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# Model Theory: Groups, Geometries and Combinatorics

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Abstract. The focus of the conference were recent interactions between model theory, group theory and combinatorics in finite geometries. In some cases, in particular in non-archimedean geometry or combinatorics in finite geometries, model theory appeared as tool. In other cases, like in ergodic theory and dynamics or in the theory of stable groups and more general neo-stable algebraic structures like valued fields, the focus was on model theoretic questions and classification results for such structures. In this way, the conference presented the broad range of topics of modern model theory.

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## Introduction by the Organizers

Recent years brought exciting developments in model theory, in particularly with respect to interactions with group theory and finite combinatorics. The classical framework of stable theories has been generalized in order to consider interesting applications beyond the stable setting, such as groups definable over Henselian valued fields or pseudofinite structures, of combinatorial nature. Thus, tools such as independence and measure, originally developed for stable theories, have now been extended to wider classes of neostable structures.

The purpose of the meeting was to cover the most relevant and active areas in the above subjects, focusing on the following four recurrent topics which lie at the core of the current developments:

(i) pseudofinite combinatorics
(ii) ergodic theory and dynamics

(iii) stable groups
(iv) neostable groups

The excellent quality of the talks, as well as the lively interaction among the participants during the meeting will certainly inspire new interactions among the different areas. We will summarize below some of the aspects which were presented during the meeting in January 2020.

Inspired by Hrushovski's work on approximate subgroups, which subsequently lead Breuillard, Green and Tao to the solution of the general inverse Freiman problem and a structure theorem for approximate groups, there has been a fruitful interaction between model theory and finite combinatorics, often in terms of pseudo-finite additive combinatorics. A pseudo-finite set is an infinite ultraproduct of finite sets. Asymptotic properties of large finite sets often translate to model-theoretic familiar properties of the corresponding ultraproducts.

A fundamental result in geometric stability shows that any form of non-linearity gives rise to a pseudo-plane, that is, a binary relation with similar properties to those of the incidence relation in a projective plane. This sheds a new light on well-known results *à la Szemeredi-Trotter*: there are no (proper) pseudo-finite pseudo-planes, unless they arise from a pseudo-finite field, which often cannot occur, for example in $\mathbb{C}$, or in finite fields at a size far below the characteristic.

Around pseudo-finite additive combinatorics, it is worth mentioning the talks of Bays, Chernikov,Hrushovski and Palacín on the model-theory side as well as those of Long and Machado on the combinatoric and geometry side, highlighting the progress done along these lines and opening new research directions.

Ultrafilters and other compactifications are common in topological dynamics. The use of continuous logic in order to treat a class of metric spaces from a first-order point of view has established as a solid approach to problems from ergodic theory, the study of sofic groups and connections to combinatorics. The talks by model-theorists Conant, Goldbring, Krupinski or Pillay, together with the talk by Thom on metric ultraproducts of groups, illustrated perfectly the different facets of this area.

In order to positively answer Tarski's original question, Sela developed algebraic geometry on the free group, a new class of stable groups, by means of a detailed study of its definable subsets. While Sela has now moved on to developing similar tools for free aglebras, his understanding of the definable sets in non-abelian free groups has had a direct impact on related questions, as work of Bestvina and Feighn shows. Model-theoretic notions such as a description of independence, exemplified during the talk of Perin on joint work with Sklinos, or a classification of finitely generated virtually free groups, up to to elementary equivalence, as in André's talk, complemented perfectly Sela's talk on non-commutative algebraic geometry.

Neostability, both as an upcoming research area per se, with the contributions of Kaplan, Peterzil or Ramsey, as well as its applications to model theory of fields with operators: valuations, as in the talks of Hempel, Jahnke, Loeser or Rideau, or Mahler operators in the talk of Scanlon, exhibited the joint effort of the model theory community in order to extend and generalise the tools and techniques known for stable strutures, and particularly groups, to wider classes of theories. These efforts will certainly lead to new applications to groups and fields definable in neostable contexts.

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## Model Theory: Groups, Geometries and Combinatorics

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# Abstracts

## Asymptotics of complex integrals via Robinson's non-archimedean field

François Loeser
(joint work with Antoine Ducros, Ehud Hrushovski)

### 1. The framework

**1.1.** A. Chambert-Loir and A. Ducros recently developed a full-fledged theory of real valued $(p, q)$-forms and currents on Berkovich spaces which is an analogue of the theory of differential forms on complex spaces [CLD]. Their forms are constructed as pullbacks under tropicalisation maps of the "superforms" introduced by Lagerberg [Lag12]. They are able to integrate compactly supported $(n, n)$-forms for $n$ the dimension of the ambient space (the output being a real number) and they obtain versions of the Poincaré-Lelong Theorem and the Stokes Theorem in this setting. Their work is guided throughout by an analogy with complex analytic geometry. The aim of the present work is to convert the analogy into a direct connection, showing how the non-archimedean theory appears as an asymptotic limit over of one-parameter families of complex (archimedean) forms and integrals.

We work over an algebraically closed field $C$ containing $\mathbf{C}$, which is a degree 2 extension of a real closed field $R$ containing $\mathbf{R}$ and is endowed at the same time with an archimedean non-standard norm $|\cdot| : C \to R_+$ and with a non-archimedean norm $|\cdot|_\flat : C \to \mathbf{R}_+$ that essentially encapsulates the "order of magnitude" of $|\cdot|$ with respect to a given infinitesimal element which should be thought of as a "complex parameter tending to zero". This presents the advantage of working on spaces that have at the same time archimedean and non-archimedean features and allows to be able to compare directly archimedean constructions and their non-archimedean counterparts. The fields $R$ and $C$ are constructed using ultrapowers. The field $R$ was introduced by A. Robinson in [Rob73], with the explicit hope that it will be useful for asymptotic analysis; see also [LR75].

**1.2.** The construction of the field $C$ goes as follows. Fix a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{C}$ containing all the neighbourhoods of the origin and consider the ultrapowers ${}^*\mathbb{C} = \prod_{t \in \mathbb{C}^\times} \mathbb{C}/\mathcal{U}$ and ${}^*\mathbb{R} = \prod_{t \in \mathbb{C}^\times} \mathbb{R}/\mathcal{U}$. We say an element $(a_t)$ in ${}^*\mathbb{C}$, resp. ${}^*\mathbb{R}$, is $t$-bounded if for some positive integer $N$, $|a_t| \leq |t|^{-N}$ along $\mathcal{U}$. Similarly, it is said to be $t$-negligible if for every positive integer $N$, $|a_t| \leq |t|^N$ along $\mathcal{U}$. The set of $t$-bounded elements in ${}^*\mathbb{C}$, resp. ${}^*\mathbb{R}$, is a local ring which we denote by $A$, resp. $A_{\mathrm{r}}$, with maximal ideal the subset of $t$-negligible elements which we denote by $\mathfrak{M}$, resp. $\mathfrak{M}_{\mathrm{r}}$. We now set $C := A/\mathfrak{M}$ and $R := A_{\mathrm{r}}/\mathfrak{M}_{\mathrm{r}}$. The field $R$ is a real closed field and $C \simeq R(i)$ is algebraically closed. The norm $|\cdot|: {}^*\mathbf{C} \to {}^*\mathbf{R}_{\geq 0}$ induces an $R$-valued norm $|\cdot| : C \to R_{\geq 0}$.

**1.3.** Any usual smooth function $\phi\colon U \to \mathbf{R}$ defined on some semi-algebraic open subset $U$ of $\mathbf{R}^n$ induces formally a map $U({}^*\mathbf{R}) \to {}^*\mathbf{R}$ which is still denoted by $\phi$. Allowing ourselves to compose these functions (which arise from *standard* smooth functions) with polynomial maps (which might have non-standard coefficients), we define for every smooth, separated ${}^*\mathbf{R}$-scheme $X$ of finite type a sheaf of so-called smooth functions for the (Grothendieck) semi-algebraic topology on $X({}^*\mathbf{R})$, which we denote by $\mathscr{C}^\infty_X$. The natural inclusion map from $X({}^*\mathbf{R})$ into the (underlying set of) the scheme $X$ underlies a morphism of locally ringed sites $\psi\colon (X({}^*\mathbf{R}, \mathscr{C}^\infty_X) \to (X, \mathscr{O}_X)$, and we can define the sheaf of smooth $p$-forms on $X({}^*\mathbf{R})$ by $\mathscr{A}^p_X := \psi^*\Omega^p_{X/{}^*\mathbf{R}}$. One has for every $p$ a natural differential $\mathrm{d}\colon \mathscr{A}^p_X \to \mathscr{A}^{p+1}_X$. We now assume $X$ is of pure dimension $n$, and that $X({}^*\mathbf{R})$ is oriented.

Let $\omega$ be a smooth $n$-form on some semi-algebraic open subset $U$ of $X({}^*\mathbf{R})$, and let $E$ be a semi-algebraic subset of $U$ whose closure in $U$ is definably compact. Choosing a description of $(X, U, \omega, E)$ through a "limited family" $(X_t, U_t, \omega_t, E_t)_t$, it is possible to define the integral $\int_E \omega$ as the class of the sequence $(\int_{E_t} \omega_t)_t$ in ${}^*\mathbf{R}$.

**1.4.** We now move from ${}^*\mathbf{R}$ to $R$, seeking to show that smooth functions, smooth forms and their integrals remains well-defined on $R$.

Let $\phi\colon U \to \mathbf{R}$ be a usual smooth function defined on some semi-algebraic open subset $U$ of $\mathbf{R}^n$. Under some boundedness assumptions on $\phi$ (which are for instance automatically fulfilled if $\phi$ is compactly supported, or more generally if all its derivatives are polynomially bounded), the induced function $\phi\colon U({}^*\mathbf{R}) \to {}^*\mathbf{R}$ in turn induces a map $U(R) \to R$, which we again denote by $\phi$.

For instance, the map $\log|\cdot|$ from $\mathbf{C}^\times \simeq \mathbf{R}^2 \setminus \{(0,0)\}$ is smooth and satisfies the boundedness conditions alluded to above; it thus induces a map $\log|\cdot|\colon C^\times \to R$, which enables us to endow the field $C$ with a real-valued non-archimedean norm $|\cdot|_\flat : C \to \mathbb{R}_{\geq 0}$ as follows. For any $z$ belonging to $C^\times$, one checks that the norm of $\frac{\log|z|}{\log|t|}$ is bounded by some positive real number in $\mathbb{R}$. One can thus consider its standard part $\alpha = \mathrm{std}\Big(\frac{\log|z|}{\log|t|}\Big) \in \mathbb{R}$. Fixing $\tau \in (0,1) \subset \mathbb{R}$, one sets $|z|_\flat := \tau^\alpha$, so that $|z|_\flat = |t|^\alpha_\flat$. With this non-archimedean norm the field $C$ is complete (even spherically complete, cf. [Lux76]).

We repeat the procedure used in 1.3: allowing ourselves to compose the functions defined at the beginning of 1.4 (which arise from *standard* smooth functions) with polynomial maps (which might have non-standard coefficients), we define for every smooth, separated $R$-scheme $X$ of finite type a sheaf of so-called smooth functions for the (Grothendieck) semi-algebraic topology on $X(R)$, which we denote by $\mathscr{C}^\infty_X$. There is a natural morphism of locally ringed sites $\psi\colon (X(R), \mathscr{C}^\infty_X) \to (X, \mathscr{O}_X)$. One then sets $\mathscr{A}^p_X := \psi^*\Omega^p_{X/R}$ and one has for every $p$ a natural differential $\mathrm{d}\colon \mathscr{A}^p_X \to \mathscr{A}^{p+1}_X$.

Assume now $X$ is of pure dimension $n$ and oriented.

**1.5. Proposition.** *Integration theory on $X(A_{\mathrm{r}})$ descends to $X(R)$.*

Namely, to a semi-algebraic subset $K$ of $X(R)$, with definably compact definable closure, and a smooth $n$-form $\omega$ on a semi-algebraic neighborhood of $K$ in $X(R)$, we assign an integral $\int_K \omega$ which is an element of $R$. This is achieved by reducing to the case when $X$ is liftable. Independence from the lifting follows from the fact that the integrals obtained from two different liftings coincide up to a $t$-negligible element. A preliminary key statement in that direction is that if $D$ is a semi-algebraic subset of $({}^*\mathbf{R})^n$ contained in $A_{\mathrm{r}}^n$, the volume of $D$ is $t$-negligible if and only if the image of $D$ in $R^n$ through the reduction map is of dimension $\leq n-1$.

**1.6.** Assume from now on that $X$ is a smooth $C$-scheme of finite type and of pure dimension $n$. One defines similarly the integral $\int_K \omega$ of a complex-valued $(n,n)$-form $\omega$ defined in a semi-algebraic neighborhood of a semi-algebraic subset $K$ of $X(C)$, assuming that there exists a semi-algebraic subset $K'$ of $K$ with definably compact closure such that $\omega$ vanishes on $K \setminus K'$. Set $\lambda := -\log|t|$. We construct, for $p \geq 0$, $q \geq 0$, a Zariski-sheaf $\mathsf{A}^{p,q}$ of (a skewed version of) Dolbeault-like forms on $X$.

To get a flavor of how sections of $\mathsf{A}^{p,q}$ look like, take $U$ an open subset of $X$ and a family $(f_1, \ldots, f_m)$ of regular invertible functions on $U$. For each subset $I$, resp. $J$, of $\{1, \ldots, n\}$ of cardinality $p$, resp. $q$, take a smooth function $\phi_{I,J}$ on $\mathbf{R}^m$. The form

$$\omega = \frac{1}{\lambda^p} \sum_{I,J} \phi_{I,J} \left( \frac{\log|f_1|}{\lambda}, \ldots, \frac{\log|f_m|}{\lambda} \right) \mathrm{dlog}|f_I| \wedge \mathrm{darg}\, f_J$$

(with $\mathrm{dlog}|f|_I$ standing for the wedge product $\mathrm{dlog}|f_{i_1}| \wedge \ldots \wedge \mathrm{dlog}|f_{i_p}|$ if $i_1 < i_2 < \ldots < i_p$ are the elements of $I$, and similarly for $\mathrm{darg}\, f_J$) is an example of a section of $\mathsf{A}^{p,q}$ on $U$.

For technical reasons we are led to introduce a larger class of forms which have the advantage of being more flexible: sections of $\mathsf{A}^{p,q}$ are locally of the form

$$\omega = \frac{1}{\lambda^p} \sum_{I,J} \phi_{I,J} \left( \frac{\log|f_1|}{\lambda}, \ldots, \frac{\log|f_m|}{\lambda} \right) \mathrm{dlog}|f_I| \wedge \mathrm{darg}\, f_J$$

where $(f_1, \ldots, f_m)$ are regular functions, where $I$, resp. $J$, is running through the set of subsets of $\{1, \ldots, n\}$ of cardinality $p$, resp. $q$, and where the functions $\phi_{I,J}$ satisfy certain conditions in addition of being smooth ensuring good behaviour when approaching the zero locus of some of the functions $f_i$. In particular they vanish around every point $(x_1, \ldots, x_m)$ of $(\mathbf{R} \cup \{-\infty\})^m$ such that $x_i = -\infty$ for some $i \in I \cup J$, and $\mathrm{dlog}|f_i|$ or $\mathrm{darg}\, f_i$ can actually appear only around points at which $f_i$ is invertible. These conditions reduce to smoothness away from the locus where some $f_i$ vanishes.

