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New Perspectives and Computational Challenges in High Dimensions

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Abstract. High-dimensional systems are frequent in mathematics and applied sciences, and the understanding of high-dimensional phenomena has become increasingly important. The mathematical subdisciplines most strongly related to such phenomena are functional analysis, convex geometry, and probability theory. In fact, a new area emerged, called asymptotic geometric analysis, which is at the very core of these disciplines and bears a number of deep connections to mathematical physics, numerical analysis, and theoretical computer science. The last two decades have seen a tremendous growth in this area. Far reaching results were obtained and various powerful techniques have been developed, which rather often have a probabilistic flavor. The purpose of this workshop was to explored these new perspectives, to reach out to other areas concerned with high-dimensional problems, and to bring together researchers having different angles on high-dimensional phenomena.

Mathematics Subject Classification (2010): Primary 52A22, 60D05, 68Q25; Secondary: 52A23, 60G55, 65C05, 65D30.

Introduction by the Organizers

This workshop was organised by Aicke Hinrichs (Linz), Joscha Prochno (Graz), Christoph Thäle (Bochum) and Elisabeth Werner (Cleveland), and was a continuation and expansion of the highly successful mini-workshop 1706c which was held at Oberwolfach in February 2017.

The special focus of the *New Perspectives and Computational Challenges in High Dimensions* workshop was to establish connections between asymptotic geometric analysis and information-based complexity (IBC), two young and until now essentially independent areas dealing with high-dimensional problems. Particular areas where these sub-fields are synthesised include high-dimensional numerical integration, discrepancy theory and dispersion of point sets in high-dimensions. IBC deals with the computational complexity of continuous problems for which available information is partial or corrupted. Questions of this form are of vital importance and in the focus of latest research as they naturally arise in physics, economics, mathematical finance, engineering, medical imaging, and weather and climate prediction. The problems and questions in these disciplines usually suffer from what is known as the curse of dimensionality and tractability of such problems lies at the heart of the theory. Very recent results reveal the impact of methods from asymptotic geometric analysis that is promising, exciting and were further pursued during this workshop.

The prior mini-workshop of 2017 stimulated a large number of new and fruitful collaborations leading to a considerable number of results in the interim period, and brought together two different groups of researchers that have different outlooks on high-dimensional phenomena, thus creating the opportunity for cross-fertilization and research synergies. In keeping with the philosophy of its predecessor workshop, and in order to take full advantage of the diversity of the group as well as of the resources of the research institute at Oberwolfach, in the scheduling of the *New Perspectives* workshop we placed a large amount of emphasis on interactive work and group discussion. The workshop kicked off on the Monday with a series of informal survey talks which introduced and explained core ideas to the non-experts in the respective fields and served as preparation for the research activities during the week. These talks were distributed across four sub-fields:

- Tractability of high-dimensional problems
- Asymptotic theory of convex bodies
- Discrepancy and dispersion of point sets in high dimensions
- Modern approximation algorithms for multivariate numerical problems

After dinner on the Monday we had an open-problem session, where each participant and survey-lecturer was invited to suggest problems for the week. The following day, the participants of the workshop congregated into three distinct directions of research: focussing in large deviations, information based complexity, and random polytopes. On the Tuesday, Wednesday and Thursday of the week we had 'special focus' talks that went deeper into particular aspects of the topics mentioned above and address potential needs and issues of the smaller working groups.

In the middle of the workshop as well as at its end, that is, on Wednesday and on Friday, the different groups were asked to present the progress they made, and to discuss potential outcomes and the (technical) problems they ran into. The second of these presentations formed the closure of the workshop, though many of the researchers who attended the workshop have continued to stay in contact and to work on the focus problems, and we expect these collaborations should produce research papers of the first quality in the near future.

Acknowledgement: The workshop organizers would like to thank the team of the MFO for providing a highly stimulating working atmosphere. The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows".

Workshop: New Perspectives and Computational Challenges in High Dimensions

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Abstracts

Large deviations and high-dimensional geometry

Kavita Ramanan
(joint work with Nina Gantert, Steven Kim, Yin-Ting Liao)

The study of high-dimensional geometric structures and measures are central themes in geometric functional analysis and asymptotic geometric analysis, and several results in these fields have shown that the presence of high dimensions often imposes a certain regularity that has a probabilistic flavor. One prominent theme is the study of high-dimensional objects by looking at their lower-dimensional projections. The study of this is often facilitated by an asymptotic analysis, as the dimension n goes to infinity. Laws of large numbers, central limit theorems and concentration results for projections of random vectors have now become a part of the classical canon, with early work by Diaconis, Freedman, Schechtman and Schmuckenschläger [2, 12, 13] supplemented by more recent results such as [10, 6]. Indeed, informally speaking, the celebrated central limit theorem for convex sets [10], says that fluctuations of the projections of (the normalized volume measure) of any isotropic convex body is approximately Gaussian. These results are universal in the sense that the asymptotic limit is typically insensitive to details of the distribution of the high-dimensional measures beyond first and second-order moments. While these universality results are elegant, they have the downside that they imply that fluctuations of lower-dimensional projections typically cannot be used to distinguish between different high-dimensional objects.

However, more recent developments have shown that large deviations analysis of random projections provides a way to capture non-universal features of high-dimensional structures, which could allow one to distinguish between high-dimensional measures. Specifically, given an n-dimensional random vector $X^{(n)}$ with distribution μ_n, a random direction $\Theta^{(n)}$ taking values $\mathbb{S}^{n-1}$, the unit $(n-1)$-dimensional sphere in $\mathbb{R}^n$, and a suitable scaling constant $\kappa_n > 0$, consider the suitably normalized scalar projection of $X^{(n)}$ onto $\Theta^{(n)}$:

$$W_\Theta^{(n)} := \kappa_n \langle X^{(n)}, \theta^{(n)} \rangle, \quad n \in \mathbb{N}.$$

Classical results in probability consider the case when each μ_n is a product measure of the form $\otimes^n \mu$ for some fixed probability measure μ on $\mathbb{R}$, and $\Theta^{(n)}$ is concentrated on the vector $\iota^{(n)} := (1, \ldots, 1)/\sqrt{n}$ and $\kappa_n = \sqrt{n}^{-1}$, the classical result of Cramér from 1938 shows that the corresponding sequence $\{W_\iota^{(n)}\}_{n\in\mathbb{N}}$ satisfies a large deviation principle with rate function Λ^*, where Λ^* is the Legendre transform of the logarithmic moment generating function of μ. Roughy speaking, this means that

$$\frac{1}{n} \log \mathbb{P}(W_\iota^{(n)} > a) \to \Lambda^*(a), \quad a \in \mathbb{R}.$$

From this perspective, it is natural to ask if there is an extension of this classical result to more general random directions $\Theta^{(n)}$ and (possibly non-product) random vectors $X^{(n)}$, for example, those uniformly distributed on a convex body. Here, one can study both quenched large deviations (conditioned on a given sequence of directions, $\Theta^{(n)} = \theta^{(n)}$, $n \in \mathbb{N}$) and annealed large deviations, which also averages over the randomness in the direction sequence.

In these lectures, we will provide a brief introduction to large deviation theory, summarize recent results on large deviations principles and sharp large deviation estimates for both one-dimensional and multi-dimensional random projections of high-dimensional measures obtained, for example, in [1, 3, 4, 7, 8, 5], and describe several potential directions of further inquiry.

References

[1] D. Alonso-Gutiérrez, J. Prochno, C. Thäle, *Large deviations for high-dimensional random projections of lpn-balls*, Advances in Applied Mathematics, **99**, (2018), 1–35.

[2] P. Diaconis and D. Freedman, *Asymptotics of graphical projection pursuit*, The Annals of Statistics, **12**, (1984), 793–815.

[3] N. Gantert, S. S. Kim, and K. Ramanan, *Cramér's theorem is atypical*, Advances in the mathematical sciences: Research from the 2015 association for women in mathematics symposium, (2015), editors G. Letzter et al, Springer International Publishing, Cham, 253–270.

[4] N. Gantert, S. S. Kim, and K. Ramanan, *Large deviations for random projections of ℓ^p balls*, Annals of Probability, **45** (2017), 4419–4476.

[5] Y.-T. Liao and K. Ramanan, *Geometric sharp large deviations for random projections of ℓ_p^n balls*, (2020), Arxiv Preprint, https://arxiv.org/abs/2001.04053.

[6] Z. Kabluchko, J. Prochno and C. Thäle, *High-dimensional limit theorems for random vectors in ℓ_p^n-balls*, Communications in Contemporary Mathematics, **21** (2019) No. 1750092.

[7] S.S. Kim and K. Ramanan, *A conditional limit theorem for high-dimensional ℓ_p spheres*, Journal of Applied Probability, **55** (2018), 1060–1077.

[8] S.S. Kim and K. Ramanan, *An asymptotic thin shell condition and large deviations for multidimensional projections*, (2019), Arxiv preprint, https://arxiv.org/abs/1912.13447.

[9] S.S. Kim, *Problems at the interface of probability and convex geometry: Random projections and constrained processes*, Ph.D. Thesis, (2017), Brown University.

[10] B. Klartag, *A central limit theorem for convex sets*, Invent. Math., **168** (2007), 91–131.

[11] E. Meckes, *Approximation of projections of random vectors*, Journal of Theoretical Probability, **25** (2012), 333–352.

[12] G. Schechtman and J. Zinn, *On the volume of the intersection of two l_p^n balls*, Proceedings of the American Mathematical Society, **110** (1990) 217–224.

[13] Schmuckenschläger, *Clt and the volume of intersections of ℓ_p^n-balls*, Geom. Dedicata, **85** (2001), 189–195.

The convex hull of random points on the boundary of a simple polytope

Carsten Schütt

(joint work with Matthias Reitzner, Elisabeth Werner)

Let P be a polytope in $\mathbb{R}^n$. Choose N random points $X_1, \dots, X_N$ on the boundary ∂P of P, and denote by $P_N = [X_1, \dots, X_N]$ the convex hull of these points. We are interested in the expected number of vertices $\mathbb{E}f_0(P_N)$, the expected number of facets $\mathbb{E}f_{n-1}(P_N)$, and the expectation of the volume difference $V_n(P) - \mathbb{E}V_n(P_N)$

of P and P_N. Since explicit results for fixed N cannot be expected we investigate the asymptotics as $N \to \infty$.
There is a vast amount of literature on random polytopes with vertices chosen *from the interior* of a convex set. Investigations started with two famous articles by Rényi and Sulanke who obtained in the planar case the asymptotic behaviour of the expected area $\mathbb{E}V_2(P_N)$ for a polygon P. In a long and intricate proof Bárány and Buchta settled the case of polytopes $P \subset \mathbb{R}^n$,

$$V_n(P) - \mathbb{E}V_n(P_N) = \frac{\text{flag}(P)}{(n+1)^{n-1}(n-1)!} N^{-1} (\ln N)^{n-1} (1 + o(1)),$$

where flag(P) is the number of flags of the polytope P. A *flag* is a sequence of i-dimensional faces F_i of P, $i = 0 \ldots, n-1$, such that $F_i \subset F_{i+1}$. The phenomenon that the expression should only depend on this combinatorial structure of the polytope had been discovered in connection with floating bodies by Schütt.
Due to Efron's identity the results on $\mathbb{E}V_n(P_N)$ can be used to determine the expected number of vertices of P_N. The general results for the number of ℓ-dimensional faces $f_\ell(P_N)$ for a polytope P is

$$\mathbb{E}f_\ell(P_N) = c(n,\ell)\ \text{flag}(P)\,(\ln N)^{n-1}(1 + o(1)).$$

Choosing random points from the interior of a convex body always produces a simplicial polytope with probability one. Yet often applications of the above mentioned results in computational geometry, the analysis of the average complexity of algorithms and optimization necessarily deal with non simplicial polytopes and it became crucial to have analogous results for random polytopes without this very specific combinatorial structure.
In this paper we are discussing the case that the points are chosen from the boundary of a polytope P. This produces random polytopes which are neither simple nor simplicial and thus our results are a huge step in taking into account the first point mentioned above. The applications in computational geometry, the analysis of the average complexity of algorithms and optimization need formulae for the combinatorial structure of the involved random polytopes and thus the question on the number of facets and vertices are of interest.
It follows immediately that for random polytopes whose points are chosen from the boundary of a polytope the expected number of vertices is

$$\mathbb{E}f_0(P_N) = c_{n,0}\,\text{flag}(P)\,(\ln N)^{n-2}(1 + o(1)).$$

Indeed, a chosen point is a vertex of a random polytope if and only if it is a vertex of the convex hull of all the points chosen in that facet. We get that the number of vertices equals

$$c_{n,0} \sum_F \text{flag}(F)(\ln N)^{n-2}(1 + o(1)),$$

where we sum over all facets. It is left to observe that flag$(P) = \sum_F \text{flag}(F)$.
For our main results we have to restrict our investigations to simple polytopes.

