreasing for large $|x|$ as $|x|^{-s}$ with $s \leq 1$ an electroneutrality condition is required to define spatially homogeneous random force fields (cf. [4], Theorem 2.13). We notice that analogous results hold also in the case of time-dependent random force fields (cf. [5]).

The dynamics of the tagged particle is defined through the solution of Newton's equation of motion

$$\frac{dx}{dt} = v \quad , \quad \frac{dv}{dt} = F_\varepsilon(x;\omega) \tag{1}$$

where $F_\varepsilon(x;\omega)$ is the generalized Holtsmark field associated to the rescaled potential $\Phi(x,\varepsilon)$. Here $\varepsilon > 0$ is a small parameter tuning the mean free path ℓ_ε. Moreover, for a given scatterer configuration ω we will denote as $S_\varepsilon^{-t}(x,v;\omega)$ the Hamiltonian flow associated to (1).

In [4] we developed a formalism which allows to obtain the kinetic equation describing the evolution of a Lorentz Gas and to identify the conditions in the interactions which allow to obtain the correct Markovian approximation. More precisely, the correct scaling limits can be defined estimating the diffusive timescale T_ε. To have a meaningful definition of kinetic limit, the interaction between the tagged particle and the scatterers must be weak enough to ensure that the mean free path ℓ_ε is much larger than the typical distance among the scatterers d, i.e. $\ell_\varepsilon \gg d$. Furthermore, a second condition is the statistical independence of the particle deflections experienced over distances of order ℓ_ε. To estimate the kinetic time-scale T_ε, the potential Φ and the corresponding random force field, can be split in the sum of a Boltzmann part and a Landau part $\Phi(x,\varepsilon) = \Phi_B(x,\varepsilon) + \Phi_L(x,\varepsilon)$ where $\Phi_B(x,\varepsilon)$ is supported in a ball of radius $M\lambda_\varepsilon$, with M of order one but large, and $\Phi_L(x,\varepsilon)$ is supported at distances larger than $\frac{M\lambda_\varepsilon}{2}$. A similar decomposition can be made also in the case of arbitrary Rayleigh gases (cf. [5], see also [6] for a rigorous derivation of the Boltzmann equation for ideal Rayleigh gases). The parameter λ_ε is the collision length, i.e. the characteristic distance at which the tagged particle experiences deflections of order one due to the interaction with one of the scatterers by means of $\Phi_B(x,\varepsilon)$. Since the scatterers are distributed according to a Poisson repartition with finite density, the time between two such consecutive collisions is of order $\frac{1}{(\lambda_\varepsilon)^2}$ that we define

as Boltzmann-Grad time scale, i.e. $T_{BG} = \frac{1}{(\lambda_\varepsilon)^2}$. The time scale in which the deflections produced by Φ_L become relevant is then denoted as Landau time scale T_L. In most of the interaction potentials considered in [4] one of the two time scales is much larger than the other as $\varepsilon \to 0$. Due to the fact that the deflections are additive in the kinetic limit, we have

$$\ell_\varepsilon = T_\varepsilon \sim \min\{T_{BG}, T_L\} \gg 1 \text{ as } \varepsilon \to 0.$$

I present some examples of families of potentials and show how different linear kinetic evolution equation arise, for a given scaling, depending on the decay s as well as on the singularities of the potential. For instance, for potentials of the form $\Phi(x,\varepsilon) = \Psi\left(\frac{|x|}{\varepsilon}\right)$, $\Phi(\cdot,\varepsilon) \in \mathcal{C}_s$, $s > 1/2$, with $\Psi \in C^2\left(\mathbb{R}^3 \setminus \{0\}\right)$ and $\Psi(y) \sim \frac{A}{|y|^s}$, $\nabla\Psi(y) \sim -\frac{sAy}{|y|^{s+2}}$ with $A \neq 0$, as $|y| \to \infty$ we obtain that, under the kinetic rescaling, the microscopic solution

$$f_\varepsilon(\ell_\varepsilon t, \ell_\varepsilon x, v) = \mathbb{E}_\omega[f_0(S_\varepsilon^{-\ell_\varepsilon t}(\ell_\varepsilon x, v; \omega))]$$

as $\varepsilon \to 0$ satisfies:

• if $s > 1$ the linear Boltzmann equation

$$(\partial_t f + v \cdot \nabla_x f)(t,x,v) = \sum_{j=1}^{L} \mu(Q_j) \int_{S^2} B(v;\omega;Q_j)\left[f(t,x,|v|\,\omega) - f(t,x,v)\right] d\omega.$$

This is due to the fact that the fastest process yielding particle deflections are binary collisions with single scatterers, namely $T_{BG} \ll T_L$ and $T_{BG} = \frac{1}{\lambda_\varepsilon^2} = \frac{1}{\varepsilon^2}$.

• if $s = 1$ the linear Landau equation

$$(\partial_t f + v\partial_x f)(x,v,t) = \kappa \Delta_{v_\perp} f(x,v,t), \quad \kappa > 0.$$

In this case the deflections due to the accumulation of a large number of small interactions yield a relevant change in the direction of the velocity before a binary collision takes place. The time scale in which such macroscopic deflections take place is the Landau time $T_L \ll T_{BG}$ and $T_L \sim \frac{1}{\varepsilon^2 \log\left(\frac{1}{\varepsilon}\right)}$. Moreover, the deflections experienced by the particle over times of order T_L must be uncorrelated.

There are examples of potentials generating strongly correlated random fields for which this lack of correlations does not take place ($\frac{1}{2} < s < 1$). In such cases, a single kinetic equation cannot be recovered in the scaling limit, but a suitable stochastic differential equation with memory is expected since the correlations between macroscopic deflections must be taken into account.

Furthermore, we emphasize that the proof of the independence of the deflections is the crucial step towards any rigorous derivation of the kinetic equations. The main difficulty towards a rigorous validation of these linear kinetic equation is the fact that the very slow (non-integrable power law in the case of Coulombian potentials) decay of correlations of the generalized Holtsmark fields implies a lack of validity of the strong mixing condition at macroscopic level. The strong mixing condition is one of the key properties that allows to obtain a rigorous validation of the linear Landau equation in the case of weakly correlated random fields (e.g. [3]).

In this direction, I conclude presenting some ideas and perspectives on the ongoing project concerning the mathematically rigorous derivation of the linear Landau equation for a Lorentz Gas with Coulombian interactions ($s = 1$).

References

[1] S. Chandrashekar, *Stochastic Problems in Physics and Astronomy*, Rev. Mod. Phys. **15** (1943).

[2] J. Holtsmark, *Uber die Verbreiterung von Spektrallinien*, Annalen der Physik, **363**(7) (1919), 577–630.

[3] H. Kesten, G. Papanicolaou, *A limit theorem for stochastic acceleration*, Comm. Math. Phys. **78** (1981), 19–63.

[4] A. Nota, S. Simonella, J.J.L. Velázquez, *On the theory of Lorentz gases with long range interactions*, Rev. Math. Phys. **30**(3) (2018), 1850007

[5] A. Nota, J.J.L. Velázquez, R. Winter, *On the theory of kinetic equations for interacting particle systems with long-range interactions*, *Preprint (2019)*

[6] A. Nota, R. Winter, B. Lods, *Kinetic description of a Rayleigh Gas with annihilation*, J. Stat. Phys., **176**(6) (2019), 1434–1462

Deterministic walks in random environment

Carlangelo Liverani

(joint work with Romain, Aimino)

Motivated by the study of the Random Lorentz gas we introduce and start the study of a class of deterministic dynamical systems which abstract the properties of the Lorentz gas while allowing for simpler dynamics. Hence, they may serve as a starting point in developing new techniques.

Indeed, while many results exist for the periodic Lorentz gas (e.g. see [2, 5, 6] for the high density regime and [18] in the Boltzmann-Grad limit) the random Lorentz gas is understood in the Boltzmann-Grad limit (e.g. see [7, 17, 16]) while in the high density situation only very partial results are available (e.g. [3, 4, 8, 9, 10, 11, 19]). In particular, the problem of establishing a CLT is wide open and very few ideas seem to be on the table.

To forward our understanding we consider the random Lorentz gas as a special case of a *deterministic walk in random environment*.

To define a deterministic walk in random environment we consider, for simplicity, a nearest neighbourhood walk in $\mathbb{Z}^d$, but similar considerations hold for a walk on more general graphs $\mathcal{G}$. Let $\mathcal{A}$ be a finite set; X be a compact manifold, $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ a set of endomorphisms of X and $\{\mathcal{P}_\alpha\}_{\alpha\in\mathcal{A}}$ a set of partitions of X with $2d+1$ elements: $\mathcal{P}_\alpha = \{p_{\alpha,1}, \dots, p_{\alpha,2d+1}\}$

By random environment we mean a translation invariant probability measure $\mathbb{P}_e$ on $\Omega = \mathcal{A}^{\mathbb{Z}^d}$. We are interested in results that hold for $\mathbb{P}_e$-almost all $\omega \in \Omega$. By determinist walk in the environment $\boldsymbol{\omega} \in \Omega$ we mean the map $\mathbb{F}_{\boldsymbol{\omega}} : X\times\mathbb{Z}^d \to X\times\mathbb{Z}^d$

defined by

$$\mathbb{F}_{\boldsymbol{\omega}}(x,z) = (f_{\boldsymbol{\omega}_{z+e(\boldsymbol{\omega}_z,x)}}(x), z + e(\boldsymbol{\omega}_z, x))$$
$$e(\omega, x) = \sum_{i=1}^{2d+1} \mathbb{1}_{p_{\omega,i}}(x) w_i.$$

$w_i \in \{\pm e_1, \dots, \pm e_d, 0\}$ and the initial condition is random in the x variable.

In the case of the Lorentz gas the maps $f_{\boldsymbol{\omega}_z}$ are the Poincarè maps of the billiard at the site z of the lattice. Their main property is the hyperbolicity. It is then natural to start by considering a case in which the maps f_α, while still being hyperbolic, have a simpler dynamics, the simplest case being piecewise expanding maps.

In such a case we can prove that the deterministic walk can be reduced to a purely probabilistic model: a random walk (with memory) in random environment [1]. The key property being that the memory is short.

More precisely, the deterministic random walk can be reduced to a *Gibbs walk in random environment*:
there exists $\nu \in (0,1)$ such that, for each path $z(n-1), \dots, z(1)$ and for all random environments $\boldsymbol{\omega}$,

$$\mathbb{P}(z(n) \mid z(n-1), \dots, z(1), \boldsymbol{\omega}) = \mathbb{P}(z(n) \mid z(n-1), \dots, z(m), \boldsymbol{\omega}) + \mathcal{O}(\nu^{n-m}).$$

Gibbs walk in random environment are a very general model as they subsume persistent random walks in random environment, hence the problem to prove a CLT for such models is still out of reach. However, the Lorentz gas case has some special features, e.g. reversibility and an explicit invariant measure for the process as seen from the environment. It is thus conceivable that some results can be obtained for models with such extra properties. This would yield the wanted result provided that one could prove the reduction to a Gibbs random walk for some classes of Lorentz gasses. This should be possible using the techniques developed in [12, 13, 15, 14].

References

[1] R. Aimino, C. Liverani, Deterministic walks in random environment. Preprint ArXiv:1804.11114.

[2] L. A. Bunimovich, Y. G. Sinaĭ, and N. I. Chernov. Statistical properties of two-dimensional hyperbolic billiards. *Uspekhi Mat. Nauk*, 46(4(280)):43–92, 192, 1991.

[3] G. Cristadoro, M. Lenci, and M. Seri. Erratum: "Recurrence for quenched random Lorentz tubes" [Chaos 20, 023115 (2010)] [mr2741899]. *Chaos*, 20(4):049903, 1, 2010.

[4] G. Cristadoro, M. Lenci, and M. Seri. Recurrence for quenched random Lorentz tubes. *Chaos*, 20(2):023115, 7, 2010.

[5] D. Dolgopyat, D. Szász, and T. Varjú. Recurrence properties of planar Lorentz process. *Duke Math. J.*, 142(2):241–281, 2008.

[6] D. Dolgopyat, D. Szász, and T. Varjú. Limit theorems for locally perturbed planar Lorentz processes. *Duke Math. J.*, 148(3):459–499, 2009.

[7] G. Gallavotti: Divergencies and the approach to equilibrium in the Lorentz and the windtree models. Phys. Rev. 185: 308-322 (1969)

[8] M. Lenci. Aperiodic Lorentz gas: recurrence and ergodicity. *Ergodic Theory Dynam. Systems*, 23(3):869–883, 2003.
[9] M. Lenci. Typicality of recurrence for Lorentz gases. *Ergodic Theory Dynam. Systems*, 26(3):799–820, 2006.
[10] M. Lenci. Central limit theorem and recurrence for random walks in bistochastic random environments. *Journal of Mathematical Physics*, 49(12):125213, 2008.
[11] M. Lenci and S. Troubetzkoy. Infinite-horizon Lorentz tubes and gases: recurrence and ergodic properties. *Phys. D*, 240(19):1510–1515, 2011.
[12] C. Liverani. Decay of correlations. *Ann. of Math. (2)*, 142(2):239–301, 1995.
[13] C. Liverani. Decay of correlations for piecewise expanding maps. *Journal of Statistical Physics*, 78(3):1111–1129, 1995.
[14] C. Liverani and V. Maume-Deschamps. Lasota-Yorke maps with holes: conditionally invariant probability measures and invariant probability measures on the survivor set. *Ann. Inst. H. Poincaré Probab. Statist.*, 39(3):385–412, 2003.
[15] C. Liverani, B. Saussol, and S. Vaienti. Conformal measure and decay of correlation for covering weighted systems. *Ergodic Theory Dynam. Systems*, 18(6):1399–1420, 1998.
[16] Christopher Lutsko, Bálint Tóth *Invariance Principle for the Random Lorentz Gas – Beyond the Boltzmann-Grad Limit* Preprint arXiv:1812.11325.
[17] J. Marklof. The low-density limit of the lorentz gas: periodic, aperiodic and random. In *Proceedings of the ICM 2014, Seoul, Vol. III*, pages 623–646. 2004.
[18] J. Marklof and A. Strömbergsson. The Boltzmann-Grad limit of the periodic Lorentz gas. *Ann. of Math. (2)*, 174(1):225–298, 2011.
[19] M. Seri, M. Lenci, M. degli Esposti, and G. Cristadoro. Recurrence and higher ergodic properties for quenched random Lorentz tubes in dimension bigger than two. *J. Stat. Phys.*, 144(1):124–138, 2011.

Geometry of Maximal Disjoint Paths in Last Passage Percolation

Riddhipratim Basu

In this talk I presented some results obtained jointly with Christopher Hoffman and Allan Sly regarding rarity of disjoint geodesics in exponential last passage percolation together with some applications; and also briefly discussed some work in preparation with Shirshendu Ganguly, Alan Hammond and Milind Hegde.

Last passage percolation on $\mathbb{Z}^2$, with vertices are equipped with i.i.d. nonnegative passage times, is a canonical model of stochastic interface growth. We shall restrict our attention to the exactly solvable case where the weight are standard exponential random variables and several exact formulae are available. Let $\{\xi_v\}_{v\in\mathbb{Z}^2}$ is a family of i.i.d. Exp(1) variables. For any up/right path γ in $\mathbb{Z}^2$, the weight of γ, denoted $\ell(\gamma)$ is define by $\ell(\gamma) := \sum_{v\in\gamma}\xi_v$. For points $u, v \in \mathbb{Z}^2$ with u coordinate wise smaller than v, we define the *last passage time* from u to v, denoted $T(u, v)$ by

$$T(u, v) := \max_{\gamma:u\to v} \ell(\gamma)$$

where the maximum is taken over all up/right paths from u to v. The almost surely unique path between u and v that attains $T(u, v)$ will be called the geodesic between u and v. For notational convenience we shall denote by T_n (resp. $T_{m,n}$) the last passage time from $(1, 1)$ to (n, n) (resp. (m, n)). A semi-infinite (resp. bi-inifinte) geodesic is a semi-infinite (resp. bi-infinite) path γ such that the restriction

of γ between any two of the vertices $u, v \in \gamma$ is the geodesic between u and v. Clearly the horizontal and vertical lines are trivially bi-inifinite geodesics. We showed in [1] that almost surely these are the only ones.

Theorem 1. *For exponential LPP on $\mathbb{Z}^2$, almost surely there does not exist any non-trivial bi-infinite geodesic.*

The main technical challenge in the proof which, at a high level, follows a heuristic of Newman for first passage percolation is to understand the rarity of multiple disjoint geodesics. Let A denote the line segment joining $(-n^{2/3}, n^{2/3})$ and $(n^{2/3}, -n^{2/3})$ and let B denote the line segment $(n, n) + A$. Let L_n denote the maximum number of pairwise disjoint geodesics with one endpoint on A and the other on B. The following theorem, also obtained in [1] is a key estimate in the proof of Theorem 1.

Theorem 2. *There exist constants $c, \alpha > 0$ such that for all $k \in \mathbb{N}$ and all n sufficiently large we have*

$$\mathbb{P}(L_n \geq k) \leq e^{-ck^{\alpha}}.$$

Let us also explain what inputs from integrable probability go into the proofs of the above results. The primary input is the exact correspondence between $T_{m,n}$ and the largest eigenvalue of a certain Wishart random matrix. For $m \geq n$, let X denote an $m \times n$ matrix of i.i.d. standard complex Gaussian entries. It was shown by Johansson [3] that the largest eigenvalue $\lambda_{m,n}$ of X^*X has the same distribution as $T_{m,n}$. Using the concentration result for the largest eigenvalues in β-ensembles, developed in [4], this immediately implies the following.

Theorem 3. *There exists an absolute constant $c > 0$ such that for all $x \leq n^{2/3}$ we have*

$$\mathbb{P}(T_n - 4n \geq xn^{1/3}) \leq e^{-cx^{3/2}};$$

$$\mathbb{P}(T_n - 4n \leq -xn^{1/3}) \leq e^{-cx^3}.$$

This, and similar concentration bounds for $T_{m,n} - (\sqrt{m} + \sqrt{n})^2$, also available in [4], together with their consequences developed in [2] play a key role in the proofs of Theorem 2 and Theorem 1, not much more is utilized from the literature on exactly solvable models.

Another context where maximal disjoint paths naturally occur is that of the so-called polymer watermelon. For $k \in \mathbb{N}$, let the weight of a collection of k disjoint paths be defined by the sum of the individual path weights. Let $\Gamma_{n,k}$ denote the collection of k-disjoint paths contained in the square $[1, n]^2 \cap \mathbb{Z}^2$ with the maximum total weight; called the k-polymer watermelon. Using the interlacing property of such paths across different values of k, and arguments similar in spirit to those in Theorem 2 we have investigated (work in preparation jointly with Ganguly, Hammond and Hegde) the geometry of the k-polymer watermelon. I also discussed certain results involving the transversal fluctuations of the polymer watermelons briefly in my talk.

