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Mini-Workshop: Degeneration Techniques in Representation Theory

Organized by
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Abstract. Modern Representation Theory has numerous applications in many mathematical areas such as algebraic geometry, combinatorics, convex geometry, mathematical physics, probability. Many of the object and problems of interest show up in a family. Degeneration techniques allow to study the properties of the whole family instead of concentrating on a single member. This idea has many incarnations in modern mathematics, including Newton-Okounkov bodies, tropical geometry, PBW degenerations, Hessenberg varieties. During the mini-workshop Degeneration Techniques in Representation Theory various sides of the existing applications of the degenerations techniques were discussed and several new possible directions were reported.

Mathematics Subject Classification (2010): 17B10, 17B35, 20G05, 14M15, 14M25, 52B20, 05E05, 05E10.

Introduction by the Organizers

The mini-workshop *Degeneration Techniques in Representation Theory*, organised by Evgeny Feigin (Moscow), Ghislain Fourier (Aachen) and Martina Lanini (Rome) took place on October 6th – October 12, 2019. It was attended by 17 participants from Italy, Germany, Russia, USA, Canada, Japan, Mexico.

The mini-workshop consisted of three mini-courses (three lectures each course), several one hour talks and several short presentations by young researchers. Special time slots were reserved for free discussions and problem sessions to facilitate the interactions between participants. All the participants were given a possibility to present their research. This was especially important for the young participants since they could get a feedback from the more mature mathematicians. We focused on two main goals: to discuss the results obtained by the participants of the mini-workshop within the last year and to lay down the foundation for the directions of the future research.

The mini-courses were given by Chris Manon (USA), Syu Kato (Japan) and Julianna Tymoczko (USA).

The talks of Chris Manon were devoted to the description of various aspects of the Khovanskii bases with applications in commutative algebra, algebraic geometry and the theory of Newton-Okounkov bodies. The lectures of Chris Manon were very important for the participants of the workshop since the language of valuations and SAGBI, Gröbner and Khovanskii bases provides a natural link between various degenerations of flag varieties, coming from the PBW-type filtrations, tropical geometry and quiver Grassmannians. This was illustrated by the talks of Valentina Kiritchenko (Russia), Peter Littelmann (Germany), Naoki Fujita (Japan), Lara Bossinger (Mexico) and Igor' Makhlin (Russia) treating various aspects of the degenerations of flag varieties and Bott-Samelson varieties. In particular, several types of toric degenerations were discussed and combinatorics of convex polytopes was touched upon.

The mini-course given by Julianna Tymoczko was devoted to the Hessenberg varieties. These are projective algebraic naturally defined as subvarieties of the flag varieties of simple finite-dimensional complex Lie groups. The geometry, representation theory and combinatorics of the Hessenberg varieties were discussed, including the Goresky-Kottwitz-MacPherson approach for the computation of the equivariant cohomology groups. The degeneration techniques play a crucial role in the story, since the Hessenberg varieties depend on an operator and it is absolutely necessary to consider these varieties in families. In particular, the famous Peterson varieties can be studied as limits of regular semi-simple Hessenberg varieties. The lectures of Julianna Tymoczko were complemented by the talk of Martha Precup (USA), who described some particular results in the theory of Hessenberg varieties and their connection to the famous Stanley-Stembridge Conjecture.

The lectures given by Syu Kato (Japan) were devoted to the theory of semi-infinite flag varieties. These are infinite-dimensional ind-schemes, which are crucial for the study of various aspects of the moduli spaces of maps to the flag varieties of simple finite-dimensional Lie groups. In particular, they carry information about degenerations of the maps into the so-called quasi-maps. In his talks Syu Kato gave an overview of the modern state-of-art in the understanding of algebro-geometric, representation-theoretic and combinatorial properties of the semi-infinite flag varieties. He also stated several very recent results and decribed open directions. The talk of Ilya Dumanski can be regarded as a companion of the Kato lectures. More precisely, Ilya Dumanski considered the semi-infinite version of the Verones curves and, generally, the Veronese embeddings of the flag varieties. The relation to the theory of affine Demazure modules was described and several conjectures were stated.

One more central topic of the mini-workshop was the theory of quiver Grassmannians. By definition, these varieties depend on a representation of a quiver and thus show up in a family. Degenerating a representation one gets a family of projective algebraic variety. Degenerations of representations of bipartite type A quivers were the focus of Jenna Rajchgot's talk, who explained how to make use of these degenerations to determine K-theory of the relevant quiver loci. Since classical flag varieties of type A can be realized as quiver Grassmannians for equi-oriented type A quiver, one gets a very natural example of such a degeneration procedure. Various PBW-type degenerations and Schubert degenerations can be obtained in this way. The talks of Xin Fang (Germany), Markus Reineke (Germany) and Johannes Flake (Germany) were devoted to various aspects of quiver Grassmannians. In particular, a sum of squares formula for the dimension of the cohomology algebra of PBW degenerate was presented and a conjecture about the cohomology algebra was formulated. Also, certain numerical results on the properties of the maximal flat degeneration were presented.

Summarizing, the workshop was very successful: not only we were able to discuss various new results obtained by several groups working in different countries, but we also paved the way for the future research, formulating open problems and discussing new directions.

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Mini-Workshop: Degeneration Techniques in Representation Theory

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Abstracts

Degenerations of flag varieties from PBW filtration

Xin Fang

In this extended abstract I present a short summary on results around FFLV (Feigin-Fourier-Littelmann-Vinberg) bases for Lie algebras of type $\mathtt{A}$ and degenerations of flag varieties from these bases. It is by no means complete.

Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ be the Lie algebra of traceless $n \times n$-matrices and $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a triangular decomposition. For a positive root $\beta \in \Delta_+$, fix $0 \neq f_\beta \in \mathfrak{n}^-$ to be a root vector of weight $-\beta$. By assigning $\deg(f_\beta) = 1$ we obtain the classical PBW (Poincaré-Birkhoff-Witt) filtration on the universal enveloping algebra $U(\mathfrak{n}^-)$ whose associated graded algebra is isomorphic to $U(\mathfrak{n}^{-,a})$ where $\mathfrak{n}^{-,a}$ is the abelianization of $\mathfrak{n}^-$ by reducing all brackets to zero. For a dominant integral weight $\lambda \in \mathcal{P}_+$, the corresponding irreducible representation $V(\lambda)$ of $\mathfrak{g}$ is $U(\mathfrak{n}^-)$-cyclic. By fixing a highest weight vector $v_\lambda \in V(\lambda)$, the PBW filtration on $U(\mathfrak{n}^-)$ induces a filtration on $V(\lambda)$. Passing to the associated graded module gives a $U(\mathfrak{n}^{-,a})$-module $V^a(\lambda)$. Let v_λ^a denote the class of v_λ therein.

Ten years ago, in [11] Feigin, Fourier and Littelmann found a basis of $V^a(\lambda)$ together with a nice parametrisation:

Theorem 1. (1) *For $\lambda \in \mathcal{P}_+$, there exists an explicit lattice polytope* $\mathrm{FFLV}(\lambda)$ *in $\mathbb{R}_{\geq 0}^{\Delta_+}$ such that $\{f^{\mathbf{a}} \cdot v_\lambda^a \mid \mathbf{a} \in \mathrm{FFLV}(\lambda)_{\mathbb{Z}}\}$ form a $\mathbb{C}$-basis of $V^a(\lambda)$, where* $\mathrm{FFLV}(\lambda)_{\mathbb{Z}} := \mathrm{FFLV}(\lambda) \cap \mathbb{Z}^{\Delta_+}$ *and for a tuple* $\mathbf{a} = (a_\beta)_{\beta \in \Delta_+} \in \mathrm{FFLV}(\lambda)_{\mathbb{Z}}$, $f^{\mathbf{a}} := \prod_{\beta \in \Delta_+} f_\beta^{a_\beta} \in U(\mathfrak{n}^{-,a})$.

(2) *For $\lambda, \mu \in \mathcal{P}_+$,* $\mathrm{FFLV}(\lambda + \mu)_{\mathbb{Z}} = \mathrm{FFLV}(\lambda)_{\mathbb{Z}} + \mathrm{FFLV}(\mu)_{\mathbb{Z}}$ *as the Minkowski sum of sets.*

According to [1], such polytopes can be looked as the marked chain polytopes associated to the root poset of $\mathfrak{g}$; the marked order counterparts are the famous Gelfand-Tsetlin polytopes. There exists a piecewise-linear and lattice preserving transfer map between them, details can be found in *loc.cit.*

Since $\mathfrak{n}^{-,a}$ acts nilpotently on $V^a(\lambda)$, the abelianized group $N^{-,a} := \exp(\mathfrak{n}^{-,a})$ acts on $V^a(\lambda)$. Feigin [9] defined the degenerate flag variety

$$\mathcal{F}_n^a := \overline{N^{-,a} \cdot [v_\rho^a]} \subseteq \mathbb{P}(V^a(\rho))$$

where ρ is the sum of fundamental weights.

Recall that the classical type $\mathtt{A}$ flag variety $\mathcal{F}_n$ admits various realizations:

(1) highest weight orbit: $\overline{\exp(\mathfrak{n}^-) \cdot [v_\rho]} \subseteq \mathbb{P}(V(\rho))$;
(2) linear subspaces: $\{(V_1, \cdots, V_{n-1}) \mid V_i \subseteq V_{i+1}\} \subseteq \prod_{k=1}^{n-1} \mathrm{Gr}_k(\mathbb{C}^n)$;
(3) vanishing locus of Plücker relations $R_{I,J}^k$ in the projectivization of the sum of fundamental representations.

A $\mathbb{C}$-basis of the homogeneous coordinate ring $\mathbb{C}[\mathcal{F}_n]$ is encoded in the semistandard Young tableaux. In [9, 10], Feigin showed that $\mathcal{F}_n^a$ similar descriptions exist for $\mathcal{F}_n^a$:

(2a) linear subspaces: $\{(V_1, \cdots, V_{n-1}) \mid \mathrm{pr}_{i+1}(V_i) \subseteq V_{i+1}\} \subseteq \prod_{k=1}^{n-1} \mathrm{Gr}_k(\mathbb{C}^n)$, where for a fixed basis $e_1, \cdots, e_n$ of $\mathbb{C}^n$, $\mathrm{pr}_k : \mathbb{C}^n \to \mathbb{C}^n$ is the linear projection along e_k;

(3a) $\mathcal{F}_n^a$ is the vanishing locus of degenerate Plücker relations $R_{I,J}^{k;a}$.

A $\mathbb{C}$-basis of $\mathbb{C}[\mathcal{F}_n^a]$ is encoded in the semi-standard PBW tableaux arising from the FFLV basis. This implies that $\mathcal{F}_n^a$ is a flat degeneration of $\mathcal{F}_n$ and is reduced.

A striking result in [5] establishes an isomorphism of projective varieties between $\mathcal{F}_n^a$ and a Schubert variety $X_w(\widetilde{\lambda})$ where $w \in \mathfrak{S}_{2n}$ and $\widetilde{\lambda}$ is a weight of $\mathfrak{sl}_{2n}(\mathbb{C})$. Later, the authors in [6] gave a representation-theoretical proof of this result by showing that the abelianized representation $V^a(\lambda)$ is isomorphic to a Demazure module for the double rank Lie algebra.

The description $(2a)$ is the starting point of [4] using the quiver Grassmannian approach. Such a description is further generalized in [2, 3] where we classify the linear maps $\mathbf{f} := (f_1, \cdots, f_{n-2}) \in \mathrm{End}(\mathbb{C}^n)^{n-2}$ such that the scheme

$$\mathcal{F}_n^{\mathbf{f}} := \{(V_1, \cdots, V_{n-1}) \mid f_i(V_i) \subseteq V_{i+1}\} \subseteq \prod_{k=1}^{n-1} \mathrm{Gr}_k(\mathbb{C}^n)$$

enjoys nice geometric properties (normal, irreducible, being a flat degeneration, *etc*).

There exists a subset of $\mathrm{End}(\mathbb{C}^n)^{n-2}$ called the PBW locus. For $\mathbf{f} = (f_1, \cdots, f_{n-2})$ coming from the PBW locus, $\mathcal{F}_n^{\mathbf{f}}$ can be realized as a highest weight orbit of some partial degenerations of $\mathfrak{n}^-$; it can be scheme-theoretically cut off by degenerations of Plücker relations and a basis of $\mathbb{C}[\mathcal{F}_n^{\mathbf{f}}]$ is encoded in the semi-standard PBW tableaux. They are isomorphic to Schubert varieties in some partial flag varieties.

