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Mini-Workshop: Operator Algebraic Quantum Groups

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Abstract. This mini-workshop brought together a rich and varied cross-section of young and active researchers working on operator algebraic aspects of quantum group theory. The primary goals of this meeting were to highlight the state-of-the-art results on the subject and to trigger new research by advertising some of the main open directions in operator algebraic quantum group theory: classification problems for C*- and von Neumann algebras, relations to free/non-commutative probability, applications in quantum information theory, and the creation of new quantum groups and potential classification results for subclasses of quantum groups.

Mathematics Subject Classification (2010): 46L05, 46L10, 46L54.

Introduction by the Organizers

The mini-workshop *Operator Algebraic Quantum Groups*, organised by Michael Brannan (College Station), Martijn Caspers (Delft), Moritz Weber (Saarbrücken), and Anna Wysoczanska-Kula (Wroclaw) was very well attended with 17 participants with broad geographic representation from Europe, North America, and Asia. The vast majority of participants in this mini-workshop were highly active early-career researchers (graduate students, postdocs, or pre-tenure faculty), resulting in a very productive and stimulating meeting for all who were present.

General background. Quantum groups form a relatively new area of mathematics. Their roots lie somewhere in the second half of the 20th century with the discovery of Hopf algebras and attempts to use them in quantum physics and quantum gravity. In mathematics, quantum groups provide the right framework for Pontrjagin duality in harmonic analysis, they give the correct notion of symmetry in Jones's subfactor theory, and they play a key role in non-commutative geometry and operator algebras.

Quantum groups appear in different guises: as Hopf algebras, tensor categories or operator algebras. Often one studies these different incarnations of quantum groups at the same time, as well as the interactions between them. In this workshop the focus was on the operator algebras generated by quantum groups and how they can be studied using free probability, C^*-tensor categories and of course C^*- and von Neumann algebras themselves. This workshop concentrated on topological quantum groups, i.e., on locally compact quantum groups in the sense of Woronowicz and Kustermans-Vaes.

Recent developments. In the early development of quantum groups ($\approx$ 1990's), operator algebras were mainly used as a tool to define what a quantum group actually is. In a nutshell a quantum group is a topological algebra (a C^*- or von Neumann algebra) equipped with a comultiplication and a coinverse (or antipode). If the algebra is commutative then it is the algebra of functions on a locally compact group with comultiplication and coinverse given by the pullbacks of the group multiplication and group inverse.

At the same time, quantum groups provide new and interesting examples of operator algebras. These are the analogues of group C^*-algebras and group von Neumann algebras, which are suitable closures of the image of the left regular representation (i.e. the group algebra). That this class of operator algebras should be studied was clear straight from the beginning: in the classical setting of groups, major questions around the Baum-Connes conjecture, Popa's deformation/rigidity programme, free probability, the Elliott programme, etc. had been studied in detail. But for the operator algebras coming from quantum groups, things are much less clear. But as examples were sparse and tools to tackle such major problems were absent not much was done; except on the somewhat isolated examples given by the free orthogonal and free unitary quantum groups ([Ban97], [VaVe07], [Wan95]).

In the past $5-7$ years, the landscape changed drastically and significant contributions to the structure, existence and sometimes even classification of quantum groups were made. The key example would again be the quantum orthogonal groups of which we know by these efforts many structural properties: approximation properties [Bra12], [CLR15], [CFY14], [Fre13], rigidity properties [Ara17], classification of semi-groups [Cas18], [CKF14], Baum-Connes conjecture and K-theory [Voi11], [VoVe13], description of MASA's [FrVe16], [Iso15], [Iso17], and many more. These results required new insights (many came from young researchers entering the area) and provided new techniques, especially concerning connections to category theory (see e.g. [Wor88], [CFY14], [NeYa17]), probability and naturally within the area of operator algebras/functional analysis itself.

This workshop focused on the state–of–the–art of our understanding of quantum group operator algebras and highlighted some of the major open directions in the field. The program was divided roughly into the following 4 themes.

1. Structure of the C^*- and von Neumann algebras of quantum groups. The free orthogonal quantum groups (mentioned above) are in a sense just a showcase example of a much larger class of quantum groups: compact matrix quantum groups. They are constructed through deformation and/or liberation techniques and give an uncountable class of examples. Outside the free orthogonal case (see again above) almost nothing is known about the structure of these quantum groups and their operator algebras (this is quite a remarkable in fact). The first focus of the workshop was on unraveling the structure of compact matrix quantum groups in general: specifically approaches to problems involving approximation properties, boundary actions, property (T), the Connes embedding problem, Baum-Connes Conjecture, free entropy dimension, L^2-Betti numbers, and so on. Work in this direction was reported on in the minicourses given by Roland Vergnioux and Christian Voigt, and also in the lectures given by Mehrdad Kalantar, Sven Raum, Alexandru Chirvasitu, Adam Skalski, Yuki Arano, and Martijn Caspers.

2. Creation of quantum groups. Whereas the class of classical groups is large, finding examples of quantum groups is a serious task. The initial work of Woronowicz and Drinfeld created deformations of compact simple Lie groups (like $SU_q(n)$ or its quantized enveloping Lie algebra $\mathfrak{su}_q(n)$) and for a long time Wang's free unitary/orthogonal quantum groups [Wan95] had been a more or less isolated example. New quantum groups were only found much later [BaSp09], [Kul15], [Web13], [RaWe16], [SpWe16]. This workshop's second focus was on on extending the class of examples even further. Recent progress in this direction was reported on in the minicourse given by Amaury Freslon, and also in the lectures by Laura Maassen, Piotr Søltan, Kari Eifler, and Moritz Weber.

3. Classification of subclasses of quantum groups. In group theory, classification programmes have been a central theme of research, for example the classification of (finite) simple groups. It is natural to try and classify certain classes of quantum groups. For the so-called easy (or partition) quantum groups, such a classification is known [BaSp09], [Web13], [RaWe16]. A central topic at the meeting was on the development of methods to extend classification results to broader classes, for example those arising in [TaWe16], [SpWe16], and describe their structure comprehensively. This material was also covered in the lectures by Amaury Freslon, Laura Maassen and Moritz Weber.

4. Connections to non-commutative probability. Quantum groups provide natural models for non-commutative probability spaces and therefore immediately provide a fruitful probabilistic playground. This interplay takes place for example if one studies a quantum group as a non-commutative space in which case suitable analogues of Lévy processes and stochastic processes have been found and sometimes classified [CKF14]. On the other hand there is a very strong connection with Voiculescu's theory of free probability (e.g. [KoSp09], [BCS12]). In many cases such connections also shed light on structural properties of quantum group operator algebras (see e.g. [BCV17]). This topic was covered in great detail by the minicourse given by Uwe Franz, and also the lectures given by Pierre Tarrago and Isabelle Baraquin.

Structure of the workshop. Each participant in the workshop had the opportunity to present a 45 minute lecture on their work. In addition, Uwe Franz, Amaury Freslon, Roland Vergnioux, and Christian Voigt each gave a 2-lecture minicourse on subjects related to the 4 topics outlined above. All lectures toook place in the morning and late afternoon, leaving plenty of time after lunch each day for small group discussions and collaborations.

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Mini-Workshop: Operator Algebraic Quantum Groups

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Abstracts

Some Properties of Discrete Quantum Group C^*-algebras

Roland Vergnioux

Following the advice of the organizers, I gave a survey talk about C^*- and von Neumann algebras associated with compact/discrete quantum groups. What follows is an introductory overview of this survey. Quite conveniently, the available space does not allow for a complete list of references, so that I prefer not to mention any reference.

The theory of compact quantum groups was initially formulated, and is actually most easily formulated, in the framework of C^*-algebras, as follows. A Woronowicz C^*-algebra is a unital C^*-algebra A equipped with a unital $*$-homomorphism $\Delta : A \to A \otimes A$ (the coproduct) such that i) $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ and ii) $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$. Woronowicz proved the existence of a unique (Haar) state $h : A \to \mathbb{C}$ such that $(h \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes h)\Delta = 1 \otimes h$. The image $A_r = \pi_h(A)$ of A under the GNS representation associated with h is again a Woronowicz C^*-algebra, and we say that A is reduced if π_h is faithful.

A compact quantum group G, and its discrete dual Γ, are given by a reduced Woronowicz C^*-algebra A_r, which is then denoted $A_r = C_r(G) = C^*_r(\Gamma)$. There are potentially other Woronowicz C^*-algebras A which admit A_r as their reduced version, and among them a universal one, denoted $A_u = C_u(G) = C^*_u(\Gamma)$. The von Neumann algebra associated with Γ and G is $\pi_h(A)'' = L^\infty(G) = \mathcal{L}(\Gamma)$. We say that Γ, G are of Kac type if h is tracial.

The notation above is motivated by the examples associated with usual groups: if G is a compact group, we can consider the algebra of continuous functions on G, $C(G) = C_r(G) = C_u(G)$, with coproduct $\Delta(f) = ((r,s) \mapsto f(rs))$, and if Γ is a discrete group we can consider the reduced and universal group C^*-algebras with coproduct $\Delta(\gamma) = \gamma \otimes \gamma$, where group elements are identified with the corresponding unitaries in the respective group algebras.

Wang's algebra $A_o(Q) = C_u(O^+_Q) = C^*_u(FO_Q)$ is a genuinely quantum and prototypical example for the properties considered in the rest of the talk. It is defined, for $Q \in GL_N(\mathbb{C})$ such that $Q\bar{Q} = \pm I_N$, by generators and relations as follows:

$$A_o(Q) = \langle 1, v_{ij} \mid vv^* = v^*v = 1 \quad \text{and} \quad v = Q\bar{v}Q^{-1} \rangle,$$

where $v = (v_{ij})$ and $v = (v^*_{ij}) \in M_N(A_o(Q))$. The corresponding quantum groups are of Kac type **iff** Q is unitary. For $Q = I_N$ one uses the shorthand notation O^+_N, FO_N. There are other interesting variants, such as the unitary ones, U^+_Q and FU_Q (a notation invented by C. Voigt), as well as the quantum permutation group S^+_N and its dual, not to mention other non crossing easy quantum groups. Also celebrated, but quite different examples are the q-deformations G_q of simple compact Lie groups G, such as $SU_q(2)$, for $q \in]0,1]$.

In the case of a usual discrete group Γ, the coproduct allows to recover the group inside its algebras as the set of group-like unitaries:

$$\Gamma \simeq \{u \in C^*(\Gamma) \mid u \text{ unitary}, \ \Delta(u) = u \otimes u\}.$$

This admits a higher-dimension version which becomes interesting in the general, quantum case: a corepresentation u of a Woronowicz C^*-algebra A on a finite-dimensional Hilbert space H_u is an invertible element $u \in B(H_u) \otimes A$ such that $(\mathrm{id} \otimes \Delta)(u) = u_{12}u_{13}$, using the leg notation.

What one recovers in this way is not a group anymore, but a rigid tensor C^*-category. Starting from $A = C_r(G) = C_r^*(\Gamma)$, we denote this category $\mathrm{Rep}(G) = \mathrm{Corep}(\Gamma)$, and we write $I = \mathrm{Irr}(G) = \mathrm{Irr}(\Gamma)$ for the set of irreducible corepresentations up to equivalence. A central tool in the theory is the bicharacter

$$V = \bigoplus_{u \in I} u \in M((\textstyle\bigoplus_{u\in I} B(H_u)) \otimes A),$$

which becomes the multiplicative unitary when represented on an appropriate Hilbert space. The left leg of V lives in the dual C^*-algebra $c_0(\Gamma) = C^*(G) = \bigoplus_{u\in I} B(H_u)$, and taking an ℓ^∞-sum instead of a c_0-sum one obtains the von Neumann algebra $\ell^\infty(\Gamma) = \mathcal{L}(G)$.

For $u \in \mathrm{Corep}(\Gamma)$ and $\omega \in B(H_u)^*$ the element $(\omega \otimes \mathrm{id})(u) \in C_r^*(\Gamma)$ is called a coefficient of u. The set of coefficients of all f.-d. corepresentations forms a canonical dense sub-$*$-algebra $\mathcal{O}(G) = \mathbb{C}[\Gamma] \subset C_r^*(\Gamma)$ which is actually a Hopf-$*$-algebra with respect to the restriction of the coproduct. One can in fact axiomatize the theory purely algebraically at the level of this Hopf-$*$-algebra.

The discrete quantum group Γ is called (strongly) amenable if the GNS representation $\pi_h : C_u^*(\Gamma) \to C_r^*(\Gamma)$ is faithful — i.e. there is only one Woronowicz C^*-algebra associated to Γ. This is equivalent to the existence of an invariant (non-normal) state on $\ell^\infty(\Gamma)$ (in the locally compact case this equivalence is still open). Amenability implies nuclearity of $C_r^*(\Gamma)$ and injectivity of $\mathcal{L}(\Gamma)$; the converse holds in the Kac case but is open in general.

We also have the following useful characterization of amenability in terms of positive-definite functions: the existence of a net of linear maps $\mu_i : \mathbb{C}[\Gamma] \to \mathbb{C}$ such that i) μ_i is a state for all i, ii) for all i we have $(\mathrm{id} \otimes \mu_i)(u) = 0$ for all but a finite number of $u \in \mathrm{Irr}(\Gamma)$, iii) for all $u \in \mathrm{Irr}(\Gamma)$ we have $(\mathrm{id} \otimes \mu_i)(u) \to \mathrm{id}$ as $i \to \infty$.

The duals of all compact groups and all q-deformations G_q are amenable — and this includes in fact FO_Q when $N = 2$. On the other hand Banica proved that FO_Q (resp. FU_Q) is not amenable when $N \geq 3$ (resp. $N \geq 2$). However, there are still values of Q for which we don't know whether $\mathcal{L}(FO_Q)$ is non-injective, see below.