References

[1] R. Basu, C. Hoffman, and A. Sly, *Nonexistence of Bigeodesics in Integrable Models of Last Passage Percolation*, arXiv:1811.04908.
[2] R. Basu, V. Sidoravicius, and A. Sly, *Last Passage Percolation with a Defect Line and the Solution of the Slow Bond Problem*, arXiv:1408.3464.
[3] K. Johansson, *Shape Fluctuations and Random Matrices*, Comm. Math. Phys., **209(2)** (2000), 437–476.
[4] M. Ledoux, and B. Rider, *Small Deviations for Beta Ensembles*, Electron. J. Probab., **15** (2010), 1319–1343.

Nonexistence of bi-infinite geodesics in exponential last passage percolation - a probabilistic way

Márton Balázs
(joint work with Ofer Busani, Timo Seppäläinen)

Take a point on the 2-dimensional integer lattice and another one North-East from the first. Place i.i.d. Exponential weights on the vertices of the lattice; the point-to-point geodesic between the two points is the a.s. unique path of North and East steps that collects the maximal sum of these weights.

A bi-infinite geodesic is a doubly infinite North-East path such that any segment between two of its points is a point-to-point geodesic. We show that this thing a.s. does not exist (except for the trivial case of the coordinate axes). A first intuition is roughly this: transversal fluctuations of a point-to-point geodesic are in the order of the 2/3rd power of its length, which becomes infinite for a bi-infinite geodesic. Hence we cannot see this path anywhere near the origin which, combined with translation-invariance, a.s. excludes its existence.

A second thought, however, makes one wonder why geodesics from other pairs of points are not seen around the origin. Coalescence of paths must play an important role here, in some sense there is a small number of geodesics compared to the pair of points that have a chance of sending them around the origin.

One needs to make this more quantitative to prove that even after taking the union for all possible directions we cannot see a bi-infinite geodesic, a program sketched by Newman [5]. Georgiou, Rassoul-Agha and Seppäläinen [4] proved that a.s. no infinite geodesics occur in any *fixed* direction, which was a significant step in the field. The full argument has recently been completed rigorously by Basu, Hoffman and Sly [3] with inputs from integrable probability.

In this work [2] we also give a full proof, this time without any input from integrable probability. Instead, we build on purely probabilistic arguments, such as couplings and maxima of drifted random walks, to arrive to this result. There are three cases to consider.

- Close to the horizontal and vertical axes a relatively easy argument, built on the limiting shape of last passage percolation, rules out that even half-infinite geodesics differ from the trivial horizontal or vertical lines.

- When we consider non-diagonal points to connect a geodesic via the origin, we can bound the probability of this using upper bounds on path exit points from our early work [1]. This works when we require paths to turn slightly more than distance $N^{2/3}$ within a square of side length N.
- When we look at close to diagonal points across the origin, that is the turn in the path is less than $N^{2/3}$, then a new insight with coupling the path with stationary geodesics, and then using simple random walk estimates bounds the probability that the geodesic goes through the origin. At the core of the argument is a comparison of fluctuations vs. drift of random walks. On the short run fluctuations win, pushing the geodesic out from near the origin.

Luckily, the above probability bounds combine to $N^{-1/24}$ after a union bound for all pairs of points, and the proof can conclude by taking N to ∞.

References

[1] M. Balázs, E. Cator, and T. Seppäläinen. Cube root fluctuations for the corner growth model associated to the exclusion process. *Electron. J. Probab.*, 11:no. 42, 1094–1132 (electronic), 2006.

[2] Márton Balázs, Ofer Busani, and Timo Seppäläinen. Non-existence of bi-infinite geodesics in the exponential corner growth model. 2019. `arXiv:1909.06883`.

[3] Riddhipratim Basu, Christopher Hoffman, and Allan Sly. Nonexistence of bigeodesics in integrable models of last passage percolation. 2018. `arXiv:1811.04908`.

[4] Nicos Georgiou, Firas Rassoul-Agha, and Timo Seppäläinen. Geodesics and the competition interface for the corner growth model. *Probab. Theory Related Fields*, 169(1-2):223–255, 2017.

[5] Charles M. Newman. A surface view of first-passage percolation. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 1017–1023, Basel, 1995. Birkhäuser.

KPZ equation in $d \geq 3$ and construction of Gaussian multiplicative chaos in the Wiener space

Chiranjib Mukherjee

(joint work with O. Shamov and O. Zeitouni, with F. Comets and C. Cosco, and with Y. Bröker)

The KPZ equation in $d \geq 3$ and the weak disorder regime. The Kardar-Parisi-Zhang (KPZ) equation enjoys immense popularity as the default model of stochastic growth in $(d+1)$-dimensions. When $d = 1$, it has now seen a huge upsurge of interest in the recent years and a vast amount of deep mathematical results are available. On the other hand, despite being ill-posed for larger dimensions, the KPZ equation still remains relevant for random surface growth and has its own appeal even in the so-called small disorder regime – a distinguishing attribute of this equation in higher dimensions. While recent progress has also been made in $d = 2$, we will report on the results obtained pertinent to $d \geq 3$. The relevant equation is

$$\frac{\partial}{\partial t}h_\varepsilon = \frac{1}{2}\Delta h_\varepsilon + \left[\frac{1}{2}|\nabla h_\varepsilon|^2 - C_\varepsilon\right] + \beta\varepsilon^{\frac{d-2}{2}}\xi_\varepsilon \tag{1}$$

Here $\beta > 0$ is a parameter called the *disorder strength*, $\xi_\varepsilon = \xi \star \phi_\varepsilon$ is a spatially smoothened (at scale ε) Gaussian space-time white noise and $C_\varepsilon = O(\varepsilon^{-2})$ is a divergent constant as $\varepsilon \to 0$. When β is sufficiently small and $\varepsilon \to 0$, it was shown in [3] that $h_\varepsilon(t,x) - \mathbf{h}_\varepsilon(t,x) \to 0$ in probability, where $\mathbf{h}_\varepsilon(t,x) = \mathbf{h}(\xi^{\varepsilon,t,x})$ and $\mathbf{h}$ is a non-degenrate random variable, or the *limiting free energy* in the weak disorder regime and $\xi^{\varepsilon,t,x}$ stands for the diffusively rescaled, time-reversed and spatially translated white noise, which possesses the same law as that of ξ. It was also shown in [3] that $\mathbf{h}$ has *sub-Gaussian tails*, and thus possesses *all negative (and positive) moments* in this regime. In [4] we focus on the space-time fluctuation of the random field

(2)

$$\big(\mathcal{H}_\varepsilon(t,x)\big)_{x\in\mathbb{R}^d,t>0} \stackrel{\text{def}}{=} \big(\varepsilon^{1-\frac{d}{2}}[h_\varepsilon(t,x) - \mathbf{h}_\varepsilon(t,x)]\big)_{x\in\mathbf{R}^d,t>0} \stackrel{(d)}{\to} \big(\mathcal{H}^{\text{GFF}}(t,x)\big)_{x\in\mathbf{R}^d,t>0}$$

which is shown to converge to a centered Gaussian field

$$\mathcal{H}^{\text{GFF}}(t,x) = \gamma(\beta)\int_0^\infty \int_{\mathbf{R}^d} \rho(\sigma+t, y-x)\,\xi(\sigma,z)\mathrm{d}\sigma\,\mathrm{d}z$$

with $\rho(\sigma,x)$ being the standard heat kernel and $\gamma(\beta)$ being an explicit constant. The limiting process $\mathcal{H}^{\text{GFF}}$ is also the (real-valued) solution of the *non-noisy heat equation* $\partial_t\mathcal{H}^{\text{GFF}} = \frac{1}{2}\Delta\mathcal{H}^{\text{GFF}}$ with a *random initial condition* $\mathcal{H}^{\text{GFF}}(0,x)$ given by a *Gaussian free field* on $\mathbf{R}^d$. It is imperative to stress that the result (2) from [4] pertaining to space-time convergence is able to glean information on the *local fluctuations* of the ambient field, which remain totally impervious to recently studied global fluctuations *alone*, i.e., spatial averages of the form

$$\varepsilon^{1-d/2}\int_{\mathbf{R}^d}\varphi(x)[h_\varepsilon(t,x) - \mathbf{E}(h_\varepsilon(t,x))] \to \int_{\mathbf{R}^d}\mathrm{d}x\varphi(x)\mathcal{H}^{\text{EW}}(t,x) \tag{3}$$

that produce averages of the solution of the stochastic heat equation with *additive noise*, or the *Edwards-Wilkinson equation* $\mathcal{H}^{\text{EW}} = \frac{1}{2}\Delta\mathcal{H}^{\text{EW}} + \gamma(\beta)\xi$. On the other hand, as a special case of our main result, global fluctuations also dropped out in [4] as a corollary: For test functions $\varphi \in \mathcal{C}_c^\infty(\mathbf{R}^d)$ it was shown there that $\int_{\mathbf{R}^d}\mathrm{d}x\,\varphi(x)\ \varepsilon^{1-\frac{d}{2}}[h_\varepsilon(t,x) - \mathbf{h}_\varepsilon(t,x)] \stackrel{(d)}{\to} \int \mathrm{d}x\,\varphi(x))\mathcal{H}^{\text{GFF}}(t,x)$. Moreover, the latter result combined with the averaged Edwards-Wilkinson fluctuation, implied global fluctuations in [4]:

$$\varepsilon^{1-d/2}\int_{\mathbf{R}^d}\varphi(x)[\widehat{\mathbf{h}}_\varepsilon(t,x) - \mathbf{E}(\widehat{\mathbf{h}}_\varepsilon(t,x)] \to \int_{\mathbf{R}^d}\mathrm{d}x\varphi(x)[\mathcal{H}^{\text{GFF}}(t,x) + \mathcal{H}^{\text{EW}}(t,x)] \tag{4}$$

corresponding to the *stationary solution* $\widehat{\mathbf{h}}_\varepsilon(t,x) = \mathbf{h}(\xi^{\varepsilon,t,x})$ of the KPZ equation (1) itself – note that $\widehat{\mathbf{h}}_\varepsilon(t,x)$ is random, but is *constant in law* for any ε, t, x and $\mathbf{h}_\varepsilon(t,x)$ solves (3) with initial condition $\widehat{\mathbf{h}}_\varepsilon(0,x)$. To compare (3) and (4), note that the law of $h_\varepsilon(t,x)$ fluctuates wildly in the space variable x, hence $h_\varepsilon(t,x) -$

$\mathbf{E}(h_\varepsilon(t,x))$ diverges at each fixed x and only spatial averaging makes cancelations to happen. On the other hand, the law of $\widehat{\mathbf{h}}_\varepsilon(t,x)$ remains the same as ε, t or x vary. Thus, the correlation structure of the limiting field (2) is not shadowed by the spatial averaging in (4), being in contrast to (3).

The strong disorder regime and the Gaussian multiplicative chaos in the Wiener space: Suppose we are given a centered Gaussian field $\{H(x)\}_{x\in X}$ indexed by a metric space X carrying an ambient measure m. A Gaussian multiplicative chaos (GMC) is the transformed measure obtained by "exponentiating" the Gaussian field and considering its "renormalized" version:

$$\widehat{m}_\beta(\mathrm{d}x) = \exp\left\{\beta H(x) - \frac{\beta^2}{2}\mathbf{E}[X^2(x)]\right\} m(\mathrm{d}x) \qquad \beta > 0.$$

When $X = D \subset \mathbf{R}^2$ and the ambient field is given by the Guassian free field (i.e., the logarithmic Green's functions) the study of GMC has seen a lot of revived interest in the recent years due to its connection to the *Liouville quantum gravity*. A natural question is to define a GMC in *infinite dimensions*, which has a curious connection to the study of the KPZ equation (1) which was observed in [5]. Indeed, via the Cole-Hopf transform $z_\varepsilon = \exp[h_\varepsilon]$ and the Feynman-Kac formula, the multiplicative noise stohastic heat equation solution $z_\varepsilon \overset{(\mathrm{d})}{=} \mathbf{Z}_{\beta,T}$ *is equal in law* to the total mass of the Gaussian multiplicative chaos

$$\widehat{\mathbf{M}}_{\beta,T}(\mathrm{d}W) = \frac{1}{\mathbf{Z}_{\beta,T}} \exp\left\{\beta H_T(W) - \frac{\beta^2}{2}\mathbf{E}[H_T^2(W)]\right\} P_0(\mathrm{d}W)$$

defined w.r.t the spatially smoothened white noise field $H_T(W) = \int_0^T (\xi \star \phi)(W_s)$ with covariance $C_T(W,W') = \int_0^T (\phi\star\phi)(W_s - W'_s)$ (here ϕ is a fixed mollifier). Note that the GMC $\mathbf{M}_{\beta,T}$ is defined as a transformed measure w.r.t. the Wiener measure P_0 and the gaussian field $\{H_T(W)\}_W$ is indexed by Wiener paths. In $d \geq 3$ and for sufficiently small β, a *quenched central limit theorem* for the distribution of $\widehat{\mathbf{M}}_{\beta,T}(W_T/\sqrt{T})^{-1}$ was shown in [1] which implies a *total disintegration of mass for the endpoint distribution*: $\widehat{\mathbf{M}}_{\beta,T}(W_T \in \mathrm{d}x) \approx CT^{-d/2}$. On the other hand, for β large, by using the covariance bound $C_T(W,W') = \int_0^T (\phi\star\phi)(W_s - W'_s) \leq T(\phi\star\phi)(0)$ and using Kahane's comparison inequality it was shown in [5] that (for β large), the total mass $\mathbf{Z}_{\beta,T}$ of the GMC $\mathbf{M}_{\beta,T}$ loses uniform integrability and eventually collapses to zero. Subsequently, in [2] these assertions were strengthened and the above decay of the total mass $\mathbf{Z}_{\beta,T}$ was shown to be exponential and the decay rate was quantified exactly by a variational formula:

$$(5) \qquad \lim_{T\to\infty} \frac{1}{T}\log \mathbf{Z}_{\beta,T} = -\inf_{\nu\in\mathbf{m}\subset\widetilde{\mathcal{X}}} \mathcal{I}_\Phi(\nu) \qquad \text{where } \mathcal{I}_\Phi(\nu) = \int_{\widetilde{\mathcal{X}}} \Phi(\xi)\nu(\mathrm{d}\xi)$$

and

$$\Phi\colon \widetilde{\mathcal{X}} \to \mathbf{R}, \quad \Phi(\xi) = C - \sum_{\widetilde{\alpha}\in\xi} \int\int (\phi\star\phi)(x-y)\alpha(\mathrm{d}x)\alpha(\mathrm{d}y), \qquad \xi = \{\widetilde{\alpha}\} \in \widetilde{\mathcal{X}}.$$

Here

$$\widetilde{\mathcal{X}} = \Big\{\xi = \big\{\widetilde{\alpha}_j\big\}_j \colon\ \alpha_j \in \mathcal{M}_{\leq 1}(\mathbb{R}^d), \sum_j \alpha_j(\mathbf{R}^d) \leq 1\Big\}. \tag{6}$$

is the space of all empty, finite or countable collections of *orbits* $\widetilde{\alpha} = \{\alpha \star \delta_x \colon x \in \mathbb{R}^d\}$ of sub-probability measures $\alpha \in \mathcal{M}_{\leq 1}(\mathbb{R}^d)$ on $\mathbb{R}^d$ (its construction was introduced in [6]), $\mathbf{m}$ is the set of fixed points of certain Markvian dynamics (or invariant measures). It is imperative to note that the the theory pertaining to the space $\widetilde{\mathcal{X}}$ which is the compactification of the quotient space of orbits of probabilty measures (under the group action of $\mathbb{R}^d$) was developed in [6] and its construction implies important properties of this Markvian dynamics (e.g. Feller continuity, existence of invariant measures). The metric structure of $\widetilde{\mathcal{X}}$ also captures continuity properties of functionals like Φ (and therefore $\mathcal{I}_\Phi$). Based on this construction, it was shown in [2] that when β is large, the endpoint distribution $\widehat{\mathbf{Q}}_{\beta,T} := \widehat{\mathbf{M}}_{\beta,T}(W_T \in \cdot)$ is *asymptotically pure atomic*, i.e.

$$\lim_T \frac{1}{T}\int_0^T \delta_{\widehat{\mathbf{Q}}_{\beta,t}(U_{t,\varepsilon_t})}\mathrm{d}t = 1 \qquad \text{almost surely,}$$

for any $\varepsilon_t \downarrow 0$ and with $U_{t,\varepsilon} = \{x \in \mathbf{R}^d \colon \widehat{\mathbf{Q}}_{\beta,t}[B_1(0)] \geq c\varepsilon\}$ being the *islands* where the endpoint distribution puts uniformly positive density of mass. In other words, the endpoint distribution accummulates *all its mass* in these islands, and no mass disintegrates in the limit. Such an atomic picture of the endpoints on these islands signify, for large β, the departure from Gaussian universality class to the *non-Gaussian nature of the endpoint GMC distributions.*

References

[1] Y. Broeker and C. Mukherjee Central limit theorem for the quenched path measures for the continuous directed polymer in $d \geq 3$ in weak disorder *In: Friz P., König W., Mukherjee C., Olla S. (eds) Probability and Analysis in Interacting Physical Systems. VAR75 2016. Springer Proceedings in Mathematics and Statistics, vol 283. Springer, Cham*, arXiv: 1710.00631

[2] Y. Broeker and C. Mukherjee Localization of the Gaussian multiplicative chaos in the Wiener space and the stochastic heat equation, *To appear: Ann. Appl. Probab.* arXiv: 1808.05202

[3] F. Comets, C. Cosco and C. Mukherjee, Renormalizing the KPZ equation in $d \geq 3$ in weak disorder *Preprint*, arXiv: 1902.04104

[4] F. Comets, C. Cosco and C. Mukherjee, Space-time fluctuation of the KPZ equation in $d \geq 3$ and the Gaussian free field *Preprint*, arXiv: 1905.03200

[5] C. Mukherjee, A. Shamov and O. Zeitouni, Weak and strong disorder for stochastic heat equation and the continuous directed polymer in $d \geq 3$ *Elec. Comm. Prob.*, **21**, (2016), arXiv: 1601.01652

[6] C. Mukherjee and S.R.S. Varadhan, Brownian Occupation measures, compactness and large deviations, *Annals of Probability*, **44**, 3934-3964, (2016), arXiv: 1404.5259

A new Universality Class for interface models in (1 + 1)-dimensions: the Brownian Castle

Giuseppe Cannizzaro
(joint work with M. Hairer)

Randomly fluctuating interfaces in (1 + 1)-dimensions are described by maps $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ whose space-time evolution is driven by a stochastic forcing. In this context, two Universality Classes have generally been considered: the Kardar-Parisi-Zhang (KPZ) and the Edwards-Wilkinson (EW). The first was introduced in [4] and is presumed to include all those models that possess three features, a smoothing mechanism, slope-dependent growth speed and an underlying space-time locally uncorrelated noise. The scaling under which any of the height functions h within this class exhibits universal fluctuations is $1 : 2 : 3$, which amounts to say that in the limit as ε goes to zero, $\varepsilon^{1/2} h(\varepsilon^{-3/2} t, \varepsilon^{-1} x) - C_\varepsilon t$, where C_ε is a model-dependent diverging constant, converges to a universal stochastic process, the so-called KPZ Fixed Point (see [5]). If a model has the same features as above but does not display any slope dependence, then it belongs instead to the EW Universality Class [3], whose universal fluctuations, visible under the $1 : 2 : 4$ scaling, are Gaussian, and given by those of the solution to the linear Stochastic Heat Equation (SHE). A remarkable aspect of this story is that there appears to be a *unique* interface model on $\mathbb{R}_+ \times \mathbb{R}$ whose large scale behaviour is regulated by the first, while its small scale statistics are those of the second. This model is given by a singular SPDE, the KPZ Equation, which can be formally written as

$$\partial_t h = \nu \Delta h + \lambda (\partial_x h)^2 + \sqrt{D}\xi \tag{1}$$

where ξ is a space–time white noise and ν, λ, $D \in \mathbb{R}$ are non-zero constants, respectively describing the strength of the smoothing, slope-dependence and noise. Moreover, the solution to the KPZ equation is universal itself, as it arises as the limit of those models in which either the strength of the growth (intermediate disorder) or the fluctuation mechanism (weak asymmetry) dominate the dynamic. This corresponds to looking at large scales while simultaneously sending D, in the first case, or λ, in the second, to 0 at a suitable rate.

