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Groups, Dynamics, and Approximation

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Abstract. The workshop covered a wide range of topics, with emphasis on Geometric Group Theory, Ergodic Theory and links with Functional Analysis on the one hand and Mathematical Logic on the other. The goal of the workshop was to bring together experts working in various fields, to foster interaction between them and to present some of the recent breakthroughs.

Mathematics Subject Classification (2010): 20xx, 37xx, 46xx, 03xx.

Introduction by the Organizers

The workshop *Groups, Dynamics, and Approximation* organized by Emmanuel Breuillard, Alex Furman, Nicolas Monod and Andreas Thom, brought together leading mathematicians working at the interface between geometric group theory, ergodic theory and operator algebras. Meant as a continuation of the workshops *Group Theory, Measure, and Asymptotic Invariants* (2013) and *Measured Group Theory* (2016) this "transversal workshop" covered a wide range of topics, this time putting more emphasis on Geometric Group Theory, Ergodic Theory and links with Functional Analysis on the one hand and Mathematical Logic on the other. The goal of the workshop was to bring together experts working in various fields, to foster interaction between them and to present some of the recent breakthroughs. The topics of the talks included: sofic groups and approximation properties of infinite groups (Lubotzky, Kun, Glebsky), word maps on finite or compact groups (Puder, Schneider), lattices and approximate lattices in Lie groups (Avni, Boutonnet, Hartnick, Machado), Benjamini-Schramm convergence, invariant random subgroups and orbit equivalence (Abert, Gaboriau, Hutchcroft), ergodic theory of group actions (Tsankov, Bjorklund, Hochman), von Neumann algebras and measure equivalence (Popa, Peterson), representation theory of infinite groups (Rosendal, Gerasimova, Ozawa, de la Salle).

Kazhdan's property (T) for countable groups was featured in several talks. Narutaka Ozawa's characterization of property (T) in terms of the solution to a set of quadratic inequalities in the group algebra over the reals has opened the way to a new method for establishing property (T) using semi-definite programming and computer help. His talk presented the recent advances that have culminated with the discovery that the automorphism groups of the free groups in 5 or more letters have property (T). Mikael de la Salle discussed how to extend this characterization to generalizations of property (T) for ℓ^p spaces and other Banach spaces in particular in the setting of groups acting on 2-dimensional simplicial complexes. Tom Hutchcroft described his recent work with Gabor Pete establishing Gaboriau's cost conjecture for property (T) groups using new ideas from percolation theory. Jesse Peterson discussed recent work in which a new notion, that of von Neumann equivalence, is introduced for discrete groups, generalizing Gromov's measure equivalence relation and for which property (T) is an invariant.

The word *approximation* in the title refers to the powerful interplay between the finite and the infinite in group theory and in dynamics. Highlighting analogies between 3-manifold groups and Galois groups of function fields, Mark Shusterman talked about the profinite structure of the étale fundamental group of a curve over a finite field and his proof that they are topologically finitely presented. The idea of approximation in infinite group theory is featured in particular in the recent body of works around the notion of soficity and its extensions to more general approximation schemes. A wide open problem is to determine whether or not every finitely generated group is sofic. Alex Lubotzky in his lecture described some (residually-finite-by-finite) counter-examples to the analogue of this question for certain other approximation schemes (defined in terms of the Schatten norms). Lev Glebsky presented recent work in which he establishes that, on the contrary, extensions of residually finite groups are always weakly sofic. Gabor Kun discussed the sofic approximations of certain non-amenable groups.

Several talks were devoted to recent advances on classical questions about the ergodic theory of group actions. Michael Hochman discussed linear actions on the torus and higher dimensional generalizations of Rudolph's theorem regarding Furstenberg's celebrated $\times 2 \times 3$-conjecture for actions on the circle. Michael Bjorklund described joint work with Vaes and Koslov regarding systems of independent by not identically distributed random variables indexed by a discrete group and the relation between the positivity of the first Betti number and the existence of an ergodic Maharam extension. Todor Tsankov presented his work with Glasner, Weiss and Zucker about Bernoulli disjointness for discrete groups and their solution to the "Ellis problem".

They were also several talks with a more operator algebraic flavor. Sorin Popa discussed two old problems in Von Neumann algebras, the free group factor problem and the single generation problem. Rémi Boutonnet described joint work with

Cyril Houdayer in which a vast generalization of Margulis' normal subgroup theorem is established for higher rank lattices in the form of a theorem asserting that every weakly mixing unitary representation of such a group weakly contains the regular representation. Maria Gerasimova discussed the unitarizability problem for discrete groups and its relation with the Littlewood exponent of a discrete group.

The notion of Invariant Random Subgroup (IRS) introduced a decade ago by Abert, Glasner and Virag, was also a theme appearing in several talks. Gaboriau discussed IRS of surface groups and fundamental groups of higher dimensional manifolds in relation to the cost of measure preserving actions. Abert described work on the Berry conjecture and its relation with Benjamini-Schramm convergence. Anna Erschler discussed the traveling salesman problem for groups and related problems.

An interesting feature of the workshop was the appearance in several talks of ideas from Mathematical Logic and Model Theory. Christian Rosendal presented a solution to an old problem on the automatic continuity of homomorphisms between Polish groups and its significance in Logic with respect to the axiom of Choice. Nir Avni explained how the first order theory of certain lattices in semisimple Lie groups can determine them entirely up to isomorphism and stressed the importance of the model-theoretic notion of bi-interpretability for this problem. Simon Machado described a classification of infinite approximate subgroups of solvable Lie groups based on the notion of good model and a variation of Hrushovski's stabilizer theorem for approximate groups. His talked was preceded by a presentation by Tobias Hartnick surveying the recent progress made about approximate lattices in Lie groups. These are non-commutative analogues of quasi-crystals, a notion he recently introduced with Michael Bjorklund.

The workshop included two excellent talks given by Ph.D. students (J. Schneider and S. Machado) and a problem session (moderated by E. Breuillard) in which many participants presented open questions and new research directions.

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Workshop: Groups, Dynamics, and Approximation

Table of Contents

Abstracts

Continuity of universally measurable homomorphisms

Christian Rosendal

The question of whether a measurable homomorphism between topological groups is continuous has a long and illustrious history. For example, in the very first issue of Fundamenta Matematicae, no less than three papers by S. Banach, W. Sierpiński and H. Steinhaus are dedicated to the question of continuity of Lebesgue measurable functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying Cauchy's functional equation

$$f(x+y) = f(x) + f(y).$$

Banach [1] and Sierpiński [12] each show that such f must be continuous, which is also established by M. Fréchet [5], while Steinhaus [14] expands on the methods of Sierpinski [11, 12] to show that, if $A \subseteq \mathbb{R}$ is a Lebesgue measurable set of positive measure, then $A - A$ contains 0 in its interior. Steinhaus' result is subsequently generalized to arbitrary locally compact groups by A. Weil [15], i.e., if A is a Haar measurable set of positive Haar measure in a locally compact group, then AA^{-1} is an identity neighborhood. In turn, this implies by a simple argument that every Haar measurable homomorphism between locally compact Polish groups is continuous.

Of course, as shown by Weil [15], in non-locally compact groups there is no notion of translation invariant σ-finite measure and, in particular, no notion of Haar measurable set. Instead, in a Polish group G, one may consider the *universally measurable* sets, i.e., sets A that are measurable with respect to every Borel probability measure μ on G. One particular reason for their interest is the construction by G. Mokobodzki [8, 7] and J. P. R. Christensen [4] of medial limits under CH.

In connection with this, Christensen [2] studies the question of whether every universally measurable homomorphism between Polish groups is continuous. He shows the following Steinhaus type principle, which turns out to be central to our study.

Theorem. *Suppose $G = \bigcup_{i=1}^{\infty} A_i$ is a covering of a Polish group G by universally measurable sets A_i and U is an identity neighborhood. Then there are a finite set $F \subseteq U$ and some i so that*

$$\bigcup_{g\in F} gA_iA_i^{-1}g^{-1}$$

is an identity neighborhood.

From this he immediately deduces that every universally measurable homomorphism $G \xrightarrow{\pi} H$ between Polish groups is continuous provided H is *SIN*, i.e., admits a bi-invariant compatible metric. In particular, this applies if either G or H is abelian and also provides an alternative proof of A. Douady's result [10] that every universally measurable linear operator between Banach spaces is continuous. However, the general problem has remained open thus far.

> Is every universally measurable homomorphism $G \xrightarrow{\pi} H$ between Polish groups continuous?

Partially motivated by this and by applications to differentiability of Lipschitz mappings, Christensen [3, 4] and other authors have developed a theory of Haar null sets and related notions of smallness in Polish groups. One of the principal aims of this theory is to find robust notions of smallness satisfying a variant of Steinhaus' theorem. For example, in [13], S. Solecki studies *left Haar null sets* and isolates a class of Polish groups G said to be *amenable at* 1 for which every universally measurable homomorphism $G \xrightarrow{\pi} H$ into an arbitrary Polish group H is continuous. In another direction, in [9] we show that the above problem has a positive answer when H is locally compact or non-Archimedean. In this talk, we present the general solution to the problem.

Theorem. *Let $G \xrightarrow{\pi} H$ be a universally measurable homomorphism from a Polish group G to a separable topological group H. Then π is continuous.*

Somewhat surprisingly, the proof proceeds by showing that the conclusion of Christensen's theorem above is already enough for the general solution and thus entirely circumvents any further considerations of universal measurability.

Theorem. *Let $G \xrightarrow{\pi} H$ be a homomorphism from a Polish group G to a separable topological group H. Assume also that, for all identity neighborhoods $U \subseteq G$ and $V \subseteq H$, there is a finite set $F \subseteq U$ so that*

$$\bigcup_{f \in F} f \cdot \pi^{-1}(V) \cdot f^{-1}$$

is a identity neighborhood in G. Then π is continuous.

For this reason, our proof also allows us to address a different but related question of logic, namely the strength of the existence of a discontinuous homomorphism between Polish groups. Therefore, the discussion that follows is relative to ZF+DC, i.e., Zermelo–Fraenkel–Skolem set theory without the full axiom of choice, but only with the principle of dependent choice. This latter principle is sufficient to establish the Baire category theorem and treat basic concepts of analysis.

Various results of the literature indicate that some amount of AC is needed to construct discontinuous homomorphisms between Polish groups. For example, P. Larson and J. Zapletal [6] show that, if there is a discontinuous additive homomorphism between two separable Banach spaces, then there is a *Vitali set*, i.e., a set $T \subseteq \mathbb{R}$ intersecting every translate of $\mathbb{Q}$ in a single point. However, without a linear structure on the groups, little is known.

In the following, for $k \geq 2$, consider the profinite group $\prod_{n=1}^{\infty} \mathbb{Z}/k\mathbb{Z}$. The *Hamming graph* on $\prod_{n=1}^{\infty} \mathbb{Z}/k\mathbb{Z}$ is then the graph with vertex set $\prod_{n=1}^{\infty} \mathbb{Z}/k\mathbb{Z}$ and so that two elements $\alpha, \beta \in \prod_{n=1}^{\infty} \mathbb{Z}/k\mathbb{Z}$ form an edge if they differ in exactly one coordinate $n \in \mathbb{N}$. Also, by $\chi(k)$ we denote the *chromatic number* of the Hamming graph on $\prod_{n=1}^{\infty} \mathbb{Z}/k\mathbb{Z}$, that is the smallest cardinality κ so that there is a graph coloring $c \colon \prod_{n=1}^{\infty} \mathbb{Z}/k\mathbb{Z} \to \kappa$, that is, so that neighboring vertices get different colors under c. Since the Hamming graph on $\prod_{n=1}^{\infty} \mathbb{Z}/k\mathbb{Z}$ has cliques of size k, we always have $\chi(k) \geq k$. Conversely, if there is a Vitali set, then the Hamming graph has chromatic number $\chi(k) = k$ for all $k \geq 2$. Also, if $\chi(k) = k$ for some k, then $\chi(k^n) = k^n$ for all $n \geq 1$. Similarly, if just some $\chi(k)$ is finite, then all the chromatic numbers $\chi(k)$ are finite.

If $G \xrightarrow{\pi} H$ is a homomorphism between Polish groups, we define a closed subgroup of H by

$$N = \bigcap_{V} \overline{\pi[V]},$$

where V ranges over identity neighborhoods in G. Then N gauges the discontinuity of π. Indeed, assuming that $\pi[G]$ is dense in H, then N is normal in H and the induced homomorphism

$$G \xrightarrow{\tilde{\pi}} H/N$$

has closed graph and thus is continuous.

Theorem. *In every model of ZF+DC, one of the following conditions hold.*

(1) Every homomorphism between Polish groups is continuous,
(2) the chromatic number $\chi(k)$ is finite for all $k \geq 2$ and, if $G \xrightarrow{\pi} H$ is a homomorphism between Polish groups, then N is compact and connected,
(3) for infinitely many $k \geq 2$, we have $\chi(k) = k$ and, if $G \xrightarrow{\pi} H$ is a homomorphism between Polish groups, then N is compact,
(4) there is a Vitali set.