A toric degeneration of $\mathcal{F}_n$ to the toric variety associated to FFLV polytopes is first constructed in [12] in an algebraic setting and later in [13] in a geometric point of view using Newton-Okounkov bodies. From a tropical geometric point of view, such a toric degeneration corresponds to a maximal prime cone in the tropical flag variety with respect to the Plücker embedding. Such a cone is explicitly described in [7] by giving all its non-redundant facets. Relations between this cone and quantum groups are discovered in the forthcoming preprint [8].

Many of the results above hold for $\mathfrak{sp}_{2n}(\mathbb{C})$, but the entire picture is far away from being complete.

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On the definition of semi-infinite flag manifolds and applications

Syu Kato

Geometry of flag manifolds reflects the representation theoretic pattern of a (simply connected) semi-simple algebraic group G and its Lie algebra $\mathfrak{g}$. It is already apparent in the Borel-Weil theorem, that states the nef line bundle of the full flag variety $\mathscr{B}$ of G is in bijection with the set of isomorphism classes of the irreducible rational representations of G. The Beilinson-Bernstein localization theorem and the Bezrukavnikov-Mirković-Ryuminin derived localization theorem amplified the Borel-Weil theorem by incorporating all the $\mathfrak{g}$-modules.

These discoveries encourage people to develop parallel description for affine Lie algebras and p-adic groups based on the geometry of affine flag manifolds. In such a development, it becomes apparent that there are three affine flag manifolds that are relevant to the representation theory of affine Lie algebras.

One of them, usually referred to as the semi-infinite flag manifolds, is defined set-theoretically as:

$$X^{\frac{\infty}{2}} := G(\mathbb{C}(\!(z)\!))/H(\mathbb{C}[\![z]\!])N(\mathbb{C}(\!(z)\!)),$$

where $B \subset G$ is a Borel subgroup, $N \subset G$ is its unipotent radical, and $H \subset G$ is a maximal torus of B. The geometry and combinatorics of $X^{\frac{\infty}{2}}$ are expected to represent the representation theory of G at the positive characteristic and representation theory of the affine Lie algebra of $\mathfrak{g}$ at the critical level [1, 2]. However, it turned out to be impossible to put $X^{\frac{\infty}{2}}$ a separated scheme structure. Taking into account of this, the geometric study of a modified version of semi-infinite flag manifold

$$\mathscr{Q}_G := G(\mathbb{C}[z^{\pm 1}])/H(\mathbb{C})N(\mathbb{C}[z^{\pm 1}])$$

was initiated in Finkelberg-Mirković [3] and Feigin-Finkelberg-Kuznetsov-Mirković [4]. One of the important observation afforded there is that $\mathscr{Q}_G$ is the union of the quasi-map spaces $\mathscr{Q}_G(\beta)$, that is a compactification of the space of maps from $\mathbb{P}^1$ to $\mathscr{B}$ of degree β, where β is an element of the non-negative span of the positive coroots of G, identified with the effective classes in $H_2(\mathscr{B}, \mathbb{Z})$. Since $\mathscr{Q}_G(\beta)$ admits a resolution of singularities by a variant of the space of stable maps, this opened a possibility to connect the theory of semi-infinite flag manifolds with the theory of quantum cohomologies and quantum K-groups of $\mathscr{B}$.

In [3], they also consider another version

$$\mathbf{Q}_G^{\mathrm{rat}} := G(\mathbb{C}(\!(z)\!))/H(\mathbb{C})N(\mathbb{C}(\!(z)\!)),$$

that we call the formal version of the semi-infinite flag manifold. It is clear that $\mathscr{Q}_G \subset \mathbf{Q}_G^{\mathrm{rat}}$ by set-theoretic consideration, and one can show that this inclusion must be Zariski dense. However, the technical diffusion between them is rather heavy as $\mathscr{Q}_G$ has countable dimension while $\mathbf{Q}_G^{\mathrm{rat}}$ has uncountable dimension. In particular, [3] exhibits that $\mathbf{Q}_G$ exists as a scheme, but it is not suited enough for an intensive algebro-geometric study except for $G = SL(2, \mathbb{C})$.

In this series of talks, we first exhibited how to capture $\mathbf{Q}_G^{\mathrm{rat}}$ and $\mathscr{Q}_G(\beta)$ more concretely in the sense one can actually characterize their homogeneous coordinate rings. More precisely, we employ the theory of extremal weight modules of quantum loop algebra associated to G to produce an ind-scheme (of infinite type) that is reduced, normal, and is the best approximation of $\mathbf{Q}_G^{\mathrm{rat}}$ by an ind-scheme. In particular, we exhibited that it is reasonable to define the ind-piece $\mathbf{Q}_G$ of $\mathbf{Q}_G^{\mathrm{rat}}$ as

$$\mathbf{Q}_G = \mathrm{mProj} \bigoplus_{\lambda \in P_+} \mathbb{W}(\lambda)^\vee,$$

where mProj is the multigraded proj, P_+ is the set of dominant weights of G, $\mathbb{W}(\lambda)$ is the global Weyl module of current algebra $\mathfrak{g}[z] = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z]$ of extremal weight λ, and $\vee$ denotes the restricted dual. In this picture, we recover $\mathscr{Q}_G(\beta)$ as a particular case of the Richardson variety of $\mathbf{Q}_G^{\mathrm{rat}}$. This makes it possible to reasonably understand the coordinate rings of $\mathbf{Q}_G^{\mathrm{rat}}$ and $\mathscr{Q}_G(\beta)$ from representation-theoretic perspective, and also exhibits the higher cohomology vanishing of their natural nef line bundles. The proofs of these results require the Frobenius splitting of the positive characteristic analogue of them ([12]), whose proof is quite untraditional and uses the Frobenius splitting of the thick flag manifolds exhibited in [9] that in turn employs an argument from [8] originally due to Olivier Mathieu.

The second topic was a definition of the equivariant K-group of $\mathbf{Q}_G^{\mathrm{rat}}$. Unfortunately, we do not know whether the scheme $\mathbf{Q}_G$ is coherent or not. In particular, a naive definition of equivariant K-group of $\mathbf{Q}_G^{\mathrm{rat}}$ using a class of finitely generated $\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}$-modules might break down heavily. To avoid this difficulty, we employ a view-point that the ring $\bigoplus_{\lambda \in P_+} \mathbb{W}(\lambda)^\vee$ is far from Noetherian, but is "graded Artin". This view-point enables us to define the equivariant K-group of $\mathbf{Q}_G^{\mathrm{rat}}$ as a "convergent" functional on P modulo the constraints/relations coming from disguises of finite generation/negligible modules. Once we have a definition of

$K_{H\times\mathbb{C}^\times}(\mathbf{Q}_G)$ or $K_{H\times\mathbb{C}^\times}(\mathbf{Q}_G^{\mathrm{rat}})$, we can prove a Pieri-Chevalley formula in this setting ([10]) by means of the analysis of the internal structure of $\mathbb{W}(\lambda)$'s. We can also interpret such coefficients in terms of Richardson varieties of $\mathbf{Q}_G^{\mathrm{rat}}$.

The third topic was the connection of $K_{H\times\mathbb{C}^\times}(\mathbf{Q}_G)$ with the equivariant quantum K-group $qK_H(\mathscr{B})$ of $\mathscr{B}$ in the sense of Givental and Lee. In fact, Braverman [5] connects $\mathscr{Q}_G(\beta)$ with quantum cohomology of $\mathscr{B}$ though the J-functions, that was further polished to the K-theoretic J-function calculation in Braverman-Finkelberg [6, 7]. There, the main geometric portion was to (essentially) guarantee that $\mathscr{Q}_G(\beta)$ is normal and has rational singularities. Thanks to the reconstruction theorem, the knowledge of the J-fuction is enough to recover the ring structure of the quantum cohomologies/K-groups in a sense. However, roughly speaking, this is a kind of statement that a commutative local Artin ring is specified as the annihilator ring of a polynomial, and it is desirable to make things more explicit. This is achieved by specifying an isomorphism

$$qK_H(\mathscr{B}) \xrightarrow{\cong} K_H(\mathbf{Q}_G)$$

as based rings ([11]). The proof of this isomorphism itself only requires the Borel-Weil-Bott theorem of $\mathbf{Q}_G$, but the proof of the preservation of the bases are more delicate. It requires to extend the above mentioned results of Braverman-Finkelberg to some particular Schubert-like subvarieties of $Q_G(\beta)$.

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An introduction to Hessenberg varieties

Julianna Tymoczko

Hessenberg varieties form a large family of subvarieties of the flag variety that arise naturally in many areas of mathematics, including quantum cohomology, numerical analysis, algebraic geometry, representation theory, and combinatorics.[1]

Hessenberg varieties can be defined in general Lie type, but we mainly consider type A_n. A flag is a set of nested subspaces $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{C}^n$ with each V_i an i-dimensional $\mathbb{C}$-vector space. The collection of flags is the flag variety $GL_n(\mathbb{C})/B$ where B consists of upper-triangular invertible matrices; the coset gB generates the flag $V_\bullet$ whose i-dimensional subspace V_i is the span of the first i columns of g.

The first parameter used to define Hessenberg varieties is a linear operator $X : \mathbb{C}^n \to \mathbb{C}^n$. (In fact, only the generalized Jordan type of X is needed [14].)

The second parameter has several equivalent descriptions, including:.

- A function $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ with $h(i) \geq h(i-1)$ for all i (nondecreasing) and $h(i) \geq i$ for all i (upper-triangular).
- A subspace H of $\mathfrak{gl}_n$ with $[H, \mathfrak{b}] \subseteq H$ and $H \supset \mathfrak{b}$.
- A subset $\mathcal{M}_H$ of roots so that if $\alpha \in \mathcal{M}_H, \beta \in \Phi^+, \alpha+\beta \in \Phi$ then $\alpha \in \mathcal{M}_H$ (nondecreasing) and $\mathcal{M}_H \supset \Phi^+$ (upper-triangular).

Hessenberg varieties were first defined by De Mari-Shayman and then generalized by De Mari-Procesi-Shayman [6, 7].

Definition 1. *Given X and h as above,* **the Hessenberg variety** $\mathcal{H}ess(X, h)$ *is*

$$\begin{aligned} \mathcal{H}ess(X,h) &= \{ \text{ Flags } V_\bullet : XV_i \subseteq V_{h(i)} \text{ for all } i\} \\ &= \{ \text{ Flags } gB : g^{-1}Xg \in H\} \end{aligned}$$

Geometry of Hessenberg varieties. Hessenberg varieties have some properties analogous to the Schubert cell decomposition of the full flag variety.

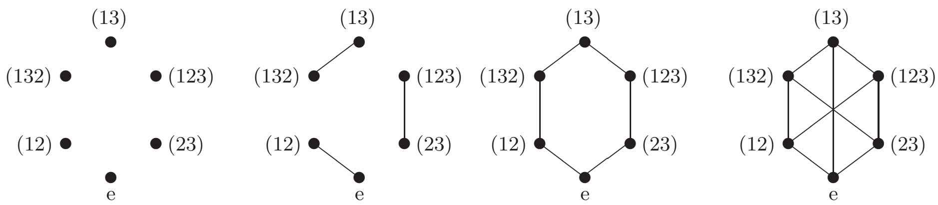
- **They are paved by affines.** This is a condition like a CW-decomposition except that closures of cells need not be a union of other cells [17].
- **If the matrix X is chosen well, the affine pieces are intersections with Schubert cells, called *Hessenberg Schubert cells.*** This holds in type A [17] and in some other cases [15].
- **The Hessenberg Schubert cells are indexed by Young tableaux that combinatorially record the dimension of the cell.** The result partly extends to other types [15]; see also when $H = \mathfrak{b}$ [8].

[1]The author was partly supported by NSF-DMS-1800773.

We give open questions, including: find a closed formula for the dimension or number of components; which components are singular and what kinds of singularities do they have; characterize the pieces of cells in each component; identify which permutation flags are in the closure of a given Hessenberg Schubert cell.

Equivariant cohomology of Hessenberg varieties. If X is regular semisimple (namely X has n distinct eigenvalues), we have special tools to compute the equivariant cohomology of $\mathcal{H}ess(X,h)$. The main tool is often called *GKM theory* after Goresky-Kottwitz-MacPherson [9], though many people contributed to its creation [2, 5, 12]. After describing GKM theory and generalizations, we specialize to the equivariant cohomology of regular semisimple Hessenberg varieties.

Let H be a Hessenberg space and let (ij) be the transposition switching i and j. Let $G_H = (V, E_H)$ be the graph whose vertices are permutations S_n and edges $w \leftrightarrow w(ij)$ if entries (i,j) and (j,i) are both free in H. Each edge $w \leftrightarrow (ij)w$ is labeled with the polynomial $t_i - t_j$. Note that *left*-multiplication gives edge-labels while *right*-multiplication determines if an edge is in the graph.



