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Mini-Workshop: Self-adjoint Extensions in New Settings

Organized by
Ugo Boscain, Paris
Aleksey Kostenko, Wien
Konstantin Pankrashkin, Orsay

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Abstract. The main focus of the workshop is on the analysis of boundary value problems for differential and difference operators in some non-classical geometric settings, such as fractal graphs, sub-Riemannian manifolds or non-elliptic transmission problems. Taking into account their importance in modern mathematical analysis, we aim at developing suitable tools in the operator theory to deal with the new problem settings.

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Introduction by the Organizers

In the mathematical physics, a wave or diffusion process inside an object occupying e.g. a domain in a Euclidian space is described by a suitable differential expression, while the interaction with the surrounding correspond to a boundary condition. In many cases, the problem under study is expected to be governed by a self-adjoint operator (e.g. if the problem is linear and shows some energy conservation), in this case the choice of a boundary condition corresponds to the choice of a self-adjoint extension of some minimal operator: as a prominent example one can mention Laplace operators with Dirichlet/Neumann boundary conditions, which coincide on functions vanishing near the boundary. In some situations no boundary condition needs to be specified (or, more precisely, the functions in the domain of the operator obey automatically some hidden boundary condition), in this case the operator is essentially self-adjoint. For example, for the Laplace operator on a geodesically complete manifold, no specific boundary condition at infinity is required, and many results of this type are available for Schödinger operators as well.

Recently, many new models were brought to the attention. The analysis of differential operators on sub-Remannian manifolds provides a great sources of challenges: so far, only a limited number of situations are understood and only a limited class of operator machineries was tested in this setting. In this context, the question of essential self-adjointness may have a special meaning as it is related to penetrability of singular sets and is strongly related to the underlying geometry, which can be understood via techniques of control theory. One may also mention some classes of transmission problems with a degeneration along a submanifold; such operators show a non-trivial link between the geometry of the interface and the regularity of the functions in the domain, which results in a number of unusual spectral properties.

The study of quantum graphs, i.e. of differential operators on systems of coupled segments, represents an extremely active domain of mathematical physics and spectral theory during the last decades. The well-established theory mostly applies if the underlying geometry has a kind of limited complexity (e.g. if suitable bounds on the length of the intervals and the degree of the nodes are available). The most general case raises a number of new questions concerning an accurate description of boundary value problems in this setting. In this connection let us mention a close relationship with the corresponding problems for (combinatorial) graph Laplacians, where the crucial role is played by various notions of graph boundaries (graph ends, Royden boundary, Martin and Poisson boundaries etc.). An additional motivating aspect in this direction is provided by recent papers appearing in the numerical analysis: if one rephrases it in the adapted language, one deals with self-adjoint realisations of Laplacians on a class of fractal networks arising from concrete modeling problems, and the construction of functional (Sobolev-type) spaces on the boundary appears to be closely related to some truncation issues for the numerical treatment. It is curious to mention that the study of random walks on trees naturally leads to a similar sort of problems. Moreover, these problems are tightly connected with the study of (ordinary) differential operators whose coefficients are self-similar measures.

The idea of the meeting arose during recent contacts amongst the organizers and the proposed participants on new classes of boundary value problems for differential operators in some non-classical geometric settings, such as fractal graphs and sub-Remannian manifolds or non-elliptic transmission problems in quantum mechanics, and which deal with a number of common topics from the operator theory. The objectives of the meeting are, on one hand, to provide the participants with a consolidated picture of the operator tools allowing to deal with large classes of boundary value problems and, on the other hand, to motivate further progress on the operator-theoretical side by providing an introduction into new potential applications.

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Mini-Workshop: Self-adjoint Extensions in New Settings

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Abstracts

An introduction to Sub-Laplacians and their self-adjointness properties

Luca Rizzi
(joint work with V. Franceschi, D. Prandi, M. Seri)

1. Introduction to sub-Laplacians

On $\mathbb{R}^n$ consider the second order differential operator

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x), \tag{1}$$

where a, b, c are smooth and real functions. We say that L is hypoelliptic if, locally, for all distributions u such that $Lu \in C^\infty$ it follows that $u \in C^\infty$. A first necessary condition for hypoellipticity is the following.

Theorem 1 (Hörmander)**.** *If L is hypoelliptic its principal symbol is semi-definite at any point, i.e. for all $x \in \mathbb{R}^n$ it holds $\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq 0$ or ≤ 0 for all $\xi \in \mathbb{R}^n$.*

This results leads us to consider a large class of operators that satisfy the above necessary condition, that is operators that can be written as a "sum of squares":

$$L = \sum_{i=1}^{N} X_i^2 + X_0 + c, \qquad \text{(Hörmander-type operator)}. \tag{2}$$

Here, N is an integer possibly greater than n, and $X_0, X_1, \ldots, X_N$ are smooth vector fields on $\mathbb{R}^n$, that is $X_\mu = \sum_{j=1}^n \alpha_{\mu j} \partial_j$. A sufficient condition for the hypoellipticity of Hörmander-type operators is given by the next result.

Theorem 2 (Hörmander)**.** *Assume that the smallest Lie algebra of vector fields generated by $X_0, X_1, \ldots, X_N$ has maximal dimension at each point $x \in \mathbb{R}^n$, that is*

$$\mathrm{Lie}(X_0, X_1, \ldots, X_N)(x) = \mathbb{R}^n, \qquad \forall x \in \mathbb{R}^n.$$

Then the operator L given in (2) *is hypoelliptic.*

Theorem 2 is a sufficient condition for hypoellipticity, but not necessary, and it is false for operators with complex coefficients, see for example [3]. Hypoellipticity is a local property, hence the above results are immediately extended to second order differential operators on smooth manifolds.

2. Geometry of Hörmander-type operators

Let L be a Hörmander-type operator as in (2) on a smooth manifold $\mathbb{M}$, and consider the family of smooth vector fields $\{X_1, \dots, X_N\}$. Notice that the drift X_0 is not included in the family, and that different families of vector fields can define the same operator. We define the set of *admissible directions* at each point

$$\mathcal{D}(x) = \operatorname{span}\{X_1(x), \dots, X_N(x)\}, \qquad \forall x \in \mathbb{M}, \tag{3}$$

and a *scalar product* g_x defined along admissible directions, induced by the norm

$$\|v\|_g^2 = \min\left\{\sum_{i=1}^N u_i^2 \mid v = \sum_{i=1}^N u_i X_i(x)\right\}, \qquad \forall v \in \mathcal{D}(x), \quad \forall x \in \mathbb{M}. \tag{4}$$

Admissible trajectories are then absolutely continuous curves $\gamma : [0,1] \to \mathbb{M}$ for which there exist $u_1, \dots, u_N \in L^\infty([0,1], \mathbb{R})$ such that

$$\dot\gamma(t) = \sum_{i=1}^N u_i(t) X_i(\gamma(t)), \qquad \text{for a.e. } t \in [0,1]. \tag{5}$$

Finally, we can define a *distance* via the usual variational formulation

$$d(x,y) = \inf\left\{\int_0^1 \|\dot\gamma(t)\|_g \, dt \mid \gamma \text{ is admissible between } x \text{ and } y\right\}. \tag{6}$$

A priori there might be no admissible curve between two given points (e.g. when $\mathcal{D}$ is an integrable distribution). However, if $X_1, \dots, X_N$ satisfy the Hörmander condition, it turns out that d is a true distance.

Theorem 3 (Chow-Rashevskii)**.** *If the family $X_1, \dots, X_N$ satisfies the Hörmander condition, then $d : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$ is finite and the topology induced by d coincides with the topology of $\mathbb{M}$. In particular d is continuous.*

In other words, a Hörmander-type hypoelliptic operator $L = \sum_{i=1}^N X_i^2 +$ lower order terms such that the elements $X_1, \dots, X_N$ satisfy the Hörmander condition induces a metric structure d on the ambient space $\mathbb{M}$, called *sub-Riemannian* (or *Carnot-Carathéodory*) metric. X_0 and c in (2) play no role in the construction of all above objects, which depend only on the principal symbol of L.

Example 1 (Riemannian)**.** Any global set of vector fields $X_1, \dots, X_N$ on $\mathbb{M}$ with maximal rank at each point (equal to $n = \dim \mathbb{M}$) satisfies the condition of Theorem 2 and hence $L = \sum_{i=1}^N X_i^2$ is trivially hypoelliptic (and also elliptic). In this case d is a Riemannian metric structure. The metric g can be obtained intrinsically by inversion of the principal symbol of L.

Example 2 (Heisenberg)**.** Consider, on $\mathbb{M} = \mathbb{R}^3$, the operator $L = X_1^2 + X_2^2$, with

$$X_1 = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}, \qquad X_2 = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}.$$

The rank of the family X_1, X_2 is constant and equal to 2. The corresponding metric structure on $\mathbb{R}^3$ is called the *Heisenberg group*.

For more details on sub-Riemannian structures we refer to [1].

3. Self-adjointness properties of L

Consider a Hörmander-type operator L as in (2), and the corresponding sub-Riemannian structure $(\mathbb{M}, d)$ as defined in Section 2. Fix a smooth measure μ on $\mathbb{M}$. The requirement that L with domain $C_c^\infty(\mathbb{M})$ is symmetric on $L^2(\mathbb{M}, \mu)$, fixes the first-order term X_0. In particular L must have the following form:

$$L = \sum_{i=1}^{N} X_i^2 + \operatorname{div}_\mu(X_i)X_i + c, \qquad \operatorname{Dom}(L) = C_c^\infty(\mathbb{M}), \tag{7}$$

where $\operatorname{div}_\mu(\cdot)$ is the divergence operator defined with respect to the measure μ. The following result is well-known, cf. for example [8].

Theorem 4. *If* $(\mathbb{M}, d)$ *is complete then* L *is essentially self-adjoint on* $L^2(\mathbb{M}, \mu)$.

The completeness of $(\mathbb{M}, d)$ is by no means a necessary condition.

Example 3 (Grushin)**.** Consider on $\mathbb{M} = \mathbb{R}^2$ the operator $L = X_1^2 + X_2^2 + X_0$, with X_0 to be fixed later and

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial z}.$$

The rank of the family X_1, X_2 is equal to 2 everywhere, with the exception of the singular set $S = \{x = 0\}$, where the rank is equal to 1. The principal symbol of L defines (cf. Section 2) a metric structure, called the *Grushin metric*. This metric is Riemannian on the set $R = \mathbb{R}^2 \setminus S$, and is given by $g = dx^2 + \frac{1}{x^2}dz^2$. The corresponding Riemannian measure is $\mu_g = \frac{1}{|x|}dxdz$. We restrict L to the regular region R, with domain $C_c^\infty(R)$. The symmetry requirement fixes X_0, so that

$$L = \frac{\partial^2}{\partial x^2} + x^2\frac{\partial^2}{\partial z^2} - \frac{1}{x}\frac{\partial}{\partial x}, \qquad \operatorname{Dom}(L) = C_c^\infty(R), \tag{8}$$

which is just the Laplace-Beltrami operator Δ_g of g. **Notice that** (R, d) **is not complete! However, the operator** (8) **is essentially self-adjoint on** $L^2(R, \mu)$. This fact can be proved by performing a Fourier decomposition w.r.t. z, and then showing that each component is essentially self-adjoint, cf. [2].

3.1. Almost-Riemannian structures.

The above is an example of a general class of sub-Riemannian structures, called *almost-Riemannian*, that we now introduce. Assume that $\{X_1, \dots, X_N\}$ are global smooth vector fields on an n-dimensional manifold $\mathbb{M}$, satisfying the Hörmander condition. Let then

- R be the set where $\dim \operatorname{span}\{X_1(x), \dots, X_N(x)\} = n$ (regular set);
- S be the set where $\dim \operatorname{span}\{X_1(x), \dots, X_N(x)\} < n$ (singular set).