In the above theorem, we see that the conclusions about continuity of homomorphisms weaken as we go from (1) to (4), while, on the other hand, the graph theoretical conclusions strengthen. For example, if (2) holds and H is a Polish group without proper compact connected subgroups, say H is non-Archimedean, then every homomorphism from a Polish group into H must have $N = \{1\}$ and thus is continuous. Similarly, if (3) holds, then every linear operator between two Banach spaces is continuous.

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On IRSs of surface groups and higher dimensional manifolds groups

Damien Gaboriau
(joint work with Alessandro Carderi, Pierre Fima, and François Le Maître)

The first goal of my talk is to explain that surface groups have plenty of IRSs. This is joint work with A. Carderi, P. Fima and F. Le Maître [3].

The fundamental group Γ of an aspherical compact surface of genus g is known to have $\text{cost}(\Gamma) = 2g - 1$ (resp. $\text{cost}(\Gamma) = g - 1$ when non-orientable) and fixed price: all its free probability measure preserving (p.m.p.) actions have the same cost [5]. Our first result is that, for non-amenable Γ, the cost of any p.m.p. *non-free* action of Γ is *strictly less* than $\text{cost}(\Gamma)$.

Let $\mathcal{R}$ is an ergodic p.m.p. standard equivalence relation (on the atomless standard probability space). This is just the "orbit equivalence relation" of some ergodic p.m.p. action of some countable (uninteresting) group G. We show: If $\text{cost}(\mathcal{R}) < \text{cost}(\Gamma)$, then there exist uncountably many highly faithful actions of Γ that define $\mathcal{R}$, and such that Γ is dense in the full group $[\mathcal{R}]$ and thus are totally non-free and highly transitive on the orbits. The natural map (given by conjugation action) from ergodic atomless IRSs of Γ to the ergodic p.m.p. equivalence relations of cost strictly less than $\text{cost}(\Gamma)$ is thus onto with uncountable fibers. This extends to surface groups certain results about the IRSs of the free groups from [2, 4, 7].

In the second part of my talk, I introduce a family of invariants of ℓ^2 type for IRSs: the *relative-L^2-Betti numbers*. More precisely, for each free-cocompact simplicial Γ-complex L, every IRS ν of Γ gets an ℓ^2-Betti number in every degree d: $\beta_d^{(2)}(\nu, L)$.

If for instance, assume the IRS ν is given by a p.m.p. action of Γ on the standard probability space (X, μ) as the distribution of stabilizers via the map $x \mapsto \mathrm{Stab}_\Gamma(x)$. Then $\beta_d^{(2)}(\nu, L)$ coincides with the L^2-Betti numbers of the laminated space $\beta_d^{(2)}(\Gamma\backslash(X \times L))$ (see [6, 1]).

This family of invariants permits to distinguish uncountably many different IRSs for certain free products by making use of a joint theorem with N. Bergeron [1, Theorem 4.1]. For instance if Γ is a residually finite group which is the fundamental group of a closed orientable d-manifold decomposable as a connected sum $M\#N$ of two manifolds of infinite fundamental groups. This works more generally, if Γ is any residually finite free product $\Gamma = G * H$ of infinite groups where G is finitely presented and H finitely generated.

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Matrix group integrals, surfaces, and mapping class groups

Doron Puder

(joint work with Michael Magee)

Since the 1970s, physicists and mathematicians who study random matrices in the standard models of GUE or GOE, are aware of intriguing connections between integrals of such random matrices and enumeration of graphs on surfaces. We establish a new aspect of this theory: For random matrices sampled from classical matrix groups such as U_n or O_n. The group structure of these matrices allows us to go further and find surprising algebraic quantities hidden in the values of these integrals.

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A class of nonamenable groups admitting no sofic approximation by expander graphs

Gábor Kun
(joint work with Andreas Thom)

A sequence of finite labeled graphs is locally convergent if for every r the isomorphism class of a rooted r-ball centered at a vertex chosen uniformly at random converges in distribution. A finitely generated group is called **sofic** if any of its labeled Cayley graphs admits a sofic approximation, this is, a local approximation by finite labeled graphs. Sofic groups were introduced by Gromov [4], see also Weiss [12]. Quite a number of classical conjectures about groups and group rings not known in general are known to hold for the class of sofic groups: Gottschalk's Conjecture [4], Kaplansky's Direct Finiteness Conjecture [1, 3], Connes' Embedding Conjecture [2], and the Kervaire–Laudenbach Conjecture [9] and its generalizations [6, 8]. For more on sofic groups see [9, 11]. It is a major open problem if every group is sofic, though it is widely believed that non-sofic groups exist. In general, we do not know much about sofic approximations. Schramm proved that the sofic approximation of an amenable group is hyperfinite [10]. On the other end of the spectrum, the first author proved Bowen's conjecture that the sofic approximation of a Kazhdan's property (T) group is essentially a vertex disjoint union of expander graphs [7]. We will build on this work in order to understand almost automorphisms of sofic approximations of Kazhdan groups: This allows us to prove inapproximability results about the direct product of a Kazhdan group and another group which is not LEF. Every finitely presented LEF group is residually finite. The first main result of this paper is the following.

Theorem 1. *Let Γ be a countable Kazhdan group and Δ a finitely generated group. Let S_Γ and S_Δ be finite generating sets of Γ and Δ. Consider a sofic approximation of $\Gamma \times \Delta$ with respect to the generating set $S_\Gamma \cup S_\Delta$. If the edges with labels in S_Γ induce an expander sequence, then Δ is* LEF*. In particular, if Δ is finitely presented then it is residually finite.*

Remark. *Note that for every sofic approximation of Γ the edges with labels in S_Γ induce a graph that is essentially a vertex disjoint union of expander graphs. The theorem requires somewhat more, i.e., that it is (essentially) an expander graph.*

As a consequence of Theorem 1, we show that certain group actions are not approximable by finite labeled graphs in the local-global sense [5]. Moreover, these actions do not weakly contain any ultra-product of a finite sequence of graphs, i.e., this half of the local-global convergence already fails.

Theorem 2. *Let Γ be a countable Kazhdan group and Δ a finitely generated group, which is not* LEF*. Consider an almost free, probability measure preserving action of $\Gamma \times \Delta$ on a probability measure space such that the restriction of the action to Γ is ergodic. Then this action does not weakly contain any sequence of finite graphs. In particular, it is not a local-global limit of finite graphs.*

The simplest example of such an action is the Bernoulli shift of $\Gamma \times \Delta$, this satisfies the conditions of the theorem.

Proof of Theorem 2. : Consider a sofic approximation of $\Gamma \times \Delta$. We may assume that its edges are labeled by $S_\Gamma \cup S_\Delta$. Restricting labels to S_Γ and using the main result of [7], we obtain a vertex disjoint union of expander graphs after making irrelevant changes to the labels, since it is a sofic approximation of Γ. Theorem 1 implies that it can not be an expander graph sequence. On the other hand, the action of Γ is strongly ergodic, since it is ergodic and Γ is a Kazhdan group. However, if the graph sequence was weakly contained by a strongly ergodic action there would be only one large component. This is a contradiction. □

Theorem 3. *Let Γ be a countable Kazhdan group and Δ a finitely generated amenable group. Assume that the group $\Gamma \times \Delta$ admits a sofic approximation by a sequence of expander graphs. Then Δ is* LEF. *In particular, if Δ is finitely presented then it is residually finite.*

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Kazhdan's property (T) and semidefinite programming

Narutaka Ozawa
(joint work with Marek Kaluba and Piotr W. Nowak)

1. Introduction

Amenability and Kazhdan's property (T) are the most important properties in analytic group theory. They are generalized notions of being finite (but into the opposite directions). Groups with property (T) have a number of applications in pure and applied mathematics. It has long been thought that groups with property (T) are rare among the "naturally-occurring" groups, but it may not be so and it may even be possible to observe this by extensive computer calculations. I will present a computer-assisted (but mathematically rigorous) method of confirming property (T) which is based on semidefinite programming with some operator algebraic input. I will report the progress recently made by M. Kaluba, P. W. Nowak, and myself [5] and by M. Kaluba, D. Kielak, and P. W. Nowak [4]. It confirms property (T) of $\mathrm{Aut}(\mathbf{F}_n)$ for $n = 5$ in [5] and $n \geq 6$ in [4], leaving the case $n = 4$ unsettled. This solves a well-known problem ([6, 10.3], [7], [1, 7.1]) in geometric group theory.

2. Kazhdan's property (T)

A (discrete) group Γ is said to have Kazhdan's property (T) if for any orthogonal representation (π, H), any almost Γ-invariant vector is close to a Γ-invariant vector: $\exists S \subseteq \Gamma$ finite and $\exists \kappa = \kappa(S) > 0$ which satisfy

$$\forall (\pi, H)\ \forall v \in H \text{ one has } \mathrm{dist}(v, H^\Gamma) \leq \kappa^{-1} \max_{s\in S} \|v - \pi(s)v\|.$$

If Γ has property (T), then S as above has to be a generating subset of Γ and so Γ is finitely generated; Moreover, for any finite generating subset S, there is a Kazhdan constant $\kappa = \kappa(S)$ that satisfies the above condition. Property (T) inherits to quotient groups and finite-index subgroups. Property (T) is similarly defined for a locally compact groups and a lattice Γ in a locally compact group G has property (T) if and only if G has it.

A group Γ is amenable if there are almost invariant vectors in $\ell_2\Gamma$: $\exists v_n \in \ell_2\Gamma$ such that $\|v_n\| = 1$ and $\|v_n - sv_n\| \to 0$ for every $s \in \Gamma$. Abelian groups (or more generally groups with subexponential growth) are amenable. Since $(\ell_2\Gamma)^\Gamma \neq 0$ only if Γ is finite, any group that satisfy both amenability and property (T) is finite. D. Kazhdan (1967) defined property (T) and proved that every simple connected Lie group with real rank ≥ 2 (e.g., $\mathrm{SL}(d \geq 3, \mathbb{R})$) has property (T) and so every lattice of it is finitely generated and has finite abelianization.

3. Algebraic characterization of property (T)

Noncommutative real algebraic geometry is a subject that deals with equations and inequalities in noncommutative algebras (over real or complex). Recall Artin's theorem (Hilbert's 17th problem) from the classical real algebraic geometry: If a polynomial f in $\mathbb{R}[x_1,\ldots,x_d]$ satisfies $f\geq 0$ on $\mathbb{R}^d$, then there are rational polynomials $g_1,\ldots,g_n$ in $\mathbb{R}(x_1,\ldots,x_d)$ such that $f=\sum_i g_i^2$. This theorem becomes trivial if one passes to the completion of the polynomial algebra, which is the continuous function algebra. Likewise in noncommutative real algebraic geometry, we solve inequalities in the completion (which is a C*-algebra) and bring down the solution to the original algebra.

Let a group Γ be given. We consider the real group algebra $\mathbb{R}[\Gamma]$ with the involution $(\sum_t \alpha_t t)^* = \sum_t \alpha_t t^{-1}$, where $\alpha_t\in\mathbb{R}$ are all zero but finitely many. The positive cone of Hermitian squares is given by

$$\Sigma^2\mathbb{R}[\Gamma] \coloneqq \{\textstyle\sum_i g_i^* g_i \,|\, g_i\in\mathbb{R}[\Gamma]\} = \{\textstyle\sum_{x,y} P_{x,y}x^{-1}y \,|\, P\in \mathbf{M}_\Gamma^+(\mathbb{R})\}.$$

Here $\mathbf{M}_\Gamma^+(\mathbb{R})$ denotes the set of finitely supported positive definite matrices indexed by Γ. We will write $a\geq b$ if $a-b\in\Sigma^2\mathbb{R}[\Gamma]$. The full group C*-algebra C$^*\Gamma$ is the universal enveloping completion of $\mathbb{R}[\Gamma]$ with respect to orthogonal representations of Γ (= $*$-representations of $\mathbb{R}[\Gamma]$ on Hilbert spaces). We assume Γ is generated by a finite symmetric subset S and consider the non-normalized Laplacian

$$\Delta \coloneqq \tfrac{1}{2}\textstyle\sum_{s\in S}(1-s)^*(1-s) = |S| - \sum_{s\in S} s \in \Sigma^2\mathbb{R}[\Gamma].$$

For any orthogonal representation (π,H) and $v\in H$, one has $\pi(\Delta)v=0$ iff v is Γ-invariant, and $\pi(\Delta)v\approx 0$ iff v is almost Γ-invariant.Hence it follows from the spectral theory that Γ has property (T) iff Δ has a spectral gap: $\exists\varepsilon>0$ such that $\mathrm{Sp}(\Delta)\subseteq\{0\}\cup[\varepsilon,\infty)$ in C$^*\Gamma$. On the other hand, by the spectral mapping theorem, one has $\mathrm{Sp}(\Delta)\subseteq\{0\}\cup[\varepsilon,\infty)$ iff $\Delta^2-\varepsilon\Delta\geq 0$ in C$^*\Gamma$.