The point of departure for this work is the attempt to make sense of the previous picture in the context of the Ballistic Deposition (BD) model. In BD, particles fall on top of a surface according to independent Poisson processes of rate 1 and then stick to the first point they touch. Although very little is rigorously known, BD is conjectured to belong to the KPZ Universality Class, and, in order to explore the crossover between EW and KPZ, we needed to modify the definition of BD so to endow it with a tunable parameter β. Let us introduce the β-Ballistic Deposition (β-BD) whose dynamic can be described as follows. At each site the height function u changes according to independent rate 1 Poisson processes, and when a Poisson clock rings at x at time t then $u(t, x)$ takes the value of its left or right neighbour

with probability $r^\beta_{-1}(x)$, $r^\beta_1(x)$, respectively, or it increases by 1 with probability $r^\beta_0(x)$, where $\beta \geq 0$ and

$$(2) \qquad r^\beta_0(x) = C(x)e^\beta \qquad \text{and} \qquad r^\beta_i(x) := C(x)e^{\beta(u(x)-u(x+i))} \quad \text{for } i = -1, 1,$$

$C(x)$ been chosen in such a way that $\sum_i r^\beta_i(x) = 1$. Notice that for $\beta > 0$ the choice of the rates favours the maximum among the three possible values $u(t,\cdot)$ can take and for $\beta = +\infty$ it reduces to BD itself. This suggests that the large scale behaviour should be analogous, i.e. for any $\beta > 0$, one expects β-BD to belong to the KPZ Universality Class. On the other hand, for $\beta = 0$ all the outcomes are equally likely and since there is no preferred direction, one might be led to think that the process belongs to the EW Universality Class.

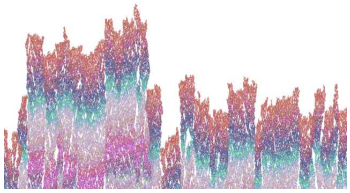

FIGURE 1. The Brownian Castle

As the simulation in Figure 1 shows, this cannot be the case since the large scale height function exhibits discontinuities absent in the EW Fixed Point (and in the KPZ one). This means that the object displayed is a *new* stochastic process that we call the Brownian Castle (BC), the name being due to the fact that the time evolution of the interface resembles the shape of a growing castle in which new towers are built, survive for a macroscopic time and then disappear.

In order to guess the scaling exponents and understand the main features of BC, it is convenient to step back and provide a description of 0-BD based on Harris graphical representation for interacting particle systems (see Figure 2). Let us fix a time horizon $T > 0$ ($T = 1$ in Figure 2) and consider $[0,T] \times \mathbb{Z}$. For all $y \in \mathbb{Z}$, draw between the lines $[0,T] \times \{y\}$ and $[0,T] \times \{y+1\}$ left and right arrows according to independent Poisson processes of intensity $\frac{1}{3}$ and, on top of the lines $[0,T] \times \{y\}$, draw dots according to a Poisson process of the same intensity. Then, it is immediate to see that, if we start with a function $u_0 : \mathbb{Z} \to \mathbb{N}$ at time 0 and let u evolve according to the rules of 0-BD, then, for every $x \in \mathbb{Z}$, $u(T,x)$ can be obtained by going downward following the arrows along the unique path connecting (T,x) to a point, say $(0,\bar{x})$, and summing $u_0(\bar{x})$ and all the dots we have met on the way from $(0,\bar{x})$ to (T,x), (the red paths in Figure 2). The family of paths connecting $\{T\} \times \mathbb{Z}$ to $\{0\} \times \mathbb{Z}$ is then a system of coalescing backward random walks on top of which we have a branching Poisson process that branches at the times the backward walks coalesce. Upon subtracting the average growth, i.e. compensating the (branching) Poisson process, we see that we need to rescale both space/time and height/time, diffusively so that the backward random

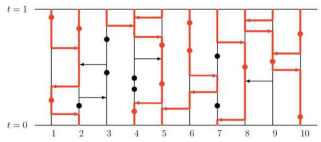

FIGURE 2. Harris graphical representation of 0-BD

walks converge to Brownian motions and the compensated Poisson process on top converges, again, to a Brownian motion. In other words, the scaling exponents should be given by $1:1:2$ as opposed to $1:2:3$ for KPZ or $1:2:4$ for EW.

Thanks to the previous representation we can actually say more, namely, we can characterise the finite dimensional distributions of the Brownian Castle, which can then be defined as follows.

Definition 1 (The Brownian Castle)**.** *Given $\mathfrak{h}_0 \in D(\mathbb{R},\mathbb{R})$, the Brownian Castle starting at $\mathfrak{h}_0$ is the process $\mathfrak{h}_{\rm bc} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ such that*

(1) $\mathfrak{h}_{\rm bc}(0,\cdot) = \mathfrak{h}_0$

(2) *for any $z_1,\dots,z_n \in (0,+\infty)\times\mathbb{R}$, the distribution of $(\mathfrak{h}_{\rm bc}(z_1),\dots,\mathfrak{h}_{\rm bc}(z_n))$ is that of $m(\leq n)$ independent branching Brownian motions starting at time 0 from $\mathfrak{h}_0(x_1),\dots,\mathfrak{h}_0(x_m)$, branching at the times n coalescing backward Brownian motions starting at $z_1,\dots,z_n$, coalesce.*

We would like to have a description of the Brownian Castle which goes beyond the finite dimensional distributions and allows to prove further properties the process enjoys. To do so, the previous definition suggests that BC is composed of two strongly interrelated parts, a family of coalescing Brownian motions starting from every point on $\mathbb{R}^2$ and a real-valued Gaussian process indexed by points in the trajectories of the coalescing family and whose covariance function is given by the time it takes two backward Brownian motions to coalesce. The first of these objects is called Brownian Web (BW) and was introduced and studied in [6], but in order to add the second on top of it, we provide a new construction of BW, describing it as a random spatial $\mathbb{R}$-tree in the language of [2]. Thanks to it we are able to show the following theorem (see [1]).

Theorem 1 (C.-Hairer, '19+)**.** *For every càdlàg funtion $\mathfrak{h}_0$, the Brownian Castle $\mathfrak{h}_{\rm bc}$ is right-continuous in time and càdlàg in space. Moreover,*

- *it admits no version which admits left limits (in time) in the Skorokhod space of càdlàg functions,*
- *it is invariant under the $1:1:2$ scaling*
- *it is Markov, strong Markov and Feller,*
- *modulo vertical translations, it admits an invariant measure to which, when restricted to the torus, it converges exponentially fast,*
- *the invariant measure is a càdlàg process whose two-point distribution is Cauchy, but whose law is singular with respect to that of the Cauchy process.*

References

[1] G. Cannizzaro, M. Hairer, *The Brownian Castle*, preprint.
[2] T. Duquesne, J.-F. Le Gall, *Probabilistic and fractal aspects of Lévy trees*, Probab. Theory Related Fields **131**(2005), no. 4, 553–603.
[3] S. Edwards, D. Wilkinson, *The surface statistics of a granular aggregate*, Proceedings of the Royal Society of London A **381**(1982), no. 1780, 17–31.
[4] M. Kardar, G. Parisi, and Y.-C. Zhang, *Dynamic Scaling of Growing Interfaces*, Phys. Rev. Lett. **56**(1986), no. 9, 889–892.
[5] K. Matetski, J. Quastel, D. Remenik, *The KPZ Fixed Point*, ArXiv e-prints, (2017).
[6] B. Tóth, W. Werner, *The true self-repelling motion*, Prob. Th. Rel. Fields **111**(1998), no. 3, 375–452.

Log-Sobolev inequality for the continuum Sine-Gordon model

Roland Bauerschmidt
(joint work with Thierry Bodineau)

We consider a probability measure on $\mathbb{R}^N$ with density written as

$$\nu_0(d\varphi) \propto e^{-H(\varphi)}\, d\varphi = e^{-\frac{1}{2}(\varphi, A\varphi) - V_0(\varphi)}\, d\varphi \tag{1}$$

where A is a positive definite $N \times N$ matrix. By the Bakry–Émery theorem [1],

$$A \geq \lambda \mathrm{id} \quad \text{and} \quad \mathrm{Hess} V_0(\varphi) \geq 0 \quad \text{as quadradic forms} \tag{2}$$

implies the Log-Sobolev inequality

$$\mathrm{Ent}_{\nu_0} F \leq \frac{2}{\lambda} \mathbb{E}_{\nu_0} (\partial \sqrt{F})^2. \tag{3}$$

We generalise this theorem to measures that are not required to be log-concave as follows. Write

$$A^{-1} = \int_0^\infty e^{-tA}\, dt, \quad C_t = \int_0^t \dot{C}_s\, ds, \quad \dot{C}_s = e^{-sA}, \tag{4}$$

and denote by E_{C_t} the Gaussian expectation with covariance C_t acting on a variable ζ. The *renormalised potential* $V_t(\varphi)$ is defined by

$$e^{-V_t(\varphi)} = \mathsf{E}_{C_t}(e^{-V_0(\varphi+\zeta)}). \tag{5}$$

Essentially equivalently, V_t solves the *Polchinski equation*

$$\partial_t V_t = \frac{1}{2} \Delta_{\dot{C}_t} V_t - \frac{1}{2} (\partial V_t)^2_{\dot{C}_t}, \tag{6}$$

where $(u, v)_w = \sum_{i,j} w_{ij} u_i v_j$, $(\partial F)^2_w = (\partial F, \partial F)_w$, and $\Delta_w F = (\partial, \partial)_w F$.

If V_0 is convex then V_t is convex for all $t \geq 0$. This follows for example from the Brascamp–Lieb inequality, but can also be seen from the maximum principle. In view of this fact, the our following theorem generalises the Bakry–Émery theorem.

Theorem [2]**.** *Suppose there are real numbers $\dot{\mu}_t$ (possibly negative) such that*

$$A \geq \lambda \mathrm{id}, \qquad Q_t \mathrm{Hess} V_t(\varphi) Q_t \geq \dot{\mu}_t \mathrm{id}, \qquad \textit{where } Q_t = e^{-tA/2}, \tag{7}$$

and define $\mu_t = \int_0^t \dot\mu_s\, ds$. *Then* ν_0 *satisfies the Log-Sobolev inequality*

$$\mathrm{Ent}_{\nu_0}(F) \le \frac{2}{\gamma}\mathbb{E}_{\nu_0}(\partial\sqrt{F})^2, \qquad \frac{1}{\gamma} = \int_0^\infty e^{-\lambda t - 2\mu_t}\, dt, \tag{8}$$

provided the integral is finite.

The proof of the theorem shares significant elements with the Bakry–Émery proof, but with the crucial difference that we do not use the canonical semigroup associated to the Dirichlet form to decompose the relative entropy, but instead a semigroup associated to the Polchinski equation defined as follows:

$$\boldsymbol{P}_{s,t}F(\varphi) = e^{V_t(\varphi)}\boldsymbol{E}_{C_t - C_s}(e^{-V_s(\varphi+\zeta)}F(\varphi+\zeta)), \tag{9}$$

where $\varphi \in X$, the expectation $\boldsymbol{E}_{C_t}$ applies to ζ. This is a time-dependent Markov semigroup (the *renormalisation semigroup*) with time-dependent generator

$$\boldsymbol{L}_t F = \frac{1}{2}\Delta_{\dot C_t} F - (\partial V_t, \partial F)_{\dot C_t}. \tag{10}$$

Defining the *renormalised measure* ν_t by

$$\mathbb{E}_{\nu_t} F = \boldsymbol{P}_{t,\infty}F(0) = e^{V_\infty(0)}\boldsymbol{E}_{C_\infty - C_t}(e^{-V_t(\zeta)}F(\zeta)), \tag{11}$$

one has $\mathbb{E}_{\nu_0}F = \mathbb{E}_{\nu_t}\boldsymbol{P}_{0,t}F$. From this, a computation shows that

$$\mathrm{E}nt_{\nu_0}(F) = \frac{1}{2}\int_0^\infty \mathbb{E}_{\nu_t}\frac{(\partial \boldsymbol{P}_{0,t}F)^2_{\dot C_t}}{\boldsymbol{P}_{0,t}F}\, dt = 2\int_0^\infty \mathbb{E}_{\nu_t}(\partial\sqrt{\boldsymbol{P}_{0,t}F})^2_{\dot C_t}\, dt. \tag{12}$$

To bound the right-hand side we proceed as in the Bakry–Émery argument, using that $\boldsymbol{P}_t$ has time-dependent generator $\boldsymbol{L}_t$. One difference compared to the standard proof is that the quadratic form also depends on time via $\dot C_t$. Interestingly, using that $\dot C_t$ satisfies the heat equation $\ddot C_t = -A\dot C_t$, this dependence interacts perfectly with the remainder of the argument.

As an application of this method to prove a Log-Sobolev inequality, we consider the continuum Sine–Gordon model. This probability measure is a fundamental example of non-Gaussian Euclidean Quantum Field Theory in two dimensions (see, e.g., [7, 3, 6]), and its Glauber dynamics is a singular SPDE recently considered using regularity structures [8, 5]. For both Glauber and Kawasaki dynamics, we apply the above criterion to obtain Log-Sobolev inequalities.

For clarity, we consider the model in a lattice approximation of a two-dimensional torus, and prove estimates uniformly in the lattice spacing and in the size of the torus. Therefore, from now on, let $d = 2$, let $\Omega_L = L\mathbb{T}^d$ be the torus of side length $L > 0$, and let $\Omega_{\epsilon,L} = \Omega_L \cap \epsilon\mathbb{Z}^d$ be its lattice approximation with mesh size $\epsilon > 0$ (where we always assume L is a multiple of ϵ). The continuum Sine-Gordon model $\nu_{\epsilon,L}$ in the lattice approximation is the probability measure on $\mathbb{R}^{\Omega_{\epsilon,L}}$ with density proportional to $e^{-H_{\epsilon,L}(\varphi)}$ where $H_{\epsilon,L}$ is defined for $\varphi : \Omega_{\epsilon,L} \to \mathbb{R}$ by

$$H_{\epsilon,L}(\varphi) = \frac{\epsilon^d}{2}\sum_{x\sim y\in\Omega_{\epsilon,L}}\frac{(\varphi_x-\varphi_y)^2}{\epsilon^2} + \epsilon^d\sum_{x\in\Omega_{\epsilon,L}}\left(\frac{m^2}{2}\varphi_x^2 + 2z\epsilon^{-\beta/4\pi}\cos(\sqrt{\beta}\varphi_x)\right), \tag{13}$$

and where the sum over $x \sim y$ is over all pairs of nearest neighbour vertices contained in $\Omega_{\epsilon,L}$. The divergent factor $\epsilon^{-\beta/4\pi}$ is a counterterm needed to obtain a non-Gaussian measure as $\epsilon \to 0$. Indeed, at least for $m^2 > 0$ and $z \neq 0$ small, this normalisation ensures that, for $0 < \beta < 8\pi$, the measures $\nu_{\epsilon,L}$ converge weakly to a non-Gaussian probability measure on $\mathcal{D}'(\mathbb{R}^2)$ as $\epsilon \to 0$ and $L \to \infty$.

Our first theorem is a uniform Log-Sobolev inequality for the Glauber dynamics of the massive Sine-Gordon measure $\nu_{\epsilon,L}$. The Glauber Dirichlet form is given by

$$\boldsymbol{D}_{\epsilon,L}(F) = \frac{1}{\epsilon^2} \sum_{x \in \Omega_{\epsilon,L}} \mathbb{E}_{\nu_{\epsilon,L}} \left(\left(\frac{\partial F}{\partial \varphi_x} \right)^2 \right), \tag{14}$$

corresponding to the system of SDEs

$$\frac{\partial}{\partial t}\varphi^\epsilon_x = (\Delta^\epsilon \varphi^\epsilon)_x + m^2 \varphi^\epsilon_x + \epsilon^{-\beta/4\pi} 2z\sqrt{\beta}\sin(\sqrt{\beta}\varphi^\epsilon_x) + \sqrt{2}\dot{W}^\epsilon_x. \tag{15}$$

.

Theorem [2]**.** *Fix $\beta < 6\pi$, and assume that $Lm \geq 1$ and that $|z|m^{-2+\beta/4\pi}$ is sufficiently small. There is $\gamma_\epsilon(\beta, z, m, L) > 0$ such that, for all $F \geq 0$,*

$$\mathrm{Ent}_{\nu_{\epsilon,L}}(F) \leq \frac{2}{\gamma_\epsilon(\beta, z, m, L)} \boldsymbol{D}_{\epsilon,L}(\sqrt{F}). \tag{16}$$

The limiting Log-Sobolev constant $\gamma(\beta, z, m, L) = \liminf_{\epsilon \downarrow 0} \gamma_\epsilon(\beta, z, m, L)$ satisfies

$$\gamma(\beta, z, m, L) \geq m^2 + O_\beta(m^{\beta/4\pi} z). \tag{17}$$

The constant $O_\beta(z)$ is uniform in $L \geq 1/m$.

There is also a version of this theorem for (conservative) Kawasaki dynamics (see [2]). Our proof relies on the approach to the Sine-Gordon model of Brydges–Kennedy [4]. The starting point is the representation of V_t as a Fourier series

$$V(\varphi) = \sum_{n=0}^{\infty} V^{(n)}(\varphi), \qquad V^{(n)}(\varphi) = \frac{1}{n!} \sum_{\xi_1,\dots,\xi_n} \tilde{V}^{(n)}(\xi_1, \dots, \xi_n) e^{i\sqrt{\beta}\sum_{i=1}^n \varphi_{x_i}\sigma_i} \tag{18}$$

where $\tilde{V}^{(n)} : (\Lambda \times \{\pm 1\})^n \to \mathbb{R}$ and

$$\xi_i = (x_i, \sigma_i) \in \Lambda \times \{\pm 1\}. \tag{19}$$

We think of ξ_i as a particle with position x_i and charge σ_i. The Duhamel formula gives a formulation of the Polchinski equation as a system of integral equations for the coefficients $\tilde{V}^{(n)}$. These can be estimated inductively to obtain strong estimates for V_t. This analysis is simplest for $\beta < 4\pi$ and requires more careful treatment of $V^{(n)}$ for $n = 2, 3, 4$ when $4\pi \leq \beta < 6\pi$.