In particular, $\mathbb{M} = R \sqcup S$. According to the construction in Section 2, these vector fields define a metric d on $\mathbb{M}$, which is Riemannian on R, i.e. locally given by a smooth Riemannian metric g. We also assume that $(\mathbb{M}, d)$ is complete. In this way, any connected component of R adjacent to S is a non-complete Riemannian manifold whose metric boundary is contained in the singular set S. We then consider the Laplace-Beltrami operator Δ_g on $L^2(R, \mu_g)$, with domain $C_c^\infty(R)$. Motivated by the example of the Grushin metric, the main question is the following:

Question. Is the Laplace-Beltrami operator Δ_g on the regular region R, with domain $C_c^\infty(R)$, essentially self-adjoint on $L^2(R,\mu_g)$?

An affirmative answer to this question means that classical particles, following the geodesic dynamics, can cross S and move between different connected components of R while quantum particles, obeying the Schrödinger equation, remain confined to different components of R, by Stone's theorem. The standing conjecture, due to Boscain and Laurent, is that the above question always has affirmative answer. The state-of-the is as follows:

2009. Boscain and Laurent, in [2], stated and proved the conjecture in the case $n = 2$, and assuming that $X_1, \dots, X_N$ verify the two-step Hörmander condition (i.e. Lie brackets of length not greater than 2 are enough to verify the Hörmander condition). This fact in particular implies that S is a smooth hypersurface. The proof relies on the machinery of normal forms, fully developed only for almost-Riemannian surfaces.
2016. Prandi, Rizzi and Seri, in [7], proved the conjecture for general n, assuming that S is a smooth hypersurface with no tangency points (i.e. S is transverse to the family $X_1, \dots, X_N$), and assuming that close to S, and letting δ the distance from S, it holds $\mu_g \sim \delta^{-a} \times$ smooth measure, for some a necessarily ≥ 1. The proof is based on the method of effective potential developed in [7] and Agmon-type estimates inspired by [6].
2017. Franceschi, Prandi and Rizzi, in [5], extended the above result under analogous assumptions to the case of rank-varying sub-Riemannian structures.

The general case is far from being understood. There are in particular two problems, of different nature, which we are not able to treat using the available methods. We illustrate them by means of two examples.

3.2. **Problem 1.** Consider the almost-Riemannian structure on $\mathbb{R}^2$ defined by

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x(x^{2\ell} + z^2)\frac{\partial}{\partial z}, \qquad \ell \in \mathbb{N}. \tag{9}$$

The singular region is $S = \{x = 0\}$ and R is its complement. The distance from S is $\delta = |x|$, which is smooth on R. The Riemannian measure there is given by

$$\mu_g = \frac{1}{|x|(x^{2\ell} + z^2)} dxdz.$$

It is not true that $\mu_g \sim \delta^{-a} \times$ smooth measure, for some constant a, so that we cannot apply the techniques from [7]. However, the case $\ell = 1$ is special and we can still prove the essential self-adjointness of the Laplace-Beltrami on the regular region, cf. [7, Example 7.7]. We expect this fact to be true for all $\ell \geq 1$, but we are not able to prove it. A natural conjecture is that the Laplace-Beltrami operator on the regular region of a real-analytic almost-Riemannian structure on a 2-dimensional manifold with no tangency points is essentially self-adjoint.

3.3. **Problem 2.** When S has tangency points the distance from S is no longer smooth, and the techniques from [7, 5] do not work. This is perhaps the hardest problem and the proof of the essential self-adjointness of Δ_g in this case would require the development of new techniques. The simplest example of such a situation is the almost-Riemannian structure on $\mathbb{R}^2$ defined by

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = (z - x^2)\frac{\partial}{\partial z}. \tag{10}$$

The singular region is the parabola $S = \{z - x^2 = 0\}$, and the origin is a tangency point. Letting $\phi(x, z) := z - x^2$, the associated Laplace-Beltrami operator reads:

$$\Delta_g = \partial_x^2 + \phi(x,z)^2\partial_z^2 + \phi(x,z)\partial_z + \frac{\partial_x\phi(x,z)}{\phi(x,z)}\partial_x, \qquad \mathrm{Dom}(\Delta_g) = C_c^\infty(R), \tag{11}$$

We stress that the natural Riemannian measure on the regular region is $\mu_g = \frac{1}{|z-x^2|}dxdz$ and therefore the above operator is symmetric on $L^2(R, \mu_g)$. More details on this example can be found in [4].

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Transparent boundary conditions for wave propagation in fractal trees

Maryna Kachanovska

(joint work with P. Joly and A. Semin)

This work is devoted to a study of the weighted wave equation on a 1D fractal tree (which models the wave propagation in human lungs). Because the problem is defined on a structurally infinite domain, we aim at truncating the computations to the sub-tree consisting of a finite number of generations, by imposing transparent boundary conditions at the truncated boundary of the tree. Construction and analysis of such transparent boundary conditions leads to exciting theoretical questions and is the main goal of the present work.

Geometric setting. We consider a p-adic infinite tree $\mathcal{T}$, defined as follows. Each edge Σ has p children edges. The set of all edges of the tree will be denoted by $E(\mathcal{T})$. We will say that the root edge belongs to the first generation; if an edge belongs to a n-th generation, its children belong to the $(n+1)$-st generation. We will denote by $\mathcal{T}^m$ a sub-tree of $\mathcal{T}$ consisting of all the edges of m first generations.

With such a tree we associate two $p-$uplets, $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^p$ and $\boldsymbol{\mu} = (\mu_i)_{i=1}^p$. If the length of an edge equals ℓ, the lengths of its p children are correspondingly $\alpha_1\ell, \alpha_2\ell, \dots, \alpha_p\ell$. We will assume that $\alpha_i < 1$ for all i. To each edge we assign a positive weight $\mu > 0$; again, if the weight of an edge equals ν, the weights of its p children are correspondingly $\mu_1\nu, \mu_2\nu, \dots, \mu_p\nu$. This defines a piecewise-constant weight function $\mu(s)$ on the tree $\mathcal{T}$. Let μ_Σ be the value of $\mu(s)$ on the edge Σ.

Problem setting. In order to define the problem in question, let us introduce a proper Sobolev space framework. First of all, provided $f : \mathcal{T} \to \mathbb{R}$, let

$$\int_{\mathcal{T}} f\mu := \sum_{\Sigma \in E(\mathcal{T})} \int_{\Sigma} f(s)\mu_\Sigma ds.$$

Let $C(\mathcal{T})$ be a space of continuous functions on $\mathcal{T}$, and

$$C_0(\mathcal{T}) := \{v \in C(\mathcal{T}) : \quad v = 0 \text{ on } \mathcal{T} \setminus \mathcal{T}^m, \quad \text{for some } m \in \mathbb{N}\}.$$

The following spaces generalize weighted Sobolev spaces on an interval:

$$\begin{aligned}
&\mathrm{L}^2_\mu(\mathcal{T}) := \{v : \|v\|_{\mathrm{L}^2_\mu(\mathcal{T})} < \infty\}, \quad \|v\|^2_{\mathrm{L}^2_\mu(\mathcal{T})} = \|v\|^2 = \int_{\mathcal{T}} \mu|v|^2,\\
&\mathrm{H}^1_\mu(\mathcal{T}) := \{v \in C(\mathcal{T}) \cap \mathrm{L}^2_\mu(\mathcal{T}) : \|\partial_s v\| < \infty\}, \quad \|v\|^2_{\mathrm{H}^1_\mu(\mathcal{T})} = \|v\|^2 + \|\partial_s v\|^2,\\
&\mathrm{H}^1_{\mu,0}(\mathcal{T}) := \overline{C_0(\mathcal{T}) \cap \mathrm{H}^1_\mu(\mathcal{T})}^{\|\cdot\|_{\mathrm{H}^1_\mu(\mathcal{T})}}.
\end{aligned}$$

Denoting by M^* the root vertex of the tree $\mathcal{T}$, let us introduce

$$V_{\mathfrak{n}}(\mathcal{T}) = \{v \in \mathrm{H}^1_\mu(\mathcal{T}) : \ v(M^*) = 0\}, \, V_{\mathfrak{d}}(\mathcal{T}) = \{v \in \mathrm{H}^1_{\mu,0}(\mathcal{T}) : \ v(M^*) = 0\}.$$

With the above, we consider the following two problems.
Neumann problem. Provided $f \in L^1([0,T]; \mathrm{L}^2_\mu(\mathcal{T}^m))$ ($f \equiv 0$ on $\mathcal{T} \setminus \mathcal{T}^m$), find

$$u_{\mathfrak{n}} \in C([0,T]; V_{\mathfrak{n}}) \cap C^1([0,T]; \mathrm{L}^2_\mu(\mathcal{T})), \text{ s.t. } u_{\mathfrak{n}}(.,0) = \partial_t u_{\mathfrak{n}}(.,0) = 0, \text{ and}$$

$$\text{(N)} \qquad \int_{\mathcal{T}} \mu \frac{d^2}{dt^2} u_{\mathfrak{n}}\, v + \int_{\mathcal{T}} \mu \partial_s u_{\mathfrak{n}} \partial_s v = \int_{\mathcal{T}} \mu f v, \qquad \forall v \in V_{\mathfrak{n}}.$$

Dirichlet problem. Provided $f \in L^1([0,T]; \mathrm{L}^2_\mu(\mathcal{T}^m))$ ($f \equiv 0$ on $\mathcal{T} \setminus \mathcal{T}^m$), find

$$u_\mathfrak{d} \in C([0,T]; V_\mathfrak{d}) \cap C^1([0,T]; \mathrm{L}^2_\mu(\mathcal{T})), \text{ s.t. } u_\mathfrak{d}(.,0) = \partial_t u_\mathfrak{d}(.,0) = 0, \text{ and}$$

$$\text{(D)} \qquad \int_{\mathcal{T}} \mu \frac{d^2}{dt^2} u_\mathfrak{d}\, v + \int_{\mathcal{T}} \mu \partial_s u_\mathfrak{d} \partial_s v = \int_{\mathcal{T}} \mu f v, \qquad \forall v \in V_\mathfrak{d}.$$

Distinction between (D) and (N). First of all, we would like to know whether the solutions to (N) and (D) differ (in particular whether $\mathrm{H}^1_\mu = \mathrm{H}^1_{\mu,0}$). Let

$$\langle \boldsymbol{\mu\alpha} \rangle := \sum_{i=1}^p \mu_i \alpha_i, \qquad \left\langle \frac{\boldsymbol{\mu}}{\boldsymbol{\alpha}} \right\rangle := \sum_{i=1}^p \frac{\mu_i}{\alpha_i}.$$

Theorem 1. *If $\langle \boldsymbol{\mu\alpha} \rangle \geq 1$ or $\left\langle \frac{\boldsymbol{\mu}}{\boldsymbol{\alpha}} \right\rangle \leq 1$, the spaces $\mathrm{H}^1_{\mu,0}(\mathcal{T})$ and $\mathrm{H}^1_\mu(\mathcal{T})$ coincide, and thus $u_\mathfrak{n} = u_\mathfrak{d}$. Otherwise, $\mathrm{H}^1_{\mu,0}(\mathcal{T}) \subsetneq \mathrm{H}^1_\mu(\mathcal{T})$, and $u_\mathfrak{n} \neq u_\mathfrak{d}$.*

The proof relies on characterizing $\mathrm{H}^1_{\mu,0}(\mathcal{T})$ as the kernel of a certain trace operator on $\mathcal{T}$, cf. [1].

Truncating the computational domain. Truncation of the computational domain to $\mathcal{T}^m$ is done by imposing transparent boundary conditions at each end point M (omitting the root M^*) of the truncated tree $\mathcal{T}^m$ in the following form:

$$\text{(1)} \qquad \partial_s u(t,M) = -\sum_{i=1}^p \mu_i \Lambda_i(\partial_t) u(t,M).$$

The operators $\Lambda_i(\partial_t)$ are defined via $\Lambda_i(\partial_t) = \ell_i^{-1} \Lambda(\ell_i \partial_t)$, where ℓ_i, $i = 1, \ldots, p$, are the lengths of the p edges whose parent vertex is M. The operator $\Lambda(\partial_t)$ is a DtN operator associated to the root edge of the tree. It is a convolution operator with a causal kernel, whose symbol (i.e. Fourier-Laplace transform of the kernel) will be denoted by $\boldsymbol{\Lambda}(\omega)$.