There exist natural differentials $d : \mathsf{A}^{p,q} \to \mathsf{A}^{p+1,q}$ and $\mathrm{d}^\sharp : \mathsf{A}^{p,q} \to \mathsf{A}^{p,q+1}$ mapping respectively a form

$$\frac{1}{\lambda^p} \phi \left( \frac{\log|f_1|}{\lambda}, \ldots, \frac{\log|f_m|}{\lambda} \right) \mathrm{dlog}|f_I| \wedge \mathrm{darg}\, f_J$$

to

$$\frac{1}{\lambda^{p+1}} \sum_{1 \leq i \leq m} \frac{\partial \phi}{\partial x_i} \left( \frac{\log|f_1|}{\lambda}, \ldots, \frac{\log|f_m|}{\lambda} \right) \mathrm{dlog}|f_i| \wedge \mathrm{dLog}|f_I| \wedge \mathrm{darg}\, f_J.$$

and to

$$\frac{1}{\lambda^{p}} \sum_{1 \leq i \leq m} \frac{\partial \phi}{\partial x_i} \left( \frac{\log|f_1|}{\lambda}, \ldots, \frac{\log|f_m|}{\lambda} \right) \mathrm{darg}\, f_i \wedge \mathrm{dlog}|f_I| \wedge \mathrm{darg}\, f_J.$$

Here the map $d$ is the usual differential, and $d^\sharp$ is designed to switch modulus and argument; it turns out to be analogous to the operator $\mathrm{d^c}$ of complex analytic geometry.

**1.7.** We now describe the non-archimedean counterparts of the previous constructions. Let us denote by $X^{\mathrm{an}}$ the Berkovich analytification of the scheme $X$ and set $\lambda_\flat := -\log|t|_\flat$. We construct, for each $p \geq 0$, $q \geq 0$, a Zariski-sheaf $\mathsf{B}^{p,q}$ on $X^{\mathrm{an}}$ whose sections, locally for the Zariski-topology of $X$, are $(p,q)$-smooth forms in the sense of [CLD] of the form

$$\omega = \sum_{I,J} \phi_{I,J} \left(\log|f_1|_\flat, \ldots, \log|f_m|_\flat\right) \mathrm{d}' \log|f_I|_\flat \wedge \mathrm{d}'' \log|f_J|_\flat$$

where $(f_1, \ldots, f_m)$ are regular functions, where $I$, resp. $J$, is running through the set of subsets of $\{1, \ldots, n\}$ of cardinality $p$, resp. $q$ (with $\mathrm{d}' \log|f_I|_\flat$ standing for the wedge product $\mathrm{d}' \log|f_{i_1}|_\flat \wedge \ldots \wedge \mathrm{d}' \log|f_{i_p}|_\flat$ if $i_1 < i_2 < \ldots < i_p$ are the elements of $I$, and similarly for $\mathrm{d}'' \log|f_J|_\flat$), and where each $\phi_{I,J}$ satisfies conditions similar to those in the definition of $\mathsf{A}^{p,q}$. These conditions reduce to smoothness when the functions $f_i$ do not vanish, and ensure good behaviour when approaching the zero locus of some of the $f_i$'s. In particular the functions $\phi_{I,J}$ vanish around every point $(x_1, \ldots, x_m)$ of $(\mathbf{R} \cup \{-\infty\})^m$ such that $x_i = -\infty$ for some $i \in I \cup J$ and $\mathrm{d}' \log|f_i|_\flat$ or $\mathrm{d}'' \log|f_i|_\flat$ can actually appear only around points at which $f_i$ is invertible.

Sections of the sheaves $\mathsf{B}^{p,q}$ are always $(p,q)$-forms in the sense of Chambert-Loir and Ducros [CLD], despite the fact that the definition of Chambert-Loir and Ducros involves only invertible functions $f_i$, while ours allow the functions $f_i$ to vanish. The reason for this apparent discrepancy is that the definition of Chambert-Loir and Ducros is local for the Berkovich topology while we work with the Zariski topology. In our case, requiring the functions $f_i$ to be invertible would be too stringent in general for the existence of enough global sections.

## 2. The main result

Our main result states that the two sheaves of bi-graded differential $\mathbb{R}$-algebras $\mathsf{A}^{\bullet,\bullet}$ and $\mathsf{B}^{\bullet,\bullet}$ on the site $X_{\mathrm{Zar}}$, consisting respectively of non-standard archimedean and non-archimedean forms, are compatible in the following sense:

**2.1. Theorem.** *There exists a unique morphism of sheaves of bi-graded differential* $\mathbb{R}$*-algebras* $\mathsf{A}^{\bullet,\bullet} \to \mathsf{B}^{\bullet,\bullet}$*, sending a non-standard archimedean form* $\omega$ *to the non-archimedean form* $\omega_\flat$*, such that if* $\omega$ *is of the form*

$$\omega = \frac{1}{\lambda^{|I|}} \phi\left(\frac{\log|f_1|}{\lambda}, \ldots, \frac{\log|f_m|}{\lambda}\right) \mathrm{dlog}|f_I| \wedge \mathrm{darg}\, f_J,$$

*with* $f_1, \ldots, f_m$ *regular functions on a Zariski-open subset* $U$ *of* $X$*,* $I$ *and* $J$ *subsets of* $\{1, \ldots, m\}$*, and* $\phi$ *a quasi-smooth function, then*

$$\omega_\flat = \frac{1}{\lambda_\flat^{|I|}} \phi\left(\frac{\log|f_1|_\flat}{\lambda_\flat}, \ldots, \frac{\log|f_m|_\flat}{\lambda_\flat}\right) \mathrm{d}' \log|f_I|_\flat \wedge \mathrm{d}'' \log|f_J|_\flat.$$

Furthermore, we also prove that the mapping $\omega \mapsto \omega_\flat$ is compatible with integration. A special case of that compatibility can be stated as follows:

**2.2. Proposition.** *Assume that* $\omega$ *is an* $(n, n)$*-form defined on some Zariski open subset* $U$ *of* $X$ *and that its support is contained in a definably compact semi-algebraic subset of* $U(C)$*, then the form* $\omega_\flat$ *on* $X^{\mathrm{an}}$ *is compactly supported,* $\int_{U(C)} |\omega|$ *is bounded by some positive real number in* $\mathbb{R}$ *and*

$$\mathrm{std}\left(\int_{U(C)} \omega\right) = (2\pi)^n \int_{U^{\mathrm{an}}} \omega_\flat,$$

*with* std *standing for the standard part.*

Compatibility with integration is used in an essential way in proving that the mapping $\omega \mapsto \omega_\flat$ is well defined. Indeed it allows us to use a result of Chambert-Loir and Ducros ([CLD], Cor. 4.3.7) stating that, in the boundaryless case, non-zero forms define non-zero currents. A key point in the proof of compatibility with integration is to show that the non-archimedean degree involved in the construction of non-archimedean integrals in [CLD] actually shows up in the asymptotics of the corresponding archimedean integrals.

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## Classification of finitely generated virtually free groups up to ∀∃-equivalence

Simon André

Around 1945, Tarski asked whether all non-abelian finitely generated free groups have the same first-order theory. This problem remained open for more than five decades, and was finally solved by Sela in [3] and by Kharlampovich and Myasnikov in [1]: all non-abelian free groups are elementarily equivalent. Then, Sela gave a classification of all torsion-free hyperbolic groups up to elementary equivalence (see [4]). He also gave sufficient conditions for a subgroup $H$ of a torsion-free hyperbolic group $G$ to be elementarily embedded in $G$. Later, Perin proved in [2] that these sufficient conditions are necessary. The combination of Sela and Perin's results provides a complete characterization of elementarily embedded subgroups of a given torsion-free hyperbolic group.

In my talk, I presented a first step towards a generalization of these results to all hyperbolic groups, possibly with torsion. A prominent subclass of hyperbolic groups is the class of finitely generated virtually free groups. Recall that a group is said to be virtually free if it has a free subgroup of finite index. For instance, it is well-known that $\mathrm{SL}_2(\mathbb{Z})$ has a subgroup of index 12 isomorphic to the free group $F_2$. I gave necessary and sufficient conditions for two virtually free groups $G$ and $G'$ to have the same $\forall\exists$-theory (recall that the $\forall\exists$-theory of a group $G$ is the set of all first-order sentences of the form $\forall \mathbf{x} \exists \mathbf{y} \phi(\mathbf{x}, \mathbf{y})$ satisfied by $G$, where $\phi(\mathbf{x}, \mathbf{y})$ is a quantifier-free formula in the language of groups, and $\mathbf{x}, \mathbf{y}$ are two tuples of variables), as well as a characterization of $\forall\exists$-elementarily embedded subgroups of a virtually free group $G$ (i.e. subgroups of $G$ such that the inclusion into $G$ preserves the validity of $\forall\exists$-formulas). In particular, one recovers the following nice result proved by Perin in [2]: an elementarily embedded subgroup of a free group is a free factor.

Here are two informal versions of our main results.

**Theorem 1.** *Two finitely generated virtually free groups $G$ and $G'$ have the same $\forall\exists$-theory if and only if there exist two isomorphic groups $\Gamma \supset G$ and $\Gamma' \supset G'$ obtained respectively from $G$ and $G'$ by performing a finite sequence of specific HNN extensions over finite groups (called legal large extensions) or replacements of virtually cyclic subgroups by virtually cyclic overgroups (called legal small extensions).*

A typical example of a legal large extension of $F_2$ is the HNN extension $F_3 = F_2 *_{\{1\}}$.

**Theorem 2.** *Let $G$ be a virtually free group, and let $H$ be a proper subgroup of $G$. The following assertions are equivalent:*

*(1) $H$ is $\forall\exists$-elementarily embedded in $G$;*
*(2) $G$ is a multiple legal large extension of $H$.*

In addition, there exist an algorithm that decides whether or not two finitely generated virtually free groups have the same $\forall\exists$-theory, and an algorithm that decides whether or not a finitely generated subgroup of a virtually free group is $\forall\exists$-elementarily embedded.

I conjecture that these results remain true with no restriction on the complexity of formulas.

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## Ramsey theory and topological dynamics for first order theories

Krzysztof Krupiński
(joint work with J. Lee, S. Moconja)

In their seminal paper [3], Kechris, Pestov and Todorčević discovered surprising interactions between dynamical properties of the group of automorphisms of a Fraïssé structure and Ramsey-theoretic properties of its age. For example, they proved that this group is extremely amenable iff the age has the structural Ramsey property and consists of rigid structures (equivalently, the age has the embedding Ramsey property in the terminology used by Zucker in [9]). This started a wide area of research of similar phenomena. Recently, Pillay and the first author [5] gave a model-theoretic account for the fundamental results of Kechris-Pestov-Todorčević (shortly KPT) theory, generalizing the context to arbitrary, possibly uncountable, structures.

However, KPT theory (including such generalizations) is not really about model-theoretic properties of the underlying theory, because: on the dynamical side, it talks about the topological dynamics of the topological group of automorphisms of a given structure, which can be expressed in terms of the action of this group on the universal ambit rather than on type spaces of the underlying theory, and, on the Ramsey-theoretic side, it considers arbitrary colorings (without any definability properties) of the finite subtuples of a given model. Definitions of Ramsey properties for a given structure stated in [5] suggest the corresponding definitions for first order theories just by applying them to a monster model. In this work, we go much further and define various "definable" versions of Ramsey properties for first order theories by restricting the class of colorings to "definable" ones. And then we find the appropriate dynamical characterizations of our "definable" Ramsey properties in terms of the dynamics of the underlying theory (in place of the dynamics of the group of automorphisms of a given model) some of which are surprising and different comparing to classical KPT theory.

The classes of amenable and extremely amenable theories introduced and studied in [2] are defined in a different way than typical Shelah-style, combinatorially defined classes of theories (such as NIP, simple, $NTP_2$). Here, we give Ramsey-theoretic characterizations of [extremely] amenable theories; these characterizations are clearly combinatorial, but still of different flavor than Shelah's definitions. Also, the new classes of theories which we introduce via some Ramsey-theoretic properties or via their dynamical characterizations do not follow the usual Shelah-style way of defining new classes of theories. This makes the whole topic rather novel in model theory.

We find the interaction between "definable" Ramsey properties and the dynamics of first order theories natural and interesting in its own right. However, our original motivation to introduce the "definable" Ramsey properties has some specific origins in model theory and topological dynamics in model theory, which we explain in the next paragraph.

Some methods of topological dynamics were introduced to model theory by Newelski in [7]. Since then a wide research on this topic has been done by Chernikov, Hrushovski, Newelski, Pillay, Rzepecki, Simon, the first author, and others. For any given theory $T$, a particularly important place in this research is reserved for the investigation of the flow $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$, where $\bar{c}$ is an enumeration of a monster model $\mathfrak{C} \models T$ and $S_{\bar{c}}(\mathfrak{C})$ is the space of global types extending $\mathrm{tp}(\bar{c}/\emptyset)$, as it turns out that topological properties of this flow carry important information about the underlying theory. In particular, in [6] it was proved that there exists a topological quotient epimorphism from the Ellis group of the flow $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$ (also called the Ellis group of $T$, as it does not depend on the choice of the monster model by [4]) to $\mathrm{Gal}_{\mathrm{KP}}(T)$ (the Kim-Pillay Galois group of $T$), and even to the larger group $\mathrm{Gal}_{\mathrm{L}}(T)$ (the Lascar Galois group of $T$); in particular, the Ellis group of $T$ captures more information about $T$ than the Galois groups of $T$. This was the starting point for our present research. Namely, from the aforementioned result of [6] one easily deduces that profiniteness of the Ellis group implies profiniteness of $\mathrm{Gal}_{\mathrm{KP}}(T)$, which in turn is known to be equivalent to the equality of the Shelah and Kim-Pillay strong types. The question for which theories the Shelah and Kim-Pillay strong types coincide is fundamental in model theory. This is known to be true in e.g. stable or supersimple theories, but remains a well-known open question in simple theories in general. This led us to the question for which theories the Ellis group is profinite, which is also interesting in its own right (keeping in mind that the Ellis group of $T$ captures more information than any of the Galois groups of $T$). And among the main outcomes of our work are results saying that various Ramsey-like properties of $T$ imply profiniteness of the Ellis group.

Let us briefly discuss the Ramsey properties which we investigate. But before that, we need to introduce the colorings that we are interested in. For a tuple $\bar{a}$ and a set $B$ (or a tuple which is treated as the set of coordinates), $\binom{B}{\bar{a}}$ denotes the set of all $\bar{a}' \subseteq B$ such that $\bar{a}' \equiv \bar{a}$.

**Definition 1.** a) A coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ is *definable* if there are formulae with parameters $\varphi_0(\bar{x}), \ldots, \varphi_{n-1}(\bar{x})$ such that:

$$c(\bar{a}')(i) = \begin{cases} 1, & \models \varphi_i(\bar{a}') \\ 0, & \models \neg\varphi_i(\bar{a}') \end{cases}$$

for any $\bar{a}' \in \binom{\mathfrak{C}}{\bar{a}}$ and $i < n$.

b) A coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ is *externally definable* if there are formulae without parameters $\varphi_0(\bar{x}, \bar{y}), \ldots, \varphi_{n-1}(\bar{x}, \bar{y})$ and types $p_0(\bar{y}), \ldots, p_{n-1}(\bar{y}) \in S_{\bar{y}}(\mathfrak{C})$ such that:

$$c(\bar{a}')(i) = \begin{cases} 1, & \varphi_i(\bar{a}', \bar{y}) \in p_i(\bar{y}) \\ 0, & \neg\varphi_i(\bar{a}', \bar{y}) \in p_i(\bar{y}) \end{cases}$$

for any $\bar{a}' \in \binom{\mathfrak{C}}{\bar{a}}$ and $i < n$.

c) If $\Delta$ is a set of formulae, then an externally definable coloring $c$ is called an *externally definable $\Delta$-coloring* if all the formulae $\varphi_i(\bar{x}, \bar{y})$'s defining $c$ are taken from $\Delta$.