References

[1] D. Bakry and M. Émery. Diffusions hypercontractives. In *Séminaire de probabilités, XIX, 1983/84*, volume 1123 of *Lecture Notes in Math.*, pages 177–206. Springer, Berlin, 1985.

[2] R. Bauerschmidt and T. Bodineau. Log-Sobolev inequality for the continuum Sine-Gordon model. Preprint, arXiv:1907.12308.

[3] G. Benfatto, G. Gallavotti, and F. Nicolò. On the massive sine-Gordon equation in the first few regions of collapse. *Comm. Math. Phys.*, 83(3):387–410, 1982.
[4] D.C. Brydges and T. Kennedy. Mayer expansions and the Hamilton-Jacobi equation. *J. Statist. Phys.*, 48(1-2):19–49, 1987.
[5] A. Chandra, M. Hairer, and H. Shen. The dynamical sine-Gordon model in the full subcritical regime. Preprint, arXiv:1808.02594.
[6] J. Dimock and T.R. Hurd. Construction of the two-dimensional sine-Gordon model for $\beta < 8\pi$. *Comm. Math. Phys.*, 156(3):547–580, 1993.
[7] J. Fröhlich and E. Seiler. The massive Thirring-Schwinger model (QED_2): convergence of perturbation theory and particle structure. *Helv. Phys. Acta*, 49(6):889–924, 1976.
[8] M. Hairer and H. Shen. The dynamical sine-Gordon model. *Comm. Math. Phys.*, 341(3):933–989, 2016.

Disordered pinning for the Gaussian Free Field

Hubert Lacoin
(joint work with Giambattista Giacomin)

We shortly review and present new results concerning the localization transition of a lattice free field $\phi = (\phi(x))_{x\in\mathbb{Z}^d}$, $d \geq 1$ in presence of a quenched disordered substrate that interacts on the interface's sites whose height is close to zero. This corresponds to a Hamiltonian

$$\sum_{x\in\mathbb{Z}^d} (\beta\omega_x - \lambda(\beta) + h)\delta_x, \tag{1}$$

where $\delta_x = \mathbf{1}_{[-1,1]}(\phi(x))$, $(\omega_x)_{x\in\mathbb{Z}^d}$ is an IID centered field and $\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega}]$ is the log-laplace transform of ω_x (which we assume to be finite for every β. A transition takes place when the average pinning potential h goes past a threshold h_c: from a delocalized phase $h < h_c$, where the field is macroscopically repelled by the substrate to a localized one $h > h_c$ where the field sticks to the substrate.

The localization transition can be identified by the study of the free-energy (whose existence in the disordered was estiablished in [6]). If $\mathbf{E}_N$ denote the distribution of the lattice free field in $[\![0, N]\!]^d$, it is defined by

$$\mathrm{F}(\beta, h) := \lim_{N\to\infty} \frac{1}{N^2} \log \mathbf{E}_N \left[e^{\sum_{x\in\mathbb{Z}^d} (\beta\omega_x - \lambda(\beta) + h)\delta_x} \right].$$

The critical point for the phase transition is given by

$$h_c(\beta) := \inf\{h \ : \ \mathrm{F}(\beta, h)\} > 0.$$

A simple annealing compution shows that $\mathrm{F}(\beta, h) \leq \mathrm{F}(0, h) =: \mathrm{F}(h)$ and hence that $h_c(\beta) \geq h_c(0)$. An interesting question is to know whether the characteristics of the disordered model are similar to the one observed for $\beta = 0$ (for which the critical behavior was identified in [1].

The case of dimension 1 was extensively studied and it is known that in this case $h_c(\beta) > 0$ and that the phase transition is at least quadratic (see e.g. the introduction of [2] for an extensive review). When $d = 2$ it was shown in [5] that

$h_c(\beta) = 0$ but that the transition is of infinite order (in the sense that for every $K \geq 2$, we have $\mathrm{F}(\beta, h) \leq C_K h^K$ for every $h \geq 0$.

Our talk focus on the case $d \geq 3$ which was first investigated in [3] and for which we obtained much more detailed results in [4]. First we obtain a sharp asymptotic of the free-energy

$$\lim_{h \searrow 0+} \frac{\mathrm{F}(\beta, h)}{h^2} = \frac{1}{2\mathrm{Var}[e^{\beta\omega - \lambda(\beta)}]} \tag{2}$$

Second, we show that in the limit when $h \to 0+$ most points are located at height $\sqrt{2\pi}\sigma_d\sqrt{\log(1/h)} + o(\sqrt{\log(1/h)})$ in absolute value where σ_d is the standard deviation of the lattice free field on $\mathbb{Z}^d$.

References

[1] E. Bolthausen and Y. Velenik, *Critical behavior of the massless free field at the depinning transition*, Comm. Math. Phys. **223** (2001), 161–203.

[2] G. Giacomin, *Disorder and critical phenomena through basic probability models*, École d'été de probablités de Saint-Flour XL-2010, Lecture Notes in Mathematics **2025**, Springer, 2011.

[3] G. Giacomin and H. Lacoin, *Pinning and disorder relevance for the lattice Gaussian free field*, J. Eur. Math. Soc. (JEMS) **20** (2018), 199–257.

[4] G. Giacomin and H. Lacoin, *Free energy and typical path behavior for disordered pinning of the Gaussian Free Field pinning*, (in preparation).

[5] H. Lacoin, *Pinning and disorder for the Gaussian free field II: the two dimensional case*, to appear in Ann. Sci. ENS.

[6] L. Coquille and P. Miłoś, *A note on the discrete Gaussian free field with disordered pinning on $\mathbb{Z}^d$, $d \geq 2$*, Stoch. Proc. and Appl. **123** (2013) 3542–3559.

From particles to fluids: study of correlations

Isabelle Gallagher

(joint work with Thierry Bodineau, Laure Saint-Raymond, Sergio Simonella)

Consider a system of $N \geq 1$ spheres of diameter $\varepsilon > 0$ in the d-dimensional space $\mathbb{T}^{dN}$ with $d \geq 2$. The positions and velocities of the particles are denoted respectively by $X_N := (x_1, \ldots, x_N) \in \mathbb{T}^{dN}$ and $V_N := (v_1, \ldots, v_N) \in \mathbb{R}^{dN}$, and we set $Z_N := (X_N, V_N)$. The positions and velocities of the particles satisfy Newton's laws

$$\frac{dx_i}{dt} = v_i\,, \quad \frac{dv_i}{dt} = 0 \quad \text{as long as } \ |x_i(t) - x_j(t)| > \varepsilon \quad \text{for } \ 1 \leq i \neq j \leq N\,,$$

with specular reflection at a collision. The probability density W_N^ε of finding N hard spheres of diameter ε at configuration Z_N at time t is governed by the Liouville equation in the $2dN$-dimensional phase space

$$\partial_t W_N^\varepsilon + V_N \cdot \nabla_{X_N} W_N^\varepsilon = 0 \quad \text{on} \quad \mathcal{D}_N^\varepsilon\,, \tag{1}$$

where $\mathcal{D}_N^\varepsilon := \left\{ Z_N \in \mathbb{R}^{2dN} \,/\, \forall i \neq j\,,\, |x_i - x_j| > \varepsilon \right\}$, and with specular reflection on the boundary. Given a continuous probability distribution f^0, exponentially

decaying in the velocity variable, the initial probability density is defined on the configurations (N, Z_N) as

$$\frac{1}{N!}W_N^{\varepsilon,0}(Z_N) := \frac{1}{\mathcal{Z}^\varepsilon}\,\frac{\mu_\varepsilon^N}{N!}\prod_{i=1}^N f^0(z_i) \quad \text{on } \mathcal{D}_N^\varepsilon \tag{2}$$

where $\mathcal{Z}^\varepsilon$ is a the normalization constant. Note that the particles are "exchangeable", in the sense that $W_N^{\varepsilon,0}$ is invariant by permutation of the particle labels in its argument. The total number of particles $\mathcal{N}$ is random and one can check that for $\mu_\varepsilon = \varepsilon^{-(d-1)}$

$$\lim_{\varepsilon\to 0}\mathbb{E}_\varepsilon\left(\mathcal{N}\right)\varepsilon^{d-1} = 1\,, \tag{3}$$

where $\mathbb{E}_\varepsilon$ stands for the expectation with respect to the measure (2). The limit (3) ensures that the *Boltzmann-Grad scaling* holds, i.e. that the inverse mean free path is of order 1. Thus we choose $\mu_\varepsilon = \varepsilon^{-(d-1)}$. The (rescaled) n-particle correlation function at time $t \geq 0$ is defined on $\mathcal{D}_n^\varepsilon$ as

$$F_n^\varepsilon(t, Z_n) := \mu_\varepsilon^{-n}\sum_{p=0}^\infty \frac{1}{p!}\int dz_{n+1}\dots dz_{n+p}\,\mathbf{1}_{\mathcal{D}_{n+p}^\varepsilon}(Z_{n+p})\,W_{n+p}^\varepsilon(t, Z_{n+p})$$

and we have

$$\mathbb{E}_\varepsilon\Big(\sum_{\substack{i_1,\dots,i_n\\ i_j\neq i_k, j\neq k}}\psi\big(z_{i_1}(t),\dots,z_{i_n}(t)\big)\Big) = \mu_\varepsilon^n\int_{\mathcal{D}_n^\varepsilon} dZ_n\,F_n^\varepsilon(t, Z_n)\,\psi\big(Z_n\big)\,.$$

Lanford's theorem [2] states that in the Boltzmann-Grad limit $\mu_\varepsilon \to \infty$, the rescaled one-particle density $F_1^\varepsilon(t)$ converges uniformly to the solution $f(t)$ of the Boltzmann equation

$$\begin{aligned}
&\partial_t f + v\cdot\nabla_x f = Q(f,f)\,,\\
&Q(f,f)(t,x,v) := \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\Big(f(t,x,w')f(t,x,v') - f(t,x,w)f(t,x,v)\Big)\\
&\qquad\qquad\qquad\qquad\times\left((v-w)\cdot\nu\right)_+ d\nu\,dw\\
&f(0,x,v) = f^0(x,v)
\end{aligned} \tag{4}$$

where the precollisional velocities (v', w') are defined by the scattering law

$$v' := v - (v-w)\cdot\nu\,\nu\,, \qquad w' := w + (v-w)\cdot\nu\,\nu\,,$$

on a time interval $[0, T^\star]$ which depends only on the size of f_0 in L^∞, and its rate of exponential decay. Furthermore for each n, the rescaled n-particle correlation function $F_n^\varepsilon(t)$ converges almost everywhere to $f^{\otimes n}(t)$ on the same time interval. This result can be seen as a law of large numbers, and we are interested in fluctuations. We therefore define the fluctuation field ζ^ε as follows for any test function h

$$\zeta_t^\varepsilon\big(h\big) := \sqrt{\mu_\varepsilon}\big(\pi_t^\varepsilon(h) - \int F_1^\varepsilon(t,z)\,h\big(z\big)\,dz\big)\,,$$

with π_t^ε the empirical measure at time t

$$\pi_t^\varepsilon(\psi) := \frac{1}{\mu_\varepsilon} \sum_{i=1}^{\mathcal{N}} \psi\left(z_i(t)\right) .$$

Initially π_0^ε starts close to the density profile f^0 and the initial fluctuation ζ_0 is a Gaussian white noise with covariance

$$\mathbb{E}\left(\zeta_0(h_1)\,\zeta_0(h_2)\right) = \int h_1(z)\, h_2(z)\, f^0(z)\, dz\,.$$

We prove that in the limit $\mu_\varepsilon \to \infty$, starting from "almost independent" hard spheres, ζ^ε converges to the generalized Ornstein-Uhlenbeck process

$$d\zeta_t = \mathcal{L}_t\, \zeta_t\, dt + d\eta_t\,, \tag{5}$$

where $\mathcal{L}_t$ is the *linearized Boltzmann operator* around the solution $f(t)$ of the Boltzmann equation (4)

$$\mathcal{L}_t\, h(x,v) := -v\cdot\nabla_x h(x,v) + \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} d\nu\, dw\, \left((v-w)\cdot\nu\right)_+ \\ \times \left(f(t,x,w')h(x,v') + f(t,x,v')h(x,w') - f(t,x,v)h(x,w) - f(t,x,w)h(x,v)\right),$$

and $d\eta_t(x,v)$ is a Gaussian noise with zero mean and covariance

$$\mathbb{E}\left(\int dt\, dz_1 h_1(z_1) d\eta_t(z_1) \int ds\, dz_2\, h_2(z_2) d\eta_s(z_2)\right) \\ = \frac{1}{2}\int dt d\mu(z_1,z_2,\omega) f(t,z_1)\, f(t,z_2) \Delta h_1\, \Delta h_2$$

denoting $d\mu(z_1,z_2,\omega) := \delta_{x_1-x_2}\left((v_1-v_2)\cdot\omega\right)_+ d\omega\, dv_1\, dv_2 dx_1 dx_2$ and defining for any h, $\Delta h(z_1,z_2,\omega) := h(z_1') + h(z_2') - h(z_1) - h(z_2)$ with $z_i' := (x_i, v_i')$.

Our result is the following. *Consider a system of hard spheres initially distributed according to the grand canonical measure (2). Assume f^0 is a Lipschitz function, with Gaussian decay in velocities. Then, in the Boltzmann-Grad limit $\mu_\varepsilon \to \infty$, the fluctuation field $(\zeta_t^\varepsilon)_{t\geq 0}$ converges in law to the the Ornstein-Uhlenbeck process $(\zeta_t)_{t\geq 0}$ defined in (5) on a time interval $[0,T^\star]$, where $T^\star$ only depends on the size of f_0 in L^∞, and on its rate of exponential decay.*

The convergence towards the limiting process (5) was conjectured by Spohn and the non-equilibrium covariance of the process at two different times was computed in [3]. The noise emerges after averaging the deterministic microscopic dynamics. It is white in time and space, but correlated in velocities so that momentum and energy are conserved.

The result is obtained by a careful study of the cumulants associated with the family F_N^ε: denoting $\mathcal{P}_n^s$ the set of parts of $\{1,\dots,n\}$ with s elements, we define

$$f_n^\varepsilon(Z_n) := \sum_{s=1}^{n}\sum_{\sigma\in\mathcal{P}_n^s} (-1)^{s-1}(s-1)!\, F_\sigma^\varepsilon\,, \quad F_{\sigma_j}^\varepsilon := F_{|\sigma_j|}^\varepsilon(Z_{\sigma_j})\,, \quad F_\sigma^\varepsilon := \prod_{j=1}^{|\sigma|} F_{\sigma_j}^\varepsilon\,.$$

By iterating the Duhamel formula for F_N^ε, in the spirit of Lanford's proof, we are able to prove that the cumulant f_n^ε is localized on particular (pseudo)-trajectories involving at least $n-1$ recollisions or overlaps between collision trees, and we can compute precisely their size: the cumulant of order n is of size $O(\mu_\varepsilon^{-(n-1)})$ in L^1. This enables us to obtain limiting equations on the cumulants, to prove that only the first two are involved in the fluctuation field, and their precise dynamics leads to the equation on the limiting fluctuation field. We refer to [1] for more.

References

[1] T. Bodineau, I. Gallagher, L. Saint-Raymond and S. Simonella, *Stastitical Dynamics of a Hard-Sphere Gas: Central Limit Theorem and Large Deviation Principle*, in progress.

[2] O. E. Lanford. Time evolution of large classical systems. In "Dynamical systems, theory and applications", *Lecture Notes in Physics*, ed. J. Moser, **38** (1975), Springer–Verlag, Berlin.

[3] H. Spohn. Fluctuations Around the Boltzmann Equation. *Journal Stat. Phys.* **26** (1981) 2.

Correlation bounds in collisional dynamics

Sergio Simonella

(joint work with Thierry Paul, Mario Pulvirenti)

This talk presents a recent result in collaboration with T. Paul and M. Pulvirenti [1] on a quantitative evaluation of the propagation of chaos for a class of models including the Kac model for the Boltzmann equation, and large quantum particle systems in the mean field regime.

Motivated by the Grad limit of hard spheres (see the companion talk by Isabelle Gallagher in the same workshop) we base our analysis on the hierarchical method, which is the most systematic approach to the quantification chaos. We show that, applying this approach to the (much easier) case of an arbitrary mean field process with bounded transport and two-body collision operator, the typical difficulties of BBGKY expansions preventing sharp bounds are simply resolved, by introducing suitable *correlation errors* (i. e. "truncated functions") which we export from previous work on the low-density gas [2].

Let the state of the system be given by a symmetric probability measure $f^N = f^N(t, Z_N)$, where $Z_N = (z_1, \cdots, z_N)$ is a configuration of particles. We assume that $f^N(0) = (f_0)^{\otimes N}$. Denoting by $\left(f_j^N = f_j^N(t, Z_j)\right)_{j\geq 1}$ the corresponding marginals, the hierarchy tells that the evolution of f_1^N depends on f_2^N, which depends on f_3^N and so on. At the level of Duhamel iterated series, this gives rise to a branching process with the number of particles proliferating backward in time.

The dynamics creates small correlations and even if at time zero the variables $(z_1, \cdots, z_N)$ are assumed to be independent, the statistical independence is lost at positive times. Our purpose is to prove that $\Delta_j^N(t) := f_j^N(t) - f(t)^{\otimes j} \to 0$ where $f(t)$ is the solution to an effective equation, and quantify the rate.

More precisely, consider the quantity

$$\mathcal{S} := \sup\left\{\alpha \geq 0 \;\middle|\; \lim_{N\to\infty} \sup_{j<N^\alpha} \|\Delta_j^N(t)\|_{L^1} = 0\right\}.$$

At fixed time $t > 0$, $\mathcal{S}$ gives a sharp information on the size of chaos. Roughly, it determines the maximum number of particles which behave as statistically independent when $N \to \infty$.

The BBGKY iteration leads to a simple bound $\|\Delta_j^N(t)\|_{L^1} \leq C^j N^{-\gamma}$, $C, \gamma > 0$ for fixed time $t > 0$. This provides no information on $\mathcal{S}$. The expected result $\|\Delta_j^N(t)\|_{L^1} \leq Cj^2/N$ can be however recovered by replacing the family $\left(f_j^N\right)_{j\geq 1}$ with the family of correlation errors $\left(E_j^N\right)_{j\geq 1}$ introduced as follows.

Suppose that for a moderately large j, say $j \approx N^\alpha$, $\alpha \in (0,1)$, $f_j^N(t) \approx (f_1^N(t))^{\otimes j}$. Then it is tempting to replace $f_j^N(t) - f(t)^{\otimes j}$ with $(f_1^N(t) - f(t))^{\otimes j}$, for which we expect much better decay. Expanding the latter quantity we find

$$\sum_{K\subset J} (-1)^{|K|} f_{J\setminus K}^N(t) f(t)^{\otimes K} =: E_j^N(t) \tag{1}$$

where $J = \{1, 2, \cdots, j\}$, $|K|$ = is the cardinality of K, $f_A^N(t)$ stands for the $|A|$-marginal $f_{|A|}^N(t)$ computed in the configuration $\{z_i\}_{i\in A}$, and $f(t)^{\otimes K} = f(t)^{\otimes |K|}$ evaluated in $\{z_i\}_{i\in K}$ (with the conventions $f_\emptyset^N = f(t)^{\otimes \emptyset} = 1$).