Characterization of the operator $\Lambda(\partial_t)$. We will need the following result [1].

Theorem 2. *The embedding of $\mathrm{H}^1_\mu(\mathcal{T})$ into $\mathrm{L}^2_\mu(\mathcal{T})$ is compact.*

This immediately leads to the following result. First of all, the bilinear form

$$a_\mathfrak{a}: \quad V_\mathfrak{a} \times V_\mathfrak{a} \to \mathbb{R}, \quad a_\mathfrak{a}(u,v) = \int_{\mathcal{T}} \mu \partial_s u\, \partial_s v, \quad \mathfrak{a} \in \{\mathfrak{n}, \mathfrak{d}\},$$

defines (in a classical way) an operator $A_\mathfrak{a} : D(A_\mathfrak{a}) \to L^2_\mu(\mathcal{T})$, $(A_\mathfrak{a} u, v) = a_\mathfrak{a}(u,v)$. By Theorem 2, the spectrum of this operator is a pure point spectrum, with eigenvalues of finite multiplicity and the only accumulation point ∞. Let

$$\text{(2)} \quad \mathcal{A}_\mathfrak{a} \phi_{\mathfrak{a},n} = \omega^2_{\mathfrak{a},n} \phi_{\mathfrak{a},n}, \qquad \|\phi_{\mathfrak{a},n}\|_{\mathrm{L}^2_\mu(\mathcal{T})} = 1, \qquad 0 < \omega^2_{\mathfrak{a},0} \leq \omega^2_{\mathfrak{a},1} \leq \ldots \to \infty.$$

The fact that the eigenvalues do not vanish was shown in [1, Remark 1.20]. With the above, we obtain the following characterization of the symbol $\boldsymbol{\Lambda}(\omega) = \boldsymbol{\Lambda}_\mathfrak{a}(\omega)$ for the Dirichlet (Neumann) problems ($\mathfrak{a} \in \{\mathfrak{n}, \mathfrak{d}\}$).

Theorem 3 ([1]). *The symbol of the reference DtN $\mathbf{\Lambda}_{\mathfrak{a}}$, $\mathfrak{a} \in \{\mathfrak{n}, \mathfrak{d}\}$ satisfies*

$$\mathbf{\Lambda}_{\mathfrak{a}}(\omega) = \mathbf{\Lambda}_{\mathfrak{a}}(0) - \sum_{n=0}^{+\infty} \frac{a_{\mathfrak{a},n}\omega^2}{(\omega_{\mathfrak{a},n})^2 - \omega^2}, \quad a_{\mathfrak{a},n} = \omega_{\mathfrak{a},n}^{-2} \left(\partial_s \phi_{\mathfrak{a},n}(M^*)\right)^2. \tag{3}$$

The above series converges uniformly on compact subsets of $\mathbb{C}$ that do not contain the poles of $\mathbf{\Lambda}_{\mathfrak{a}}$. Moreover, $\mathbf{\Lambda}_{\mathfrak{a}}(0) \geq 0$.

Approximation of the operator $\Lambda(\partial_t)$. Theorem 3 provides a convenient way to approximate $\Lambda(\partial_t)$ by truncating the respective series (we omit the index $\mathfrak{a}$):

$$\mathbf{\Lambda}_N(\omega) = \mathbf{\Lambda}(0) - \sum_{n=0}^{N-1} \frac{a_n \omega^2}{\omega_n^2 - \omega^2}.$$

A realization of $\Lambda_N(\partial_t)$ in the time domain is rather simple and can be found in [2]. The respective approximated transparent boundary conditions (1) rewrite

$$\partial_s u(t, M) = -\sum_{i=1}^{p} \mu_i \Lambda_i^N(\partial_t) u(t, M), \qquad \Lambda_i^N(\partial_t) = \ell_i^{-1} \Lambda_N(\ell_i \partial_t). \tag{4}$$

The error induced by (4) is quantified below and depends on the following quantity:

$$r_N = \sum_{n=N}^{\infty} \frac{a_n}{\omega_n^2}.$$

Theorem 4 ([2]). *Let u solve the Dirichlet (Neumann) problem (D) (resp. (N)), and let u_N solve the respective problem on the truncated tree $\mathcal{T}^m$, with the approximated boundary conditions (4). Then, with $c > 0$,*

$$\|\partial_s(u - u_N)(t)\|_{\mathcal{T}^m} + \|\partial_t(u - u_N)(t)\|_{\mathcal{T}^m} \leq c\, r_N\, t \|\partial_s u\|_{W^{4,1}(0,t;\mathrm{L}^2_\mu(\mathcal{T}))}.$$

Precise bounds on r_N depend on the (Weyl) asymptotic estimates on the eigenvalues (2) and estimates on the derivatives of the eigenfunctions $\partial_s \phi_n(M^*)$. This is summarized below.

Theorem 5 ([2]). *If $\sum\limits_{i=1}^{p} \alpha_i < 1$, $r_N = O(N^{-1})$.*
If $\sum\limits_{i=1}^{p} \alpha_i = 1$, $r_N = O(N^{-1} \log N)$.
If $\sum\limits_{i=1}^{p} \alpha_i > 1$, $r_N = O(N^{-\frac{1}{d_s}})$, where $d_s > 1$ is a unique number s.t. $\sum\limits_{i=1}^{p} \alpha_i^{d_s} = 1$.

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Dirichlet forms and boundaries of graphs I and II

Daniel Lenz, Matthias Keller

The study of graphs and their Laplacians has a long history. A systematic approach of certain features within the context of Dirichlet forms was pursued in recent years in various works starting with [6]. The point of view of Dirichlet forms gives both manifolds and graphs an equal footing. Indeed in this context graphs and their Laplacians present the paradigm of the non-local situation whereas Laplace Beltrami operators on manifolds present the paradigm of a local situation. Whereas earlier investigations of graphs have often focused on the normalized Laplacian, which is a bounded operator, this setting allows for unbounded operators and these give rise to a variety of new phenomena, see e.g. the survey articles [7, 5].

A *graph* on a set X consists of a pair (b, c) with $c : X \longrightarrow [0, \infty)$ arbitrary and $b : X \times X \longrightarrow [0, \infty)$ symmetric, vanishing on the diagonal with

$$\sum_{y \in X} b(x, y) < \infty$$

for all $x \in X$. In order to set up a proper theory one needs a measure m on X giving rise to the Hilbert space $\ell^2(X, m)$. Each graph gives rise to the form

$$\mathcal{Q}(f) := \frac{1}{2} \sum_{x,y \in X} b(x, y)|f(x) - f(y)|^2 + \sum_{x \in X} c(x)|f(x)|^2$$

for complex valued f on X. The restriction of $\mathcal{Q}$ to suitable subspaces of $\ell^2(X, m)$ are then the associated *Dirichlet forms* of the graph. A basic result then asserts that there is a one-to-one correspondence between so-called regular Dirichlet forms on $\ell^2(X, m)$ and graphs in our sense above, see e.g. [6].

Dirichlet forms on a graph allow one to set up and study *intrinsic geometry* of the graph as given by

- intrinsic metrics [1, 8],
- canonical compactness and uniform transience [2, 10],
- Royden boundary [2, 10].

A multitude of results has been obtained in these directions in the last years. In the context of the topic of the workshop we highlight in particular the following results. The first result focuses on the graph as a metric space and the second result focuses on the graph as a measure space.

Theorem. (Huang/Keller/Masamune/Wojciechowski '13 [3]) The Laplacian on a graph is essentially selfadjoint (on the functions with finite support) if the graph is complete with respect to an intrinsic metric.

Theorem. (Keller/Lenz '12 [6]) The Laplacian on a graph is essentially selfadjoint (on the functions with finite support) if all infinite paths have infinite measure.

Although not mentioned explicitly it is assumed in both theorems that the Laplacian maps the compactly supported functions $C_c(X)$ into $\ell^2(X, m)$ to be a symmetric operator on $\ell^2(X, m)$ to begin with.

The two results above make it clear that one should look at incomplete graphs with finite measure in order to study self-adjoint extensions. Clearly, it is an nearly impossible task to study all self-adjoint extensions in a general setting. Hence, we focus on Markov extensions in the following.

Specifically, we look at forms Q which lie between $Q^{(D)}$ and $Q^{(N)}$. Here $Q^{(D)}$ is the restriction of $\mathcal{Q}$ to the form closure of $C_c(X)$ in $\ell^2(X, m)$ and $Q^{(N)}$ is the restriction to $\mathcal{D} \cap \ell^2(X, m)$ where

$$\mathcal{D} = \{f : X \to \mathbb{C} \mid \mathcal{Q}(f) < \infty\}$$

is called the space of *functions of finite energy*.

We employ Gelfand theory to define a boundary arising from $\mathcal{Q}$. This is done by considering the commutative C*-algebra of the uniform closure of bounded functions of finite energy as the continuous functions on a compact Hausdorff space R, i.e.,

$$\overline{\mathcal{D} \cap \ell^\infty(X)}^{\|\cdot\|_\infty} \simeq C(R)$$

whenever c is summable (otherwise one has to replace $C(R)$ by $C_0(R)$). The space R is called the *Royden compactification* and X embeds into R via identifying elements of X by the corresponding point evaluation, which is a character on the C^*-algebra. The complement

$$\partial X := R \setminus X$$

is called the *Royden boundary*.

Typically the Royden boundary is a rather monstrous object, [11, 12]. However, there are cases of graphs where the Royden boundary is rather accessible to analysis. Two large classes of such cases are

- canonically compactifiable graphs, i.e., when $\mathcal{D} \subseteq \ell^\infty(X)$, [2],
- uniform transient graphs, i.e., when $\mathcal{D}_0 \subseteq C_0(X)$, [8],

where $\mathcal{D}_0$ is the closure of $C_c(X)$ with respect to $\mathcal{Q}$ and pointwise convergence and $C_0(X)$ is the uniform closure of $C_c(X)$. Indeed, all canonically compactifiable graphs are uniformly transient. In more special situations one can even show that the Royden boundary is homeomorphic to a metric boundary with respect to a certain metric.

When the underlying measure is finite, it can be seen that on these graphs the Laplacian is not essentially self-adjoint and has not even a unique Markov extension. It turns out that one can characterize all Markov extensions via Dirichlet forms on the Royden boundary [9]. To this end one equips the Royden boundary with a harmonic measure $\mu = \mu_x$ for some $x \in X$ which is characterized via the equality for all bounded harmonic functions h in $\mathcal{D}$

$$h(x) = \int_{\partial X} (\mathrm{Tr}\, h) d\mu$$

where $\mathrm{Tr} f$ is the restriction to ∂X of the continuous extension of f to R. We can extend the trace operator to an operator $\mathrm{Tr} : \mathcal{D} \to L^2(\partial X, \mu)$. Furthermore, the trace $\mathrm{Tr} Q$ of a Dirichlet form Q on $L^2(\partial X, \mu)$ is defined by

$$\mathrm{Tr}\, Q(f) = \mathcal{Q}(Hf)$$

where Hf is the harmonic extension of a function on ∂X (and the domain is given by the traces of functions in the extended Dirichlet space). The trace of $Q^{(N)}$ is called the *Dirichlet-to-Neumann form* and it is denoted by $q^{(DN)}$.

In [9] it is shown that for any Q between $Q^{(D)}$ and $Q^{(N)}$ there is a Dirichlet form (in the wide sense) q on $L^2(\partial X, \mu)$ such that

$$Q(f) = Q^{(D)}(f_0) + q(\mathrm{Tr} f) = Q^{(N)}(f) + (q - q^{(DN)})(\mathrm{Tr} f),$$

where f_0 is the projection of f to $D(Q^{(D)})$ and $q = \mathrm{Tr}\, Q$. Furthermore, the form $q - q^{(DN)}$ is Markovian, i.e., compatible with normal contractions. Indeed, the following theorem holds.