Our Ramsey properties are defined with respect to a monster model $\mathfrak{C}$ of a first-order theory $T$, but we show that they do not depend on the choice of $\mathfrak{C}$, so they are really properties of $T$. We say that $T$ has *separately finite elementary embedding Ramsey degree (sep. fin. EERdeg)* if for every finite $\bar{a} \subseteq \mathfrak{C}$ there exists $l < \omega$ such that for every finite $\bar{b} \supseteq \bar{a}$, $r < \omega$, and coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to r$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\#c[\binom{\bar{b}'}{\bar{a}}] \leqslant l$. If $l$ above can be taken to be 1 for every finite $\bar{a}$, we say that $T$ has the *elementary embedding Ramsey property (EERP)*. If $l$ can be taken to be 1 and we restrict ourselves to considering only [externally] definable colorings, we say that $T$ has the *[externally] definable elementary embedding Ramsey property ([E]DEERP)*. If for every finite set of formulae $\Delta$ and every finite $\bar{a}$ the above holds (for some $l$) for the externally definable $\Delta$-colorings, then we say that $T$ has *separately finite externally definable elementary embedding Ramsey degree (sep. fin. EDEERdeg)*.

Theories with *EERP* and sep. fin. *EERdeg* are generalizations of the classical notions of embedding Ramsey property and finite embedding Ramsey degree in the following sense: If $K$ is an $\aleph_0$-saturated locally finite Fraïssé structure, then its age has the embedding Ramsey property [sep. fin. embedding Ramsey degree] iff $\mathrm{Th}(K)$ has *EERP* [sep. fin. *EERdeg*].

We also consider the following convex Ramsey-like properties. We say that that $T$ has the *elementary embedding convex Ramsey property (EECRP)* if for every $\epsilon \geqslant 0$ and finite $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, $n < \omega$, and coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ there exist $k < \omega$, $\lambda_0, \ldots, \lambda_{k-1} \in [0, 1]$ with $\lambda_0 + \cdots + \lambda_{k-1} = 1$, and $\sigma_0, \ldots, \sigma_{k-1} \in \mathrm{Aut}(\mathfrak{C})$ such that for any two tuples $\bar{a}', \bar{a}'' \in \binom{\bar{b}}{\bar{a}}$ the convex combinations $\sum_{j<k} \lambda_j c(\sigma_j(\bar{a}'))(i)$ and $\sum_{j<k} \lambda_j c(\sigma_j(\bar{a}''))(i)$ differ by at most $\epsilon$ for every $i < n$. If we restrict ourselves to definable colorings, we say that $T$ has the *definable elementary embedding convex Ramsey property (DEECRP)*.

To state our main results, we need to use a natural variant of the usual space of $\Delta$-types, denoted by $S_{\bar{c},\Delta}(\bar{p})$ for a finite set of formulae $\Delta = \{\varphi_0(\bar{x},\bar{y}),...,\varphi_{k-1}(\bar{x},\bar{y})\}$ and a finite set (or sequence) of types $\bar{p} = \{p_0(\bar{y}),\dots,p_{m-1}(\bar{y})\} \subseteq S_{\bar{y}}(\emptyset)$. The space $S_{\bar{c},\Delta}(\bar{p})$ is defined as the space of all complete $\Delta$-types over $\bigsqcup_{j<m} p_j(\mathfrak{C})$ consistent with $\mathrm{tp}(\bar{c})$. For a flow $(G,X)$, by $EL(X)$ we denote the Ellis semigroup of this flow. By $\mathrm{Inv}_{\bar{c}}(\mathfrak{C})$, we denote the space of global invariant types extending $\mathrm{tp}(\bar{c}/\emptyset)$. Our main result yields dynamical characterizations of the introduced Ramsey properties.

**Theorem 2.** *Let $T$ be a complete first-order theory and $\mathfrak{C}$ its monster model. Then:*

*(i) $T$ has DEERP iff $T$ is extremely amenable (in the sense of* [2]*).*
*(ii) $T$ has EDEERP iff there exists $\eta \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ such that $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C})$.*
*(iii) $T$ has sep. fin. EDEERdeg iff for every finite set of formulae $\Delta$ and finite sequence of types $\bar{p}$ there exists $\eta \in \mathrm{EL}(S_{\bar{c},\Delta}(\bar{p}))$ such that $\mathrm{Im}(\eta)$ is finite.*
*(iv) $T$ has DEECRP iff $T$ is amenable (in the sense of* [2]*).*

How is it related to the Ellis group of the theory? The answer is given by the next corollary.

**Corollary 3.** *(i) Each theory with EDEERP has trivial Ellis group.*
*(ii) Each theory with sep. fin. EDEERdeg has profinite Ellis group.*

Item (i) is an easy consequence of Theorem 2(ii). Item (ii) follows from Theorem 2(iii) and the implication (D) $\implies$ (A) in Theorem 4 below.

In the next theorem, $\mathcal{M}$ denotes a minimal left ideal in $\mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ and $u$ an idempotent in this ideal, so $u\mathcal{M}$ is the Ellis group of $T$; $u\mathcal{M}/H(u\mathcal{M})$ is the canonical Hausdorff quotient of $u\mathcal{M}$. Analogously, $u_{\Delta,\bar{p}}\mathcal{M}_{\Delta,\bar{p}}$ is the Ellis group of the flow $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c},\Delta}(\bar{p}))$. The main idea behind the next result is that a natural way to obtain that the Ellis group of $T$ is profinite is to present the flow $S_{\bar{c}}(\mathfrak{C})$ as the inverse limit of some flows each of which has finite Ellis group, and if it works, it should also work for the standard presentation of $S_{\bar{c}}(\mathfrak{C})$ as the inverse limit of the flows $S_{\bar{c},\Delta}(\bar{p})$ (where $\Delta$ and $\bar{p}$ vary).

**Theorem 4.** *Consider the following conditions:*

*(A") $\mathrm{Gal}_{\mathrm{KP}}(T)$ is profinite;*
*(A') $u\mathcal{M}/H(u\mathcal{M})$ is profinite;*
*(A) $u\mathcal{M}$ is profinite;*
*(B) The $\mathrm{Aut}(\mathfrak{C})$-flow $S_{\bar{c}}(\mathfrak{C})$ is isomorphic to the inverse limit $\varprojlim_{i\in I} X_i$ of some $\mathrm{Aut}(\mathfrak{C})$-flows $X_i$ each of which has finite Ellis group;*
*(C) for every finite sets of formulae $\Delta$ and types $\bar{p} \subseteq S(\emptyset)$, $u_{\Delta,\bar{p}}\mathcal{M}_{\Delta,\bar{p}}$ is finite;*
*(D) for every finite sets of formulae $\Delta$ and types $\bar{p} \subseteq S(\emptyset)$, there exists $\eta \in \mathrm{EL}(S_{\bar{c},\Delta}(\bar{p}))$ with $\mathrm{Im}(\eta)$ finite.*

*Then (D) $\implies$ (C) $\iff$ (B) $\implies$ (A) $\implies$ (A') $\implies$ (A").*

We also find several other criteria for [pro]finiteness of the Ellis group. Applying Corollary 3 or our other criteria together with some well-known theorems from structural Ramsey theory (saying that various Fraïssé classes have the appropriate Ramsey properties), we get wide classes of examples of theories with [pro]finite or sometimes even trivial Ellis groups. But we also find some specific examples illustrating interesting phenomena, e.g. we give examples showing that in Theorem 4: (A") does not imply (A'), and (A') does not imply (B). The example showing that (A") does not imply (A') is supersimple of SU-rank 1, so it shows that even for supersimple theories the Ellis group of the theory need not be profinite. We have not found examples showing that (C) does not imply (D), and (A) does not imply (B), which we leave as open problems.

We also give a precise computation of the Ellis group of the theory of the random hypergraph with one binary and one 4-ary relation. This group turns out to be the cyclic two-element group. This example is interesting for various reasons. Firstly, by classical KPT theory, we know that it has sep. finite *EERdeg*, so the Ellis group is profinite by the above results (in fact, it satisfies the assumptions of some other criteria that we found, which implies that the Ellis group is finite), and the example shows that it may be non-trivial. A variation of this example yields an infinite Ellis group, which shows that in some of our criteria for profiniteness, we cannot expect to get finiteness of the Ellis group. Finally, this example is easily seen to be extremely amenable in the sense of [2], so its KP-Galois group is trivial. But the Ellis group is non-trivial. Hence, the epimorphism (found in [6]) from the Ellis group to the KP-Galois group is not an isomorphism. On the other hand, by [4, Theorem 0.7], we know that under NIP, even amenability of the theory is sufficient for this epimorphism to be an isomorphism. So our example shows that one cannot drop the NIP assumption in [4, Theorem 0.7], which was not known so far.

Using our observations that both properties $EERP$ and $EECRP$ do not depend on the choice of the monster model, or even an $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model $M \models T$, and the results from [5] saying that $EERP$ (defined in terms of $M$) is equivalent to extreme amenability of the topological group $\mathrm{Aut}(M)$, and $EECRP$ (defined in terms of $M$) is equivalent to amenability of $\mathrm{Aut}(M)$, we get the following corollary.

**Corollary 5.** *Let $T$ be a complete first-order theory. The group $\mathrm{Aut}(M)$ is [extremely] amenable as a topological group for some $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model $M \models T$ iff it is [extremely] amenable as a topological group for all $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous models $M \models T$.*

This means that [extreme] amenability of the group of automorphisms of an $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous structure is actually a property of its theory, which seems to be a new observation.

Some "definable" versions of Ramsey properties were also introduced and considered in a recent paper by Nguyen Van Thé [8]; also, Ehud Hrushovski has very recently written an interesting paper [1], where he introduces some version of Ramsey properties in a first-order setting. But all these notions seem to be different and they are introduced for different reasons. It would be interesting to see in the future if there are any relationships.

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## On uniform definability of types over finite sets for NIP formulas


Itay Kaplan

(joint work with Shlomo Eshel)


Let $L$ be any language and let $T$ be any $L$-theory. An $L$-formula $\varphi(x, y)$ has *uniform definability of types over finite sets* (*UDTFS*) in $T$ iff there is a formula $\psi(y, z)$ which uniformly (in any model of $T$) defines $\varphi$-types over finite sets of size $\geq 2$. If $\varphi$ has UDTFS, then for any finite $A \subseteq M^y \models T$, the number of $\varphi$-types over $A$ is bounded by $|A|^{|z|}$, which immediately implies that $\varphi$ has finite VC-dimension in $T$, i.e., $\varphi$ is NIP in $T$. This raises the question, asked by Laskowski, of whether these two notions (UDTFS and NIP) are equivalent. Note that in that case, this also implies the Sauer-Shelah lemma in the sense of counting types

This question was first addressed in [JL10] where it was proved assuming that $T$ is weakly o-minimal. Later, [Gui12] extended this result to dp-minimal theories. Finally, in [CS15, Theorem 15] it was proved in the level of the theory $T$: a (complete) theory is NIP iff every formula has UDTFS. They actually proved something stronger: in NIP theories, every formula has uniform *honest* definitions.

The main theorem we presented solves Laskowski's question (and thus answers all the questions in the final paragraph of [Gui12]).

**Theorem 1.** *The following are equivalent for an $L$-theory $T$ and an $L$-formula $\varphi(x, y)$.*

*(1) $\varphi$ is NIP in $T$ (i.e., NIP in any completion of $T$).*
*(2) $\varphi$ has UDTFS in $T$.*

The proof has two ingredients, both from machine learning theory.

The first is the proof of the main result in [MY16]: the existence of *sample compression schemes* for concept classes of finite VC-dimension $d$ whose sizes are bounded in terms of $d$ (answering a question of Littlestone and Warmuth). Roughly speaking, this result says that there is some number $k$ depending only on $d$ such that for any finite set of labeled examples (*concepts*), it is possible to recover our knowledge on that concept by considering a specific subset of size $k$.

The second ingredient is [CCT16] where an upper bound for the *Recursive Teaching Dimension* (*RTD*) is given for concept classes of finite VC-dimension $d$. Roughly speaking this means that there is some number $t$ (depending only on $d$) such that every concept can be identified by at most $t$ samples according to the recursive teaching model. This results translates in our language to the existence of $\varphi$-types which are isolated by their restriction to a set of bounded size.

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## The almost sure theory of finite metric spaces

Isaac Goldbring
(joint work with Bradd Hart)

Recall that the *Urysohn sphere* $\mathbb{U}$ is the unique Polish metric space of diameter 1 satisfying two properties: *universality:* all Polish metric spaces of diameter at most 1 embed into $\mathbb{U}$; and *ultrahomogeneity:* any isometry between finite subspaces of $\mathbb{U}$ extends to a self-isometry of $\mathbb{U}$. From the model-theoretic perspective, $\mathbb{U}$ is the Fraïsse limit of the class of all finite metric spaces of diameter at most 1 and is the model completion of the pure theory of metric spaces.

A lingering question about $\mathbb{U}$ is whether or not it is *pseudofinite*, that is, elementarily equivalent to an ultraproduct of finite metric spaces, or, equivalently, whether or not, given a sentence $\sigma$ for which $\sigma^{\mathbb{U}} = 0$ and $\epsilon > 0$, there is a finite metric space $X$ of diameter at most 1 such that $\sigma^X < \epsilon$. In an earlier preprint, we claimed that not only is $\mathbb{U}$ pseudofinite, but indeed a stronger result is true, namely $\mathrm{Th}(\mathbb{U})$ is the almost-sure theory of finite metric spaces, which means, given any sentence $\sigma$ and any $\epsilon > 0$, almost all sufficiently large metric spaces $X$ of diameter at most 1 satisfy $|\sigma^X - \sigma^{\mathbb{U}}| < \epsilon$.

However, a serious flaw in our argument was discovered by Alex Kruckman and thus the pseudofiniteness of the Urysohn sphere is still in question. In this talk, we rescue the latter fact, namely we show that there is an almost-sure theory of finite metric spaces of diameter at most 1. The motivation for the definition of this theory comes from the fact that almost all sufficiently large metric spaces of diameter at most 1 have all nontrivial distances at least $\frac{1}{2} - O(n^C)$ (see [1] and [2]). This led us to consider a space, denoted $\mathbb{AS}$, defined just like $\mathbb{U}$ except with all nontrivial distances being at least $\frac{1}{2}$. Since any assignment of distances between distinct points taking values at least $\frac{1}{2}$ automatically satisfies the triangle inequality, this allowed us to salvage a version of our argument in this context and combine them with the results of [1] to show that the almost-sure theory of finite metric spaces of diameter at most 1 is the theory of $\mathbb{AS}$. We also discuss some model-theoretic properties of $\mathbb{AS}$, in particular that it is simple, unstable.