Technically we treat this again by a crude Duhamel expansion. Now the family $\left(E_j^N\right)_{j\geq 1}$ satisfies a hierarchy of equations where the evolution of E_j^N depends on E_{j+1}^N, but *also* on E_{j-1}^N, E_{j-2}^N. At variance with the BBGKY, where the collision operator creates one extra particle, the new hierarchy includes annihilation operators. As a consequence, the branching process associated to the BBGKY is replaced by a random walk on the number of particles, with positive and negative jumps. Since $E_j^N(0) = \delta_{j,0}$, such a random walk is killed at the origin and the series expansion is virtually finite. This leads to the improved rate $\|E_1^N(t)\|_{L^1} \leq C/N$ and

$$\|E_j^N(t)\|_{L^1} \leq \left(\frac{Cj}{\sqrt{N}}\right)^j , \qquad j \geq 2$$

for some positive C. In particular (after inverting formula (1)) it follows that

$$\|\Delta_j^N(t)\|_{L^1} \leq \frac{C}{N} + \sum_{k=2}^{j} \binom{j}{k} \left(\frac{Ck}{\sqrt{N}}\right)^k \leq C\frac{j^2}{N} .$$

References

[1] T. Paul, M. Pulvirenti and S. Simonella. *On the Size of Chaos in the Mean Field Dynamics.* Arch. Rat. Mech. and Anal. **231** (2019) 1, 285–317.

[2] M. Pulvirenti and S. Simonella. *The Boltzmann–Grad limit of a hard sphere system: analysis of the correlation error.* Inventiones **207** (2017) 3, 1135–1237.

On the size of backward clusters in a low-density regime for Hard-sphere systems

Mario Pulvirenti
(joint work with S. Simonella)

We consider a system of N hard spheres of diameter $\epsilon > 0$ in the 3-D torus $\mathbb{T}^3$. We are interested in a low-density regime, namely when $N \to \infty, \epsilon \to 0, N\epsilon^2 < +\infty$.

Fixed a time $t > 0$, for a given particle (say particle i), we consider its backward cluster $BC(i)$, which is the minimal subset of particles which really influence the motion of this particle up to the time t.

This notion is interesting because as far as the behaviour of particle 1 is concerned, we can consider a much simpler dynamics. Therefore it would be interesting to bound the mean size of a backward cluster, for a given measure on the configurational space of the system. At moment only local (in time) estimates are available. Now I present an estimate global in time, uniform in ϵ, N, when the averages are computed by means of an equilibrium measure at a given temperature. This is a work in progress, in collaboration with S. Simonella.

Diffusive and superdiffusive asymptotics of a linear kinetic equation with interface

Tomasz Komorowski

We consider the limit of a linear phonon Boltzmann equation, with reflection-transmission-absorption at an interface, with a degenerate scattering kernel. An equation of this type arises from a microscopic harmonic chain of oscillators whose dynamics is perturbed by a stochastic term, conserving energy and momentum. The system is in contact, via one oscillator, with a heat bath at temperature T. It is known that in the absence of the interface, the solution of the kinetic equation of this type exhibits superdiffusive, or diffusive behavior in the proper long time - large scale limit, depending on the behavior of the group velocity $\bar{\omega}'(k)$ of the phonons corresponding to the wavenumbers $|k| \ll 1$. In the acoustic case ($\bar{\omega}'(k) \sim \operatorname{sign} k$ for $|k| \ll 1$) and when the absorption probability of phonons, corresponding to $|k| \ll 1$, at the interface does not vanish, the superdifusive limit is the unique solution of a version of the fractional in space heat equation, with reflection-transmission-absorption at the interface. The limit problem corresponds to a certain symmetric stable process that is either absorbed, reflected, or transmitted upon crossing the interface.

If the dispersion relation is optical ($\bar{\omega}'(k) \sim k$ for $|k| \ll 1$), then, under the diffusive scaling, the solutions of the kinetic equation tend to the solution of a heat equation with the Dirichlet boundary condition determined by temperature T. The presented results have been obtained in collaboration with G. Basile (Univ. Roma I), S. Olla (Univ. Paris-Dauphine) and L. Ryzhik (Stanford Univ.).

Time-time covariance for last passage percolation with generic initial profile

Patrik L. Ferrari
(joint work with Alessandra Occelli and Herbert Spohn)

This talk is based on the papers [10] with H. Spohn and [9] with A. Occelli. In [10] we made some well-funded predictions on the time-time correlations for models in the Kardar-Parisi-Zhang universality class, while in [9] we provide mathematical proofs of the results presented below.

Stochastic growth models in the Kardar-Parisi-Zhang (KPZ) universality class [14] on a one-dimensional substrate are described by a height function $h(x,t)$ with x denoting space and t time. The height function evolves microscopically according to a random and local dynamics, while on a macroscopic scale the evolution is a deterministic PDE and the limit shape is non-random. In particular, if the speed of growth as a function of the gradient of the interface is a strictly convex or concave function, then the model is in the KPZ universality class. One expects large time universality under an appropriate scaling limit.

Away from shocks, the fluctuations of the height functions growth as $t^{1/3}$, while the spatial correlations are of order $t^{2/3}$. Furthermore, along space-time trajectories given by the characteristic lines of the PDE for the macroscopic evolution, non-trivial correlations survive on the macroscopic time scale, i.e., on scales of order t [8, 5].

The study of the time-time process is recent. On the experimental and numerical simulation side observables like the persistence probability or the covariance of an appropriately rescaled height function have been studied [19, 17, 18, 16]. On the analytic and rigorous side, the two-time joint distribution of the height function is known for special initial conditions: Johansson, later with Rahman, analyzed a model on full space [11, 12, 13], while Baik and Liu considered a model on a torus [2, 1]. For general (random) initial conditions exact formulas on the joint distributions are not yet available. Also, the analysis of the covariance starting from the available formulas [13, 12, 1] seems to be a very difficult task.

In [10] and [9] we consider the last passage percolation as model with generic initial conditions. The predictions of [10] are made under the assumptions that the exchange of the large time limit and maximum over sums of Airy processes and that the covariances of the rescaled processes converges holds. In [9] we provide mathematical proofs, using (1) the method of Corwin, Liu and Wang [7], who lifted the finite-dimensional slow-decorrelation result of [8, 5] to a functional slow-decorrelation statement, and (2) the method of comparison with stationarity developed by Cator and Pimentel [4, 15].

The model is the following. Consider a collection of i.i.d. random variables $\omega_{i,j}, i,j \in \mathbb{Z}$ with exponential distribution of parameter one. An *up-right path* $\pi = (\pi(0), \pi(1), \dots, \pi(n))$ on $\mathbb{Z}^2$ from a point A to a point E is a sequence of points in $\mathbb{Z}^2$ with $\pi(k+1) - \pi(k) \in \{(0,1),(1,0)\}$, with $\pi(0) = A$ and $\pi(n) = E$, and n is called the length $\ell(\pi)$ of π. Given a set of points S_A with some random

variables (not necessarily independent) h^0 on S_A, but independent of the ω's, and given a point E, one defines the last passage time $L_{S_A\to E}$ as

$$L_{S_A\to E} = \max_{\substack{\pi:A\to E\\ A\in S_A}} \left(h^0(\pi(0)) + \sum_{1\le k\le n} \omega_{\pi(k)} \right). \tag{1}$$

Here we consider $S_A = \mathcal{L} := \{(i,j) \in \mathbb{Z}^2 \,|\, i+j = 0\}$ for the point-to-point geometry and $S_A = \{(0,0)\}$, $E = (\tau N, \tau N)$ for the remaining geometries. Define the limiting rescaled LPP time

$$\tau \mapsto \chi^\star(\tau) = \lim_{N\to\infty} \frac{L^\star_{S_A\to E} - 4\tau N}{2^{4/3}N^{1/3}}, \tag{2}$$

where the superscript $\star$ denotes the different configurations, point-to-point ($\bullet$), point-to-line ($\diagdown$), stationary ($\mathcal{B}$) and random (σ), which are given as follows:

- *Point-to-point:* $S_A = \{(0,0)\}$ and $h^0 = 0$,
- *Point-to-line:* $S_A = \mathcal{L}$ and $h^0 = 0$.
- *Stationary:* $S_A = \mathcal{L}$ and h^0 as follows. Let $\{X_k, Y_k\}_{k\in\mathbb{Z}}$ be i.i.d. random variable Exp(1/2)-distributed. Then define

$$h^0(x,-x) = \begin{cases} \sum_{k=1}^{x}(X_k - Y_k), & \text{for } x \ge 1,\\ 0, & \text{for } x = 0,\\ -\sum_{k=x+1}^{0}(X_k - Y_k), & \text{for } x \le -1. \end{cases} \tag{3}$$

- *A family of random initial conditions.* We consider the case where for a given $\sigma \ge 0$, h^0 is given by (3) multiplied by σ. Clearly, the cases $\sigma = 0$ and $\sigma = 1$ correspond to the flat and to the stationary cases.

The main results proven in [9] are the following (in [9] the results are extended to points in neighborhoods of the characteristics as well):

Theorem 1. *For the stationary LPP, the covariance of the limiting height function for all $\tau \in (0,1)$ can be expressed as*

$$\mathrm{Cov}\left(\chi^{\mathcal{B}}(\tau), \chi^{\mathcal{B}}(1)\right) = \frac{1+\tau^{2/3}-(1-\tau)^{2/3}}{2}\mathrm{Var}(\xi_{\mathrm{BR}}), \tag{4}$$

where ξ_{BR} is a Baik-Rains distributed random variable.

Theorem 2. *As $\tau \to 1$ we have, for $\star = \{\bullet, \diagdown, \mathcal{B}\}$,*

$$\mathrm{Cov}\left(\chi^\star(\tau), \chi^\star(1)\right) = \frac{1+\tau^{2/3}}{2}\mathrm{Var}\left(\chi^\star(1)\right) - \frac{(1-\tau)^{2/3}}{2}\mathrm{Var}\left(\xi_{\mathrm{BR}}\right) + \mathcal{O}(1-\tau)^{1^-}. \tag{5}$$

Here $\chi^\bullet(1)$ (resp. $2^{2/3}\chi^\diagdown(1)$) is distributed according to a GUE (resp. GOE) Tracy-Widom law.

Shortly after finishing our paper, for the point-to-point geometry, Basu and Ganguly obtained the same exponents for the behaviour at close or far away points [3]. Unlike in our paper, they did not identify the prefactor, but on the other hand, their result are non-asymptotic as well. Very recently, a result analogue to [3] for the KPZ equation with sharp wedge initial condition has been obtained in [6].

References

[1] J. Baik and Z. Liu, *Multi-point distribution of periodic TASEP*, J. Amer. Math. Soc. **32** (2019), 609–674.

[2] J. Baik and Z. Liu, *TASEP on a ring in sub-relaxation time scale*, Comm. Pure Appl. Math. **71** (2018), 747–813.

[3] R. Basu and S. Ganguly, *Time correlation exponents in last passage percolation*, preprint: arXiv:1807.09260 (2018).

[4] E. Cator and L. Pimentel, *On the local fluctuations of last-passage percolation models*, Stoch. Proc. Appl. **125** (2015), 879–903.

[5] I. Corwin, P.L. Ferrari, and S. Péché, *Universality of slow decorrelation in KPZ models*, Ann. Inst. H. Poincaré Probab. Statist. **48** (2012), 134–150.

[6] I. Corwin, P. Ghosal, and A. Hammond, *KPZ equation correlations in time*, preprint: arXiv:1907.09317 (2019).

[7] I. Corwin, Z. Liu, and D. Wang, *Fluctuations of TASEP and LPP with general initial data*, Ann. Appl. Probab. **26** (2016), 2030–2082.

[8] P.L. Ferrari, *Slow decorrelations in KPZ growth*, J. Stat. Mech. (2008), P07022.

[9] P.L. Ferrari and A. Occelli, *Time-time covariance for last passage percolation with generic initial profile*, Math. Phys. Anal. Geom. **22** (2019), 1.

[10] P.L. Ferrari and H. Spohn, *On time correlations for KPZ growth in one dimension*, SIGMA **12** (2016), 074.

[11] K. Johansson, *Two time distribution in Brownian directed percolation*, Comm. Math. Phys. **351** (2017), 441–492.

[12] K. Johansson, *The two-time distribution in geometric last-passage percolation*, Probab. Theory Relat. Fields (2019), online first.

[13] K. Johansson and M. Rahman, *Multi-time distribution in discrete polynuclear growth*, preprint, arXiv:1906.01053 (2019).

[14] M. Kardar, G. Parisi, and Y.Z. Zhang, *Dynamic scaling of growing interfaces*, Phys. Rev. Lett. **56** (1986), 889–892.

[15] L.P.R. Pimentel, *Local Behavior of Airy processes*, J. Stat. Phys. **173** (2018), 1614–1638.

[16] K.A. Takeuchi, *Statistics of circular interface fluctuations in an off-lattice Eden model*, J. Stat. Mech. (2012), P05007.

[17] K.A. Takeuchi, *Crossover from Growing to Stationary Interfaces in the Kardar-Parisi-Zhang Class*, Phys. Rev. Lett. **110** (2013), 210604.

[18] K.A. Takeuchi and T. Akimoto, *Characteristic Sign Renewals of Kardar-Parisi-Zhang Fluctuations*, J. Stat. Phys. **164** (2016), 1167–1182.

[19] K.A. Takeuchi and M. Sano, *Evidence for geometry-dependent universal fluctuations of the Kardar-Parisi-Zhang interfaces in liquid-crystal turbulence*, J. Stat. Phys. **147** (2012), 853–890.

The local geometry at the rough-smooth interface in the two-periodic Aztec diamond

Sunil Chhita
(joint work with Vincent Beffara and Kurt Johansson)

An *Aztec diamond graph of size* n is a bipartite graph which contains white vertices given by

$$\mathtt{W} = \{(i,j) : i \mod 2 = 1, j \mod 2 = 0, 1 \leq i \leq 2n-1, 0 \leq j \leq 2n\} \tag{1}$$

and black vertices given by

$$\mathtt{B} = \{(i,j) : i \mod 2 = 0, j \mod 2 = 1, 0 \leq i \leq 2n, 1 \leq j \leq 2n-1\}. \tag{2}$$

The edges of the Aztec diamond graph are given by $\mathtt{b} - \mathtt{w} = \pm e_1, \pm e_2$ for $\mathtt{b} \in \mathtt{B}$ and $\mathtt{w} \in \mathtt{W}$, where $e_1 = (1,1)$ and $e_2 = (-1,1)$. The coordinate of a face in the graph is defined to be the coordinate of its center. For an Aztec diamond graph of size $n = 4m$ with $m \in \mathbb{N}$, define the *two-periodic Aztec diamond* to be an Aztec diamond graph with edge weights a for all edges around the faces (i,j) with $(i+j) \mod 4 = 2$ and edge weights b for all the edges around the faces (i,j) with $(i+j) \mod 4 = 0$. A dimer covering is an arrangement of edges so that each vertex is incident to exactly one edge. Each dimer covering is picked with probability proportional to the product of the edge weights.

For large two-periodic Aztec diamonds, it is well known that a *limit shape* emerges whose curves are described by an 8 degree curve [5], separating the random tilings into three macroscopic regions. These macroscopic regions are called *frozen*, where the tiling is supposed to be deterministic; *rough*, where the correlations between dimers should decay polynomially with distance; *smooth*, where the correlations between dimers should decay exponentially with distance. These measures were characterized in [6].

It is well known that dimers on bipartite graphs form a determinantal point process, with the correlation kernel given by entries from the inverse of the so-called Kasteleyn matrix. In general, computing the inverse of the Kasteleyn matrix is challenging. For the two-periodic Aztec diamond, the inverse of the Kasteleyn matrix was computed in [4]. Using these formulas, Chhita and Johansson in [2] were able to find formulas for probabilities of the dominoes at the rough-smooth interface, finding that there was a complicated mixture of full-plane smooth terms and Airy kernel correction terms. Similar results of this form were also obtained by Duits and Kuijlaars in [3], who found a new approach for computing the correlation kernel via Riemann Hilbert analysis. Later, Beffara, Chhita and Johansson in [1] were able to show that a very specific averaging of the height function for dominoes converges weakly to the Airy kernel point process under suitable scaling and centering.

None of these results, however, showed whether there is a set of lattice paths whose last path converges to the Airy process, which is similar to what is observed at the frozen-rough interface. Indeed, this is what is expected from simulations. In this talk, we describe a way to formulate paths, which in some sense are the

level lines of the height function. Moreover, we show that the point process built from these paths converges to the Airy kernel point process provided that $a < 1/3$. This restriction in the parameter is purely technical. A surprising feature of these paths is that they are allowed to have small back tracks, which is different from what is observed at the frozen-rough interface. It remains open to show that there is a last path which converges to the Airy process, as one would expect.

References

[1] Vincent Beffara, Sunil Chhita, and Kurt Johansson. Airy point process at the liquid-gas boundary. *Ann. Probab.*, 46(5):2973–3013, 2018.

[2] Sunil Chhita and Kurt Johansson. Domino statistics of the two-periodic Aztec diamond. *Adv. Math.*, 294:37–149, 2016.

[3] Maurice Duits and Arno Kuijlaars. The two periodic Aztec diamond and matrix valued orthogonal polynomials. *JEMS*, 2017. To Appear.

[4] Sunil Chhita and Benjamin Young. Coupling functions of domino tilings of the Aztec diamond. *Adv. Math.* 259:173–251, 2014.

[5] Richard Kenyon and Andrei Okounkov. Limit shapes and the complex Burgers equation. *Acta Math.*, 199(2):263–302, 2007.

[6] Richard Kenyon, Andrei Okounkov, and Scott Sheffield. Dimers and amoebae. *Ann. of Math. (2)*, 163(3):1019–1056, 2006.

Multi-particle diffusion limited aggregation

Alexandre Stauffer

We consider the following process for the growth of an aggregate on $\mathbb{Z}^d$. Initially, the aggregate occupies only the origin of $\mathbb{Z}^d$, and each site $x \in \mathbb{Z}^d \setminus \{0\}$ hosts one particle with probability μ or no particle with probability $1-\mu$, independently of one another. Then, particles perform a simple exclusion process on $\mathbb{Z}^d$; that is, particles jump at rate 1 to a neighboring site chosen uniformly at random, with jumps towards sites already occupied by another particle being suppressed. In addition, whenever a particle jumps from a site x to a site already occupied by the aggregate, the jump is suppressed, x is added to the aggregate and the particle at x will stop moving, remaining indefinitely at x. In other words, the aggregate grows by attaching particles at its boundary whenever a particle attempts to jump onto the aggregate. We let $A_t \subset \mathbb{Z}^d$ denote the set of sites occupied by the aggregate at time t.

This process, which we refer to as *multi-particle diffusion limited aggregation* (MDLA), was introduced in 1980 by Rosenstock and Marquardt [5] and popularized by Voss [8]. Previous rigorous results on MDLA were restricted to the case of dimension 1 [2, 4]. In particular, Kesten and Sidoravicius [4] showed that in $d = 1$ the aggregate grows with speed zero and reaches to distance of order $\sqrt{t}$ by time t.