Theorem. (Keller/Lenz/Schmidt/Schwarz '19 [9]) Assume the graph is uniformly transient with finite measure. Then there is a bijection between Dirichlet forms Q such that $Q^{(D)} \geq Q \geq Q^{(N)}$ and the Dirichlet forms (in the wide sense) q on $L^2(\partial X, \mu)$ such that $q - q^{(DN)}$ is Markovian.

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An introduction to Martin boundary theory

Wolfgang Woess

Upon invitation by the workshop organisers, I gave a classroom-style introduction to the Martin boundary theory for infinite networks (denumerable Markov chains). This goes back to the classical work of Doob [1] and Hunt [2]. See my textbook [3].

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Brownian motion at a family of geometric singularities

Robert Neel
(joint work with U. Boscain)

Let $\alpha \in \mathbb{R}$. On $\mathbb{R} \times \mathbb{S}^1$ with coordinates (x, θ), consider the pair of vector fields

$$X = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 \\ |x|^\alpha \end{pmatrix}.$$

Letting these be an orthonormal frame on $M = \big(\mathbb{R}\backslash\{0\}\big) \times \mathbb{S}^1$ gives the Riemannian metric

$$g = dx^2 + |x|^{-2\alpha} d\theta^2,$$

which degenerates on the singularity $\mathcal{Z} = \{x = 0\}$. We let $M^- = \{x < 0\}$ and $M^+ = \{x > 0\}$ be the two Riemannian components. We also let ω denote the Riemannian volume measure on $M = M^+ \cap M^-$.

There is a (possibly degenerate) distance associated to this geometry that extends across $\mathcal{Z}$. Note the case $\alpha = 1$ gives the (fairly well-studied) Grushin cylinder (the natural quotient of the Grushin plane). We define

$$M_{\text{cylinder}} = \mathbb{R} \times \mathbb{S}^1, \qquad M_{\text{cone}} = M_{\text{cylinder}} / \sim,$$

where $(x_1, \theta_1) \sim (x_2, \theta_2)$ if and only if $x_1 = x_2 = 0$. From Boscain-Prandi [2], when $\alpha \geq 0$ (resp. $\alpha < 0$) this distance makes M_{cylinder} (resp. M_{cone}) into a metric space in a way that induces on M_{cylinder} (resp. M_{cone}) its original topology; that is, M_{cylinder} (resp. M_{cone}) gives the metric compactification of M. We denote these metric spaces by M_α.

Except for the standard cylinder ($\alpha = 0$), the Riemannian structure on M is singular at $\mathcal{Z}$. Thus the heat and Schroedinger equations, as well as Brownian motion, are a priori not well defined at $\mathcal{Z}$, and we can consider how they might extend. Boscain-Prandi [2], via Fourier transform methods, showed that there is a unique Markovian extension of $\Delta|_{C_c^\infty(M)}$ in $L^2(M, \omega)$ if and only if $\alpha \in (-\infty, -1] \cup$

$[1, \infty)$, and it does not permit any heat flow between $M^+$and M^-. When $\alpha \in (-1, 1)$, there are Markovian self-adjoint extensions of Δ that permit heat to flow between M^+ and M^-, and in particular, there is a self-adjoint extension called the *bridging extension* realizing the "maximal communication."

Here (see [1] for the details), we take the alternative, and complementary, route of extending Brownian motion on M to a diffusion on M_α. Note that, under Brownian motion,

$$dx_t = dW_t^1 - \frac{\alpha}{2x}\, dt \quad \text{and} \quad d\theta_t = |x|^\alpha\, dW_t^2,$$

until the first hitting time of $\mathcal{Z}$. So x_t is a $1-\alpha$ dimensional Bessel process. Thus,

- the process doesn't explode to infinity (in finite time) for any value of α,
- for $\alpha \leq -1$, 0 is an entrance-only boundary (in the standard Feller classification for one-dimensional diffusions),
- for $\alpha \in (-1, 1)$, 0 is a regular boundary, and thus one needs to specify boundary conditions,
- for $\alpha \geq 1$, 0 is an exit-only boundary.

This gives a probabilistically natural explanation of the dependence of the uniqueness of Markovian extensions on α. Moreover, the only interesting cases to consider are when $\alpha \in (-1, 1)$.

When $\alpha \in (-1, 0)$, $\mathcal{Z}$ is a single point. Thus for any diffusion, x_t is independent of θ_t, which allows us to determine all diffusions extending Brownian motion on M using mostly classical methods. In particular, we first determine the possible diffusions for x_t that extend Brownian motion in terms of the classical 1-dimensional theory of Feller and Ito-McKean (see, for example, [4] for a readable account, and the references therein for some of the history). Given the possibilities for x_t, we can then find the θ_t diffusions they support. Moreover, of these extensions, there is a natural candidate for the "best" one (which is also the unique symmetric one-point extension of Brownian motion on M in the sense of Chen-Fukushima [3]), which we now describe.

Theorem 1 ([1]). *Let M_α and ω be as above, for $-1 < \alpha < 0$. Then the unique (conservative) diffusion on M_α that extends Brownian motion on M, spends time 0 at $\mathcal{Z}$, and is ω-symmetric is given as follows. Let (x_t^2, θ_t) be the diffusion on $[0, \infty) \times \mathbb{S}^1$ that undergoes instantaneous normal reflection at the boundary and letting x_t be constructed from x_t^2 by giving each excursion a positive or negative sign with probability $1/2$. Then let θ_t take initial value uniformly on $\mathbb{S}^1$ every time the process enters M from $\mathcal{Z}$ (each time independent of the previous, of course). This diffusion is also the unique extension of Brownian motion spending time 0 at $\mathcal{Z}$ that is invariant under the isometry group of M_α.*

By conformal methods, we can show that the Martin boundary of M "at $\mathcal{Z}$" is two copies of $\mathbb{S}^1$, which is much larger than $\mathcal{Z}$. Indeed, several of the self-adjoint extensions of Δ considered in [2] (in particular, the Neumann extension and the bridging extension) give diffusions that are carried by the Martin compactification, not by M_α.

Next, we discuss the case $\alpha \in [0,1)$. Now $\mathcal{Z} = \mathbb{S}^1$, and an exact classification of diffusions would be messy, if possible. However, we can again determine a natural candidate for the "best" diffusion (in particular, in this case, the bridging extension is carried by M_α).

Theorem 2 ([1])**.** *For $0 \le \alpha < 1$, the only (conservative) diffusion on M_α extending Brownian on M that spends 0 time at $\mathcal{Z}$ and is invariant under the isometry group of M_α is given by letting (x_t^2, θ_t) be the diffusion on $[0,\infty) \times \mathbb{S}^1$ that undergoes instantaneous normal reflection at the boundary and letting x_t be constructed from x_t^2 by giving each excursion a positive or negative sign with probability $1/2$. Moreover, this is the diffusion associated to the bridging extension.*

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Density and trace results in generalized fractal networks

ADRIEN SEMIN
(joint work with S. Nicaise)

We presented some part of the work we published in [1] concerning the density and trace results in generalized fractal networks and that are a generalization of some results published in [2]. The first aim of this talk was to give different and necessary sufficient conditions that guarantee the density of the set of compactly supported functions into the Sobolev space of order one in infinite p-adic weighted trees. The second goal is to define properly a trace operator in this Sobolev space if it makes sense.

One question that remained open in [1] is the characterization of the closure space of the compactly supported functions for the weighted norm on the derivative by the trace at infinity. Although the compactly supported functions have a null trace at infinity, it is not obvious that a limit of sequence of compactly supported functions will have a null trace at infinity as well. Discussion in this workshop with Prof. Dr. Daniel Lenz and Prof. Dr. Matthias Keller lended to the assumption that a limit of sequence of compactly supported functions will have a null trace at infinity on the Royden boundary [3] and conversely.

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Essential self-adjointness of sub-Laplacians

Valentina Franceschi, Dario Prandi
(joint work with R. Adami, U. Boscain, L. Rizzi)

In this talk we present some results about the essential self-adjointness of sub-Laplacians on *non-complete* sub-Riemannian manifolds. We refer to Luca Rizzi's talk for the introduction to sub-Laplacians and sub-Riemannian manifolds, and we will follow his notation in the present abstract.

1. sub-Riemannian geometry

Let M be a smooth manifold of dimension $n \in \mathbb{N}$. We recall that, given a family of smooth vector fields $\{X_1, \dots, X_N\} \subset TM$ satisfying *Hörmander condition*, and a *smooth measure* ω on M, the *sub-Laplacian* is the operator

$$\Delta_\omega = \sum_{i=1}^{N} X_i^2 + \operatorname{div}_\omega(X_i)X_i + c, \quad \operatorname{Dom}(\Delta_\omega) = C_c^\infty(M).$$

Essential self-adjointness of Δ_ω holds if (M, d) is complete as a metric space, where d is the associated sub-Riemannian distance (see [13] and Theorem 4 in Luca Rizzi's talk). In this seminar, we address the case where (1) either the measure (chosen to be intrinsic) is non-smooth (see Section 2), or (2) the metric structure is non-complete (see Section 3).

1.1. Examples of sub-Riemannian manifolds.

- The *Heisenberg Group* $\mathbb{H}^1$ is $\mathbb{R}^3$ endowed with the non-commutative group law defined for $(x, y, z), (x', y', z') \in \mathbb{H}^1$ as

$$(x, y, z) * (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)\right). \tag{1}$$

This is a sub-Riemannian structure on $\mathbb{R}^3$ when the distribution of admissible directions $\mathcal{D}$ is given by the Lie algebra of left-invariant vector fields

$$\mathcal{D}(x, y, z) = \operatorname{span}\left\{X_1 := \partial_x - \frac{y}{2}\partial_z,\ X_2 := \partial_y + \frac{y}{2}\partial_z\right\}, \quad (x, y, z) \in \mathbb{H}^1.$$

In this case, the Hörmander condition is satisfied at step 2: $[X_1, X_2] \equiv \partial_z$.

- The *Grushin plane* is $\mathbb{R}^2$ endowed with the following distribution of admissible directions
$$\mathcal{D}(x,y) = \operatorname{span}\{X_1 := \partial_x,\ X_2 := x\partial_y\}, \quad (x,y) \in \mathbb{R}^2.$$
In this case, the Hörmander condition is satisfied at different steps, depending on the point $(x,y) \in \mathbb{R}^2$, namely:
$$\begin{cases} \dim \mathcal{D}(x,y) = 2 & \text{if } x \neq 0, \\ \dim \mathcal{D}(x,y) = 1 & \text{if } x = 0 \text{ and in this case } [X_1, X_2](x,y) = \partial_z, \end{cases}$$
so that Hörmander condition is globally satisfied at step 2, but a difference in dimension is recorded in the distribution, depending on the position.
- The *Martinet distribution* in $\mathbb{R}^3$ is
$$\mathcal{D}(x,y,z) = \operatorname{span}\left\{X_1 := \partial_x,\ X_2 := \partial_y + \frac{x^2}{2}\partial_z\right\}.$$
Observe that $\dim \mathcal{D} \equiv 2$, but a difference in dimension is recorded by computing the first order commutator:
$$[X_1, X_2] = x\partial_z,$$
so that
$$\dim\big(\mathcal{D}(x,y,z) + \operatorname{span}\{X_1(x,y,z),\ X_2(x,y,z)\}\big) = \begin{cases} 3 & \text{if } x \neq 0, \\ 2 & \text{if } x = 0. \end{cases}$$
In conclusion, the Martinet distribution defines a sub-Riemannian structure on $\mathbb{R}^3$, since Hörmander condition is globally satisfied at step 3, but the step changes depending on the position.

The Grushin plane and the Martinet distribution constitute exemples of *non-equiregular* sub-Riemannian manifolds, see [3] for a precise definition.

2. Singular intrinsic sub-Laplacians

2.1. **The choice of the measure.** In a Riemannian manifold (M,g) of dimension n there is a canonical choice of the volume measure ω, that can be explicitly computed in coordinates in terms of g and coincides up to a constant factor with the n-Hausdorff measure with respect to the Riemannian distance on M.