Theorem ([9])**.** *A finitely generated group $\Gamma=\langle S\rangle$ has property (T) iff $\exists\varepsilon>0$ such that $\Delta^2-\varepsilon\Delta\geq 0$ in $\mathbb{R}[\Gamma]$. If this is the case, one has $\kappa(S)^2\geq 2|S|^{-1}\varepsilon$.*

This theorem tells that property (T) is semidecidable (an observation made earlier by a different method by L. Silberman), i.e., there is an algorithm which stops iff Γ has property (T). However, property (T) is not decidable, i.e., there is no a priori estimate of the stopping time.

4. Semidefinite programming

The following algorithm was first implemented by T. Netzer and A. Thom in [8]. For the computer verification of property (T) of a given group Γ, we fix a finite subset $E\subseteq\Gamma$ and restrict the search area from $\mathbf{M}_\Gamma^+(\mathbb{R})$ to $\mathbf{M}_E^+(\mathbb{R})$. This results in the semidefinite programming (SDP):

$$\begin{aligned}&\texttt{maximize } \varepsilon\\ &\texttt{subj. to } \exists P\in\mathbf{M}_E^+(\mathbb{R}) \texttt{ such that } \Delta^2-\varepsilon\Delta = \sum_{x,y\in E} P_{x,y}x^{-1}y \texttt{ in } \mathbb{R}[\Gamma]\end{aligned}$$

If $\varepsilon>0$, then we conclude that Γ has property (T), but since we have restricted the search area, the converse need not hold. By the way, we will ignore in this text the word problem of identifying elements in Γ.

Suppose that a hypothetical solution (ε_0,P_0) to the above SDP is given. We describe here how to ensure existence of a rigorous solution to the inequality $\Delta^2-\varepsilon\Delta\in\Sigma^2\mathbb{R}[\Gamma]$ out of it. We factorize P_0 as $P_0\approx Q^TQ$ for some Q with

$Q\mathbf{1} = 0$. We utilize the fact that $\|f\|\Delta - f \in \Sigma^2\mathbb{R}[\Gamma]$ for every $f \in \mathbb{R}[\Gamma]$, $f = f^*$ and $\sum_x f(x) = 0$. Here $\|f\|$ is a weighted ℓ_1-norm which is explicitly calculable. Thus property (T) of Γ is guaranteed if one sees

$$\|\Delta^2 - \varepsilon_0\Delta - \textstyle\sum_{x,y}(Q^TQ)_{x,y}x^{-1}y\| < \varepsilon_0$$

by a computer calculation with guaranteed accuracy (rational arithmetic or interval arithmetic). We remark that finding a solution is practically difficult but verifying a given solution is relatively easy.

5. The size of SDP

Due to computer capacity limitation, we almost always take E to be the ball Ball(2) of radius 2. So the dimension of SDP is $\dim \mathbf{M}_E = |\mathrm{Ball}(2)|^2 \approx |S|^4$ and the number of constraints is $|E^{-1}E| = |\mathrm{Ball}(4)| \approx |S|^4$. The ball of radius 2 may appear too small, but property (T) has been be confirmed on Ball(2) in many cases, by Netzer–Thom [8], Fujiwara–Kabaya [2], and Kaluba–Nowak [3]. We were a lot encouraged by these success. People often complain that we do not learn anything (besides it is true) from a computer-assisted proof, and indeed we do not learn why it is true, but in fact we can learn how the truth can be verified.

The group $\mathrm{SAut}(\mathbf{F}_n)$ is an index-two subgroup of $\mathrm{Aut}(\mathbf{F}_n)$ and is generated by left and right transvections $S = \{L^{\pm}_{i,j},\ R^{\pm}_{i,j}\}$ with $|S| = 4n(n-1)$. It was too large for currently existing computers to run the above algorithm. So, we exploited the $\Sigma := \{\alpha \in \mathrm{Aut}(\Gamma) \mid \alpha(S) = S\}$ symmetry of the problem and carried out the invariant SDP. Fortunately, since $\Sigma = (\bigoplus_{i=1}^{n} \mathbb{Z}/2) \rtimes \mathfrak{S}_n$ is quite large in the case of $\mathrm{SAut}(\mathbf{F}_n)$, this greatly facilitates the SDP.

6. Results in [5]

We were able to verify property (T) of $\mathrm{SAut}(\mathbf{F}_n)$ for $n = 5$. One can certify our solution with a reasonably good desktop computer. It is known $\mathrm{SAut}(\mathbf{F}_n)$ does not have (T) for $n \leq 3$. For $n = 4$, we did not find a solution in Ball(2). I think we can have a definitive result/conjecture (depending on the outcome) if we are able to run the algorithm on Ball(3). We were not able to run the algorithm for $n = 6$ because the symmetrization process was beyond the computer's capacity.

7. Infinitely many cases ([4])

The above algorithm can check (T) only for one group at each time. We want to see all $\Gamma_n := \mathrm{SAut}(\mathbf{F}_n)$ have (T). Put $S_n := \{R^{\pm}_{i,j},\ L^{\pm}_{i,j} \mid i \neq j\}$ with $\Gamma_n = \langle S_n \rangle$. It suffices to show $\Delta_n = \sum_{s\in S_n} 1 - s$ satisfies $\Delta_n - \varepsilon_n\Delta_n^2 \geq 0$ in $\mathbb{R}[\Gamma_n]$. Consider $\mathrm{E}_n := \{\{i,j\} \mid i \neq j\}$ and observe

$$\begin{aligned}\Delta_n &= \textstyle\sum_{\mathrm{e}\in \mathrm{E}_n} \Delta_{\mathrm{e}},\\ \Delta_n^2 &= \textstyle\sum_{\mathrm{e}} \Delta_{\mathrm{e}}^2 + \sum_{\mathrm{e}\sim \mathrm{f}} \Delta_{\mathrm{e}}\Delta_{\mathrm{f}} + \sum_{\mathrm{e}\perp \mathrm{f}} \Delta_{\mathrm{e}}\Delta_{\mathrm{f}}\\ &=: \ \mathrm{Sq}_n \ + \ \mathrm{Adj}_n \ + \ \mathrm{Op}_n.\end{aligned}$$

Here Sq_n and Op_n are positive (because Δ_e and Δ_f commutes when $e = f$ or $e \perp f$), but Adj_n may not (and so it causes quite a bit of problem). For $n > m$,

$$\sum_{\sigma\in\mathfrak{S}_n}\sigma(\Delta_m)=m(m-1)\cdot(n-2)!\cdot\Delta_n,$$
$$\sum_{\sigma\in\mathfrak{S}_n}\sigma(\mathrm{Adj}_m)=m(m-1)(m-2)\cdot(n-3)!\cdot\mathrm{Adj}_n,$$
$$\sum_{\sigma\in\mathfrak{S}_n}\sigma(\mathrm{Op}_m)=m(m-1)(m-2)(m-3)\cdot(n-4)!\cdot\mathrm{Op}_n.$$

A computer has confirmed

$$\mathrm{Adj}_5+\alpha\mathrm{Op}_5-\varepsilon\Delta_5\geq 0$$

with $\alpha=2$ and $\varepsilon=0.13$. It follows that

$$0\leq 60(n-3)!\big(\mathrm{Adj}_n+\tfrac{2\alpha}{n-3}\mathrm{Op}_n-\tfrac{n-2}{3}\varepsilon\Delta_n\big)\leq 60(n-3)!\big(\Delta_n^2-\tfrac{n-2}{3}\varepsilon\Delta_n\big),$$

provided $\frac{2\alpha}{n-3}\leq 1$. This confirms (T) for $\mathrm{SAut}(\mathbf{F}_n)$, $n>6$.

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Group approximation and stability

Alexander Lubotzky

Let $\mathcal{G}=(G_n,d_n)_{n\in\mathbb{N}}$ be a family of groups G_n with bi-invariant metrics d_n, and let Γ be any group. An *almost homomorphism* from Γ to $\mathcal{G}$ is a family of (set theoretic) maps $\varphi_n\colon\Gamma\to G_n$ satisfying the following:

(i) For every $a,b\in\Gamma$, we have
$$\lim_{n\to\infty}d_n(\varphi_n(ab),\varphi_n(a)\varphi_n(b))=0.$$
We say that Γ is $\mathcal{G}$-*approximated* if there exists an almost homomorphism $(\varphi_n)_{n\in\mathbb{N}}$ from Γ to $\mathcal{G}$ which satisfies:

(ii) For every $1\neq a\in\Gamma$, we have $\limsup_{n\to\infty}d_n(\varphi_n(a),1_{G_n})>0$, i.e. the almost homomorphism "separates the points of Γ".

The following examples of $\mathcal{G}$'s are of special interest:

(a) $\mathcal{G} = \mathcal{P} = (\mathfrak{S}_n, d_{\mathrm{H}})_{n\in\mathbb{N}}$, where the *normalized Hamming metric* d_{H} on the symmetric group $\mathfrak{S}_n$ is given by $d_{\mathrm{H}}(\sigma,\tau) \coloneqq |\{x \in \{1,\dots,n\} \mid x.\sigma \neq x.\tau\}|/n$. The $\mathcal{P}$-approximated groups are exactly the so-called *sofic groups*. It is a seminal problem due to Gromov and Weiss whether every group is sofic.

In all other examples, the groups will be the unitary groups U_n, the various different metrics are all defined by using a norm $\|\bullet\|$ in the following way: $d(g,h) = \|g-h\|$. To define the various norms, for $T \in \mathbf{M}_n(\mathbb{C})$ set $|T| \coloneqq \sqrt{T^*T}$. We now have:

(b) $\mathcal{G}^{\mathrm{HS}} = (\mathrm{U}_n, d_{\mathrm{HS}})$ with $\|T\|_{\mathrm{HS}} = \sqrt{\mathrm{tr}(|T|^2)/n}$ – the *normalized Hilbert–Schmidt norm*. The $\mathcal{G}^{\mathrm{HS}}$-approximated groups are usually called in the literature *hyperlinear groups* or *Connes embeddable groups*. A well known open problem due to Alan Connes is whether every group has this property.

(c) Let $1 \le p < \infty$ and let $\mathcal{G}^{(p)} = \big(\mathrm{U}_n, d_n^p\big)$ where $\|T\|_p = (\mathrm{tr}(|T|^p))^{1/p}$, the *p-Schatten norm*. Of special interest is $p = 2$ which is also called the *Frobenius norm*.

(d) $\mathcal{G}^{\infty} = \mathcal{G}^{\mathrm{op}} = \big(\mathrm{U}_n, d_{\mathrm{op}}\big)$ when $\|T\|_{\mathrm{op}} = \max\{\|Tv\| \,\big|\, v \in \mathbb{C}^n, \|v\| = 1\}$. The $\mathcal{G}^{\mathrm{op}}$-approximated groups are called in the literature *MF-groups* and it is an open problem due to Kirchberg whether every group is MF.

In this talk we show:

Theorem 1 ([2]). *There exist finitely presented groups Γ which are not $\mathcal{G}^{(2)}$-approximated (i.e. not Frobenius-approximated).*

Theorem 2 ([5]). *There exists a finitely presented group Γ which is not $\mathcal{G}^{(p)}$-approximated for every $1 < p < \infty$.*

So, while the results solve uncountably many open problems, they leave open 4 cases including the 3 most important ones . . . The method of proof is based on *stability*, so we need another definition:

The group Γ is said to be *$\mathcal{G}$-stable* if for every almost homomorphism $(\varphi_n)_{n\in\mathbb{N}}$ from Γ to $\mathcal{G}$, there exists a sequence of *true* homomorphisms $\Psi_n \colon \Gamma \to G_n$ such that for every $a \in \Gamma$, $\lim_{n\to\infty} d_n\big(\Psi_n(a), \varphi_n\big) = 0$.

The following proposition observed in [4] is the initial step for the proofs of the above Theorems:

Proposition. *If $\Gamma, \mathcal{G}$ are as above and Γ is $\mathcal{G}$-stable and $\mathcal{G}$-approximated then Γ is residually finite.*

Thus, to find a non-$\mathcal{G}$-approximated group, it suffices to find a non-residually-finite $\mathcal{G}$-stable group. A criterion for $\mathcal{G}^{\mathrm{Frob}} = \mathcal{G}^{(2)}$-stability is given in [2]:

Theorem 3. *If Γ is a finitely generated group satisfying $H^2(\Gamma, V) = 0$ for every unitary representation of Γ on any Hilbert space V, then Γ is $\mathcal{G}^{(2)}$-stable.*

Garland's method [3] gives many examples of lattices in p-adic simple Lie groups satisfying the needed vanishing second cohomology, but all of them are residually finite. But one can imitate a construction of Deligne [1] of some non-residually-finite central extensions of some lattices in real Lie groups, to obtain similar p-adic examples. This will give Theorem 1. Theorem 2 is proved by a similar method extending the vanishing cohomology form Hilbert spaces to some Banach spaces.

For more on approximation and stability, see Thom's ICM 2018 lecture [6].