In a joint work with Vladas Sidoravicius [6], we showed that in dimensions $d \geq 2$ the aggregate has a regime of positive speed of growth. We also showed that when μ is large enough the aggregated produces a bulky set. More precisely, let $\bar{A}_t$ be the set of sites disconnected from infinity by the aggregate (in other words, $\bar{A}_t$

is composed of A_t and the finite components of its complement). We show the following result.

Theorem 1 ([6]). *For any $d \geq 2$, there exists $\mu_0 \in (0,1)$ such that for all $\mu \in (\mu_0, 1)$ there is a constant $c > 0$ for which*

$$\mathbb{P}(\bar{A}_t \supset \mathcal{B}_{ct} \text{ for all } t \geq 0) > 0,$$

where $\mathcal{B}_r$ stands for the ball of radius r centered at the origin of $\mathbb{Z}^d$.

The above result is expected to hold for all large enough t *almost surely*. Recently, Allan Sly [7] obtained partial results on this question. If we let $\|A_t\| = \sup_{x \in A_t} \|x\|_1$ stand for the furthest away distance the aggregate reaches by time t, then [7] shows that almost surely $\|A_t\|$ is of order t for all large enough t. But, unfortunately, the result in [7] does not give any further information regarding the geometry of the aggregate.

In order to prove Theorem 1, we introduced a novel growth process with competition, which we describe now. There are two growth processes, which we call type 1 and type 2. Type 1 starts from the origin, whereas for each site in $\mathbb{Z}^d \setminus \{0\}$ with probability p we place a so-called *type 2 seed*, otherwise we leave the site empty. Then type 1 starts spreading through $\mathbb{Z}^d$ as a first passage percolation process at rate 1 (that is, with passage times distributed as independent exponential random variables of rate 1). Type 2 initially does nothing. Whenever a process (type 1 or type 2) attempts to occupy a site which hosts a type 2 seed, the occupation does not happen and that seed is *activated*. From that moment on, type 2 starts spreading from that seed as a first passage percolation process of rate λ. As other type 2 seeds are activated, more and more clusters of type 2 will start to grow. Each site of $\mathbb{Z}^d$ will be occupied by the type that arrives to each first, and will never switch types afterwards. The above process is refereed to as *first passage percolation in a hostile envronment* (FPPHE).

FPPHE is related to MDLA as type 1 represents the growth of the aggregate. The locations of the seeds represent the sites of $\mathbb{Z}^d$ where the aggregate will discover a *hole*, which is a site that does not host a particle of the exclusion process of MDLA. Thus, intuitively, we should have $\mu = 1 - p$. Then, once a seed is activated at a site x in FPPHE, which represents the event that the aggregate just discovered the presence of a hole at x in MDLA, the cluster of type 2 that grows from x in FPPHE represents the region inside which the hole will be located as it moves. The above description is just intuitive, and we must emphasize that there is no coupling between FPPHE and MDLA that shows, for example, that the aggregate in MDLA contains the sites occupied by type 1 in FPPHE. The proof of Theorem 1 using FPPHE is rather delicate, and we refer the reader to [6] for the details.

There are three possible outcomes for FPPHE: *extinction* (meaning that type 1 stops growing in finite time almost surely), *strong survival* (meaning that, with positive probability, type 1 produces an infinite cluster and all clusters of type 2 are finite), and *coexistence* (meaning that both types 1 and 2 produce an infinite cluster with positive probability). We refer to a cluster of type 2 as a maximal set

of sites that are occupied by type 2, regardless of whether they were all occupied from the activation of the same type 2 seed or from several type 2 seeds.

Let η_t^1 and η_t^2 be the set of sites occupied by type 1 and type 2, respectively, at time t (where non-active seeds of type 2 are not included in η_t^2). With Vladas Sidoravicius we showed the following result, whose proof can be adapted to establish Theorem 1.

Theorem 2 ([6]). *For any $d \geq 2$ and any $\lambda < 1$, there exists $p_0 > 0$ such that for all $p \in (0, p_0)$ there exists a constant $c > 0$ for which*

$$\mathbb{P}(\bar{\eta}_t^1 \supset \mathcal{B}_{ct} \text{ for all } t \geq 0) > 0.$$

In other words, the above theorem shows the existence of a strong survival regime for all $\lambda < 1$, provided p is small enough.

Intuitively, one expects that increasing p or λ favors type 2, however there is no proof of monotonicity for FPPHE (neither is there a proof of monotonicity for MDLA). Yet, some further results can be obtained. It is not difficult to see that when $p > 1 - p_c^{\text{site}}$, where p_c^{site} is the critical probability for site percolation on $\mathbb{Z}^d$, there is extinction because almost surely the origin is confined to a finite cluster of sites not hosting a type 2 seed. One can also show that when $\lambda \geq 1$ there is extinction for all p. In a joint work with T. Finn [3] we study such extreme cases and show strong survival and coexistence when λ approaches zero while p remains fixed.

When the underlying graph is not $\mathbb{Z}^d$, but a hyperbolic (and non-amenable) graph, the results drastically change. In a joint work with E. Candellero [1], we showed the following two theorems.

Theorem 3 ([1]). *For any hyperbolic, vertex-transitive graph, any $\lambda > 0$ and any $p \in (0, 1)$, we have*

$$\mathbb{P}(\textit{type 2 produces an infinite cluster}) = 1.$$

Theorem 4 ([1]). *For any hyperbolic, non-amenable graph and any $\lambda > 0$, there exists $p_0 > 0$ such that for all $p \in (0, p_0)$ we have*

$$\mathbb{P}(\textit{type 1 produces an infinite cluster}) > 0.$$

Theorem 4 above gives a somehow analogous result as Theorem 2 but with $\mathbb{Z}^d$ replaced by a hyperbolic, non-amenable graph. But there is one interesting difference. In $\mathbb{Z}^d$, we know that type 1 cannot produce an infinite cluster when $\lambda \geq 1$; however, in a hyperbolic, non-amenable graph type 1 has a positive probability of producing an infinite cluster for all $\lambda > 0$, provided p is small enough. Theorems 3 and 4 together imply that there is a regime *coexistence* when the graph is hyperbolic, non-amenable and vertex transitive.

References

[1] E. Candellero and A. Stauffer, *Coexistence of competing first passage percolation on hyperbolic graphs*, available from arXiv:1810.04593.

[2] L. Chayes and G. Swindle, *Hydrodynamic limits for one-dimensional particle systems with moving boundaries*, Annals of Probability **24** (1996), 559–598.

[3] T. Finn and A. Stauffer, *Coexistence of first passage percolation in hostile environment in* $d \geq 3$, in preparation.
[4] H. Kesten and V. Sidoravicius, *A problem in one-dimensional diffusion-limited aggregation (DLA) and positive recurrence of Markov chains*, Annals of Probability **36** (2008), 1838–1879.
[5] H. Rosenstock and C. Marquardt, *Cluster formation in two-dimensional random walks: application to photolysis of silver halides*, Physical Review B **22** (1980), 5797–5809.
[6] V. Sidoravicius and A. Stauffer, *Multi-particle diffusion limited aggregation*, Inventiones Mathematicae **218** (2019), 491–571.
[7] A. Sly, *On one-dimensional multi-particle diffusion limited aggregation*, available from arXiv:1609.08107.
[8] R. Voss, *Multiparticle fractal aggregation*, Journal of Statistical Physics **36** (1984), 861–872.

A kinetically constrained model in a random environment

Assaf Shapira

Kinetically constrained models are a family of interacting particle systems that were introduced in order to study glassy materials. They are based on the hypothesis, that glassy behavior is caused by an out of equilibrium dynamic effect, and that interactions don't play an important role. More specifically, when a liquid is cooled down very quickly, and becomes very dense, molecules are not able to move and relax to equilibrium. In kinetically constrained models, these features are translated to a Markov process reversible with respect to a non-interacting equilibrium (i.e. a product measure), and sites are blocked whenever their neighborhood is too dense.

The model I've talked about is one example of a kinetically constrained model in a random environment, where the notion of "too dense" is determined independently at random for each site. For simplicity, take the dynamics on $\mathbb{Z}^2$. Before starting the dynamics, choose for each site whether it's *easy* or *difficult*, independently with probabilities π and $1-\pi$ (for some $\pi \in [0,1]$). These are the quenched variables (denoted ω), and they'll be kept frozen when the process runs. The measure from which we choose them is called ν. The dynamic variables, which will change with time, are given by the sites' occupation. Each site could be either *empty* or *occupied*, defining the configuration η. Initially η is chosen according to the measure μ, setting independently each site to "empty" with probability q and "occupied" with probability $1-q$. Since the glass is dense, we think of the limit $q \to 0$. We then run the following dynamics – on each site we put a clock ringing with rate 1. Whenever the clock rings, we need to decide whether the ring is legal. For an easy site, the ring is legal if at least one of its four neighbors is empty, and for a difficult site if at least two neighbors are empty. When the ring is not legal we don't do anything, and when it is legal we toss a coin (independent of everything), giving "empty" with probability q and "occupied" with probability $1-q$. We then set the site's occupation to the result of the coin toss. This is a reversibly process with respect to μ, and we denote the measure of rings and tosses by $\mathbb{P}_\mu, \mathbb{E}_\mu$ (the subscript μ indicates stationary initial state).

When q is small many rings are illegal, and time scales diverge. The goal of the talk was to compare the divergence of different time scales, and notice some new phenomena that we don't see in homogeneous environments. The first time scale that I mentioned was the relaxation time τ_{rel}, which is the best constant for which all centered local functions f satisfy

$$\mathbb{E}_\mu[f(\eta(0))f(\eta(t))] \leq \text{Var}(f)\, e^{-t/\tau_{\text{rel}}}.$$

In the homogeneous cases, [3, 4] show that $\tau_{\text{rel}} \approx q^{-2}$ when $\pi = 1$ and $e^{1/q}$ when $\pi = 0$ (the $\approx$ symbol is used quite freely throughout). In the inhomogeneous case, since τ_{rel} describes the uniform decay of all functions, it will focus on remote areas of the lattice that have many difficult sites. This is the reason that the relaxation time is $\approx e^{1/q}$ when $\pi \in (0, 1)$. That's not very satisfying if we want to understand the typical behavior at a typical position, e.g., near the origin. One observable that will describe the true behavior near the origin is its emptying time

$$\tau_0 = \inf\{t : \eta_0(t) = \text{empty}\}.$$

In the homogeneous case it scales like the relaxation time ($\approx q^{-2}$ for $\pi = 1$ and $e^{1/q}$ for $\pi = 0$). When $\pi \in (0, 1)$ this will no longer be true.

One way to analyze the emptying time is to consider the *bootstrap percolation time* τ_0^{BP}, which is the time it would take the origin to become empty if all coin tosses gave "empty". This is a very optimistic assumption, but it does give an upper bound on τ_0, and it's an interesting quantity on its own. In the homogeneous case we know that $\tau_0^{\text{BP}} \approx q^{-1/2}$ for $\pi = 1$ and $e^{1/q}$ for $\pi = 0$ [2].

The results that I presented (theorems 2.1 and 2.2 of [1]) state (loosely) that in the inhomogeneous case the bootstrap percolation time scales like $q^{-1/2}$, and that τ_0 scales like $q^{-\alpha}$, for some ν-random α (so different realizations of the quenched variables will give different values of α). The reason for this difference is that the bootstrap percolation is dominated by long range effects, which are self-averaging, while the kinetically constrained model is affected by the local neighborhood of the origin.

There are many open questions concerning kinetically constrained models in random environments. First, studying more random environments would be interesting (e.g., random graphs). Many of them have been studied for bootstrap percolation, but not much is known about kinetically constrained models. One more question is to consider other time scales, and in particular the relaxation of local correlations. A simple example is to understand for which time t the correlation becomes small, i.e., $\frac{\mathbb{E}[\eta_0(0)\eta_0(t)]-\mu(\eta_0)^2}{\text{Var}(\eta_0)} < \epsilon$. Finally, going beyond the stationary dynamics is also a very interesting (and challenging) problem.

References

[1] Assaf Shapira, *Kinetically constrained models with random constraints*, arXiv preprint arXiv:1812.00774 (2018).

[2] Michael Aizenman and Joel Lebowitz, *Metastability effects in bootstrap percolation*, Journal of Physics A: Mathematical and General **21** (1988).

[3] Nicoletta Cancrini, Fabio Martinelli, Cyril Roberto and Cristina Toninelli, *Kinetically constrained spin models*, Probability Theory and Related Fields **140** (2008).
[4] Fabio Martinelli and Cristina Toninelli, *Towards a universality picture for the relaxation to equilibrium of kinetically constrained models*, The Annals of Probability **47** (2019).

Frozen percolation on the binary tree is nonendogenous

Balázs Ráth
(joint work with Jan M. Swart, Tamás Terpai)

Let (T, E) be a regular tree where each vertex has degree 3, and let $\mathcal{U} = (U_e)_{e\in E}$ be an i.i.d. collection of uniformly distributed $[0, 1]$-valued random variables, indexed by the edges of the tree. We write $E_t := \{e \in E : U_e \leq t\}$ $(t \in [0, 1])$. Aldous [Ald00] has proved the following theorem.

Theorem 1. *It is possible to couple $\mathcal{U}$ to a random subset $F \subseteq E$ with the following properties:*

(1) *$e \notin F$ if and only if no endvertex of e is part of an infinite cluster of $E_{U_e} \setminus (F \cup \{e\})$.*
(2) *The law of $(\mathcal{U}, F)$ is invariant under automorphisms of the tree.*

At time $t \in [0, 1]$, we call edges in $E_t \setminus F$ *open*, edges in $E_t \cap F$ *frozen*, and all other edges *closed*. Then property (i) can be described in word as follows. Initially all edges are closed. At time U_e, the edge e opens provided neither of its endvertices is at the time part of an infinite open cluster; in the opposite case, it freezes.

In [Ald00, Section 5.7], Aldous asks whether for the pair $(\mathcal{U}, F)$ that he constructs, F is measurable w.r.t. the σ-field generated by $\mathcal{U}$. Our main result is that almost sure uniqueness does not hold.

Theorem 2. *There exists a triple $(\mathcal{U}, F, F')$ such that $\mathcal{U} = (U_e)_{e\in E}$ is an i.i.d. collection of uniformly distributed $[0, 1]$-valued random variables, F and F' are random subsets of E satisfying property (i) of Theorem 1, the law of $(\mathcal{U}, F, F')$ is invariant under automorphisms of the tree, and $F \neq F'$ a.s.*

The construction of Theorem 1 uses a so-called *recursive tree process* (RTP), c.f. [AB05, Section 2.3], and we prove Theorem 2 by showing that this RTP is *nonendogenous*, c.f. [AB05, Section 2.4]. In fact, we prove Theorem 2 by explicitly constructing a non-diagonal fixed point of the bivariate *recursive distributional equation* (RDE) associated to the frozen percolation RTP, c.f. [AB05, Theorem 11]. An essential role in our proofs is played by a frozen percolation process on a continuous-time binary Galton Watson tree that has nice scale invariant properties.

References

[AB05] D.J. Aldous and A. Bandyopadhyay. A survey of max-type recursive distributional equations. *Ann. Appl. Probab.* 15(2) (2005), 1047–1110.
[Ald00] D.J. Aldous. The percolation process on a tree where infinite clusters are frozen. *Math. Proc. Cambridge Philos. Soc.* 128 (2000), 465–477.

Split-and-Merge in Stationary Random Stirring on Lattice Torus

Dmitry Ioffe
(joint work with Bálint Tóth)

For $n \in \mathbb{N}$ and $N = n^d$ let $\mathbb{T}^N = (\mathbb{Z}/n)^d$ be the d-dimensional lattice torus of linear size n and, accordingly, of volume N, and $\mathbb{B}^N$ the set of nearest neighbour unoriented edges b of $\mathbb{T}^N$. Let Σ^N be the symmetric group of all permutations of $\{1, \dots, N\}$

The random stirring (or, random transposition) process on $\mathbb{T}^N$ is the continuous time Markov process $t \mapsto \eta^N(t)$ on the state-space Σ^N, generated by independent Poisson flows (of rate one) of elementary transpositions τ_{b} along the unoriented edges $\mathsf{b} \in \mathbb{B}^N$. Its infinitesimal generator, acting on functions $f : \Sigma^N \to \mathbb{R}$, is

$$\mathcal{L}^N f(\sigma) = \sum_{\mathsf{b} \in \mathbb{B}^N} \left(f(\tau_{\mathsf{b}}\sigma) - f(\sigma) \right).$$

The uniform distribution of permutations, μ^N, is the unique invariant measure of the Markov process $t \mapsto \eta^N(t)$ which is also reversible under this measure.

Given $\eta \in \Sigma^N$ denote by $\mathcal{C}(\eta) = (\mathcal{C}_i(\sigma))_{i \geq 1}$ the cycle decomposition of the permutation η, listed in decreasing order of their sizes, so that in case of ties the order of cycles is given by the decreasing lexicographic order of their largest element. The cycle structure $\xi = \mathsf{p}(\eta)$ of η is the ordered list

$$\xi_i = p_i(\eta) := \frac{|\mathcal{C}_i(\eta)|}{N};\ i = 1, 2 \dots$$

By construction, $\xi \in \Omega$, where Ω is the set of ordered partitions,

$$\Omega = \left\{ \mathsf{p} = (p_i)_{i \geq 1} : p_i \in [0,1], \quad p_1 \geq p_2 \geq \cdots \geq 0, \quad \sum_i p_i = 1 \right\}$$

We use $\mathcal{L}(\eta) = \max\{i \ : \ p_i > 0\}$ for the number of cycles of η.

Define the slowed down cycle structure process $\xi^N(t)$ of η^N via

$$\xi^N(t) = \eta^N\left(\frac{t}{dN}\right),$$

where η^N is the above random transposition process on $\mathbb{T}^N$. Note that the time scaling we have chosen leads to the unit total rate of change of ξ^N. Note also that, since η^N feels the geometry of $\mathbb{T}^N$, ξ^N is not Markovian. Nevertheless, our main result asserts that in any dimension $d \geq 1$, the equilibrium random cycle structure ξ^N induced by the random transposition process on $\mathbb{T}^N$ converges, as the size of the torus $N \to \infty$ tends to infinity, to the canonical Markovian split and merge process ζ on Ω. The latter is specified by the generator

$$\mathcal{G}f(\mathsf{p}) = 2 \sum_{i<j} p_i p_j \left(f(\mathsf{M}_{ij}\mathsf{p}) - f(\mathsf{p}) \right) + \sum_i p_i^2 \int_0^1 \left(f(\mathsf{S}_i^u \mathsf{p}) - f(\mathsf{p}) \right) du,$$

where, for $1 \leq i < j$, the map $\mathsf{M}_{ij} : \Omega \to \Omega$ merges the partition elements p_i and p_j into one of size $p_i + p_j$, and subsequently rearranges the partition elements in

decreasing order, whereas, for $1 \leq i$ and $u \in [0,1)$, the map $\mathsf{S}_i^u : \Omega \to \Omega$ splits the partition element p_i into two pieces of size up_i, respectively, $(1-u)p_i$ and subsequently rearranges the partition elements in decreasing order. This canonical process is much studied and well understood. In particular, it is a known fact – see [4], [2] – that the Poisson-Dirichlet law $\mathsf{PD}(1)$ on Ω is the unique stationary measure for the process $t \mapsto \zeta(t)$ which is also reversible under this measure.