This is not the case in sub-Riemannian manifolds, where the problem of finding canonical measures is still open in the general case of (possibly non-equiregular) sub-Riemannian manifolds. In [10] a canonical volume measure has been proposed: this is the *Popp's measure* and it turns out to be smooth on the equiregular regions of M, cf. also [5]. [1] For instance, Popp's measure on the Heisenberg group (which is an equiregular structure) is proportional to the Lebesgue measure on $\mathbb{R}^3$. On the other hand, on the *singular region* where the sub-Riemannian structure "looses its regularity", Popp's measure blows up. For instance, in the Martinet case, Popp's

[1] To conclude the picture, we mention that Popp's measure and the Hausdorff measure with respect to the sub-Riemannian distance are in general not equivalent, as it is shown in [2].

measure is proportional to $|x|^{-1}dx\,dy\,dz$, and in the Grushin plane Popp's measure reads $|x|^{-1}dx\,dy$.

2.2. **Essential self-adjointness of singular sub-Laplacians.** Motivated by Section 2.1, we study essential-self adjointness properties of sub-Laplacians in the case where the intrinsic Popp's measure ω_{Popp} is smooth only outside a singular region $\mathcal{Z} \subset M$. To this purpose, in [8] we proved the following result, that generalizes [4, 11] and has been mentioned in Luca Rizzi's talk in the present workshop.

Theorem 1. *Let M be a smooth manifold and $\mathcal{Z} \subset M$ be a smooth compact hypersurface such that $T_q\mathcal{Z}$ is transversal to $\mathcal{D}(q)$ for any $q \in \mathcal{Z}$. Assume that locally near $\mathcal{Z}$, Popp's measure reads $\omega_{Popp} \simeq \delta^{-a} \times \sigma$, where $\delta(\cdot) = d(\cdot, \mathcal{Z})$ is the sub-Riemannian distance from $\mathcal{Z}$, σ is a smooth measure and $a \geq 1$. Then $\Delta_{\omega_{Popp}}$ with domain $C_c^\infty(M \setminus \mathcal{Z})$ (or any of its connected components) is essentially self-adjoint in $L^2(M, \omega_{Popp})$.*

The key ingredients of the proof are Hardy-type inequalities combined with Agmon-type estimates. A comment on these techniques is presented (for a different case) in section 3.2.

3. Non-complete sub-Laplacians

A simple way to obtain non-complete manifolds from a complete one is by removing a point. In the Euclidean setting, the *pointed Laplacian* $\Delta = \partial_1^2 + \cdots + \partial_n^2$ defined on $C_c^\infty(\mathbb{R}^n \setminus \{0\})$ is essentially self-adjoint if and only if $n \geq 4$, and the correspondent result still holds true in the Riemannian case (see for instance [6]).

In the last part of this talk, we want to understand essential self-adjointness properties of *pointed sub-Laplacians on* $3D$ *equiregular sub-Riemannian structures*, whose local model is the Heisenberg group.

We henceforth focus on the Heisenberg sub-Laplacian $\Delta_H = X_1^2 + X_2^2$, $\text{Dom}(\Delta_H) = C_c^\infty(\mathbb{R}^3)$ that is essentially self-adjoint by Strichartz' theorem and on its pointed counterpart $\mathring{\Delta}_H = X_1^2 + X_2^2$, $\text{Dom}(\mathring{\Delta}_H) = C_c^\infty(\mathbb{R}^3 \setminus \{0\})$. (Here we fixed the volume measure ω to be the Lebesgue measure, that corresponds to Popp's measure and to the Haar measure.)

3.1. **Relevant dimensions on the Heisenberg group.** Two important properties of the Heisenberg group $\mathbb{H}^1$ are the following

- The sub-Riemannian distance d is left-invariant on $\mathbb{H}^1$:
$$d(p * q_1, p * q_2) = d(q_1, q_2), \quad q_1, q_2 \in \mathbb{H}^1.$$
- The sub-Riemannian distance is 1-homogeneous w.r.t. the family of anisotropic dilations $\delta_\lambda(x, y, z) = (\lambda x, \lambda y, \lambda^2 z)$, $\lambda > 0$:
$$d(\delta_\lambda(q_1), \delta_\lambda(q_2)) = \lambda d(q_1, q_2), \quad q_1, q_2 \in \mathbb{H}^1.$$

According to the Ball-Box theorem (see for instance [3] for details), one can then deduce that the metric ball of radius $\varepsilon << 1$ w.r.t. the sub-Riemannian distance d is equivalent to a box $[-\varepsilon, \varepsilon]^2 \times [-\varepsilon^2, \varepsilon^2]$. From all these facts one can deduce that the *Hausdorff dimension* (also called Homogeneous dimension) is 4. We are then in presence of three relevant dimensions

- The topological dimension is 3,
- The dimension of the distribution of admissible directions is 2,
- The Hausdorff dimension is 4.

Recalling that in the Riemannian setting essential self-adjointed of the pointed Laplace-Beltrami operator change between dimension 3 and 4, the following natural question arises. What is the relevant dimension in terms of essential self-adjointness of the associated pointed sub-Laplacian?

3.2. **Possible strategies: Hardy-type inequalities.** We plan to exploit the classical essential self-adjointness criterion [12, Thm X.I and Corollary] ensuring that $\mathring{\Delta}_H$ is essentially self-adjoint if and only if

$$-\mathring{\Delta}_H^* \psi = E\psi,\ \psi \in L^2(\mathbb{H}^1),\ E < 0 \implies \psi \equiv 0. \tag{2}$$

One can show that $\Delta_H = \operatorname{div}(\nabla_H)$, where for a smooth function u we let $\nabla_H u = (X_1 u)X_1 + (X_2 u)X_2$. In particular, Δ_H is naturally associated with the following Dirichlet energy:

$$\int_{\mathbb{H}^1} \|\nabla_H u\|^2 \, dp = -(u, \Delta_H u)_{L^2(\mathbb{H}^1)}. \tag{3}$$

A first attempt to prove (2) would be via Hardy-type inequality combined with Agmon type estimate. We briefly show the idea, and then explain why this does not work.

Let $\delta(\cdot) = d(\cdot, 0)$ be the sub-Riemannian distance from the origin. Suppose that the following Hardy inequality holds true in a neighborhood $\mathcal{O}$ of the origin:

$$\int_{\mathcal{O}} \|\nabla_H u\|^2 \geq \int_{\mathcal{O}} \frac{u^2}{|\delta|^2}, \qquad u \in C_c^\infty(\mathbb{H}^1 \setminus \{0\}). \tag{4}$$

Observe that since the Heisenberg sub-Laplacian Δ_H is hypoelliptic, the equation on the left-hand side of (2) implies that ψ is a smooth function. We multiply it by a function $f = F \circ \delta$ compactly supported outside the origin and only depending on δ. Plugging $u = f\psi$ into (4), through (3), we obtain

$$\int_{\mathcal{O}} \frac{|f\psi|^2}{\delta^2} \leq \|\nabla_H (f\psi)\|_2^2 = E\|f\psi\|_2^2 + \langle \psi, \|\nabla_H f\|^2 \psi \rangle.$$

Rearranging the terms and using $f = F \circ \delta$ we get

$$-E\|f\psi\| \leq \int_{\mathcal{O}} \left(F'(\delta)^2 - \frac{F(\delta)^2}{\delta^2} \right) |\psi|^2$$

and for a suitable choice of F that is only possible if (4) holds with a constant ≥ 1 on the right-hand side, this implies the statement (see [8, Rmk. 4.2] for more details).

Seeking for Hardy-type inequalities as (4), one discovers that an optimal Hardy inequality with constant 1 for the Heisenberg operator has been proved in 1990 by

Garofalo an Lanconelli in [9]. Unfortunately, instead of δ^2, the latter inequality involves a singular weight that prevents to apply the previous technique.

In [7] we proved that the optimal constant for the Heisenberg Hardy inequality with respect to the non-weighted distance from the origin δ is strictly smaller than one. This concludes our attempts to prove (2) via Hardy inequalities.

3.3. **Essential self-adjointness of $\mathring{\Delta}_H$.** In [1] we proved the following

Theorem 2. *$\mathring{\Delta}_H$ is essentially self-adjoint.*

Motivated by Section 3.2 the proof of the above does not involve Hardy-type inequalities.

3.3.1. *Idea of the proof.* We start by proving that the essential self-adjointness criterion (2) can be equivalently formulated by saying that there does not exists $\theta \in L^2$, $\theta \not\equiv 0$ solving the following equation in the distributional sense

$$(-\Delta_H + i)\theta = \sum_{|\alpha|<N} c_\alpha D^\alpha \delta_0, \quad (c_\alpha)_{\alpha\in\mathbb{N}^3} \not\equiv 0. \tag{5}$$

We present here the ideas to prove absence of L^2 solutions to the latter equation in the Euclidean n-dimensional setting, by exploiting the classical Fourier transform. A suitable adaptation of these arguments involving non-commutative Fourier analysis leads to the conclusion in the Heisenberg group.

Assume that there exists $\theta \in L^2$, $\theta \not\equiv 0$ solving (5) $c_{(0,0,0)} = 1$, $c_\alpha = 0$ for $\alpha \neq (0,0,0)$ (that is, we only have δ_0 on the r.h.s.). We compute the Fourier transform of both sides obtaining

$$(|\lambda|^2 + i)\hat{\theta}(\lambda) = 1 \iff \hat{\theta}(\lambda) = \frac{1}{|\lambda|^2 + i}.$$

By Plancherel theorem we then get

$$\|\theta\|_{L^2(\mathbb{R}^n)} = \|\hat{\theta}\|_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \frac{1}{(|\lambda|^2+i)^2}\, d\lambda \gtrsim \int_0^\infty \frac{\rho^{n-1}\, d\rho}{(\rho^2+1)^2} \gtrsim \int_1^\infty \rho^{n-5}\, d\rho.$$

Namely, if $n \geq 4$ we are contradicting the fact that $\theta \in L^2$, while for $n \leq 3$ we do not get any contradiction. This proves that for $n \geq 4$ there are no L^2 solutions to (5), implying in turn that the Euclidean Laplacian is essentially self-adjoint if $n \geq 4$. Since it is then easy to prove that for $n \leq 3$ equation (5) has non-trivial solutions in L^2, we obtain essential self-adjointness of the Euclidean Laplacian if and only if $n \geq 4$.

Remark 1. In [1] we show that Theorem 2 can be generalized to any sub-Laplacian Δ_ω on a $3D$ contact manifold M (i.e.,the distribution of admissible directions $\mathcal{D}$ is of rank 2 and the step in Hörmander condition is 3 at every point).

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Self-adjoint extensions of infinite quantum graphs

Noema Nicolussi

(joint work with A. Kostenko, D. Mugnolo)

This talk is concerned with developing extension theory for *quantum graphs*. Quantum graphs are Schrödinger operators on metric graphs, that is combinatorial graphs where edges are considered as intervals with certain lengths. The *Kirchhoff Laplacian* **H** provides the analog of the Laplace-Beltrami in this setting. The main goal of this talk is to address the self-adjointness problem for **H**, which classically consists of the following three questions:

- Is the Kirchhoff Laplacian **H** self-adjoint?
- If **H** is not self-adjoint, what are the deficiency indices $n_\pm(\mathbf{H})$?
- ... and how can obtain a description of all self-adjoint extensions?

Whereas on finite metric graphs, the Kirchhoff Laplacian **H** is always self-adjoint, the question is more complicated for *graphs with infinitely many edges* since their geometrical structure can be quite complex. The search for self-adjointness criteria in this case is an open problem. For instance, a uniform lower bound for the edge lengths guarantees self-adjointness [1], but this commonly used condition is to some extent unsatisfactory (e.g., it is independent of the combinatorial graph structure).

Based on connections with discrete Laplacians on graphs, several Gaffney-type criteria were obtained recently in [5]. However, by the example of *antitrees* (a particular class of highly connected graphs) we demonstrate that these sufficient criteria are in general very far from being necessary (see also [6]). Moreover, we provide a simple example of an antitree which allows to realize all possible values $N \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ as deficiency indices by different choices of edge lengths.