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## Forking and JSJ decompositions in the free group

Chloé Perin

(joint work with Rizos Sklinos)

In [Sel13], Sela proves that the first order theory of torsion free hyperbolic groups, and thus in particular that of free groups, is stable. This amazing result implies the existence of a good notion of independence between tuples of elements over any set of parameters, akin to that of algebraic independence in algebraically closed fields. It was thus natural to ask whether there was a way to characterize this notion of independence (called forking independence) in a purely group theoretic way.

In [PS16] we gave such a description for two families of parameter sets: those which are free factors of the free group $\mathbb{F}$, and at the other extreme, those which are not contained in any proper free factor of $\mathbb{F}$.

In the case where the parameter set $A$ is a free factor of $\mathbb{F}$, we showed

**Theorem 1.** *Let $\bar{b}, \bar{c}$ be tuples of elements in the free group $\mathbb{F}_n$ and let $A$ be a free factor of $\mathbb{F}_n$. Then $\bar{b}$ and $\bar{c}$ are independent over $A$ if and only if $\mathbb{F}_n$ admits a free decomposition $\mathbb{F}_n = \mathbb{F} * A * \mathbb{F}'$ with $\bar{b} \in \mathbb{F} * A$ and $\bar{c} \in A * \mathbb{F}'$.*

In other words, the tuples are independent if and only if there is a Grushko decomposition for $\mathbb{F}$ relative to $A$ for which $b$ and $c$ live in "different parts".

The essential ingredients of the proof of our first result is the homogeneity of non abelian free groups and a result of independent interest concerning the stationarity of types in the theory of non abelian free groups.

The relative Grushko decomposition of a group with respect to a set of parameters is a way to see all the splittings of the group as a free product in which the set of parameters is contained in one of the factors. The relative cyclic JSJ decomposition is a generalization of this: it is a graph of groups decomposition which encodes all the splittings of the group as an amalgamated product or an HNN extension over a cyclic group, for which the parameter set is contained in one of the factors.

In the case where $A$ is not contained in any proper free factor of $\mathbb{F}$, there is a canonical JSJ decomposition for $\mathbb{F}$ relative to $A$, namely the tree of cylinders of the deformation space. In this setting, we prove

**Theorem 2.** *Let $\mathbb{F}_n$ be freely indecomposable with respect to $A$.*

*Let $(\Lambda, v_A)$ be the pointed cyclic JSJ decomposition of $\mathbb{F}_n$ with respect to $A$. Let $\bar{b}$ and $\bar{c}$ be tuples in $\mathbb{F}_n$, and denote by $\Lambda_{A\bar{b}}$ (respectively $\Lambda_{A\bar{c}}$) the minimal subgraphs of groups of $\Lambda$ whose fundamental group contains the subgroups $\langle A, \bar{b}\rangle$ (respectively $\langle A, \bar{c}\rangle$) of $\mathbb{F}_n$.*

*Then $\bar{b}$ and $\bar{c}$ are independent over $A$ if and only if $\Lambda_{A\bar{b}} \cap \Lambda_{A\bar{c}}$ is contained in a disjoint union of envelopes of rigid vertices.*

In other words, tuples $b$ and $c$ are independent over $A$ if and only if they live in "different parts" of the pointed cyclic JSJ decomposition of $\mathbb{F}$ relative to $A$.

In this second result, the middle step between the purely model theoretic notion of forking independence and the purely geometric one of JSJ decomposition is that of understanding the automorphism group of $\mathbb{F}_n$ relative to $A$. Indeed, the JSJ decomposition enables us to give in this setting a very good description (up to a finite index) of the group of automorphisms which fix $A$ pointwise. On the other hand, in many cases the model theoretic definitions bear a strong relation to properties of invariance under automorphisms.

In [PS18] we complete this description for any set of parameters. In this setting, there is no canonical JSJ decomposition - we thus give a condition in terms of all the **normalized** pointed cyclic JSJ decompositions. Our main result is:

**Theorem 3.** *Let $A \subset \mathbb{F}$ be a set of parameters and $b, c$ be tuples from $\mathbb{F}$. Then $b$ is independent from $c$ over $A$ if and only if there exists a normalized cyclic JSJ decomposition $\Lambda$ of $\mathbb{F}$ relative to $A$ in which any two blocks of the minimal subgraphs $\Lambda^{min}_{Ab}, \Lambda^{min}_{Ac}$ of $\langle A, b\rangle$ and $\langle A, c\rangle$ respectively intersect at most in a disjoint union of envelopes of rigid vertices.*

Before Sela's work proving the stability of torsion free hyperbolic groups, only the families of abelian groups and algebraic groups (over algebraically closed fields) were known to be stable. For these families it was fairly easy to understand and characterize forking independence. But there is a qualitative difference between the already known examples of stable groups and torsion-free hyperbolic groups: a group elementarily equivalent to an abelian group is an abelian group and likewise a group elementarily equivalent to an algebraic group is an algebraic group. This is certainly false in the case of torsion-free hyperbolic groups: an ultrapower of a non abelian free group under a non-principal ultrafilter is **not** free. Even more, by definition, a non finitely generated group cannot be hyperbolic.

The difficulty in describing forking independence arising from this easy observation is significant. To emphasize this fact we recall the definition of forking independence.

**Definition.** Let $M$ be a stable structure and let $A$ be a subset of $M$. Tuples $b$ and $c$ of elements of $M$ fork over $A$ (in other words, are NOT independent over $A$) if and only if there exists a set $X$ definable over $Ac$ which contains $b$, and a sequence of automorphisms $\theta_n \in \mathrm{Aut}_A(\hat{M})$ for some elementary extension $\hat{M}$ of $M$, such that the translates $\theta_n(X)$ are $k$-wise disjoint for some $k \in \mathbb{N}$.

(For some intuition on the notion of forking, see Section 2 of [LPS13]).

Thus a priori, even for understanding the independence relation provided by stability in natural models of our theory, one has to move to a *saturated model* of the theory. The main problem with that is that we have very little knowledge of what a saturated model of the theory of nonabelian free groups look like. In [PS16] we managed to overcome this difficulty by using the assumptions we imposed on the parameter set $A$. When $A$ is not contained in any proper free factor, there are a number of useful results available. Model theoretically: under this assumption, $\mathbb{F}$ is atomic over $A$, i.e. every type can in fact be defined by a single formula (see [PS16], [OH11]). This enables us on the one hand to transfer a sequence witnessing forking from a big model to $\mathbb{F}$, but more importantly it gives us a natural candidate for a formula witnessing forking: by homogeneity (see [PS12] and [OH11]), types in $\mathbb{F}$ correspond to orbits under the automorphism group, so in fact the orbit of a tuple under $\mathrm{Aut}_A(\mathbb{F})$ is definable. Geometrically, under the assumption that $A$ is not contained in any proper free factor, we have a very good understanding of $\mathrm{Aut}_A(\mathbb{F})$: the canonical JSJ decomposition of $\mathbb{F}$ relative to $A$ enables one to describe up to finite index the automorphisms fixing $A$ (one understands the modular automorpshism group $\mathrm{Mod}_A(\mathbb{F})$).

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## Elementary Ramsey theory

Ehud Hrushovski

**Definition.** A complete 1st-order theory $T$ is *Ramsey* (at a sort $S$) if any completion of $T$ in the language $L_P$ has a model $N$ and $M \preceq N|_L$, such that $P^N \cap M$ is 0-definable in $M$.
Here $L_P = L \cup \{P\}$, where $P$ is a predicate of sort $S$.

When $T$ is $\aleph_0$-categorical, this is equivalent to the *structural ramsey theory* of Graham-Leeb-Rothschild, Nešetřil-Rödl, Abramson-Harrington, with a slight change or refinement of the terminology (the usual notion is equivalent to Ramseyness in the sort of $n$-element substructures, for each $n$.)

**Theorem.** *Any $T$ has a unique minimal expansion $T^{ram}$ that is Ramsey in every sort.*

Minimality means that if $T'$ is an everywhere Ramsey expansion of $T$ and $N' \models T'$, then there exists an $L$- embedding $j : N' \to N$ with $N \models \widetilde{T}$, and so that the pullback of any definable subset of $N$ is definable in $N'$.

A more general, sort-by-sort version is valid for irreducible universal theories. The uniqueness is up to a non-unique bi-interpretation over $T$. The group of auto-bi-interpretations of $T^{ram}$ over $T$ appears to be an interesting invariant of the situation, and can sometimes be seen to control the functoriality of $T^{ram}$ applied to different sorts.

**Examples.** Describing $T^{ram}$ where $T$ is the theory of affine spaces over a given field $k$, or the theory of Hilbert spaces, or the theory of atomless Boolean algebras, in their home sorts, or ACF presented in the affine line sort, reveals and is equivalent to a string of combinatorial theorems, including the finite affine space Ramsey, Van den Waerden's theorem on arithmetic progressions (and some polynomial versions), the Dvoretzky-Milman theorem, dual Ramsey and Hales-Jewett.

The theorem is a soft version of earlier results and conjectures of Melleray-Nguyen Van Thé-Tsankov and Melleray-Tsankov-Ben Yaacov in the line of Kechris-Pestov-Todorčević theory. Constructions of Evans-Hubička-Nešetřil show that the statement cannot be formulated within the $\aleph_0$-categorical framework.

## An o-minimalist view of the group configuration

Ya'acov Peterzil

The group configuration is one of the most important tools of Geometric Model Theory. It allows to extract a (type) definable group from a small number of model theoretic (in)dependencies. Logicians working in o-minimal structures are often asked by other model theorists whether the group configuration theorem holds in o-minimal structures. Here I will state the precise result which one can prove in the o-minimal setting and sketch its proof. The details can be found in [6].

**Preliminaries.** For basic reference on o-minimality, see [2] and also [5].

Fix $\mathcal{M} = \langle M; <, \ldots \rangle$ an $|L|^+$-saturated o-minimal structure. Because of the linear order, we have $acl = dcl$. For $A, B \subseteq M$ small sets, one defines

$$\dim(A/B) = \{|A_0| : A_0 \subseteq A \text{ is maximal } dcl\text{-independent over } B\},$$

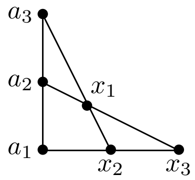
(a set $A \subseteq M$ is *dcl-independent over B* if for every $a \in A$, $a \notin dcl((A \setminus \{a\}) \cup B)$).

**Definition 1.** *An $(m,k)$- homogenous space configuration in $\mathcal{M}$ over $A \subseteq M$*, or just *an $(m,k)$-configuration over $A$*, is 6-tuple of tuples $(a_1, a_2, a_3, x_1, x_2, x_3)$ from $\mathcal{M}$, such that (we allow redundancies in the clauses below)

(i) For $i = 1, 2, 3$, $\dim(a_i/A) = m$ and $\dim(x_i/A) = k$.
(ii) $\dim(x_1, x_2, x_3/A) = 3k$ and for $i \neq j$, $\dim(a_i, a_j/A) = \dim(a_1, a_2, a_3/A) = 2m$.
(iii) For distinct $i, j, k$, $\dim(x_i, x_j/a_k A) = k$.
(iv) For $i \neq j$, and any $k$, $\dim(a_i, a_j, x_k/A) = 2m + k$ and $\dim(a_i, x_i, x_j/A) = m + 2k$.

The configuration is called *minimal* if for every $B \supseteq A$ independent from $V$ over $A$, and for every $b_1, b_2, b_3$, with $b_i \in dcl(Ba_i)$, if $(b_1, b_2, b_3, x_{1,2}, x_3)$ is an $(m', k)$-configuration over $B$ (so in particular, $m' \leq m$) then $m' = m$.

It is common to read-off the above information from the diagram below as follows: In addition to (i), every triples which is not co-linear is independent over $A$ and also, whenever $x_i, x_j, a_k$, $i \neq j$, is a co-linear triple then $x_i \in dcl(x_j, a_k, A)$, and each element of the co-linear triple $a_1, a_2, a_3$ is in the definable closure over $A$ of the other two.




(1)

Recall that *a type-definable group* is a partial type $G(x)$ (i.e. a collection of formulas in $x$ of cardinality at most $\kappa = |L|$) together with a definable binary function whose restriction to $G \times G$ is a group operation. A group action of $G$ on a type $\Sigma$ is called *definable* if it is the restriction to $G \times \Sigma$ of a definable function.

**Theorem.** *Let $\mathcal{M}$ be an $|L|^+$-saturated o-minimal structure which eliminates imaginaries. Assume that $V = (a_1, a_2, a_3, x_1, x_2, x_3)$ is an $(m,k)$-configuration over $A$.*

*Then there exists an $m'$-dimensional type-definable group $(G, \star)$ over $M$, with $k \leq m' \leq m$, acting definably and transitively on a $k$-dimensional partial type over $M$.*

*The configuration and group action are related as follows: There is $B \supseteq A$ independent from $V$ over $A$, and there are $g_1, g_2, g_3 \in G$, and $y_1, y_2, y_3 \models \Sigma$ such that for each $i = \{1,2,3\}$, we have $g_i \in dcl(a_i B)$, and $dcl(x_i B) = dcl(y_i B)$, and* *in addition* $(g_1, g_2, g_3, y_1, y_2, y_3)$ *is an* $(m', k)$*-configuration in* $G$*, with* $g_3 = g_2 g_1$*,* $y_3 = g_1 y_2$ *and* $y_1 = g_2 g_1 y_2$*.*

*If, in addition, the configuration is minimal then* $m' = m$ *and each* $g_i$ *is interdefinable with* $a_i$ *over* $B$*.*

As in the stable case, an additional node in the configuration yields an abelian group. Also, if $m = k$ then one can identify the group with the homogeneous space.

**Some references.**

The Group Configuration Theorem in the stable setting and its model theoretic proof and framing is due to Hrushovski, [3] and [4]. Accounts of Hrushovski's theorem in the superstable setting appear in Bouscaren's article [1], and in the stable setting in Pillay's book [7, Theorem 5.4.5].

**Infinitesimal neighborhood and infinitesimal loci.**

**Definition 2.** For $a \in M^n$, *the* $\mathcal{M}$*-infinitesimal neighborhood of* $a$ is the partial type over $M$ consisting of all $M$-definable open subset of $M^n$ which contain $a$. For $\mathcal{N}$ an $|M|^+$-saturated elementary extension of $\mathcal{M}$, it is often convenient to identify this type with its realization in $\mathcal{N}$. Suppressing the dependence on $\mathcal{M}$ the type is denoted by $\mu_a$ and its realization by $\mu_a(\mathcal{N})$. It is easy to check that if $a = (a_1, \ldots, a_n)$ then $\mu_a = \mu_{a_1} \times \cdots \times \mu_{a_n}$.

If $X \subseteq M^n$ is an $\mathcal{M}$-definable set and $a \in X$ then let $\mu_a(X)$ denote the set $\mu_a(\mathcal{N}) \cap X(\mathcal{N})$ (namely, the partial type $\mu_a \cup \{\varphi(\bar{x})\}$, where $\varphi$ defines $X$).

**Fact 3.** Assume that $Q$ is $A$-definable and $a$ is generic in $Q$ over $A$. Then for every $A$-definable set $R$, if $a \in R$ then $\mu_a(Q) \subseteq \mu_a(R)$. In particular, if $\dim Q = \dim R$ then $\mu_a(Q) = \mu_a(R)$.

This fact gives rise to the following definition.