In the case of transient dimensions $d \geq 3$ the problem is motivated by attempts to understand the onset of long range order in quantum Heisenberg models via random loop representations of the latter.

The random loop representation of the latter was developed in [7, 8], see also [1, 3, 5] for the related material. However, for finite temperatures $\beta < \infty$ one should consider the random stirring process $t \mapsto \eta^N(t)$ on $\mathbb{T}^N$ which starts from the initial state $\eta^N(0) = \mathsf{id}$ and runs β units of time on the original time scale of unit stirring rate per edge, rather than being stationary. We use subscripts $0, \beta$ in $\mathbf{P}_{0,\beta}(\cdot)$ to stipulate this initial condition and finite horizon. One should the consider the modified path measures $\mathbf{P}^{\theta}_{0,\beta}(\cdot)$;

$$\mathbf{P}^{\theta}_{0,\beta}\left(\mathrm{d}\eta^N\right) \propto \theta^{\mathcal{L}(\eta^N(\beta))}\mathbf{P}_{0,\beta}\left(\mathrm{d}\eta^N\right),$$

with $\theta = 2$ (measures $\mathbf{P}^{\theta}_{0,\beta}$ with other values of $\theta \neq 2$ are perfectly well defined, but only integer values $\theta = 2, 3, 4, \ldots$ are related to stochastic representations of quantum spin systems with spin $s = \frac{\theta-1}{2}$ with pair interactions - see [6] for details).

The basic and best known conjecture in the above context of (modified) random stirrings on $\mathbb{T}^N$ is the "long cycle conjecture" of Tóth [8] which states that, once $d \geq 3$ and β is sufficiently large, then under $\mathbf{P}^{\theta}_{0,\beta}$ the eventual random permutation $\eta^N(\beta)$ contains macroscopic cycles. Affirmative settling of this conjecture for $\theta = 1/2$ would be essentially equivalent to proving existence of off-diagonal long range order at low temperatures for the isotropic spin-$\frac{1}{2}$ quantum Heisenberg ferromagnet, in dimensions $d \geq 3$ – a Holy Grail of mathematically rigorous quantum statistical physics. For details see [8, 5]. The conjecture is, however, open for any $\theta \in \mathbb{N}$, including the case $\theta = 1$ of pure random stirring on $\mathbb{T}^N$ considered here.

References

[1] M. Aizenman, B. Nachtergaele: Geometric aspects of quantum spin states. *Commun. Math. Phys.* **164:** 17–63 (1994)

[2] P. Diaconis, E. Mayer-Wolf, O. Zeitouni, M.P.W. Zerner: The Poisson-Dirichlet law is the unique invariant distribution for uniform split-merge transformations. *Ann. Probab.* **32:** 915–938 (2004)

[3] D. Ioffe: Stochastic geometry of classical and quantum Ising models. In: *Methods of contemporary mathematical statistical physics*, *Lecture Notes in Mathematics* **1970:** 87–127, Springer, 2009

[4] E. Mayer-Wolf, O. Zeitouni, M.P.W. Zerner: Asymptotics of certain coagulation-fragmentation processes and invariant Poisson-Dirichlet measures. *Electr. J. Probab.* **7:** paper no. 8. 1–25 (2002)

[5] C. Goldschmidt, D. Ueltschi, P. Windridge: Quantum Heisenberg models and their probabilistic representations. In: Entropy and the quantum II *Contemp. Math.* **552** pp. 177–224 (2011)
[6] D. Ueltschi: Random loop representations for quantum spin systems. *J. Math. Phys.* **54:** (8) pp. 1–40. (2013)
[7] B. Tóth: Probabilistic representation of the quantum Heisenberg ferromagnet. In: A. Truman, I.M. Davies (eds), *Stochastics and Quantum Mechanics, Swansea 1990*, World Scientific, Singapore, 1991
[8] B. Tóth: Improved lower bound on the thermodynamical pressure of the spin-1/2 Heisenberg ferromagnet. *Lett. Math. Physics* **28:** 75-84 (1993)

Coagulation equations with a source term

Jani Lukkarinen
(joint work with Marina Ferreira, Alessia Nota, Juan J. L. Velázquez)

We study coagulation equations under non-equilibrium conditions which are induced by addition of a source term for small cluster sizes, resulting in the evolution equation

$$\partial_t f(x,t) = \frac{1}{2}\int_0^x K(x-y,y)\, f(x-y,t)\, f(y,t)\, dy - \int_0^\infty K(x,y,t)\, f(x,t)\, f(y,t)\, dy + \eta(x)\,. \tag{1}$$

We consider measure valued solutions to the equation which allows to study both discrete and continuous coagulation equations simultaneously. We also allow for a large class of coagulation rate kernels, aiming to capture all kernels relevant to atmospheric coagulation phenomena; reviews of the related physics and chemistry may be found in [1] and of mathematical properties in [2]. To this end, we consider continuous kernel functions which are bounded from above and below by

$$w(x,y) = x^{\gamma+\lambda}y^{-\lambda} + y^{\gamma+\lambda}x^{-\lambda}\,, \tag{2}$$

where γ and $\lambda \in \mathbb{R}$. That is, we assume that there are $c_1, c_2 > 0$ such that $c_1 w(x,y) \le K(x,y) \le c_2 w(x,y)$ for all $x, y > 0$. The assumptions cover, in particular, the commonly used free molecular (aka ballistic) and diffusion limited aggregation coagulation kernels.

Our main result, described in detail in Ref. [3], is to show that, when the source term is present, the parameters γ and λ determine whether the system can have stationary solutions. Namely,

(1) if $|\gamma + 2\lambda| < 1$, then there exists at least one stationary solution, but
(2) if $|\gamma + 2\lambda| \ge 1$, then there are no stationary solutions.

In particular, we find that the diffusive kernel allows for stationary solutions while there can be no such solutions for the free molecular kernel. The argument to prove the non-existence of solutions relies on a novel power law lower bound, valid in the appropriate parameter regime, for the decay of stationary solutions with a constant flux. We obtain optimal lower and upper estimates of the solutions

for large cluster sizes, and prove that the solutions of the discrete model behave asymptotically as solutions of the continuous model.

Many applications of coagulation phenomena involve several coagulating particle species. For d species, this can be taken into account by considering in (1) clusters which are labelled by their composition vectors, $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ where $x_i \geq 0$, $i = 1, 2, \ldots, d$, and $\mathbf{x} \neq 0$. The solutions to the multicomponent case, with or without a source term, exhibit unusual "spontaneous localization" phenomena. This is evident for instance from the explicit solutions available for the stationary constant kernel case in Ref. [4]. However, the effect appears to be quite generic, and we can prove that there are examples of coagulation kernels, covering both $\gamma = 0$ and $\gamma = 1$ above, for which nearly all clusters have a composition which lies ever closer to a straight line for large times. The position of the line is determined by the source terms when they are present, and by the initial data if there are no source terms. The extend of generality of this statement is still a work in progress.

References

[1] H. Vehkamäki and I. Riipinen, *Thermodynamics and kinetics of atmospheric aerosol particle formation and growth*, Chem. Soc. Rev. **41**(15) (2012) 5160.

[2] F. P. da Costa, *Mathematical Aspects of Coagulation-Fragmentation Equations.* In J. P. Bourguignon, R. Jeltsch, A. Pinto, M. Viana (Eds.), Mathematics of Energy and Climate Change. CIM Series in Mathematical Sciences, vol. 2, Springer, (2015), pp. 83–162.

[3] M. Ferreira, J. Lukkarinen, A. Nota, and J. J. L. Velázquez, *Stationary non-equilibrium solutions for coagulation systems*, preprint (2019), `https://arxiv.org/abs/1909.10608`.

[4] P. L. Krapivsky and E. Ben-Naim, *Aggregation with multiple conservation laws*, Phys. Rev. E **53** (1996) 291–298.

Discrete classical integrable systems and generalized Pitman's transform

Makiko Sasada

(joint work with David Croydon, Tsuyoshi Kato and Satoshi Tsujimoto)

The Korteweg-de Vries equation (KdV equation) and the Toda lattice are typical and well-known classical integrable systems. For the KdV equation, Killip, Murphy and Visan constructed a solution of the KdV equation started from white noise, and further showed the invariance in distribution of this solution under the KdV dynamics recently [2], but the (almost-sure) well-posedness of a solution starting from a general ergodic random field on $\mathbb{R}$ is still an open problem. On the other hand, for the infinite Toda lattice, the invariance under the generalized Gibbs ensembles (GGE) are standard. Recently, Spohn shows that the generalized Gibbs free energy of the Toda chain is related to the β-ensembles of random matrix theory in the mean-field regime and obtained an exact variational formula for the density of states of the Lax matrix, when its matrix elements are distributed according to some GGE [3].

In this talk, I present our recent results on discrete time versions of the KdV equation and the Toda lattice starting from random initial conditions. In particular, we first need to study the (almost-sure) well-posedness of a solution.

First, I introduce the box-ball system (BBS), which is a cellular automaton that exhibits solitonic behaviour and show results obtained in our recent paper [1]. The box-ball system can be understood as a special class of ultra-discrete KdV equation and also a special class of ultra-discrete Toda lattice. We study the BBS dynamics using the transformation of a nearest neighbour path encoding of the particle configuration given by 'reflection in the past maximum', which was famously shown by Pitman to connect Brownian motion and a three-dimensional Bessel process. We use this to characterise the set of configurations for which the dynamics are well-defined and reversible (i.e. can be inverted) for all times. The techniques developed to understand the deterministic dynamics are subsequently applied to study the evolution of the BBS from a random initial configuration. Specifically, we give simple sufficient conditions for random initial conditions to be invariant in distribution under the BBS dynamics, which we check in several natural examples, and also investigate the ergodicity of the relevant transformation. Furthermore, we analyse various probabilistic properties of the BBS that are commonly studied for interacting particle systems, such as the asymptotic behavior of the integrated current of particles and of a tagged particle.

Next, we introduce some generalization of Pitman's transform and show that the dynamics of several discrete integrable systems, such as the discrete KdV equation, the ultra-discrete KdV equation, the discrete Toda lattice and the ultra-discrete Toda lattice are given by them. We apply this observation to define the dynamics uniquely on the infinite product space and study their invariant measures.

References

[1] D. A. Croydon, T. Kato, M. Sasada and S. Tsujimoto, *Dynamics of the box-ball system with random initial conditions via Pitman's transformation*, arXiv:1806.02147, (2018).
[2] R. Killip, J. Murphy and M. Visan, *Invariance of white noise for KdV on the line*, arXiv:1904.11910, (2019).
[3] H. Spohn, *Generalized Gibbs Ensembles of the Classical Toda Chain*, J. Stat. Phys. (2019), https://doi.org/10.1007/s10955-019-02320-5.

Universality for Critical Kinetically Constrained Models

Ivailo Hartarsky

(joint work with Laure Marêché, Fabio Martinelli, and Cristina Toninelli[5, 6])

Kinetically constrained models (KCM) are interacting particle systems on the integer lattice $\mathbb{Z}^d$, introduced in the physics literature to model the liquid-glass transition. A generic KCM is a continuous-time Markov process of Glauber type characterised by a finite collection $\mathcal{U}$ of finite nonempty subsets of $\mathbb{Z}^d \setminus \{0\}$, its *update family*. A configuration ω is defined by assigning to each site $x \in \mathbb{Z}^d$ an *occupation variable* $\omega_x \in \{0,1\}$, corresponding to an *empty* or *occupied* site

respectively. Each site $x \in \mathbb{Z}^d$ waits an independent, mean one, exponential time and then, iff there exists $U \in \mathcal{U}$ such that $\omega_y = 0$ for all $y \in U + x$, then site x is updated to empty with probability q and to occupied with probability $1 - q$. Since each $U \in \mathcal{U}$ is contained in $\mathbb{Z}^d \setminus \{0\}$, the constraint to allow the update does not depend on the state of the to-be-updated site. As a consequence, the product Bernoulli$(1 - q)$ measure, μ, is reversible and invariant.

Both from a physical and from a mathematical point of view, a central issue for KCM is to determine the speed of divergence of the characteristic time scales as $q \to 0$. Two key quantities are: (i) the *relaxation time* $T_{\rm rel}$, i.e. the inverse of the spectral gap of the Markov generator and (ii) the *mean infection time* $\mathbb{E}(\tau_0)$, i.e. the mean over the stationary process started at μ of the first time at which the origin becomes empty. These quantities can have very different scalings for different models. A question that naturally emerges, and that was first addressed in [8], is whether it is possible to group all possible update families into distinct *universality classes*, so that all models of the same class display the same divergence of time scales.

Before presenting the results and the conjectures of [8], we should describe the key connection of KCM with a class of monotone cellular automata known as $\mathcal{U}$-bootstrap percolation (or simply bootstrap percolation). $\mathcal{U}$-bootstrap percolation on $\mathbb{Z}^d$ can be defined as the discrete time version of the corresponding $\mathcal{U}$-KCM with $q = 1$, but starting from a product measure with a different parameter, which we will still denote $1-q$. In other words at each time step we empty all sites whose constraint is satisfied and never occupy additional sites. A key time scale for this dynamics is the first time at which the origin is empty (infected), $\tau_{\rm BP}$. In order to study it for models on $\mathbb{Z}^2$, the update families were classified by Bollobás, Smith and Uzzell [4] into three universality classes: *supercritical*, *critical* and *subcritical*, according to a simple geometric criterion involving *stable directions*. For supercritical update families, [4] proved that $\tau_{\rm BP} = q^{-\Theta(1)}$ w.h.p. as $q \to 0$, while in the critical case $\tau_{\rm BP} = \exp(q^{-\Theta(1)})$. Finally, in the subcritical case Balister, Bollobás, Przykucki and Smith [2] showed that $\tau_{\rm BP} = \infty$ with positive probability for q sufficiently small. The result for critical families was later improved by Bollobás, Duminil-Copin, Morris and Smith [3], who identified the critical exponent $\alpha = \alpha(\mathcal{U})$ such that $\tau_{\rm BP} = \exp(q^{-\alpha+o(1)})$.

A natural question is whether there is a direct connection between the infection mechanism of bootstrap percolation and the relaxation mechanism for KCM, and, more precisely, whether the scaling of $T_{\rm rel}$ and $\mathbb{E}(\tau_0)$ is connected to the typical value of $\tau_{\rm BP}$ when the law of the initial infections is μ. It is not difficult to establish that $\mu(\tau_{\rm BP})$ provides a lower bound for $\mathbb{E}(\tau_0)$ and $T_{\rm rel}$ [9], but in general, as we will explain, this lower bound fails to provide the correct behaviour.

In [8], Martinelli, Morris and Toninelli proposed that the supercritical class should be refined into *unrooted* supercritical and *rooted* supercritical models in order to capture the richer behavior of KCM. For unrooted models the scaling is of the same type as for bootstrap percolation, $T_{\rm rel} \sim \mathbb{E}(\tau_0) = q^{-\Theta(1)}$ as $q \to 0$ [8],

while for rooted models the divergence is much faster, $\mathbb{E}(\tau_0) \sim T_{\rm rel} = e^{\Theta((\log q)^2)}$ [7, 8].

Concerning the critical class, the lower bound with $\mu(\tau_{\rm BP})$ mentioned above and the results of [4] on bootstrap percolation imply that $T_{\rm rel}$ and $\mathbb{E}(\tau_0)$ diverge at least as $\exp(q^{-\Omega(1)})$. In [8] an upper bound of the same form was established and a conjecture was put forward on the value of the critical exponent ν such that both $\mathbb{E}(\tau_0)$ and $T_{\rm rel}$ scale as $\exp(|\log q|^{O(1)}/q^{\nu})$, with ν in general different from the exponent of the corresponding bootstrap percolation process and matching the upper bound. Furthermore, a toolbox was developed for the study of the upper bounds. The main issue left open in [8] was to develop tools to establish sharp lower bounds. A first step in this direction was made by Martinelli, Marêché and Toninelli [7] by analyzing a specific critical model known as the Duarte model. They established that the divergence is again much faster than for the corresponding bootstrap percolation model, namely the critical exponent for the Duarte KCM is twice the critical exponent for the Duarte bootstrap percolation.

In [5], with Marêché and Toninelli, we extend this result that $\nu = 2\alpha$ to all critical models for which it holds – those with an infinite number of stable directions – by proving a lower bound matching the upper one of [8]. The sharper divergence of time scales for KCM is due to the fact that τ_0 is not well approximated by the infection mechanism of the monotone bootstrap percolation process, but is instead the result of a much more complex emptying/occupying mechanism. Indeed, visiting regions of the configuration space with an anomalous amount of empty sites is heavily penalised and requires a very long time to actually take place. The basic underlying idea is that the dominant relaxation mechanism is an *East-like dynamics* for large *droplets* of empty sites. Here East-like means that the presence of an empty droplet allows to empty (or fill) another adjacent droplet but only in a certain direction (or, more precisely, in a limited cone of directions). This is reminiscent of the relaxation mechanism for the East model, a prototype one-dimensional KCM for which x can be updated iff $x-1$ is empty, thus, a single empty site allows to create/remove an empty site only on its right. For critical models, droplets have a size that diverges as $\ell = q^{-\alpha}$ and thus an equilibrium density $q_{\rm eff} = q^{\ell} \sim e^{-q^{-\alpha}}$. Then a (very) rough understanding of the results of [5, 7, 8] is obtained by replacing q with $q_{\rm eff}$ in the time scale for the East model $T_{\rm rel}^{\rm East} = e^{\Theta((\log q)^2)}$ [1]. The main technical difficulty, along with the lack of orientation of general models, to translate this intuition into a lower bound is that the droplets cannot be identified with a rigid structure. In [7] this difficulty for the Duarte model was overcome by an algorithmic construction that allows to sequentially scan the system in search of sets of empty sites that *could* (without violating the constraint) empty a certain rigid structure. For general models, however, we introduce a much more flexible notion of droplet playing the role of the empty sites for the East dynamics.

In [6], with Martinelli and Toninelli, we prove that for all other critical models, those with a finite number of stable directions, we have $\nu = \alpha$. This completes the universality partition for critical KCM and disproves the conjecture of [8].

In order to establish this stronger upper bound, we introduce a novel and very peculiar mechanism mixing oriented dynamics on small scales and non-oriented dynamics on large scales. The first step is to consider a droplet as above and move it in an East-like way. However, this is done only for a distance $q^{-O(1)}$, until it reaches the help required for moving in a transversal direction. At that point the droplet is extended transversally and then one retraces the East dynamics, bringing the extension back to the original position. This effectively results in a step in a difficult direction requiring only $O(\log(1/q))$ droplets at any single time. Iterating this procedure one can move the droplet in any direction by this quasi-local mechanism. Then one can simply move a remote droplet to the origin without ever creating more than logarthmically many droplets – it suffices to repetitively make a step towards the origin and remove the previous droplet as described.

References

[1] D. Aldous, P. Diaconis, *The asymmetric one-dimensional constrained Ising model: rigorous results*, J. Stat. Phys. **107** (2002), no. 5-6, 945–975.