$\mathbf{h} = (1,2,3)$ $\mathbf{h} = (2,2,3)$ $\mathbf{h} = (2,3,3)$ $\mathbf{h} = (3,3,3)$

FIGURE 1. Graphs for $H_T^*(\mathcal{H}ess(X,h))$. Label slope 1 edges by $t_1 - t_2$, slope -1 by $t_2 - t_3$, and vertical by $t_1 - t_3$

Theorem 1 (Tymoczko, [18]). *Let G_H be the edge-labeled graph above. The equivariant cohomology of the regular semisimple Hessenberg variety $\mathcal{H}ess(X,H)$ is:*

$$\mathcal{H}ess(X,H) \cong \left\{ p \in \mathbb{C}[t_1,\ldots,t_n]^{n!} : \begin{array}{l} \text{if } w \leftrightarrow (ij)w \text{ is an edge in } E_H \\ \text{then } (t_i - t_j) | (p_w - p_{(ij)w}) \end{array} \right\}$$

We describe methods and questions about bases for $H_T^*(\mathcal{H}ess(X,h)$ like:

- how to interpret "upper-triangular bases" for equivariant cohomology [11]
- a formula for the upper-triangular (Schubert) basis of $H_T^*(G/B)$ [1, Appendix D], [3]
- when there is a unique homogeneous upper-triangular basis [16, 19]
- **Question:** what can be said about other kinds of bases, e.g. symmetrized?
- **Question:** what is a formula for a basis of $\mathcal{H}ess(X,h)$?

Representations and Hessenberg varieties. There are two different S_n actions on the equivariant cohomology of $GL_n(\mathbb{C})/B$ arising from left-multiplication and right-multiplication by permutations on the vertices of the graph $G_{\mathfrak{g}}$ [18]. These give two different representations on equivariant cohomology. Only one restricts to Hessenberg varieties: the left-multiplication action [18].

A simple transposition $(i, i+1)$ acts on an arbitrary $p \in H_T^*(\mathcal{H}ess(X, h))$ by

$$((i, i+1) \cdot p)_w = (i, i+1)(p_{(i,i+1)w})$$

where $(i, i+1)$ acts on polynomials by exchanging t_i and t_{i+1}. Extend to the action of an arbitrary permutation by composing simple transpositions.

The left-action and right-action of S_n on $H_T^*(GL_n(\mathbb{C})/B)$ induce different representations on ordinary cohomology: one is trivial while the other contains the sign representation. The right action is called the *Springer representation* and is associated to $\mathcal{H}ess(X, \mathfrak{b})$. MacDonald describes the Springer representation as:

$$\text{(1)} \quad \text{Poincaré polynomial of } \mathcal{H}ess(N, \mathfrak{b}) = \sum_{\text{partitions } \lambda} (\text{rank of irrep. } \lambda)\, \widetilde{K}_{\lambda\mu(N)}(q)$$

where N is nilpotent, $\mu(N)$ is its Jordan type, and $\widetilde{K}_{\lambda\mu(N)}(q)$ is a particular normalization of Kostka-Foulkes polynomials [13, III. Sect.7, 7.11 and Ex. 6].

The left action, which descends to Hessenberg varieties, is the *monodromy action.* A version of Equation (1) applies, replacing $\mathfrak{b}$ with H and replacing the rank of λ with the multiplicity of λ in the S_n-representation on $H_T^*(\mathcal{H}ess(X, H))$ [14].

This representation on $H_T^*(\mathcal{H}ess(X, H))$ is particularly important because it coincides with a combinatorial representation at the heart of the so-called Stanley-Stembridge conjecture. Brosnan-Chow and Guay-Paquet [4, 10] recently proved this, after a conjecture of Shareshian-Wachs. It suggests a geometric approach to the Stanley-Stembridge conjecture through analysis of the representation on $H_T^*(\mathcal{H}ess(X, H))$. Another talk in this workshop will give more details.

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Khovanskii bases in three settings

Christopher Manon

In this series of three lectures we discuss computational aspects of valuations, the existence of a degeneration to a variety of complexity one for any irreducible, reduced projective variety, and a classification theorem for flat toric families of finite type. These three topics share Khovanskii bases as a common feature.

Khovanskii bases and computations in algebras

(joint with Kiumars Kaveh)

The theory of Khovanskii bases generalizes SAGBI bases, which is itself an analogue of the theory of Gröbner bases for subalgebras of polynomial rings. Let $\mathbf{k}[\mathbf{x}] = \mathbf{k}[x_1, \ldots, x_n]$ be a polynomial ring over an algebraically closed field $\mathbf{k}$, and let $\prec$ be a monomial order. Following a standard treatment of Gröbner theory ([3], [11]), we have the initial form $in_\prec(f)$ of a polynomial $f = \sum_{i=1}^{\ell} c_i \mathbf{x}^{\alpha_i}$, and the initial ideal $in_\prec(I) = \langle \{\mathbf{x}^\alpha \mid c\mathbf{x}^\alpha = in_\prec(f), f \in I\} \rangle$ of an ideal $I \subset \mathbf{k}[\mathbf{x}]$. In what follows we assume that I is homogeneous with respect to some positive grading on $\mathbf{k}[\mathbf{x}]$ (say x_i has degree $d_i \in \mathbb{Z}_{>0}$). Gröbner bases and the Gröbner fan are defined as usual. A Gröbner basis allows for the resolution of the ideal membership problem, and Buchberger's algorithm allows us to procedurally expand any generating set of I to a Gröbner basis. In [10], Robbiano and Sweedler defined a SAGBI basis $\mathcal{B} \subset A$ for a subalgebra $A \subset \mathbf{k}[\mathbf{x}]$ to be a set whose initial forms generate the initial algebra $in_\prec(A)$. If a finite SAGBI basis $\mathcal{B} \subset A$ exists, the subduction algorithm ([11]) represents any $p \in A$ as a polynomial in $\mathcal{B}$. Moreover, any generating set of A can be expanded to a SAGBI basis ([11]); these are the SAGBI analogues of the division algorithm and Buchberger's algorithm, respectively. Notably, $in_\prec(A)$ is an affine semigroup algebra, so $\mathrm{Proj}(in_\prec(A))$ is a toric variety.

Unfortunately there are subalgebras and monomial orders with no finite SAGBI basis. To remedy this situation, we generalize to the setting of a valuation $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^d$, where $\mathbb{Z}^d$ is equipped with a group order $\prec$. The associated graded algebra $gr_\mathfrak{v}(A)$ plays the role of $in_\prec(A)$. We define a Khovanskii basis to be a set $\mathcal{B} \subset A$ whose image $\bar{\mathcal{B}} \subset gr_\mathfrak{v}(A)$ is a generating set. In [7] it is shown that the subduction algorithm exists in this setting (actually for quasivaluations), and that any generating set can be expanded to a finite Khovanskii basis, provided one exists. Moreover, a theorem of Anderson [2] shows that a toric degeneration also exists in this setting, provided $\mathfrak{v}$ is a full rank valuation.

The structure of the set of quasivaluations with finite Khovanskii bases comes from Gröbner theory. Let $I \subset \mathbf{k}[\mathbf{x}]$ be the ideal which vanishes on a generating set $\mathcal{B} = \{b_1, \ldots, b_n\} \subset A$. We let $\Gamma = (\mathbb{Z}^d, \prec)$, and $\Sigma_\Gamma(I) \subset \Gamma^n$ be the Gröbner complex of I [7]. Furthermore, we take $K_\Gamma(I) \subseteq \Sigma_\Gamma(I)$ to be the set of points of the form $w = (\mathfrak{v}(b_1), \ldots, \mathfrak{v}(b_n))$, where $\mathfrak{v} : A \to \Gamma$ is some quasivaluation. Finally, we let $\mathrm{Trop}_\Gamma(I)$ be the Γ-tropical variety of I, this is the set of $w \in \Gamma^n$ where $in_w(I)$ contains no monomials (the *tropical variety*). We have the following inclusions of complexes: $\mathrm{Trop}_\Gamma(I) \subseteq K_\Gamma(I) \subseteq \Sigma_\Gamma(I)$.

It is shown in [7] that $K_\Gamma(I)$ can be identified with the set of quasivaluations with Khovanskii basis $\mathcal{B}$. Likewise, the points $w \in \mathrm{Trop}_\Gamma(I)$ with $in_w(I)$ a prime ideal correspond precisely to the valuations on A with Khovanskii basis $\mathcal{B}$. In [7] it is also shown that the initial ideals found in $\Sigma_\Gamma(I)$ are the same as those found by classical Gröbner theory (ie $\mathbb{Z}^d = \mathbb{Z}$). The following is a consequence.

Theorem 1. *[Kaveh-M] A positively graded domain A has a full rank valuation $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^d$ with finite Khovanskii basis $\mathcal{B} \subset A$ if and only if the tropical variety* $\mathrm{Trop}(I)$ *contains a full dimensional open cell C with $in_C(I)$ a prime ideal.*

Existence of finite Khovanskii bases
(joint with Kiumars Kaveh and Takuya Murata)

By Theorem 1, finding a full rank valuation on an algebra A is equivalent to finding a full-dimensional prime cone $C \subset \mathrm{Trop}(I)$. Moreover, Anderson's theorem [2] and [9] imply that these conditions are both equivalent to the existence of a homogeneous flat degeneration $\mathrm{Spec}(A) \to \mathrm{Spec}(\mathbf{k}[S])$, where $S \subset \mathbb{Z}^d$ is a finitely generated semigroup. In this context, a toric degeneration $X \to X_0$ is a flat family $\pi : \mathcal{X} \to \mathbb{A}^1(\mathbf{k})$, where $\mathcal{X}$ is equipped with a $\mathbb{G}_m(\mathbf{k})$ action which is intertwined with the standard action on $\mathbb{A}^1(\mathbf{k})$ by π, with $\pi^{-1}(0) = X_0$ a toric variety, and $\pi^{-1}(C) = X$ for any $C \neq 0$.

There are positively graded algebras with no toric degenerations. Following [9, Section 3], the section ring $R_D = \bigoplus_{n \geq 0} H^0(\mathcal{C}, \mathcal{O}(nD))$ of a divisor D on a smooth curve C carries a homogeneous rank 2 valuation with finite Khovanskii basis if and only if $nD \sim mP$ for some $n, m > 0$ and $P \in \mathcal{C}$. Ilten and Wröbel [5] have also constructed examples out of non-normal rational curves.

In response, we let the special fiber of the degeneration belong to a larger class of varieties. The complexity of a variety X equipped with an effective action by an algebraic torus T is $dim(X) - rank(T)$. In this way, complexity-1 varieties are a natural relaxation of toric varieties. Altmann and Hausen [1] have constructed a combinatorial theory of varieties of arbitrary complexity; in particular complexity-1 varieties are roughly captured by polyhedral data and the geometry of curves. The following theorem from [9] says that the coordinate ring of projective variety can be degenerated to the coordinate ring of a complexity-1 variety.

Theorem 2. *[Kaveh-M-Murata] Let A be a positively graded* $\mathbf{k}$*-domain of dimension d, then there is a homogeneous valuation $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^{d-1}$ of rank $d-1$ with a finite Khovanskii basis.*

Roughly speaking, the two main ingredients of Theorem 2 are a technical result on the finite generation of Rees algebras of symbolic powers of height 1 prime ideals, and Bertini's theorem.

Khovanskii bases for toric families
(joint with Kiumars Kaveh)

Fix a valuation $\mathfrak{v} : A \to \mathbb{Z}^d$ with finite Khovanskii basis, let $\sigma \subset \mathrm{Trop}(I)$ be the corresponding prime cone, and let N be the lattice generated by the integer points in $\sigma \cap \mathbb{Z}^n$. A construction from [8] shows that there is a flat T_N-family $\pi : \mathcal{X} \to Y_\sigma$, where Y_σ is the affine toric variety associated to $\sigma \subset N_\mathbb{Q}$. Just as a toric degeneration over $\mathbb{A}^1(\mathbf{k})$ corresponds to the rank 1 valuation given by a point in a prime cone in $\mathrm{Trop}(I)$, the family $\mathcal{X}$ corresponds to a valuation $\mathfrak{w} : A \to \mathcal{O}_\sigma$, where $\mathcal{O}_\sigma$ is the semialgebra of N-integral piecewise-linear functions on the cone σ. If two such prime cones σ_1 and σ_2 share a facet $\sigma_1 \cap \sigma_2 = \tau$, one can then consider a corresponding valuation into $\mathcal{O}_\Sigma$, where Σ is the fan defined by σ_1, σ_2, τ; this is the natural setting for considering mutations between the Newton-Okounkov bodies associated to the prime cones σ_1, σ_2. With these constructions as motivation, it is natural to consider valuations $\mathfrak{w} : A \to \mathcal{O}_N$, where $\mathcal{O}_N$ is the semifield of piecewise-linear functions on the lattice N.