As in the classical case, it is possible and interesting to study weaker versions of amenability. Weak amenability with constant $\leq C$ (also called CBAP) is obtained by relaxing condition i) into i') $(\mu_i \otimes \mathrm{id})\Delta$ extends to a completely bounded map $m_i : C_r^*(\Gamma) \to C_r^*(\Gamma)$ with $\|m_i\|_{cb} \leq C$ for all i. The Haagerup approximation property (HAP) is obtained by relaxing condition ii) into ii') for all i we have $(\mathrm{id} \otimes \mu_i)(u) \to 0$ in norm as $u \to \infty$ in $\mathrm{Irr}(\Gamma)$.

These properties imply the corresponding ones for the von Neumann algebra $\mathcal{L}(\Gamma)$, defined as approximation properties (in the pointwise $*$-weak sense) of the identity map $\mathrm{id} : \mathcal{L}(\Gamma) \to \mathcal{L}(\Gamma)$ by uniformly completely bounded (resp. completely positive), finite rank (resp. L^2-compact) maps. The converse is again known in the Kac case but open in general.

Thank to the work of many agile hands (most of them participating to the mini-workshop) it was shown that all (non-classical) discrete quantum groups mentioned in this survey have the HAP and the CBAP with constant 1. In fact, they even have these properties *centrally*, meaning that the μ_i's can be chosen so that $(\mathrm{id} \otimes \mu_i)(u)$ is a scalar multiple of the identity matrix for all $u \in \mathrm{Irr}(\Gamma)$.

The advantage of central approximation properties is that they transfer through monoidal equivalence, a major tool in the study of compact quantum groups. Examples of discrete quantum groups which do not have such central approximation properties — even satisfying the central Property (T), but however (non-centrally) amenable — have been discussed in Y. Arano's lecture.

We now arrive to the structure of $C_r^*(\Gamma)$ and $\mathcal{L}(\Gamma)$. In the case of FU_Q $(N \geq 2)$ it is relatively easy to show, using the free combinatorics of the fusion rules between irreducible corepresentations, that $C_r^*(FU_Q)$ is simple with unique KMS state (with respect to the KMS group of the Haar state), and that $\mathcal{L}(FU_Q)$ is a full factor, of type II_1 in the Kac case, III_λ $(0 < \lambda \leq 1)$ otherwise.

In the case of FO_Q $(N \geq 3)$ the fusion rules are commutative and one has to use a delicate spectral gap property for the operator of "conjugation by generators", so that simplicity, uniqueness of KMS state, factoriality, fullness and type as above are only known for certain matrices Q, including all Q sufficiently close to unitaries. In theses cases fullness entails non-injectivity of $\mathcal{L}(FO_Q)$. It is of course tempting to seek a better method that would prove these properties for all parameters Q.

One can then go further and prove that $M = \mathcal{L}(FO_Q)$, $\mathcal{L}(FU_Q)$ are strongly solid von Neumann algebras i.e., for every diffuse amenable subalgebra $Q \subset M$ with expectation, the normalizer $\mathcal{N}_M(Q) = \{u \in \mathcal{U}(M) \mid uQu^* = Q\}''$ is amenable. When M is a non-injective factor, this implies that it cannot be decomposed as a tensor product of non-type I factors, nor as a group measure space factor. Note that $\mathcal{L}(FO_Q)$ was the first known example of a solid type III factor, back in 2005.

Strong solidity clearly implies, in the non-injective case, the absence of Cartan subalgebras, i.e. maximal abelian subalgebras $A \subset M$ such that $\mathcal{N}_M(A) = M$. It remains however interesting to study general maximal abelian subalgebras (MASA). One can show for instance that the subalgebra $A = \chi_1'' \subset \mathcal{L}(FO_N)$ generated by the fundamental character $\chi_1 = \sum_{i=1}^N v_{ii}$ is a singular MASA, i.e. $\mathcal{N}_M(A) = A$. This is an analogue of the radial MASA in free group factors, and it would be interesting to know whether the subalgebras generated by one generator, e.g. $A = u_{11}''$, are maximal abelian, or maybe even maximal amenable as in the free group case.

In view of all results mentioned above, it is natural to ask whether $\mathcal{L}(FO_N)$, $\mathcal{L}(FU_N)$ are isomorphic to free group factors. It was proved recently, using deep results from the theory of free entropy, that this is not the case for $\mathcal{L}(FO_N)$. In the unitary case Banica proved that $\mathcal{L}(FU_2) \simeq \mathcal{L}(F_2)$, and the question remains wide open for higher values of N.

As far as the reduced C^*-algebras $C_r^*(\Gamma)$ are concerned, the first task beyond simplicity and nuclearity is to compute their K-theory groups. Since this is the subject of C. Voigt's lecture, let us just mention the values $K_*(C_r^*(FO_N)) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(1)}$ and $K_*(C_r^*(FU_N)) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(1)}^2$. In particular K-theory does not allow to recover the value of N as in the free group case, and it remains open to know whether $C_r^*(FO_M) \simeq C_r^*(FO_N)$ for $M \neq N$.

In the last part of the lecture I discussed the Property of Rapid Decay (RD), which is a basic tool for the study of reduced C^*- and von Neumann algebras of discrete (quantum) groups, and is actually needed to prove some of the results mentioned above.

In the case of a classical discrete group Γ, Property RD amounts to controlling the norm of $C_r^*(\Gamma)$ from above by the 2-norm. More precisely a discrete group Γ has Property RD if there exists a polynomial P such that

$$\|x\|_{C_r^*(\Gamma)} \leq P(k)\|x\|_2 \tag{1}$$

for all $k \in \mathbb{N}$ and all $x \in C_r^*(\Gamma)$ supported on group elements of length k (with respect to some fixed length, for instance a word length if Γ is finitely generated). Note that the reverse inequality $\|x\|_2 \leq \|x\|_{C_r^*(\Gamma)}$ is always true.

I introduced in 2007 a quantum generalization of Property RD by means of the same inequality, with appropriate notions of length and support (and the 2-norm associated with the Haar state). It was shown in that article that Property RD holds for FO_N and FU_N but fails for any discrete quantum group which is not of Kac type.

Then a modification of the definition was proposed by Bhowmick, Voigt and Zacharias so as to accommodate non-Kac examples such as $SU_q(2)$, G_q, and more generally all discrete quantum groups with polynomial (classical) growth. This modification is obtained by replacing the 2-norm on the right-hand side of (1) by a still "easily computable" twisted 2-norm, $\|x\|_{2,\sqrt{C}} = \|x * \varphi_{\sqrt{C}}\|_2$, where $x * \varphi = (\varphi \otimes \mathrm{id})\Delta(x)$ for $x \in \mathbb{C}[\Gamma]$, $\varphi \in \mathbb{C}[\Gamma]^*$, and $\varphi_{\sqrt{C}}$ is determined by

$$(\mathrm{id} \otimes \varphi_{\sqrt{C}})(u) = \sqrt{C_u} \quad \text{with} \quad C_u = \frac{\mathrm{qdim}(u)}{\dim(u)} F_u$$

for all $u \in \mathrm{Irr}(\Gamma)$. Here the $F_u \in B(H_u)_+$ are Woronowicz' modular matrices.

However we could prove recently, with M. Brannan and S.-G. Youn, that the non-Kac type, non-amenable discrete quantum groups FO_Q still do not satisfy this twisted Property RD. It is then possible, and sometimes useful, to formulate an even weaker statement, where the matrices $\sqrt{C_u}$ above are replaced with the diagonal matrices $D_u = \|F_u\|\,\mathrm{id}$. The right-hand side of (1) is then exponentially growing with k in the non-Kac case, but at least the statement holds true for all quantum groups FO_Q and all discrete quantum groups with polynomial growth.

On the Baum-Connes conjecture for complex semisimple quantum groups

Christian Voigt

Let G be a locally compact group. The Baum-Connes conjecture [2] asserts that the assembly map

$$\mu : K_*^{top}(G) \to K_*(C_r^*(G))$$

is an isomorphism. Here $K_*(C_r^*(G))$ denotes the K-theory of the reduced group C^*-algebra of G, and the so-called topological K-theory $K_*^{top}(G)$ is defined as the equivariant K-homology with G-compact supports of the universal proper G-space. The Baum-Connes conjecture is known to hold for large classes of groups, including in particular all connected Lie groups.

We study an analogue of this conjecture for complex semisimple quantum groups, that is, Drinfeld doubles of q-deformations of compact semisimple Lie groups [6]. Here the definition of $K_*^{top}(G)$ in terms of the universal proper G-space does not make sense a priori, but a general framework for formulating an assembly map in our situation has been developed by Meyer and Nest [3].

We fix $q \in (0,1)$ and consider the standard q-deformation K_q of a simply connected compact semisimple Lie group K. The complex semisimple quantum group G_q corresponding to the complexification of K is defined in terms of its C^*-algebra of functions

$$C_0(G_q) = C(K_q) \otimes C^*(K_q),$$

which becomes a Hopf C^*-algebra with the comultiplication

$$\Delta_{G_q} = (\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes ad(W) \otimes \mathrm{id})(\Delta \otimes \hat{\Delta}).$$

Here $ad(W)$ denotes conjugation with the fundamental multiplicative unitary $W \in M(C(K_q) \otimes C^*(K_q))$ associated with the compact quantum group K_q, and σ is the flip map.

The equivariant Kasparov category KK^{G_q} has a objects all separable G_q-C^*-algebras, that is, separable C^*-algebras with a continuous injective coaction of the Hopf C^*-algebra $C_0(G_q)$. Morphisms in KK^{G_q} are given by the equivariant Kasparov groups $KK^{G_q}(A, B)$, and composition of morphisms is given by Kasparov product, see [1]. It is known that the equivariant Kasparov category is a triangulated category in a natural way. The analogous category KK^{K_q} associated to the quantum subgroup $K_q \subset G_q$ is linked to KK^{G_q} via induction and restriction functors.

In analogy to the classical situation we consider the full subcategories $\mathcal{CC}$ and $\mathcal{CI}$ of KK^{G_q} defined by

$$\mathcal{CC} = \{A \in KK^{G_q} \mid \mathrm{res}^{G_q}_{K_q}(A) \cong 0 \text{ in } KK^{K_q}\}$$

and

$$\mathcal{CI} = \{A \in KK^{G_q} \mid A \cong \mathrm{ind}_{K_q}^{G_q}(B) \text{ for some } B \in KK^{K_q}\},$$

respectively. The category $\mathcal{CC}$ is localising, and we let $\langle \mathcal{CI} \rangle$ be the localising subcategory generated by $\mathcal{CI}$.

Using the fact that $K_q \subset G_q$ is an open quantum subgroup, it follows from the results in [4] that the categories $\mathcal{CC}$ and $\langle \mathcal{CI} \rangle$ are complementary. Hence the general machinery from [4] shows that there exists an exact triangle in KK^{G_q} of the form

$$\mathcal{P} \longrightarrow \mathbb{C} \longrightarrow N \longrightarrow \mathcal{P}[1]$$

with $\mathcal{P} \in \langle \mathcal{CI} \rangle$ and $N \in \mathcal{CC}$. Taking reduced crossed products with G_q gives an exact triangle in KK, and in particular a morphism $G_q \ltimes_r \mathcal{P} \to G_q \ltimes_r \mathbb{C} = C^*_r(G_q)$. The assembly map for G_q is the map on the level of K-theory induced by this morphism.

Our main result is the following theorem.

Theorem 1. *Let $q \in (0,1)$ and let G_q be a complex semisimple quantum group. Then G_q satisfies the Baum-Connes conjecture in the sense that the assembly map*

$$\mu_q : K_*(G_q \ltimes_r \mathcal{P}) \to K_*(C^*_r(G_q))$$

is an isomorphism.

Our proof proceeds by constructing a model for the algebra $\mathcal{P}$ obtained from the Koszul complex for the representation ring of K_q.

We also show that the the assembly map in the deformation picture obtained in [5] is canonically isomorphic to the assembly map μ_q obtained from the abstract categorical setting as above.

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Probability on Quantum Groups

Uwe Franz

1. Lévy processes

We give a short introduction to the theory of Lévy processes on quantum groups and dual groups. In the first section we present the basic definitions. In the second section we recall a recent result that characterizes the Haagerup property of discrete quantum groups in terms of their Schürmann triples.

1.1. **On topological semigroups.** In classical probability, a Lévy processes is a stochastic process with independent and stationary increments. They are exactly the Markov processes that are time and space homogeneous. Here is a general definition for Lévy processes with values in a (topological) semigroup.

Definition. Let $(G, \cdot, e)$ be a semigroup with unit e. A stochastic process $(X_{st})_{0\leq s\leq t}$ of G-valued random variables is called a *Lévy process*, if it has the following properties:

(i) (Increment property) $X_{st} \cdot X_{tu} = X_{su}$ almost surely for all $0 \leq s \leq t \leq u$;
(ii) (Independence) The increments $X_{s_1t_1}, \ldots, X_{s_nt_n}$ are stochastically independent for all $x \in \mathbb{N}$ and $0 \leq s_1 \leq t_1 \leq \cdots \leq s_n \leq t_n$;
(iii) (Stationarity) The marginal distributions $P_{X_{st}}$ depend only on $t - s$, i.e., X_{st} and $X_{s+h,t+h}$ are identically distributed for all $0 \leq s \leq t$ and $h \geq 0$;
(iv) (Weak continuity) $X_{st} \xrightarrow{t\searrow s} e$ in probability for all $s \geq 0$.

1.2. **On involutive bialgebras.** In quantum probability [12, 15], probability spaces are replaced by algebraic (or quantum) probability spaces which are pairs (A, Φ), where A is a unital *-algebra and $\Phi : A \to \mathbb{C}$ a normalized positive linear functional, i.e., a state. Random variables are replaced by unital *-homomorphisms $j : B \to A$ taking values in the *-algebra of some quantum probability space.