Theorem 1. *Choose N uniform random points on the boundary of a simple polytope P. For the expected number of facets of the random polytope P_N, we have*

$$\mathbb{E}f_{n-1}(P_N) \quad = \quad c_{n,n-1}f_0(P)(\ln N)^{n-2}(1+O((\ln N)^{-1})),$$

with some $c_{n,n-1} > 0$.

Theorem 2. *For the expected volume difference between a simple polytope $P \subset \mathbb{R}^n$ and the random polytope P_N with vertices chosen from the boundary of P, we have*

$$\mathbb{E}(V_n(P) - V_n(P_N)) = c_{n,P}N^{-\frac{n}{n-1}}(1+O(N^{-\frac{1}{(n-1)(n-2)}}))$$

with some $c_{n,P} > 0$.

Optimal algorithms for some high dimensional problems

Erich Novak

(joint work with Aicke Hinrichs, David Krieg, Joscha Prochno, Mario Ullrich and Henryk Woźniakowski)

We consider functions $f : D_d \to \mathbf{R}$, where $D_d \subset \mathbf{R}^d$, that are unknown to us, it is assumed that we can compute function values

$$f(x_1), f(x_2), \ldots, f(x_n).$$

We want to compute something, here we consider

$$S(f) = \mathrm{INT}(f) = \int_{D_d} f(x)\,\mathrm{d}x \qquad \text{or} \qquad S(f) = \mathrm{APP}(f) = f.$$

Algorithms A_n use n function values of f and D_d is a bounded Lipschitz domain with volume 1.

How many values $f(x_i)$ do we need to compute $S(f)$ up to an error $\varepsilon > 0$? Makes only sense if we know that $f \in F$ for a given class F of "possible" functions; the class F is the input class or problem class for algorithms. Information complexity

$$n(\varepsilon, S, F)$$

is the smallest n needed to solve the problem up to an error ε. Sometimes more convenient is the nth minimal error

$$e_n(F, S) = \inf_{A_n} \sup_{f \in F} \|S(f) - A_n(f)\|.$$

For APP we take (mainly) the L_∞-norm.

Known fact: Assume that $F = F(D_d)$ is a unit ball for a norm. Then

$$e_n(F(D_d), \mathrm{APP}) = \inf_{x_1,\ldots,x_n \in D_d} \sup_{f \in F(D_d),\, f(x_i)=0} \|f\|_\infty,$$

and

$$e_n(F(D_d), \mathrm{INT}) = \inf_{x_1,\ldots,x_n \in D_d} \sup_{f \in F(D_d),\, f(x_i)=0} \int_{D_d} f(x)\,\mathrm{d}x.$$

Consider a unit ball F in $C^k([0,1]^d)$ or $C^k(D_d)$. Well known, for fixed k and d,

$$e_n(F, \mathrm{APP}) \asymp e_n(F, \mathrm{INT}) \asymp n^{-k/d}.$$

Questions:

- How does the complexity n depend on d if we fix ε?
- What about $k = \infty$? Is the problem easy for large d?
- How do the results depend on D_d?

We present results and open problems (OPs) for the following classes of functions:
$C^k(D_d)$: all partial derivatives, up to order k, are bounded by 1.
$C^\infty(D_d)$: all partial derivatives are bounded by 1.
$\widetilde{C}^k(D_d)$: all derivatives, up to order k, are bounded by 1.
$\widetilde{C}^\infty(D_d)$: all derivatives are bounded by 1.

Example: $f\colon [0,1]^d \to \mathbf{R}$, given by $f(x) = x_1 x_2 x_3 \dots x_d$. Then all partial derivatives are bounded by 1 but $f(x, x, \dots, x) = x^d$ and hence some directional derivatives are huge. Hence the classes with tilde are much smaller.

Theorem: For all k there exist c_k and $\tilde{c}_k$ such that for all $n, d \in \mathbf{N}$ we have

$$\min(1/2,\, c_k d n^{-k/d}) \le e_n(C^k([0,1]^d), \mathrm{INT}) \le \min(1,\, \tilde{c}_k d n^{-k/d}).$$

Hence the complexity is roughly $(d/\varepsilon)^{d/k}$, super-exponential in d.

(1) The lower bound holds for all n and D_d with the same c_k.
(2) OP: For which D_d does the upper bound hold?
(3) OP: Does the asymptotic constant $\limsup_{n\to\infty} e_n(C^k(D_d), \mathrm{INT}) \cdot n^{k/d}$ depend on D_d? Is it always a lim? For $k = 1$: $\lim_n e_n(C^1(D_d), \mathrm{INT}) \cdot n^{1/d} = \zeta_d\, \lambda^d(D_d) \cdot \left[\frac{d!\,\lambda^d(D_d)}{2^d}\right]^{1/d}$, where $\zeta_d \to 1$.

Theorem: For all k there exist c_k and $\tilde{c}_k$ such that for all $n, d \in \mathbf{N}$ we have

$$\min(1/2,\, c_k d^{1/2} n^{-k/d}) \le e_n(\tilde{C}^k([0,1]^d), \mathrm{INT}) \le \min(1,\, \tilde{c}_k d n^{-k/d}).$$

OP: Close the gap. (For $k = 1$ the lower bound is sharp and it seems that for $k = 2$ the upper bound is sharp.)

Consider the integration problem for the class

$$\bar{C}^2(D_d) = \{f \in C^2(D_d) \mid \|f\|_\infty \le 1,\, \mathrm{Lip}(f) \le d^{-1/2},\, \mathrm{Lip}(D^\Theta f) \le d^{-1}\},$$

as in several recent papers. We conjecture that there exists a constant $C > 0$ (independent on d and D_d) such that

$$e_n(\bar{C}^2(D_d), \mathrm{INT}) \ge C \cdot n^{-2/d}.$$

OP: Prove/disprove this. The lower bound is true (and sharp) for $D_d = [0,1]^d$ if we take a grid.

For $C^\infty([0,1]^d) = \{f : [0,1]^d \to \mathbf{R} \mid \|D^\alpha f\|_\infty \le 1 \text{ for all } \alpha \in \mathbf{N}_0^d\}$ we do not know much about the complexity. The best lower bound (unpublished) is linear, i.e., $n(\varepsilon, d) \ge c \cdot d$ for small ε while all known upper bounds are exponential in d.

For the class $\widetilde{C}^\infty([0,1]^d)$ we know that the problem is weakly tractable, where: A problem is weakly tractable iff

$$\lim_{\varepsilon^{-1}+d\to\infty} \frac{\ln n(\varepsilon,d)}{\varepsilon^{-1}+d} \;=\; 0\,.$$

OP: Improve these bounds.

Now we consider the approximation problem and, in particular, the question: When is it more difficult than integration?

For $C^\infty([0,1]^d) = \{f : [0,1]^d \to \mathbf{R} \mid \|D^\alpha f\|_\infty \le 1 \text{ for all } \alpha \in \mathbf{N}_0^d\}$ and L_∞-approximation, the order of convergence is excellent. Nevertheless:

Theorem: For L_∞-approximation we have

$$e_n(C^\infty([0,1]^d), \mathrm{APP}) = 1 \quad \text{for all} \quad n < 2^{\lfloor d/2 \rfloor}$$

or

$$n(\varepsilon,d) \ge 2^{\lfloor d/2 \rfloor} \quad \text{for all } \varepsilon < 1.$$

OP: It is not known whether the curse holds for L_∞-approximation and the smaller class $\widetilde{C}^\infty([0,1]^d)$. Guiqiao Xu (2015) proved that for L_q-approximation and $q < \infty$ the problem is weakly tractable for this class. For the proof he used the Smolyak algorithm of Barthelmann, N. and Ritter (2000).

Theorem: [Krieg 2019] For all k there exist c_k and $\tilde{c}_k$ and ε_k such that for all $n, d \in \mathbf{N}$ we have

$$\min(\varepsilon_k,\, c_k d^{k/2} n^{-k/d}) \le e_n(C^k([0,1]^d), \mathrm{APP}) \le \min(1,\, \tilde{c}_k d^{k/2} n^{-k/d})$$

if k is even and

$$\min(\varepsilon_k,\, c_k d^{k/2} n^{-k/d}) \le e_n(C^k([0,1]^d), \mathrm{APP}) \le \min(1,\, \tilde{c}_k d^{(k+1)/2} n^{-k/d})$$

if $k \ge 3$ is odd.

OP: Close the gap for odd $k \ge 3$.

OP: Close the gap for "large" errors or "small" n.

It follows that approximation is essentially harder than integration iff $k \ge 3$.

The lower bounds hold for general domains, the upper bounds use the geometry of the cube.

Theorem: [Krieg 2019] For all k there exist c_k and $\tilde{c}_k$ and ε_k such that for all $n, d \in \mathbf{N}$ we have

$$\min(\varepsilon_k,\, c_k d^{k/2} n^{-k/d}) \le e_n(\tilde{C}^k([0,1]^d), \mathrm{APP}) \le \min(1,\, \tilde{c}_k d^{k/2} n^{-k/d}).$$

OP: Close the gap for "small" n or "large" errors.

Approximation is essentially harder than integration if $k \ge 3$; the case $k = 2$ is still open.

The last result, where we consider APP for the classes $\tilde{C}^k([0,1]^d)$, can be generalized (the lower and the upper bound) to rather general domains. What we need for the upper bounds is that D_d is "locally star-shaped" (unpublished).

Random information and high dimensional geometry

David Krieg
(joint work with Aicke Hinrichs, Erich Novak, Joscha Prochno, Mario Ullrich)

We consider problems that are given by a solution operator $S : F \to G$ between normed spaces F and G. We want to compute $S(f)$ for some $f \in F$, but we only have incomplete information about f. The information is given by n measurements $L_1, \ldots, L_n : F \to \mathbb{R}$. The radius of information

$$e(L_1, \ldots, L_n) = \inf_{\Phi\colon \mathbb{R}^n \to G} \sup_{\|f\|_F \le 1} \left\| S(f) - \Phi(L_1(f), \ldots, L_n(f)) \right\|_G$$

is the minimal worst case error that can be achieved with these measurements. It measures the quality of the given information. In particular, the nth minimal worst case error

$$e(n) = \inf_{L_1, \ldots, L_n \in \Lambda} e(L_1, \ldots, L_n)$$

describes the quality of optimal information within a class Λ of allowed measurements. We consider approximation and integration problems where either $S(f) = f$ or $S(f)$ is the integral of a function f. Our information is given by function values or other linear measurements. We ask the following questions:

- How good is optimal information? What is the rate of convergence of the nth minimal error $e(n)$ as n tends to infinity?
- What does optimal information look like?
- How good is random information? How much worse are random measurements in comparison to optimal measurements?

We start with the problem of L_q-Approximation of functions from the Sobolev space $W_p^k([0,1]^d)$ with smoothness $k > d/2$ and integrability $p \in [1, \infty]$, i.e.,

$$S : W_p^k([0,1]^d) \to L_q([0,1]^d), \quad S(f) = f.$$

The information is given by function values, that is, $L_i(f) = f(x_i)$ for some $x_i \in [0,1]^d$. It is well known that optimal information satisfies

$$e(n) \asymp n^{-k/d + (1/p - 1/q)_+}.$$

The information is optimal under the condition that the volume of the largest empty ball amidst the point set $\{x_1, \ldots, x_n\}$ is of order n^{-1}. This leads to the observation that $n \log n$ random points (taken independently and uniformly distributed on the domain) are at least as good as n optimal points, i.e.,

$$\mathbb{E}\, e(L_1, \ldots, L_{n \log n}) \preceq e(n).$$

We refer to [2]. In general, the upper bound cannot be improved. However, we are optimistic that the logarithmic oversampling is not necessary for integration and approximation in the case that $q < p$, i.e.,

$$\mathbb{E}\, e(L_1, \ldots, L_n) \asymp e(n).$$

This would mean that random information is optimal. Moreover, we believe that analogous results can be shown for much more general domains than the cube.

For L_q-approximation of functions from Sobolev spaces $W_p^{k,\mathrm{mix}}([0,1]^d)$ with mixed smoothness $k > 1/2$ the situation seems to be more difficult. The rate of convergence of the minimal error $e(n)$ is not known, even for $p = q = 2$. It was believed by many that Smolyak's algorithm is optimal. This algorithm uses function values on a sparse grid. In [3], we disprove this conjecture and show that i.i.d. random points are better if the dimension d is larger than $2k + 1$.