A valuation $\mathfrak{w} : A \to \mathcal{O}_N$ defines a rank 1 valuation for each $\rho \in N$ given by evaluation: $\mathfrak{w}_\rho(f) = \mathfrak{w}(f)[\rho]$. We say $\mathcal{B} \subset A$ is a Khovanskii basis of $\mathfrak{w}$ if $\mathcal{B}$ is a Khovanskii basis of $\mathfrak{w}_\rho$ for each $\rho \in N$. The following is a result in [8], it is a generalization of the equivalence between toric degenerations and valuations with finite Khovanskii bases in the classical setting.

Theorem 3. *[Kaveh-M] For every valuation $\mathfrak{w} : A \to \mathcal{O}_N$ with finite Khovanskii basis, there is finite complete fan Σ and a corresponding flat affine T_N-family $\pi : \mathcal{X}(\mathfrak{v}) \to Y_\Sigma$ of finite type with reduced, irreducible fibers and general fiber* $\mathrm{Spec}(A)$*. Moreover, every such family arrises this way.*

Theorem 3 is not stated as $1-1$ correspondence between flat toric families and valuations because there is an indeterminacy in the choice of the fan Σ. If we specialize to the case where A is a polynomial ring and the Khovanskii basis is a set of linear forms, Theorem 3 recovers Klyachko's classification of toric vector bundles [6]. In particular, one can take the Khovanskii basis to be the representable matroid constructed by Di Rocco, Jabbusch, and Smith in [4].

Theorem 3 and its corollaries suggest a new way to construct toric vector bundles, and other toric families. Let $\mathbf{k}[T_N]$ be the coordinate ring of the torus T_N and $\mathfrak{w}_N : \mathbf{k}[T_N] \to \mathcal{O}_N$ be the canonical valuation which sends a Laurent polynomial $p(\mathbf{t}) = \sum_{i=1}^{\ell} c_i \mathbf{t}^{\alpha_i}$ to the support function $\mathfrak{w}_N(p) = \min\{\alpha_i \mid c_i \neq 0\}$ of its Newton polytope. This valuation immediately extends to the rational functions $\mathbf{k}(T_N)$. Now, any affine variety X equipped with a $\mathbf{k}(T_N)$ point $i : \mathbf{k}[X] \to \mathbf{k}(T_N)$ has a valuation $i^*\mathfrak{w}_N : \mathbf{k}[X] \to \mathcal{O}_N$ which potentially defines a toric family with general fiber X. In this way, we produce toric vector bundles by solving linear equations in the field $\mathbf{k}(T_N)$ and evaluating the result with $\mathfrak{w}_N$. One can use similar techniques to show that the support of functions of the m-faces of any n-simplex are a $\mathcal{O}_N$-point on the tropical Grassmannian variety of m-planes in n-space, see [8].

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Some open problems on degerate flag varieties

Markus Reineke

We review some of the main results of [1, 2, 3] on so-called linear degenerations of SL_{n+1}-flag varieties. We formulate several open problems concerning their geometry, and speculate on potential convolution-type constructions and their representation-theoretic properties.

Let V be a complex vector space of dimension $n+1$ with a fixed basis $v_1, \ldots, v_{n+1}$. For a subset $I \subset \{1, \ldots, n+1\}$, we denote by $\mathrm{pr}_i \in \mathrm{End}(V)$ the projection operator defined by

$$\mathrm{pr}_I(v_i) = 0 \text{ for } i \in I \text{ and } \mathrm{pr}_I(v_i) = v_i \text{ for } i \notin I.$$

We define the degenerate flag variety

$$\mathrm{Fl}^a(V) = \{(U_1, \ldots, U_n) \in \prod_{i=1}^{n} \mathrm{Gr}_i(V) \,|\, \mathrm{pr}_i(U_i) \subset U_{i+1} \text{ for } i = 1, \ldots, n-1\}.$$

It is an irreducible normal variety of dimension $n(n+1)/2$ which is a flat degeneration of the variety $\mathrm{Fl}(V)$ of complete flags in V. It is acted upon by a unipotent group G^a with finitely many orbits. The maximal torus T of $\mathrm{GL}(V)$ scaling the given basis of V admits a one-parameter subgroup whose fixed points are tuples of coordinate subspaces indexed by

$$\{I_* = (I_1, \ldots, I_n) \,|\, I_i \subset \{1, \ldots, n+1\}, |I_i| = i, I_i \setminus \{i+1\} \subset I_{i+1}\},$$

and such that the attractors of the fixed points are precisely the G^a-orbits.

Problem 1: Describe the closure relation between G^a-orbits in their parametrization via tuples of sets I_*. What are the (minimal) singularities in the closures of orbits? How to describe the intersection cohomology complexes on orbit closures (see Problem 3below)?

The Euler characteristic of $\mathrm{Fl}^a(V)$ can be computed as the number of cells, which equals the $(n+1)$-st Genocchi number. One compact formula for this is

$$\chi(\mathrm{Fl}^a(V)) = \sum_{f_*} \left(\frac{\prod_i (f_i + 1)}{2^{r(f)}} \right),$$

where the sum ranges over all Motzkin paths of length $n+1$, that is, tuples $f_* = (0 = f_0, f_1, \ldots, f_n, f_{n+1} = 0)$ of non-negative integers such that $f_{i+1} - f_i \in \{-1, 0, 1\}$ for all i, and $r(f)$ denotes the number of rises of f_*, that is, indices i such that $f_{i+1} = f_i + 1$. This sum-of-squares type formula suggests the following:

Problem 2: Define a convolution-type algebra structure on $H_*^{\mathrm{BM}}(\mathrm{Fl}^a(V))$, making it into a semisimple algebra whose irreducible representations are naturally parametrized by Motzkin paths.

The variety $\mathrm{Fl}^a(V)$ is a special case of a principal quiver Grassmannian, that is, a variety parametrizing subrepresentations U of a representation $P \oplus I$ of a Dynkin quiver Q, where P is a projective representation, I is an injective representation, and the class $[U]$ of U in the Grothendieck group of Q equals $[P]$. Namely, the variety $\mathrm{Fl}^a(V)$ arises when Q is a linearly oriented type A_n quiver, $P = \mathbb{C}Q$ and $I = (\mathbb{C}Q)^*$. For every principal quiver Grassmannian, the group $\mathrm{Aut}_Q(P \oplus I)$ acts on $\mathrm{Gr}_{[P]}(P \oplus I)$ with finitely many orbits. Assuming that all these facts already work over finite fields, we can consider spaces of functions invariant with respect to this group action:

Problem 3: Define a Hall-algebra type multiplication on

$$A = \bigoplus_{[P],[I],\mathbf{e}} \mathbb{Q}^{\mathrm{Aut}_Q(P\oplus I)}\mathrm{Gr}_{\mathbf{d}}(P \oplus I)$$

making it into a graded associative algebra. Let $\mathcal{U}_q(\mathfrak{b}^+)$ be the quantized enveloping algebra of the Borel subalgebra of $\mathfrak{sl}_{n+1}(\mathbb{C})$; then multiplication should induce an isomorphism of graded vector spaces $\mathcal{U}_q(\mathfrak{b}^+) \otimes \mathcal{U}_q(\mathfrak{b}^+) \to A$. For every orbit $\mathcal{O}$ in some $\mathrm{Gr}_{\mathbf{d}}(P \oplus I)$, the function associating to a point U in $\mathcal{O}$ the Poincaré polynomial of the stalk over U of the intersection cohomology sheaves on the orbit closure of $\mathcal{O}$ gives an element $f_{\mathcal{O}}$ of A; the collection of all $f_{\mathcal{O}}$ should provide A with a canonical basis. How is this basis related to variants of Lusztig's canonical basis in quantized enveloping algebras?

The maximal flat linear degeneration of the flag variety $\mathrm{Fl}(V)$ is defined by

$$\mathrm{FL}^{mf}(V) = \{(U_1, \dots, U_n) \in \prod_{i=1}^{n} \mathrm{Gr}_i(V) \,|\, \mathrm{pr}_{i,i+1}(U_i) \subset U_{i+1} \text{ for } i = 1, \dots, n-1\}.$$

It is a locally complete intersection variety which is equidimensional, and it is a flat degeneration of both $\mathrm{Fl}(V)$ and of $\mathrm{Fl}^a(V)$.

Its irreducible components are naturally parametrized by non-crossing arc diagrams: an arc diagram is a subset A of $\{(i,j) \,|\, 1 \le i < j \le n\}$; it is called non-crossing if there is no pair $(i,j),(k,l) \in A$ such that $i \le k < j \le l$. Non-crossing arc diagrams are counted by the n-th Catalan number. Given a non-crossing arc diagram A, define

$$r(A)_{ij} = i - |\text{arcs starting in } \{1,\dots,i\} \text{ and ending in } \{i+1,\dots,j\}|.$$

Then the closure $C(A)$ of the set of all $(U_1, \dots, U_n) \in \mathrm{Fl}^{mf}(V)$ such that

$$\mathrm{rk}(\mathrm{pr}_{[i,j]} : U_i \to U_j) = r(A)_{i,j} \text{ for all } i < j$$

is an irreducible component of $\mathrm{Fl}^{mf}(V)$, and all components arise in this way.

Problem 4: Are the components $C(A)$ normal? What is their singular locus? Do they admit natural closed embeddings into Schubert varieties? When are two such components isomorphic?

As for $\mathrm{Fl}^a(V)$, also $\mathrm{Fl}^{mf}(V)$ can be realized as a quiver Grassmannian, namely as $\mathrm{Gr}_{[\mathbb{C}Q]}(M)$ for $M = \mathrm{rad}(\mathbb{C}Q) \oplus \mathbb{C}Q/\mathrm{rad}(\mathbb{C}Q) \oplus (\mathbb{C}Q)^*$, thus the automorphism group $\mathrm{Aut}_Q(M)$ acts naturally on $\mathrm{Fl}^{mf}(V)$.

Problem 5: Are there only finitely many orbits for this action? If yes, how can they be parametrized? What is the closure relation?

Both $\mathrm{Fl}^a(V)$ and $\mathrm{Fl}^{mf}(V)$ are members of a flat family of so-called linear degenerations of $\mathrm{Fl}(V)$: Let U be the variety of tuples $(f_1, \ldots, f_{n-1}) \in \mathrm{End}(V)^{n-1}$ of linear operators such that

$$\mathrm{rk}(f_{j-1} \circ \ldots \circ f_i) \geq n + i - j \text{ for all } i < j.$$

Let $\mathcal{F}_U$ be the variety of tuples

$$((U_1, \ldots, U_n), (f_1, \ldots, f_{n-1})) \in \prod_{i=1}^{n} \mathrm{Gr}_i(V) \times U$$

such that $f_i(U_i) \subset U_{i+1}$ for all $i = 1, \ldots, n-1$. Then $\pi_U : \mathcal{F}_U \to U$ is a flat family with generic fibre isomorphic to $\mathrm{Fl}(V)$, and the most degenerate fibre isomorphic to $\mathrm{Fl}^{mf}(V)$.

We can stratify $\mathcal{F}_U$ according to a kind of attractor flow into the irreducible components of its most degenerate fibre by defining, for every non-crossing arc diagram A as above, a stratum $S(A)$ consisting of all (U_*, f_*) such that

$$\mathrm{rk}(f_{j-1} \circ \ldots \circ f_i : U_i \to U_j) = r(A)_{ij} \text{ for all } i < j.$$

Problem 6: Study this stratification of $\mathcal{F}_U$.

The derived pushforward $R(\pi_U)_* \mathbb{Q}_{\mathcal{F}_U}$ decomposes into a direct sum of shifts of simple perverse sheaves on U whose supports are indexed by Motzkin paths of length n. This suggests:

Problem 7: Describe the convolution algebra $H_*^{\mathrm{BM}}(\mathcal{F}_U \times_U \mathcal{F}_U)$ as a quotient of a quiver Hecke/KLR algebra, and describe its standard representations indexed by Motzkin paths.