To define Lévy processes in the context of quantum probability, we need furthermore a notion of independence and a composition of random variables. The approach by Accardi, Schürmann, and von Waldenfels [1] uses tensor independence, see (ii) in the Definition below, which corresponds to the independence of observables in quantum mechanics. They also require B to be a *-bialgebra, so that quantum random variables $j, k : B \to A$ can be composed using the convolution $j \star k = m_A \circ (j \otimes k) \circ \Delta_B$. If j and k are independent, so that their ranges commute, then $j \star k$ is again a unital *-homomorphism.

Definition. Let (A, Φ) be an algebraic probability space and B an involutive bialgebra. A family of *-homomorphisms $(j_{st} : B \to A)_{0\leq s\leq t}$ is a *Lévy process* on B over (A, Φ), if it satisfies the following conditions.

(i) (Increment property) $j_{ss} = \varepsilon \mathbf{1}_A$ and $j_{st} \star j_{tu} = j_{su}$ for all $0 \leq s \leq t \leq u$;
(ii) (Independence) $j_{s_1t_1}(a)j_{s_2t_2}(b) = j_{s_2t_2}(b)j_{s_1t_1}(a)$ for all $a, b \in B$, $0 \leq s_i \leq t_1$, whenever $(s_1, t_1) \cap (s_2, t_2) = \emptyset$, and

$$\Phi\big(j_{s_1t_2}(b_1) \cdots j_{s_nt_n}(b_n)\big) = \Phi\big(j_{s_1t_2}(b_1)\big) \cdots \Phi\big(j_{s_nt_n}(b_n)\big)$$

for all $n \geq 1$, $0 \leq s_1 \leq t_1 \leq le \cdots \leq s_n \leq t_n$, $b_1, \ldots, b_n \in B$;

(iii) (Stationarity) $\Phi\big(j_{st}(b)\big) = \Phi\big(j_{s+h,t+h}(b)\big)$ for all $0 \leq s \leq t$ and $h \geq 0$;

(iv) (Weak Continuity) $\lim_{t\searrow s} \Phi\big(j_{st}(b)\big) = \Phi\big(j_{ss}(b)\big) = \varepsilon(b)$ for all $b \in B$, $s \geq 0$.

It follows that $\varphi_{t-s}(b) = \Phi\big(j_{st}(b)\big)$ defines a convolution semigroup of states on B and Schürmann [16] showed that Lévy processes and their convolution semigroups are uniquely determined (up to stochastic equivalence) by their *generating functional*, which can be defined as

$$\psi(b) = \lim_{t\searrow 0} \frac{1}{t}\big(\varphi_t(b) - \varepsilon(b)\big) \qquad b \in B.$$

Definition. Let B be a unital *-algebra and $\varepsilon : B \to \mathbb{B}$ a unital *-homomorphism. A linear functional $\psi : B \to \mathbb{C}$ is called a *generating functional* (w.r.t. ε), it

(i) (Normalization) $\psi(\mathbf{1}_B) = 0$;

(ii) (Hermitianity) $\psi(b^*) = \overline{\psi(b)}$ for all $b \in B$;

(iii) (Conditional positivity) $\psi(b^*b) \geq 0$ for all $b \in \ker\varepsilon$.

Schürmann's Schönberg correspondence [16, Theorem 3.2.7] states that for an involutive bialgebra B, a linear functional $\psi : B \to \mathbb{C}$ is a generating functional (w.r.t. the counit) if and only if $\exp_\star t\psi$ is a state for all $t \geq 0$.

1.3. **On dual groups.** Let $\mathcal{C}$ be a category that has coproducts for any finite collection of objects $B_1, \ldots, B_n \in \mathrm{Ob}\,\mathcal{C}$. A coproduct for the empty collection is an initial object, i.e., an object $I \in \mathrm{Ob}\,\mathcal{C}$ that has a unique morphism $i_B : I \to B$ for any $B \in \mathrm{Ob}\,\mathcal{C}$. It is characterized by the universal property that for any morphism $f : A \to B$ we have $i_B = f \circ i_A$. The coproduct of a single object $B \in \mathrm{Ob}\,\mathcal{C}$ is simply the object itself, $(B, \mathrm{id}_B : B \to B)$. The coproduct $(A \coprod B, j_A : A \to A \coprod B . j_B : B \to A \coprod B)$ of two objectis is also characterized by a universal property: for any pair of morphisms $h : A \to C$ and $k \to C$ there exists a unique morphism $h \Box k : A \coprod B \to C$ such that $h = (h\Box k) \circ j_B$ and $k = (h \Box k) \circ j_B$. All other finite coproducts can be construction from the initial object and the coproducts of pairs, see [11, Proposition III.5.1].

To arrive at the definition of a co-group in a category with finite products, on reformulates the axioms for the unit element, the product, and the inverse of a group, as commutative diagrams, and one reverses the direction of the arrows.

Definition. (Cf. [2, Definition 2.1], [14, Theorem 2.2]) Let $\mathcal{C}$ be a cateory with finite coproducts. A *co-group* in $\mathcal{C}$ is a quadruple $(A, \varepsilon, \Delta, S)$, where

(i) $A \in \mathrm{Ob}\,\mathcal{C}$;

(ii) (Coassociativity) $\Delta : A \to A \coprod A$ is a morphism s.t.

$$(\Delta \coprod \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \coprod \Delta) \circ \Delta;$$

(iii) (Unit property) $\varepsilon : A \to I$ is a morphism s.t.

$$(\varepsilon \coprod \mathrm{id}_A) \circ \Delta = \mathrm{id}_A = (\mathrm{id}_A \coprod \varepsilon) \circ \Delta$$

(where we identify $I \coprod A \cong A \cong A \coprod I$);

(iv) (Inverse or antipode property) $S : A \to A$ is a morphism such that

$$(S\Box\mathrm{id}_A) \circ \Delta = i_A \circ \varepsilon = (\mathrm{id}_A\Box S) \circ \Delta.$$

Commutative Hopf algebras are co-groups in the category of unital associative algebras. Voiculescu [18] defined and studied *dual groups* as co-groups in the category of pro-C*-algebras. Schürmann defined *dual groups* and *dual semigroups* as co-groups and co-semigroups (defined similarly as co-groups, but without the antipode S), resp., in the category of unital *-algebras, and then defined Lévy processes on them.

To define independence for quantum random variables he used natural products of algebraic probability spaces. A product on the category of algebraic probability spaces is called *natural*, if it is of the form $(A, \Phi) \bullet (B, \psi) = (A \coprod B, \Phi \bullet \Psi)$, i.e., the underlying algebra of the product is given by the free product, and $\bullet$: States$(A) \times$ States$(B) \to$ States$(A \coprod B)$ is associative and 'functorial.' See, e.g., [13] for details.

Definition. Let $\bullet$ be a natural product on the category of algebraic probability spaces. Then two random variables $j : B \to (A, \Phi)$ and $k : C \to (A, \Phi)$ with values in the same algebraic probability space (A, Φ) are called *independent*, if

$$\Phi \circ (j\Box k) = (\Phi \circ j) \bullet (\Phi \circ k).$$

In [4] it is explained how this definition can be motivated by dualizing the definition of stochastic independence in classical probability.

Muraki [13] has shown that there exist (essentially) five natural products: the tensor product, the free product, the boolean product, the monotone product, and the anti-monotone product. See also [8].

Definition. Let (B, ε, Δ) be a dual semigroup in the category of unital *-algebras, $\bullet$ a natural product on the category of quantum probability spaces, and (A, Φ) an algebraic probability space. A family of *-homomorphisms $(j_{st} : B \to A)_{0\leq s\leq t}$ is a *Lévy process* (w.r.t. $\bullet$) on B over (A, Φ), if the following hold:

(i) (Increment property) $j_{ss} = \varepsilon \mathbf{1}_A$ and $(j_{st}\Box j_{tu}) \circ \Delta = j_{su}$ for all $0 \leq s \leq t \leq u$;

(ii) (Independence) $j_{s_1 t_1}, \ldots, j_{s_n t_n}$ are independent (w.r.t. $\bullet$) for all $n \geq 1$, $0 \leq s_1 \leq t_1 \leq le \cdots \leq s_n \leq t_n$, $b_1, \ldots, b_n \in B$;

(iii) (Stationarity) $\Phi\big(j_{st}(b)\big) = \Phi\big(j_{s+h,t+h}(b)\big)$ for all $0 \leq s \leq t$ and $h \geq 0$;

(iv) (Weak Continuity) $\lim_{t\searrow s} \Phi\big(j_{st}(b)\big) = \Phi\big(j_{ss}(b)\big) = \varepsilon(b)$ for all $b \in B$, $s \geq 0$.

When working with dual semigroups, one has to distinguish between the convolution of random variables, $j \star k = (j\Box k) \circ \Delta$ that is used in the increment property, and the convolution of states $\phi \star_\bullet \psi = (\phi \bullet \psi) \circ \Delta$ which depends on the choice of the natural product $\bullet$.

Schürmann [17] showed that the marginal states $\phi_{t-s} = \Phi \circ j_{st}$ of a Lévy process form again a convolution semigroup, and that the Lévy processes and their convolution semigroups of states are again uniquely characterizes by their generating functional. Franz [5] gave a different proof of these results for the tensor, boolean, monotone, and anti-monotone product, and construction of the processes on the symmetric Fock space, using a 'reduction' (or 'unification') of independences.

Question. Does their exist a 'reduction' of the free product of algebraic probability spaces?

Gilliers [9] has introduced a construction of noncommutative gauge theories based on Lévy on dual semigroups (called Zhang algebras in his paper, for Zhang's contribution [19])

2. Generating functionals and Hochschild cohomology

Lévy processes on involutive bialgebras and dual semigroups turned out to be uniquely characterized by their generating functionals, which are normalized hermitian linear functional that are positive on the kernel of the counit. Therefore it becomes interesting to classify these functionals for given algebras, e.g., for the Hopf *-algebras of compact quantum groups.

In the first step, on shows that one can complete any generating functional to a Schürmann triple, by a kind of GNS construction.

Definition. Let D be a pre-Hilbert space and denote by $L(D)$ the *-algebra of linear adjointable operators on D. A triple $(\pi : B \to L(D), \eta : B \to D, \psi : B \to \mathbb{C})$ is called a *Schürmann triple on* (B, ε) *over* D if

(i) π is a unital $*$-representation,
(ii) η is a π-ε-cocycle, i.e., $\eta(ab) = \pi(a)\eta(b) + \eta(a)\varepsilon(b)$ for $a, b \in B$
(iii) ψ is a hermitian linear functional with $B \otimes B \ni (a \otimes b) \mapsto -\langle \eta(a^*), \eta(b) \rangle \in \mathbb{C}$ as coboundary, i.e.,

$$\varepsilon(a)\psi(b) - \psi(ab) + \psi(a)\varepsilon(b) = -\langle \eta(a^*), \eta(b) \rangle, \qquad a, b \in B.$$

A crucial step in classifying generating functionals on a given *-algebra B (w.r.t. to a fixed *-hom. $\varepsilon : B \to \mathbb{C}$) is deciding if a given *-representation π and a given π-ε-cocyle can be completed to a Schürmann triple, i.e., whether the bilinear map in condition (iii) in the Definition above is a coboundary. (It is always a cocycle, cf. [7, Proposition 3.1]).

[3, Theorem 2.8] gives a positive answer to this question for so-called α-real cocycles, thereby generalizing a result by Vergnioux (first published in [10]). This allowed us to prove the following characterisation of the Haagerup property for discrete quantum groups.

Theorem. [3, Theorem 3.6] A discrete quantum group $\Gamma = \hat{\mathbb{G}}$ has the Haagerup property if and only if $\mathbb{C}\Gamma \cong \mathrm{Pol}(\mathbb{G})$ admits a proper cocycle.

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On the classification of non-crossing partition quantum groups

Amaury Freslon

The connection between compact quantum groups and the combinatorics of partitions has been known since the founding works of T. Banica on representation theory. It was formalized in a systematic way by T. Banica and R. Speicher in the seminal paper [2], under the name of *easy quantum groups*. This was followed by several generalisations, and we will focus on one called *partition quantum groups* and introduced in [3]. Classifying these compact quantum groups is a natural and fundamental problem which has attracted a lot of attention in the past ten years and the purpose of this mini-course is to describe some of the results obtained with a perspective which is different from the original one.

Let us start by recalling the fundamental construction. A *category of partitions* is a collection $\mathcal{C}$ of partitions of finite sets which is stable under a set of operations defined through a graphical representation : horizontal concatenation, vertical concatenation, reflection and rotation. Given any integer N, there is a canonical way to turn $\mathcal{C}$ into a concrete rigid C*-tensor category, and the machinery of Tannaka-Krein duality [7] then produces a compact quantum group $\mathbb{G}_N(\mathcal{C})$. This can be extended to the setting where the partitions are coloured in a suitable sense by a set $\mathcal{A}$, and the resulting objects are called *partition quantum groups*.

We focus on the case of non-crossing partitions, that is to say those which can be drawn in such a way that lines do not cross. The uncoloured case was settled by M. Weber in [6], building on the results of [2]. One first defines a *local invariant* $BS(\mathcal{C})$ recording the size of blocks of partitions in $\mathcal{C}$. This already yields a set of four examples, namely

$$\mathcal{S} = \{O_N^+, B_N^+ * \mathbb{Z}_2, H_N^+, S_N^+ \times \mathbb{Z}_2\}.$$

Then, a *global invariant* $BN(\mathcal{C})$ counting the number of odd blocks encodes the possibility of making the non-trivial one-dimensional representation trivial, yielding B_N^+ and S_N^+. Eventually, considering the presence or absence in $\mathcal{C}$ of the so-called *positioner partition*

$$p = \{\{1\}, \{2, 4\}, \{3\}\}$$

enables to make the non-trivial one-dimensional representation of $B_N^+ * \mathbb{Z}_2$ central, yielding $B_N^+ \times \mathbb{Z}_2$.