[2] P. Balister, B. Bollobás, M. Przykucki, and P. Smith, *Subcritical $\mathcal{U}$-bootstrap percolation models have non-trivial phase transitions*, Trans. Amer. Math. Soc.**368** (2016), no. 10, 7385–7411.

[3] B. Bollobás, H. Duminil-Copin, R. Morris, and P. Smith, *Universality of two-dimensional critical cellular automata*, Proc. Lond. Math. Soc. (to appear).

[4] B. Bollobás, P. Smith, and A. Uzzell, *Monotone cellular automata in a random environment*, Combin. Probab. Comput. **24** (2015), no. 4, 687–722.

[5] I. Hartarsky, L. Marêché, and C. Toninelli, *Universality for critical kinetically constrained models: infinite number of stable directions*, preprint arXiv:1904.09145.

[6] I. Hartarsky, F. Martinelli, and C. Toninelli, *Universality for critical kinetically constrained models: finite number of stable directions*, in preparation.

[7] L. Marêché, F. Martinelli, and C. Toninelli, *Exact asymptotics for Duarte and supercritical rooted kinetically constrained models*, Ann. Probab. (to appear).

[8] F. Martinelli, R. Morris, and C. Toninelli, *Universality results for kinetically constrained spin models in two dimensions*, Comm. Math. Phys. **369**(2019), no. 2, 761–809.

[9] F. Martinelli and C. Toninelli, *Towards a universality picture for the relaxation to equilibrium of kinetically constrained models*, Ann. Probab. **47**(2019), no. 1,324–361.

Low-temperature Sampling Algorithms

Tyler Helmuth

(joint work with Christian Borgs, Jennifer Chayes, Will Perkins, Guus Regts, Prasad Tetali)

Associated to any discrete equilibrium statistical mechanics model are at least two fundamental computational tasks: *(approximate) counting* and *(approximate) sampling*. To describe these tasks and our results precisely, we begin by briefly recalling two well-known models of interest. Throughout we let $G = (V, E)$ denote a finite graph.

To describe the first model, let $\lambda \geq 0$, and let $\mathcal{I}_G$ denote the set of independent sets on G, i.e., sets $I \subset V$ such that $x, y \in I$ implies $\{x, y\} \notin E$. The *hard-core*

model on G is the probability distribution $\mathbb{P}_G$ on $\mathcal{I}_G$ defined by

$$\mathbb{P}_G(I) = \frac{1}{Z_G(\lambda)}\lambda^{|I|}, \qquad Z_G(\lambda) = \sum_{I\in\mathcal{I}_G} \lambda^{|I|}. \tag{1}$$

The parameter λ is called the *activity*, and intuitively a larger value of λ biases $\mathbb{P}_G$ to favour independent sets with a higher density.

To describe the second model, let $q \in \mathbb{N}$ and $\beta \geq 0$. The *q-state Potts model* is the probability distribution $\mathbb{P}_G$ on colourings $\sigma \in [q]^V$ defined by

$$\mathbb{P}_G(\sigma) = \frac{1}{Z_G(\beta, q)}e^{-\beta H(\sigma)}, \qquad Z_G(\beta, q) = \sum_{\sigma\in[q]^V} e^{-\beta H(\sigma)}. \tag{2}$$

Here $H(\sigma)$ is the number of bichromatic edges under the colouring σ, i.e., $H(\sigma) = \sum_{\{x,y\}\in E} 1_{\sigma(x)=\sigma(y)}$. The parameter β is called the *inverse temperature*. Lower temperatures (i.e., larger values of β) biases $\mathbb{P}_G$ to favour colourings with fewer bichromatic edges.

With these examples in mind, we now define the computational tasks named in the first paragraph. For definiteness, consider the Potts model with q and β fixed. Given $G = (V, E)$ and $\epsilon > 0$,

- *Counting* is the task of producing a number $\hat{Z}_G$ such that $e^{-\epsilon}Z_G \leq \hat{Z}_G \leq e^{\epsilon}Z_G$, and
- *Sampling* is the task of producing a configuration $\hat{\sigma} \in [q]^V$ with law $\hat{\mathbb{P}}$ such that $\|\hat{\mathbb{P}} - \mathbb{P}_G\|_{\mathrm{TV}} < \epsilon$.

Interest is focused on *efficient* algorithms for these tasks, meaning ones that run in time polynomial in $|V|$ and ϵ^{-1}. Modern research is focused on approximation algorithms (meaning $\epsilon > 0$) as *exact* counting and sampling (meaning $\epsilon = 0$) is typically #P-hard. It is widely believed there is no polynomial time algorithm for #P-hard problems. In what follows we will hence omit the adjective approximate.

We note that while this notion of efficiency may not conform to the everyday use of the word when the runtime is a polynomial of high degree, it has proven to be a useful and accepted notion in theoretical computer science. Moreover, once a polynomial time algorithm is found, it is often the case that a low-degree polynomial time algorithm is found soon after [11, Section 3.2].

Recall that the hard-core model has a phase transition on d-regular trees at the parameter value $\lambda_c(d) = (d-1)^{d-1}/(d-2)^d$. A tantalising link between equilibrium statistical mechanics and theoretical computer science is contained in the following results.

Theorem 1 (Weitz [10])**.** *Fix $d \geq 3$ and $0 \leq \lambda < \lambda_c(d)$. There is an efficient sampling and counting algorithm for the hard-core model with activity λ on d-regular graphs.*

Theorem 2 (Sly [8], Sly–Sun [9], Galanis–Štefankovič–Vigoda [4])**.** *Fix $d \geq 3$ and $\lambda > \lambda_c(d)$. Unless $RP = NP$, there does not exist an efficient sampling or counting algorithm for the hard-core model with activity λ on d-regular graphs.*

Thus there is a *computational* phase transition for the hard-core model on d-regular graphs. A natural, if imprecise, question arises: which input graphs G cause difficulty for efficient counting and sampling? Particularly interesting for theoretical computer science is the restriction to *bipartite* inputs [5].

Motivated by this question, consider the restriction to G being a discrete torus of side-length n, i.e., $G = (\mathbb{Z}/n\mathbb{Z})^d$. For the low density hard-core and high temperature Potts models it is known that the Glauber dynamics provides efficient counting and sampling algorithms for these input graphs for all $d \in \mathbb{N}$. Conversely, it is also known that the Glauber dynamics fail to mix rapidly at high densities and low temperatures. The following theorem provides the first known efficient algorithm for the hard-core model on $\mathbb{Z}^d$ at high densities.

Theorem 3 (Helmuth–Perkins–Regts [6]). *Fix $d \geq 2$ and $n \in 2\mathbb{N}$. There are $\lambda_0(d) > 0$ and $c(d) > 0$ such that for $\lambda \geq \lambda_0(d)$ there exists efficient counting and sampling algorithms for the hard-core model on $(\mathbb{Z}/n\mathbb{Z})^d$ provided $\epsilon > e^{-c(d)n}$.*

The algorithms of [6] also apply for any $\epsilon > 0$ for relatively general connected subgraphs of $\mathbb{Z}^d$, provided one considers the hard-core model with all-even or all-odd boundary conditions. The proof of Theorem 3 involves using the cluster expansion and Pirogov–Sinai theory for algorithmic purposes. This approach was inspired by work of Barvinok and of Patel and Regts [1, 7].

In fact, Pirogov–Sinai theory is very general, and as a result the results of [6] are very general. Namely, they apply deep inside the low temperature (or high density) phase of essentially any discrete statistical mechanics model that has finitely many ground states, all of which are stable. An example of such a model is the Potts model at low temperatures $\beta \geq \beta_0(d, q)$.

Given the existence of algorithms at very low temperatures, it is natural to wonder if the difficulty in designing efficient algorithms has to do with the behaviour of models precisely at their critical points. This question is in general open, but the next theorem provides some progress for the q-state Potts model. Recall that the q-state Potts model on $\mathbb{Z}^d$ undergoes a phase transition at a critical value $0 < \beta_c(q, d) < \infty$.

Theorem 4 (Borgs–Chayes–Helmuth–Perkins–Tetali [2]). *Fix $d \geq 2$. There is a $q_0(d)$ such that for $q \geq q_0(d)$ and any $\beta \geq 0$, there are efficient counting and sampling algorithms for the Potts model on $(\mathbb{Z}/n\mathbb{Z})^d$.*

Thus when q is large the algorithmic tasks associated to the Potts model can be performed efficiently at *all* temperatures. This result also applies to a large class of connected subgraphs of $\mathbb{Z}^d$ with appropriate boundary conditions.

The proof of Theorem 4 is via an extension of the methods used to prove Theorem 3, with the key advance being a technique to handle *unstable* ground states. An important input was the refined Pirogov–Sinai theory for the Potts model developed in [3].

An intriguing open problem is to use the Glauber dynamics to obtain fast algorithms at low temperatures. This should be possible by making use of a well-chosen (random) initial state, e.g., a fair coin flip to decide between the all-even and all-odd configurations for the hard-core model on $(\mathbb{Z}/n\mathbb{Z})^d$.

References

[1] A. Barvinok. Combinatorics and complexity of partition functions. *Algorithms and Combinatorics*, 30, 2017.

[2] C. Borgs, J. Chayes, T. Helmuth, W. Perkins, and P. Tetali. Efficient sampling and counting algorithms for the Potts model on $\mathbb{Z}^d$ at all temperatures. 2019.

[3] C. Borgs, J. T. Chayes, and P. Tetali. Tight bounds for mixing of the Swendsen-Wang algorithm at the Potts transition point. *Probab. Theory Relat. Fields*, 152(3-4):509–557, 2012.

[4] A. Galanis, D. Štefankovič, and E. Vigoda. Inapproximability of the partition function for the antiferromagnetic Ising and hard-core models. *Combinatorics, Probability and Computing*, 25(4):500–559, 2016.

[5] L. A. Goldberg and M. Jerrum. Approximating the partition function of the ferromagnetic Potts model. *J. ACM*, 59(5):Art. 25, 31, 2012.

[6] T. Helmuth, W. Perkins, and G. Regts. Algorithmic Pirogov-Sinai theory. *Probab. Theory Relat. Fields*, 2019.

[7] V. Patel and G. Regts. Deterministic polynomial-time approximation algorithms for partition functions and graph polynomials. *SIAM Journal on Computing*, 46(6):1893–1919, 2017.

[8] A. Sly. Computational transition at the uniqueness threshold. In *2010 IEEE 51st Annual Symposium on Foundations of Computer Science(FOCS)*, volume 00, pages 287–296, 10 2010.

[9] A. Sly and N. Sun. Counting in two-spin models on d-regular graphs. *The Annals of Probability*, 42(6):2383–2416, 2014.

[10] D. Weitz. Counting independent sets up to the tree threshold. In *Proceedings of the Thirty-Eighth Annual ACM Symposium on Theory of Computing*, STOC 2006, pages 140–149. ACM, 2006.

[11] A. Wigderson. *Mathematics and Computation: A Theory Revolutionizing Technology and Science*. Princeton University Press, 2019.

Reporter: Clément Erignoux

Participants

Dr. Márton Balázs
School of Mathematics
University of Bristol
Fry Building
Woodland Road
Bristol BS8 1UG
UNITED KINGDOM

Prof. Dr. Giada Basile
Dipartimento di Matematica
Università degli Studi di Roma I
"La Sapienza"
Piazzale Aldo Moro, 2
00185 Roma
ITALY

Dr. Riddhipratim Basu
International Centre for Theoretical Sciences
Tata Institute of Fundamental Research
Survey No. 151, Shivakote, Hesaraghatta
Bangalore, Bengaluru 560 089
INDIA

Dr. Roland Bauerschmidt
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Prof. Dr. Cédric Bernardin
Laboratoire J.-A. Dieudonné
UMR CNRS - UNS No. 7351
Université de Nice Sophia-Antipolis
Parc Valrose
06108 Nice Cedex 02
FRANCE

Dr. Oriane Blondel
Institut Camille Jordan
Université Claude Bernard Lyon I
43, Boulevard du 11 novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Thierry Bodineau
Centre de Mathématiques Appliquées
École Polytechnique
Route de Saclay
91128 Palaiseau Cedex
FRANCE

Prof. Dr. Erwin Bolthausen
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
8057 Zürich
SWITZERLAND

Dr. Giuseppe Cannizzaro
Department of Statistics
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Pietro Caputo
Dipartimento di Matematica e Fisica
Università di Roma III
Largo S. Leonardo Murialdo 1
00146 Roma
ITALY

Prof. Dr. Ana Patrícia Carvalho Goncalves
Instituto Superior Técnico
Universidade de Lisboa
Avenida Rovisco Pais, 1
Lisboa 1049-001
PORTUGAL

Dr. Sunil Chhita
Department of Mathematical Sciences
Durham University
Science Laboratories
South Road
Durham DH1 3LE
UNITED KINGDOM

Benoit Dagallier
CMAP
École Polytechnique
Plateau de Palaiseau
Route de Saclay
91120 Palaiseau Cedex
FRANCE

Dr. Esther Daus
Institut für Analysis und
Scientific Computing
Technische Universität Wien
Wiedner Hauptstrasse 8-10
1040 Wien
AUSTRIA

Dr. Clément Erignoux
Dipartimento di Matematica
Università degli Studi Roma Tre
Largo S. L. Murialdo, 1
00146 Roma
ITALY

Prof. Dr. Alessandra Faggionato
Dipartimento di Matematica
Università di Roma "La Sapienza"
Istituto "Guido Castelnuovo"
Piazzale Aldo Moro, 2
00185 Roma
ITALY

Dr. Alejandro Fernandez-Montero
CMAP
École Polytechnique
Plateau de Palaiseau
Route de Saclay
91120 Palaiseau Cedex
FRANCE

Prof. Dr. Patrik L. Ferrari
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Tadahisa Funaki
Graduate School of Mathematical
Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Isabelle Gallagher
UFR de Mathématiques
Université Paris Diderot (Paris VII)
Bâtiment Sophie Germain
5 Rue Thomas Mann
75013 Paris Cedex
FRANCE

Dante Grevino
Departamento de Matemática - FCEN
Universidad de Buenos Aires
Ciudad Universitaria
Pabellón 1
Buenos Aires C 1428 EGA
ARGENTINA

Amirali Hannani
CEREMADE
Université Paris Dauphine
Place du Maréchal de Lattre de Tassigny
75775 Paris Cedex 16
FRANCE

Ivailo Hartarsky
CEREMADE
Université Paris Dauphine
Place du Maréchal de Lattre de Tassigny
75775 Paris Cedex 16
FRANCE

Dr. Tyler Helmuth
Department of Mathematics
University of Bristol
University Walk
Bristol BS8 1TW
UNITED KINGDOM

Francois Huveneers
CEREMADE
Université Paris Dauphine
Place du Maréchal de Lattre de Tassigny
75775 Paris Cedex 16
FRANCE

Prof. Dr. Dmitri Ioffe
Faculty of Industrial Engineering
and Management, TECHNION
Israel Institute of Technology
Haifa 3200003
ISRAEL

Tom Klose
Fachbereich Mathematik
Technische Universität Berlin
Sekr. MA 7-1
Strasse des 17. Juni 135
10623 Berlin
GERMANY

Prof. Dr. Tomasz Komorowski
Institute of Mathematics
IMPAN
ul. Sniadeckich 8
00-656 Warszawa
POLAND

Dr. Hubert Lacoin
Instituto de Matematica Pura e Aplicada
IMPA
Jardim Botanico
Estrada Dona Castorina, 110
Rio de Janeiro, RJ 22460-320
BRAZIL

Prof. Dr. Claudio Landim
Instituto de Matematica Pura e Aplicada
IMPA
Jardim Botanico
Estrada Dona Castorina, 110
Rio de Janeiro, RJ 22460-320
BRAZIL

Dr. Benoit Laslier
UFR de Mathématiques
Université Paris Diderot (Paris VII)
Bâtiment Sophie Germain
5 Rue Thomas Mann
75013 Paris Cedex
FRANCE

Prof. Dr. Raphaël Lefévère
UFR de Mathématiques
Stochastic Models Group
Laboratoire de Probabilités et Modèles
Aléatoires
Boite Courrier 7012
75205 Paris Cedex 13
FRANCE

Vincent Lerouvillois
Institut Camille Jordan
Université Claude Bernard Lyon 1
43, Boulevard du 11 novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Carlangelo Liverani
Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1
00133 Roma
ITALY

Prof. Dr. Jani Lukkarinen
Department of Mathematics and
Statistics
University of Helsinki
P.O. Box 68
00014 Helsinki
FINLAND

Christopher Lutsko
Department of Mathematics
University of Bristol
University Walk
Bristol BS8 1TW
UNITED KINGDOM

Dr. Stefano Marchesani
Gran Sasso Science Institute
Viale Francesco Crispi, 7
67100 L'Aquila (AQ)
ITALY

Dr. Fabio Martinelli
Dipartimento di Matematica
Università Roma III
Largo S. Leonardo Murialdo, 1
00146 Roma
ITALY

Dr. Chiranjib Mukherjee
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Dr. Alessia Nota
Hausdorff Center for Mathematics
Institute for Applied Mathematics
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Stefano Olla
CEREMADE
Université Paris Dauphine - PSL
Place du Maréchal de Lattre de Tassigny
75775 Paris Cedex 16
FRANCE

Prof. Dr. Mario Pulvirenti
Facolta di Scienze Matematiche
Università di Roma
Piazza A. Moro, 1
00185 Roma
ITALY

Dr. Balázs Ráth
Institute of Mathematics
Budapest University of Technology
and Economics
Building H
1, Egry Jozsef U.
1111 Budapest
HUNGARY

Dr. Makiko Sasada
Graduate School of Mathematical
Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Assaf Shapira
UFR de Mathématiques
Université Paris-Diderot (Paris VII)
Bâtiment Sophie Germain
Bureau 5032
5 Rue Thomas Mann
75013 Paris Cedex
FRANCE

Marielle Simon
INRIA Lille - Nord Europe
Parc Scientifique de la Haute Borne
Bâtiment A
40, Avenue Halley
59650 Villeneuve d'Ascq
FRANCE

Dr. Sergio Simonella
ENS - UMPA
UMR CNRS 5669
46, Allée d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Herbert Spohn
Zentrum Mathematik
Technische Universität München
Boltzmannstrasse 3
85748 Garching bei München
GERMANY

Prof. Dr. Alexandre Stauffer
Dipartimento di Matematica e Fisica
Università degli Studi Roma Tre
Largo S. L. Murialdo, 1
00146 Roma
ITALY

Hayate Suda
Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Réka Szabó
CEREMADE
Université Paris Dauphine
Place du Maréchal de Lattre de Tassigny
75775 Paris Cedex 16
FRANCE

Prof. Dr. Domokos Szász
Mathematical Institute
Budapest University of Technology and Economics
P.O. Box 91
1521 Budapest
HUNGARY

Prof. Dr. Fabio Toninelli
Institut Camille Jordan
Université Claude Bernard Lyon I
43, Boulevard du 11 novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Bálint Tóth
School of Mathematics
University of Bristol
Fry Building
Woodland Road
Bristol BS8 1UG
UNITED KINGDOM

Dr. Lu Xu
CEREMADE
Université Paris Dauphine
Place du Maréchal de Lattre de Tassigny
75775 Paris Cedex 16
FRANCE