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Low-dimensional Topology

Organized by
Stefan Friedl, Regensburg
Yoav Moriah, Haifa
Jessica Purcell, Melbourne
Saul Schleimer, Warwick

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Abstract. The workshop brought together experts from across all areas of low-dimensional topology, including knot theory, mapping class groups, three-manifolds and four-manifolds. In addition to the standard research talks we had five survey talks by Burton, Minsky, Powell, Reid, and Roberts leading to discussions of open problems. Furthermore we had three sessions of five-minute talks by a total of thirty-five participants.

Mathematics Subject Classification (2020): 57K10, 57K20, 57K30, 57K40.

Introduction by the Organizers

The workshop *Low-dimensional Topology* (2020) was organized by Stefan Friedl (Regensburg), Yoav Moriah (Haifa), Jessica Purcell (Melbourne) and Saul Schleimer (Coventry). Unfortunately Jessica Purcell could not attend due to logistical problems. The workshop was attended by over 50 researchers from countries including Australia, Canada, Denmark, France, Germany, Hungary, Israel, Japan, UK, and USA.

We had sixteen research talks: speaking broadly, five lectures in knot theory, two lectures on mapping class groups, five lectures on three-manifolds, and two lectures on four-manifolds. Furthermore we had two talks by algebraic topologists on the closely related topic of the homotopy types of embedding spaces. Also, we had five survey talks on, respectively, hyperbolic geometry, four-manifolds, computational low-dimensional topology, arithmetic hyperbolic three-manifolds, and foliation theory. Finally we had three lively sessions of five-minute talks. Altogether thirty-five participants spoke in these three sessions.

Even though the talks covered a wide range of topics, the speakers were very successful in establishing connections between the various subfields of low-dimensional topology.

One of the highlights were the talks by Pinsky and Li on their recent proof, with Moriah, of the Berge conjecture for tunnel number one knots. In addition to the delicate analysis of the combinatorics of curves on surfaces the proof contains a subtlety: a tunnel number one knot may be doubly primitive on one of its Heegaard splittings while not being so on another. This requires "switching" Heegaard splittings to complete the proof.

Another exciting talk, given by Rasmussen, covered her work-in-progress on the Boyer-Gordon-Watson L-space conjecture; the innovation here is to decompose the three-manifold after the fashion of Heegaard-Floer homology. This allows her to transform the left-orderability hypothesis into a collection of explicit holonomies in $\mathrm{Homeo}^+(\mathbb{R})$ and thus construct a foliation.

Another spectacular result presented was the Bowden-Hensel-Webb theorem that $\mathrm{Homeo}^+(S_g)$ admits uncountably many unbounded quasimorphisms. This talk, given by Webb, was particularly notable for its beautiful, much-praised, exposition.

The conference concluded by a beautifully illustrated talk by Sakuma on his classification, with various coauthors, of all Kleinian groups generated by two parabolic transformations. This completes a research program initiated by Adams in the 1990s and advanced by Agol in the 2000s.

A remarkable feature of the workshop were the computationally focused talks by Burton, Dunfield, Owens, Schleimer and Segerman. They demonstrated how increasingly sophisticated programs, such as Regina and SnapPy, have become an indispensable tool in low-dimensional topology. We expect that such example driven research will become an increasingly important feature the field.

In addition to the research and survey talks we initially planned to have a single session of five-minute talks in the late afternoon. However, due to the enthusiastic response to the first session we then scheduled another two sessions. Altogether thirty-five participants gave five-minute presentations on a wide range of topics. The very large majority of the speakers managed to get their key idea across. This format was also a highly efficient means for younger participants to introduce themselves, and their work, to the community.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Rachel Roberts and Henry Segerman in the "Simons Visiting Professors" program at the MFO.

Workshop: Low-dimensional Topology

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Abstracts

Necklace Theory and Maximal Cusps of Hyperbolic 3-Manifolds

David Gabai
(joint work with Robert Haraway, Robert Meyerhoff, Nathaniel Thurston, Andrew Yarmola)

Theorem 1. *Let N be a complete finite volume hyperbolic 3-manifold with a maximal cusp of volume ≤ 2.62. Then N is obtained by filling either the 3-cusped manifold s776 or is obtained by filling one of 15 explicit 2-cusped hyperbolic 3-manifolds. Further, the figure-8 knot complement and its sister are the two 1-cusped hyperbolic 3-manifolds with a minimal volume maximal cusp. The maximal cusp has volume $\sqrt{3} = 1.73\cdots$.*

The statement of this result was explained in more detail as well as basic background.

The main result can be viewed as a solution, for cusped hyperbolic 3-manifolds, to the *Hyperbolic Complexity Conjecture* of Thurston, Matveev - Fomenko, Hodgson - Weeks that was formulated in various forms in the 70's and 80's.

Conjecture 2. *Low volume hyperbolic 3-manifolds are obtained by filling cusped hyperbolic 3-manifolds of low topological complexity.*

This conjecture is open ended and somewhat vague. The challenge includes finding the right notion of topological complexity for the case at hand. Classically, volume meant volume of the manifold and relevant measures of complexity included number of 3-simplices, the standard spine number and the mom number. Here volume refers to the volume of a maximal cusp. The proof shows that a manifold with a maximal cusp of volume ≤ 2.62 is obtained by filling a manifold obtained from $T^2 \times I$ by adding a single 1-handle and a 2-handle that goes over the 1-handle at most 7 times. This simple handle structure is what we mean by low topological complexity in this context.

We went on to describe one application. Tom Crawford made a contribution to the proof.