The previous quantum groups are *orthogonal* in the sense that their defining representation is self-conjugate. The general unitary case is obtained by considering two colours which are in a sense inverse to one another and implement the conjugation operation. These were classified by P. Tarrago and M. Weber in [5] using a family of complexification operations inspired by the free complexification of T. Banica in [1], called the *d-free*, *d-tensor* and *r-self-adjoint d-free* complexifications. It then turns out that all non-crossing partition quantum groups can be obtained from $\mathcal{S}$ by applying at most once each of these three operations.

We then consider the case of two colours which are their own conjugates and use a different approach based on the connection between the combinatorial structure of a category of partitions $\mathcal{C}$ and the representation theory of the corresponding compact quantum group $\mathbb{G}_N(\mathcal{C})$ established in [4]. Using this, three natural operations on the class of compact quantum groups appear to play a central role, namely

- Quotienting by relations inside the group of one-dimensional representations,
- Quotienting by commutation relations between one-dimensional representations and higher-dimensional ones,
- Making amalgamated free product in a twisted way (see [4, Def 3.12]).

This is however not enough to describe all non-crossing partition quantum groups on two colours. More precisely, this class can be divided into two subclasses, one which is obtained from the generating set $\mathcal{S}$ using the three operations, and one which consists in all *free wreath products of pairs* (see [4, Sec 3.2]), a generalization of free wreath products allowing for non-trivial one-dimensional representations. That this yields everything is the main result of [4].

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Non-local games and the graph isomorphism game

Kari Eifler

(joint work with Michael Brannan, Alexandru Chirvasitu, Samuel Harris, Vern Paulsen, Xiaoyu Su, Mateusz Wasilewski)

Non-local games give us a way of observing quantum behaviour through the observation of only classical data. A non-local game consists of two players, Alice and Bob, cooperating to win a round of the game. The referee gives each player a question from a set list, and the two players then resond with an answer from a set list of outputs. The referee uses the rules of the game to determine whether the two players win that round or not. Their goal, of course, is to win every round of the game. The catch is that the the players are separated and are unable to communicate by any classical means during each round of the game, leading to the term 'non-local'. They can, however, agree upon a shared strategy for producing winning answers.

For certain games, the two players cannot win using only classical means. In order to win, the players must utilize some shared resource of quantum entanglement between the players, which we will call a quantum strategy. There a number of mathematical strategies to describe these quantum strategies: quantum, quantum spacial, quantum approximate, and quantum commuting. For one class of non-local games, called synchronous games, each game $\mathcal{G}$ has an associative algebra $\mathcal{A}(\mathcal{G})$ associated to the game whose structure completely characterizes the existence of winning strategies for the game.

In this talk, I will focus on the graph isomorphism game, $\mathrm{Iso}(X, Y)$. In this game, two graphs X and Y are fixed and the disjoint union of the vertices will be the questions and answers; the rules of the game come from the structure of the two graphs. Winning conditions require that

- the input and output vertices for Alice belong to different graphs
- ditto for Bob

- the relation between the inputs of the two players is the same as the relation between the two outputs, where the relation may be equal, adjoint, or distinct and disconnected

As shown in [2], the rules are such that a winning strategy exists if and only if the two graphs are isomorphic. For the graph isomorphism game, the game $*$-algebra $\mathcal{A}(\mathrm{Iso}(X,Y))$ can be viewed as a non-commutative analogue of the space of isomorphisms from X to Y. If a quantum winning strategy exists, then we call the two graphs "quantum isomorphic". In [1], we show that

Theorem 1. *Given two graphs, X and Y, the following are equivalent:*

1. $\mathcal{A}(Iso(X,Y))$ *is non-zero (in this case, we call X and Y algebraically quantum isomorphic)*
2. $\mathcal{A}(Iso(X,Y))$ *admits a non-zero C^*-representation*
3. $Iso(X,Y)$ *has a perfect quantum commuting strategy*

We consider quantum graphs (finite-dimensional C^*-algebras equipped with some additional structure mimicking an adjacency matrix). Every quantum graph X has a quantum automorphism group G_X, which as the name suggests, is a quantum group.

Theorem 2. *Let X and Y be two quantum graphs. If the quantum isomorphism space $\mathcal{A}(Iso(X,Y))$ is non-trivial, then it admits a faithful state and the two quantum groups G_X and G_Y are monoidally equivalent.*

We have a sort of converse to this statement:

Theorem 3. *If a quantum graph G is monoidally equivalent to G_X for a classical or quantum graph X, then there exists a quantum graph Y such that $G \cong G_Y$ and X and Y are algebraically quantum isomorphic as graphs.*

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Deligne categories and easy quantum groups

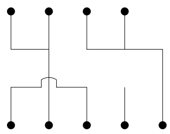
Laura Maassen

(joint work with Johannes Flake, Moritz Weber)

The symmetric group $\mathsf{S}_n \subseteq \mathbb{C}^{n\times n}$ and the partition algebra $P_k(n)$ centralise one another in their actions on $(\mathbb{C}^n)^{\otimes k}$. Hence, we can describe the category of finite-dimensional representations of the symmetric group $\mathrm{Rep}(\mathsf{S}_n)$ using set partitions. This was the starting point for Deligne's work [4] in which he defined the categories $\underline{\mathrm{Rep}}(\mathsf{S}_t)$ for arbitrary $t \in \mathbb{C}$ which interpolate in a certain way the representation categories $\mathrm{Rep}(\mathsf{S}_n), n \in \mathbb{N}$. An analogous construction has also been considered for groups other than S_n, like O_n and GL_n (Deligne, Milne [4], [5]) or wreath products $\mathsf{S}_n \wr \Gamma$, for Γ a finite group (Knop [9]).

The symmetric group S_n and the orthogonal group O_n are special cases of easy quantum groups, a class of compact matrix quantum groups introduced by Banica and Speicher [1]. Representation categories of easy quantum groups are always described by set partitions. This allows us to generalise Deligne's construction to all easy quantum groups, see [7]. We will study these interpolation categories with regards to semisimplicity and indecomposable objects.

The categories $\underline{\mathbf{Rep}}(\mathcal{C}, t)$
For all $k, l \in \mathbb{N}$, we consider $P(k,l) := \{\text{set partitions of } \{1, \ldots, k, 1', \ldots, l'\}\}$. These partitions can be pictured by diagrams, as for example

$p =$  $\in P(4,5)$,

where connected components are parts of a partition. A *category of partitions* is a set $\mathcal{C} \subseteq \bigcup_{k,l \in \mathbb{N}_0} P(k,l)$ such that $| \in P(1,1)$, $\sqcup \in P(2,0)$ and $\mathcal{C}$ is closed under composition (vertical concatenation), taking tensor products (horizontal concatenation) and involution (flip along the horizontal axis) of partitions. For any such category of partitions $\mathcal{C}$ and $t \in \mathbb{C}$ we define the category $\underline{\mathrm{Rep}}_0(\mathcal{C}, t)$ via:

Objects:	$[k], k \in \mathbb{N}_0$,
Morphisms:	$\mathrm{Hom}([k],[l]) = \mathbb{C}\mathcal{C}(k,l)$,
Composition:	$q \circ p = t^{l(q,p)}\ qp$.

Here qp denotes the composition of p and q and $l(q,p)$ denotes the number of connected components in the vertical concatenation of p and q, which are not connected to any upper point of p or lower point of q.

The *generalised Deligne category* or *partition C^*-tensor category* $\underline{\mathrm{Rep}}(\mathcal{C}, t)$ is the Karoubi envelope of $\underline{\mathrm{Rep}}_0(\mathcal{C}, t)$, i.e. the additive and idempotent completition of $\underline{\mathrm{Rep}}_0(\mathcal{C}, t)$ (see [7]). It is a rigid monoidal category with the Krull-Schmidt property, i.e. every object can be decomposed as a finite direct sum of indecomposable objects which is unique up to a permutation of the summands. If $\mathcal{C}$ is the set of all partitions, we recover Deligne's category $\underline{\mathrm{Rep}}(\mathsf{S}_t)$.

Representation categories of easy quantum groups
In order to show that the categories $\underline{\mathrm{Rep}}(\mathcal{C}, t), t \in \mathbb{C}$ interpolate representation categories of easy quantum groups, let us briefly recall some facts about easy quantum groups. A *compact matrix quantum group* $G = (A, u)$ consists of a unital C^*-algebra A and a matrix $u \in A^{n \times n}$ (fundamental corepresentation) such that

- A is generated by the matrix entries $u_{ij}, 1 \leq i, j \leq n$,
- u and u^t are invertible matrices,
- and $\Delta : A \to A \otimes_{\min} A,\ u_{ij} \mapsto \sum_{j=1}^{n} u_{ik} \otimes u_{jk}$ is a *-homomorphism.

A finite-dimensional representation of G is a matrix $v \in A^{m\times m}$ such that for all $1 \leq i, j \leq n$ we have $\Delta(v_{ij}) = \sum_{j=1}^{n} v_{ik} \otimes v_{jk}$. The intertwiner space of two representations $v \in A^{m\times m}$ and $v' \in A^{m'\times m'}$ is defined as $\mathrm{Hom}_G(v, v') = \{\phi : \mathbb{C}^m \to \mathbb{C}^{m'} \mid \phi v = v'\phi\}$ (see [11]).

There exists a linear map $T : \mathbb{C}P(k, l) \to \mathrm{Hom}((\mathbb{C}^n)^{\otimes k}, (\mathbb{C}^n)^{\otimes l})$ which respects the operations compostition, taking tensor products and involution/dualisation (see [1] for more details). A compact matrix quantum group $G = (A, u)$ is called *(orthogonal) easy quantum group* if $\mathsf{S}_n \subseteq G \subseteq \mathsf{O}_n^+$ and if there exists a category of partitions $\mathcal{C}$ such that $\mathrm{Hom}_G(u^{\otimes k}, u^{\otimes l}) = \{T(p) \mid p \in \mathbb{C}\mathcal{C}(k, l)\}$.
By a Tannaka-Krein type result of Woronowicz [12] there exists an easy quantum group $G_n(\mathcal{C})$ for any category of partitions $\mathcal{C}$ and $n \in \mathbb{N}$, whose fundamental corepresentation is of size n and whose intertwiner spaces are described by $\mathcal{C}$ as above.

If $\mathcal{C}$ is the set of all partitions, then $G_n(\mathcal{C})$ corresponds to the C^*-algebra of complex valued functions on the symmetric group $C(\mathsf{S}_n)$. Hence in this case the representation category $\mathrm{Rep}(G_n(\mathcal{C}))$ is the representation category of the symmetric group S_n.

The interpolation functor
For any $n \in \mathbb{N}$ we consider the functor:

$$\mathcal{F} : \underline{\mathrm{Rep}}(\mathcal{C}, n) \to \mathrm{Rep}(G_n(\mathcal{C})), \quad [k] \mapsto u^{\otimes k}, \quad p \mapsto T(p).$$

Then $\mathcal{F}$ is full and essentially surjective, but it is in general not faithful. We consider the quotient category $\widehat{\underline{\mathrm{Rep}}}(\mathcal{C}, t) := \underline{\mathrm{Rep}}(\mathcal{C}, t)/\mathcal{N}$ by the tensor ideal of negligible morphism

$$\mathcal{N} := \{f : X \to Y \mid tr(f \circ g) = 0 \text{ for all } g : Y \to X\}.$$

Then the induced functor $\widehat{\mathcal{F}} : \widehat{\underline{\mathrm{Rep}}(\mathcal{C}, n)} \to \mathrm{Rep}(G_n(\mathcal{C}))$ is an equivalence of monoidal categories. Furthermore, a result of Etingof and Ostrik [6] implies that $\underline{\mathrm{Rep}}(\mathcal{C}, t)$ is semisimple if and only if the ideal of negligible morphisms $\mathcal{N}$ is trivial. Hence the categories $\underline{\mathrm{Rep}}(\mathcal{C}, t), t \in \mathbb{C}$, interpolate the representation categories $\mathrm{Rep}(G_n(\mathcal{C})), n \in \mathbb{N}$, in the sense that for each $n \in \mathbb{N}$ the semisimplification of $\underline{\mathrm{Rep}}(\mathcal{C}, n)$ is equivalent to $\mathrm{Rep}(G_n(\mathcal{C}))$.

Results
The categories $\underline{\mathrm{Rep}}(\mathsf{S}_t)$ and $\underline{\mathrm{Rep}}(\mathsf{O}_t)$ are semisimple if and only if $t \in \mathbb{N}$ (see [4]). Moreover, in both cases the indecomposable objects up to isomorphism are in correspondence with Young diagrams of arbitrary size (see [2],[3]).
We consider the categories $\underline{\mathrm{Rep}}(NC_2, t)$ and $\underline{\mathrm{Rep}}(NC, t)$, which interpolate the representation categories of the free symmetric quantum group S_n^+ and the free orthogonal quantum group O_n^+. Using that the endomorphism algebras of $\underline{\mathrm{Rep}}(NC_2, t)$ are Temperley-Lieb algebras, we can show that $\underline{\mathrm{Rep}}(NC_2, t)$ is semisimple if and only if $t \neq 2 \cdot \cos(k\pi/j),\ j \in \mathbb{N}, k \in \{1, \ldots, k-1\}$ and that the indecomposable objects up to isomorphism are in correspondence with the Jones-Wenzel idempotents. We can further show that $\underline{\mathrm{Rep}}(NC, t^2)$ can be embedded into $\underline{\mathrm{Rep}}(NC_2, t)$.

This embedding allows us to determine the semisimplicity and indecomposable objects of $\underline{\text{Rep}}(NC, t)$.

$\mathcal{C}$	P = all partitions	P_2 = pair partitions	NC_2 = non-crossing pair partitions	NC = non-crossing partitions
$G_n(\mathcal{C})$	S_n	O_n	O_n^+	S_n^+
End([k])	Partition algebras	Brauer algebras	Temperley-Lieb algebras	
Semisimple	$t \notin \mathbb{N}$	$t \notin \mathbb{N}$	$t \neq 2 \cdot \cos(k\pi/j)$	$t \neq (2 \cdot \cos(k\pi/j))^2$
Indecomp. objects up to isomorphism corr. to	Young diagrams of arbitrary size	Young diagrams of arbitrary size	Jones-Wenzel idempotents	"Jones-Wenzel idempotents"

All categories of partitions have been classified by Raum and Weber [10] and their representation catgeories have been studied by Freslon and Weber [8]. We aim to use these combinatorial results to study $\underline{\text{Rep}}(\mathcal{C}, t)$ for other examples of $\mathcal{C}$.