We also consider the problem of recovering vectors in a convex and symmetric body $F_0 \subset \mathbb{R}^m$ from $n \ll m$ coordinates, where the coordinates are computed in the directions of $y_1, \ldots, y_n \in \mathbb{S}_{m-1}$ and the approximation error is measured in the Euclidean norm. Note that F_0 is the unit ball of a norm in $\mathbb{R}^m$, i.e., we consider the solution operator

$$S : \mathbb{R}^m \to \mathbb{R}^m, \quad S(x) = x,$$

where the input space is equipped with the F_0-norm and the target space is equipped with the Euclidean norm. We compare optimal directions with i.i.d. random directions that are uniformly distributed on the sphere. In the case of random directions, the radius of information equals the radius of the intersection of F_0 with a random subspace of codimension n. By famous results of Kashin, Garnaev and Gluskin from 1977 and 1984, random directions are optimal if F_0 is a cross-polytope. In [1] we consider the case that F_0 is an ellipsoid. Roughly speaking, it turns out that random directions are optimal if and only if the lengths of the semi-axes decay fast enough. If the ellipsoid is too thick, random information is almost useless. For other (weighted) ℓ_p-balls and other error norms, the question for the quality of random information is an open problem.

References

[1] A. Hinrichs, D. Krieg, E. Novak, J. Prochno, and M. Ullrich, *Random sections of ellipsoids and the power of random information*, arXiv: 1901.06639.

[2] A. Hinrichs, D. Krieg, E. Novak, J. Prochno, and M. Ullrich, *On the power of random information.* In F.J. Hickernell, P. Kritzer (eds.): Multivariate Algorithms and Information-Based Complexity. Berlin/Boston: De Gruyter, accepted for publication, 2019. arXiv: 1903.00681.

[3] D. Krieg, M. Ullrich, *Function values are enough for L_2-approximation*, arXiv: 1905.02516.

Singularity of sparse Bernoulli matrices

Alexander Litvak

(joint work with Konstantin Tikhomirov)

Invertibility of discrete random matrices attracts considerable attention in the literature. The problem of estimating the singularity probability of a square random matrix B_n with i.i.d. ± 1 entries was first addressed by Komlós in the 1960-es. An old folklore conjecture states that

$$\mathbb{P}\{B_n \text{ is singular}\} = (1/2 + o_n(1))^n = (1/2)^{(1+o_n(1))n}, \tag{1}$$

which roughly says that the main reason for singularity is that two rows or columns are the same (up to a sign). Komlós showed that $\mathbb{P}\{B_n \text{ is singular}\}$ decays to zero as the dimension grows to infinity. Then Kahn–Komlós–Szemerédi confirmed that the singularity probability of B_n is exponentially small in dimension, namely 0.999^n. Further improvements on the singularity probability were obtained by Tao and Vu to $(3/4)^n$ and by Bourgain, Vu, and P.M. Wood to $2^{-n/2}$. A recent breakthrough result of K. Tikhomirov [9] provides the affirmative answer to this conjecture (due to the length constraints we provide references only to recent papers, for the more detailed history and all other related references see [2, 8, 9]).

A more precise calculation of the probability that two rows or two columns of the matrix B_n are equal up to a sign leads to a slightly stronger conjecture that

$$\mathbb{P}\{B_n \text{ is singular}\} = 2(1+o_n(1))n^2\, 2^{-n}.$$

This conjecture can be naturally extended to the model with 0/1 Bernoulli matrices. Given $n \geq 1$ and $p = p_n \in (0,1)$, we say that an $n \times n$ matrix M_n is a p-Bernoulli matrix, if its entries are i.i.d. 0/1 Bernoulli r.v. taking value 1 with probability p_n. A quantity which goes to 0 as n grows is denoted by $o_n(1)$.

Conjecture. *For each $n \geq 1$, let $p_n \in (0, 1/2]$ and M_n be a p_n-Bernoulli matrix. Let $\mathcal{E}_{zero}$ be the event that a row or a column of M_n equals zero and $\mathcal{E}_{equal}$ be the event that two rows or two columns are equal to each other (up to a sign). Then*

$$\mathbb{P}\{M_n \textit{ is singular}\} = (1+o_n(1))\mathbb{P}\left(\mathcal{E}_{zero} \cup \mathcal{E}_{equal}\right).$$

In particular, if $\limsup_{n\to\infty} p_n < 1/2$ *then*

$$\mathbb{P}\{M_n \textit{ is singular}\} = (1+o_n(1))\mathbb{P}\left(\mathcal{E}_{zero}\right) = 2n(1+o_n(1))(1-p)^n.$$

The case of p independent of n was addressed in [9] as well. For $p = 1/2$, K. Tikhomirov proved the same bound as in (1), while for every fixed $p \in (0, 1/2)$ (independent of n) and for large enough n (that is, $n \geq C_p$, where C_p is a constant depending only on p), he proved that

$$\mathbb{P}\{M_n \text{ is singular}\} \leq (1-p+o_n(1))^n.$$

This almost solves the Conjecture in the case of constant p. Such a case we call by the *dense* regime of sparse matrices. Moreover, K. Tikhomirov obtained a sharp bound on the smallest singular value s_n of M_n, showing that for every $\varepsilon > 0$ and every large enough n (that is, $n \geq C_{p,\varepsilon}$),

$$\mathbb{P}\left\{s_n(M_n) \leq t\,\sqrt{p/n}\right\} \leq C(p,\varepsilon)t + (1-p+\varepsilon)^n.$$

In the *sparse* regime, that is, when $p = p_n \to 0$ as $n \to \infty$, the best known bounds were obtained by Basak and Rudelson who first proved [1] that

$$\mathbb{P}\{M_n \text{ is singular}\} \leq \exp(-cnp) \quad \text{for} \quad p = p(n) \geq (C \ln n)/n,$$

which gives the exponential decay in pn as we want, but the constant $c > 0$ in the exponent is much less than 1 suggested by the Conjecture. They also proved almost sharp bound on the smallest singular value, namely

$$\mathbb{P}\left\{s_n(B_p) \le c\, t\, C_{p,n}\, \sqrt{p/n}\right\} \le t + \exp(-cnp),$$

where $C_{p,n} = \exp(-C \ln(1/p)/\ln(np))$. We expect that the right bound for $s_n(M_n)$ is $c\sqrt{p/n}$, that is, $C_{p,n}$ in the above formula is a constant (note that this holds if p is at least polynomial in n). We would also like to note that the restriction $p = p_n \ge (C \ln n)/n$ is natural, as in the case $p \le (\ln n)/n$ the matrix M_n has a zero row with probability at least half.

In the subsequent work [2], Basak and Rudelson solved the Conjecture when $p = p_n$ is around the threshold value $(\ln n)/n$. More precisely, they proved that the Conjecture holds whenever $np_n \le \ln n + o_n(\ln \ln n)$.

The purpose of this talk is to present the solution for $C(\ln n)/n \le p_n \le 1/C$.

Theorem 1 (Litvak, K. Tikhomirov, [8])**.** *There is an absolute constant $C > 1$ with the following property. Let $n \ge 1$ and $C(\ln n)/n \le p = p_n \le 1/C$. Let M_n be a p-Bernoulli matrix. Then*

$$\mathbb{P}\{M_n \textit{ is singular}\} = (1 + o_n(1))\mathbb{P}\,(\mathcal{E}_{zero}) = 2n(1 + o_n(1))(1 - p)^n.$$

Moreover, for every $t > 0$,

$$\mathbb{P}\big\{s_n(M_n) \le t\, \exp(-2\ln^2(2n))\big\} \le t + 2(1 + o_n(1))n\,(1 - p)^n.$$

In the case of constant p (independent of n) we have a better estimate.

Theorem 2. *There exists an absolute constant $c > 0$ with the following property. Let $q \in (0, c)$ be a parameter (independent of n). Then there exists C_q and $n_q \ge 1$ (both depend only on q), such that for every $n \ge n_q$ and every $p \in (q, c_1)$ a p-Bernoulli random matrix M_n satisfies for every $t > 0$,*

$$\mathbb{P}\big\{s_n(M_n) \le C_q \sqrt{p}\, n^{-2.5}\, t\big\} \le t + 2(1 + o_n(1))n\,(1 - p)^n.$$

We would like to mention that it is natural to compare the model of Bernoulli 0/1 matrices with so-called d-regular matrices. Let $1 \le d \le n$. We say that an $n \times n$ matrix R_n is d-regular if in every row and in every column it has exactly d ones and $n - d$ zeros. We endow the set of such matrices with the uniform probability. Note that a p-Bernoulli random matrix is the adjacency matrix of Erdős–Rényi directed graph, while R_n is the adjacency matrix of a random d-regular directed graph. Clearly, a p-Bernoulli random matrix M_n has in average pn ones in every row and every column, therefore, intuitively, one expects that two models behave similarly when $d = pn \ge C(\ln n)/n$. The restriction on the lower bound on $d = pn$ comes from the fact that M_n has a zero row with probability at least half when $pn \le \ln n$, while R_n has no zero rows. A conjecture of Van Vu (repeated in a paper by Costello–Vu and in ICM talks by Vu and by Frieze), originally stated for undirected graphs, claims that $\mathbb{P}\{R_n \text{ is singular}\} \to 0$ as $n \to \infty$ for every $3 \le d = d_n \le n - 3$. Vu's conjecture was confirmed by N.A. Cook [3] for $d \ge \ln^2 n$ and by Litvak, Lytova, Tikhomirov, Tomczak-Jaegermann, and Youssef in [5, 6] for any $d = d_n \to \infty$ as $n \to \infty$. Moreover, in subsequent works the following bounds on the smallest singular number were obtained.

Theorem 3 (N.A. Cook, [4]). *There is an absolute constant $C>0$ such that for $C\ln^{11} n \le d \le n/2$ one has*

$$\mathbb{P}\big\{s_n(R_n) \le \exp(-C(\ln n)^2/\ln d)\big\} \le C(\ln n)^{5.5}/\sqrt{d}.$$

Theorem 4 (Litvak, Lytova, K. Tikhomirov, Tomczak-Jaegermann, Youssef, [7]). *There is an absolute constant $C>0$ such that for $C\le d\le n/\ln^2 n$ one has*

$$\mathbb{P}\big\{s_n(R_n) \le n^{-6}\big\} \le C(\ln n)^2/\sqrt{d}.$$

Recently Vu's conjecture (in both, directed and undirected cases) was confirmed in works by J. Huang, by A. Mészaros, and by H.H. Nguyen and M.M.Wood (however without bounds on the smallest singular number).

References

[1] A. Basak, M. Rudelson, *Invertibility of sparse non-Hermitian matrices*, Adv. Math. **310** (2017), 426–483.

[2] A. Basak, M. Rudelson, *Sharp transition of the invertibility of the adjacency matrices of random graphs,* arXiv:1809.08454

[3] N.A. Cook, *On the singularity of adjacency matrices for random regular digraphs,* PTRF **167** (2017), 143–200.

[4] N.A. Cook, *The circular law for random regular digraphs*, Ann. Inst. H. Poincaré, Probab. Statist., **55** (2019), 2111–2167.

[5] A.E. Litvak, A. Lytova, K. Tikhomirov, N. Tomczak-Jaegermann, P. Youssef, *Anti-concentration property for random digraphs and invertibility of their adjacency matrices*, C. R. Math. Acad. Sci. Paris **354** (2016), 121–124.

[6] A.E. Litvak, A. Lytova, K. Tikhomirov, N. Tomczak-Jaegermann, P. Youssef, *Adjacency matrices of random digraphs: singularity and anti-concentration,* J. Math. Anal. Appl. **445** (2017), 1447–1491.

[7] A.E. Litvak, A. Lytova, K. Tikhomirov, N. Tomczak-Jaegermann, P. Youssef, *The smallest singular value of a shifted d-regular random square matrix*, PTRF **173** (2019), 1301–1347.

[8] A.E. Litvak, K. Tikhomirov, *Singularity of sparse Bernoulli matrices*, arXiv:2004.03131.

[9] K. Tikhomirov, *Singularity of random Bernoulli matrices,* Annals of Math., to appear.

A central limit theorem and large deviation principles for ℓ_p^n- balls

David Alonso-Gutiérrez

The classical central limit theorem states that given a sequence $(X_n)_{n=1}^\infty$ of independent copies of a centered random variables X with $\mathbb{E}X^2=\sigma^2$, and denoting by $S_n := \sum_{i=1}^n X_i$, one has that the sequence of random variables $\left(\frac{S_n}{\sigma\sqrt{n}}\right)_{n=1}^\infty$ converges in distribution to a standard Gaussian random variable. Therefore, $\mathbb{P}\{S_n > t\sigma\sqrt{n}\}$, for some $t\in\mathbb{R}$, shows a universal behavior, which does not depend on the distribution of the random variables X.