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Types A and D quiver representation varieties

Jenna Rajchgot

(joint work with Ryan Kinser, Allen Knutson)

Given a quiver Q with vertex set Q_0, arrow set Q_1, and dimension vector $\mathbf{d} : Q_0 \to \mathbb{Z}_{\geq 0}$, there is a *representation space* $\mathrm{rep}_Q(\mathbf{d}) := \prod_{a \in Q_1} \mathrm{Mat}(\mathbf{d}(ta), \mathbf{d}(ha))$, where ta and ha denote the tail and head vertices of the arrow a, and $\mathrm{Mat}(m, n)$ denotes the space of $m \times n$ matrices with entries in a field K. The product of general linear groups $\mathrm{GL}(\mathbf{d}) := \prod_{z\in Q_0} \mathrm{GL}_{\mathbf{d}(z)}(K)$ acts on $\mathrm{rep}_Q(\mathbf{d})$ on the right by conjugation, that is, $(V_a)_{a\in Q_1} \cdot (g_z)_{z\in Q_0} := (g_{ta}^{-1} V_a g_{ha})_{a\in Q_1}$, for $(V_a)_{a\in Q_1} \in \mathrm{rep}_Q(\mathbf{d})$, and $(g_z)_{z\in Q_0} \in \mathrm{GL}(\mathbf{d})$. The closure of a $\mathrm{GL}(\mathbf{d})$ orbit in $\mathrm{rep}_Q(\mathbf{d})$ is called a *quiver locus*.

Motivations for the study of quiver loci come from various disciplines including algebraic geometry, commutative algebra, representation theory, and algebraic combinatorics. For example, through the study of quiver loci, one encounters well-known ideals from commutative algebra, including classical determinantal ideals and defining ideals of varieties of complexes. The primary motivation for the work discussed herein is from algebraic geometry, through study of *degeneracy loci*: given a non-singular algebraic variety X and a map of vector bundles $\phi : V \to W$ on X, there is a degeneracy locus $\Omega_r := \{x \in X \mid \text{rank } \phi_x \leq r\}$, where $\phi_x : V_x \to W_x$ is the induced map on fibers. The locus Ω_r is a closed subvariety of X, defined locally by the vanishing of minors of a matrix. When ϕ is sufficiently general, an expression for its fundamental class in the cohomology ring of X is given by the Giambelli-Thom-Porteous formula. A. Buch and W. Fulton generalized this to sequences of vector bundle maps $V_1 \to V_2 \to \cdots \to V_n$ in [6]. Related formulas were subsequently produced in works such as [6, 2, 8, 5, 13, 18, 15, 3, 7].

The problem of producing formulas for degeneracy loci is closely related to that of finding formulas for *multidegrees* of associated quiver loci (see [13]). Indeed, in [13], A. Knutson, E. Miller, and M. Shimozono produced multiple formulas for the multidegrees and *K-polynomials* of quiver loci of equioriented type A quivers (i.e. all arrows point in the same direction). One important ingredient in this work was the *Zelevinsky map*, which identifies an equioriented type A quiver locus with an open subvariety of a Schubert variety [19, 14], thereby allowing for importation of results from Schubert calculus to the equioriented type A quiver setting.

In joint work with R. Kinser and A. Knutson [10], we generalized three of Knutson, Miller, and Shimozono's formulas from [13] to all type A orientations. Our work also generalized or recovered formulas from [16, 4, 7]. Our main result was a proof of A. Buch and R. Rimányi's conjectured K-theoretic component formula from [7]. In analogy with the equioriented setting, an explicit connection to Schubert varieties (from [11]) was important to our work.

This extended abstract focuses on algebro-geometric results on type A quiver loci important to the proofs of the formulas in [10], as well as analogs of some of these algebro-geometric results for type D loci. The latter part is based on recent joint work with Kinser [12].

1. Type A

In this section we discuss some geometric results in type A, including the bipartite Zelevinsky map from [11], and a degeneration from [10] which was central to the proof of the K-theoretic type A quiver component formula.

1.1. **Bipartite Zelevinsky map.** Let Q be a bipartite (i.e. alternating) type A quiver and $\mathbf{d}$ a dimension vector for Q. There is an associated general linear group

$\mathrm{GL}_d(K)$, parabolic subgroup $P \subseteq \mathrm{GL}_d(K)$, opposite Schubert cell $Y \subseteq P\backslash\mathrm{GL}_d$, and closed immersion $\zeta : \mathrm{rep}_Q(\mathbf{d}) \to Y$ which restricts to an isomorphism from each quiver locus in $\mathrm{rep}_Q(\mathbf{d})$ to a Schubert variety intersected with Y [11]. We refer to ζ as the *bipartite Zelevinsky map.* For example, the type A_5 representation

$$K^{d_5} \xrightarrow{D} K^{d_4} \xleftarrow{C} K^{d_3} \xrightarrow{B} K^{d_2} \xleftarrow{A} K^{d_1}$$

maps, via ζ, to the block matrix on the left below:

$$\begin{bmatrix} 0 & A & 1 & 0 & 0 \\ C & B & 0 & 1 & 0 \\ D & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \in \begin{bmatrix} * & * & 1 & 0 & 0 \\ * & * & 0 & 1 & 0 \\ * & * & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \cong Y.$$

Note that 1s denote identity matrices of appropriate sizes, and the matrix on the right denotes the space of all matrices of the given form (i.e. with arbitrary elements of K in the locations with stars). The ranks of the matrices

$$A, B, C, D, \begin{bmatrix} A \\ B \end{bmatrix}, \begin{bmatrix} C & B \end{bmatrix}, \begin{bmatrix} C \\ D \end{bmatrix}, \begin{bmatrix} 0 & A \\ C & B \end{bmatrix}, \begin{bmatrix} C & B \\ D & 0 \end{bmatrix}, \begin{bmatrix} 0 & A \\ C & B \\ D & 0 \end{bmatrix}$$

characterize the points in the $\mathrm{GL}(\mathbf{d})$ orbit containing (D, C, B, A). This list of ranks is equivalent to a list of ranks of certain North-West justified submatrices of $\zeta(D, C, B, A)$. Using this, one can show that the closure of the orbit through (D, C, B, A) is isomorphic, via ζ, to a Schubert variety intersected with Y.

The bipartite Zelevinsky map is useful beyond the bipartite setting because of the following: given a type A quiver locus $\Omega \subseteq \mathrm{rep}_Q(\mathbf{d})$ (for Q of arbitrary orientation), there is as associated bipartite type A quiver locus $\widetilde{\Omega}$ and product of general linear groups G^* such that $\Omega \times G^*$ is isomorphic to an open subvariety of $\widetilde{\Omega}$ [11]. It follows from this result, the bipartite Zelevinsky map, and [9, Lemma A.4] that each type A quiver locus is isomorphic, up to a smooth factor, to an open subvariety of a Schubert variety. This gives a uniform way of obtaining results such as type A quiver loci are normal and Cohen-Macaulay with rational singularities (also proved earlier via other methods by G. Bobiński and G. Zwara [1]); quiver locus containment is governed by Bruhat order on the symmetric group; there is a Frobenius splitting (in positive characteristic) of each representation space of a type A quiver that compatibly splits all quiver loci; certain generalized determinantal ideals are prime, and scheme-theoretically define type A quiver loci (C. Riedtmann and G. Zwara [17] also obtained this result via other methods).

1.2. **Degenerations.** Certain degenerations of bipartite type A quiver loci are important in the proofs of the component formulas found in [10]. To describe these degenerations (also given in [10]), we start with a general set-up: let G be an algebraic group over K, $H \leq G$ a closed subgroup, X a G-variety, and $Y \subseteq X$ an H-stable closed subvariety. Let $\mu : K^\times \to G$ be a group homomorphism and consider the right action of $K^\times$ on G by $g \cdot t = \mu(t^{-1})g\mu(t)$ and the right action of $K^\times$ on X by $x \cdot t = x \cdot \mu(t)$. With this set-up, we get two families $\widetilde{H}$ and $\widetilde{Y}$ over $\mathbb{A}^1 - \{0\}$ where the fiber $H \cdot t$ in the first family is a subgroup which acts on the fiber $Y \cdot t$ in the second family.

In our case, X is a bipartite type A quiver representation space $\mathrm{rep}_Q(\mathbf{d})$, $Y \subseteq X$ is a quiver locus, $G = \mathrm{GL}(\mathbf{d}) \times \mathrm{GL}(\mathbf{d})$, and $H = \mathrm{GL}(\mathbf{d})^\Delta$, a copy of $\mathrm{GL}(\mathbf{d})$ embedded diagonally in G. The action of G on X is a conjugation action where if $((g_z, h_z))_{z \in Q_0} \in G$ then g_z (respectively h_z) acts on the map over the arrow to the left of z (respectively to the right of z). The induced action of H on X is then the usual action of $\mathrm{GL}(\mathbf{d})$ on $\mathrm{rep}_Q(\mathbf{d})$. Letting $\rho_z(t)$ denote the $\mathbf{d}(z) \times \mathbf{d}(z)$ diagonal matrix with $t, t^2, \ldots, t^{\mathbf{d}(z)}$ down the diagonal, the homomorphism μ is defined by $t \mapsto (\ (\rho_z(t^{-1}), \rho_z(t))\ )_{z \in Q_0}$. The families $\widetilde{H}$ and $\widetilde{Y}$ extend to flat families over $\mathbb{A}^1$ and the special fiber of the first family acts on the special fiber of the second family. From this, one deduces that a bipartite type A quiver locus degenerates to a union of products of matrix Schubert varieties, up to radical. One can further prove that this degeneration is reduced. See [10] for details. Similar degenerations appeared previously in the equioriented setting in [13].

2. Type D

In recent joint work with Kinser [12], we obtain results in type D which are analogous to the type A results from [11]. Indeed, we unify aspects of the equivariant geometry of three classes of varieties: type D quiver representation varieties, double Grassmannians $Gr(a, n) \times Gr(b, n)$, and symmetric varieties $GL(a + b)/(GL(a) \times GL(b))$. In particular, we translate results about singularities of orbit closures, combinatorics of orbit closure containment, and torus equivariant K-theory between these three families. This is accomplished by producing explicit embeddings of homogeneous fiber bundles over type D quiver representation spaces into symmetric varieties. Immediate consequences of these embeddings, together with results on symmetric varieties, include type D quiver loci are normal and Cohen-Macaulay with rational singularities (recovering work from [1] obtained by other methods); the poset of orbit closures in a type D representation space (and also the poset of diagonal B-orbit closures in a double Grassmannian) is isomorphic to a subposet of a poset of *clans*, which are involutions in the symmetric group with signed fixed points.

A next step in the investigation of type D quiver loci is to make use of the explicit embeddings in [12] to help produce formulas for multidegrees and K-polynomials, in analogy with what was done in type A.

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Convex geometric push-pull operators and FFLV polytopes

Valentina Kiritchenko

Let X be a smooth algebraic variety, and $E \to X$ a vector bundle of rank two on X. Define the projective line fibration $Y = \mathbb{P}(E)$ as the variety of all lines in E. The natural projection $\pi : Y \to X$ induces pull-back $\pi^* : A^*(X) \to A^*(Y)$ and push-forward $\pi_* : A^*(Y) \to A^{*-1}(X)$ (aka *transfer* or *Gysin map*) in the (generalized) cohomology rings of X and Y. The *push-pull* operator $\pi^*\pi_* : A^*(Y) \to A^{*-1}(Y)$ is a homomorphism of $A^*(X)$-modules, and can be described explicitly via Quillen–Vishik formula for any algebraic oriented cohomology theory A^* (such as Chow ring, K-theory or algebraic cobordism). Push-pull operators are used extensively in representation theory (Demazure operators) and in Schubert calculus (divided difference operators). We discuss convex geometric counterparts of push-pull operators and their applications in the theory of Newton–Okounkov convex bodies.

We focus on the case where $Y = G/B$ is the complete flag variety for a connected reductive group G. Let $P_i \subset G$ be the minimal parabolic subgroup associated with a simple root α_i, and $X = G/P_i$ the corresponding partial flag variety. Clearly,

$\pi_i : Y \to X$ is a projective line fibration. For instance, if $G = GL_n(\mathbb{C})$ then points in G/B can be identified with complete flags $(V^1 \subset V^2 \subset \ldots \subset V^{n-1} \subset \mathbb{C}^n)$ of subspaces, and the map π_i forgets the subspace V^i. The corresponding push-pull operator $\partial_i : CH^*(Y) \to CH^{*-1}(Y)$ for Chow rings is often called *divided difference operator*, while the push-pull operator $D_i : K^*(Y) \to K^{*-1}(Y)$ for the K-theory is usually called *Demazure operator*.

A classical result of Schubert calculus [BGG, D] is an inductive description of Schubert cycles $[X_w] \in CH^*(Y)$ for all elements $w \in W$ in the Weyl group of G. Namely, if $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ is a reduced decomposition of w into the product of simple reflections, then

$$[X_w] = \partial_{i_\ell} \ldots \partial_{i_2}\partial_{i_1}[X_{\mathrm{id}}]. \tag{1}$$

A classical result in representation theory [A] is an inductive description of the Demazure character $\chi_w(\lambda)$ for every Schubert variety X_w and a dominant weight λ of G:

$$\chi_w(\lambda) = D_1 D_2 \ldots D_\ell(e^\lambda). \tag{2}$$

While formulas (1) and (2) look similar, there is no direct relation between them since in (1) operators are applied in the order opposite to that of (2).