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Random walks on Sekine finite quantum groups

Isabelle Baraquin

We study convergence of random walks, arising from linear combination of irreducible characters, on the Sekine family of finite quantum groups.

First, let us present a family of finite quantum groups defined in 1996 by Sekine [1]. They are given by their algebra $\mathcal{A}_n = \bigoplus_{i,j\in\mathbb{Z}_n} \mathbb{C}e_{(i,j)} \oplus M_n(\mathbb{C})$, the direct sum of n^2 copies of $\mathbb{C}$ and the matrices of size $n\times n$, and the coproduct Δ_n satisfying the following formulas:

$$\Delta_n(e_{(i,j)}) = \sum_{k,l\in\mathbb{Z}_n} e_{(k,l)} \otimes e_{(i-k,j-l)} + \frac{1}{n}\sum_{k,l=1}^{n} \eta^{i(k-l)} E_{k,l}\otimes E_{k+j,l+j}$$

$$\Delta_n(E_{i,j}) = \sum_{k,l\in\mathbb{Z}_n} \eta^{-k(i-j)} e_{(k,l)} \otimes E_{i+l,j+l} + \sum_{k,l\in\mathbb{Z}_n} \eta^{-k(i-j)} E_{i-l,j-l}\otimes e_{(k,l)}$$

where $\eta = e^{\frac{2i\pi}{n}}$ is a primitive nth root of unity and the $E_{i,j}$'s are the image in $\mathcal{A}_n$ of the elementary matrices in $M_n(\mathbb{C})$. As any finite quantum groups, $(\mathcal{A}_n, \Delta_n)$ admits a tracial Haar state h_n, whose explicit form is

$$h_n\left(\sum_{i,j\in\mathbb{Z}_n} x_{(i,j)}e_{(i,j)} + \sum_{i,j=1}^{n} X_{ij}E_{i,j}\right) = \frac{1}{2n^2}\left(\sum_{i,j\in\mathbb{Z}_n} x_{(i,j)} + n\sum_{i=1}^{n} X_{ii}\right)$$

for any complex numbers $x_{(i,j)}$ and any matrix $X = (X_{ij})_{1\le i,j\le n}$. Moreover, the representation theory of these groups can be determined [2, 3]. In particular, the irreducible characters are, up to equivalence:

- $\rho_l^{\pm} = \sum_{i,j\in\mathbb{Z}_n} \eta^{il} e_{(i,j)} \pm \sum_{i=1}^{n} E_{i,i+l}$
- if n is even, $\sigma_l^{\pm} = \sum_{i,j\in\mathbb{Z}_n} (-1)^j \eta^{il} e_{(i,j)} \pm \sum_{i=1}^{n} (-1)^i E_{i,i+l}$
- $\chi(X^{u,v}) = 2\sum_{i,j\in\mathbb{Z}_n} \eta^{iu} \cos\left(\frac{2\pi jv}{n}\right) e_{(i,j)}$

for any $l,u\in\mathbb{Z}_n$ and $v\in\left\{1,2,\dots,\left\lfloor\frac{n-1}{2}\right\rfloor\right\}$. Note that $\rho_0^+ = 1_{\mathcal{A}_n}$ is the trivial representation.

A random walk on a quantum group is defined as a convolution semigroup of state $(\phi^{\star k})_{k\ge1}$, where $\phi\star\psi = (\phi\otimes\psi)\Delta$ is the convolution of linear functionals. The Quantum Diaconis-Shahshahani Theory, introduced by McCarthy [2], allows us to bound from above and below the distance, in total variation, between the kth step and the Haar measure h_n.

Let us fix $a = \sum_\alpha a_\alpha \chi(\alpha)$ a linear combination of irreducible characters in $\mathcal{A}_n$. Then the Fourier transform of a, denoted $\mathcal{F}(a)$ and given for any $x\in\mathcal{A}_n$ by

$$\mathcal{F}(a)(x) = h_n(xa)$$

is a linear functional on $\mathcal{A}_n$. This is a state if the coefficients of a in the canonical basis satisfy some conditions [4]. In particular, we have to set the coefficient of the trivial representation $a_{\rho_0^+}$ to 1. Moreover, if $\mathcal{F}(a)$ is a state, then $|a_\alpha|\le d_\alpha$ for any non trivial representation α.

Assume now that a satisfies the conditions such that $\mathcal{F}(a)$ is a state. Then the random walk is called central, because it commutes with any other linear functional for the convolution product.

The quantum Diaconis-Shahshahani lower and upper bounds give that:

Theorem 1. *The random walk* $\left(\mathcal{F}(a)^{\star k}\right)_{k\geq 1}$ *converges to the Haar state* h_n *if and only if, for any non trivial irreducible representation* α, $|a_\alpha| < d_\alpha$.

Once we know this necessary and sufficient condition for the random walk to converge to the Haar state, we can ask what happens when, for some non trivial irreducible representation, $|a_\alpha| = d_\alpha$.

Note that, in this case, the limit state, if it exists, is a central idempotent state. Thanks to the study of idempotent states on Sekine quantum groups by Zhang [5], we are able to obtain a classification of the asymptotic behaviour of the central random walks:

Theorem 2 ([3])**.** *Assume that* n *is odd. Let us denote by* μ *the limit state of the random walk* $\left(\mathcal{F}(a)^{\star k}\right)_{k\geq 1}$ *if it exists. Then, there are four possible situations:*

- $\mu = h_h$, *which is equivalent to*
$$\forall \alpha \neq \rho_0^+, |a_\alpha| < d_\alpha$$
- $\mu = \frac{2n}{\#\Gamma}\sum_{(i,j)\in\Gamma}\mathcal{F}(e_{(i,j)})$ *for* Γ *a subgroup of* $\mathbb{Z}_n \times \mathbb{Z}_n$ *such that*
$$(k,l)\in\Gamma \;\Leftrightarrow\; (k,-l)\in\Gamma$$
which is equivalent to
$$\forall \alpha \neq \rho_0^+, |a_\alpha| < d_\alpha \text{ or } a_\alpha = d_\alpha, \text{ and } a_{\rho_0^-} = 1$$
- $\mu = n\sum_{j\in\mathbb{Z}_n}\mathcal{F}(e_{(0,j)}) + \sum_{i,j=1}^{n}\mathcal{F}(E_{i,j})$, *which is equivalent to*
$$\forall \alpha \neq \rho_0^+, |a_\alpha| < d_\alpha \text{ or } a_\alpha = d_\alpha, \text{ but } a_{\rho_0^-} \neq 1, \text{ and } \exists \beta \neq \rho_0^+, a_\beta = d_\beta = 1$$
- *the random walk diverges, which is equivalent to*
$$\exists \alpha, |a_\alpha| = d_\alpha \text{ and } a_\alpha \neq d_\alpha$$

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Quantum harmonic functions

Adam Skalski
(joint work with Matthias Neufang, Pekka Salmi and Nico Spronk)

Convolution operators associated to measures on locally compact groups appear in a variety of contexts: from purely operator-theoretic considerations, via partial differential equations, to the study of random walks on the groups in question. They act boundedly on several function spaces: if say ω is a bounded measure on a locally compact group G, then the (suitably interpreted) formula

$$L_\omega(f)(g) = \int_G f(h^{-1}g)d\omega(h)$$

yields a contractive operator on $C_0(G)$, $L^p(G)$ ($p \in [1,\infty]$), etc. In the study of such operators an important role is often played by their fixed point spaces, which can be, at least in the case of positive measures, interpreted as collections of harmonic functions with respect to a given measure or as the means to realize the measure-theoretic boundary of an associated random walk.

Questions about the structure and nature of the fixed point spaces have been in recent years studied also in the dual context, that is that of the Herz-Schur multipliers acting on a von Neumann algebra of a locally compact group ([ChL]). The unified view on these two frameworks (and a far-reaching generalization of both) is provided by the notion of convolution operators on locally compact quantum groups in the sense of Kustermans and Vaes ([KV]).

Thus in what follows we will assume that $\mathbb{G}$ is a locally compact quantum group, $M^u(\mathbb{G})$ denotes the space of 'universal' quantum measures on $\mathbb{G}$ and for $\omega \in M^u(\mathbb{G})$ by $L_\omega : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ we will denote the associated (completely bounded, normal) convolution operator acting on the von Neumann algebra representing 'bounded measurable functions on $\mathbb{G}$. We might also consider its universal counterpart, L^u_ω, acting on the universal version of the C^*-algebra of continuous functions, $C_0^u(\mathbb{G})$, and on its multiplier algebra, $C_b^u(\mathbb{G})$.

The composition of convolution operators corresponds to the convolution $\star$ of (quantum) measures; thus the initial case to be studied is that of idempotent measures, for which the relevant fixed point space coincides with the range of the operator and the convergence questions become trivial. This was addressed in the positive case (for example) in [SaS], and for general contractive idempotents in [NSSS$_1$] (see also [Kas]). It is worth to notice that even in the classical case the structure of idempotent measures is fully known only under the contractivity assumptions ([Gre]); thus everywhere below we will consider only the quantum measures in the unit ball, $M^u(\mathbb{G})_1$.

The first result describes when a contractive convolution operator has non-trivial fixed points at all. It turns out that nice tools to express this are provided by $RUC(\mathbb{G})$ and $RUC^u(\mathbb{G})$, the reduced and universal versions of spaces of right uniformly continuous functions on $\mathbb{G}$, which are certain operator systems contained respectively in $C_b(\mathbb{G})$ and $C_b^u(\mathbb{G})$. These possess also natural left versions.

Theorem 1. *Let $\omega \in M^u(\mathbb{G})_1$. The following are equivalent:*

(i) *L_ω has no non-zero fixed points in $L^\infty(\mathbb{G})$;*
(ii) *L_ω has no non-zero fixed points in $RUC(\mathbb{G})$;*
(iii) *$\frac{1}{n}\sum_{k=1}^{n} \omega^{\star k} \to 0$ weak* in $RUC^u(\mathbb{G})^*$.*

The fixed point spaces for convolution operators associated to states were already studied for example in [KNR]. The positivity assumption simplifies many considerations, and it is very desirable to relate the quantum harmonic functions for a general contractive measure to these related to a state. Naturally any element in $M^u(\mathbb{G})$ decomposes into a combination of positive ones, but this seems to give no information on the fixed point spaces. A more useful tool is that of a polar decomposition: the *right absolute value* of $\omega \in M^u(\mathbb{G})$ is a positive functional $|\omega|$ in $M^u(\mathbb{G})$ defined through the polar decomposition $\omega = |\omega|(\cdot u)$ where $u \in C_0^u(\mathbb{G})^{**}$ is a partial isometry satisfying some extra properties (see Definition III.4.3 of [Tak]). We call ω non-degenerate if its both right and left absolute values are non-degenerate in the sense of [KNR], so that in particular for every non-zero positive $a \in C_0^u(\mathbb{G})$ there exists $k \in \mathbb{N}$ such that $|\omega|^{\star k}(a) > 0$.

Theorem 2. *Suppose that $\omega \in M^u(\mathbb{G})_1$ is non-degenerate. If L_ω^u has a non-zero fixed point in $LUC^u(\mathbb{G})$, then there is a unitary $v \in LUC(\mathbb{G})$ such that*

$$\Delta(v) = v \otimes v,$$

and $Fix\, L_\omega = (Fix\, L_{|\omega|})v^*$.

The next result describes the situation where non-zero fixed points exist already in $C_0(\mathbb{G})$.

Theorem 3. *Let $\omega \in M^u(\mathbb{G})_1$. Then the following are equivalent:*

(i) *Cesàro sums $\frac{1}{n}\sum_{k=1}^{n} \omega^{\star k} \to 0$ do not converge to 0 weak* on $C_0^u(\mathbb{G})$;*
(ii) *L_ω^u has a non-zero fixed point in $C_0^u(\mathbb{G})$;*
(iii) *L_ω has a non-zero fixed point in $C_0(\mathbb{G})$;*
(iv) *there is a non-zero $\tau \in M^u(\mathbb{G})$ such that $\tau \star \omega = \tau$;*
(v) *L_ω has a non-zero fixed point in $L^\infty(\mathbb{G})$ and there exists e in $C_0^u(\mathbb{G})_+$ such that $|\omega|^{\star k}.e = e.|\omega|^{\star k} = |\omega|^{\star k}$ and $|\omega|^{\star k}((ae-a)^*(ae-a)) = 0$ for every $k \in \mathbb{N}$ and $a \in C_0^u(\mathbb{G})$.*

The last condition above should be interpreted as a compactness requirement for the subsemigroup generated by the support of ω; and the theorem above after some extra work yields also certain triviality results for fixed points of quantum convolution operators acting on non-commutative L^p-spaces, generalising several statements of [Kal].

We conclude this sample of results with an example showing how the properties of the fixed points of convolution operators, or rather their pre-annihilators, can be used to characterise (co-)amenability of the quantum group in question. Denote by $P(\mathbb{G})$ the state space of $C_0^u(\mathbb{G})$ and for ω in $P(\mathbb{G})$ write I_ω for the set of these elements of $L^1(\mathbb{G})$ which vanish on each fixed point of the operator L_ω. The proof of the theorem below borrows several ideas from [Wil].