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Bernoulli disjointness

TODOR TSANKOV

(joint work with Eli Glasner, Benjamin Weiss, and Andy Zucker)

The concept of disjointness of dynamical systems (both topological and measure-theoretic) was introduced by Furstenberg [1] in the 60s and has since then become a fundamental tool in dynamics. Generalizing a theorem of Furstenberg (who proved the result for the group of integers), we show that for any discrete group G, the Bernoulli shift 2^G is disjoint from any minimal dynamical system. This result, together with techniques of Furstenberg, some tools from the theory of strongly irreducible subshifts, and Baire category methods, allows us to answer several open questions in topological dynamics: we solve the so-called "Ellis problem" for discrete groups and characterize the underlying topological space for the universal minimal flow of discrete groups.

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First order rigidity and bi-interpretation

Nir Avni
(joint work with Alexander Lubotzky and Chen Meiri)

A finitely generated group Γ is called first order rigid if every finitely generated group that is elementarily equivalent to Γ is isomorphic to Γ. We prove that any non-uniform lattice in a higher rank group is first order rigid and we give many examples of higher rank uniform lattices which are first order rigid.

Stationary characters on lattices in semi-simple groups

Rémi Boutonnet

In this talk I presented a result about unitary representations of certain semi-simple lattices by using "non-commutative ergodic theory". It was based on joint work with Cyril Houdayer.

As is well known a group is often best understood when it is represented as a group of transformations. In our case, we are interested in representations by unitary transformations of a Hilbert space. More precisely, we study the relationship between various such unitary representations. While the space of all unitary representations of a group, up to unitary conjugacy/containment, is too wild to be understood in general, things can be said about the space of all unitary representations of a group, up to weak equivalence, and also about the partial order given by weak containment of representations.

For example, recent work on C*-simplicity provides characterizations of groups whose regular representation is minimal with respect to weak containment, we refer to [1] for an overview of existing results and recent advances in this direction. We show that in some cases, the regular representation is actually a *smallest element* among all weakly mixing representations (i.e. those not admitting an invariant finite dimensional subspace).

Our framework is that of *strictly higher rank lattices*; meaning irreducible lattices in connected, center-free, semi-simple Lie groups all of whose simple factors have rank at least 2.

Theorem 1. *Let Γ be a strictly higher rank lattice. Then any weakly mixing unitary representation weakly contains the regular representation on $\ell^2(\Gamma)$.*

The strictly higher rank assumption is crucial in our proof, but we expect our results to hold for more general higher rank lattices.

Let us point out that this statement implies Margulis normal subgroup theorem for strictly higher rank lattices [6, Theorem IV.4.10]. Indeed, if Γ is such a lattice and $\Lambda < \Gamma$ is a normal subgroup with infinite quotient Σ, then the quasi-regular representation $\Gamma \to \Sigma \to \mathcal{U}(\ell^2(\Sigma))$ is weakly mixing. So it must weakly contain the regular representation, and in particular, it must be faithful, which implies that Λ trivial.

Here is another corollary based on quasi-regular representations, answering a question of Glasner and Weiss [4].

Corollary. *Consider a minimal action $\Gamma \curvearrowright K$ of a strictly higher rank lattice by homeomorphisms of the compact space K. If K is infinite, then the action is topologically free, i.e. the fixed point set of every non-trivial element of Γ has empty interior.*

In fact our main theorem and its proof are connected with Peterson's character rigidity theorem [8]. Recall that a *trace* on a C^*-algebra A is a positive unital linear functional $\tau\colon A \to \mathbb{C}$ which is invariant under unitary conjugacy. A *character* on a group Γ is a normalized positive definite function on Γ which is conjugacy invariant. So, given any unitary representation π of Γ, a trace on $C^*(\pi(\Gamma))$ gives rise to a character $g \mapsto \tau(\pi(g))$ on Γ. For example, if π is the regular representation λ of Γ, the C^*-algebra $C^*(\lambda(\Gamma))$ always carries a canonical trace, and the corresponding character is the Dirac function δ_e at the trivial element $e \in \Gamma$.

Peterson's character rigidity theorem [8] classifies all the characters of strictly higher rank lattices (and of other groups): they are all convex combinations of the regular character δ_e and of characters associated with finite dimensional unitary representations (and composing with some trace). Thanks to the GNS construction, this implies that any weakly mixing representation π of Γ such that $C^*(\pi(\Gamma))$ admits a trace must weakly contain the regular representation. So Theorem 1 follows by combining Peterson's character rigidity result with the following.

Theorem 2. *Let Γ be a strictly higher rank lattice, and let π be any unitary representation of Γ. Then there exists a trace on $C^*(\pi(\Gamma))$.*

Inspired from [5], our strategy to find such a trace is to study a stationary analogue of traces. This notion involves a group action on a C^*-algebra $\Gamma \curvearrowright A$. Given a probability measure μ on Γ, a *μ-stationary state* on A will be a unital positive linear functional $\phi\colon A \to \mathbb{C}$ such that

$$\sum_{g\in\Gamma} \mu(g)\phi(g^{-1}a) = \phi(a), \text{ for all } a \in A.$$

While a Γ-invariant state on A needs not exist, there is always a μ-stationary state for any probability measure μ on Γ.

Note that in the special case of a conjugacy action $\Gamma \curvearrowright C^*(\pi(\Gamma))$ by the unitaries $\pi(g)$, $g \in \Gamma$, associated with some group representation π, a trace is exactly an invariant state. So Theorem 2 will follow if we can prove that for such conjugacy actions any μ-stationary state is Γ-invariant, for a suitable choice of the probability measure μ.

This is the strategy that we apply. We choose the measure μ as constructed by Furstenberg in [3]: we require that the Poisson boundary of (Γ, μ) is equal to the Poisson boundary of $(G, \tilde{\mu})$ for some suitable probability measure $\tilde{\mu}$ on G. Then our argument is based on a combination of C^*-algebraic techniques imported from the recent approach to C^*-simplicity, and of a non-commutative generalization of a theorem of Nevo and Zimmer [7], which is worth mentioning.

Theorem 3. *Assume that* Γ *is a strictly higher rank lattice and let* μ *be a "Furstenberg" probability measure on* Γ *as described above. Consider an action of* Γ *on a von Neumann algebra* M*, whose fixed point algebra is trivial:* $M^\Gamma = \mathbb{C}$*. Assume that* ϕ *is a weakly continuous state on* M*, which is* μ*-stationary. Then we have a dichotomy:*

- *either* ϕ *is* Γ*-invariant.*
- *or there exists a proper parabolic subgroup* $Q \subsetneq G$ *and a* Γ*-equivariant normal unital* $*$*-embedding* $\theta\colon L^\infty(G/Q) \to M$*.*

Note that we have passed to the von Neumann algebraic setting as opposed to the C^*-algebraic setting. This transition is made through the so-called GNS construction. The commutative analogue of this transition is always implicit: given a Borel probability measure ν on a compact space X, we may forget about the topological information, and only focus on the measure space (X, ν). This change of point of view is often used in dynamical systems.

There are two major differences with the initial result of Nevo and Zimmer [7], both in the statement and in the techniques of proof. Besides the fact that our version is for actions on *non-commutative* spaces (and all the difficulties that this brings), the other main difference is that our result applies to actions of the lattice and not just to actions of the Lie group. This novelty is based on an induction procedure for stationary measures, which is both easy and very useful but surprisingly had remained unnoticed.

To conclude, let us mention that Theorem 3 also implies Peterson's result on character rigidity. At first glance, this may seem surprising since we already know that a character is conjugacy invariant, so there is no point in applying directly Theorem 3 to the GNS von Neumann algebra generated by a character. Instead, we apply it to the so-called *non-commutative Poisson boundary* considered by Peterson [8] (see also [2]).

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Isoperimetry, Littlewood functions, and unitarizability of groups

Maria Gerasimova
(joint work with Dominik Gruber, Nicolas Monod, and Andreas Thom)

Let us assume that Γ is a discrete group. A representation $\pi : \Gamma \to B(H)$, where H is a Hilbert space, is called uniformly bounded, if $\sup_{g\in\Gamma} \|\pi(g)\| < \infty$. A representation $\pi : \Gamma \to B(H)$ is called unitarizable, if there exists an operator $S : H \to H$ such that $S^{-1}\pi(g)S$ is a unitary representation for any $g \in \Gamma$. A group Γ is called unitarizable if any uniformly bounded representation is unitarizable. The first question which arises in this context if all groups are unitarizable. The answer to this is negative and the main non-example is a non-abelian free group $\mathbb{F}_n$ with n generators for $n \geq 2$ ([4]). The next classical result says that amenable groups are unitarizable. It has been open ever since whether this is a characterization of unitarizability (this question is called the Dixmier's problem [2]). The question remains open only for non-amenable groups without free subgroups.

One of the approaches to study unitarizability and amenability is related to the space of the Littlewood functions $T_1(\Gamma)$.The latter is the space of all functions $f : \Gamma \to \mathbb{C}$ admitting a decomposition

$$f(x^{-1}y) = f_1(x, y) + f_2(x, y) \qquad \forall x, y \in \Gamma$$

with $f_i : \Gamma \times \Gamma \to \mathbb{C}$ such that both of the following are finite:

$$\sup_x \sum_y |f_1(x, y)| \quad \text{and} \quad \sup_y \sum_x |f_2(x, y)|.$$

The connection is as follows. First, Γ is amenable if and only if $T_1(\Gamma) \subseteq \ell^1(\Gamma)$ [5]. Secondly, if Γ is unitarizable, then $T_1(\Gamma) \subseteq \ell^2(\Gamma)$ [1]. Thirdly, if Γ contains a non-abelian free subgroup, then $T_1(\Gamma) \nsubseteq \ell^p(\Gamma)$ for all $p < \infty$.
It turned out that we can say something more about non-amenable groups.

Theorem 1 (GGMT,[3])**.** *For any non-amenable group Γ there exists $p > 1$ such that*

$$T_1(\Gamma) \nsubseteq \ell^p(\Gamma).$$

This result inspired us to define the Littlewood exponent $\mathrm{Lit}(\Gamma) \in [0, \infty]$ of a group Γ as follows:

$$\mathrm{Lit}(\Gamma) = \inf \left\{p : T_1(\Gamma) \subseteq \ell^p(\Gamma)\right\}.$$

The main results about the Littlewood exponent are listed in the theorem below.
Theorem 2 (GGMT,[3])**.**

(1) $\mathrm{Lit}(\Gamma) = 0$ *if and only if* Γ *is finite.*
(2) $\mathrm{Lit}(\Gamma) = 1$ *if and only if* Γ *is infinite amenable.*
(3) $\mathrm{Lit}(\Gamma) \leq 2$ *if* Γ *is unitarizable.*
(4) $\mathrm{Lit}(\Gamma)$ *is outside the interval* $(1, 2)$ *if* Γ *has the rapid decay property.*
(5) $\mathrm{Lit}(\Gamma) = \infty$ *if* Γ *contains a non-abelian free subgroup.*

Unfortunately, the last statement is not a characterization of the existence of a free non-abelian subgroup.

Theorem 3 (GGMT,[3])**.** *There exists a torsion group* Λ *with* $\mathrm{Lit}(\Lambda) = \infty$.

There is also a connection between $\mathrm{Lit}(\Gamma)$ and the geometry of a group Γ, more precisely, between $\mathrm{Lit}(\Gamma)$ and the asymptotics of isopermetric quantities attached to Γ as follows. Given a finite symmetric subset $S \subseteq \Gamma$, consider the (possibly disconnected) Cayley graph $\mathrm{Cay}(\Gamma, S)$. Recall that the Cheeger constant $h(\Gamma, S)$ is defined by

$$h(\Gamma, S) = \inf_{F} \frac{|\partial_S(F)|}{|F|},$$

where the infimum runs over all non-empty finite subsets $F \subseteq \Gamma$. Define the relative maximal average degree $e(\Gamma, S)$ by

$$e(\Gamma, S) = 1 - \frac{h(\Gamma, S)}{|S|}.$$

Finally, our asymptotic invariant is

$$\eta(\Gamma) = - \liminf_{S} \frac{\ln e(\Gamma, S)}{\ln |S|},$$

where the limes inferior is taken over all symmetric finite subsets S of Γ.

Then we can prove the following result.

Theorem 4 (GGMT,[3])**.** *For any group* Γ *we have* $\eta(\Gamma) = 1 - \frac{1}{\mathrm{Lit}(\Gamma)}$.

This result allows us to prove that the invariant $\mathrm{Lit}(\Gamma)$ is not trivial (that is $\mathrm{Lit}(\Gamma) \notin \{0, 1, \infty\}$) and construct a group Γ with $1 < \mathrm{Lit}(\Gamma) < \infty$. It also allows us to estimate this invariant for some complicated groups (i.e. for Burnside groups of the large exponent) and find some geometric applications.

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Ordering ratio function, gap problem for orders and traveling salesman girth of groups

Anna Erschler
(joint work with Ivan Mitrofanov)

We define and study asymptotic invariants of metric spaces and infinite groups related to the universal traveling salesman problem.