In [Ki16], we defined convex geometric analogs of Demazure operators. They can be used to construct inductively polytopes P_λ such that the sum of exponentials over lattice points in P_λ yields the Demazure character $\chi_w(\lambda)$. Recently, Naoki Fujita showed that the Nakashima–Zelevinsky polyhedral realizations of crystal bases for a special reduced decomposition of the longest element $w_0 \in W$ can be constructed inductively using convex geometric Demazure operators in types A_n, B_n, C_n, D_n, and G_2 [Fu18]. In this setting, the convex geometric Demazure operators are applied in the same order as in (2).

Below we define different convex geometric analogs of push-pull operators that are more natural from the perspective of (1). When computing Newton–Okounkov polytopes of Schubert or Bott–Samelson varieties one often needs an effective tool for comparing the degrees of varieties with the volumes of resulting polytopes. For a reduced decomposition $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$, convex geometric push-pull operators produce inductively polytopes such that their volume polynomials coincide with the degrees of Bott–Samelson varieties corresponding to collections of simple roots (α_{i_1}), $(\alpha_{i_1}, \alpha_{i_2})$,..., $(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_\ell})$.

Let $P \subset \mathbb{R}^n$ be a convex polytope, and $I \subset \mathbb{R}^n$ a segment. Let $Q \subset \mathbb{R}^n$ be a polytope analogous to P (i.e., having the same normal fan). Define the *push-pull polytope* $\Delta(P, Q, I) \subset \mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ as the convex hull of the following set:

$$(1 \times P) \cup (0 \times (Q + I)).$$

When $I = \{0\}$, this construction produces the Newton polytope of a projective line fibration $Y = \mathbb{P}^1(E)$ over the toric variety X corresponding to the polytope P. We assume that $E = \mathcal{O} \oplus \mathcal{L}$ is split so Y is also a toric variety. In this case, there is a simple relation between the polytope Q and the first Chern class of $\mathcal{L}$. In particular, construction of Grossberg–Karshon cubes [GK] (done originally in the spirit of (2)) for Bott–Samelson varieties can also be reproduced in the spirit of (1) using convex-geometric push-pull operators with $I = \{0\}$.

While Grossberg–Karshon cubes can be realized as Newton–Okounkov polytopes of Bott–Samelson varieties for some line bundles [Fu15, HY16, HY17], they do not degenerate to Newton–Okounkov polytopes of flag varieties (instead they turn into twisted cubes, which are not true polytopes). Using segment I of positive length allows us to produce polytopes that do not break in the limit when passing to flag varieties. The following example produces the Newton–Okounkov polytope of a Bott–Samelson variety in type A_2 with the desired degeneration.

Example 1. *cf.* [An13, Section 6.4] *Let $n = 2$, and $P \subset \mathbb{R}^2$ a trapezoid with the vertices $(0,0)$, $(2,0)$, $(0,1)$, $(1,1)$. Take $I = [(0,0),(0,1)]$, and $Q = P$. Then the push-pull polytope $\Delta(P,Q,I) \subset \mathbb{R}^3$ is the Minkowski sum of the FFLV polytope in type A_2 corresponding to the weight ρ and the segment $J = [(0,0,0),(1,0,0)]$ (see* [FFL] *for the definition of FFLV polytopes in type A and their representation-theoretic meaning). By shrinking J we get the degeneration to the FFLV polytope.*

We plan to use convex geometric push-pull operators as a tool for proving the following conjecture on Newton–Okounkov polytopes of Bott–Samelson varieties in type A and generalized FFLV polytopes. We use notation of [Ki18].

Conjecture 1. *Let $\Lambda_i = (\lambda^i_i, \ldots, \lambda^i_n)$. If $j > i$, put $\lambda^j_i = \lambda^j_j$. The Newton–Okounkov polytope $\Delta_{v_0}(X_{\overline{w_0}}, L(\Lambda_1, \ldots, \Lambda_{n-1}))$ is given by inequalities: $u^l_m \geq 0$ and*

$$\sum_{(l,m)\in D} u^l_m \leq \sum_{s=1}^{j} (\lambda^s_k - \lambda^s_{i+j})$$

for all Dyck paths going from λ_k to u^i_j in table (FFLV) where $1 < k \leq j < n$ and $1 \leq i \leq n-j$.

$$\begin{array}{ccccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & & \ldots & & & & \lambda_n \\
& u^1_1 & & u^1_2 & & \ldots & & & & u^1_{n-1} & \\
& & u^2_1 & & \ldots & & & & u^2_{n-2} & & \\
& & & \ddots & & \ddots & & & & & \\
& & & & u^{n-2}_1 & & & u^{n-2}_2 & & & \\
& & & & & & u^{n-1}_1 & & & &
\end{array} \qquad (FFLV)$$

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Hessenberg Varieties and the Stanley–Stembridge Conjecture

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(joint work with Megumi Harada)

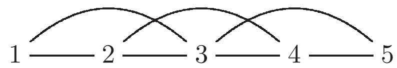
Recent results have forged exciting new connections between algebraic combinatorics and the geometry and topology of certain subvarieties of the flag variety called Hessenberg varieties. In particular, the Shareshian–Wachs conjecture [4], proven in 2015 by Brosnan and Chow [1], established a new connection between Hessenberg varieties and the Stanley–Stembridge conjecture in combinatorics. This talk gives a brief introduction to that story, and discusses a recent theorem of the author and Harada which uses the topology of Hessenberg varieties to give an inductive approach to the Stanley–Stembridge conjecture in a special case.

Let n be a positive integer. A *Hessenberg function* is an increasing sequence $h = (h(1), h(2), \dots, h(n))$ such that $h(i) \geq i$ for all $i \in [n]$. Let h be a Hessenberg function and X be an $n \times n$ matrix in $\mathfrak{gl}(n, \mathbb{C})$. The *Hessenberg variety* $\mathcal{B}(X, h)$ associated to h and X is the subvariety of the flag variety $\mathcal{F}\ell ags(\mathbb{C}^n)$ defined by

$$\mathcal{B}(X, h) := \{V_\bullet \in \mathcal{F}\ell ags(\mathbb{C}^n) \mid X(V_i) \subset V_{h(i)} \text{ for all } i \in [n]\}.$$

Let $S \in \mathfrak{gl}_n(\mathbb{C})$ be a regular semisimple matrix. The Hessenberg variety $\mathcal{B}(S, h)$ is called a *regular semisimple Hessenberg variety*. Tymoczko proved that there is a graded symmetric group representation on the cohomology $H^*(\mathcal{B}(S, h))$ for each Hessenberg function h, called the *dot action representation* [10]. We will see below that the dot action provides the essential link between Hessenberg varieties and the Stanley–Stembridge conjecture.

Another way to interpret the data of a Hessenberg function is as a graph. We define the *incomparability graph* $\Gamma_h = (V, E_h)$ associated to a Hessenberg function h as follows. The vertex set V is $[n] = \{1, 2, \dots, n\}$ and the edge set E_h is defined as: $\{i, j\} \in E_h$ if $1 \leq j < i \leq n$ and $i \leq h(j)$. For example, the incomparability graph for $h = (3, 4, 5, 5, 5)$ is given below.



Let Γ be a (simple) graph with vertex set $[n]$. A *proper coloring* of Γ is a function $\kappa : [n] \to \mathbb{N}$ such that $\kappa(v) \neq \kappa(w)$ whenever $\{v, w\}$ is an edge. We define a monomial x^κ in variables $\mathbf{x} = x_1, x_2, \ldots$ by $x^\kappa := \prod_{v \in [n]} x_{\kappa(v)}$. Stanley's *chromatic symmetric function* is

$$X_\Gamma(\mathbf{x}) := \sum_\kappa x^\kappa$$

where the sum is taken over all proper colorings of Γ. This symmetric function was defined by Stanley in [8] and has subsequently received considerable attention from combinatorists. We can now state the Stanley–Stembridge conjecture.

Conjecture 2. *Let h be a Hessenberg function. Then $X_{\Gamma_h}(\mathbf{x})$ is e-positive, that is, it is a nonnegative integer combination of the elementary symmetric functions.*

The original statement of the Stanley–Stembridge conjecture asserts that the chromatic symmetric function of the incomparability graph of a (3+1)-free poset is e-positive [9]. We will not define the incomparability graph of a poset here. Rather, we note that work of Guay-Paquet [3] and Shareshian–Wachs [4] shows that the original conjecture is implied by Conjecture 2 above.

One might suspect that X_{Γ_h} is the image under the Frobenius character map of some naturally occurring S_n-representation. We have already seen an example of such a representation, namely the dot action representation. The following conjecture of Shareshian and Wachs [4] first established the connection between the dot action and $X_{\Gamma_h}(\mathbf{x})$. It is now a theorem thanks to the work of Brosnan and Chow [1, Theorem 129].

Theorem 1. *For each Hessenberg function h, $X_{\Gamma_h}(\mathbf{x}) = \omega(\mathrm{ch}(H^*(\mathcal{B}(S, h))))$, where* ch *denotes the Frobenius character map and ω is the usual involution on symmetric functions.*

Theorem 1 can be used to recover information about the dot action representation. For example, the Schur-basis expansion of $X_{\Gamma_h}(\mathbf{x})$ is known; there is a combinatorial formula for the coefficients due to Gasharov obtained by enumerating P_h-tableaux [2]. Gasharov's formula determines the decomposition of the dot action representation into irreducible S_n-representations.

It is well known that the set of representations $\{M^\lambda := \mathrm{Ind}_{S_\lambda}^{S_n}(1) \mid \lambda \vdash n\}$ form a $\mathbb{Z}$-basis for the representation ring $\mathcal{R}ep(S_n)$ of S_n. Thus there exist unique integers c_λ such that

$$H^*(\mathcal{B}(\mathsf{S}, h)) = \sum_{\lambda \vdash n} c_\lambda M^\lambda \qquad (1)$$

as elements in $\mathcal{R}ep(S_n)$. Since $\omega(\mathrm{ch}(M^\lambda)) = e_\lambda$, Theorem 1 implies that in order to prove the Stanley–Stembridge conjecture it suffices to show $c_\lambda \geq 0$ for all λ.

In [6], the author and Harada use Theorem 1 to prove a new inductive formula for certain coefficients appearing in (1). Recall that a subset of vertices $I \subseteq [n]$ in the graph Γ_h is called *independent* if they are pairwise nonadjacent. We obtain a 'smaller' graph $\Gamma_{h_I} := \Gamma_h \setminus I$ from each such subset I by deleting the vertices in I and all adjacent edges. The graph Γ_{h_I} uniquely determines a new Hessenberg function $h_I = (h_I(1), \dots, h_I(n - |I|))$.

Theorem 2 (Harada–Precup)**.** *Let h be a Hessenberg function and $\lambda \vdash n$ be a partition of n with exactly ℓ parts, where ℓ is the independence number of Γ_h. Let $\mu = (\mu_1, \dots, \mu_\ell) \vdash (n - \ell)$ such that $\lambda = (\mu_1 + 1, \dots, \mu_\ell + 1)$. Then,*

$$c_\lambda = \sum_{I \in \mathcal{I}_\ell(\Gamma_h)} c_\mu^I$$

where $\mathcal{I}_\ell(\Gamma_h)$ is the set of all independent sets of vertices of size ℓ in Γ_h and for each $I \in \mathcal{I}_\ell(\Gamma_h)$, the coefficients c_μ^I are the coefficients as in (1) associated to the semisimple Hessenberg variety $\mathcal{B}(S, h_I)$ in the flag variety of $GL_{n-\ell}(\mathbb{C})$.

As a corollary, we obtain a special case of the Stanley–Stembridge conjecture by induction [5].

Corollary 1. *Let h be a Hessenberg function such that the independence number of Γ_h is at most 2. Then the integers c_λ appearing in (1) are non-negative.*

The technical details of the induction argument leading to Theorem 2 use– among other things– Brosnan and Chow's proof of the Shareshian–Wachs Conjecture. In particular, Theorem 76 of [1] states that the dot action representation is completely determined by the Betti numbers of *regular Hessenberg varieties*, that is, varieties of the form $\mathcal{B}(X_\lambda, h)$ where X_λ denotes a regular matrix of Jordan type λ. Combining [1, Theorem 76] with results of author on the geometry and combinatorics of Hessenberg varieties [7] provides the necessary tools to prove Theorem 2.

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Positive initial ideals and the FFLV polytope

Lara Bossinger

The following abstract summarizes a short talk presenting an open problem about *Feigin-Fourier-Littelmann-Vinberg polytopes* (short: *FFLV*) [3] and Newton-Okounkov bodies from cluster algebras.