**Definition 4.** Given $a \in M^n$, $A \subseteq M$, and $Q \subseteq M^n$ an $A$-definable set such that $a$ is generic in $Q$ over $A$, one calls $\mu_a(Q)$ *the infinitesimal locus of* $a$ *over* $A$*, with respect to* $\mathcal{M}$, and denote it by $\mu(a/A)$.

Let us sketch the idea of the proof of the main theorem.

The group configuration, together with the fact that $acl = dcl$, give rise to four definable functions, $H, F, K, L$ such that:

$$H(F(a_2, a_1), x_2) = K(a_2, L(a_1, x_2)) \tag{2}$$

The function $H$, when restricted to $\mu(a_3/A) \times \mu(x_2/A)$, takes values in $\mu(x_1/A)$. The other functions acts similarly on infinitesimal loci. Each binary function gives rise to a definable family of unary functions, for example, for $a'_3 \in \mu(a_3/A)$, let $h_{a'_3}(-) = H(a'_3, -) : \mu(x_2/A) \to \mu(x_1/A)$. Similarly, one defines $k_{a'_2}$ and $\ell_{a'_1}$. Thus we have $k_{a_2} \circ \ell_{a_1} = h_{a_3}$, as functions on $\mu(x_2/A)$, and in fact for each $k_{a'_2}, \ell_{a'_1}$ there is $h_{a'_3}$ such that $k_{a'_2} \circ \ell_{a'_1} = h_{a'_3}$. Similarly, for each choice of any of the two functions there exists a third one in the family which respects the same equation.

Let $\mathcal{L} = \{\ell_{a_1'} : a_1' \in \mu(a_1/A)\}$ and let $G = \{\ell_1^{-1}\ell_2 : \ell_i \in \mathcal{L}\}$, a family of permutations of $\mu(x_2/A)$. One can show that, up to a definable equivalence relation (identifying two functions in $G$ if the agree on $\mu(a_1/A)$), $G$ is a type definable-group under composition and it acts transitively and definably on $\mu(a_2/A)$.

An additional question arises:

**Question**: Can every definably connected type-definable group in an o-minimal structure be definably embedded into a definable group, possibly of the same dimension?

Hrushovski proved the analogous result in the stable case using the uniqueness of generic in types in connected stable groups. That proof does not carry over to the o-minimal setting, but still in the one dimensional case, and when the Lie algebra of $G$ is simple (when $\mathcal{M}$ expands a field), the answer is positive. We leave the general question open.

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## Partial Associativity in Latin Squares

Jason Long

(joint work with W.T. Gowers)

Let $\circ$ be a binary operation defined on a finite set $X$. We assume that it is a bijection in each variable separately, and that there exists a constant $c > 0$, independent of the size of $X$, such that the number of triples $(x, y, z) \in X^3$ with $x \circ (y \circ z) = (x \circ y) \circ z$ is at least $c|X|^3$. It is easy to see that if $c = 1$ then these conditions are equivalent to the group axioms, so it is natural to ask whether if a binary operation has this property for some smaller $c$, then there must be some underlying group structure that 'explains' the prevalence of associative triples.

The '99% case' was dealt with by Elad Levi [2], who proved that if $c$ is close to 1, then there must be a group $G$ of size approximately equal to $|X|$ and an injection $\phi : X \to G$ such that $\phi(x \circ y) = \phi(x)\phi(y)$ for almost all pairs $x, y \in X^2$. In other words, the multiplication table of $\circ$ agrees almost everywhere with a group operation. In our work we look at the '1% case' – that is, the case where $c$ is a small fixed constant. We also weaken the hypothesis in a small way by considering binary operations that are only partially defined: this has no significant effect on our arguments, but it is convenient when discussing examples not to have to worry about whether they are defined everywhere.

One way to create an operation with many associative triples is based on structures that are approximately groups in a metric sense. For concreteness, we discuss a specific example. Let $\delta > 0$ and let $X$ be a maximal $\delta$-separated subset of SO(3). Now define a partial binary operation as follows. Let $\theta > 0$ be a suitable absolute constant (as opposed to $\delta$, which is comparable to $|X|^{-1/3}$) and then for $x, y, z \in X$ let $x \circ y = z$ if and only if $d(xy, z) \leq \theta\delta$. We in fact show that however $X$ is chosen there will necessarily be many associative triples, but there is no obvious way of passing to a subset of $X^2$ where the operation is isomorphic to a restriction of a group operation. Indeed, we conjectured that there was no such subset, and that conjecture has been proved by Ben Green [3].

This example shows that a natural conjecture – that a partially associative binary operation agrees on a substantial set of pairs with a group operation – is false. However, the example has structure that suggests an appropriate weakening of the conjecture. Our main result will be that if an operation has many associative triples (and is injective in each variable separately), then it agrees on a large set of pairs with a restriction of a small perturbation of the binary operation on a metric group. The precise statement is as follows.

**Theorem 1.** *Let $c > 0$, let $X$ be a finite set and let $\circ$ be a partially defined binary operation on $X$ that is injective in each variable separately. Suppose that there are at least $c|X|^3$ triples $(x, y, z) \in X^3$ such that $x\circ(y\circ z) = (x\circ y)\circ z$ (where this means in particular that all expressions and subexpressions are defined). Then for every positive integer $b$ there exist $\delta(c, b) \geq c^{b^{26b}}$, a subset $A \subset X^2$ of density at least $\delta$, a metric group $G$, and maps $\phi : X \to G$, $\psi : Y \to G$ and $\omega : Z \to G$, such that the images $\phi(X), \psi(Y)$ and $\omega(Z)$ are 1-separated, and $d(\phi(x)\psi(y), \omega(z)) \leq b^{-1}$ for every $(x, y, z) \in X \times Y \times Z$ such that $(x, y) \in A$ and $x \circ y = z$.*

In fact, $G$ is a free group and the metric $d$ is obtained from the restriction $\circ_A$ of $\circ$ to $A$ by appropriately defining a 'length of proof'. More specifically, the distance $d$ between words $w_1$ and $w_2$ of $G$ is given by the smallest number of relations of the form $x \circ y = z$ which could be used to prove that $w_1 = w_2$ (this could be infinite).

It is important to stress that although algebraically $G$ is just a free group, the metric gives it a much more interesting structure. Indeed, one can think of this metric as an approximate group presentation: instead of declaring that certain words are equal to the identity, we declare that they are *close* to the identity, and then we take the distance to be the largest one that is compatible with these 'approximate relations'.

Theorem 1 gives us in particular a metric group $G$ and three 1-separated subsets $X, Y, Z$ of $G$ of comparable size with the property that for a constant proportion of pairs $(x, y) \in X \times Y$ there exists $z \in Z$ such that $d(xy, z) \leq \delta$, where $\delta = b^{-1}$. If we replace the condition $d(xy, z) \leq \delta$ by the condition that $xy = z$, we obtain a condition that is very closely related to the definition of an approximate group. From such a condition it is possible to conclude that there is an approximate group $H$ of size not much larger than $|X|$ and translates $xH$ and $Hy$ of $H$ such that a constant proportion of the points of $X$ belong to $xH$ and a constant proportion of the points of $Y$ belong to $Hy$. We show that a suitable 'metric entropy version' of this result holds, which allows us to replace equality by approximate equality and obtain an appropriate conclusion, where the notion of an approximate group is replaced by that of an approximate group that is also approximate in a metric sense. We call these structures 'rough approximate groups'. (To the best of our knowledge, this concept was first formulated by Tao [5], and a slight adaptation of it has been introduced and studied by Hrushovski [4], who called it a metrically approximate subgroup.)

It would be very interesting to go further and describe in a more concrete way the structure of rough approximate groups, ideally obtaining an analogue of the results of Breuillard, Green and Tao on approximate groups [1].

It is also natural to ask whether there is an analogue of the results of this paper for Abelian groups. In forthcoming work we will address this question, identifying a certain structure which simultaneously witnesses associativity and commutativity, in the sense that if the number of copies of that structure in a partial Latin square is within a constant of maximal, then the partial Latin square has Abelian-group-like behaviour.

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## Preliminary results in non-commutative algebraic geometry

Zlil Sela

Algebraic geometry studies the structure of sets of solutions to systems of equations usually over fields or commutative rings. The developments and the considerable abstraction that currently exist in the study of varieties over commutative rings, still resist to apply to the study of varieties over non-abelian rings or over other non-abelian algebraic structures.

Since about 1960 ring theorists, P. M. Cohn, G. M. Bergman, and others, have tried to study varieties over non-abelian rings, notably free associative algebras (and other free rings). However, the pathologies that they tackled and the lack of unique factorization that they study in detail, prevented any attempt to prove or even speculate what can be the structure of varieties over free associative algebras.

We suggest to study varieties over free associative algebras using techniques and analogies of structural results from the study of varieties over free groups and semigroups. Over free groups and semigroups geometric techniques as well as low dimensional topology play an essential role in the structure of varieties. These include Makanin's algorithm for solving equations, Razborov's analysis of sets of solutions over a free group, the concepts and techniques that were used to construct and analyze the JSJ decomposition, and the applicability of the JSJ machinery to study varieties over free groups and semigroups. Our main goal is to demonstrate that these techniques and concepts can be modified to be applicable over free associative algebras as well.

We start the analysis of systems of equations over a free associative algebra with what we call monomial systems of equations. These are systems of equations over a free associative algebra, in which every polynomial in the system contains two monomials. We analyze the case of homogeneous solutions to homogeneous monomial systems of equations. In this case it is possible to apply the techniques that were used in analyzing varieties over free semigroups, and associate a Makanin-Razborov diagram that encodes all the homogeneous solutions to homogeneous monomial system of equations.

We introduce *limit algebras* that are a natural analogue of a *limit group*, and prove that such algebras are always embedded in (limit) division algebras. The automorphism (modular) groups of these division algebras are what is needed in the sequel in order to modify and shorten solutions to monomial systems of equations.

We further present a combinatorial approach to (some cases of) the celebrated Bergman's centralizer theorem, and finally use this combinatorial approach to analyze the set of solutions to a monomial system of equations with a single variable. The results that we obtain are analogous to the well known structure of the set of solutions to systems of equations with a single variable over a free group or semigroup.

## $n$-dependent groups and fields

Nadja Hempel

1-dependent theories, better known as NIP theories, are the first class of the hierarchy of $n$-dependent structures. The random $n$-hypergraph is the canonical object which is $n$-dependent but not $(n-1)$-dependent. Thus the hierarchy is strict. But one might ask if there are any algebraic objects (groups, rings, fields) which are strictly $n$-dependent for every $n$. We will start by introducing the $n$-dependent hierarchy and present the known results on $n$-dependent groups and (valued) fields. These were inspired by the above question.

## Characterizing NIP henselian valued fields

Franziska Jahnke
(joint work with Sylvy Anscombe)

In this talk we characterize NIP henselian valued fields modulo the theory of their residue field. In particular, we show that every NIP henselian valued field satisfies some instance of the Ax-Kochen/Ersov principle. We then discuss several consequences of the theorem, including a characterization of all NIP fields when one assumes the conjecture that every infinite NIP field is either separably closed, real closed or admits a henselian valuation. We also discuss open questions arising from these results.

Since Macintyre showed in the early seventies that infinite $\omega$-stable fields are algebraically closed ([Mac71]), much research has gone into the question whether key model-theoretic tameness properties coming from Shelah's classification theory (like stability, simplicity, NIP) correspond to natural algebraic definitions when applied to fields. The most prominent of these is the Stable Fields Conjecture, predicting that any infinite stable field is separably closed. In 1980, Cherlin and Shelah generalized Macintyre's result to superstable fields ([CS80]), but despite much effort, no further progress was made. Although the Stable Fields Conjecture still seems to be far beyond our reach, its generalization to NIP fields has recently received much attention:

**Conjecture 1** (Conjecture on NIP fields)**.** *Let $K$ be an infinite NIP field. Then $K$ is separably closed, real closed or admits a non-trivial henselian valuation.*

Conjecture 1 has many variants but no clear origin, and is usually attributed to Shelah (who stated a closely related conjecture on strongly NIP fields and asked for a similar description of NIP fields in [She14]). It was recently proven in the special cases of dp-minimal fields and for positive characteristic fields of finite dp-rank by Johnson ([Joh16] and [Joh19]).

Apart from separably closed fields, real closed fields and the $p$-adics plus their finite extensions, the only way we know to date of how to construct NIP fields are by NIP transfer theorems in the spirit of Ax-Kochen/Ershov: under certain algebraic assumptions, if $(K, v)$ is a henselian valued field such that the residue field $Kv$ is NIP, then $(K, v)$ is NIP. The first such theorem was shown by Delon:

**Fact** (Delon, [Del81])**.** Let $(K, v)$ be a henselian valued field of equicharacteristic 0. Then,

$$(K, v) \text{ is NIP in } \mathcal{L}_{\text{val}} \iff Kv \text{ is NIP in } \mathcal{L}_{\text{ring}}.$$

Here, $\mathcal{L}_{\text{ring}} = \{0, 1, +, \cdot\}$ denotes the language of rings and $\mathcal{L}_{\text{val}}$ is the usual three-sorted language with sorts for the field, the residue field and the value group.

Note that Delon originally proved the theorem under the additional assumption that the value group $vK$ is NIP as an ordered abelian group. It was later shown by Gurevich and Schmidt that this holds for any ordered abelian group ([GS84, Theorem 3.1]). Several variants of Delon's theorem were proven in mixed and positive characteristic, first by Bélair ([Bél99]) and more recently by Jahnke and Simon

([JS19]). Bélair showed that an algebraically maximal Kaplansky field $(K, v)$ of positive characteristic is NIP in $\mathcal{L}_{\text{val}}$ if and only if its residue field $Kv$ is NIP in $\mathcal{L}_{\text{ring}}$, and that the same holds if $(K, v)$ is finitely ramified with perfect residue field. Jahnke and Simon generalize Bélair's result to separably algebraically maximal Kaplansky fields of finite degree of imperfection and arbitrary characteristic. Conversely, they use the theorem by Kaplan, Scanlon and Wagner stating that NIP fields of positive characteristic admit no Artin-Schreier extensions ([KSW11]) to show that NIP henselian valued fields of positive characteristic are separably algebraically maximal. The approach used by Jahnke and Simon builds on machinery developed by Chernikov and Hils in the $\text{NTP}_2$ context ([CH14]). Also following this route, we prove what one might consider as the ultimate transfer theorem: our main result is that a henselian valued field $(K, v)$ is NIP (in $\mathcal{L}_{\text{val}}$) if and only if its residue field $Kv$ is NIP (in $\mathcal{L}_{\text{ring}}$) and the valued field satisfies a list of purely algebraic conditions (all of which are preserved under elementary equivalence). More precisely, we show the following

**Main Theorem.** *Let $(K, v)$ be a henselian valued field. Then $(K, v)$ is NIP in $\mathcal{L}_{\text{val}}$ if and only if the following hold:*

**(1)** *$Kv$ is NIP.*

**(2)** *Either*

**(a)**
- **(a.i)** *$(K, v)$ is of equal characteristic and*
- **(a.ii)** *$(K, v)$ is trivial or separably defectless Kaplansky;*

*or*

**(b)**
- **(b.i)** *$(K, v)$ has mixed characteristic $(0, p)$,*
- **(b.ii)** *$(K, v_p)$ is finitely ramified, and*
- **(b.iii)** *$(Kv_p, \bar{v})$ is trivial or separably defectless Kaplansky;*

*or*

**(c)**
- **(c.i)** *$(K, v)$ has mixed characteristic $(0, p)$,*
- **(c.ii)** *$(Kv_0, \bar{v})$ is defectless Kaplansky.*

Here, for a valuation $v$ of mixed characteristic $(0, p)$, we use $v_0$ to denote the finest coarsening of $v$ with residue characteristic 0 and $v_p$ to denote the coarsest coarsening of $v$ with residue characteristic $p$.