In order to simplify the problem, we restrict our attention to a suitable subclass of self-adjoint extensions and consider the class of *finite energy extensions* $\widetilde{\mathbf{H}}$: that is, the functions in the domain of $\widetilde{\mathbf{H}}$ belong to the Sobolev space H^1 (=continuous L^2-functions of finite energy). On the one hand, finite energy extensions should intuitively have good properties. Indeed, it turns out that their resolvents and semigroups (under some additional technical assumption) are integral operators with a bounded, continuous kernel and belong to the trace class in case of finite total volume. On the other hand, all *Markovian extensions* (i.e. the corresponding quadratic is a Dirichlet form) are of finite energy, and hence this class contains all extensions which typically are considered in context with the heat equation.

The problem of extension theory is closely related to finding suitable boundary notions for infinite graphs. In this context, a natural idea is to consider rays (i.e., infinite self-avoiding paths), which intuitively should lead to different directions at infinity. This approach is formalized in the concept of *graph ends* introduced independently by Freudenthal [3] and Halin [4]. Graph ends are in bijection with the topological ends of the graph and hence they coincide with the boundary in the sense of the Freudenthal compactification [2]. However, the definition of graph ends is purely combinatorial and hence must be modified to capture the additional metric structure of our setting. Based on the correspondence between graph ends and topological ends we introduce the concept of *graph ends of finite volume* [7]. It turns out that this notion is indeed well suited for the study of the H^1-space and finite energy extensions. For instance, it leads to a geometrical characterization of uniqueness of Markovian extensions:

$\mathbf{H}$ has a unique Markovian extension if and only if
all graph ends have infinite volume.

Moreover, returning to the question of deficiency indices, we provide a lower estimate for $n_\pm(\mathbf{H})$ in terms of the number of finite volume graph ends. This estimate is sharp and we also find a necessary and sufficient condition for the equality between the number of finite volume graph ends and $n_\pm(\mathbf{H})$ to hold. Finally, under the additional assumption that the number of finite volume graph ends is finite, we obtain a complete description of all finite energy extensions of $\mathbf{H}$ in terms of self-adjoint linear relations on the corresponding boundary space. This also leads to a description of all Markovian extensions in this case. Related results on Markovian extensions of discrete Laplacians on graphs (in terms of the Royden boundary) were proven recently in [8].

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Brownian motion and curvature in sub-Riemannian geometry

Anton Thalmaier
(joint work with L.-J. Cheng, E. Grong)

The goal of sub-Riemannian geometry is the investigation of geometric structures intrinsically induced by the sub-Riemannian data (M, H, g_H) where M is a smooth manifold, H a subbundle of the tangent bundle (describing the "horizontal" directions), and g_H a metric tensor defined on the "horizontal" subbundle H. The subbundle H is assumed to be bracket-generating, meaning that its sections and their iterated brackets span the entire tangent bundle. We describe recent work related to the concept of "horizontal Ricci curvature". Our approach relies on a study of sub-Riemannian Brownian motions and stochastic analysis on path space over sub-Riemannian manifolds. Analogously to the work of Aaron Naber [6] (see also [5]) we show that certain functional inequalities and gradient estimates on path space are equivalent to boundedness of the horizontal Ricci tensor [3]. For the proofs we adopt the methods of [1, 2] to the sub-Riemannian setting.

We work with a connection ∇ on M which is compatible with (H, g_H) in the sense that parallel transport along smooth curves in M takes orthonormal frames in H to orthonormal frames in H. Since H is bracket-generating, compatible connections ∇ always have torsion $\mathbf{T}$:

$$\nabla_A B - \nabla_B A - [A, B] = \mathbf{T}(A, B), \quad A, B \in \Gamma(H).$$

To construct canonical connections one starts with a partial connection $\nabla\colon \Gamma(H) \times \Gamma(H) \to \Gamma(H)$, $(A, B) \mapsto \nabla_A B$ on H and extends it to a full connection in a canonical way. A connection ∇ on M compatible with (H, g_H) is uniquely determined by its torsion, and choosing a complement V for H, that is $TM = H \oplus V$, there is a unique such connection with $\mathbf{T}(H, H) \subset V$.

Let R be the curvature of a compatible connection ∇ and $\mathrm{Ric}\colon TM \to TM$ the corresponding Ricci operator given by

$$\mathrm{Ric}(v) = \mathrm{trace}_H \mathbf{R}(v, \times)\times$$

where the trace is taken over H with respect to the inner product g_H. Our object of interest is the horizontal Ricci curvature $\mathrm{Ric}^H = \mathrm{Ric}|_H \in \mathrm{End}(H)$ defined as the restriction Ric of H. We consider the corresponding sub-Laplacian

$$\Delta^H = \mathrm{trace}_H \nabla^2_{\times,\times}$$

defined as horizontal trace of the Hessian ∇^2. Diffusion processes on M with generator $\frac{1}{2}\Delta^H$ are called sub-Riemannian Brownian motions, cf. [4]. For fixed $T > 0$, let $W^T = C([0,T]; M)$ be the path space over M equipped with the measure induced by the sub-Riemannian Brownian motion with starting point $x \in M$, and let

$$\mathcal{F}C^\infty_{0,T} = \big\{ W^T \ni \gamma \mapsto f(\gamma_{t_1}, \dots, \gamma_{t_n})\colon \ 0 < t_1 < \ldots < t_n \le T, \ f \in C^\infty_c(M^n) \big\}$$

be the class of smooth cylindrical functions on W^T. Consider the Cameron-Martin space

$$\mathbb{H} = \left\{ h\colon [0,T] \to H_x \text{ absolutely continuous} \;\middle|\; \int_0^T |\dot{h}(t)|^2_{g_H}\, dt < \infty \right\}$$

which becomes a Hilbert space with inner product

$$\langle h_1, h_2 \rangle_{\mathbb{H}} = \int_0^T \langle \dot{h}_1(t), \dot{h}_2(t) \rangle_{g_H}\, dt.$$

For $F \in \mathcal{F}C^\infty_{0,T}$ we define a directional derivative $D_h F$ in the direction of $h \in \mathbb{H}$ and associated derivative operators D_t on $\mathcal{F}C^\infty_{0,T}$ such that

$$D_h F = \int_0^T \langle D_t F, \dot{h}_t \rangle_{g_H}\, dt.$$

The definition of D_h incorporates explicitly the torsion of the connection.

Theorem. *(Characterization of Ric^H by functional inequalities)*
For a non-negative constant K the following conditions are equivalent:

(1) the horizontal Ricci curvature Ric^H is bounded by K, i.e.

$$-K \le \mathrm{Ric}^H \le K;$$

(2) *(Gradient estimate)* for any smooth cylindrical function $F \in \mathcal{F}C^\infty_{0,T}$ on path space the following estimate holds:

$$|D_0 \mathbb{E}_x[F]|_{g_H} \le \mathbb{E}_x\Big[|D_0 F|_{g_H} + \frac{K}{2} \int_0^T e^{\frac{K}{2}s} |D_s F|_{g_H}\, ds \Big];$$

(3) *(Log-Sobolev inequality)* for any $F \in \mathcal{F}C^\infty_{0,T}$ and $t > 0$ in $[0, T]$,

$$\mathbb{E}_x\Big[\mathbb{E}_x[F^2|\mathcal{F}_t]\log\mathbb{E}_x[F^2|\mathcal{F}_t]\Big] - \mathbb{E}_x[F^2]\log\mathbb{E}_x[F^2]$$
$$\leq 2\int_0^t e^{\frac{K}{2}(T-r)}\left(\mathbb{E}_x|D_rF|^2_{g_H} + \frac{K}{2}\int_r^T e^{\frac{K}{2}(s-r)}\mathbb{E}_x|D_sF|^2_{g_H}\,ds\right)dr;$$

(4) *(Poincaré inequality)* for any $F \in \mathcal{F}C^\infty_{0,T}$ and $t > 0$ in $[0, T]$,

$$\mathbb{E}_x\Big[\mathbb{E}_x[F|\mathcal{F}_t]^2\Big] - \mathbb{E}_x[F]^2$$
$$\leq \int_0^t e^{\frac{K}{2}(T-r)}\left(\mathbb{E}_x|D_rF|^2_{g_H} + \frac{K}{2}\int_r^T e^{\frac{K}{2}(s-r)}\mathbb{E}_x|D_sF|^2_{g_H}\,ds\right)dr.$$

Here $\mathbb{E}_x$ denotes the expectation with respect to the probability measure on path space induced by the sub-Riemannian Brownian motion on M starting at $x \in M$, and $(\mathcal{F}_t)$ denotes its natural filtration.

The theorem above can be extended to a characterization of $K_1 \leq \mathrm{Ric}^H \leq K_2$ with arbitrary constants $K_1 \leq K_2$ by redefining D_h appropriately.

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Fractal string and its spectral properties

Igor Sheipak

We consider the spectral boundary value problem

$$-y'' = \lambda P' y, \tag{1}$$

$$y(0) = y(1) = 0 \tag{2}$$

with singular weight function P, $y \in \overset{\circ}{W}{}^1_2[0;1]$. The derivative is understood in the sense of the distributions. M.G. Krein considered this problem in the case of nondecreasing function P and in [1] obtained following eigenvalues asymptotic formula

$$\lim_{n\to\infty}\frac{n}{\sqrt{\lambda_n}} = \frac{1}{\pi}\int_0^1\sqrt{P'(x)}\,dx.$$

In terms of counting function of eigenvalues this asymptotics can be presented as $N(\lambda) = \frac{\sqrt{\lambda}}{\pi} \int_0^1 \sqrt{P'(x)}\, dx$.

So if function P is singular or equivalently its absolutely continuous part equals to zero, then $P' = 0$ almost everywhere and Krein formula gives no information on eigenvalues asymptotics.

In [2] the class of self-similar non-decreasing continuous functions P was considered. In this case dP defines self-similar measure singular with respect to Lebesgue measure. The main result of [2] is following: the asymptotic of counting function has the form

$$N(\lambda) = \lambda^D (s(\ln \lambda) + o(1)),$$

where $D \in \left(0; \frac{1}{2}\right)$ can be calculated via characteristics of self-similarity of measure dP. Function s is a periodic in the case of arithmetical self-similarity of the measure dP and is a constant in the case of non-arithmetical self-similarity of the measure dP.

The results in [2] were generalized later in [3] where non-sign-definite weight functions were considered. The class of functions P is defined as follows. We consider integer $n \geq 2$ and positive numbers a_1, a_2, ..., a_n satisfying conditions $\sum\limits_{k=1}^{n} a_k = 1$. Further we introduce numbers $\alpha_1 = 0$, $\alpha_k = \sum_{j=1}^{k-1} a_j$, $k = 2, 3, \dots, n+1$ and for all $k = 1, 2, \dots, n$ define affine transformations of the segment $[0, 1]$ on segments $[\alpha_k; \alpha_{k+1}]$: $S_k(x) = a_k x + \alpha_k$.

For some sets of real numbers $\{d_k\}_{k=1}^n$ $\{\beta_k\}_{k=1}^n$, $k = 1, \dots, n$ we define operator

$$[G(f)](x) = \sum_{k=1}^{n} \left(d_k \cdot f\left(S_k^{-1}(x)\right) + \beta_k\right) \cdot I_{(\alpha_k;\alpha_{k+1})}, \tag{3}$$

where $I_{(a,b)}$ is an indicator of the interval (a, b).

If $\sum_{k=1}^n a_k (d_k)^2 < 1$, then G is compressive in $L_2[0; 1]$. The unique function P that is a fixed point of the operator G is called self-similar. For $P \in L_2[0; 1]$ the problem (1), (2) is understood in in the sense of a quadratic form

$$\forall y \in \overset{\circ}{W}{}_2^1[0; 1] \int_0^1 |y'(x)|^2\, dx - \lambda \int_0^1 P(x)(|y(x)|^2)'\, dx = 0.$$

In common case of non-monotonic function $P \in L_2[0; 1]$ the problem (1), (2) has both negative and positive eigenvalues.