Theorem 4. *Assume that $\mathbb{G}$ is second countable and consider the following list of conditions:*

(i) $\mathbb{G}$ *is coamenable;*
(ii) $\mathbb{G}$ *is amenable;*
(iii) *for each $\omega \in P(\mathbb{G})$, the right ideal I_ω admits a bounded left approximate identity;*
(iv) *the collection $\mathcal{I} := \{I_\omega : \omega \in P(\mathbb{G})\}$ admits a unique maximal element.*

Then the following implications/equivalences hold: (ii)$\Longleftrightarrow$(iv), (i)$\Longrightarrow$(iii) and (i)+(ii)$\Longleftrightarrow$(iii)+(iv). Moreover if (iv) holds then $I_{\max} = L^1_0(\mathbb{G})$, where the right hand side denotes the augmentation ideal of $L^1(\mathbb{G})$.

Theorems 1-3 can be significantly strengthened and given parallel interpretations in the classical and dual-to-classical cases mentioned in the beginning of this report. For the details we refer to the preprint [NSSS$_2$], where for example certain extensions of classical Choquet-Deny ([ChD]) and Derriennic-Mukherjea ([Der], [Muk]) theorems are established.

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Superrigidity of group operator algebras

Sven Raum

In this talk, I presented different aspects of the problem to recover a discrete group from one of its group rings or group operator algebras. Further, I shared suggestions of superrigidity problems around the free discrete quantum group $\mathbb{F}\mathrm{U}_N$.

The classical group ring isomorphism problem asks, given a group G and a ring R, when the group G is remembered by the R-algebra RG. Formally, given any other group H, does $RG \cong RH$ imply $G \cong H$? The conjecture that this problem should have a positive solution for every domain R and every torsion-free discrete group goes back to the PhD thesis of Higman [2]. In analogy with the group ring isomorphism problem, similar problems have been considered for group operator algebras. A discrete group G is called C*-superrigid if for any other group H the existence of an isomorphism $\mathrm{C}^*_{\mathrm{red}}(G) \cong \mathrm{C}^*_{\mathrm{red}}(H)$ implies the existence of an isomorphism $G \cong H$. Replacing reduced group C*-algebras by group von Neumann algebras, we arrive at the definition of W*-superrigidity. Taking instead the reduced group L^p-operator algebra, we arrive at the definition of L^p-superrigidity.

In the first part of the talk, I discussed existing methods of proof for a positive solution to the group ring isomorphism problem for torsion-free groups. In the focus stood the so-called unique product property, which was explained with the examples of the group of integers and the free group. I then suggested to study analogue problems for the free discrete quantum group $\mathbb{F}\mathrm{U}_N$. Given any other discrete quantum group $\mathbb{H}$ such that $\mathbb{C}(\mathbb{F}\mathrm{U}_N) \cong \mathbb{C}(\mathbb{H})$ as $\mathbb{C}$-algebras, does it follow that $\mathbb{H} \cong \mathbb{F}\mathrm{U}_N$?

The second part of the talk reviewed results from recent years on C*-superrigidity, W*-superrigidity and L^p-superrigidity. Extending the above mentioned problem, I finished by suggesting the problem to prove an L^p-superrigidity result for $\mathbb{F}\mathrm{U}_N$ in view of the L^p-superrigidity results of Gardella-Thiel [1], which hold for arbitrary locally compact groups and all $p \in (1, \infty)$, $p \neq 2$.

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Universal coefficient theorem for discrete quantum groups

Yuki Arano

In [4], Rosenberg and Schochet has introduced a property of C*-algebras called the Universal Coefficient Theorem (UCT in short) for K-theory of C*-algebras and has shown it for C*-algebras in the bootstrap class. The UCT gives a formula computing the KK-groups only from the K-groups. This property plays an important role in the classification of nuclear C*-algebras.

The UCT for group C*-algebras is related to (a variation of) the Baum–Connes conjecture of groups. In [5], Tu proved that the group C*-algebra of a discrete group with Haagerup property satisfies the UCT using the Higson–Kasparov type argument for groupoids. It is a more subtle problem whether the UCT is preserved under taking crossed products. This is true when the group is torsion-free, but when the group is finite, the problem is equivalent to the famous open problem, namely, all nuclear C*-algebras satisfy the UCT or not.

The Baum–Connes conjecture for quantum groups first appeared in the series of works of Meyer and Nest [2], [3]. Even though there is no unified method proving the Baum–Connes conjecture for fairly general quantum groups, it is proven for many known examples of discrete quantum groups [1], [6], [7],[8].

In this talk, I first summarize the current status of the Baum–Connes property for discrete quantum groups. Next I explain how to apply the Baum–Connes conjecture to obtain the UCT for group C*-algebras and the crossed products by discrete quantum groups. First we study the general theory of the Baum–Connes conjecture for discrete quantum group with possible torsions, which is studied in many special cases. As a byproduct of the general theory, we observe that the group C*-algebra of a discrete quantum group with the strong Baum–Connes conjecture satisfies the UCT. Furthermore, for torsion-free discrete quantum groups with the Baum–Connes conjecture, we observe that the UCT is preserved under taking crossed products.

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The Furstenberg boundary of discrete quantum groups

Mehrdad Kalantar
(joint work with Paweł Kasprzak, Adam Skalski, Roland Vergnioux)

The notion of topological boundary actions has recently found striking applications in the study of operator algebras associated to discrete groups. In view of this, it is very natural to develop a boundary theory in the setting of quantum groups. In joint work with Paweł Kasprzak, Adam Skalski and Roland Vergnioux, we introduce and develop this notion, and in particular the Furstenberg boundary of a discrete quantum group $\mathbb{G}$. Motivated by the identification result of [2] we define a unital $\mathbb{G}$-C^*-algebra $\mathcal{A}$ to be a $\mathbb{G}$-boundary if for every state ν on $\mathcal{A}$ the Poisson transform $\mathcal{P}_\nu : \mathcal{A} \to \ell^\infty(\mathbb{G})$ defined by $\mathcal{P}_\nu(x) = (\nu \otimes \mathrm{id})\alpha(x)$ is completely isometric. Then, we can prove that every discrete quantum group $\mathbb{G}$ admits a unique (up to $\mathbb{G}$-isomorphism) universal $\mathbb{G}$-boundary $C(\partial\mathbb{G}_F)$, in the sense that for any $\mathbb{G}$-boundary $\mathcal{A}$ there is a completely isometric $\mathbb{G}$-map $\mathcal{A} \to C(\partial\mathbb{G}_F)$. We call $C(\partial\mathbb{G}_F)$ the *Furstenberg boundary* of $\mathbb{G}$.

The construction of the universal object above is similar to Hamana's original work [1], but the fact that this object is a $\mathbb{G}$-C^*-algebra is not obvious at all.

It follows that a discrete quantum group $\mathbb{G}$ is amenable iff its Furstenberg boundary $C(\partial\mathbb{G}_F)$ is trivial. We further prove the following important facts:

(1) Let $\mathbb{G}$ be a unimodular discrete quantum group. If the action of $\mathbb{G}$ on its Furstenberg boundary $C(\partial\mathbb{G}_F)$ is faithful, then $\mathbb{G}$ has the unique trace property.

(2) If $\mathbb{G}$ is a C^*-simple discrete quantum group or a unimodular discrete quantum group with the unique trace property, then $\mathbb{G}$ has no non-trivial normal amenable quantum subgroups.

We prove a general result, which provides a systematic way to obtain non-trivial examples of boundary actions, namely we prove that with some additional mild conditions, unique stationarity implies boundary actions.

In particular, for the Van Daele and Wang's free orthogonal discrete quantum group $\mathbb{F}O_Q$ with $Q \in M_N(\mathbb{C})$ such that $Q\bar{Q} = \pm I_N$, $N \geq 2$ we prove the Gromov boundary $\mathcal{B}_\infty$ (in the sense of [3]) admits a unique μ-stationary state for a generating positive state $\mu \in \ell^\infty(\mathbb{G})_*$. As a consequence, we conclude that $\mathcal{B}_\infty$ is a $\mathbb{G}$-boundary.

In combination of general properties of boundary actions, the latter result allow us to prove the simplicity of the reduced crossed product of the action of a C^*-simple free orthogonal discrete quantum group $\mathbb{G}$ on its Gromov boundary $\mathcal{B}_\infty$.

Furthermore, the unique stationarity result further allow us to prove Ozawa's nuclear embedding conjecture for the free orthogonal quantum groups, namely we prove the C^*-embedding of the reduced crossed product of the action of a free orthogonal discrete quantum group $\mathbb{G}$ on its Gromov boundary $\mathcal{B}_\infty$ into the injective envelope $I(C(\widehat{\mathbb{G}}))$ of the reduced C^*-algebra $C(\widehat{\mathbb{G}})$ of the dual quantum group $\widehat{\mathbb{G}}$.

There are still several important problems that remain open. In particular, we do not know whether C^*-simplicity implies the unique trace property in the discrete quantum group case. We also do not know if the unique trace property implies faithfulness of the Furstenberg boundary action.

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Vectorial cumulants for compact quantum group

Pierre Tarrago

(joint work with Jonas Wahl)

The goal of this talk is to explain some similarities that appear between the combinatorics of free probability and the representation theory of certain compact quantum groups. The starting point of this project is the following conjecture of Banica and Bichon [1] :

Suppose that $(\mathbb{F}, u)$ *and* $(\mathbb{G}, v)$ *are non-commutative permutation quantum group and let* $\mathbb{F} \wr_* \mathbb{G}$ *denote the free wreath product of both quantum groups. Then, under the irreducibility assumption* $\dim(\mathrm{Fix}(u)) = \dim(\mathrm{Fix}(v)) = 1$*, the distribution of the fundamental character* $\chi_{\mathbb{F}\wr_*\mathbb{G}}$ *with respect to the Haar measure satisfies the relation*

$$\chi_{\mathbb{F}\wr_*\mathbb{G}} = \chi_{\mathbb{F}} \boxtimes \chi_{\mathbb{G}},$$

where $\boxtimes$ *denotes the free multiplicative convolution.*

Let us recall some basic facts from free probability theory. We consider a $*$-algebra A (the algebra of random variables) together with a positive linear functional ϕ, which represents the expectation in the classical case. Once given a self-adjoint element $a \in A$, we are interested in the sequence of moments of a, namely the sequence $(\phi(a^n))_{n\geq 1}$. When we consider two variables a and b which are free in (A, ϕ), it is often hard to express directly mixed moments of a and b. Following Speicher's idea [5], the way to circumvent this problem is to introduce the free cumulants $(k_n(a))_{n\geq 1}$, defined implicitly by the formula

$$\phi(a^n) = \sum_{\pi\in NC(n)} \prod_{S\in\pi} k_{|S|}(a). \tag{1}$$

The free cumulants satisfy in particular the simple relation $k_n(a+b) = k_n(a)+k_n(b)$ when a and b are free. The formula for the product of two free variables is slightly more involved, since we have

$$k_{ab}(n) = \sum_{\pi \in NC(n)} \prod_{S \in \pi} k_{|S|}(a) \prod_{S \in \mathrm{kr}(\pi)} k_{|S|}(b),$$

where $\mathrm{kr}(\pi)$ denotes the Kreweras complement of the partition π.

Since a compact matrix quantum group $(\mathbb{G}, u)$ is a non-commutative probability space with respect to the Haar state h, we can define free cumulants for the fundamental characters $\chi_{\mathbb{G}}$. The main property of the character of a representation u of $\mathbb{G}$ is that $h(\chi_u^n) = \dim \mathrm{Fix}(u^{\otimes n})$, where $u^{\otimes n}$ denotes the n-th tensor product of u and $\mathrm{Fix}(v)$ denote the vector space of fixed vectors in a representation v of $\mathbb{G}$. It is then natural to ask whether the free cumulants of χ_u are also dimensions of some vector spaces.

A tempting approach is to define directly free cumulants at the level of the collection of vector spaces $\{\mathrm{Fix}(u^{\otimes n})\}_{n\geq 1}$. Mimicking the situation in the scalar case, we are looking for a sequence of vector spaces $(K_n(u))_{n\geq 1}$, such that $K_n \subset \mathrm{Fix}(u^{\otimes n})$ and such that we have the decomposition

$$\mathrm{Fix}(u^{\otimes n}) \simeq \bigoplus_{\pi \in NC(n)} \bigotimes_{S \in \pi} K_{|S|}(u)$$

for each n. When such a relation holds for any $n \geq 1$, we say that $(K_n)_{n\geq 1}$ is a free decomposition of $(\mathrm{Fix}(u^{\otimes n}))_{n\geq 1}$, and the vector space $K_n(u)$ is called the n-th free vectorial cumulant of u. This approach works fairly well in the setting of a free wreath product of a orthogonal matrix compact quantum group $(\mathbb{G}, u)$ with S_n^+. In the latter case, a result of Lemeux and myself [4] shows that when w is the fundamental representation of $\mathbb{G} \wr_* S_n^+$, we have indeed the decomposition

$$\mathrm{Fix}(w^{\otimes n}) \simeq \bigoplus_{\pi \in NC(n)} \bigotimes_{S \in \pi} \mathrm{Fix}(u^{|S|},$$

so that setting $K_n(w) = \mathrm{Fix}(u^{\otimes n})$ gives a free decomposition $(K_n(w))_{n\geq 1}$ of $(\mathbb{G} \wr_* \mathbb{F}, w)$. In general however, there is no hope to get such decomposition : indeed, we easily see that if we had a free decomposition $(K_n(u))_{n\geq 1}$ for a compact matrix quantum group $(\mathbb{G}, u)$, we would have $k_n(\chi_{\mathbb{G}}) = \dim K_n(u)$. But in general, $k_n(\chi_{\mathbb{G}})$ can be negative, which prevents the existence of a free decomposition (see for example the case of $\mathbb{Z}_2$ with the one-dimensional non-trivial representation). Two first open questions are thus the following :

- *When do there exist a free decomposition of a compact matrix quantum group $(\mathbb{G}, u)$? Is it the case if and only if $\mathbb{G}$ has a free fusion semi-ring ?*
- *How to generalize the definition of a free decomposition, in order to take into account the possibly negative values of free cumulants?*

See [3] for some backgrounds and interesting results on free fusion semi-rings. Since there is in general no free decomposition of the sequence $(\mathrm{Fix}(u^{\otimes n}))_{n\geq 1}$, we can therefore look for other type of cumulants. There exist actually four kinds of cumulants, and among them one, the Boolean cumulant, shares many good properties with the free cumulants. The Boolean cumulants of a random variable

a is the sequence $(b_n(a))_{n\geq 1}$ such that the decomposition (1) is replaced by

$$\phi(a^n) = \sum_{I\in\mathcal{I}(n)} \prod_{S\in I} b_{|S|}(a),$$

where $\mathcal{I}(n)$ denotes the set of interval partition of n. Like in the free case, one can define a Boolean decomposition for the sequence of vector spaces $(\mathrm{Fix}(u^{\otimes n}))_{n\geq 1}$. In the Boolean case, the situation is much better, since we can prove that a Boolean decomposition always exists for any compact matrix quantum group $(\mathbb{G}, u)$. Moreover, using planar algebras results, we also show in [6] that if $(B_n(u))_{n\geq 1}$ and $(B_n(v))_{n\geq 1}$ are Boolean decompositions for irreducible non-commutative permutation groups $(\mathbb{F}, u)$ and $(\mathbb{G}, v)$, then we get a Boolean decomposition $(B_n(u \wr_* v))_{n\geq 1}$ of $(\mathbb{F} \wr_* \mathbb{G}, u \wr_* v)$, with the formula

$$B_n(u \wr_* v) \simeq \bigoplus_{\pi\in NC(n)} \left(\bigotimes_{S\in\pi} B_{|S|}(u)\right) \left(\bigotimes_{S'\in \mathrm{kr}(\pi)} B_{|S'|}(v)\right).$$

This formula is exactly the vectorial version of the formula discovered by Belinschi and Nica [2]

$$b_n(ab) = \sum_{\pi\in NC(n)} \left(\prod_{S\in\pi} b_{|S|}(a)\right) \left(\prod_{S'\in \mathrm{kr}(\pi)} b_{|S'|}\right)$$

for a and b free. This proves in particular the conjecture of Banica and Bichon mentioned at the beginning of this report.