On a different scale, Cramér's theorem state that if 0 is in the interior of the effective domain of the cumulant generating function

$$\Lambda(u) := \log \mathbb{E}e^{uX},$$

then the sequence $\left(\frac{S_n}{n}\right)_{n=1}^{\infty}$ satisfies a large deviation principle with speed n and rate function Λ^*, providing some information about $\mathbb{P}\{S_n > t\sigma n\}$, for some $t \in \mathbb{R}$, which depends on the distribution of X.
In the context of convex geometry, the central limit theorem shows that when X_n is a random vector uniformly distributed on the n-dimensional cube $K_n = [-\sqrt{3}, \sqrt{3}]^n$, then the one dimensional marginal $\langle X,\theta\rangle$, with $\theta = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)$ is almost Gaussian when the dimension n is large.
The question of whether the same is true when considering a different sequence of n-dimensional convex bodies K_n and different directions has been widely studied and Klartag proved that, whenever K_n is a centered convex body such that a random vector uniformly distributed on K_n has covariance matrix I_n (i.e., X_n is isotropic), and $1 \le k \le n^{\kappa}$ for some absolute constant κ, then the subset of k-dimensional subspaces $E \in G_{n,k}$ for which

$$d_{TV}(P_E(X_n), G_E) \le \frac{1}{n^{\kappa}}$$

has measure greater than $1 - e^{-c\sqrt{n}}$, where c is an absolute constant.
However, the theory of large deviation principles (LDP) had left no traces in the context of convex geometry until the work of Gantert, Kim, and Ramanan, who proved LDP's for the sequence of random variables $\left(n^{\frac{1}{p}-\frac{1}{2}}\langle X_n, \theta_n\rangle\right)_{n=1}^{\infty}$, where X_n is a random vector uniformly distributed on B_p^n, the unit ball of ℓ_p^n, and $(\theta_n)_{n=1}^{\infty}$ is a sequence of vectors in S^{n-1} which can be fixed (quenched) or random (annealed).
In this talk we will extend the annealed case to higher-dimensional projections. More precisely, we take X_n, random vectors uniformly distributed on B_p^n, E_n a random k_n-dimensional subspace, with $\frac{k_n}{n} \to \lambda \in [0,1]$ and we will study the behavior of the Euclidean norm of $P_{E_n}X_n$, showing that

$$\mathcal{X}_{n,p} := n^{1/p}\sqrt{\frac{\Gamma\left(\frac{1}{p}\right)}{p^{2/p}\Gamma\left(\frac{3}{p}\right)}}\,\|P_{E_n}X_n\|_2 - \sqrt{k_n}$$

converges in distribution to a centered Gaussian random variable with variance

$$\sigma^2 = \frac{\lambda}{4}\frac{\Gamma(\frac{1}{p})\Gamma(\frac{5}{p})}{\Gamma(\frac{3}{p})^2} - \lambda\Big(\frac{3}{4} + \frac{1}{p}\Big) + \frac{1}{2},$$

and that

$$Y_n := n^{\frac{1}{p}-\frac{1}{2}}\|P_{E_n}X_n\|_2$$

satisfies an LDP with speed n if $p \ge 2$ and with speed $n^{\frac{p}{2}}$ if $1 \le p < 2$ and $\lambda > 0$.

Polytopal approximation in non-Euclidean geometries

Florian Besau

1. Random Polytope Models

The classical random polytope model can be described as follows: Let $K \subset \mathbb{R}^d$, $d \geq 2$, be a compact convex subset with non-empty interior. Choose $n \in \mathbb{N}$, $n \geq d+1$, random points $X_1, \ldots, X_n$ from K uniformly and independently. Then the random polytope K_n is defined as the convex hull of $X_1, \ldots, X_n$, i.e.,

$$K_n := [X_1, \ldots, X_n].$$

Here we will focus on the volume of K_n as $n \to \infty$ and refer to the surveys [1, 15] for a wider context.

From an even more general point of view, we may consider a Borel probability measure μ on $\mathbb{R}^d$ with convex support and let $X_1, \ldots, X_n$ be independent random points distributed on $\mathbb{R}^d$ with respect to μ. Then, the random polytope $[\mu]_n$ is defined by

$$[\mu]_n := [X_1, \ldots, X_n].$$

We will further assume that μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ (full dimensional measures) and such that that the density function $\varphi : \operatorname{supp} \mu \to [0, +\infty)$ is continuous. Following the ideas of C. Borell [14] (see also [12]), for $\kappa \in [-\infty, \infty]$ we call such a measure μ κ-concave, if for all Borel $A, B \subset \mathbb{R}^d$ and $t \in (0,1)$ we have that

$$\mu(tA + (1-t)B) \geq \left(t\mu(A)^\kappa + (1-t)\mu(B)^\kappa\right)^{1/\kappa}.$$

In the following we briefly attempt to classify the different random polytope models that can be found in the literature by κ-concavity:

a) **Random polytopes inside a convex body ($\kappa = 1/d$):** In this case $K = \operatorname{supp} \mu$ is a convex body. In particular, if φ is constant on K, then we are in the classical model of A. Rényi & R. Sulanke, see e.g. the survey of Bárány [1]. If φ is continuous and uniformly bounded, i.e., there is $c > 1$ such that $1/c \leq \varphi(x) \leq c$ for all $x \in K$, then $K_n^\varphi := [\mu]_n$ has been studied as the weighted random polytope inside K, see e.g. [8, 13]. Note that in this case μ is, up to a constant, κ-concave for $\kappa \leq 1/d$, since

$$\mu(tA + (1-t)B) \geq c^{-2} \left(t\mu(A)^\kappa + (1-t)\mu(B)^\kappa\right)^{1/\kappa}.$$

b) **Gaussian polytopes ($\kappa = 0$):** In this case $\mu = \gamma_d$ is the standard Gaussian measure with density $\varphi(x) = (2\pi)^{d/2} \exp(-\|x\|_2^2/2)$, where $\|x\|_2 = \sqrt{x \cdot x}$ is the Euclidean norm of $x \in \mathbb{R}^d$, and we set $G_n := [\gamma_d]_n$. Note that γ_d is log-concave and $\operatorname{supp} \gamma_d = \mathbb{R}^d$, i.e., $\operatorname{supp} \mu$ is convex but not bounded. This model has been studied fairly extensive, see e.g. [6, 7, 16, 17].

c) **Beta polytopes ($0 < \kappa \leq 1/d$):** For $\beta > -1$ we consider the measure μ_β on the unit ball $B_2^d = \{x \in \mathbb{R}^d : \|x\|_2^2 \leq 1\}$ with density

$$\varphi_\beta(x) = \frac{\Gamma(\frac{d}{2}+1+\beta)}{\pi^{d/2}\Gamma(\beta+1)}\,(1-\|x\|_2^2)^\beta, \qquad x \in \operatorname{int} B_2^d.$$

Then, $P_n^\beta = [\mu_\beta]_n$ is a random polytope in B_2^d. Note that μ_β is κ-concave with $\kappa \leq \frac{1}{\beta+d}$ for $\beta \geq 0$ and therefore also log-concave.

Beta distributions were first considered by Miles and random beta and beta-prime polytopes have very recently become a focus of interest [18, 19, 20].

For $\beta \in (-1,0)$, μ_β is log-convex. For $K \subset \operatorname{int} B_2^d$, we may also consider $\mu_{\beta,K}$ for $\beta \leq -1$ as a probability measure restricted to K, i.e, $\mu_{\beta,K}$ has density $\varphi_{\beta,K}(x) = c_{\beta,K}(1-\|x\|_2^2)^\beta$, where $c_{\beta,K} > 0$ is the normalizing constant such that $\int_K \varphi_{\beta,K} = 1$. Then $\mu_{\beta,K}$ is also log-convex on K for $\beta \leq -1$.

d) **Beta-prime polytopes ($\kappa < 0$):** For $\beta' > \frac{d}{2}$ the beta-prime measure $\mu_{\beta'}$ has density

$$\varphi_{\beta'}(x) = \frac{\Gamma(\beta')}{\pi^{d/2}\Gamma(\beta'-\frac{d}{2})}\,(1+\|x\|_2^2)^{-\beta'}, \qquad x \in \mathbb{R}^d.$$

The beta-prime random polytope is defined by $P_n^{\beta'} = [\mu_{\beta'}]_n$. Note that $\mu_{\beta'}$ is κ-concave where $\kappa \leq -\frac{1}{2\beta'-d} < 0$ for $\beta > \frac{d}{2}$.

e) **Random polytopes in spherical d-space ($\kappa = -1$):** Let $\mathbb{S}^d = \{u \in \mathbb{R}^{d+1} : \|u\|_2 = 1\}$ be the Euclidean unit sphere in $\mathbb{R}^{d+1}$. Let further $\mathbb{S}_+^d = \{(u_1,\dots,u_{d+1}) \in \mathbb{S}^d : u_{d+1} > 0\}$ be the open half-sphere. We consider the random spherical polytope $S_n := [\sigma_d]_n$, where σ_d is the uniform probability measure on the half-sphere $\mathbb{S}_+^d$. Using the gnomonic projection $g : \mathbb{S}_+^d \to \mathbb{R}^d$, defined by $g(u) = (\frac{u_1}{u_{d+1}},\dots,\frac{u_d}{u_{d+1}})$, we find that $g(S_n)$ is equal in distribution to $P_n^{\beta'}$ for $\beta' = \frac{d+1}{2}$, see e.g. [11]. The random polytopes S_n have been investigated more recently in [3], see also [18, 19].

f) **Random polytopes in hyperbolic d-space (log-convex measures):** Let $\mathbb{H}^d = \{u \in \mathbb{R}^{d+1} : u_1^2 + \ldots + u_d^2 - u_{d+1}^2 = -1, u_{d+1} > 0\}$ be the hyperbolid model of hyperbolic d-space in $\mathbb{R}^{d+1}$. We may fix a bounded hyperbolic convex domain K in $\mathbb{H}^d$ and consider the random polytope $H_n := [\nu_d]_n$, where ν_d is the uniform probability measure on $K \subset \mathbb{H}^d$. Then H_n is a hyperbolic convex random polytope and using gnomonic projection $g : \mathbb{H}^d \to \mathbb{R}^d$, defined by $g(u) = (\frac{u_1}{u_{d+1}},\dots,\frac{u_d}{u_{d+1}})$, we find that $g(H_n)$ is equal in distribution to $P_n^\beta = [\mu_{\beta,g(K)}]_n$ for $\beta = -\frac{d+1}{2}$, see e.g. [11]. Not much seems to be known in the hyperbolic setting, but see e.g. [8] for asymptotic results on the expected (hyperbolic) volume of H_n.

2. Floating Body of a Measure and Random Polytopes

Let μ be a Borel probability measure on $\mathbb{R}^d$ with convex support that is absolutely continuous with respect to the Lebesgue measure. For $\delta \in (0, \frac{1}{2})$ the convex floating body $[\mu]_\delta$, see [22, 25], is defined by

$$[\mu]_\delta = \bigcap \{H^-(u,t) : u \in \mathbb{S}^{d-1} \text{ and } t \in \mathbb{R} \text{ such that } \mu(H^+(u,t)) \leq \delta\},$$

where $H(u,t) = \{x \in \mathbb{R}^d : x \cdot u = t\}$ is the affine hyperplane with normal direction $u \in \mathbb{S}^{d-1}$ and distance t from the origin, $H^+(u,t) = \{x \in \mathbb{R}^d : x \cdot u \geq t\}$ is the closed half-space bounded by $H(u,t)$ and $H^-(u,t) = H^+(-u,-t)$.

Let us also give an equivalent definition:

we define the half-space density function $\eta_\mu : \operatorname{supp} \mu \to [0, +\infty)$ by

$$\eta_\mu(x) = \min_{u \in \mathbb{S}^{d-1}} \mu(H^+(u, u \cdot x)),$$

i.e., $\eta_\mu(x)$ is the minimal measure of a half-space that contains x. Then the convex floating body is the superlevel set of η_μ, that is,

$$[\mu]_\delta = \{x \in \mathbb{R}^d : \eta_\mu(x) \geq \delta\}.$$

Following the ideas of Bárány & Larman [4], for the random polytope $[\mu]_n = [X_1, \ldots, X_n]$ we have that

$$\mathbb{E}\mu(\mathbb{R}^d \setminus [\mu]_n) \geq \frac{1}{4}\mu(\mathbb{R}^d \setminus [\mu]_{\delta(n)}) \qquad \text{for all } n \geq d+1,$$

where $\delta(n) = \frac{1}{n}$.