Theorem 1. ([3]) Under the requirements that among numbers d_i $(i = 1, 2, \dots, n)$ at least two are nonzero, and among numbers β_i $(i = 1, 2, \dots, n)$ — at least one, then the corresponding counting functions $N_\pm(\lambda) := \#\{\lambda_n : 0 < \lambda_n \leq \lambda\}$ have asymptotics

$$N_\pm(\lambda) = |\lambda|^D (s_\pm(\ln |\lambda|) + o(1),$$

where $D \in (0; 1)$ is the unique solution of the equation $\sum_{k=1}^n (a_k |d_k|)^t = 1$. For any $D \in (0; 1)$ it is possible to build an example of a function P so that the asymptotics of the problem (1), (2) has the desired power D.

The following studies are related to discrete weights: only one number in set $\{d_i\}$ and at least one in set $\{\beta_i\}$ $(i = 1, 2, \ldots, n)$ is nonzero. In this case the self-similar function P is piecewise constant, weight P' has representation $P' = \sum_{k=1} m_k \delta(x - x_k)$ with special numbers m_k and x_k. Such self-similar functions are called *self-similar functions with of zero spectral order.*

Denote by $M \in \{1, 2, \ldots, n\}$ the only index for which d_M is not zero and define quantity $q := \frac{1}{a_M |d_M|} > 1$. The condition $P \in L_2[0; 1]$ is equivalent to the inequality $a_m(d_M)^2 < 1$. Consider quantities ζ_k, $k = 2, \ldots n$ defined by

$$\zeta_k := \begin{cases} \beta_M - \beta_{M-1} + d_M \beta_1 & \text{if } k = M, \\ \beta_{M+1} - \beta_M - d_M \beta_n & \text{if } k = M + 1, \\ \beta_k - \beta_{k-1} & \text{otherwise.} \end{cases}$$

Also, let $Z_\pm$ denote two quantities $Z_\pm := \#\{k \in \{2, 3, \ldots, n\} : \ \pm\zeta_k > 0\}$. Some results are given in the following two theorems

Theorem 2. ([4]) Suppose that the relations $d_M > 0$, $Z_+ > 0$ and $Z_+ + Z_- = n-1$ hold. Then there exist real numbers $c_l > 0$, where $l = 1, 2, \ldots, Z_+$, for which the sequence $\{\lambda_k\}_{k=1}^\infty$ of positive eigenvalues of problem (1), (2) numbered in increasing order satisfies the asymptotics

$$\lambda_{l+kZ_+} = c_l q^k (1 + o(1)) \text{ as } k \to +\infty.$$

Theorem 3. ([4]) Suppose that the relations $d_M > 0$, $Z_- > 0$ and $Z_+ + Z_- = n-1$ hold. Then there exist real numbers $c_l > 0$, where $l = 1, 2, \ldots, Z_-$, for which the sequence $\{\lambda_{-k}\}_{k=1}^\infty$ of negative eigenvalues of problem (1), (2) numbered in decreasing order satisfies the asymptotics

$$\lambda_{-(l+kZ_-)} = -c_l q^k (1 + o(1)) \text{ as } k \to +\infty.$$

The boundary conditions (2) do not affect the asymptotics in theorems above. Any selfadjoint boundary conditions can be considered. However, Dirichlet boundary conditions allow us to consider more singular weights than $P \in L_2[0; 1]$ or equivalently $P' \in \overset{\circ}{W}{}_2^{-1}[0; 1]$. In [5] the weight P' from the space of the multipliers $\mathcal{M} := \mathcal{M}[\overset{\circ}{W}{}_2^1[0; 1], \overset{\circ}{W}{}_2^{-1}[0; 1]]$ was considered.

Theorem 4. ([5]) If $a_M|d_M| < 1$, then the generalized derivative P' of a self-similar function P of zero spectral order is compact multiplier. In this case the spectrum of the problem (1), (2) is discrete and its behavior is completely described by theorems 2,3.

Theorem 5. ([5]) If $a_M|d_M| = 1$, then the generalized derivative P' of a self-similar function P of zero spectral order is non-compact multiplier.

The complete description of the spectrum of the problem (1), (2) with non-compact multiplier as a weight is an open problem.

Let us show that the spectrum of the problem (1), (2) is absolutely continuous in the case of two-term self-similar functions $(n = 2)$ provided that $a_1 d_1 = 1$. For brevity, we put $a_1 = 1-a$, $a_2 = a$, $d_2 = d$. Recall that $d_1 = 0$ and $ad = 1$. In this notation, the formula (3) for the similarity operator G takes a simpler form:

$$G(P)(x) = \beta_1 I_{[0;1-a)}(x) + d \cdot \left(P\left(\frac{x-1+a}{a}\right) + \beta_2 \right) I_{[1-a;1]}(x).$$

Theorem 6. Suppose that $n = 2$, $ad = 1$ and the self-similar function P is non-constant. Then the spectrum of the problem (1), (2) is absolutely continuous and equals the closed interval

$$\left[\frac{(1-\sqrt{a})^2}{a(d\beta_1+\beta_2-\beta_1)}, \frac{(1+\sqrt{a})^2}{a(d\beta_1+\beta_2-\beta_1)} \right].$$

If $a_M|d_M| > 1$ and P is not equal to a constant, then P' does not belong to $\mathcal{M}$.

Conjecture. Let function P is n-term self-similar function of zero spectral order. Then the spectrum of the problem (1), (2) with a non-compact multiplier P' as a weight consists of $n-1$ segments of a continuous spectrum that can overlap. In each gap in continuous spectrum there can be no more than one eigenvalue and correspondingly no more than n-2 eigenvalues in total.

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Self-adjoint indefinite Laplacians

Konstantin Pankrashkin
(joint work with C. Cacciapuoti, A. Posilicano)

The mathematical study of metamaterials of negative refraction index is closely related to the study of differential operators of the form $-\nabla \cdot (h\nabla)$, where ∇ is the gradient and h is a real-valued function: usually one has $h > 0$ in the regions occupied by a dielectric and $h < 0$ in the regions occupied by a negative metamaterial [8]. In the simplest model situation one has an open set $\Omega \subset \mathbb{R}^n$, its open subset Ω_- (we then denote $\Omega_+ := \Omega \setminus \overline{\Omega_-}$) and the function h given by

$$h(x) = \begin{cases} 1, & x \in \Omega_+, \\ -\mu, & x \in \Omega_-, \end{cases}$$

where $\mu > 0$ is a parameter called *contrast*. One is then interested in the operator L acting as $u \mapsto -\nabla \cdot (h\nabla u)$ on the functions u satisfying the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. In view of the differential expression for L one can naturally expect its self-adjointness in $L^2(\Omega)$ on a suitable domain. A possible approach to find a self-adjoint realization is to consider the quadratic form

$$H_0^1(\Omega) \ni u \mapsto q(u) = \int_\Omega h|\nabla u|^2 dx,$$

and to find the associated operator, i.e. a self-adjoint operator L in $L^2(\Omega)$ whose domain $\mathcal{D}(L)$ is contained in $H_0^1(\Omega)$ and such that

$$\int_\Omega u\, Lu\, dx = q(u) \text{ for all } u \in \mathcal{D}(L).$$

The condition $h \geq c$ with some constant $c > 0$ would guarantee the existence and uniqueness of such an operator by the standard representation theorems. Nevertheless, this initial assumption on the sign of h does not hold anymore, and additional arguments are needed. In the present talk we discuss the self-adjointness of such operators under suitable assumptions on $\Omega_\pm$.

Namely, assume that both Ω and Ω_- are bounded domains in $\mathbb{R}^n$ with smooth boundaries and such that $\overline{\Omega_-} \subset \Omega$, which implies that $\Omega_+ := \Omega \setminus \overline{\Omega_-}$ is also a bounded domain with a smooth boundary, moreover,

$$\partial\Omega_+ = \partial\Omega \cup \Sigma, \quad \Sigma := \partial\Omega_- = \overline{\Omega_+} \cap \overline{\Omega_-}, \quad \partial\Omega \cap \Sigma = \emptyset.$$

Remark that the discontinuity of h along the *interface* Σ implies that the functions u from the domain of L should satisfy, in a suitable sense, a specific transmission condition along Σ in order to have a compensation of singularities guaranteeing $\nabla \cdot (h\nabla u) \in L^2(\Omega)$. We identify $L^2(\Omega)$ with $L^2(\Omega_+) \oplus L^2(\Omega_-)$ by $u \simeq (u_+, u_-)$, $u_\pm = u|_{\Omega_\pm}$, and denote

$$\begin{aligned}
\mathcal{D}_{\max} &:= \Big\{ u \in L^2(\Omega) : \, \Delta u_\pm \in L^2(\Omega_\pm)\,, \\
&\qquad u_+ = u_- \text{ and } \frac{\partial u_+}{\partial N_+} = \mu \frac{\partial u_-}{\partial N_-} \text{ on } \Sigma\,, \quad u_+ = 0 \text{ on } \partial\Omega \Big\}, \\
\mathcal{D}_s &:= \mathcal{D}_{\max} \cap H^s(\Omega \setminus \Sigma), \quad s \geq 0,
\end{aligned}$$

where $N_\pm$ stands for the unit normal on Σ pointing to the exterior of $\Omega_\pm$. Recall that for $u_\pm \in L^2(\Omega_\pm)$ with $\Delta u_\pm \in L^2(\Omega_\pm)$, the boundary traces on Σ and $\partial\Omega$ are defined in suitable negative Sobolev spaces using the duality with the help of the Green formula. We further denote by L the linear operator in $L^2(\Omega)$ acting on the domain $\mathcal{D}_{\max}$ by $(u_+, u_-) \mapsto (-\Delta u_+, \mu\Delta_-)$, and by S we denote its restriction to $\mathcal{D}_2$. The standard integration by parts shows that the operator S is symmetric. Moreover, it is known for a long time that for $\mu \neq 1$ one simply has $S = L$, and L is self-adjoint with compact resolvent [2, 5]. We are going to discuss the case $\mu = 1$.

Theorem. *Assume that $\mu = 1$, then the operator S is not closed but is essentially self-adjoint, and its unique self-adjoint extension is L.*

- *If $n = 2$, then the essential spectrum of L is $\{0\}$, and $\mathcal{D}(L) \not\subset \mathcal{D}_s$ for any $s > 0$.*
- *If $n \geq 3$, then $\mathcal{D}_1 \subset \mathcal{D}(L)$, and*
  - *if each connected component of Σ is strictly convex, then $\mathcal{D}(L) = \mathcal{D}_1$, and L has compact resolvent,*
  - *if a part of Σ is flat, then the essential spectrum of L contains the point 0, and $\mathcal{D}(L) \not\subset \mathcal{D}_s$ for any $s > 0$.*

The above theorem is proved in [4], and the proof combines the theory of boundary triples [3, 7] with some pseudodifferential tools. In particular, an important role is played by the subprincipal parts of suitably defined Dirichlet-to-Neumann maps on the both sides of Σ. We remark that the presence of a non-empty essential spectrum is a rather cutious fact as we deal with a differential operator in a bounded domain. Before this effect was only observed in very particular cases [1].

Let us formulate some open questions which can be viewed as a program for future study:

(1) In the above setting with $n \geq 3$ and $\mu = 1$, can the essential spectrum of L be larger than just one point? In particular, can it contain a non-empty interval?
(2) If the essential spectrum of L consists of a single point, can one estimate the accumulation rate of the eigenvalues to this point in geometric terms? For a particular configuration with separated variables, an exponential accumulation was obtained in [9].
(3) Can one develop an approach based on boundary triples for the case when the relative position of Ω_+ and Ω_- is different, i.e. without assuming $\overline{\Omega_-} \subset \Omega$?
(4) Can one develop an approach based on boundary triples for the case of a non-smooth interface Σ? Remark that a direct analog of the operator S is known to have non-zero defficiency indices for some particular geometries [2, 6].