Consider the full flag variety $\mathcal{F}\ell_n$ of type $\mathtt{A}$, i.e. points are in correspondence with full flags of subspaces $\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n$ with $\dim(V_i) = i$. We further assume $n \geq 5$ and use the notation $[n] := \{1, \dots, n\}$. We embed it into the product of Grassmannains $\mathrm{Gr}_1(\mathbb{C}^n) \times \cdots \times \mathrm{Gr}_{n-1}(\mathbb{C}^n)$ and further every Grassmannain in a projective space with respect to its Plücker embedding. Hence, $\mathcal{F}\ell_n \hookrightarrow \mathbb{P}^{\binom{n}{1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{n-1}-1}$. The (multi-)homogeneous coordinate ring of $\mathcal{F}\ell_n$ with respect to this embedding is $A_n := \mathbb{C}[p_J | J \subset [n]]/I_n$, where I_n is a (multi-)homogeneous ideal generated by the Plücker relations.

On A_n we have a family of valuations coming from the cluster structure of $\mathcal{F}\ell_n$. To be precise, by [1] the algebra A_n is a *cluster algebra* meaning that it can be generated recursively by *seeds* (certain maximally algebraically independent subsets of algebra generators for A_n) and an algebraic operation called *mutation* (a procedure to construct new seeds from a given one). For every seed s we have a full-rank homogeneous valuation, denoted by $\nu_s : A_n \setminus \{0\} \to \mathbb{Z}^{d+(n-1)}$, where $d = \dim(\mathcal{F}\ell_n)$.

These valuations have the special property that their associated initial ideals (in the sense of Kaveh-Manon [4] as presented in Christopher Manon's lecture series on Khovanskii basis in this Mini-Workshop) are *positive*, i.e. for every seed s the ideal $\mathrm{in}_{\nu_s}(I_n)$ does not contain any non-zero element of $\mathbb{R}_+[p_J | J \subset [n]]$.

Following Kiritchenko [5] or Fang-Fourier-Littelmann [2] the FFLV-polytope (that as been discussed in detail during this Mini-Workshop) can be realized as a Newton-Okounkov body associated with a full-rank homogeneous valuation $\nu_{\mathrm{FFLV}} : A_n \setminus \{0\} \to \mathbb{Z}^{d+(n-1)}$. We have $\mathrm{FFLV}(\rho) = \Delta(A_n, \nu_{\mathrm{FFLV}})$. Since a few years the following question has been studied but not yet answered:

Question: Does the FFLV-polytope occur as a Newton-Okounkov polytope $\Delta(A_n, \nu_s)$ for some seed s?

While studying this question the following observation might give some new insights.

Proposition: The initial ideal $\mathrm{in}_{\nu_{\mathrm{FFLV}}}(I_n)$ associated with the FFLV-valuation is not positive.

This suggests that the answer to the above question might be no. However, it should be noted that positivity of ideals is not preserved under isomorphisms. Already isomorphisms induced by the action of the symmetric group on Plücker coordinates does not preserve positivity on the level of ideals.

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Global Demazure modules and semi-infinite Veronese curves

Ilya Dumanski

Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be a simple Lie algebra and let $\mathfrak{g}[t]$ be its current algebra. Fix a dominant weight $\lambda = \sum_{i=1}^r m_i\omega_i$. The global Weyl module $\mathbb{W}_\lambda$ of weight λ over $\mathfrak{g}[t]$ is defined as a cyclic module with cyclic vector v with defining relations $\mathfrak{n}^+[t].v = 0$, $ht^0.v = \lambda(h)v$ and $(f_\alpha t^0)^{(\lambda,\alpha)}.v = 0$ for any positive root α.

We also define the local Weyl module W_λ as a cyclic $\mathfrak{g}[t]$-module with a cyclic vector v and defining relations $\mathfrak{n}^+[t].v = 0$, $t\mathfrak{h}[t].v = 0$, $ht^0.v = \lambda(h)v$ and $(f_\alpha t^0)^{(\lambda,\alpha)}.v = 0$ for any positive root α.

For arbitrary $\mathfrak{g}[t]$-module W and $c \in \mathbb{C}$ we define the $\mathfrak{g}[t]$-module $W(c)$, which is isomprphic to W as a vector space, and xt^m act as $x(t+c)^m$ on it.

The following important theorem was proved in [2], [4], [6]:

Theorem 1. (1) *$\mathbb{W}_\lambda$ admits an $\mathfrak{h}[t]$-action, induced by $ht^m.uv = uht^m v$ (for $u \in U(\mathfrak{g}[t])$).*

Define $\mathcal{A}_\lambda = U(\mathfrak{h}[t])/\mathrm{Ann}_{U(\mathfrak{h}[t])}v$.

(2) *As algebra $\mathcal{A}_\lambda$ is isomorphic to $\mathbb{C}[z_1, \dots, z_{m_1+\dots+m_r}]^{S_{m_1}\times\dots\times S_{m_r}}$.*

(3) *For $x = (x_{1,1}, \dots x_{r,m_r}) \in \mathbb{A}^{m_+\dots+m_r}$ with pairwise distinct coordinates we have*

$$\mathbb{W}_\lambda \otimes_{\mathcal{A}_\lambda} \mathbb{C}_x \simeq \bigotimes_{i=1}^{r}\bigotimes_{i=1}^{m_i} W_{\omega_i}(x_{i,j}).$$

(4) $\mathbb{W}_\lambda \otimes_{\mathcal{A}_\lambda} \mathbb{C}_0 \simeq W_\lambda$

(5) $\dim W_\lambda = \prod_{i=1}^r (\dim W_{\omega_i})^{m_i}$.

This theorem implies that the specialisation of $\mathbb{W}_\lambda$ as $\mathcal{A}_\lambda$-module have equal dimension every point. Hense, $\mathbb{W}_\lambda$ is projective $\mathcal{A}_\lambda$-module, and taking into account the Quillen-Suslin theorem we state that it is free $\mathcal{A}_\lambda$-module.

Geometric realisation of $\mathbb{W}_\lambda$ was given in [1], [5]. It was proved that the homogeneous coordinate ring of Drinfeld-Plücker embedding of semi-infinite flag variety to $\prod_{i=1}^r \mathbb{P}(V_{\omega_i}[t])$ is isomorphic to $\bigoplus_\lambda \mathbb{W}_\lambda^*$.

Kato [5] also proved that the natural map $\mathbb{W}_{\lambda+\mu} \to \mathbb{W}_\lambda \otimes \mathbb{W}_\mu$ is injective. Note also that for type A one has $\mathbb{W}_\lambda = V_\lambda \otimes \mathbb{C}[t]$, where V_λ is an irreducible $\mathfrak{g}$-module of highest weight λ.

Taking it into account we conclude that in type A we have

$$\mathbb{W}_\lambda \simeq U(\mathfrak{g}[t]).v \subset \bigotimes_{i=1}^r V_{\omega_i}[t]^{\otimes m_i},$$

where v is a tensor product of the highest vectors of $V_{\omega_i}[t]$.

Fix some $l \in \mathbb{Z}_{>0}$. Our idea is to consider the $\mathfrak{sl}_{r+1}[t]$-module

$$\mathbb{D}_{l,\lambda} \simeq U(\mathfrak{g}[t]).v \subset \bigotimes_{i=1}^r V_{l\omega_i}[t]^{\otimes m_i},$$

where v is a tensor product of the highest vectors of $V_{l\omega_i}[t]$.

We are trying to prove the analogue of Theorem 1 for arbitrary l. So far we can prove the following steps:

Theorem 2. (1) *$\mathbb{D}_{l,\lambda}$ admits an $\mathfrak{h}[t]$-action, induced by $ht^m.uv = uht^m v$ (for $u \in U(\mathfrak{g}[t])$).*

Define $\mathcal{A}_{l,\lambda} = U(\mathfrak{h}[t])/\mathrm{Ann}_{U(\mathfrak{h}[t])}v$.

(2) *As algebra $\mathcal{A}_{l,\lambda}$ is isomorphic to $\mathbb{C}[z_1, \dots, z_{m_1+\dots+m_r}]^{S_{m_1}\times\dots\times S_{m_r}}$.*

(3) *For $x = (x_{i,1}, \dots x_{r,m_r}) \in \mathbb{A}^{m_+\dots+m_r}$ with pairwise distinct coordinates we have*

$$\mathbb{D}_{l,\lambda} \otimes_{\mathcal{A}_{l,\lambda}} \mathbb{C}_x \simeq \bigotimes_{i=1}^r \bigotimes_{i=1}^{m_i} D_{l,\omega_i}(x_{i,j}),$$

Where $D_{l,\lambda}$ is the $\mathfrak{sl}_{r+1}$-stable Demazure module of level l and highest weight $l\lambda$ in the integrable $\widehat{\mathfrak{sl}_{r+1}}$-module $L(l\Lambda_0)$.

It suffices to study $\mathbb{D}_{l,\lambda} \otimes_{\mathcal{A}_{l,\lambda}} \mathbb{C}_0$ to finish the theorem.

Geometric realisation of $\mathbb{D}_{l,\lambda}$ is the following. Consider the embedding of semi-infinite flag variety into $\prod_{i=1}^r \mathbb{P}(V_{l\omega_i}[t])$, which can be thought as a composition of Drinfeld-Plücker embedding and r-tuple of semi-infinite Veronese maps of degree l, $\nu_i : \mathbb{P}(V_{\omega_i}[t]) \to \mathbb{P}((S^l V_{\omega_i})[t]) = \mathbb{P}(V_{l\omega_i}[t])$. The homoheneous coordinate ring of this variety is isomorphic to $\bigoplus_\lambda \mathbb{D}_{l,\lambda}^*$.

We study the case of $\mathfrak{sl}_2$ more precisely. In this case the described embedding is just an arc scheme of finite-dimensional Veronese curve of degree l, $\mathbb{P}(V_\omega[t]) \to \mathbb{P}(V_{l\omega}[t])$.

Knowing representation-theoretic facts about $\mathbb{D}_{l,\lambda}$ for $\mathfrak{sl}_2$ we deduce the reduced structure of this scheme. Let $(x_0 : \dots : x_l)$ be standard homogeneous coordinates on $\mathbb{P}^l$. To obtain the corresponding arc scheme one uses the infinite set of coordinates $x_a^{(k)}$, $k \geq 0$ packed into the formal series $x_a(t) = \sum_{k\geq 0} x_a^{(k)} t^k$.

Theorem 3. *The reduced scheme structure of arc sheme of Veronese curve of degree l is given by the quadratic ideal generated by all the coefficients of certain linear combinations of expressions of the form* $\frac{d^i x_a(t)}{dx^i} x_b(t)$.

Note that relations with derivatives first appeared in [3] in the homogeneous coordinate ring of usual Drinfeld-Plücker embedding of the semi-infinite flag variety in type A.

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Newton-Okounkov bodies of Schubert varieties and tropicalized cluster mutations

Naoki Fujita

(joint work with Hironori Oya)

A Newton-Okounkov body $\Delta(X, \mathcal{L}, v)$ is a convex body constructed from a polarized variety $(X, \mathcal{L})$ with a valuation v on the function field $\mathbb{C}(X)$, which gives a systematic method of constructing toric degenerations of X by [1]. In the case of flag varieties and Schubert varieties, their Newton-Okounkov bodies realize the following representation-theoretic polytopes:

(1) Berenstein-Littelmann-Zelevinsky's string polytopes [8],
(2) Nakashima-Zelevinsky polytopes [4],
(3) Feigin-Fourier-Littelmann-Vinberg polytopes [3, 9].

One motivating problem is to relate these polytopes by using the framework of cluster algebras. The theory of cluster algebras also gives a general framework to obtain toric degenerations of projective varieties, following Gross-Hacking-Keel-Kontsevich [6]. They introduced the notion of positive polytopes, and showed that it gives toric degenerations of compactified $\mathcal{A}$-cluster varieties. Our aim in this talk is to study relations between these two constructions of toric degenerations.

To be more precise, let

$$\mathcal{A} = \bigcup_t \mathcal{A}_t = \bigcup_t \mathrm{Spec}(\mathbb{C}[x_1(t)^{\pm 1}, \ldots, x_m(t)^{\pm 1}])$$

be a cluster variety, where t runs over a set of seeds which are all mutation equivalent, and $x_1(t), \ldots, x_m(t)$ are the corresponding cluster variables. Assuming that

X is birational to the cluster variety $\mathcal{A}$, we have the following identification for each t:

$$\mathbb{C}(X) \simeq \mathbb{C}(x_1(t), \ldots, x_m(t)).$$

Fix a total order $\leq_t$ on $\mathbb{Z}^m$ refining the dominance order introduced by Qin [11]; this $\leq_t$ induces a total order on the set of Laurent monomials in $x_1(t), \ldots, x_m(t)$. We define a valuation

$$v_t : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^m$$

to be the associated lowest term valuation. By definition, this valuation v_t generalizes g-vectors of cluster monomials and of the theta function basis, which implies that Gross-Hacking-Keel-Kontsevich's positive polytopes are identical to Newton-Okounkov bodies associated with v_t. Let us compute the Newton-Okounkov body $\Delta(X, \mathcal{L}, v_t)$ when X is a Schubert variety.