V. Vu [26] showed that for $\gamma > 1$ there are constants $c_1, c_2 > 0$ such that the random polytope $[\mu]_n$ contains $[\mu]_{c_1 \frac{\ln n}{n}}$ with probability $n^{-\gamma}$ for all $n \geq c_2$. We may therefore derive the upper bound

$$\mathbb{E}\mu(\mathbb{R}^d \setminus [\mu]_n) \leq \mu(\mathbb{R}^d \setminus [\mu]_{c_1 \frac{\ln n}{n}}) + \mathbb{P}([\mu]_{c_1 \frac{\ln n}{n}} \not\subset [\mu]_n) \leq \mu(\mathbb{R}^d \setminus [\mu]_{c_1 \frac{\ln n}{n}}) + n^{-\gamma}.$$

Hence, it seems natural to ask the following question.

Question: *Let μ be a Borel probability measure on $\mathbb{R}^d$ that is absolutely continuous with respect to the Lebesgue measure and such that* $\operatorname{supp} \mu$ *is convex. Do there exist constants $c_1, c_2 > 0$ such that, if μ is κ-concave for $\kappa < 1/d$, then*

$$\mathbb{E}\mu(\mathbb{R}^d \setminus [\mu]_n) \leq c_1 \mu(\mathbb{R}^d \setminus [\mu]_{\delta(n)}) \qquad \textit{for all } n \geq c_2,$$

where $\delta(n) = \frac{1}{n}$?

This is known to be true for the uniform case, i.e., if μ is κ-concave for $\kappa = 1/d$ by Bárány & Larman [4].

For the uniformly bounded case we also have more general and precise information about the limit, that is, if $\operatorname{supp} \mu = K$ is bounded and convex and $\varphi : K \to (0, +\infty)$ is continuous on K and uniformly bounded, then for $K_\delta^\varphi = [\mu]_\delta$ we have that

$$\mu(\mathbb{R}^d \setminus K_\delta^\varphi) = c(d) \left(\int_{\operatorname{bd} K} \varphi(x)^{\frac{d-1}{d+1}} H_{d-1}(K,x)^{\frac{1}{d+1}} \, \mathcal{H}^{d-1}(\mathrm{d}x) \right) \delta^{\frac{2}{d+1}} (1 + o_\delta(1)),$$

see [8, 25]. Here $H_{d-1}(K,\cdot)$ denotes the generalized Gauss–Kronecker curvature, $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure restricted to $\operatorname{bd} K$ and $c(d)$ will denote here and in the following some constant that only depends on the space dimension d. If $K = P$ is actually a convex polytope, then

$$\mu(\mathbb{R}^d \setminus P_\delta) = c(d)|\operatorname{flag} P|\,\delta(-\ln\delta)^{d-1}(1 + o_\delta(1)),$$

see [9, 24], where $|\operatorname{flag} P|$ is the total number of maximal chains (complete flags) in the face lattice of P.

3. Expectation, Variance and Central Limit Theorems

In the following we briefly collect results on the expected volume of $[\mu]_n$.

a) **Random polytopes inside a convex body:** We assume that $\operatorname{supp}\mu = K$ is a convex body with C^2_+ boundary and $\varphi : K \to (0,+\infty)$ is continuous and uniformly bounded. We know that

$$\mathbb{E}\mu(K \setminus K_n^\varphi) = c(d)\left(\int_{\operatorname{bd} K} \varphi(x)^{\frac{d-1}{d+1}} H_{d-1}(K,x)^{\frac{1}{d+1}}\, \mathcal{H}^{d-1}(\mathrm{d}x)\right) \times n^{-\frac{2}{d+1}}(1 + o_n(1)),$$

see [13] and [8]. If φ is constant, then asymptotic bounds on the variance of $\operatorname{Vol}(K_n)$ were obtaind by Reitzner [23] and a lower bound for the variance of $\mu(K_n^\varphi)$ was derived in [10], where also a central limit theorem for $\mu(K_n^\varphi)$ was derived, which extends the central limit theorem for $\operatorname{Vol}(K_n)$ obtained by Reitzner [23]

If $K = P$ is a polytope and φ is constant, $P_n = [\mu]_n$, then

$$\mathbb{E}\operatorname{Vol}(P \setminus P_n) = c(d)|\operatorname{flag} P|\, n^{-1}(\ln n)^{d-1}(1 + o_n(1)),$$

see [2]. Bounds on the variance of $\operatorname{Vol}(P_n)$ were established in [5].

b) **Gaussian polytopes:** As mentioned before, the case of Gaussian random polytopes $G_n = [\gamma_d]_n$ is fairly well investigated. We know that

$$\mathbb{E}\operatorname{Vol}(G_n) = c(d)(\ln n)^{\frac{d}{2}}(1 + o_n(1)),$$

and

$$\mathbb{E}\gamma_d(\mathbb{R}^d \setminus G_n) = c(d)n^{-1}(\ln n)^{\frac{d-1}{2}}(1 + o(1)).$$

Bounds on the variance of $\operatorname{Vol}(G_n)$ and a central limit theorem were obtained in [7, 17], see also [6].

c) **Beta and beta-prime polytopes:** For beta polytopes $P_n^\beta = [\mu_\beta]_n$ and beta-prime polytopes $P_n^{\beta'} = [\mu_{\beta'}]_n$ we have that

$$\mathbb{E}\mu_\beta(B_2^d \setminus P_n^\beta) = c(d)n^{-1+\frac{d-1}{d+1+2\beta}}(1 + o_n(1)),$$

and

$$\mathbb{E}\mu_{\beta'}(\mathbb{R}^d \setminus P_n^{\beta'}) = c(d)n^{-1}(1 + o_n(1)),$$

where we refer to [18, 20] for more precise results and also some non-asymptotic formulas.

References

[1] I. Bárány, *Random points and lattice points in convex bodies*, Bull. Amer. Math. Soc. **45** (2008), 339–365.

[2] I. Bárány, C. Buchta, *Random polytopes in a convex polytope, independence of shape, and concentration of vertices*, Math. Ann. **297** (1993), 467–497.

[3] I. Bárány, D. Hug, M. Reitzner, R. Schneider, *Random points in halfspheres*, Random Struct. Algorithms **50** (2017), 3–22.

[4] I. Bárány, D. G. Larman, *Convex bodies, economic cap coverings, random polytopes*, Mathematika **35** (1988), 274–291.

[5] I. Bárány, M. Reitzner, *On the variance of random polytopes*, Adv. Math. **225** (2010), 1986–2001.

[6] I. Bárány, C. Thäle, *Intrinsic volumes and Gaussian polytopes: the missing piece of the jigsaw*, Doc. Math. **22** (2017), 1323–1335.

[7] I. Bárány, V. Vu, *Central limit theorems for Gaussian polytopes*, Ann. Probab. **35** (2007), 1593–1621. Correction: Ann. Probab. **36** (2008).

[8] F. Besau, M. Ludwig, E. M. Werner, *Weighted floating bodies and polytopal approximation*, Trans. Amer. Math. Soc. **370** (2018), 7129–7148.

[9] F. Besau, C. Schütt, E. M. Werner, *Flag numbers and floating bodies*, Adv. Math. **338** (2018), 912–952.

[10] F. Besau, C. Thäle, *Asymptotic normality for random polytopes in non-Euclidean geometries*, preprint (2019), arXiv:1909.05607.

[11] F. Besau, E. M. Werner, *The floating body in real space forms*, J. Differential Geom. **110** (2018), 187–220.

[12] S. G. Bobkov, *Convex bodies and norms associated to convex measures*, Probab. Theory Related Fields **147** (2010), 303–332.

[13] K. J. Böröczky, F. Fodor, D. Hug, *The mean width of random polytopes circumscribed around a convex body*, J. Lond. Math. Soc. **81** (2010), 499–523.

[14] C. Borell, *Convex set functions in d-space*, Period. Math. Hungar. **6** (1975), 111–136.

[15] P. Calka, *Some classical problems in random geometry*, Lecture Notes in Math. **2237** (2019), 1–43.

[16] I. Hueter, *Limit theorems for the convex hull of random points in higher dimensions*, Trans. Amer. Math. Soc. **351** (1999), 4337–4363.

[17] D. Hug, M. Reitzner, *Gaussian polytopes: variances and limit theorems*, Adv. in Appl. Probab. **37** (2005), 297–320.

[18] Z. Kabluchko, *Angles of Random Simplices and Face Numbers of Random Polytopes*, preprint (2019), arXiv:1909.13335.

[19] Z. Kabluchko, A. Marynych, D. Temesvari, C. Thäle, *Cones generated by random points on half-spheres and convex hulls of Poisson point processes*, Probab. Theory Relat. Fields **175** (2019), 1021–1061.

[20] Z. Kabluchko, D. Temesvari, C. Thäle, *Expected intrinsic volumes and facet numbers of random beta-polytopes*, Math. Nachr. **292** (2019), 79–105.

[21] M. Meyer, S. Reisner, *Characterizations of affinely-rotation-invariant log-concave measures by section-centroid location*, Lecture Notes in Math. **1469** (1991), 145–152.

[22] S. Nagy, C. Schütt, E. M. Werner, *Halfspace depth and floating body*, Stat. Surv. **13** (2019), 52–118.

[23] M. Reitzner, *Central limit theorems for random polytopes*, Probab. Theory Related Fields **133** (2005), 483–507.

[24] C. Schütt, *The convex floating body and polyhedral approximation*, Israel J. Math. **73** (1991), 65–77.

[25] C. Schütt, E. M. Werner, *The convex floating body*, Math. Scand. **66** (1990), 275–290.

[26] V.H. Vu, *Sharp concentration of random polytopes*, Geom. Funct. Anal. **15** (2005), 1284–1318.

Floating bodies and geometry of random polytopes

Olivier Guédon

In this talk, I have presented recent results from [3] and I refer to this paper for more detailed explanation. Let X be a symmetric random vector in $\mathbb{R}^n$ and let $X_1, \dots, X_N$ be independent copies of X. Our goal is to study the geometry of the random polytope $\mathrm{absconv}(X_1, \dots, X_N)$, that is, the convex hull of the points $\pm X_1, \dots, \pm X_N$. We study the following questions:

(1) Is it possible to find a set K that is naturally associated with X and is contained in $\mathrm{absconv}(X_1, \dots, X_N)$ with high probability?
(2) If the answer to (1) is yes, when does K contain large (intersections of) ℓ_p balls as, classical results in the Gaussian or Rademacher cases ?

The geometric features of X that are significant in this context are reflected by the natural *floating bodies* associated with X. For $p \geq 1$, we define the associated floating body

$$K_p(X) := \{t \in \mathbb{R}^n : \mathbb{P}(\langle X, t\rangle \geq 1) \leq \exp(-p)\}.$$

The notion of floating bodies plays a crucial role in the study of approximation of convex bodies by polytopes, see, e.g., [8, 6, 1], where X is distributed according to the uniform probability measure on the given convex body. It is known how to identify The description of floating bodies associated to a Gaussian vector is easy to get while the case of a Rademacher random vector follows from a result of [7]. We give a complete answer to the above question under minimal assumptions on X that we now describe. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. The random vector X is said to satisfy a small-ball condition with respect to the norm $\|\cdot\|$ with constants γ and δ if for every $t \in \mathbb{R}^n$,

$$\mathbb{P}(|\langle X, t\rangle| \geq \gamma\|t\|) \geq \delta.$$

Also, for some $r > 0$, X is said to satisfy an L_r condition with respect to the norm $\|\cdot\|$ and with constant L if for every $t \in \mathbb{R}^n$,

$$(\mathbb{E}|\langle X, t\rangle|^r)^{1/r} \leq L\|t\|.$$

Our main result is the following

Theorem 1. *Let X be a symmetric random vector. Assume that X satisfies a small-ball condition with constants $\gamma > 0$ and $\delta > 0$, and an L_r condition with constant L for some $r > 0$ with respect to the same norm $\|\cdot\|$.*

Let $0 < \alpha < 1$ and set $p = \alpha \log(eN/n)$ and assume that $N \geq c_0 n$ for a constant $c_0 = c_0(\alpha, \delta, r, L/\gamma)$. Let $X_1, \dots, X_N$ be independent copies of X then with probability at least $1 - 2\exp(-c_1 N^{1-\alpha} n^\alpha)$,

$$\mathrm{absconv}(X_1, \dots, X_N) \supset \frac{1}{2}\big(K_p(X)\big)^\circ,$$

where c_1 is an absolute constant.