We prove that spaces with doubling property, in particular all virtually nilpotent groups, admit a gap for ordering ratio functions: any order on these spaces satisfies either $\mathrm{OR}(k) = k-1$, for all k (in other words traveling salesman girth is infinite), or $\mathrm{OR}(k) \leq C \ln(k)$ (where C depends only on the traveling salesman girth g, the value $(g-1) - \mathrm{OR}(g)$ and the doubling constant of the space). We characterize groups with traveling salesman girth ≤ 4 as virtually free ones.

We show that the ordering ratio function is constant (which is the best possible function) for all hyperbolic groups, and more generally, for all uniformly discrete bounded geometry δ-hyperbolic spaces. We prove that the ordering ratio function of any group, containing weakly a sequence of arbitrarily large cubes (for example, any group admitting a uniform embedding of $\mathbb{Z}^d$, for all d) has infinite traveling salesman girth; this means that any order on such spaces satisfies $\mathrm{OR}(s) = s-1$ for all s. This is the worst possible case for ordering ratio functions. We show that any metric space of finite Assouad–Nagata dimension admits an order satisfying $\mathrm{OR}(s) \leq C \ln(s)$.

Equidistribution for toral endomorphisms

Michael Hochman

Host's theorem [3] is a pointwise strengthening of Rudolph's measure rigidity result [7, 5]. It states that if $a, b \in \mathbb{N}$ are relatively prime (or, more generally, multiplicatively independent) integers, and $a, b \geq 2$, then for every measure μ on $\mathbb{R}/\mathbb{Z}$ that is invariant, ergodic and has positive entropy under the endomorphism $\times a$, almost every point x (w.r.t. μ) equidistributes for Lebesgue measure under $\times b$.

In the late 1990s and early 2000s, some extensions were proved in higher dimensions, by D. Meiri [6] and in greater generality by B. Host [4], but these results have restrictions due to which they do not apply to groups of automorphisms (making it weaker than the analogous measure rigidity results on $\mathbb{T}^d$), and requiring more arithmetic independence than what is expected. Other recent work by Algom [1] has dealt with some cases involving diagonal matrices.

In my talk I presented recent work which extends Host's results to (almost) their natural generality. Similarly to Host's results, our work applies in both the commuting and non-commuting case:

Theorem. *Let A, B be non-singular integer matrices. Let μ be an A-invariant and ergodic probability measure on $\mathbb{T}^d$ with positive entropy. Suppose that*

(1) The characteristic polynomial of B^n is irreducible over $\mathbb{Q}$ for all $n \geq 1$.

(2) No power of B is conjugate to a power of A.

Then μ is B-normal in the sense that $\frac{1}{N}\sum_{n=1}^{N} B^n\mu \to$ Lebesgue.
If, in addition, any one of the following holds:

(3a) No expanded eigenvector of A is in the central subspace of B.
(3b) $\mathbb{Q}(\Lambda_A) \cap \mathbb{Q}(\Lambda_B) = \mathbb{Q}$, where Λ_A, Λ_B are the sets of eigenvalues of A, B respectively.
(3c) B is hyperbolic.
(3d) A is expanding.

Then μ-a.e. point equidistributes for Lebesgue measure under B.
If A, B commute, we can weaken assumption (1) to total irreducibility of the joint action of A, B, and (2) to the condition that $A^m \neq B^n$ for all $m, n \in \mathbb{N}$.

It remains open whether conditions like (3a)–(3c) are needed for the pointwise result (as indicated, they are not needed for mean equidistribution). These conditions are there to ensure that B does not act in an isometric way on the leafwise measures of μ on the A-expanded foliation. One should note that the additional conditions are quite mild: For general pairs of matrices A, B, the eigenspaces are in general position with respect to each other, so (3a) holds.

The idea of the proof is an outgrowth of my work with Shmerkin [2] on Host's theorem on $\mathbb{R}/\mathbb{Z}$, but that argument was too specialized to work in other contexts. The proof differs from the standard proofs of measure rigidity in the following way: Usually in measure rigidity proofs, one acts on the measure μ by $B^n A^{-cn}$ where $c > 0$ is chosen so that the action is isometric in some direction (one must extend the action from an $\mathbb{Z}^2$ action to an $\mathbb{R}^2$ to make sense of this), and then one uses recurrence of leafwise measure to themselves under this isometric action to derive translation invariance. In contrast we consider compositions $B^n A^{-[cn]}$, which no longer act isometrically; in fact, the point is that these compositions act in a rich way on the leafwise measures, for example, their expansion constants may be dense in an interval. We utilize the smoothing that arises from this rich family of maps to deduce equidistribution.

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Banach space actions and L_2-spectral gap

Mikael de la Salle
(joint work with Tim de Laat and Amine Marrakchi)

Let $\Gamma = \langle S \rangle$ be a countable group with a finite generating set S. If X is a Banach space, Bader, Furman, Gelander, and Monod [1] defined the *fixed point property on* X as follows: Γ has (F_X) if every action by affine isometries on X has a fixed point. When X is a Hilbert space, we recover Kazhdan's property (T), and it is widely felt that the less close to a Hilbert space X is, the stronger (F_X) is. Of course, this statement has to be taken with care, as a typical Banach space has essentially no isometry and therefore any group with trivial abelianization has (F_X) for such X. The situation for an L_p space illustrates this feeling quite well. If one defines the fixed point spectrum $\mathcal{F}(\Gamma)$ (respectively proper spectrum $\mathcal{P}(\Gamma)$) of Γ as the set of values $p \in [1, \infty]$ such that Γ has F_{L_p} (respectively Γ has a metrically proper action on an L_p space), then it was shown in [1] that $\mathcal{F}(\Gamma)$ is open, and is non-empty if and only if Γ has (T), in which case it contains $[1, 2]$. On the other hand, Guioliang Yu proved that if Γ is hyperbolic, then $\mathcal{P}(\Gamma)$ contains a neighborhood of ∞. Answering a conjecture of Druţu [2], Amine Marrakchi and I recently proved (work in progress) that $\mathcal{F}(\Gamma)$ and $\mathcal{P}(\Gamma)$ are moreover intervals.

Theorem (Marrakchi-dlS)**.** *For an infinite group* Γ*, there are critical parameters* $1 \leq p_{\mathcal{F}}(\Gamma) \leq p_{\mathcal{P}}(\Gamma) \leq \infty$ *such that* $\mathcal{F}(\Gamma) = [1, p_{\mathcal{F}}(\Gamma))$ *and* $\mathcal{P}(\Gamma) = (p_{\mathcal{P}}(\Gamma), \infty]$ *or* $\mathcal{P}(\Gamma) = [p_{\mathcal{P}}(\Gamma), \infty]$.

According to a recent result by Druţu and Minasyan, $p_{\mathcal{P}} - p_{\mathcal{F}}$ can be arbitrarily large for property (T) groups.

But all the preceding relies very heavily on the explicit description of the isometry group of $L_p(\Omega, \mu)$ as $L_0(\Omega; \mathbf{T}) \rtimes \mathrm{Aut}(X, [\mu])$ (the Banach-Lamperti theorem). A very interesting question is to study how the geometry/algebra of the group interacts with the geometry of the Banach space, without having an explicit knowledge of its isometry group. An intriguing conjecture [1] is for example that lattices in connected simple Lie groups of rank at least two have (F_X) for every uniformly convex X. This conjecture is known for many specific Banach spaces (for example L_p spaces [1]) as well as for groups over non-Archimedean local fields [5], but remains open in general despite a lot of efforts [3, 7]. A stronger form of it is probably also true, when one replaces uniformly convex by non-trivial type.

With Tim de Laat, we proved the following generalization to uniformly curved spaces of Żuk's celebrated criterion [8], which asserts that a group, acting geometrically on a two dimensional simplicial complex whose links have spectral gap $> \frac{1}{2}$, has property (T). Uniform curvedness is an intriguing property of a Banach space which refines uniform convexity. It was defined and studied by Pisier in [6]. Another Banach space extension of Żuk's criterion has also been previously obtained by Bourdon for L_p spaces, but in terms of p-Laplacian.

Theorem ([4])**.** *Let* X *be a uniformly curved Banach space. Then there exists an* $\varepsilon > 0$ *(depending on* X*) such that the following holds: If* Γ *is a group that admits* *a proper cocompact action by simplicial automorphisms on a simplicial 2-complex M such that the spectra of all its links are contained in $[-\varepsilon,\varepsilon]\cup\{1\}$, then Γ has (F_X).*

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Session for presentation of open problems

Collected by Jakob Schneider

During the workshop at this point we had one evening session for the presentation of open problems and questions. We list here the problems and questions which were presented during this slot together with the initiator.

- Let G be a group acting on the Cantor set X. The topological full group of the action is the group of all homeomorphisms $f\colon X\to X$ such that for every $x\in X$ there exists a neighborhood U of x such that $f|_U$ is equal to $g|_U$ for some $g\in G$. If U_1,U_2,U_3 are three pairwise disjoint clopen subsets of X, and g_1,g_2 are such that $g_1(U_1)=U_2$ and $g_2(U_2)=U_3$, then the homeomorphism equal to g_1 on U_1, to g_2 on U_2, to $g_1^{-1}g_2^{-1}$ on U_2, and identity on the complement of $U_1\cup U_2\cup U_3$, is an element of the topological full group. Define the alternating full group as the group generated by all such elements. There are no known examples of minimal actions for which the alternating full group is different from the derived subgroup of the topological full group (it is easy to see that the alternating full group is a subgroup of the derived subgroup). The question is to find some examples, or to prove that they do not exist. *V. Nekrashevych*
- Let (X,μ) be a standard probability space and denote by $\mathrm{Aut}(X,\mu)$ its automorphism group, the group of measure preserving and measurable bijections of X up to null-sets. A homomorphim $\rho\colon \mathrm{Aut}(X,\mu)\to\mathrm{Aut}(X,\mu)$ is called an *essentially free quasi-action* if for every $T\in\mathrm{Aut}(X,\mu)$ which is not the identity, the set of points $x\in X$ such that $\rho(T)x=x$ is

a null-set. Observe that there always exists an essentially free quasi-action. Indeed we can define $\rho_\infty\colon \mathrm{Aut}(X,\mu) \to \mathrm{Aut}(X^{\mathbb{N}}, \mu^{\mathbb{N}})$ by declaring $\rho_\infty(T)(x_n)_n = (Tx_n)_n$. Remark that $(X^{\mathbb{N}}, \mu^{\mathbb{N}})$ is also a standard probability space and hence isomorphic to (X,μ). How many essentially free quasi-actions does $\mathrm{Aut}(X,\mu)$ have up to conjugacy? Are all such quasi-actions factors of ρ_∞? *A. Carderi*

- *"Are Cayley graphs of subgroups of the free group with large girth almost Ramanujan?"*: Is it true that the Cayley graph of a k-generated subgroup of a free group that has large girth must be almost Ramanujan? In other words, given an integer $k \geq 2$ and some $\varepsilon > 0$ does there exist $C > 0$ such that for every $x_1, ..., x_k$ in a non-abelian free group F such that $w(x_1, \ldots, x_k) \neq 1$ for every non-trivial reduced word w in k letters, one has:
$$\left\| \frac{1}{2k} \sum_{i=1}^{k} \lambda_{x_i} + \lambda_{x_i^{-1}} \right\| < \frac{\sqrt{2k-1}}{k} + \varepsilon,$$
where the norm is the operator norm on $\ell^2(F)$ and λ_g the regular representation of F on $\ell^2(F)$. *E. Breuillard*

Extension of a residually finite group by a residually finite group is weakly sofic

Lev Glebsky

Sofic groups have been defined in relation with Gottschalk's surjunctivity conjecture, see [1, 2]. It is an open question if all groups are sofic. There is a hope that a non-sofic group may be constructed as an extension of a residually finite group by a finite one, [3, 4]. (It is known, however, that an extension of an amenable group by a sofic (resp. a hyperlinear or a weakly sofic) group is sofic (resp. hyperlinear or weakly sofic), [5, 6]. The main result of [7] is an example of a non-approximable by $(\mathrm{U}_n, \|\bullet\|_2)$ group. This example is a residually-finite-by-finite extension. It is a kind of a subtle support to the above mentioned hope as sofic groups may be defined through metric approximation by symmetric groups [8]. I show that residually-finite-by-finite extensions are weakly sofic:

- Let H be a normal subgroup of a group K. If H and $G = K/H$ are residually finite, then K is weakly sofic.

W.l.o.g. we may consider finitely generated H and G. Then, as any extension of G by H is contained in the wreath product $H \wr G$, it suffices to show that $H \wr G$ is weakly sofic. (Recall that a wreath product $H \wr G$ is a semidirect product of H^G and G with an action $(g.f)(x) = f(xg)$, for $g \in G$ and $f \in H^G$. Particularly, $(f,g)(f',g') = (f(g.f'), gg')$.) Third, a homomorphism $H_1 \to H_2$ naturally defines a homomorphism $H_1 \wr G \to H_2 \wr G$. Moreover, a residually weakly sofic group is weakly sofic. So, it suffices to show that $H \wr G$ is weakly sofic for finite H and residually finite G. To this end we use the following characterization of weakly sofic groups, see [9]:

- A group K is weakly sofic if and only if every system of equations solvable in all finite groups is solvable over K.