Let G be a simply-connected semisimple algebraic group over $\mathbb{C}$, B a Borel subgroup of G, W the Weyl group, and P_+ the set of dominant integral weights. We denote by $X(w) \subset G/B$ the Schubert variety corresponding to $w \in W$, and by $\mathcal{L}_\lambda$ the globally generated line bundle on $X(w)$ associated with $\lambda \in P_+$. We consider the cluster structure on the unipotent cell which is naturally birational to $X(w)$. Let $\Delta_{\mathbf{i}}(\lambda)$ (resp., $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$) denote the string polytope (resp., the Nakashima-Zelevinsky polytope) associated with a reduced word $\mathbf{i}$ for $w \in W$ and $\lambda \in P_+$.

Theorem 1. *Let $\mathbf{i}$ be a reduced word for $w \in W$. Then, there exists a seed t (resp., $\tilde{t}$) such that $\Delta(X(w), \mathcal{L}_\lambda, v_t)$ (resp., $\Delta(X(w), \mathcal{L}_\lambda, v_{\tilde{t}})$) is unimodularly equivalent to $\Delta_{\mathbf{i}}(\lambda)$ (resp., $\widetilde{\Delta}_{\mathbf{i}}(\lambda)$) for all $\lambda \in P_+$.*

Analogous relations between string polytopes and cluster varieties were previously given by Magee [10] and Bossinger-Fourier [2] in type A, and by Genz-Koshevoy-Schumann [5] in simply-laced case.

Let $\mathcal{A}^\vee$ denote the Fock-Goncharov dual of $\mathcal{A}$. By tropicalizing the cluster mutation μ_k for $\mathcal{A}^\vee$, we obtain the tropicalized cluster mutation μ_k^T. Using Kang-Kashiwara-Kim-Oh's monoidal categorification [7] of the cluster algebra, we deduce the following in simply-laced case.

Theorem 2. *If G is of simply-laced, then the following hold for all $w \in W$ and $\lambda \in P_+$.*

(1) *For each t, the Newton-Okounkov body $\Delta(X(w), \mathcal{L}_\lambda, v_t)$ is a rational convex polytope, which induces a toric degeneration of $X(w)$.*
(2) *If t' is obtained from t by the mutation μ_k, then the following equality holds:*

$$\Delta(X(w), \mathcal{L}_\lambda, v_{t'}) = \mu_k^T(\Delta(X(w), \mathcal{L}_\lambda, v_t)).$$

By combining the theorems above, we see that string polytopes and Nakashima-Zelevinsky polytopes are all related by tropicalized cluster mutations.

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Non-abelian PBW degenerations

Igor Makhlin

Originally, PBW degenerations of highest weight representations of semisimple Lie algebras were defined as associated graded spaces for filtrations given by standard PBW degree. These spaces are acted upon by the associated graded algebra of $\mathcal{U}(\mathfrak{n}_-)$ (which in this case is just a polynomial algebra).

Here we are evidently dealing with a filtration by the ordered group $\mathbb{Z}$. Other filtrations by ordered groups were considered in, for instance, [1] and [2], however, in all previous cases the ordered group in consideration was abelian. The first idea we wish to express is that one may, in fact, consider filtrations by arbitrary totally ordered semigroups, not necessarily abelian, and still obtain associated graded representations acted upon by the associated graded algebra. All one needs to do is to choose a map from the set of negative roots to a totally ordered semigroup.

Now it must be said that in this generality the theory is, apparently, not very rich, since it is unclear how to define one of the main ingredients: the degenerate flag variety. In the cases considered previously the associated graded algebra was always itself a universal enveloping algebra and one could define the variety as an orbit closure for the corresponding Lie group.

In general, the associated graded algebra is not a universal enveloping algebra. Nevertheless, in certain situations the degenerate variety can still be defined and this does produce degenerations which could not be obtained from abelian filtrations. We focus on a particular well-known example in type A: the toric variety associated with the Gelfand-Tsetlin polytope.

We describe an alternative approach to the definition which does not rely on a Lie group action. This is done by noting that if we have a monoidal structure on degenerate representations and maps $\mathrm{gr}L_{\lambda+\mu} \to \mathrm{gr}L_\lambda \otimes \mathrm{gr}L_\lambda$ (where $\mathrm{gr}L_\lambda$ is the degeneration of the irreducible representation with highest weight λ), then the sum $\bigoplus_\lambda (\mathrm{gr}L_\lambda)^*$ turns into a commutative ring. If this ring is finitely generated, then we may define the degenerate flag variety as its "multigraded Proj". (This is equivalent to the original definition in the known cases.)

Now we are left to provide a specific ordered semigroup and a filtration thereon which meet the above requirements and produce the Gelfand-Tsetlin toric degeneration as a result. To do so one may, in fact, take the free monoid in generators corresponding to the negative roots together with a rather natural graded lexicographic order on this monoid.

A detailed description of this construction can be found in the addendum of [3].

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Flag varieties, Valuations and Standard Monomial Theory

Peter Littelmann

(joint work with Rocco Chirivì, Xin Fang)

The theory of standard monomials on flag varieties is closely related to semi-toric degenerations. The starting point of this project is the conviction that the theory of valuations and Newton-Okounkov bodies provides a new and more algebraic-geometric approach towards the construction of a standard monomial theory, compared to the construction in [4]. Another aim is to get an algebraic-geometric version of the LS-algebras introduced in [1]. This is a report on work in progress.

1. The setting

Let $X \subset \mathbb{P}(V)$ be an embedded projective variety with homogeneous coordinate ring $R = \bigoplus_{i\geq 0} R_i = \mathbb{K}[X]$. We assume throughout the following that $\mathbb{K}$ is an algebraically closed field. In addition we assume that we have

- a finite partially ordered set A. The partial order is graded, and A has a unique minimal element p_{min} and a unique maximal element p_{max};
- $\{Y_p\}_{p\in A}$ is a family of projective subvarieties of X, such that $Y_{p_{min}} = pt$, and p covers q if and only if $Y_q \subset Y_p$ of codimension one;
- $\{f_p\}_{p\in A}$ is a family of homogeneous functions (on V) such that
  - $f_p|_{Y_p} \not\equiv 0$, and $Y_p = \{x \in X \mid f_q(x) = 0\ \forall q \not\leq p\}$ (set theoretically)

– let H_p be the hypersurface $\{[v] \in \mathbb{P}(V) \mid f_p(v) = 0\}$, then:

$$H_p \cap Y_p = \bigcup_{q \text{ covered by } p} Y_q \quad \text{(set theoretically)}.$$

For simplicity we assume that the Y_p are projectively normal. In the proofs this condition can often be replaced by other properties like smooth in codimension 1, or the existence of open affine patches.

Example 1. Let $X = G/B \subset \mathbb{P}(V(\lambda))$ be the (generalized) flag variety, where G is a semisimple algebraic group and λ is a regular dominant weight. The partially ordered set is the Weyl group W, endowed with the Bruhat order. The family of subvarieties is given by the Schubert varieties $X(\tau)$, $\tau \in W$, and the functions f_p are given by extremal weight vectors $\{p_\tau\}_{\tau \in W} \subset V(\lambda)^* = R_1$.

Let $\mathcal{G}_A$ be the Hasse graph of A. We put weights on the edges as follows: we write $p \to^b q$ if p covers q and $f_p|_{Y_p}$ vanishes with multiplicity b on Y_q. In case $X = G/B \subseteq \mathbb{P}(V(\lambda))$, the weights are fixed by the Pieri-Chevalley formula.

2. Valuations and a quasi-valuation

Let $\mathfrak{C}$ be a maximal chain in A. The associated sequence of subvarieties (of codimension one) $X = Y_{p_r} \supset \cdots \supset Y_{p_0} = pt$, $p_i \in \mathfrak{C}$, induces a $\mathbb{Z}^{r+1}$-valued valuation. Using ideas of Rees [6], we add to this valuation asymptotic considerations and attach to $\mathfrak{C}$ a new full rank valuation

$$\mathcal{V}_{\mathfrak{C}} : R \to \mathbb{Q}^{r+1}.$$

Note that $r + 1 = \dim \hat{X}$ is the dimension of the affine cone $\hat{X}$ over X. Even in the case $X = G/B$, it is in general not known whether the associated semigroup $S_{\mathfrak{C}} = \mathcal{V}_{\mathfrak{C}}(R) \subset \mathbb{Q}^{r+1}$ is finitely generated. The modification introduced in the construction allows us to replace the valuations by a non-negative quasi-valuation: we fix on $\mathbb{Q}^{r+1}$ the lexicographic order, and for a given function h we choose a maximal chain such that $\mathcal{V}_{\mathfrak{C}}(h) = \min\{\mathcal{V}_{\mathfrak{C}'}(h) \mid \mathfrak{C}' \text{ maximal chain}\}$. We replace then $\mathbb{Q}^{r+1}$ by $\mathbb{Q}^{|A|}$ with basis $\{e_p\}_{p \in A}$ and set:

$$\mathcal{V} : R \to \mathbb{Q}^{|A|}_{\geq 0}, \quad h \mapsto \sum_{p_i \in \mathfrak{C}} (\mathcal{V}_{\mathfrak{C}}(h))_i e_{p_i}$$

3. Results

Theorem 1. *i) The quasi-valuation $\mathcal{V}$ induces a filtration on R such that the associated graded algebra $gr_{\mathcal{V}} R$ is finitely generated. The graded components are at most one-dimensional.*

ii) The irreducible components of the associated projective variety X_0 are in bijection with maximal chains in the partially ordered set A. There exists a flat stepwise degeneration of X into the semitoric variety X_0.

iii) The irreducible component of X_0 associated to a maximal chain $\mathfrak{C}$ is the toric variety associated to the finitely generated semigroup:

$$\Gamma_{\mathfrak{C}} := \{\mathcal{V}(h) \mid h \in R \textit{ homogeneous}, \mathcal{V}_{\mathfrak{C}}(h) \textit{ is minimal}\} \subset \mathbb{Q}^{r+1}_{\geq 0}$$

4. Flag variety

In case $X = G/B \subseteq \mathbb{P}(V(\lambda))$, the semigroup can be made explicit. Let $b_1, \ldots, b_r$ be the weights on the edges in the maximal chain $\mathfrak{C}$. Then

Theorem 2.

$$\Gamma_{\mathfrak{C}} = \left\{ v = \begin{pmatrix} a_r \\ \vdots \\ a_0 \end{pmatrix} \in \mathbb{Q}_{\geq 0}^{r+1} \;\middle|\; \begin{array}{r} b_r a_r \in \mathbb{Z} \\ b_{r-1}(a_r + a_{r-1}) \in \mathbb{Z} \\ \ldots \\ b_1(a_r + a_{r-1} + \ldots + a_1) \in \mathbb{Z} \\ a_0 + a_1 + \ldots + a_r \in \mathbb{N} \end{array} \right\}$$

The construction via valuations yields in a natural way an algebraic-geometric interpretation of the path model [5]. Let $\mathfrak{C} = (\tau_r, \ldots, \tau_0)$:

Proposition 1. *i) The map which associates to* $(a_r, \ldots, a_0) \in \Gamma_{\mathfrak{C}}$ *the pair* $(\tau_r, \ldots, \tau_0; a_r, a_{r-1} + a_r, \ldots, a_0 + \ldots + a_r)$ *induces a bijection between the elements of* $\Gamma_{\mathfrak{C}}$ *and the LS-paths of shape* $n\lambda$, $n \in \mathbb{N}$, *having support in* $\mathfrak{C}$.

ii) The Newton-Okounkov body $\Delta_{\mathcal{V}}(R) \subset \mathbb{Q}^{|A|}$ *associated to* $\mathcal{V}$ *is the (generalized) polytope with integral structure described in* [2].

Conjecture 3. We conjecture that the description of the semigroup $\Gamma_{\mathfrak{C}}$ (which only uses data which can be read off the weighted Hasse graph) is valid in general if the functions $\{f_p\}_{p \in A}$ satisfy the following additional condition: let p cover q and let b be the weight on the edge joining the two. Then there exists a rational function $\eta \in \mathbb{K}(\hat{Y}_p)$ such that $\eta^b = \frac{f_p}{f_q}$ in $\mathbb{K}(\hat{Y}_p)$.

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