The second part of the question can be answered by identifying those floating bodies for a variety of choices of X—thus recovering, and at times improving, previously known results, as well as establishing new estimates in cases that were out of reach before. For example, if $X = G \sim \mathcal{N}(0, Id)$ then for $p = \alpha \log(eN/n)$

$$\big(K_p(G)\big)^\circ \supset c\sqrt{p}B_2^n$$

and if $X = \mathcal{E} = (\varepsilon_1, \ldots, \varepsilon_n)$ where ε_i are iid Rademacher entries then

$$\big(K_p(\mathcal{E})\big)^\circ \supset c(B_\infty^n \cap \sqrt{p}B_2^n)$$

recovering known estimates from [2] and [5]. Using the concept of stochastic domination, it is not difficult to show that if $\xi_1, \ldots, \xi_n$ are independent copies of a symmetric random variable ξ such that $\mathbb{P}(|\xi| \geq \gamma_0) \geq \delta_0$ and if $X = (x_i)_{i=1}^n$, then X stochastically dominates the Rademacher vector $\mathcal{E}$ with constants λ_1 and λ_2 that depend only on γ_0 and δ_0. A direct consequence is that

$$K_p(X) \subset \lambda_2 K_{p'}(\mathcal{E}),$$

where $p' = p - \log(1/\lambda_1)$. Thanks to the characterization of $K_p(\mathcal{E})$ and Theorem 1 one immediately recovers the main result from [4]. The result can be pushed much further and the fact that X has i.i.d. coordinates can be relaxed to an unconditional assumption. Thanks to the universality of Theorem 1, one may establish various new outcomes that were previously completely out of reach like also when X has iid q-stable random entries for $1 \leq q < 2$ (e.g., a Cauchy random vector).

The second outcome of Theorem 1 is related to a fundamental question in the area of *compressive sensing*: can sparse signals be recovered efficiently when the given data consist of a few measurements that are noisy, *but the 'noise level' is not known*. We refer to [3] for a more detailed exposition.

References

[1] I. Bárány, *Random polytopes, convex bodies, and approximation*, Stochastic geometry, Lecture Notes in Math. 1892, (2007), 77–118.

[2] E.D. Gluskin, *Extremal Properties of Orthogonal Parallelepipeds and Their Applications to the Geometry of Banach Spaces*, Sb. Math. **64**-1 (1989).

[3] O. Guédon, F. Krahmer, C. Kümmerle, S. Mendelson, O. Rauhut, *On the geometry of polytopes generated by heavy-tailed random vectors*, submitted, arXiv:1907.07258.

[4] O. Guédon, A.E. Litvak, K. Tatarko, *Random polytopes obtained by matrices with heavy tailed entries*, to appear in Commun. Contemp. Math. doi 10.1142/S0219199719500275

[5] A.E. Litvak, A. Pajor, M. Rudelson, N. Tomczak-Jaegermann, *Smallest singular value of random matrices and geometry of random polytopes*, Adv. Math. **195**-2 (2005), 491–523.

[6] M. Meyer, S. Reisner, *Characterizations of affinely-rotation-invariant log-concave measures by section-centroid location*, Geometric aspects of functional analysis (1989–90), Lecture Notes in Math. 1469, (1991), 145–152.

[7] S.J. Montgomery-Smith, *The distribution of Rademacher sums*, Proc. Amer. Math. Soc. **109**-2 (1990), 517–522.

[8] C. Schütt, E. Werner, *The convex floating body*, Math. Scand. **66**-2 (1990), 275–290.

Discrepancy and integration, anpOpen IBC problem

Henryk Woźniakowski

In the three volume book [2] we presented about 150 open IBC (Information-Based Complexity) problems on tractability of multivariate problems. In this talk we concentrate on one of them which is related to weighted L_2-discrepancy and multivariate integration.

It is well known that if multivariate integration is defined on the specific Sobolev space then the nth minimal weighted discrepancy for the d-variate case and multivariate integration errors coincide. For $\varepsilon \in (0,1)$, an arbitrary integer d, let $\gamma = \{\gamma_j\}_{j=1,2,\dots,d}$ denote the so called product weights in $(0,1]$ introduced in [4]. Let $n_\gamma(\varepsilon, d)$ be the minimal number of discrepancy points, or equivalently the minimal number of function values to approximate the nth minimal weighted discrepancy or multivariate integration to within ε times the initial error. For the unweighted case, $\gamma_j = 1$ for all integer j, we have the curse of dimensionality. More precisely, there exists a positive number c such that

$$c(1.022)^d \le n_\gamma(\varepsilon, d) \le (1.1144)^d\, \varepsilon^{-2}.$$

The lower bound is proved in [1], and the upper bound in [3].

To vanquish the curse of dimensionality we must use product weights tending to zero. It is known that

$$n_\gamma(\varepsilon, d) = \mathcal{O}(\varepsilon^{-2}) \quad \text{iff} \quad \sum_{j=1}^{\infty} \gamma_j < \infty,$$

with the factors in the big $\mathcal{O}$ notation independent of d.

These bounds were proved in [4] for quasi-Monte Carlo algorithms, where the discrepancy or integration coefficients weights are all $1/n$, and in full generality in [1] by the use of the concept of decomposable reproducing kernels.

There are many improvements on the bound of $n_\gamma(\varepsilon, d)$ on the expense of more strict conditions on product weights. Let $p = p_\gamma$ denote the infimum of the exponent q such that $n_\gamma(\varepsilon, d)$ is uniformly (in d) bounded by ε^{-1}. Clearly, $p = p_\gamma \ge 1$ since even for $d = 1$ $n_\gamma(\varepsilon, 1)$ is of order ε^{-1}. It is known, see [5], that

$$\sum_{j=1}^{\infty} \gamma_j^{1/2} < \infty \quad \Longrightarrow \quad p_\gamma = 1.$$

It is however not known if this bound is sharp, and this is an open problem which we want to introduce the the audience of the workshop.

There are many related open problems and we always want to find iff condition on weights to obtain specific behaviour of $n_\gamma(\varepsilon, d)$. Obviously the question should be studied not only for discrepancy or multivariate integration but for general multivariate problems.

References

[1] E. Novak and H. Woźniakowski, Intractability results for integration and discrepancy, *J. Complexity, 17, 388–441, 2001.*

[2] E. Novak and H. Woźniakowski, *Tractability of Multivariate Problems. Volume I, Linear Information, 2006, Volume II, Standard Information for Functionals, 2008, Volume III, Standard Information for Operators, 2010,* European Mathematical Society, Z`urich.

[3] L. Plaskota, G.W. Wasilkowski and Y Zhao, New averaging technique for approximating weighed integrals, *J. Complexity, 25, 268–291, 2009.*

[4] I. H. Sloan and H. Wozńiakowski, When are quasi-Monte Carlo algorithms efficient for high dimensional integrals? *J. Complexity, 14, 1–33, 1998.*

[5] G.W. Wasilkowski and H. Wozńiakowski, Weighted tensor-product algorithms for linear multivariate problems, *J. Complexity¡ 15, 402–447, 1999.*

Numerical integration and Schur's product theorem

Jan Vybíral

(joint work with Aicke Hinrichs, David Krieg and Erich Novak)

The aim of this contribution is to resolve the conjecture of E. Novak, which reads as follows.

Conjecture 1 (E. Novak [2]**).** *The matrix*

$$\Big\{\prod_{i=1}^{d}\frac{1+\cos(x_{j,i}-x_{k,i})}{2}-\frac{1}{n}\Big\}_{j,k=1}^{n}$$

is positive semidefinite for all $n, d \geq 2$ and all choices of $x_1, \dots, x_n \in \mathbb{R}^d$.

Erich Novak published this conjecture also in NA Digest in November 1997 and tested it numerically. It also appeared as Open Problem 3 in [3]. The conjecture was recently proven to be true in [4] with further applications to numerical integration given in [1].

We start with certain problem from numerical analysis, which was actually the starting point for Conjecture 1. We study the quadrature formulas

$$Q_n(f)=\sum_{i=1}^{n}c_i f(x_i),\qquad c_i\in\mathbb{R},\quad x_i\in[0,1]^d \tag{1}$$

and how well are they suited to approximate the integral $\mathrm{INT}_d(f)=\int_{[0,1]^d}f(x)dx$. Here f belongs to a unit ball of a Hilbert space F_d, which is defined as a d-fold tensor product of a space F_1, which in turn is a three dimensional Hilbert space with an orthonormal basis given by the functions

$$e_1(x)=1,\quad e_2(x)=\cos(2\pi x),\quad e_3(x)=\sin(2\pi x),\quad x\in[0,1].$$

Hence F_d is a 3^d-dimensional Hilbert space. The point evaluation $\delta_x : f \to f(x)$ may be written in the form

$$f(x)=\langle f,\delta_x\rangle_{F_d}\quad\text{with}\quad \delta_x(z)=\prod_{j=1}^{d}[1+\cos(2\pi(x_j-z_j))].$$

In this way, F_d becomes a reproducing kernel Hilbert space with the kernel

$$K_d(x,y) = \langle \delta_x, \delta_y \rangle_{F_d} = \prod_{j=1}^{d} [1 + \cos(2\pi(x_j - y_j))], \quad x, y \in [0,1]^d.$$

If all the c_j's are positive, a simple calculation shows that the worst-case error of Q_n given by (1) can be estimated as

$$e^{\mathrm{wor}}(Q_n)^2 \geq 1 - 2\sum_{j=1}^{n} c_j + \sum_{j=1}^{n} c_j^2 2^d$$

and for the optimal choice $c_j = 2^{-d}$ this becomes

$$e^{\mathrm{wor}}(Q_n)^2 \geq \max(1 - n2^{-d}, 0). \tag{2}$$

This estimate shows the intractability of numerical integration on F_d with quadrature formulas with positive weights since for a fixed error the number n of sample points needs to grow exponentially with the dimension d.

If the signs of c_j's are not fixed, we can still write

$$e^{\mathrm{wor}}(Q_n)^2 = 1 - \sup_{c_j, x_j} \frac{\left(\sum_{j=1}^{n} c_j\right)^2}{\sum_{j,k=1}^{n} c_j c_k K_d(x_j, x_k)}. \tag{3}$$

Erich Novak conjectured, that the estimate (2) applies also for quadrature formulas (1) with general weights, which is by (3) equivalent to Conjecture 1.

The main tool in the solution of Conjecture 1 is the following property of the Schur matrix product.

Theorem 1. *Let $M \in \mathbb{R}^{n\times n}$ be a positive semidefinite matrix with $M_{j,j} = 1$ for all $j = 1, \dots, n$. Then*

$$M \circ M \succeq \frac{1}{n} \cdot E_n,$$

where $E_n \in \mathbb{R}^{n\times n}$ is a matrix with all entries equal to one.

Finally, let us show how this Theorem implies the positive answer to Conjecture 1. We define matrices $M^1, \dots, M^d$ by

$$M^i_{j,k} = \cos\Big(\frac{x_{j,i} - x_{k,i}}{2}\Big), \quad i = 1, \dots, d, \quad j,k = 1, \dots, n.$$

By Bochner's theorem, the matrices M^i are all positive semidefinite and (by the classical Schur product theorem) so is their Hadamard product $M = M^1 \circ \dots \circ M^d$. Obviously, M has all its diagonal elements equal to one. Finally, the Theorem shows that the matrix $M \circ M - \frac{1}{n}E_n$ with entries

$$M_{j,k}^2 - \frac{1}{n} = \prod_{i=1}^{d} \cos^2\Big(\frac{x_{j,i} - x_{k,i}}{2}\Big) - \frac{1}{n} = \prod_{i=1}^{d} \frac{1 + \cos(x_{j,i} - x_{k,i})}{2} - \frac{1}{n}$$

is also positive semidefinite.

Hence, the integration problem on F_d is intractable even when we allow negative weights c_j's in the quadrature formula (1).

Remark: It was noted in a discussion in Oberwolfach by Dmitriy Bilyk (University of Minnesota), that the Theorem also follows from the theory of Gegenbauer polynomials. We refer to [1], where we give more details about this connection.

References

[1] A. Hinrich, D. Krieg, E. Novak, and J. Vybíral, *Lower bounds for the error of quadrature formulas for Hilbert spaces*, preprint under preparation

[2] E. Novak, *Intractability results for positive quadrature formulas and extremal problems for trigonometric polynomials*, J. Complexity 15 (1999), 299–316

[3] E. Novak and H. Woźniakowski, Tractability of Multivariate Problems, Volume I: Linear Information, EMS Tracts in Mathematics 6, European Math. Soc. Publ. House, Zürich, 2008.