Let me describe in more details the systems of equations considered here. For a set X we use the notation $X^* = \bigcup_{n\in\mathbb{N}} X^n$. Let $\overline{y} = (y_1, y_2, \ldots, y_j, \ldots)$ and $\overline{x} = (x_1, x_2, \ldots, x_j, \ldots)$ be countable sets of symbols for constants and variables, respectively. Let $F = F(\overline{y}, \overline{x})$ be the free group freely generated by the variables $\overline{y}$ and $\overline{x}$. Let $\overline{w} \in F^*$. Notice that $\overline{w} \in F^r(y_1, \ldots, y_k, x_1, \ldots, x_n)$ for some $k, n, r \in \mathbb{N}$. By substitution, $\overline{w}$ defines a map $G^k \times G^n \to G^r$. Consider the system of equations $\overline{w} = 1$. We say that $\overline{w}$ is solvable in a group G if the sentence

$$\forall \overline{y}\ \exists \overline{x}\ \overline{w}(\overline{y}, \overline{x}) = 1$$

is valid in G. We say that a system $\overline{w}$ is solvable over group G if for some overgroup H of G the sentence

$$\forall \overline{y} \in G^*\ \exists \overline{x} \in H^*\ \overline{w}(\overline{y}, \overline{x}) = 1$$

is valid.

Let Sys(Fin) be the set of systems of equations solvable in all finite groups. The consideration from above shows that it suffices to prove the following statement:

- Let H be a finite and G a finitely generated residually finite group. Let $\overline{w} \in \mathrm{Sys(Fin)}$. Then $\overline{w}$ is solvable over $H \wr G$.

Let $\hat{G}$ be a profinite completion of G and $(\overline{f}, \overline{a}) \in (H \wr G)^k$. We find a solution of $\overline{w}((\overline{f}, \overline{a}), \overline{x}) = 1$ in $H \wr \hat{G}$, where $H \wr \hat{G}$ is an abstract wreath product (we consider as well discontinuous functions.) More precisely, we find a solution in $H \wr \Gamma$ where $\Gamma < \hat{G}$ is a finitely generated group. Our proof is somehow topological and uses different topologies. We need the following definition:

- Let $\overline{a} \in \hat{G}^k$ and $\overline{w} \in \mathrm{Sys(Fin)}$. A solution $\overline{u} \in \hat{G}^n$ of $\overline{w}(\overline{a}, \overline{u})$ is called $(H, \overline{a})$-*universal* if for every finite continuous quotient of $\hat{G}$, $G_N = \hat{G}/N$, the following statement is true
$$\forall \overline{f} \in (H^{G_N})^k\ \exists \overline{\phi} \in (H^{G_N})^n\ \ \overline{w}((\overline{f}, \overline{a}_N), (\overline{\phi}, \overline{u}_N)) = 1$$
Here we denote by x_N the image of $x \in \hat{G}$ in the quotient group G_N.

First, we show the existence of an $(H, \overline{a})$-universal solution for $\overline{w}$. It uses the profinite structure of $\hat{G}$. Then Γ is generated by G and an $(H, \overline{a})$-universal solution $\overline{u} = (u_1, \ldots, u_n)$. To show the existence of a solution in $H \wr \Gamma$, we use the Tychonoff (direct product) topology on H^Γ. Precisely, we equipped H^Γ with the Tychonoff topology, Γ with the discrete topology and use the fact that $H \wr \Gamma$ is a topological group with respect to the product topology of these topologies.

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Berry's conjecture and limits of eigenfunctions

Miklós Abért
(joint work with Nicolas Bergeron and Etienne Le Masson)

We investigate the asymptotic behavior of eigenfunctions of the Laplacian on compact Riemannian manifolds. We show that Benjamini–Schramm convergence provides a unified language for the level and eigenvalue aspects of the theory. As a result, we present a mathematically precise formulation of Berry's conjecture for a compact negatively curved manifold and formulate a Berry-type conjecture for sequences of locally symmetric spaces. This allows us to ask the simplest case, which is to show that the sine wave is not a limit of eigenfunctions for a compact negatively curved manifold. We prove some weak versions of these conjectures. Using ergodic theory, we also show that the Berry conjecture implies Quantum Unique Ergodicity.

Balanced presentations of étale fundamental groups of curves over finite fields

Mark Shusterman

1. Complex Geometry, Fundamental groups, Representation Varieties

Let $X = \Sigma_g$ be a compact orientable surface of genus $g \geq 1$. It can be viewed as a complex curve, and a geometer is interested in local systems of rank $n \geq 1$ on X over $\mathbb{C}$. These are in bijection with representations (homomorphisms)

$$\rho\colon \pi_1(X) \to \mathrm{GL}_n(\mathbb{C}).$$

We know that

$$\pi_1(X) \cong \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \dots [x_g, y_g] = 1\rangle, \quad [x, y] = xyx^{-1}y^{-1}$$

which means that our representations are just collections of $2g$ matrices $n \times n$ satisfying one specific algebraic relation. This gives rise to the representation variety, and to a fruitful study of deformations of representations. The situation is similar for more general X, as long as $\pi_1(X)$ is finitely presented.

2. Algebraic Geometry, Number Theory, Fundamental groups

Let X be a smooth projective curve over a finite field k of characteristic p. Algebraic geometers and number theorists are interested in local systems on X. Indeed, the Langlands program studies continuous representations of the étale (or algebraic) fundamental group $\pi_1^{\text{ét}}(X)$. As in the previous section, it would be desirable to have a description of $\pi_1^{\text{ét}}(X)$ in terms of generators and relations.

3. The étale Fundamental group

Subgroups of the usual fundamental group correspond to covers of the topological space. Similarly, open subgroups of the étale fundamental group (which is a profinite group) correspond to étale covers of the variety. Moreover, for an algebraic variety X over $\mathbb{C}$, the profinite completion of $\pi_1(X(\mathbb{C}))$ is topologically isomorphic to $\pi_1^{\text{ét}}(X)$. See [3, Exposé XII, Section 5].

The above suggests that étale fundamental groups over algebraically closed fields are more manageable, so for a smooth projective curve X over a finite field k of characteristic p, it is desirable to consider the short exact sequence

$$1 \to \pi_1^{\text{ét}}(X_{\bar{k}}) \to \pi_1^{\text{ét}}(X) \to \mathrm{Gal}(\bar{k}/k) \to 1$$

where $\bar{k}$ is an algebraic closure of k, and $X_{\bar{k}}$ is the base change of X to $\bar{k}$.

In some sense, this reduces the study of the étale fundamental group to the geometric situation (over the algebraic closure), and to the action of $\mathrm{Gal}(\bar{k}/k)$ (topologically generated by $\mathrm{Frob}_{|k|}$) on the geometric étale fundamental group. The latter action encodes the answer to arithmetic questions about the curve such as the Weil conjectures on the number of points on X over finite extensions of k.

By lifting to characteristic zero, Grothendieck has shown in [3, Exposé XIII] that $\pi_1^{\text{ét}}(X_{\bar{k}})$, and thus also $\pi_1^{\text{ét}}(X)$, is topologically finitely generated. Moreover, he shows that for every prime $\ell \neq p$, the maximal pro-ℓ quotient of $\pi_1^{\text{ét}}(X_{\bar{k}})$ admits the pro-ℓ presentation

$$\langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \dots [x_g, y_g] = 1 \rangle$$

where g is the genus of X. This result is complemented by the work [10] of Shafarevich, showing that the maximal pro-p quotient of $\pi_1^{\text{ét}}(X_{\bar{k}})$ is free pro-p.

In case $g = 1$ we have

$$\pi_1^{\text{ét}}(X) \cong \mathbb{Z}_p^{\alpha} \times \prod_{\ell \neq p} \mathbb{Z}_\ell^2 \rtimes \widehat{\mathbb{Z}}$$

where $\alpha = 1$ if X is regular and $\alpha = 0$ if X is supersingular. The possible actions of $\widehat{\mathbb{Z}} \cong \mathrm{Gal}(\overline{k}/k)$ are implicit already in the Hasse bound.

In a sharp contrast, once $g \geq 2$, the group $\pi_1^{\text{ét}}(X)$ encompasses the structure of X in a non-trivial way. One evidence for this is the work [11] of Tamagawa which shows that there are only finitely many smooth projective curves Y over $\bar{k}$ with

$\pi_1(Y) \cong \pi_1(X_{\bar{k}})$. Another result in this vein, obtained by Mochizuki in [6], says (roughly speaking) that X can be (functorially) reconstructed from $\pi_1(X)$.

4. Balanced Presentations – Results

Our main result is the following.

Theorem 1. *The étale fundamental group of a smooth projective curve over a finite field is topologically finitely presented. Moreover, the number of relations is at most the number of generators (in some presentation).*

In fact, we show that for every topological presentation

$$1 \to N \to F \to \pi_1^{\text{ét}}(X) \to 1$$

of our étale fundamental group, the number of generators of N as a closed normal subgroup of the free profinite group F equals the rank of F. Moreover, our arguments apply not only to $\pi_1^{\text{ét}}(X)$ itself, but to any closed topologically finitely generated subgroup of it.

Corollary 1. *The group $\pi_1^{ét}(X)$ is topologically coherent.*

Coherence means that all the closed topologically finitely generated subgroups of $\pi_1(X)$ are topologically finitely presented.

Theorem 1 brings the algebraic situation discussed in Section 2 closer to the geometric situation of Section 1. However, as opposed to the situation in Section 1, from the proof of Theorem 1 we gain very little insight into the nature of the relations (beyond their number).

5. Étale fundamental group as a Galois Group

Recall that if K is the function field of X, then $\pi_1^{\text{ét}}(X)$ can be identified with $\text{Gal}(K^{\text{ur}}/K)$, where K^{ur} is the maximal unramified extension of K. As a result, the group $\pi_1^{\text{ét}}(X)$ is also studied as a function field analog of the generalized class group $\text{Gal}(L^{\text{ur}}/L)$ of a number field L. Examples include the works [1, 12] by Boston and Wood. These works suggest, vaguely speaking, that for a randomly chosen X, the group $\pi_1(X)$ is given by a random balanced presentation (that is, a presentation where the number of generators is at least the number of relations). Motivated by this, Liu and Wood in [4] study random balanced presentations and their variants. Theorem 1 may thus be viewed as a deterministic counterpart of these works.

6. Arithmetic Topology

Arithmetic topology, as studied for instance in [7], postulates that the group $\text{Gal}(L^{\text{ur}}/L)$ should exhibit properties similar to those of the fundamental group of a 3-manifold. Indeed, balanced presentations are attributes of some 3-manifold groups (see [2]), and Pardon asks in [8] whether an analog of this can be established for $\text{Gal}(L^{\text{ur}}/L)$. This serves as an additional motivation for Theorem 1. What's more, Corollary 1 is analogous to Scott's coherence results from [9].

7. Tools from Proof

Our proof of Theorem 1 starts from a formula proved by Lubotzky in [5]. The latter, once combined with Grothendieck's finite generation result mentioned above, reduces our task to an estimation of the dimensions of several cohomology groups. We perform these estimations using the Lyndon–Hochschild–Serre spectral sequence. An important input is duality in cohomology, deduced from the aforementioned results on maximal pro-ℓ and pro-p quotients by Grothendieck and Shafarevich. Those cohomology groups which we do not know how to compute, can be shown to cancel the contribution of one another.

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Infinite approximate groups

Tobias Hartnick

Given a group G and a positive integer k, a subset $\Lambda \subseteq G$ is called a *k-approximate subgroup* of G if it is symmetric (i.e. $\Lambda = \Lambda^{-1}$) and contains the identity e_G, and if moreover

$$\Lambda \cdot \Lambda \subseteq \Lambda \cdot F$$

for a finite subset $F \subseteq G$ of cardinality k [12]. We then call the group Λ^∞ generated by Λ the *enveloping group* of Λ and the pair $(\Lambda, \Lambda^\infty)$ a *k-approximate group.* A *global morphism* $(\Lambda, \Lambda^\infty) \to (\Xi, \Xi^\infty)$ between k-approximate groups is a map of pairs which is also a homomorphisms of the enveloping groups. If G is a group, then a global morphism into (G, G) is also called a *representation*.

Note that subgroups are precisely the 1-approximate subgroups and that every finite set Λ is a $|\Lambda|$-approximate subgroup. However, if one fixes k and considers large but finite k-approximate subgroup, then there is a rich structure theory of such finite approximate subgroups [5], with many deep applications, in particular to additive combinatorics. On the contrary, relatively little is known about the structure of countably infinite approximate subgroups so far. The goal of this note is to survey some recent developments towards a general structure theory of infinite approximate groups.