We then apply our main theorem in two different ways: we show that the henselization $(K^h, v^h)$ of any NIP valued field $(K, v)$ is again NIP and give a classification of NIP fields assuming that Conjecture 1 holds.

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## An imaginary Ax-Kochen-Ershov principle


Silvain Rideau-Kikuchi

(joint work with Martin Hils)


In their seminal work on the model theory of henselian valued fields, Ax and Kochen [1], and independently Ershov [2], proved that the theory of an equicharacteristic zero valued fields is entirely determined by that of its value group $\Gamma$ (as an ordered group) and of its residue field k (as a ring). This principle has since then permeated the model theory of (enriched) valued fields, and, as it turns out, the description of interpretable sets, also know as imaginaries, provides a new incarnation of this principle.

The imaginaries of the model completion of valued fields, the theory of non-trivially valued algebraically closed valued fields, have been entirely described in work of Haskell, Hrushovski and Macpherson [3]. They have been showed to essentially only consist of the so-called geometric sorts $\mathcal{G}$ : the field K itself, the quotients $\mathrm{S}_n := \mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$, where $\mathcal{O}$ is the valuation ring of K, which are the moduli spaces of free rank $n$ $\mathcal{O}$-submodules of $\mathrm{K}^n$, and the moduli spaces of the reducts modulo the maximal ideal $\mathfrak{m}$ of $\mathcal{O}$, and $\mathrm{T}_n := \bigsqcup_{s\in \mathrm{S}_n} s/\mathfrak{m}s$.

This result generalizes to characteristic zero Henselian:

**Theorem 1** (Imaginary Ax-Kochen-Ershov principle)**.** *The theory of any equicharacteristic zero Henselian valued fields whose residue field eliminates $\exists^\infty$ and with definably complete value group weakly eliminates imaginaries relative to the sorts* K*,* $\Gamma^{\mathrm{eq}}$ *and* ${\mathrm{Lin}_{\mathrm{k}}}^{\mathrm{eq}}$*.*

The sorts ${\mathrm{Lin}_{\mathrm{k}}}^{\mathrm{eq}}$ are obtained in the following manner: for every $\emptyset$-interpretable set $X$ in the two sorted theory of vector spaces $(k, V)$, and every $n \in \mathbb{Z}_{>0}$, we consider $\mathrm{T}_{n,X} := \bigsqcup_s X(\mathrm{k}, s/\mathfrak{m}s)$, where $X(\mathrm{k}, s/\mathfrak{m}s)$ denotes $X$ interpreted in the structure $(\mathrm{k}, s/\mathfrak{m}s)$. Then ${\mathrm{Lin}_{\mathrm{k}}}^{\mathrm{eq}} := \bigsqcup_{n,X} \mathrm{T}_{n,X}$.

This theorem can be generalized to unramified mixed characteristic and valued fields with certain operators. The theorem is also resplendent in $\Gamma$ and k and remains true when adding an angular component. But these generalizations will not be covered here.

Following Hrushovski's recent improvement on the proof of elimination of imaginaries in algebraically closed valued fields, definable types play a fundamental role in our proof. However, in such a general setting, it is not true that definable types are dense, or for that matter, that there are definable types other than the realized ones. But if we restrict ourselves to finding quantifier free definable types, this obstruction is lifted:

**Proposition 2.** *Let $K$ be equicharacteristic zero Henselian valued fields. Assume that $\Gamma$ is definable complete and that* $\mathrm{k}$ *eliminates $\exists^\infty$. Then for every $K$-definable $X$, there exists $F \succcurlyeq K$ and $a \in X(F)$ whose quantifier free type over $K^{\mathrm{a}}$ is $\mathcal{G}(\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner))$-definable.*

On the other hand, finding complete types is still possible provided one allows for a weaker notion than almost definability. Let $\mathrm{RV} := \mathrm{K}^\times/1 + \mathfrak{m}$ and, for all $A \subseteq \mathcal{G}$, $\mathrm{Lin}_A := \bigsqcup_{s \in \mathrm{S}_n(A)} s/\mathfrak{m}s$. We can then prove the following:

**Proposition 3.** *Let $K$ be equicharacteristic zero Henselian valued fields, $A = \mathrm{dcl}(A) \subseteq \mathcal{G}(K)$ and $a \in F \succcurlyeq K$ a tuple whose quantifier free type over $K$ is $A$-definable. Then the type of $a$ over $K$ is invariant over $A \cup \mathrm{RV}(K) \cup \mathrm{Lin}_A(K)$.*

These two propositions, along with the fact that $\mathrm{RV}(K) \cup \mathrm{Lin}_A(K)$ is stably embedded, reduces the classification of imaginaries to that of $\mathrm{RV}(K) \cup \mathrm{Lin}_A(K)$. Recall that $\mathrm{Lin}_A$ is a collection of k-vector spaces and RV is an extension of $\Gamma$ by $\mathrm{k}^\times$. Understanding imaginaries in short exact sequences allows us to prove that they essentially come from k and $\Gamma$:

**Proposition 4.** *The imaginaries in* $\mathrm{RV} \cup \mathrm{Lin}_A$ *can be eliminated relative to* $\Gamma^{\mathrm{eq}} \cup {\mathrm{Lin}_{\mathrm{k}}}^{\mathrm{eq}}$.

Together these are the three main ingredients to prove the imaginary Ax-Kochen-Ershov principle.

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**On Some Classes of Infinite Approximate Subgroups**

Simon Machado

We study infinite approximate subgroups with a particular focus on Approximate lattices. Approximate lattices were first defined by Michael Björklund and Tobias Hartnick in [BH18]. They were inspired by the work of Yves Meyer about a type of approximate subgroups that later came to be known as mathematical quasi-crystals ([Mey72]). Approximate lattices also generalise lattices of locally compact groups (i.e. discrete subgroups of locally compact groups with finite co-volume).

When $G$ is a locally compact group, we say that $X$ is *uniformly discrete* if there is a neighbourhood of the identity $U$ such that $(xU)_{x\in X}$ is a family of disjoint sets. We say that $X$ is *relatively dense* if there exists a compact subset $K \subset G$ such that $\Lambda K = G$. An approximate subgroup $\Lambda$ of a locally compact group $G$ is a *uniform approximate lattice* if $\Lambda$ is uniformly discrete and relatively dense. The approximate subgroup condition arises naturally from the combination of discreteness and co-compactness. Indeed, any symmetric subset $\Lambda \subset G$, such that $\Lambda$ is relatively dense and $\Lambda^6$ is discrete (with respect to the induced topology), is an approximate subgroup, and hence a uniform approximate lattice.

Examples of uniform approximate lattices are given by *cut-and-project schemes.* A cut-and-project scheme $(G, H, \Gamma)$ is the datum of two locally compact groups $G$ and $H$, and a uniform lattice $\Gamma$ in $G\times H$ such that $\Gamma \cap (\{e_G\} \times H) = \{e_{G\times H}\}$ and $\Gamma$ projects densely to $H$. Given a cut-and-project scheme $(G, H, \Gamma)$ and a symmetric relatively compact neighbourhood $W_0$ of $e_H$ in $H$, one gets a uniform approximate lattice when considering the projection $\Lambda$ of $(G \times W_0) \cap \Gamma$ to $G$. Any such set is called a *model set* and any approximate subgroup of $G$ which is commensurable to and contained in a model set is called a *Meyer set* of $G$. This construction was first introduced by Yves Meyer in the abelian case [Mey72] and extended by Michael Björklund and Tobias Hartnick [BH18].

In [Mey72] Yves Meyer proved a structure theorem for mathematical quasi-crystals. Quasi-crystals correspond to uniform approximate lattices in locally compact abelian groups. Rephrased in our terminology he proved that all approximate lattices of locally compact abelian groups are Meyer sets ([Mey72, Theorem 3.2]). Motivated by this result the authors of [BH18] asked whether similar results would hold for other classes of locally compact groups. This question, and more generally the structure theory of Approximate lattices, is the main motivation of this talk.

Using methods from algebraic group theory we can prove a first generalisation of Meyer's theorem. Namely, all approximate lattices of soluble Lie groups are Meyer sets. To further extend Meyer's theorem we introduce *good models* for approximate subgroups. The definition of good models and the following results come from an article in preparation ([Mac19a]). A *good model for* an approximate subgroup $\Lambda$ of a group $G$ is a group homomorphism $f : \Lambda^\infty \to H$, with $H$ a locally compact group, such that: (i) The set $f(\Lambda)$ is relatively compact and (ii) there is $U \subset H$ a neighbourhood of the identity such that $f^{-1}(U) \subset \Lambda$.

The definition of good models is closely related to [BGT12, Definition 3.5], [Hru12, Theorem 4.2] and [BH18, Definition 2.12]. However, each of these prior definitions asks for stronger hypotheses whereas we try to keep the definition of good models as simple as possible. This allows for a simple and handy characterisation of approximate subgroups that have a good model. Indeed, an approximate subgroup $\Lambda$ of a group $G$ has a good model if and only if there exists a decreasing sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of approximate subgroups commensurable to $\Lambda$ with $\Lambda_0 = \Lambda$ adn $\Lambda_{n+1}^2 \subset \Lambda_n$. This criterion is inspired by the construction from [BGT12, Section 6]. It is a handy criterion that finds applications in a variety of situations. Some approximate subgroups however do not have a good model.

Coming back to approximate lattices, one can see that good models generalise cut-and-project schemes. In particular an approximate lattice has a good model if and only if it is a model set. Building up on the criterion mentioned above and an argument due to Tom Sanders and Massicot–Wagner we give a new generalisation to Meyer's theorem; we prove that if $\Lambda$ an approximate lattice of an amenable locally compact group $G$, then $\Lambda^4$ is a model set. The method used actually yields a much stronger result about all uniformly discrete approximate subgroups of amenable locally compact groups.

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## Stability in Freiman-Ruzsa theorem


Daniel Palacín

(joint work with Amador Martin-Pizarro, Julia Wolf)


A finite subset $A$ of a group $G$ is said to have *doubling* $K$ if the product set $A \cdot A = \{a \cdot b \,|\, a, b \in A\}$ has size at most $K|A|$. Archetypal examples of sets with small doubling (where $K$ is constant as the size of the group $G$, and the set $A$, tend to infinity) are cosets of subgroups.

Theorems of Freiman-Ruzsa type assert that sets with small doubling are "not too far" from being subgroups in a suitable sense. Specifically, the so called Freiman-Ruzsa theorem due to Ruzsa [4] asserts that a subset $A$ of doubling $K$ in $\mathbb{F}_2^\infty$ can be covered by $C_1(K)$ many cosets of some subspace of size $C_2(K)|A|$, where both $C_1(K)$ and $C_2(K)$ only depend on $K$. Or equivalently, that there are constants $C_3(K)$ and $C_4(K)$, each depending on $K$, such that for some coset

$v+H$ of a subspace $H$ of size $C_3(K)|A|$, we have that $A\cap(v+H)$ has size at least $|A|/C_4(K)$.

The latter formulation of the Freiman-Ruzsa theorem resonates with a classical phenomenon in model theory: every type over a model is the generic type of a coset of its stabilizer subgroup. Roughly speaking, a large proportion of a given definable set intersects a coset of a definable group, so they are *commensurable*.

Phenomena of Freiman-Ruzsa type are present in recent work of Hrushovski [3], who showed that a set of small tripling in a (possibly infinite) group of bounded exponent is commensurable with a subgroup, inspired by classical results and techniques from stability theory in a non-standard setting.

Motivated by Hrushovski's work, we shall explain how to use the local approach to stability of Hrushovski and Pillay in [2] in order to obtain a non-quantitative version of the Freiman-Ruzsa theorem for arbitrary (possibly infinite) groups under the assumption of stability. We say that a subset $A$ of $G$ is *$r$-stable* if there are no elements $a_1,\dots,a_r,b_1,\dots,b_r$ in $G$ such that $b_j\cdot a_i$ belongs to $A$ if and only if $i\leq j$. In particular, we will prove the following result.

**Theorem A.** *Given real numbers $K\geq 1$ and $\epsilon>0$ and a natural number $r\geq 2$, there exists a natural number $n=n(K,\epsilon,r)$ such that for any (possibly infinite) group $G$ and any finite $r$-stable subset $A\subseteq G$ with tripling $K$, there is a subgroup $H\subseteq A\cdot A^{-1}$ of $G$ with $A\subseteq C\cdot H$ for some $C\subseteq A$ of size at most $n$. Moreover, there exists $C'\subseteq C$ such that*

$$|A\triangle(C'\cdot H)|\leq\epsilon|A|.$$

In the case when $G$ is abelian, it suffices to assume that $A$ has doubling $K$. Furthermore, we shall see that, when $G$ is abelian, the subgroup $H$ can be taken to be a boolean combination (of complexity only depending on $K$, $\epsilon$ and $r$) of translates of $A$.

In particular, on choosing $\epsilon=1$, Theorem A yields the existence of a natural number $n_0=n(K,1,r)$ such that any finite $r$-stable subset $A$ of tripling $K$ is contained in $n_0$ translates of a subgroup $H\subseteq A\cdot A^{-1}$ of $G$. It then follows that $|A\cap g\cdot H|\geq|A|/n_0$ for some subgroup $H\leqslant G$ and some $g\in A$, which is a qualitative result in the spirit of Freiman-Ruzsa. As in the previous paragraph, when $G$ is abelian, the complexity of such a subgroup $H$ as a boolean combination of translates of $A$ can be bounded solely in terms of $K$ and $r$.

The above result dovetails with a suite of arithmetic regularity lemmas under the additional assumption of stability that have been obtained recently by Terry and the Wolf [6, 7], as well as by Conant, Pillay and Terry [1]. However, without the assumption of small doubling/tripling, the bound on the symmetric difference is at best $\epsilon|H|$. Theorem A is also reminiscent of work of Sisask [5, Theorem 5.4], who combined the assumption of small doubling with that of bounded VC-dimension in vector spaces over finite fields.

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## Density of compressibility, and incidence bounds in $\mathbb{F}_p(t)$

Martin Bays

A type $p(x) \in S(A)$ is compressible if for every $\phi(x, y)$ there is $\zeta(x, z)$ such that for every finite $A_0 \subseteq A$ there is $c \in A^{|z|}$ such that $p(x) \ni \zeta(x, c) \vdash (p_\phi)|_{A_0}(x)$. This is a weakening of the notion of *l-isolation* from classical stability theory, in which $c$ is constant. In a stable theory, the only compressible types are the l-isolated types. An l-isolated type over a model must be realised. Compressibility of types is therefore opposed to stability, and indeed they first arose in the Chernikov-Simon "strong honest definitions" characterisation of distality: a theory is distal if and only if all types over arbitrary sets are compressible. Applying the $(p, q)$-theorem, they moreover obtain uniformity in a distal theory, namely that $\zeta$ depends only on $\phi$ and not on $p(x)$. In later work [CGS16] [CS18], this property of a formula $\phi(x, y)$ was isolated; $\zeta$ is said to be a *distal cell decomposition* for $\phi$, and the existence of such a $\zeta$ (in any language) implies strong combinatorial properties of the binary relation defined by $\phi(x, y)$, in particular incidence bounds of Szemeredi-Trotter type.