[4] J. Vybíral, *A variant of Schur's product theorem and its applications*, preprint, 2019, arXiv:1909.11726

Affine processes in the cone of positive Hilbert-Schmidt operators: a volatility model

Sonja Cox

(joint work with Sven Karbach, Asma Khedher)

Affine processes, i.e., processes of which the characteristic function is exponentially affine in the initial value, have gained a lot of attention in the finance literature in the past two decades due to their good tractability and their ability to cover a reasonably wide range of relevant processes. In particular, affine processes are used to model volatility processes. A full characterisation of affine processes taking values in the cone of positive semidefinite matrices has been obtained in [4]. However, various models, e.g. in the HJM framework, require infinite-dimensional volatility processes. In this report we explain explain the challenges of identifying affine processes in the cone of positive Hilbert-Schmidt operators and discuss some well-posedness results.

Roughly speaking[1], an H-valued Markov process $(X_t)_{t\in[0,\infty)}$ (where H is a Hilbert space) is called *affine* if there exist functions $\phi\colon [0,\infty) \times H \to \mathbb{R}$ and $\psi\colon [0,\infty) \times H \to H$ such that for every $u, h \in H$ it holds that

$$\mathbb{E}\left[e^{i\langle X(t),h\rangle} \mid X(0) = u\right] = e^{\phi(t,u)+\langle\psi(t,u),h\rangle_H}.$$

[1] Some additional regularity assumptions are typically neccessary in order to provide a full characterisation of affine processes.

Theorem 2.4 in [4] gives a characterisation of affine processes with values in the space of positive semidefinite matrices S_d^+ in terms of the generator of the process, which is of the form

$$\begin{aligned}\mathcal{A}f(x) = \tfrac{1}{2}\sum_{i,j,k,l} A_{i,j,k,l}\,x\,\tfrac{\partial^2 f(x)}{\partial x_{i,j}\partial x_{k,l}} + \sum_{i,j}(b_{i,j}+B_{i,j}x)\tfrac{\partial f(x)}{\partial x_{i,j}} - (c+\langle\gamma,x\rangle)f(x)\\ + \int_{S_d^+\setminus\{0\}}(f(x+\xi)-f(x))\,m(d\xi)\\ + \int_{S_d^+\setminus\{0\}}(f(x+\xi)-f(x)-1_{\{\|\xi\|\le 1\}}\langle\xi,\nabla f(x)\rangle)M(x,d\xi),\end{aligned} \tag{1}$$

where $A, B \in L(S_d^+)$, $b, \gamma \in S_d^+$, $c \in [0,\infty)$, $m\colon \mathcal{B}(S_d^+) \to [0,\infty]$ and $M\colon \mathcal{B}(S_d^+) \to S_d^+$ satisfy certain *admissability conditions*. Via the generator one obtains a system of Riccati differential equations for the functions ϕ and ψ. (For details, see [4, Section 2].)

Our aim is to extend these results to the setting of positive self-adjoint Hilbert-Schmidt operators $S^+(H)$ on a Hilbert space H. In doing so, we encounter the following difficulties, which are all related to the fact that we are in the infinite-dimensional setting:

- The cone $S^+(H)$ is not locally compact, hence the classical Feller theory is not applicable.
- Weak convergence of measures on Hilbert spaces is laborious due to a lack of Lévy's continuity theorem: one has to prove tightness 'by hand'.
- The cone $S^+(H)$ has empty interior, hence it is unclear whether one can allow for a diffusion term (i.e., whether when can take $\mathcal{A}$ in (1) to be non-zero).

Indeed, to the best of our knowledge the only known example of an affine process taking values in the space of positive self-adjoint Hilbert-Schmidt operators is in [1], where the authors consider the following equation:

$$dX_t = BX_t dt + dL_t,$$

where $(L_t)_{t\ge 0}$ is a square-integrable operator valued Lévy process and $B \in L(S^+(H))$. This model is relatively easy to treat as it is well-posed in a probabilistic and analytic strong sense.

Using the theory of *generalized Feller processes* developed in [2], we have extended [1] and [4] to prove (see [3]) that under certain conditions (comparable to those in [4]) on $b \in S^+(H)$, $B \in L(S^+(H))$, $m\colon \mathcal{B}(S^+(H)) \to [0,\infty]$ and

$M\colon \mathcal{B}(S^+(H)) \to S^+(H)$ there exists an $S^+(H)$-valued Markov process with generator

$$\begin{aligned}\mathcal{A}f(x) &= \sum_{i,j}(b_{i,j} + B_{i,j}x)\tfrac{\partial f(x)}{\partial x_{i,j}} \\ &\quad + \int_{S^+(H)\setminus\{0\}} (f(x+\xi) - f(x))\, m(d\xi) \\ &\quad + \int_{S^+(H)\setminus\{0\}} (f(x+\xi) - f(x) - 1_{\{\|\xi\|\le 1\}}\langle \xi, \nabla f(x)\rangle) M(x, d\xi). \qquad (2)\end{aligned}$$

We hope to tackle various related open questions, such as:

- Is the affine process a weak solution to an associated stochastic differential equation?
- Can we allow for a diffusion term (i.e., can we allow for $A \neq 0$)?
- Are our 'admissibility conditions' necessary, i.e., can we provide a full characterisation of $S^+(H)$-valued affine processes?

References

[1] Benth, F.-E., B. Rüdiger, and A. Süss, *Ornstein-Uhlenbeck Process in Hilbert Space with Non-Gaussian Stochastic Volatility* Stochastic Processes and their Applications (2018).

[2] Dörsek, Ph., and J. Teichmann, *Efficient simulation and calibration of general HJM models by splitting schemes.* SIAM Journal on Financial Mathematics (2013).

[3] Cox, S., S. Karbach, and A. Khedher, *Affine processes on positive self-adjoint operators*, work in progress.

[4] Cuchiero, C., D. Filipovic, B. Mayerhofer, and J. Teichmann, *Affine processes on positive semi-definite matrices*, Annals of Applied Probability (2011).

Random polytopes: Report of the group work

Florian Besau, Anna Gusakova, Matthias Reitzner, Carsten Schütt, Christoph Thäle, Elisabeth Werner

To motivate the problem our group was working on during the workshop, we shall rephrase some key results in the theory of random polytopes. To construct random polytopes, one classically starts with a fixed convex body $K \subset \mathbb{R}^d$ and a sequence $(X_i)_{i\ge 1}$ of independent random points uniformly distributed in K. For $n \ge d+1$ we denote by

$$K_n := [X_1, \dots, X_n]$$

the random convex hull of $X_1, \dots, X_n$. It can be thought of as a random polytopal approximation of the set K, which approaches K, as $n \to \infty$. There are several interesting random variables connected to the random polytopes K_n, which measure the degree of approximation of K by K_n as well as its combinatorial complexity. These are

(i) the volume $V_d(K_n)$ of K_n, and, more generally, the k-th intrinsic volume $V_k(K_n)$ of K_n, $k = \{0, 1, \dots, d\}$,

(ii) the number $f_{d-1}(K_n)$ of facets (i.e. $(d-1)$-dimensional faces) of K_n, and, more generally, the number $f_k(K_n)$ of k-dimensional faces of K_n, $k = \{0, 1, \dots, d-1\}$.

From now on, we concentrate on first-order properties of the combinatorial structure of K_n, more precisely, on the expected number $\mathbb{E}f_k(K_n)$ of k-dimensional faces of K_n.

Since the 1990ies it is well known that the asymptotic behaviour of $\mathbb{E}f_k(K_n)$, as $n \to \infty$, is depending on the geometry of the underlying convex body K. Indeed, if K is of class C^2_+, that is, if K has a boundary that is a C^2-submanifold of $\mathbb{R}^d$ with strictly positive Gaussian curvature $\kappa(x)$ at every point $x \in \partial K$, then

$$\mathbb{E}f_k(K_n) = c_{d,k}\Omega(K)n^{\frac{d-1}{d+1}}(1 + o_n(1)), \tag{1}$$

as $n \to \infty$, where $c_{d,k}$ is a constant only depending on d and on k and $\Omega(K) = \int_{\partial K} \kappa(x)^{1/(d+1)}\,\mathrm{d}x$ is known to be the affine surface area of K. On the other hand, if $K = P$ is a d-dimensional polytope, then

$$\mathbb{E}f_k(K_n) = \hat{c}_{d,k}\mathrm{flag}(P)(\log n)^{d-1}(1 + o_n(1)), \tag{2}$$

as $n \to \infty$, where $\hat{c}_{d,k}$ is another constant only depending on d and on k, while $\mathrm{flag}(P)$ is the number of flags of P, that is, the number of chains $F_0 \subset F_1 \subset \dots \subset F_{d-1}$, where for each $i \in \{0, 1, \dots, d-1\}$, F_i is an i-dimensional face of P, see e.g. [3]. On different levels of generality these results can be found in [7] (for $d = 2$), [1] (for $k \in \{0, d-1\}$) and [6] (for general d and k).

Now, let us consider the following setup, which has been introduced in [2]. We assume that $(X_i)_{i \geq 1}$ is a sequence of independent random points uniformly distributed on the d-dimensional upper halfsphere $\mathbb{S}^d_+ := \mathbb{S}^d \cap \{x_{d+1} \geq 0\} \subset \mathbb{R}^{d+1}$. For $n \geq d+1$ the positive hull of $X_1, \dots, X_n$ cuts out a spherical random polytope $K_n^{(s)} \subset \mathbb{S}^d_+$, whose expected number of k-dimensional spherical faces has the remarkable property that

$$\lim_{n \to \infty} \mathbb{E}f_k(K_n^{(s)}) = \tilde{c}_{d,k} \tag{3}$$

where $\tilde{c}_{d,k}$ is another (finite!) constant only depending on d and k, see [2] (for $k \in \{0, d-1\}$) and [5] (for general k). Also see [4] for an analog to (1) on the unit sphere for $k = 0$.

After gnomonic projection of $\mathbb{S}^d_+$, the spherical random convex hull $K_n^{(s)}$ might be interpreted as the convex hull of uniform random points in a d-dimensional convex 'polytope with a single facet'. Given these facts the work in our group concentrated on the following question.

Question: *Are there models for random polytopes that interpolate between the behaviour of* (2) *and* (3)*?*

To attack this question we focused on the case $k = d-1$ and on the following construction, which generalizes the approach in [5]. Given $j \in \{1, \dots, d\}$ hyperplanes

$H_1, \ldots, H_j$ passing through the origin of $\mathbb{R}^{d+1}$ that are in general position, define the set

$$\mathbb{S}^d_{j,+} := \mathbb{S}^d \cap H_1^+ \cap \ldots \cap H_j^+.$$

Then $\mathbb{S}^d_{j,+}$ is a d-dimensional spherically convex subset of $\mathbb{S}^d$, which contains a great subsphere of dimension $d - j$ and its shape is determined by the angles between $H_1, \ldots, H_j$. Let further $(X_i)_{i\geq 1}$ be independent random points uniformly distributed on $\mathbb{S}^d_{j,+}$ and for $n \geq d+1$ let $K_n^{(s,j)}$ be the spherical convex hull of $X_1, \ldots, X_n$. Note that for $j = 1$, up to a rotation, $\mathbb{S}^d_{1,+}$ can be identified with $\mathbb{S}^d_+$ and the number of facets of the spherical random polytope $K_n^{(s,1)}$ has the same distribution as that of $K_n^{(s)}$ studied in [5]. On the other hand, if $j = d$, after gnomonic projection, one can think of $K_n^{(s,d)}$ as the convex hull of random points chosen from a d-dimensional simplex in $\mathbb{R}^d$ with a beta prime distribution of parameter $\beta' = \frac{d+1}{2}$ restricted to the domain of the simplex.

Conjecture: *For $j \in \{1, \ldots, d\}$ one has that*

$$\mathbb{E} f_{d-1}(K_n^{(s,j)}) = c_{d,j} (\log n)^{j-1} (1 + o_n(1)), \tag{4}$$

where $c_{d,j}$ is a constant that depends on the parameters d and j. In particular, the first-order asymptotic expansion does not depend on the actual angles between $H_1, \ldots, H_j$ (if they are in general position).

Given the above discussion we note that this behaviour would naturally interpolate between (2) (corresponding to the choice $j = d$) and (3) (which arises by taking $j = 1$).