One of the early empirical discoveries in the study of infinite approximate groups was that there exist "tame" examples which admit nice representations into locally compact groups, and "wild" examples which do not admit any such representations. A prototypical example of a tame approximate group is given by the vertex set Λ_A of a symmetric Penrose tiling, which comes equipped with a natural representation into $\mathbb{R}^2$. As an example of a wild approximate group, consider the function $f : F_2 \to \mathbb{Z}$ from the free group on two generators a and b, which with every reduced word associates the number of subwords of the form ab minus the number of subwords of the form $b^{-1}a^{-1}$. The set

$$\Lambda_B := \{g \in F_2 \mid f(g) \in \{-1, 0, 1\}\}$$

is a 6-approximate subgroup of F_2 which does not admit any nice representation in a sense made precise below.

Model sets in the sense of Meyer [11] are a vast generalization of example Λ_A. If Γ is an irreducible lattice in a product $G \times H$ and $W \subseteq H$ is a compact identity neighborhood, then the associated model set is defined by the cut-and-project construction

$$\Lambda := \Lambda(G, H, \Gamma, W) := \mathrm{pr}_G(\Gamma \cap (G \times W)) \subseteq G.$$

Any model set Λ is an approximate subgroup of G as well as a Delone subset with respect to some (hence any) proper continuous left-invariant metric on G; we refer to such Delone approximate subgroups as *uniform approximate lattices* [1]. If G is abelian and compactly-generated, then thee uniform approximate lattices are precisely the syndetic subsets of model sets by a classical theorem of Meyer [11]. Recently, this result was extended to general amenable groups by Machado (see his contribution in the present volume).

Uniform approximate lattices are tame in the sense that they admit embeddings into locally compact groups with Delone image. Slightly more general one can try to construct representations into locally compact groups with Delone image such that the kernel intersects each of the sets Λ^k (equivalently, the set Λ^2) in a finite set. Since locally compact groups are essentially isometry groups of proper metric spaces, constructing such representations is equivalent to constructing geometric actions of approximate groups on proper metric spaces.

One can try to construct such actions by a version of geometric group theory, but this runs into several problems. Every countable approximate group gives rise to a canonical coarse equivalence class of metric spaces, but there are two different ways to single out a quasi-isometry type within this coarse equivalence class: *Extrinsically*, by assuming that Λ^∞ is finitely-generated and restricting word metrics on Λ^∞ to Λ (as in [1]) or *intrinsically*, by assuming that the coarse class admits a large-scale geodesic representative (which is then unique up to quasi-isometry), and considering its quasi-isometry type (as in [7]). If both constructions apply, then they may still yield different results, in which case we call the approximate group *distorted*. Uniform approximate lattices are always undistorted [7], hence distortion is an obstruction to admitting nice representations. Quasi-kernels of quasimorphisms on hyperbolic groups, such as Example Λ_B above, are always distorted (by unpublished work of Heuer and Kielak), hence they do not admit nice representations. Even for undistorted approximate groups one obtains in general only a quasi-isometric quasi-action on a suitable model space (rather than an isometric action), and additional assumptions are required to obtain a nice representation. We refer the reader to Machado's contribution in this volume for a different approach to constructing nice representations (in the amenable context), which avoids geometry altogether and instead uses tools from additive combinatorics, such as variants of Sanders' lemma.

The geometric approach to infinite approximate groups has some merits in its own rights though. In particular, it allows to associate asymptotic invariants - such as growth, Dehn functions or asymptotic dimension - to approximate groups which suitable finiteness properties. Some sample results in this direction are:

- Every undistorted approximate group of polynomial growth is essentially nilpotent. (This is an application of the Breuillard–Green–Tao theorem to a suitable sequence of balls in such groups.)
- The asymptotic dimension of a hyperbolic approximate group is given by the topological dimension of its Gromov boundary +1. (More generally, this holds for visual, quasi-cobounded, Morse-hyperbolic spaces, [7].)
- An approximate subgroup which is quasi-isometric to a higher rank building or symmetric space is essentially a uniform approximate lattice in its isometry group. (This is an extension of the QI rigidity theorem of Kleiner–Leeb [9] from [1].)

For tame infinite approximate subgroups, a far-reaching theory can be developed by dynamical methods. Namely, if Λ is a uniform approximate lattice in a locally compact group G, then the orbit closure of Λ in the Chabauty space of G is an interesting dynamical system for G, called the *hull system*. This system is particularly useful, if it admits a G-invariant measure (as is the case in the amenable case, but also for general model sets [3]), in which case Λ is called a *strong uniform approximate lattice*. By a variant of this construction, one can also define *non-uniform strong approximate lattices* [1]. This is important to be able to conclude model sets arising from arithmetic and S-arithmetic lattices, which are typically non-uniform. Some sample results are the following:

- Envelopes of strong or uniform approximate lattices are unimodular [1, 4].
- Approximate lattices in nilpotent Lie groups are uniform [1].

- In the context of algebraic groups, the density theorems of Borel and Dani–Shalom generalize in the obvious way [4].
- Analytic properties such as property (T) and the Haagerup property pass between strong uniform approximate lattices and their envelopes [2].

While the theory of strong approximate lattices is developing rapidly, our understanding of general discrete approximate subgroups of locally compact groups is still rather poor. However, there are also some interesting first results concerning smaller discrete approximate subgroups. For instance:

- Approximate subgroups of certain amenable groups are compactly-close to actual subgroups [8, 10].
- If Λ is an approximate subgroup of $\mathrm{Is}(\mathbb{H}^n)$, whose limit set $\mathcal{L}_\Lambda \subseteq \partial\mathbb{H}^n$ has at least two points and consists entirely of conical limit points, then either Λ is commensurable to a convex-cocompact subgroup, or $\mathcal{L}_\Lambda$ is the boundary of a totally geodesic copy of some $\mathbb{H}^k$ and Λ acts on the latter as a uniform approximate lattice in $\mathrm{Is}(\mathbb{H}^k)$ [6].

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Word images in symmetric and classical groups of Lie type are dense

Jakob Schneider
(joint work with Andreas Thom)

Let $w \in \mathbf{F}_k = \langle x_1, \dots, x_k \rangle$ be a non-trivial word and denote by $w(G) \subseteq G$ the image of the associated word map $w\colon G^k \to G$. Let G be one of the finite groups $\mathfrak{S}_n$, $\mathrm{GL}_n(q)$, $\mathrm{Sp}_{2m}(q)$, $\mathrm{GO}^{\pm}_{2m}(q)$, $\mathrm{GO}_{2m+1}(q)$, $\mathrm{GU}_n(q)$ (q a prime power, $n \geq 2$, $m \geq 1$), or the unitary group U_n over $\mathbb{C}$. Let d_G be the normalized Hamming metric $d_{\mathrm{H}}(\sigma, \tau) \coloneqq |\{x \in \{1, \dots, n\} | x.\sigma \neq x.\tau\}|/n$ resp. the normalized rank metric $d_{\mathrm{rk}}(g, h) \coloneqq \mathrm{rk}(g - h)/n$ on G when G is a symmetric group $\mathfrak{S}_n$ resp. one of the other classical linear groups and write $n(G)$ for the degree of G.

For $\varepsilon > 0$ arbitrary, we prove that there exists an integer $N(\varepsilon, w)$ such that $w(G)$ is ε-dense in G with respect to the metric d_G if $n(G) \geq N(\varepsilon, w)$, i.e. $d_G(g, w(G)) \leq \varepsilon$ for all $g \in G$. This confirms metric versions of conjectures by Shalev [2, Conjecture 8.3] and Larsen at the 2008 meeting of the AMS in Bloomington. Equivalently, we prove that any non-trivial word map is surjective on a metric ultraproduct of groups G from above such that $n(G) \to \infty$ along the ultrafilter.

The method of the proof is cohomological. We consider the Cayley complex X of the one-relator group $K \coloneqq \langle x_1, \dots, x_k | w \rangle$ and the corresponding quotient complex $X(\pi)$ under a given surjective homomorphism $\pi\colon K \twoheadrightarrow G$ to a finite group of order n. Then we draw a connection between the above density question and the second cohomology group $H^2(X(\pi), A)$, where A is a suitable abelian group which is chosen according to the type of G. Roughly speaking, we study the set $w(A \wr G)$ (where $G \curvearrowright G$ by left-multiplication) by plugging in the tuple $(a_1.g_1, \dots, a_k.g_k) \in (A \wr G)^k$ into w, where $a_i \in A^G$ and $g_i = \pi(x_i)$ for $i = 1, \dots, k$. Let us sketch the proof in the case $G = \mathrm{U}_n$ In this case we use the group $A = \mathbb{T}$ and that $\mathbb{T} \wr G \subseteq \mathrm{U}_n$ as monomial matrices, so that $w(\mathbb{T} \wr G) \subseteq w(\mathrm{U}_n)$ (here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle). Then $w(a_1.g_1, \dots, a_k.g_k) = b.1_G \in \mathbb{T}^G \subseteq \mathbb{T} \wr G \subseteq \mathrm{U}_n$ is a diagonal matrix. One can then show that one can approximate a given diagonal matrix $c \in \mathrm{U}_n$ by a word value $w(a_1.g_1, \dots, a_k.g_k)$ up to $\dim H^2(X(\pi), \mathbb{T})$ diagonal entries (for suitable $a_1, \dots, a_k \in \mathbb{T}^G$). The remaining task is then to construct "good" homomorphism π such that $\dim H^2(X(\pi), \mathbb{T})/n$ is "small".

As a consequence of our methods, we also obtain an alternative proof of the result of Hui, Larsen, and Shalev [3, Theorem 2.3] that $w_1(\mathrm{SU}_n)w_2(\mathrm{SU}_n) = \mathrm{SU}_n$ for non-trivial words $w_1, w_2 \in \mathbf{F}_k$ and n sufficiently large.

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Good Models for Infinite Approximate Subgroups

Simon Machado

We study infinite approximate subgroups with a particular focus on Approximate lattices. Approximate lattices were first defined by Michael Björklund and Tobias Hartnick in [2]. They were inspired by the work of Yves Meyer about a type of approximate subgroups that later came to be known as mathematical quasi-crystals ([6]). Approximate lattices also generalize lattices of locally compact groups (i.e. discrete subgroups of locally compact groups with finite co-volume).

When G is a locally compact group, we say that X is *uniformly discrete* if there is a neighborhood of the identity U such that $(xU)_{x\in X}$ is a family of disjoint sets. We say that X is *relatively dense* if there exists a compact subset $K \subseteq G$ such that $\Lambda K = G$. An approximate subgroup Λ of a locally compact group G is a *uniform approximate lattice* if Λ is uniformly discrete and relatively dense. The approximate subgroup condition arises naturally from the combination of discreteness and co-compactness. Indeed, any symmetric subset $\Lambda \subseteq G$, such that Λ is relatively dense and Λ^6 is discrete (with respect to the induced topology), is an approximate subgroup, and hence a uniform approximate lattice.

Examples of uniform approximate lattices are given by *cut-and-project schemes.* A cut-and-project scheme (G, H, Γ) is the datum of two locally compact groups G and H, and a uniform lattice Γ in $G\times H$ such that $\Gamma\cap(\{e_G\} \times H) = \{e_{G\times H}\}$ and Γ projects densely to H. Given a cut-and-project scheme (G, H, Γ) and a symmetric relatively compact neighborhood W_0 of e_H in H, one gets a uniform approximate lattice when considering the projection Λ of $(G \times W_0) \cap \Gamma$ to G. Any such set is called a *model set* and any approximate subgroup of G which is commensurable to and contained in a model set is called a *Meyer set* of G. This construction was first introduced by Yves Meyer in the abelian case [6] and extended by Michael Björklund and Tobias Hartnick [2].

In [6] Yves Meyer proved a structure theorem for mathematical quasi-crystals. Quasi-crystals correspond to uniform approximate lattices in locally compact abelian groups. Rephrased in our terminology he proved that all approximate lattices of locally compact abelian groups are Meyer sets ([6, Theorem 3.2]). Motivated by this result the authors of [2] asked whether similar results would hold for other classes of locally compact groups. This question, and more generally the structure theory of Approximate lattices, is the main motivation of this talk.

Using methods from algebraic group theory we can prove a first generalization of Meyer's theorem. Namely, all approximate lattices of soluble Lie groups are Meyer sets. To further extend Meyer's theorem we introduce *good models* for approximate subgroups. The definition of good models and the following results come from an article in preparation ([4]). A *good model for* an approximate subgroup Λ of a group G is a group homomorphism $f\colon \Lambda^\infty \to H$, with H a locally compact group, such that: (i) The set $f(\Lambda)$ is relatively compact and (ii) there is $U \subseteq H$ a neighborhood of the identity such that $f^{-1}(U) \subseteq \Lambda$.