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Braided Quantum Spheres

Piotr M. Sołtan

Quantum deformation of the compact Lie group SU(2) was introduced by S.L. Woronowicz in the seminal paper [5]. The C*-algebra $\mathrm{C}(\mathrm{SU}_q(2))$ playing the role of the algebra of all continuous functions on this quantum group is the universal C*-algebra generated by two elements α and γ satisfying the relations

$$\alpha^*\alpha+\gamma^*\gamma=\mathbb{1},\quad \alpha\gamma=\overline{q}\gamma\alpha,$$
$$\alpha\alpha^*+|q|^2\gamma^*\gamma=\mathbb{1},\quad \gamma\gamma^*=\gamma^*\gamma$$

with q a real parameter in the interval $[-1,1]$. One can easily check that there exists a unital $*$-homomorphism $\Delta:\mathrm{C}(\mathrm{SU}_q(2))\to\mathrm{C}(\mathrm{SU}_q(2))\otimes\mathrm{C}(\mathrm{SU}_q(2))$ such that

$$\Delta(\alpha)=\alpha\otimes\alpha-q\gamma^*\otimes\alpha,\quad \Delta(\gamma)=\gamma\otimes\alpha+\alpha^*\otimes\gamma$$

which for $q\neq 0$ endows the quantum space $\mathrm{SU}_q(2)$ with the structure of a compact quantum group.

It has been noted in [1] that this C*-algebra can also be defined for any complex q with $|q|<1$, but it is also easily checked that a morphism Δ as described above exists only when q is real.

Nevertheless in [2] P. Kasprzak, R. Meyer, S. Roy and S.L. Woronowicz introduced a way to define a comultiplication on $\mathrm{C}(\mathrm{SU}_q(2))$ for complex q. They proposed to have Δ take values not in the tensor product $\mathrm{C}(\mathrm{SU}_q(2))\otimes\mathrm{C}(\mathrm{SU}_q(2))$, but in a *braided tensor product* $\mathrm{C}(\mathrm{SU}_q(2))\boxtimes_\zeta\mathrm{C}(\mathrm{SU}_q(2))$, where ζ is a certain complex parameter related to q.

This braided tensor product is defined for C*-algebras endowed with an action of the circle group $\mathbb{T}$ in the following way: let A and B be such C*-algebras and let ζ be a complex number of modulus 1. Furthermore let $\mathrm{C}(\mathbb{T}^2_\zeta)$ be the C*-algebra of functions on the quantum torus (generated by unitary u and v such that $uv=\zeta vu$). There are unique unital $*$-homomorphisms $j_1:\mathsf{A}\to\mathrm{C}(\mathbb{T}^2_\zeta)\otimes\mathsf{A}\otimes\mathsf{B}$ and $j_2:\mathsf{B}\to\mathrm{C}(\mathbb{T}^2_\zeta)\otimes\mathsf{A}\otimes\mathsf{B}$ such that

$$j_1(a)=u^{\deg(a)}\otimes a\otimes\mathbb{1},\quad j_2(b)=u^{\deg(a)}\otimes\mathbb{1}\otimes b$$

for all homogeneous elements $a\in\mathsf{A}$ and $b\in\mathsf{B}$. The braided tensor product $\mathsf{A}\boxtimes_\zeta\mathsf{B}$ is defined as the closed linear span of all products of the form $j_1(a)j_2(b)$ for all $a\in\mathsf{A}$ and $b\in\mathsf{B}$ (often denoted $\big[j_1(\mathsf{A})j_2(\mathsf{B})\big]$). Letting $\jmath_1$ and $\jmath_2$ be the maps j_1 and j_2 treated as homomorphisms $\mathsf{A}\to\mathsf{A}\boxtimes_\zeta\mathsf{B}$ and $\mathsf{B}\to\mathsf{A}\boxtimes_\zeta\mathsf{B}$ we obtain embeddings whose ranges do not necessarily commute. They replace the mappings $a\mapsto a\otimes\mathbb{1}$ and $b\mapsto\mathbb{1}\otimes b$ into the standard tensor product.

The braided product $\boxtimes_\zeta$ is a bifunctor. Its action on morphisms is uniquely determined by the following: if $\Phi:\mathsf{A}\to\mathsf{A}'$ and $\Psi:\mathsf{B}\to\mathsf{B}'$ are $\mathbb{T}$-equivariant then $\Phi\boxtimes_\zeta\Psi$ is the unique unital $*$-homomorphism $\mathsf{A}\boxtimes_\zeta\mathsf{B}\to\mathsf{A}'\boxtimes_\zeta\mathsf{B}'$ such that

$$\Phi\boxtimes_\zeta\Psi\big(\jmath_1(a)\jmath_2(b)\big)=\jmath_1\big(\Phi(a)\big)\jmath_2\big(\Psi(b)\big),\qquad a\in\mathsf{A},\ b\in\mathsf{B}.$$

The algebra $\mathrm{C}(\mathrm{SU}_q(2))$ carries a unique action of $\mathbb{T}$ such that $\deg(\alpha)=0$ and $\deg(\gamma)=1$. With this action we have

Theorem 1 ([2])**.** *Let* $\zeta = \frac{q}{\bar{q}}$*. Then there exists a unique* $\Delta : \mathrm{C}(\mathrm{SU}_q(2)) \to \mathrm{C}(\mathrm{SU}_q(2)) \boxtimes_\zeta \mathrm{C}(\mathrm{SU}_q(2))$ *such that*

$$\Delta(\alpha) = \jmath_1(\alpha)\jmath_2(\alpha) - q\jmath_1(\gamma)^*\jmath_2(\alpha), \quad \Delta(\gamma) = \jmath_1(\gamma)\jmath_2(\alpha) + \jmath_1(\alpha)^*\jmath_2(\gamma).$$

Moreover we have $(\Delta \boxtimes_\zeta \mathrm{id}) \circ \Delta = (\mathrm{id} \boxtimes_\zeta \Delta) \circ \Delta$ *and*

$$\begin{aligned}&\big[\Delta\big(\mathrm{C}(\mathrm{SU}_q(2))\big)\jmath_2\big(\mathrm{C}(\mathrm{SU}_q(2))\big)\big]\\ &\quad= \big[\jmath_1\big(\mathrm{C}(\mathrm{SU}_q(2))\big)\Delta\big(\mathrm{C}(\mathrm{SU}_q(2))\big)\big] = \mathrm{C}(\mathrm{SU}_q(2)) \boxtimes_\zeta \mathrm{C}(\mathrm{SU}_q(2)).\end{aligned}$$

The classical group $\mathbb{T}$ with trivial action of $\mathbb{T}$ is a "braided quantum subgroup" of $\mathrm{SU}_q(2)$ via the unital $*$-homomorphism $\pi : \mathrm{C}(\mathrm{SU}_q(2)) \to \mathrm{C}(\mathbb{T})$ given by $\pi(\alpha) = \boldsymbol{z}$ (the canonical generator of $\mathrm{C}(\mathbb{T})$) and $\pi(\gamma) = 0$. We call the associated "quotient space" the *braided quotient sphere* $\mathbb{S}^2_q$:

$$\mathrm{C}(\mathbb{S}^2_q) = \big\{a \in \mathrm{C}(\mathrm{SU}_q(2)) \,\big|\, (\mathrm{id} \boxtimes_\zeta \pi)\Delta(a) = \jmath_1(a)\big\}.$$

Theorem 2. *The* C^**-algebra* $\mathrm{C}(\mathbb{S}^2_q)$ *coincides with the unital* C^**-subalgebra of* $\mathrm{C}(\mathrm{SU}_q(2))$ *generated by* $\gamma^*\gamma$ *and* $\alpha\gamma^*$*. It is isomorphic the the quotient quantum sphere* $\mathrm{SU}_{|q|}(2)/\mathbb{T}$*.*

Let $\Gamma = \Delta\big|_{\mathrm{C}(\mathbb{S}^2_q)}$*. Then* Γ *is* $\mathbb{T}$*-equivariant and*

1. $(\mathrm{id} \boxtimes_\zeta \Gamma) \circ \Gamma = (\Delta \boxtimes_\zeta \mathrm{id}) \circ \Gamma$,
2. $\big[\jmath_1\big(\mathrm{C}(\mathrm{SU}_q(2))\big)\Gamma\big(\mathrm{C}(\mathbb{S}^2_q)\big)\big] = \mathrm{C}(\mathrm{SU}_q(2)) \boxtimes_\zeta \mathrm{C}(\mathbb{S}^2_q)$,
3. *for* $x \in \mathrm{C}(\mathbb{S}^2_q)$ *we have* $\Gamma(x) = \jmath_2(x)$ *if and only if* $x \in \mathbb{C}\mathbb{1}$.

Put

$$\boldsymbol{V} = \begin{bmatrix} v_{-1,-1} & v_{-1,0} & v_{-1,1} \\ v_{0,-1} & v_{0,0} & v_{0,1} \\ v_{1,-1} & v_{1,0} & v_{1,1} \end{bmatrix} = \begin{bmatrix} \alpha^2 & -(1+|q|^2)^2\gamma^*\alpha & -q\gamma^{*2} \\ \zeta\alpha\gamma & \mathbb{1} - (1+|q|^2)^2\gamma^*\gamma & \gamma^*\alpha^* \\ -q\zeta\gamma^2 & -(1+|q|^2)^2\alpha^*\gamma & \alpha^{*2} \end{bmatrix}.$$

Then $\boldsymbol{V}$ can be interpreted as a unitary representation of the braided quantum $\mathrm{SU}_q(2)$ (with appropriate action of $\mathbb{T}$ on $\mathbb{C}^3$, cf. [2, Section 5]). Clearly $\mathrm{C}(\mathbb{S}^2_q)$ is generated by the middle column of $\boldsymbol{V}$. Moreover one can check that

- putting $e_i = v_{i,0}$ we have $\Gamma(e_i) = \sum\limits_k \jmath_1(v_{i,j})\jmath_2(e_j)$,
- $\deg(e_i) = i$ for all i,
- $\operatorname{span} e_{-1}, e_0, e_1$ is the unique subspace of $\mathrm{C}(\mathbb{S}^2_q)$ possessing a basis which transforms in this manner under the action of $\mathrm{SU}_q(2)$.

Theorem 3. *Let* $\mathbb{X}$ *be a compact quantum space with an action of* $\mathbb{T}$ *such that there is an action* Γ *of the braided quantum group* $\mathrm{SU}_q(2)$ *such that*

1. *if* $\Gamma(x) = \jmath_2(x)$ *then* $x \in \mathbb{C}\mathbb{1}$,
2. *there exist linearly independent elements* $e_{-1}, e_0, e_1 \in \mathrm{C}(\mathbb{X})$ *such that* $\Gamma(e_i) = \sum\limits_k \jmath_1(v_{i,j})\jmath_2(e_j)$,
3. $\deg(e_i) = i$ *for all* i,
4. $\operatorname{span} e_{-1}, e_0, e_1$ *is the unique subspace of* $\mathrm{C}(\mathbb{S}^2_q)$ *possessing a basis which transforms in this manner under the action of* $\mathrm{SU}_q(2)$.

Then (possibly after rescaling by a non-zero constant) we have ${e_i}^* = e_{-i}$ *for all* i *and there exist* $\lambda, \rho \in \mathbb{R}$ *such that*

$$\begin{aligned} e_{-1}e_1 + (1+|q|^2)e_0^2 + |q|^2 e_1 e_{-1} &= \rho\mathbb{1}, \\ (1+|q|^2)(e_{-1}e_0 - |q|^2 e_0 e_{-1}) &= \lambda e_{-1}, \\ |q|^2(e_1e_{-1} - e_{-1}e_1) + (1-|q|^4)e_0^2 &= \lambda e_0, \\ (1+|q|^2)(e_0e_1 - |q|^2 e_1 e_0) &= \lambda e_1. \end{aligned}$$

In other words the existence of an action with the properties which we described for the quotient sphere implies strong algebraic relations on the generators.