In on-going joint work with Itay Kaplan and Pierre Simon, we consider compressibility as an isolation notion and study its behaviour in arbitrary NIP theories. It is a classical result in stability theory that l-isolated types are dense in the Stone spaces $S(A)$ in countable stable theories, and so it is natural to ask whether the same holds of compressibility in countable NIP theories. This turns out to be the case, but to prove it we had to adapt a recent result [CCT16] in the statistical learning literature bounding the "recursive teaching dimension" of a finite set system in terms of its vc-dimension. We obtain some corollaries for a countable NIP theory: stability is *equivalent* to compressibility coinciding with l-isolation; one can extend models without growing a stable stably embedded part; models which are atomic with respect to compressibility exist over a set $A$, at least as long as $|A| \leq \aleph_1$.

In work-in-progress joint with Jean-François Martin, we consider Szemeredi-Trotter type incidence bounds for $\mathbb{F}_p(t)$ and similar fields. The simplest case of such a bound on such a field $k$ would be an $\epsilon$ such that for any $N$ points in the plane $k^2$ and any $N$ lines defined over $k$, the number of point-line incidences is bounded as $O(N^{\frac{3}{2}-\epsilon})$. One is also interested (e.g. for Elekes-Szabó purposes) in versions for arbitrary polynomially defined binary relations with a bound on pairwise intersections of the "lines". A general conjecture of Hrushovski [Hru13] indicates that pseudofinite fields should be the only obstruction to such Szemeredi-Trotter-like bounds, which implies in particular that they should hold unrestrictedly for a field, such as $\mathbb{F}_p(t)$, which does not contain unboundedly large finite subfields. The results of Chernikov-Galvin-Starchenko mentioned above imply such bounds whenever the incidence relation admits a distal cell decomposition. We show that polynomially defined relations in $\mathbb{F}_p(t)$ do admit a distal cell decomposition, and similarly for any field $k$ admitting a non-trivial valuation with finite residue field, by using some elementary model theory of ACVF and the yoga of compressibility to obtain that suitable ACVF-types corresponding to this valued subfield are compressible, and using a $(p, q)$ argument to obtain uniformity.

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## Amenability and definability


Anand Pillay

(joint work with E. Hrushovski, K. Krupinski)


We introduce the notion of *first order (extreme) amenability*, as a property of a first order theory $T$: every complete type over $\emptyset$, in possibly infinitely many variables, extends to an automorphism-invariant global Keisler measure (type) in the same variables. (Extreme) amenability of $T$ will follow from (extreme) amenability of the (topological) group $Aut(M)$ for all sufficiently large $\aleph_0$-homogeneous countable models $M$ of $T$ (assuming $T$ to be countable), but is radically less restrictive. First, we study basic properties of amenable theories, giving many equivalent conditions. Then, applying a version of the stabilizer theorem from our joint paper "Amenability, connected components, and definable actions" we prove that if $T$ is amenable, then $T$ is G-compact, namely Lascar strong types and Kim-Pillay strong types over $\emptyset$ coincide. In the special case when amenability is witnessed by $\emptyset$-definable global Keisler measures (which is for example the case for amenable $\omega$-categorical theories), we also give a different proof, using stability in continuous logic.

## Stability in a group

Gabriel Conant

One of the most well established and fruitful areas of model theory is the study of groups definable in *stable* first-order theories, which connects mathematical logic to algebraic geometry, topological dynamics, and combinatorial number theory. At the heart of this connection is the notion of a *generic* subset of a group. Given a group $G$, we say $A \subseteq G$ is **generic** if $G$ can be covered by finitely many left translates of $A$. An important fact in stable group theory is that the generic definable subsets of a stable group are *partition regular*, i.e., the union of two non-generic definable sets is non-generic. Consequently, there exist *generic types* (i.e., ultrafilters of generic definable sets), which provide a model theoretic analogue of *generic points* in the sense of algebraic groups and of group actions on compact Hausdorff spaces.

Let us recall the definition of a *stable group* in model theory. Given a complete first-order theory $T$ in a language $\mathcal{L}$, we say that an $\mathcal{L}$-formula $\phi(x;y)$ is **stable in** $T$ if there is some $k \geq 1$ such that, for any model $M \models T$, there do not exist tuples $a_1, \ldots, a_k, b_1, \ldots, b_k$ from $M$ such that $M \models \phi(a_i, b_j)$ if and only if $i \leq j$ (in this case, we also say $\phi(x;y)$ is $k$**-stable in** $T$). A theory $T$ is **stable** if every formula is stable in $T$. A **stable group** is a group whose underlying set and group operation are definable in some model of a stable theory.

A canonical example of a stable theory is the complete theory of an algebraically closed field. So algebraic groups provide natural examples of stable groups. The model-theoretic study of algebraic groups motivated much of the early work in stable group theory, and has now developed into an entire industry focusing on groups of "finite Morley rank". Another important example of a stable group is any abelian group (in the pure group language). These are special cases of "1-based" groups, whose theory was developed by Hrushovski and Pillay [9]. This notion is related to the Mordell-Lang Conjecture which, in model-theoretic language, says that if $G$ is a finite rank subgroup of a semiabelian complex variety, then the first-order structure on $H$ induced from the complex field is stable and 1-based. In [6], Hrushovski combined the study of 1-based groups and groups of finite Morley rank to prove the Mordell-Lang Conjecture for function fields in all characteristics.

A great deal of stability theory can also be developed "locally", i.e., for a single stable formula (see, e.g., Shelah's "Unstable Formula Theorem" [12, Theorem 2.2]). In [10], Hrushovski and Pillay use stable formulas to prove that any group definable in a pseudofinite field $F$ (whose complete theory is necessarily unstable) is virtually isogenous with $G(F)$, where $G$ is an algebraic group defined over $F$. An especially spectacular display of the effectiveness of local stability is Hrushovski's work from [7] on the structure of approximate groups, which uses a very general "Stabilizer Theorem" modeled after early work of Zilber on groups of finite Morley rank.

More recently, interactions with functional analysis have motivated the study of stability "in a model". Given a first-order structure $M$, we say that a formula $\phi(x;y)$ is **stable in** $M$ if there do not exist sequences $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ from

$M$, indexed by an infinite linear order $I$, such that $M \models \phi(a_i; b_j)$ if and only if $i \leq j$. This is weaker than stability of $\phi(x; y)$ in a theory $T$ (which is equivalent to stability in an $\aleph_0$-*saturated* model of $T$). In [1], Ben Yaacov established a direct connection between stability in a model and Grothendieck's [5] characterization of relatively weakly compact sets in certain Banach spaces. A natural question, which we address here, is how this connection applies to stable group theory.

Topological dynamics has played a major role in the model theory of groups since the work of Newelski [11], which provided a model-theoretic interpretation of the "Ellis semigroup" of a *G-flow* (i.e., a compact space with an action of $G$ by homeomorphisms). Moreover, certain parts of a recent preprint of Hrushovski, Krupiński, and Pillay [8] indicate a close connection between stable group theory and results of Ellis and Nerurkar [4] on *weakly almost periodic* $G$-flows. Our main goal here is to develop stable group theory entirely from [4] and in the more general setting of local stability "in a model". It is interesting to note that the original development of (global) stable group theory was roughly contemporaneous with [4] and related work on almost periodic minimal flows.

Given a group $G$, we call a set $A \subseteq G$ **stable in** $G$ if there do not exist sequences $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ from $G$, indexed by an infinite linear order $I$, such that $a_i b_j \in A$ if and only if $i \leq j$ (i.e., the "formula" $xy \in A$ is stable in the structure $(G, A)$ obtained by expanding $G$ with a predicate naming $A$). Let $\mathcal{B}^{\mathrm{st}}_G$ be the collection of *all* subsets of $G$ that are stable in $G$. Then $\mathcal{B}^{\mathrm{st}}_G$ is bi-invariant Boolean algebra of subsets of $G$, which contains all subgroups of $G$. One of our main results is the following structure theorem for stable subsets of groups.

**Theorem 1.** *Let $G$ be a group. Then there is a bi-invariant finitely additive probability measure $\mu^{\mathrm{st}}_G$ on $\mathcal{B}^{\mathrm{st}}_G$, which satisfies the following properties.*

(a) *If $\mathcal{B}$ is a left-invariant sub-algebra of $\mathcal{B}^{\mathrm{st}}_G$, then $\mu^{\mathrm{st}}_G{\restriction}\mathcal{B}$ is the unique left-invariant finitely additive probability measure on $\mathcal{B}$.*
(b) *If $A \in \mathcal{B}^{\mathrm{st}}_G$ then $\mu^{\mathrm{st}}_G(A) > 0$ if and only if $A$ is generic. Moreover, $A$ is generic if and only if $G$ can be covered by finitely many right translates of $A$.*
(c) *Let $\mathcal{B}$ be a bi-invariant sub-algebra of $\mathcal{B}^{\mathrm{st}}_G$, and suppose $A \in \mathcal{B}$. Then there is a finite-index normal subgroup $H \leq G$, which is in $\mathcal{B}$, and a set $Y \subseteq G$, which is a union of cosets of $H$, such that $\mu^{\mathrm{st}}_G(A \triangle Y) = 0$. Consequently, if $\mathcal{C}$ is the set of cosets of $H$ contained in $Y$, then $\mu^{\mathrm{st}}_G(A) = \mu^{\mathrm{st}}_G(Y) = |\mathcal{C}|/[G : H]$.*

This theorem is modeled after [3, Theorem 2.3], which focuses on the special case of a single $k$-stable invariant formula (further discussed below). The motivation in [3] was to prove a "stable arithmetic regularity lemma" for finite groups, which qualitatively generalized a combinatorial result of Terry and Wolf [13] on $\mathbb{F}_p^n$.

To prove Theorem 1, we apply various results from [4] to the action of $G$ on the *Stone space* $S(\mathcal{B})$ of ultrafilters (or "types") over a fixed left-invariant sub-algebra $\mathcal{B}$ of $\mathcal{B}^{\mathrm{st}}_G$. In this case, if $\mathcal{B}^\sharp$ denotes the smallest bi-invariant Boolean algebra containing $\mathcal{B}$ then, using "definability of types", we show that $S(\mathcal{B}^\sharp)$ a semigroup under the usual operation of ultrafilters, and is canonically isomorphic to the Ellis semigroup of $S(\mathcal{B})$. Moreover, by "symmetry of forking" for definable types, every function in the Ellis semigroup of $S(\mathcal{B})$ is continuous, and so $S(\mathcal{B})$ is weakly almost periodic by the characterization in [4] of such actions. Thus $S(\mathcal{B})$ is uniquely ergodic and has a unique minimal closed $G$-invariant subset, which is precisely the space $\mathrm{Gen}(\mathcal{B})$ of *generic* types in $S(\mathcal{B})$ (i.e., types containing only generic sets in $\mathcal{B}$). This forms the foundation for the proof of Theorem 1.

Note that Theorem 1 includes the "global" setting where $G$ is a group definable in a stable theory and $\mathcal{B}$ is the Boolean algebra of definable subsets of $G$. It also includes the case where $\mathcal{B}$ is the Boolean algebra of sets $A \subseteq G$ that are $k$-stable for some $k \geq 1$ (i.e., $xy \in A$ is $k$-stable with respect to the theory of $(G, A)$). The "$k$-stable case" generalizes the local setting of Hrushovski and Pillay [10], which applies to a sub-algebra of $\mathcal{B}^{\mathrm{st}}_G$ generated by the instances of a single left-invariant stable formula in an ambient theory $T$. This is also the setting for the weaker version of Theorem 1 from [3] mentioned above. Thus our results extend (and in some sense complete) the work in [3] on $k$-stable subsets of groups. In addition to Theorem 1, we also analyze local connected components, stabilizers of generic types, and measure-stabilizers of sets, along the lines of the results obtained in [2] for "$k$-NIP" sets in pseudofinite groups. Among other things, we prove:

**Theorem 2.** *Let $G$ be a group, and let $\mathcal{B}$ be a left-invariant sub-algebra of $\mathcal{B}^{\mathrm{st}}_G$. Let $G^0_{\mathcal{B}}$ be the intersection of all finite-index subgroups of $G$ in $\mathcal{B}$. Given $p \in S(\mathcal{B})$, let $\mathrm{Stab}(p) = \{g \in G : gp = p\}$. Let $\mathcal{N}$ be the collection of finite-index normal subgroups of $G$ in $\mathcal{B}^\sharp$.*

(*a*) $\mathrm{Gen}(\mathcal{B}^\sharp)$ *is a profinite group isomorphic to* $\varprojlim_{\mathcal{N}} G/N$, *and* $\mathrm{Gen}(\mathcal{B})$ *is a profinite homogeneous space under a transitive continuous action of* $\mathrm{Gen}(\mathcal{B}^\sharp)$.

(*b*) *Suppose $G^0_{\mathcal{B}}$ has finite index in $G$. Then $G^0_{\mathcal{B}} \in \mathcal{B}$ and $G^0_{\mathcal{B}^\sharp} = \bigcap_{a\in G} aG^0_{\mathcal{B}}a^{-1} \in \mathcal{N}$. Moreover,* $\mathrm{Gen}(\mathcal{B}^\sharp) \cong G/G^0_{\mathcal{B}^\sharp}$ *and* $\mathrm{Gen}(\mathcal{B}) \cong G/G^0_{\mathcal{B}}$ *via the maps taking $p$ to the unique coset $C$ of $G^0_{\mathcal{B}^\sharp}$ or $G^0_{\mathcal{B}}$ in $p$. Also, if $p \in \mathrm{Gen}(\mathcal{B}^\sharp)$ then $\mathrm{Stab}(p) = G^0_{\mathcal{B}^\sharp}$; and if $p \in \mathrm{Gen}(\mathcal{B})$ and $aG^0_{\mathcal{B}} \in p$, then $\mathrm{Stab}(p) = aG^0_{\mathcal{B}}a^{-1}$.*

Beyond this, we show that part ($b$) of Theorem 2 includes the case when $\mathcal{B}$ is defined from a single left-invariant relation (or "formula") that is stable in $G$.

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## Model-theoretic Elekes-Szabó

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(joint work with Ya'acov Peterzil and Sergei Starchenko)

### 1. Introduction

Erdős and Szemerédi [1] observed the following sum-product phenomenon: there is some $c \in \mathbb{R}_{>0}$ such that for any finite set $A \subseteq \mathbb{R}$, $\max\{|A+A|, |A \cdot A|\} \geq |A|^{1+c}$. Later, Elekes and Rónyai [2] generalized this by showing that for any polynomial $f(x,y)$ we must have $|f(A \times A)| \geq |A|^{1+c}$, unless $f$ is either additive or multiplicative (i.e. of the form $g(h(x) + i(y))$ or $g(h(x) \cdot i(y))$ for some univariate polynomials $g, h, i$ respectively). Elekes and Szabó [3] provide a conceptual generalization, showing that for any irreducible polynomial $F(x,y,z)$ depending on all of its coordinates such that its set zero set has dimension 2, if $F$ has a maximal possible number of zeroes $n^2$ on finite $n \times n \times n$ grids, then $F$ is in a finite-to-finite correspondence with the graph of multiplication of an algebraic group (in the special case above, either the additive or the multiplicative group of the field). Recently, several generalizations were obtained for relations of higher dimension and arity that we review in the next section. Here we announce a generalization of this result to hypergraphs of any arity and dimension definable in a large class of stable structures which includes differentially closed fields and compact complex manifolds, as well as for arbitrary *o*-minimal structures (to appear in [4]).