The definition of good models is closely related to [1, Definition 3.5], [3, Theorem 4.2] and [2, Definition 2.12]. However, each of these prior definitions asks for stronger hypotheses whereas we try to keep the definition of good models as simple as possible. This allows for a simple and handy characterization of approximate subgroups that have a good model. Indeed, an approximate subgroup Ł of a group G has a good model if and only if there exists a decreasing sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of approximate subgroups commensurable to Λ with $Ł_0 = Ł$ and $\Lambda_{n+1}^2 \subseteq \Lambda_n$. This criterion is inspired by the construction from [1, Section 6]. It is a handy criterion that finds applications in a variety of situations. Some approximate subgroups however do not have a good model.

Coming back to approximate lattices, one can see that good models generalize cut-and-project schemes. In particular an approximate lattice has a good model if and only if it is a model set. Building up on the criterion mentioned above and an argument due to Tom Sanders and Wagner–Massicot we give a new generalization to Meyer's theorem; we prove that if Λ an approximate lattice of an amenable locally compact group G, then Λ^4 is a model set. The method used actually yields a much stronger result about all uniformly discrete approximate subgroups of amenable locally compact groups.

We mention yet another result: a generalization of theorems of Auslander and Mostow about lattices and radicals of Lie groups. In particular, this result asserts that the theory of approximate lattices of Lie groups reduces to two simpler bits: the theory of approximate lattices of amenable Lie groups and the theory of lattices of semi-simple groups.

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On the "free group factor" and the "single generation" problem

Sorin Popa

A well known open problem in operator algebras, originating in Murray's and von Neumann's "Rings af operators IV" paper from 1943, asks whether the group II_1 factors $L(\mathbf{F}_n)$ of the free groups on n generators ($2 \leq n \leq \infty$) are isomorphic or not. Another open problem going back to a paper by Chevalley ??? from 1954 asks whether any II_1 factor is simply generated (or generated by two hermitians). I will explain an approach to solving these two problems, based on the idea of constructing "tight pairs" of embeddings of the amenable II_1 factor $\mathcal{R}$ (the so-called hyperfinite II_1 factor) into any II_1 factor that is stably single generated.

Fun with i.(ni).d. random variables

Michael Björklund
(joint work with Zemer Kosloff and Stefaan Vaes)

Let G be a countable group and consider the G-action on $X \coloneqq \{0,1\}^G$ defined by

$$(g.x)_h = x_{g^{-1}h}, \quad \text{for } h, g \in G.$$

Given a map $g \mapsto \mu_g \in \mathrm{Prob}(\{0,1\})$, we define

$$\mu = \prod_{g \in G} \mu_g \in \mathrm{Prob}(X).$$

If we assume that there exists $0 < \delta < 1/2$ such that $\mu_g(0) \in [\delta, 1-\delta]$ for all $g \in G$, then μ is quasi-invariant if and only if

$$\sum_{h \in G} \big(\mu_{g^{-1}h}(0) - \mu_h(0)\big)^2 < \infty, \quad \text{for all } g \in G,$$

or equivalently, if $g \mapsto c_g$, where $c_g(h) = \mu_{g^{-1}h}(0) - \mu_h(0)$ is an $\ell^2(G)$-cocycle (for the left-regular G-representation). We note that if c is cohomologically trivial, then μ is equivalent to a G-invariant probability measure.

During this talk, we discussed the following question: Suppose that G admits a non-trivial ℓ^2-cocyle. Can we then construct a map $g \mapsto \mu_g$ as above, so that the resulting measure μ is quasi-invariant, but "very far from being G-invariant"? By this we mean: Consider the Maharam extension of the G-action on (X,μ), that is to say, the G-action on $X \times \mathbb{R}$, given by

$$g.(x,t) = (g.x, t - \log \frac{dg^{-1}\mu}{d\mu}(x)), \quad \text{for } (x,t) \in X \times \mathbb{R},$$

endowed with the G-invariant (non-finite) measure $\mu \otimes e^{-t}\,dt$. We note that if (X,μ) is equivalent to a σ-finite G-invariant measure, then the Maharam extension admits a G-invariant Borel function. Hence, to exclude the possibility of a coordinate change to a G-invariant measure, we ask when the Maharam extension is ergodic (or even weakly mixing) – in this case we say that (X,μ) is type III_1. We prove in [1] that every group which admits a non-trivial $\ell^2(G)$-cocycle, has measures μ with type III_1.

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Kazhdan groups have cost 1

Thomas Hutchcroft
(joint work with Gábor Pete)

The *cost* of a free, probability measure preserving (p.m.p.) action of a group is an orbit-equivalence invariant that was introduced by Levitt [10] and studied extensively by Gaboriau [4, 5, 6]. Gaboriau used the notion of cost to prove several remarkable theorems, including that free groups of different ranks cannot have orbit equivalent actions. This result is in stark contrast with the amenable case, in which Ornstein and Weiss [11] proved that any two free p.m.p. actions are orbit equivalent.

The cost of a *group* is defined to be the infimal cost of all free, ergodic p.m.p. actions of the group. We will employ here the following probabilistic definition, which is shown to be equivalent to the classical definition in [9, Proposition 29.5]. Let Γ be a countable group. We define $\mathcal{S}(\Gamma) \subseteq \{0,1\}^{\Gamma^2}$ to be the set of (undirected) *connected graphs* with vertex set Γ. For each $\omega \in \mathcal{S}(\Gamma)$ and $\gamma \in \Gamma$ we define $\gamma\omega$ by setting $\gamma\omega(u,v) = \omega(\gamma u, \gamma v)$. We say that a probability measure on $\mathcal{S}(\Gamma)$ is Γ-*invariant* if $\mu(\mathscr{A}) = \mu(\gamma^{-1}\mathscr{A})$ for every Borel set $\mathscr{A} \subseteq \mathcal{S}(\Gamma)$, and write $M(\Gamma, \mathcal{S}(\Gamma))$ for the set of Γ-invariant probability measures on $\mathcal{S}(\Gamma)$. The *cost of the group* Γ can be defined to be

$$\text{(1)} \qquad \operatorname{cost}(\Gamma) = \frac{1}{2} \inf \left\{ \int_{\omega \in \mathcal{S}(\Gamma)} \deg_\omega(o) d\mu(\omega) : \mu \in M(\Gamma, \mathcal{S}(\Gamma)) \right\},$$

where o is the identity element of Γ and $\deg_\omega(o)$ is the degree of o in the graph $\omega \in \mathcal{S}(\Gamma)$.

Many properties of the cost remain poorly understood. One of the most important questions is as follows: Gaboriau proved that for any finitely generated group Γ, the cost of Γ satisfies $\operatorname{cost}(\Gamma) \geq 1 + \beta_1(\Gamma)$, where $\beta_1(\Gamma)$ is the first ℓ^2-Betti number of Γ. It is conjectured that this inequality is in fact an equality for every finitely generated group Γ.

For groups with Kazhdan's property (T), it was proven in 1997 by Bekka and Valette [1] that $\beta_1 = 0$. However, the *cost* of Kazhdan groups has remained elusive [6, Question 6.4], and has thus become a notable test example for the above question. In this talk, I will sketch a proof that Kazhdan groups do indeed have cost 1, following our work [8].

Our proof uses probabilistic techniques from percolation theory and the probabilistic characterization of property (T) due to Glasner and Weiss [7]. (This proof fact obtained in Oberwolfach at the meeting "Scaling Limits in Models of Statistical Mechanics" in September 2018.)

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Von Neumann Equivalence

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(joint work with Ishan Ishan, Lauren Ruth)

Two countable groups Γ and Λ are measure equivalent if they have commuting measure-preserving actions on a σ-finite measure space (Ω, m) such that the actions of Γ and Λ individually admit a finite-measure fundamental domain. This notion was introduced by Gromov in [4, 0.5.E] as an analogy to the topological notion of being quasi-isometric for finitely generated groups. The basic example of measure equivalent groups is when Γ and Λ are lattices in the same locally compact group G. In this case, Γ and Λ act on the left and right of G respectively, and these actions preserve the Haar measure on G.

Two groups Γ and Λ are W^*-equivalent if they have isomorphic group von Neumann algebras $L\Gamma \cong L\Lambda$. This is somewhat analogous to measure equivalence (although a closer analogy is made between measure equivalence and virtual W^*-equivalence, which for ICC groups asks for $L\Gamma$ and $L\Lambda$ to be virtually isomorphic in the sense that each factor is stably isomorphic to a finite index subfactor in the other factor [5, Section 1.4]) and both equivalence relations preserve many of the same "approximation type" properties. These similarities led Shlyakhtenko to ask whether measure equivalence implied W^*-equivalence in the setting of ICC groups. It was shown in [2] that this is not the case, although the converse implication of whether W^*-equivalence implies measure equivalence is still open.

Returning to measure equivalence, if Γ and Λ have commuting actions on (Ω, μ) and if $F \subset \Omega$ is a Borel fundamental domain for the action of Γ, then on the level of function spaces, the characteristic function 1_F gives a projection in $L^\infty(\Omega, m)$ such that the collection $\{1_{\gamma F}\}_{\gamma\in\Gamma}$ forms a partition of unity, i.e., $\sum_{\gamma\in\Gamma} 1_{\gamma F} = 1$. This notion generalizes quite nicely to the non-commutative setting where we will say that a fundamental domain for an action on a von Neumann algebra $\Gamma \curvearrowright^\sigma \mathcal{M}$ consists of a projection $p \in \mathcal{M}$ such that $\sum_{\gamma\in\Gamma} \sigma_\gamma(p) = 1$, where the convergence is in the strong operator topology.

Using this perspective for a fundamental domain we may then generalize the notion of measure equivalence by simply considering actions on non-commutative spaces.

Definition 1. *Two groups Γ and Λ are von Neumann equivalent, written $\Gamma \sim_{vNE} \Lambda$, if there exists a von Neumann algebra $\mathcal{M}$ with a semi-finite normal faithful trace* Tr *and commuting, trace-preserving, actions of Γ and Λ on $\mathcal{M}$ such that the Γ and Λ-actions individually admit a finite-trace fundamental domain.*

The proof of transitivity for measure equivalence is adapted to show that von Neumann equivalence is a transitive relation. It is also clearly reflexive and symmetric, so that von Neumann equivalence is indeed an equivalence relation.

Von Neumann equivalence is clearly implied by measure equivalence, and, in fact, von Neumann equivalence is also implied by W^*-equivalence. Indeed, if $\theta : L\Gamma \to L\Lambda$ is a von Neumann algebra isomorphism then we may consider $\mathcal{M} = \mathcal{B}(\ell^2\Lambda)$ where we have a trace-preserving action $\sigma : \Gamma \times \Lambda \to \mathrm{Aut}(\mathcal{M})$ given by $\sigma_{(s,t)}(T) = \theta(\lambda_s)\rho_t T \rho_t^* \theta(\lambda_s^*)$, where $\rho : \Lambda \to \mathcal{U}(\ell^2\Lambda)$ is the right regular representation, which commutes with operators in $L\Lambda$. It's then not difficult to see that the rank one projection p onto the subspace $\mathbb{C}\delta_e$ is a common fundamental domain for the actions of both Γ and Λ. In fact, we'll show below that virtual W^*-equivalence also implies von Neumann equivalence.

Using a general induction procedure for inducing representations via von Neumann equivalence we obtain the following theorem.

Theorem. *Amenability, property (T), and the Haagerup property are all von Neumann equivalence invariants.*

A group Γ is properly proximal if there does not exist a left-invariant state on the C^*-algebra $(\ell^\infty\Gamma/c_0\Gamma)^{\Gamma_r}$ consisting of elements in $\ell^\infty\Gamma/c_0\Gamma$ that are invariant under the right action of the group. Properly proximal groups were introduced in [1], where a number of classes of groups were shown to be properly proximal, including non-elementary hyperbolic groups, convergence groups, bi-exact groups, groups admitting proper 1-cocycles into non-amenable representations, and lattices in non-compact semi-simple Lie groups of arbitrary rank. It is also shown that the class of properly proximal groups is stable under commensurability up to finite kernels, and it was then asked if this class was also stable under measure equivalence [1, Question 1(b)].

Proper proximality also has a dynamical formulation [1, Theorem 4.3], and using this, together with our induction technique applied to isometric representations on dual Banach spaces, we show that the class of properly proximal groups is not only closed under measure equivalence but also under von Neumann equivalence.

Theorem. *If* $\Gamma \sim_{vNE} \Lambda$ *then* Γ *is properly proximal if and only if* Λ *is properly proximal.*

An example of Caprace, which appears in Section 5.C of [3], shows that the class of inner amenable groups is not closed under measure equivalence. Specifically, if p is a prime and F_p denotes the finite field with p elements, then the group $\mathrm{SL}_3(F_p[t^{-1}]) \ltimes F_p[t, t^{-1}]^3$ is not inner amenable, although is measure equivalent to the inner amenable group $\mathrm{SL}_3(F_p[t^{-1}]) \ltimes F_p[t^{-1}]^3) \times F_p[t]^3$. Using the previous theorem we then answer another question from [1] by providing with $\mathrm{SL}_3(F_p[t^{-1}]) \ltimes F_p[t, t^{-1}]^3$ an example of a non-inner amenable group that is also not properly proximal.

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