The final result can be summarized in the next theorem.

Theorem 4. *For* $\rho, \lambda \in \mathbb{R}$ *define the compact quantum space* $\mathbb{X}_{q,\rho,\lambda}$ *by setting* $\mathrm{C}(\mathbb{X}_{q,\rho,\lambda})$ *to be the universal* C^**-algebra generated by* e_{-1}, e_0, e_1 *such that* ${e_i}^* = e_{-i}$ *for all* i *and*

$$\begin{aligned} e_{-1}e_1 + (1+|q|^2)e_0^2 + |q|^2 e_1 e_{-1} &= \rho\mathbb{1}, \\ (1+|q|^2)(e_{-1}e_0 - |q|^2 e_0 e_{-1}) &= \lambda e_{-1}, \\ |q|^2(e_1e_{-1} - e_{-1}e_1) + (1-|q|^4)e_0^2 &= \lambda e_0, \\ (1+|q|^2)(e_0e_1 - |q|^2 e_1 e_0) &= \lambda e_1. \end{aligned}$$

Then $\mathrm{C}(\mathbb{X}_{q,\rho,\lambda})$ *carries an action of* $\mathbb{T}$ *such that* $\deg(e_i) = i$ *for all* i *and there is an action* $\Gamma_{q,\rho,\lambda} : \mathrm{C}(\mathbb{X}_{q,\rho,\lambda}) \to \mathrm{C}(\mathrm{SU}_q(2)) \boxtimes_\zeta (\mathbb{X}_{q,\rho,\lambda})$ *of* $\mathrm{SU}_q(2)$ *on* $\mathbb{X}_{q,\rho,\lambda}$ *such that conditions* (1), (2) *and* (4) *from theorem 3 hold.*

Moreover for $\lambda' = \frac{\lambda}{|q|(1+|q|^2)}$ *and* $\rho' = \frac{\rho}{|q|^2(1+|q|^2)}$ *the* C^**-algebra* $\mathrm{C}(\mathbb{X}_{q,\rho,\lambda})$ *is isomorphic to* $\mathrm{C}(X_{|q|\lambda'\rho'})$ *defined by Podleś in* [3, Section 3].

The classification of the quantum spaces $X_{|q|\lambda'\rho'}$ was already performed in [3] and it follows from this classification that for a fixed q each element of the family of quantum spaces $\{\mathbb{X}_{q,\rho,\lambda}\}$ is $\mathrm{SU}_q(2)$-equivariantly isomorphic to one member of the family of Podleś spheres $\{\mathbb{S}^2_{|q|,c}\}_{c\in\mathbb{R}\cup\infty}$ and they are not $\mathrm{SU}_q(2)$-equivariantly isomorphic for different c.

For all details we refer the reader to the preprint [4].

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Quantum isometries

Alexandru Chirvasitu

The talk is concerned with quantum symmetry groups of compact metric spaces (X, d) in the sense of [3, Definition 3.1]: an action

$$\rho : C(X) \to C(X) \otimes A$$

of a compact quantum group A with bounded antipode κ is *isometric* if, denoting

$$d_p(-) = d(p, -),\ \forall p \in X,$$

we have

$$\rho(d_x)(y) = \kappa\left(\rho(d_y)(x)\right),\ \forall x, y \in X.$$

While *finite* metric spaces always have quantum isometry groups (i.e. universal isometric compact quantum group actions in the sense above) [1], it is unclear whether *all* compact metric spaces (X, d) do.

[3, Corollary 4.9] says that (X, d) does indeed have a compact quantum isometry group provided it admits what we will refer to as a *loose embedding* in some Euclidean metric space $(\mathbb{R}^n, d_{\mathbb{R}^n})$: a continuous map

$$\phi : X \to \mathbb{R}^n$$

that preserves equality and difference of distances, i.e. such that

$$d_{\mathbb{R}^n}(\phi x, \phi y) = d_{\mathbb{R}^n}(\phi x', \phi y') \iff d(x, y) = d(x', y'),\ \forall\ x, y, x', y' \in X.$$

We then say that (X, d) is *loosely embeddable* in $\mathbb{R}^n$. The talk poses a number of questions that arise naturally in the present context:

- Let (X, d) be a compact metric space all of whose finite subspaces are loosely embeddable in $\mathbb{R}^N$ for the same uniformly-chosen N. Is (X, d) itself uniformly embeddable in $\mathbb{R}^{\mathbb{N}}$?
- Does every compact Riemannian manifold, equipped with the geodesic distance, satisfy the hypothesis of the preceding question?

While the machinery developed in [3] requires loose embeddability in *finite-dimensional* Hilbert spaces, one can pose the same problem for the countably infinite-dimensional Hilbert space ℓ^2. In that case, I do not know even a single example of a compact metric space that is not loosely embeddable.

- Is *every* compact metric space (X, d) loosely embeddable in the countably infinite-dimensional Hilbert space?

Finally, moving back to compact quantum groups, the question that motivated the work of [3] remains:

- Does every compact metric space have a compact quantum isometry group?

It is an unpublished result [2] that *countable* compact metric spaces do.

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L_2-cohomology, derivations and quantum Markov semi-groups on q-Gaussian algebras

Martijn Caspers
(joint work with Yusuke Isono, Mateusz Wasilewski)

The aim of this talk is to connect the theory of Markov semi-groups in quantum probability to cohomology theory. This gives fundamentally new links between these two worlds with surprising new applications to rigidity questions of von Neumann algebras. Let us illuminate this in more detail.

Markov processes plays a central role in classical probability. They are time-continuous probabilistic processes in which the future state of a system depends on the current state only and in particular does not depend on any information from the past. Where probability deals with probability spaces (normalized measure spaces), *quantum* probability deals with quantum probability spaces (a matrix algebra or von Neumann algebra with a tracial state). The right notion of a quantum Markov process is then a time-continuous semi-group of trace preserving unital completely positive maps.

Now take such a quantum Markov semi-group (QMS) and write it as a semi-group $(\Phi_t = \exp(-t\Delta))_{t\geq 0}$ of trace preserving unital completely positive maps with generator Δ. Δ then plays the role of a non-commutative Laplacian. There has been an incredible study of such semi-groups in several contexts; classically in the theory of harmonic analysis and PDE's, differential (Riemannian) geometry (see the work of Ledoux, Bakry and many others) and in the non-commutative setting in quantum probability with very recent links to the structure of von Neumann algebras. In the latter context we refer especially to the recent results by Cipriani-Sauvageot [3], [4] and the author [1], [2].

A crucial first connection between cohomology theory and quantum Markov semi-groups was found in [4]. They showed that every non-commutative Laplacian Δ (i.e. generator of a QMS) admits a derivation as its square root. So there exists some algebra $\mathcal{A}$ in the domain of $\Delta^{\frac{1}{2}}$ and a derivation $\partial : \mathcal{A} \to H$ into some $\mathcal{A} - \mathcal{A}$-bimodule H such that,

$$\langle \partial(a), \partial(b) \rangle = \langle \Delta^{\frac{1}{2}}(a), \Delta^{\frac{1}{2}}(b) \rangle.$$

Ignoring technicalities of domains, this says that $\Delta = \partial^* \overline{\partial}$ so that indeed ∂ is a square root. Since derivations are 1-cocycles in Hochschild cohomology, this is the first evidence that QMS's are related to cohomology.

In [2] we study the consequences for higher order cohomology theory. We show that from a QMS one can naturally construct n-cocycles in Hochschild cohomology at the expense of changing the bimodule. More precisely, let $\mathcal{A}$ be an algebra and assume $\Delta(\mathcal{A}) \subseteq \mathcal{A}$ (plus some extra conditions of technical nature). Let H be a Hilbert space equipped with an $\mathcal{A}$-$\mathcal{A}$-bimodule structure. There exists a chain map:

$$G_\bullet : H_\bullet(\mathcal{A}, H) \to H_{\bullet+1}(\mathcal{A}, \mathcal{A} \otimes_\Delta H)$$

where $\mathcal{A} \otimes_\Delta H$ is the natural completion of $\mathcal{A} \otimes H$ with pre-inner product

$$\langle a \odot \xi, b \odot \eta \rangle = \langle \Gamma(a,b)\xi, \eta \rangle, \tag{1}$$

and

$$\Gamma(a,b) = \Delta(b)^* a + b^* \Delta(a) - \Delta(b^* a).$$

The chain map is given by

$$G\partial(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes_\Delta \partial(a_1 \otimes \ldots \otimes a_n).$$

Algebraically this construction is a well-known method from cohomology theory. Here an analytic part enters the scene as we use Hilbert spaces and the inner product (1). Note that $G^n(\xi)$ for any vector ξ in a Hilbert space K yields an n-cocycle with values in $\mathcal{A}^{\otimes_\Delta n} \otimes_\Delta K$. So we have a natural way of constructing cocycles at the expense of changing bimodules. We remedy this change of bimodules to some extend using the following theorem.

Theorem A. Suppose that $\mathcal{A}$ is contained in a finite von Neumann algebra M. Suppose that for every $a, b \in \mathcal{A}$ the mapping

$$x \mapsto \Delta(axb) + a\Delta(x)b - \Delta(ax)b - a\Delta(xb)$$

is in the Schatten-von Neumann $\mathcal{S}_{2n}$-class of $L_2(M)$. Then $\mathcal{A}^{\otimes_\Delta n} \otimes_\Delta L_2(M)$ is quasi-contained in the coarse bimodule of M.

Combining this, Theorem A gives tools to construct higher order cocycles that are quasi-contained in the coarse bimodule of M. It is nice that such natural cocycles exist, however on the level of cohomology Theorem A may not give much. In fact for q-Gaussian algebras, the n-th cohomology for $n \geq 2$ vanishes ([2]), so everything we construct is a coboundary. Nevertheless, for the 1-cocycles we find important applications. We show the following using the machinery developed by Ozawa-Popa [5], [6] and Peterson [7].

Theorem B. Suppose that the conditions of Theorem A are fulfilled. Suppose further that Δ has compact resolvent, the eigenvalues $\lambda_1 < \lambda_2 < \ldots$ of Δ have subexponential growth and further that the eigenspaces $\mathcal{A}(\lambda)$ of an eigenvalue λ are in $\mathcal{A}$ and satisfy the filtering condition

$$\mathcal{A}(\lambda_k)\mathcal{A}(\lambda_l) \subseteq \oplus_{i=0}^{k+l} \mathcal{A}(\lambda_{k+l}).$$

Suppose further that $\overline{\mathcal{A}}^{\| \ \|}$ is locally reflexive. Then M has the Akemann-Ostrand property. If M is moreover weakly amenable, then it is strongly solid.

As an example, we obtain the following improvement of Shlyakhtenko's result [8] in low dimensions. The upper bound $\sqrt{2}-1$ comes from [8].

Theorem C. Bozejko-Speicher q-Gaussian algebras have the Akemann-Ostrand property for $0 < q < \max(\dim(H)^{-1/2}, \sqrt{2}-1)$.

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Quantum mathematics and quantum groups

Moritz Weber

In the recent decades, a number of mathematical theories have been developed dealing with quantum phenomena in a broad sense. A common theme is a formulation which is compatible with noncommutativity, i.e. with algebraic structures whose multiplication is noncommutative. One might view these theories as a kind of *quantum mathematics* containing the following branches amongst others:

- Quantum topology (C^*-algebras)
- Quantum measure theory (von Neumann algebras)
- Quantum probability (Voiculescu's free probability and other quantum probability theories)
- Quantum groups (Woronowicz's compact quantum groups and Kustermans-Vaes's locally compact quantum groups)
- Quantum information theory
- Quantum complex analysis (free analysis)

Several of these theories are based on Gelfand-Naimark duality, but not all of them. However, there are links between these fields and we are just in the beginning of finding and exploring them and it seems that quantum groups provide exactly the right notion of symmetry in this context. It appears to be a fruitful task to explore the interplay within quantum mathematics as well as the links between quantum mathematics and other domains of mathematics.

Besides discussing this general framework a bit, we reported on two examples for the latter kinds of links arising from the subclass of quantum groups G with $S_n \subset G \subset U_n^+$:

(1) The categories of partitions underlying easy quantum groups [1, 2] provide examples of new kinds of categories in the sense of Deligne's categories interpolating the representation categories of the symmetric group. See the talk by Laura Maaßen.

(2) The unitary easy half-liberations are indexed by cyclic groups on the one hand and by subsemigroups $D \subset (\mathbb{N}_0, +)$ on the other [3, 4]. They provide an operator algebraic and a combinatorial model for numerical semigroups, a link to be explored further.

Furthermore, we sketched the outcomes of joint work with Daniel Gromada (funded by *SFB-TRR 195*):

(a) Using computer experiments, we found a couple of new non-easy linear categories of partitions, some of which don't containing singletons. [5]

(b) While seeking for an interpretation, we discovered a functor $\mathcal{V}$ which describes subrepresentations of easy quantum groups. More precisely, given an easy quantum group $S_n \subset G \subset O_n^+$ with category $\mathcal{C}$ containing the double singleton, it is isomorphic to a quantum group $S_{n-1} \subset G_{\mathrm{irr}} \subset O_{n-1}^+$. Its irreducible part, and its intertwiner spaces may be described as

$$\mathrm{Hom}_{G_{\mathrm{irr}}} = \mathrm{span}\{T_{\mathcal{V}p} \mid p \in \mathcal{C}\}.$$

The definition of the functor $\mathcal{V}$ is derived from the above mentioned examples of non-easy quantum groups: in a way it is "killing" singletons. [6]

(c) Going one step further, we take the converse perspective and construct "superrepresentations" by adding extra singletons to the category. We then obtain a number of new products of quantum groups interpolating the free product and the direct product [7].

It will be interesting to study properties of quantum groups which pass through the free product but not through the direct product (or vice versa) and to check for which of the interpolating products the property holds, such as weak amenability. One could also try to make observations on the variation of ℓ^2-Betti numbers along the interpolating products.

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