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To the spectral theory of infinite quantum graphs

Mark Malamud

During the last two decades, quantum graphs became an extremely popular subject because of numerous applications in mathematical physics, chemistry and engineering. Indeed, the literature on quantum graphs is vast and extensive and there is no chance to give even a brief overview of the subject here. We only mention recent monographs [1], [2], with a comprehensive bibliography.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a (combinatorial) graph with finite or countably infinite sets of vertices $\mathcal{V}$ and edges $\mathcal{E}$. For two different vertices $u, v \in \mathcal{V}$, we shall write $u \sim v$ if there is an edge $e \in \mathcal{E}$ connecting u with v.

We investigate quantum graphs with infinitely many vertices and edges without the common restriction on the geometry of the underlying metric graph that there is a positive lower bound on the lengths of its edges. To simplify our considerations, we assume that the graph $\mathcal{G}$ is connected and there are no loops or multiple edges.

Our central result is a close connection between spectral properties of a quantum graph and the corresponding properties of a certain weighted discrete Laplacian on the underlying discrete graph. Emphasize that the biggest part of results are new even in the case $\inf_{e\in\mathcal{E}} |e| > 0$ while our main results are valid without this assumption, i.e. in the case

$$\inf_{e\in\mathcal{E}} |e| \geq 0. \tag{1}$$

Turning to a more specific problem, we need to make further assumptions on the geometry of a connected metric graph $\mathcal{G}$.

In what follows we assume that $\mathcal{G}$ is locally finite, that is, every vertex $v \in \mathcal{V}$ has finitely many neighbors. Moreover, there is a finite upper bound on the lengths of edges,

$$\sup_{e\in\mathcal{E}} |e| < \infty. \tag{2}$$

Let $\alpha\colon \mathcal{V} \to \mathbb{R}$ be given and equip every vertex $v \in \mathcal{V}$ with the so-called δ-type vertex condition:

$$\begin{cases} f \text{ is continuous at } v, \\ \sum_{e\in\mathcal{E}_v} f_e'(v) = \alpha(v) f(v), \end{cases} \tag{3}$$

Let us define the operator H_α as a closure of the operator H_α^0 given by

$$H_\alpha^0 = H_{\max}|\mathrm{dom}(H_\alpha^0),$$
$$\mathrm{dom}(H_\alpha^0) = \{f \in \mathrm{dom}(H_{\max}) \cap L_c^2(\mathcal{G})\colon f \text{ satisfies } (3),\ v \in \mathcal{V}\}. \tag{4}$$

Alongside the operator H_α we consider the following minimal difference operator h_α defined in $\ell^2(\mathcal{V})$ by

(5)
$$(\tau_{\mathcal{G},\alpha} f)(v) = \frac{1}{\sqrt{m(v)}}\left(\sum_{u\in\mathcal{V}} b(v,u)\Big(\frac{f(v)}{\sqrt{m(v)}} - \frac{f(u)}{\sqrt{m(u)}}\Big) + \frac{\alpha(v)}{\sqrt{m(v)}} f(v)\right), \quad v \in \mathcal{V},$$

where $b\colon \mathcal{V}\times\mathcal{V} \to [0,\infty)$ is given by

$$b(v,u) = \begin{cases} |e_{v,u}|^{-1}, & v \sim u, \\ 0, & v \not\sim u, \end{cases} \tag{6}$$

and the function $m\colon \mathcal{V} \to (0,\infty)$ is defined by

$$m\colon v \mapsto \sum_{e\in\mathcal{E}_v} |e|, \quad v \in \mathcal{V}. \tag{7}$$

More precisely, we define the operator h_α in $\ell^2(\mathcal{V})$ as the closure of the pre-minimal symmetric operator

$$h^0_\alpha\colon \mathrm{dom}(h^0_\alpha) \to \ell^2(\mathcal{V}), \qquad f \mapsto \tau_{\mathcal{G},\alpha} f, \qquad \mathrm{dom}(h^0_\alpha) := \ell^2_c(\mathcal{V})$$

Notice that the assumption that $\mathcal{G}$ is locally finite ensures that h^0_α is well defined since $\tau_{\mathcal{G},\alpha} f \in \ell^2(\mathcal{V})$ for every $f \in \ell^2_c(\mathcal{V})$.

We also need another discrete Laplacian. Namely, in the weighted Hilbert space $\ell^2(\mathcal{V}; m)$ consider the minimal operator defined by the difference expression

$$(\widetilde{\tau}_{\mathcal{G},\alpha} f)(v) := \frac{1}{m(v)}\left(\sum_{u\in\mathcal{V}} b(v,u)(f(v) - f(u)) + \alpha(v) f(v)\right), \quad v \in \mathcal{V}. \tag{8}$$

In the following we shall use h_α to denote the closures of both operators. Now we are ready to formulate the main result.

Theorem 1 ([3])**.** *Assume that the graph $\mathcal{G}$ is connected, there are no loops or multiple edges, and condition* (2) *holds. Let $\alpha : \mathcal{V} \to \mathbb{R}$ and let H_α be a closed symmetric operator associated with the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and equipped with the δ-type coupling conditions* (3) *at the vertices. Let also h_α be the discrete Laplacian defined either by* (5) *in $\ell^2(\mathcal{V})$ or by* (8) *in $\ell^2(\mathcal{V}; m)$, where the functions $m\colon \mathcal{V} \to (0,\infty)$ and $b\colon \mathcal{V}\times\mathcal{V} \to [0,\infty)$ are given by* (7) *and* (6)*, respectively. Then:*

(i) *The deficiency indices of H_α and h_α are equal and*

$$n_+(H_\alpha) = n_-(H_\alpha) = n_\pm(h_\alpha) \le \infty. \tag{9}$$

In particular, H_α is self-adjoint if and only if h_α is self-adjoint. Assume in addition that H_α (and hence h_α) is self-adjoint. Then:

(ii) *The operator H_α is lower semibounded if and only if the operator h_α is lower semibounded.*

(iii) *The operator H_α is nonnegative (positive definite) if and only if the operator h_α is nonnegative (positive definite).*

(iv) *The total multiplicities of negative spectra of H_α and h_α coincide,*

$$\kappa_-(H_\alpha) = \kappa_-(h_\alpha). \tag{10}$$

(v) *Moreover, for the negative parts* H_α^- *and* h_α^- *of the operators* H_α *and* h_α*, respectively, the following equivalence holds*

$$H_\alpha^- \in \mathcal{S}_p(L^2(\mathcal{G})) \Longleftrightarrow h_\alpha^- \in \mathcal{S}_p(\ell^2(\mathcal{V};m)), \qquad p \in (0,\infty]. \tag{11}$$

In particular, negative spectra of H_α *and* h_α *are discrete (with the only accumulation point zero) simultaneously.*

(vi) *If* $h_\alpha^- \in \mathcal{S}_\infty(\ell^2(\mathcal{V};m))$*, then for each* $p \in (0,\infty)$ *the following equivalence holds*

$$\lambda_j(H_\alpha) = j^{-p}(a + o(1)) \iff \lambda_j(h_\alpha) = j^{-p}(b + o(1)), \tag{12}$$

as $j \to \infty$*, where either* $ab \neq 0$ *or* $a = b = 0$*.*

(vii) *If in addition* h_α *is lower semibounded, then* $\inf \sigma_{ess}(H_\alpha) > 0$ ($\inf \sigma_{ess}(H_\alpha) = 0$) *if and only if* $\inf \sigma_{ess}(h_\alpha) > 0$ ($\inf \sigma_{ess}(h_\alpha) = 0$)*.*

(viii) *The spectrum of* H_α *is purely discrete if and only if the number* $\#\{e \in \mathcal{E} : |e| > \varepsilon\}$ *is finite for every* $\varepsilon > 0$ *and the spectrum of the operator* h_α *is purely discrete.*

(ix) *Let* $\widetilde{\alpha} : \mathcal{V} \to \mathbb{R}$ *and let* H_α *be the corresponding Hamiltonian and let* $h_{\widetilde{\alpha}}$ *be the corresponding difference operator. Then for every* $p \in (0,\infty]$ *the following equivalence holds*

$$(H_\alpha - \lambda)^{-1} - (H_{\widetilde{\alpha}} - \lambda)^{-1} \in \mathcal{S}_p(L^2(\mathcal{G})) \Longleftrightarrow (h_\alpha - \lambda)^{-1} - (h_{\widetilde{\alpha}} - \lambda)^{-1} \in \mathcal{S}_p(\ell^2(\mathcal{V})).$$

The proof of this result as well as its numerous applications can be found in [3]. Note that for Schrodinger operators with δ-interactions on a discrete subset $X = \{x_n\}_{n=1}^\infty$ of the half-line (line) Theorem 1 was proved in [4]. In this case δ-type coupling conditions (3) at the points $\{x_n\}$ turn into the conditions of δ-interactions and the discrete Laplacian h_α turns into the minimal operator generated in $l^2(\mathbb{N})$ by a special the Jacobi matrix.

Using the connection described in Theorem 1 together with spectral theory of (unbounded) discrete Laplacians on infinite graphs, it is proved in [3] a number of new results on spectral properties of quantum graphs. For instance, using Theorem 1 it is proved several self-adjointness results on Hamiltonians H_α including a Gaffney type result. Several spectral estimates (bounds for the bottom of spectra and essential spectra of quantum graphs, CLR-type estimates) are also investigated in [3].

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Reporter: Valentina Franceschi

Participants

Prof. Dr. Ugo Boscain
Laboratoire Jacques-Louis Lions
Sorbonne Université
4, Place Jussieu
75005 Paris Cedex
FRANCE

Dr. Valentina Franceschi
Laboratoire Jacques-Louis Lions
Sorbonne Université
4, Place Jussieu
75005 Paris Cedex
FRANCE

Dr. Maryna Kachanovska
Laboratoire POEMS
CNRS, ENSTA, INRIA
828, Boulevard des Maréchaux
91762 Palaiseau Cedex
FRANCE

Prof. Dr. Matthias Keller
Institut für Mathematik
Universität Potsdam
Postfach 601553
14415 Potsdam
GERMANY

Dr. Aleksey Kostenko
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Wien
AUSTRIA

Prof. Dr. Daniel Lenz
Institut für Mathematik
Friedrich-Schiller-Universität Jena
07743 Jena
GERMANY

Prof. Dr. Mark Malamud
Rossiyskiy Universitet Druzhby Narodov
S. M. Nikol'skii Mathematical Institute
Miklukho-Maklaya str. 6
117 198 Moscow
RUSSIAN FEDERATION

Dr. Robert W. Neel
Department of Mathematics
Lehigh University
Chandler-Ullmann Hall
17 Memorial Drive East
Bethlehem, PA 18015-3174
UNITED STATES

Noema Nicolussi
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Wien
AUSTRIA

Prof. Dr. Konstantin Pankrashkin
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Bâtiment 307
91405 Orsay Cedex
FRANCE

Dr. Dario Prandi
CNRS, L2S, Centrale Supélec
Université Paris-Saclay
3, Rue Joliot-Curie
91192 Gif-sur-Yvette Cedex
FRANCE

Dr. Luca Rizzi
Institut Fourier
Université Grenoble Alpes
100, Rue des Maths
38610 Gières
FRANCE

Dr. Adrien Semin
Fachbereich Mathematik
Technische Universität Darmstadt
Rm S4/1022
Dolivostrasse 15
64293 Darmstadt
GERMANY

Prof. Dr. Anton Thalmaier
Unite de Recherche en Mathématiques
Université du Luxembourg
Campus Belval, Maison du Nombre
6, Avenue de la Fonte
4364 Esch-sur-Alzette
LUXEMBOURG

Dr. Igor A. Sheipak
Department of Mathematics
Lomonosov Moscow State University
Vorobyovy gory
119 899 Moscow
RUSSIAN FEDERATION

Prof. Dr. Wolfgang Woess
Institut für Diskrete Mathematik
Technische Universität Graz
Steyrergasse 30
8010 Graz
AUSTRIA