### 2. Elekes-Szabó principle

We fix a structure $\mathcal{M}$, definable sets $X_1, \ldots, X_s$, and a definable relation $Q \subseteq \bar{X} = X_1 \times \ldots \times X_s$. We write $A_i \subseteq_n X_i$ if $A_i \subseteq X_i$ with $|A_i| \leq n$. By a *grid on* $\bar{X}$ we mean a set $\bar{A} \subseteq \bar{X}$ with $\bar{A} = A_1 \times \ldots \times A_s$ and $A_i \subseteq X_i$. By an *n-grid on* $\bar{X}$ we mean a grid $\bar{A} = A_1 \times \ldots \times A_s$ with $A_i \subseteq_n X_i$.

2.1. **Fiber-algebraic relations.** A relation $Q \subseteq \bar{X}$ is *fiber-algebraic* if there is some $d \in \mathbb{N}$ such that for any $1 \leq i \leq s$ we have

$$\mathcal{M} \models \forall x_1 \ldots x_{i-1} x_{i+1} \ldots x_s \exists^{\leq d} x_i \, Q\left(x_1, \ldots, x_s\right).$$

For example, if $Q \subseteq X_1 \times X_2 \times X_3$ is fiber-algebraic, then for any $A_i \subseteq_n X_i$ we have $|Q \cap A_1 \times A_2 \times A_3| = dn^2$. Conversely, let $Q \subseteq \mathbb{C}^3$ be given by $x_1 + x_2 - x_3 = 0$, and let $A_1 = A_2 = A_3 = \{0, \ldots, n-1\}$. Then $|Q \cap A_1 \times A_2 \times A_3| = \frac{n(n+1)}{2} = \Omega\left(n^2\right)$. This indicates that the upper and lower bounds match for the graph of addition in an abelian group (up to a constant) — and the Elekes-Szabó principle suggests that in many situations this is the only possibility. Before making this precise, we introduce some notation.

2.2. **Grids in general position.** From now on we will assume that $\mathcal{M}$ is equipped with some notion of integer-valued dimension on definable sets, to be specified later. A good example to keep in mind is Zariski dimension on constructible subsets of $\mathcal{M} := (\mathbb{C}, +, \times) \models \mathrm{ACF}_0$, the theory of algebraically closed fields of characteristic 0.

Let $X$ be an $\mathcal{M}$-definable set and let $\mathcal{F}$ be a (uniformly) $\mathcal{M}$-definable family of subset of $X$. For $l \in \mathbb{N}$ we say that a set $A \subseteq X$ is *in* $(\mathcal{F}, l)$*-general position* if $|A \cap F| \leq l$ for every $F \in \mathcal{F}$ with $\dim(F) < \dim(X)$.

Let $X_i$, $i = 1, \ldots, s$, be $\mathcal{M}$-definable sets. Let $\bar{\mathcal{F}} = (\mathcal{F}_1, \ldots, \mathcal{F}_s)$, where $\mathcal{F}_i$ is a definable family of subsets of $X_i$. For $l \in \mathbb{N}$ we say that a grid $\bar{A}$ on $\bar{X}$ is in $(\bar{\mathcal{F}}, l)$*-general position* if each $A_i$ is in $(\mathcal{F}_i, l)$-general position.

For example, if $X$ is strongly minimal and $\mathcal{F}$ is any definable family of subsets of $X$, then for any large enough $l = l(\mathcal{F}) \in \mathbb{N}$, every $A \subseteq X$ is in $(\mathcal{F}, l)$-general position. On the other hand, let $X = \mathbb{C}^2$ and let $\mathcal{F}_d$ be the family of algebraic curves of degree $d$. If $l < d$, then any set $A \subseteq X$ is not in $(\mathcal{F}_d, l)$-general position.

2.3. **Generic correspondence with group multiplication.** Let $Q \subseteq \bar{X}$ be a definable relation and $(G, \cdot)$ a type-definable group in $\mathbb{M}^{\mathrm{eq}}$ which is connected (i.e. $G = G^0$). We say that $Q$ is in a *generic correspondence with multiplication in* $G$ if there exist elements $g_1, \ldots, g_s \in G(\mathbb{M})$, where $\mathbb{M}$ is a saturated elementary extension of $\mathcal{M}$, such that:

(1) $g_1 \cdot \ldots \cdot g_s = e$;
(2) $g_1, \ldots, g_{s-1}$ are independent generics in $G$ over $\mathcal{M}$, i.e. each $g_i$ doesn't belong to any definable set of dimension smaller than $G$ definable over $\mathcal{M} \cup \{g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_{s-1}\}$;
(3) For each $i = 1, \ldots, s$ there is a generic element $a_i \in X_i$ interalgebraic with $g_i$ over $\mathcal{M}$, such that $\models Q(a_1, \ldots, a_s)$.

If $X_i$ are irreducible, then (3) holds for all $g_1, \ldots, g_s \in G$ satisfying (1) and (2), providing a generic finite-to-finite correspondence between $Q$ and the graph of $(s-1)$-fold multiplication in $G$.

2.4. **The Elekes-Szabó principle.** Let $X_1, \dots, X_s$ be definable sets in $\mathcal{M}$ with $\dim(X_i) = k$ and $X_i$ irreducible (i.e. can't be split into two disjoint definable sets of full dimension). We say that they satisfy the *Elekes-Szabó principle* if for any irreducible fiber-algebraic definable relation $Q \subseteq \bar{X}$, one of the following holds:

(1) $Q$ admits power saving: there exist some $\varepsilon \in \mathbb{R}_{>0}$ and some definable families $\mathcal{F}_i$ on $X_i$ such that: for any $l \in \mathbb{N}$ and any $n$-grid $\bar{A} \subseteq \bar{X}$ in $(\bar{F}, l)$-general position, we have $|Q \cap \bar{A}| = O_l(n^{(s-1)-\varepsilon})$;
(2) $Q$ is in a generic correspondence with multiplication in a type-definable *abelian* group of dimension $k$.

Below are the previously known cases of the Elekes-Szabó principle:

- [3] $\mathcal{M} \models \mathrm{ACF}_0$, $s = 3$, $k$ arbitrary;
- [5] $\mathcal{M} \models \mathrm{ACF}_0$, $s = 4$, $k = 1$;
- [6] $\mathcal{M} \models \mathrm{ACF}_0$, $s$ and $k$ arbitrary, recongized that the arising groups are abelian (they work with a more relaxed notion of general position and arbitrary codimension, however no bounds on $\varepsilon$);
- [7] $\mathcal{M}$ is any strongly minimal structure interpretable in a distal structure, $s = 3$, $k = 1$.

**Theorem 1.** [4] *The Elekes-Szabó principle holds in the following two cases:*

*(1) $\mathcal{M}$ is a stable structure interpretable in a distal structure, with respect to $\mathfrak{p}$-dimension (see below).*
*(2) $\mathcal{M}$ is an o-minimal structure, with respect to the usual dimension (in this case, on a type-definable generic subset of $\bar{X}$, we get a definable coordinate-wise bijection of $Q$ with the graph of multiplication of $G$).*

Here we only discuss the stable case (1). Examples of structures satisfying the assumption of (1) are models of $\mathrm{ACF}_0$, $\mathrm{DCF}_{0,m}$ (i.e. differentially closed fields with $m$ commuting derivations), CCM (the theory of compact complex manifolds). Our method provides explicit bounds on $\varepsilon$ for power saving in these cases.

## 3. Ingredients of the proof in the stable case

3.1. **$\mathfrak{p}$-dimension.** We choose a saturated elementary extension $\mathbb{M}$ of a *stable* structure $\mathcal{M}$. By *a $\mathfrak{p}$-pair* we mean a pair $(X, \mathfrak{p}_X)$, where $X$ is an $\mathcal{M}$-definable set and $\mathfrak{p}_X \in S(\mathcal{M})$ is a complete stationary type on $X$. Assume we are given $\mathfrak{p}$-pairs $(X_i, \mathfrak{p}_i)$ for $1 \leq i \leq s$. We say that a definable $Y \subseteq X_1 \times \ldots \times X_s$ is $\mathfrak{p}$-generic if $Y \in \mathfrak{p}_1 \otimes \ldots \otimes \mathfrak{p}_s|_{\mathbb{M}}$. Finally, we define the $\mathfrak{p}$-dimension via $\dim_{\mathfrak{p}}(Y) \geq k$ if for some projection $\pi$ of $\bar{X}$ onto $k$ components, $\pi(Y)$ is $\mathfrak{p}$-generic. This $\mathfrak{p}$-dimension enjoys definability and additivity properties crucial for our arguments that may fail for Morley rank in general $\omega$-stable theories such as $\mathrm{DCF}_0$. However, if $X$ is a definable subset of finite Morley rank $k$ and degree one, taking $\mathfrak{p}_X$ to be the unique type on $X$ of Morley rank $k$, we have that $k \cdot \dim_{\mathfrak{p}} = \mathrm{MR}$, and Theorem 1(1) implies that the Elekes-Szabó principle holds with respect to Morley rank in this case.

3.2. **Distality and incidence bounds.** Distal structures were introduced in [8], and connections with combinatorics and generalized incidence bounds were established in [9, 10, 11]. The key result for us is the following generalized "Szemerédi-Trotter" theorem:

**Theorem 2.** [11, 4] *If $E \subseteq U \times V$ is a binary relation definable in a distal structure and $E$ is $K_{s,2}$-free for some $s \in \mathbb{N}$, then there is some $\delta > 0$ such that: for all $A \subseteq_n U, B \subseteq_n V$ we have $|E \cap A \times B| = O(n^{\frac{3}{2}-\delta})$.*

3.3. **Recovering groups from abelian $m$-gons.** Working in a stable theory, an *$m$-gon* over $A$ is a tuple $a_1, \ldots, a_m$ such that any $m-1$ of its elements are independent over $A$, and any element in it is in the algebraic closure of the other ones and $A$. We say that an $m$-gon is *abelian* if, after any permutation of its elements, we have $a_1 a_2 \downarrow_{\mathrm{acl}_A(a_1 a_2) \cap \mathrm{acl}_A(a_3 \ldots a_m)} a_3 \ldots a_m$.

If $(G, \cdot)$ is a type-definable abelian group, $g_1, \ldots, g_{m-1}$ are independent generics in $G$ and $g_m := g_1 \cdot \ldots \cdot g_{m-1}$, then $g_1, \ldots, g_m$ is an abelian $m$-gon (associated to $G$). Conversely,

**Theorem 3.** [4] *Let $a_1, \ldots, a_m$ be an abelian $m$-gon. Then there is a type-definable (in $\mathcal{M}^{eq}$) connected abelian group $(G, \cdot)$ and an abelian $m$-gon $g_1, \ldots, g_m$ associated to $G$, such that after a base change each $g_i$ is inter-algebraic with $a_i$.*

An analogous result was obtained independently by Hrushovski.

3.4. **Distinction of cases in Theorem 1.** We may assume $\dim(Q) = s - 1$, and let $\bar{a} = (a_1, \ldots, a_s)$ be a generic tuple in $Q$ over $\mathcal{M}$. As $Q$ is fiber-algebraic, $\bar{a}$ is an $s$-gon over $\mathcal{M}$.

**Theorem 4.** [4] *One of the following is true:*

*(1) For $u = (a_1, a_2)$ and $v = (a_3, \ldots, a_s)$ we have $u \downarrow_{\mathrm{acl}_M(u) \cap \mathrm{acl}_M(v)} v$.*

*(2) $Q$, as a relation on $U \times V$, for $U = X_1 \times X_2$ and $V = X_3 \times \ldots \times X_s$, is a "pseudo plane".*

In case (2) the incidence bound from Theorem 2 can be applied inductively to obtain power saving for $Q$. Thus we may assume that that for any permutation of $\{1, \ldots, s\}$ we have

$$a_1 a_2 \downarrow_{\mathrm{acl}_M(a_1 a_2) \cap \mathrm{acl}_M(a_3 \ldots a_s)} a_3 \ldots a_s,$$

i.e. the $s$-gon $\bar{a}$ is abelian, and Theorem 3 can be applied to establish generic correspondence with a type-definable abelian group.

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## Cardinal Invariants and Tree Properties

Nicholas Ramsey

In *Classification Theory*, Shelah induced the invariants $\kappa_{\mathrm{cdt}}(T)$, $\kappa_{\mathrm{sct}}(T)$, and $\kappa_{\mathrm{inp}}(T)$ which bound approximations to the tree property (TP), the tree property of the first kind ($\mathrm{TP}_1$), and the tree property of the second kind ($\mathrm{TP}_2$), respectively. Eventually, the local condition that a theory does not have the tree property (*simplicity*), and the global condition that $\kappa(T) = \kappa_{cdt}(T) = \aleph_0$ (*supersimplicity*) proved to mark substantial dividing lines. These invariants provide a coarse measure of the complexity of the theory, providing a "quantitative" description of the patterns that can arise among forking formulas. Motivated by some questions from [She90], we explored the question of which relationships known to hold between the *local* properties TP, $\mathrm{TP}_1$, and $\mathrm{TP}_2$ also hold for the *global* invariants $\kappa_{\mathrm{cdt}}(T)$, $\kappa_{\mathrm{sct}}(T)$, and $\kappa_{\mathrm{inp}}(T)$. This continues the work done in [CR16], where, with Artem Chernikov, we considered a global analogue of the following theorem of Shelah:

**Theorem.** [She90, III.7.11] *For complete theory $T$, $\kappa_{cdt}(T) = \infty$ and only if $\kappa_{sct}(T) = \infty$ or $\kappa_{inp}(T) = \infty$. That is, $T$ has the tree property if and only if it has the tree property of the first kind or the tree property of the second kind.*

Shelah then asked if $\kappa_{\mathrm{cdt}}(T) = \kappa_{\mathrm{sct}}(T) + \kappa_{\mathrm{inp}}(T)$ in general [She90, Question III.7.14]. In [CR16], we showed that is true under the assumption that $T$ is countable. For a countable theory $T$, the only possible values of these invariants are $\aleph_0, \aleph_1$, and $\infty$—our proof handled each cardinal separately using a different argument in each case. In [Ram15], we considered this question without any hypothesis on the cardinality of $T$, answering the general question negatively :

**Theorem.** [Ram15] *There is a stable theory $T$ such that $\kappa_{cdt}(T) > \kappa_{sct}(T) + \kappa_{inp}(T)$. Moreover, it is consistent with ZFC that for every regular uncountable $\kappa$, there is a stable theory $T$ with $|T| = \kappa$ and $\kappa_{cdt}(T) > \kappa_{sct}(T) + \kappa_{inp}(T)$.*

To construct a theory $T$ so that $\kappa_{\mathrm{cdt}}(T) \neq \kappa_{\mathrm{sct}}(T) + \kappa_{\mathrm{inp}}(T)$, we use results on *strong colorings* constructed by Galvin under GCH [Gal80] and later by Shelah in ZFC [She97]. This was accomplished by a method inspired by Medvedev's $\mathbb{Q}$ACFA construction [Med15], realizing the theory as a union of theories in a system of finite reducts each of which is the theory of a Fraïssé limit.

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