During the week we were able to make progress on the special case $j = 2$ and under the additional randomization that the number $n = N(n)$ of generating points is Poisson distributed with parameter n. Then, using the multivariate Mecke formula for Poisson point processes, we have that

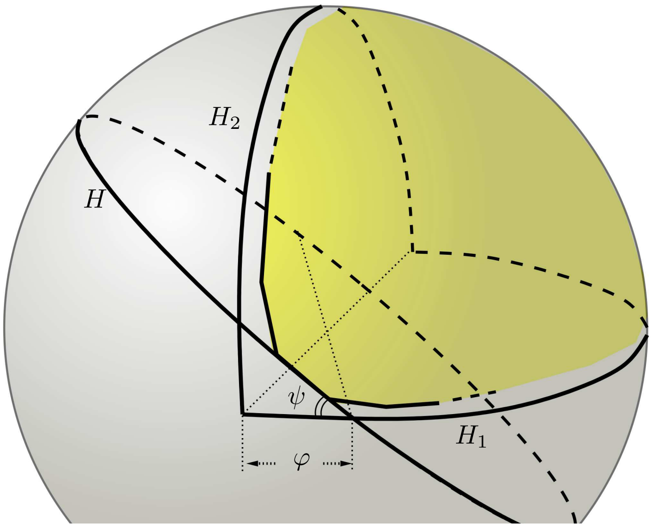
$$\begin{aligned}\mathbb{E} f_{d-1}(K_n^{(s,2)}) &= \frac{1}{d!} \mathbb{E} \sum_{1 \leq i_1 < \ldots < i_d \leq N} \mathbf{1}\{x_{i_1}, \ldots, x_{i_d} \text{ generate a facet}\} \\ &= \frac{n^d}{d!} \int_{\mathbb{S}^d_{2,+}} \cdots \int_{\mathbb{S}^d_{2,+}} \mathbb{P}(x_1, \ldots, x_d \text{ generate a facet}) \, \mathrm{d}x_1 \ldots \mathrm{d}x_d.\end{aligned}$$

Now, using the Blaschke–Petkantschin formula from spherical integral geometry, this transforms into

$$c_d n^d \int_{G(d+1,d)} \left(\int_{\mathbb{S}^d_{2,+} \cap H} \cdots \int_{\mathbb{S}^d_{2,+} \cap H} \Delta_d(x_1, \ldots, x_d) \, \mathrm{d}x_1 \ldots \mathrm{d}x_d \right) \times \exp(-n \sigma_d(\mathbb{S}^d_{2,+} \cap H^+)) \, \mathrm{d}H,$$

where $G(d+1, d)$ is the Grassmannian of d-dimensional linear subspaces of $\mathbb{R}^{d+1}$ and $\Delta_d(x_1, \ldots, x_d)$ is the volume of the parallelotope spanned by $x_1, \ldots, x_d$. Here and in what follows, c_d will denote a constant only depending on the dimension d whose value might change from occasion to occasion. To proceed, the following two steps are necessary:

(i) Evaluate the integral in brackets.
(ii) Derive asymptotic estimates for $\sigma_d(\mathbb{S}^d_{2,+} \cap H^+)$.



For this we noted that, up to isometry, the shape of the intersection

$$\mathbb{S}^d_{2,+} \cap H^+ = \mathbb{S}^d \cap H_1^+ \cap H_2^+ \cap H^+ = \mathbb{S}^d_{3,+},$$

is controlled by only two parameters–the angle ψ between H_1 and H, and the angle φ between $H_1 \cap H_2$ and $H_1 \cap H$ (measured within H_1). By choosing coordinates in the right way, we may reduce asymptotically the above integral expression to a double integral of the form

$$c_d n^d \int_0^a \int_0^b \frac{\varphi^{d+1}}{(\cos\psi)^{d+1}} (\sin\psi)^{d-1} (\sin\varphi)^{d-2} \exp\Big(-\frac{n}{2}\varphi^2 \tan\psi\Big)\, \mathrm{d}\psi \mathrm{d}\varphi,$$

for some fixed $a, b > 0$.

By appropriate substitutions one can see that, as $n \to \infty$, this behaves like a constant multiple of $\log n$, where the constant only depends on d and not on a and b. Summarizing, we conclude that

$$\mathbb{E} f_{d-1}(K_n^{(s,2)}) = c_d (\log n)(1 + o_n(1))$$

for an appropriate constant c_d only depending on d. This establishes the conjecture for the special case $j = 2$.

References

[1] I. Bárány, C. Buchta, *Random polytopes in a convex polytope, independence of shape, and concentration of vertices*, Math. Ann. **297** (1993), 467–497.
[2] I. Bárány, D. Hug, M. Reitzner, R. Schneider, *Random points in halfspheres*, Random Struct. Algorithms **50** (2017), 3–22.

[3] F. Besau, C. Schütt, E. M. Werner, *Flag numbers and floating bodies*, Adv. Math. **338** (2018), 912–952.
[4] F. Besau, M. Ludwig, E. M. Werner, *Weighted floating bodies and polytopal approximation*, Trans. Amer. Math. Soc. **370** (2018), 7129–7148.
[5] Z. Kabluchko, A. Marynych, D. Temesvari, C. Thäle, *Cones generated by random points on half-spheres and convex hulls of Poisson point processes*, Probab. Theory Relat. Fields **175** (2019), 1021–1061.
[6] M. Reitzner, *The combinatorial structure of random polytopes*, Adv. Math. **191** (2005), 178–208.
[7] A. Rényi, R. Sulanke, *Über die konvexe Hülle von n zufällig gewählten Punkten*, Z. Wahrscheinlichkeitstheor. Verw. Geb. **2** (1963), 75–84.

Large deviations for random measures associated with certain projections: Report of the group work

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Suppose under a probability measures $\mathbb{P}$ we have a random variable X uniformly distributed on the unit cube $[-1,1]^N$ in N dimensions. For elements θ of the Euclidean unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$, let μ_θ be the probability measure on $\mathbb{R}$ defined by setting

$$\mu_\theta(A) := \mathbb{P}\left(\langle \theta, X\rangle \in A\right),$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{R}^N$.

Now suppose we choose $\theta = \Theta^N$ according to the uniform measure on $\mathbb{S}^{N-1}$ so that $\nu_N := \mu_{\Theta^N}$ is a random measure on $\mathbb{R}$. We are interested in studying these random measures as the underlying dimension N tends to infinity, with a particular emphasis on the large deviation behaviour.

First, let us make a few remarks on the typical behaviour of the random measures ν_N as N tends to infinity. The random variable $\langle \theta, X\rangle$ whose law is given by ν_N may be expressed as a sum of independent random variables

$$\langle \theta, X\rangle = \theta_1 X_1 + \dots \theta_n X_n,$$

where the X_i are independent random variables uniformly distributed on $[-1,1]$ and from this fact we may expect Gaussian behaviour in the limit. We make the first observation that as long as no θ_i is large compared to the sum $\sum_{j=1}^N \theta_j^2 = 1$, for large N the random variable $\langle \theta, X\rangle$ is close to a Gaussian random variable with variance

$$\left(\sum_{i=1}^N \theta_i^2\right) \mathbb{E}[X_1^2] = 1/3.$$

We now make the second observation that for large N, when Θ^N is uniformly distributed on the unit sphere, all of the coordinates of Θ^N are small with high probability. Let $\gamma_{\sigma^2} := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}s^2} ds$ be the Gaussian measure with variance σ^2. Pairing these two observations, we naturally expect that

For large N, ν_N is typically concentrates around the Gaussian measure $\gamma_{1/3}$ of variance $1/3$.

With this picture in mind we now turn to looking at the large deviations of the sequence of random measures $(\nu_N)_{N\geq 1}$. Below we state a conjecture giving a precise prediction for these large deviations, which is based on the idea that the contributions in large deviation come from coordinates of the random vector Θ^N taking atypically large values. Before stating this conjecture, we would first like to furnish some motivation by gaining an understanding on the asymptotic probabilities that some of the coordinates of Θ^N take on aytpically large values.

A straightforward calculation tells us that the marginal distribution of the first k coordinates of an element chosen uniformly from the unit sphere $\mathbb{S}^{N-1}$ is given by

$$g_{N,k}(s_1,\ldots,s_k) = \frac{\Gamma(N/2)}{\pi^{k/2}\Gamma(\frac{N-k}{2})}\left(1-\sum_{i=1}^k s_i^2\right)^{\frac{N-k-1}{2}}$$

It is clear that as N becomes large (with k fixed), the measure $g_{N,k}$ on $\mathbb{R}^k$ concentrates around zero, and from here it is straightforward to study the deviations of $g_{N,k}$ away from the origin. Indeed, for a fixed vector $t := (t_1,\ldots,t_k)$ of $\mathbb{R}^k$ we have

$$(1) \qquad \lim_{N\to\infty}\frac{1}{N}\log g_{N,k}(t_1,\ldots,t_k) = \begin{cases} -\frac{1}{2}\left(1-||t||_2^2\right) & \text{if } ||t||_2<1, \\ \infty & \text{otherwise.}\end{cases}$$

Now consider reordering the coordinates in a non-increasing in terms of their absolute value. Rough calculations based around Markov's inequality as well as the simple observation that $\lim_{N\to\infty}\frac{1}{N}\log\binom{N}{k} = 0$ suggest that k coordinates with largest modulus of a uniformly chosen element θ^N of the unit sphere $\mathbb{S}^{N-1}$ also satisfy this same large deviation principle (1).

Let $\tilde{\Theta}^N$ be an element of $\mathbb{S}^{N-1}$ obtained by listing the components in decreasing order of magnitude (with an arbitrary choice made in the case of a tie). By taking k to be arbitrarily large in our previous calculation, evidence leads us towards making the following statement about the large deviations of the largest components of Θ^N:

Theorem 1. *The random vector $\tilde{\Theta}^N$ satisfies a large deviation principle with speed N and rate function*

$$I(\alpha) = -\frac{1}{2}\log\left(1-||\alpha||_2^2\right)$$

for $\alpha\in\ell^2$ whose components are listed in nonincreasing order of magnitude.

We now relate this observation to the random measures $\nu_N = \mu_{\Theta^N}$. To set this up, given an element α in the unit ball of ℓ^2, let κ_α be the probability measure associated with the random variable

$$\langle\alpha, X\rangle + \sqrt{\frac{1-||\alpha||_2^2}{3}}Z,$$

where $X = (X_1, X_2, \ldots)$ is a sequence of random variables uniformly distributed on $[-1, 1]$ and Z is a standard Gaussian random variable.

Now when N is large, if we condition on Θ^N having k atypically large coordinates of sizes $(\alpha_1, \ldots, \alpha_k)$, the remaining $N - k$ coordinates tend to be small compared to 1, and the sum of squares of these coordinates is equal to $\left(1 - ||\alpha||_2^2\right)$. It follows that the conditional law of the random variable $\langle \Theta^N, X \rangle$ given that the largest coordinates of θ^N have sizes $(\alpha_1, \ldots, \alpha_k)$ is close to κ_α when N is large. Based on Theorem 1, which characterises the large deviations of Θ^N, we are lead to make the following conjecture about the large deviations of $(\nu_N)_{N \geq 1}$.

Conjecture 1. *Let $\Theta^{(N)}$ be a sequence of elements of $\mathbb{R}^N$ uniformly distributed on the unit sphere $\mathbb{S}^{N-1}$. Then the sequence of random measure $\nu_N := \mu_{\Theta^{(N)}}$ satisfy a large deviation principle with rate N and rate function*

$$I(\nu) = \begin{cases} -\frac{1}{2} \log \left(1 - ||\alpha||_2^2\right) & \text{if } \nu = \kappa_\alpha \text{ where } ||\alpha||_2 < 1, \\ \infty & \text{otherwise.} \end{cases}$$

One of the key intermediate results that must necessarily hold in order for Conjecture 1 to be true is the following technical lemma about the limits of certain sequences of probability distributions.

Lemma 1. *For $N \geq 1$, let E_N denote the set of probability laws of random variables of the form $\sum_{i=1}^N \alpha_i^{(N)} U_i$, where $\alpha^{(N)} \in \mathbb{S}^{N-1}$. Clearly $E_N \subseteq E_{N+1}$. Suppose $\nu^{(N)}$ is a sequence of measures such that each $\nu^{(N)}$ is contained in E_N and the sequence $\nu^{(N)}$ converge (in Prohorov topology) to some measure ν on $\mathbb{R}$. Then the set of all possible limits is given by*

$$E^* := \left\{ \kappa_\alpha : \alpha \text{ is an element of the unit ball in } \ell^2 \right\}.$$

We have so far investigated two contrasting approaches towards proving this lemma, one involving characteristic functions and another involving the Berry-Esseén theorem.

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