Theorem 3. *The figure-8 knot complement is the unique 1-cusped hyperbolic 3-manifold with nine or more non hyperbolic fillings.*

This gives a positive solution to a well known conjecture of Cameron Gordon's from the 1990's. Our main result also answers a question of Ian Agol from 2010 who asked what are the 1-cusped hyperbolic 3-manifolds with a maximal cusp of volume $\leq 24/7 = 2.54\cdots$. Agol noted that manifolds with larger cusps have the property that the distance between non hyperbolic fillings is at most 5 and such manifolds have at most 8 non hyperbolic fillings. Thus, the main result together with some hyperbolicity / non hyperbolicity checking using the six theorem of Agol and Lackenby and work of Robert Haraway and others, the proof is completed.

We went on to outline the proof of the theorem, summarized as follows.

Step 1: A bicusped subgroup of $\pi_1(N)$ has g-exponent length ≤ 7. (This step required rigorous computer assistance.)

Step 2: N has a handle decomposition of the form $T^2 \times I \cup$ 1-handle $\cup$ 2-handle, where the 2-handle runs over the 1-handle at most 7 times.

Step 3: The enumeration of such hyperbolic 3-manifolds. (This step also required rigorous computer assistance.)

Next we explained the structure of bi-cusped subgroups. We remarked that a consequence of Step 2 is that such a bicusped group is in fact index-1. Also, our results were close to being sharp in that SnapPy shows that the manifold $m135$ has a maximal cusp of volume approximately 2.82. It's bicusped group has (according to SnapPy) g-exponent length 8.

Question 4. *Does $m135$ have tunnel number 2.*

We noted that a positive solution to this question would imply that $m135$ does not have the structure of Step 2. We went on to explain how Step 1 implies that there exists a cycle of $k \leq 7$ horoballs with disjoint interiors, one tangent to the next. If this cycle bounds an embedded 2-disc whose $\pi_1(N)$-orbit is a union of non intersecting discs, then we obtain the desired handle decomposition. We then drew the picture of the necklace for $m135$ and indicated how it was blocked by another horoball. We indicated that a key step towards proving Step 2 is showing that minimal ≤ 7-necklaces arising from a cusped hyperbolic 3-manifold is unblocked.

Geometry of graphs of multicurves

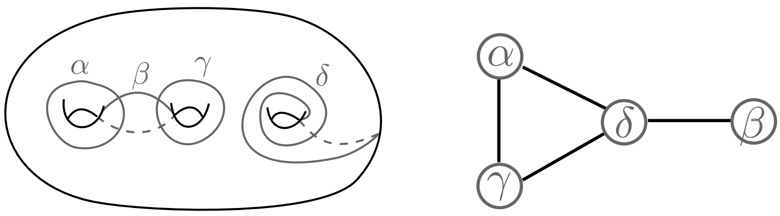
Katie Vokes

(joint work with Jacob Russell)

Let S be a connected, compact, orientable surface. The **curve graph**, $\mathcal{C}(S)$, has a vertex for every isotopy class of (essential simple closed) curves in S (that is, homotopically non-trivial embedded copies of S^1 in S), and an edge joining two vertices if they have disjoint representatives. For any sufficiently complicated surface, the curve graph has a countably infinite set of vertices, and in fact each vertex has infinite degree.
We may introduce a metric in the curve graph by setting each edge to have length 1. A key property of this graph is that it has a type of coarse negative curvature.

Theorem (Masur–Minsky [8])**.** *For any surface S of sufficient complexity, the curve graph of S has infinite diameter and is Gromov hyperbolic.*

It is significant that the mapping class group, $\mathrm{MCG}(S)$, that is, the group of isotopy classes of orientation-preserving self-homeomorphisms of S, acts on $\mathcal{C}(S)$ by isometries. We can use such an isometric action of a group on a metric space to study the **large scale geometry** of the group. In this way, the curve graph has been an important tool in the study of the geometry of the mapping class



(A) An example of four simple closed curves on the genus 3 surface.

(B) The corresponding full subgraph of the curve graph.

FIGURE 1

group. Another topic where the curve graph has proved to be a natural object is the geometry of 3-manifolds, playing a key role in the proof by Brock, Canary and Minsky of Thurston's Ending Lamination Conjecture [7].

In addition to the curve graph, we may define a whole family of graphs associated to surfaces which have multicurves (collections of pairwise disjoint curves) as vertices and a natural isometric action of the mapping class group. Many of these have been found to arise in natural situations; for example, the graph of pants decompositions of a surface was used by Brock to study the geometry of quasi-Fuchsian 3-manifolds [6].

In a paper of 2000, Masur and Minsky described the large scale geometry of $\mathrm{MCG}(S)$ by using a family of Lipschitz **subsurface projection** maps from $\mathrm{MCG}(S)$ to curve graphs of subsurfaces of S [9]. Various properties of, and relations between, these projections, as well as the Gromov hyperbolicity of the curve graphs, are combined to prove that the mapping class group has properties typical of non-positive curvature. This work was axiomatised by Behrstock, Hagen and Sisto to make $\mathrm{MCG}(S)$ the prototype of a kind of non-positively curved space called a **hierarchically hyperbolic space** [2, 3]. I showed in [12] that many graphs of multicurves also have this property.

Theorem 1. *Let $\mathcal{G}(S)$ be a graph whose vertices are multicurves in S, satisfying:*

- *$\mathcal{G}(S)$ is connected,*
- *the action of $\mathrm{MCG}(S)$ on the set of curves in S induces a cocompact isometric action on $\mathcal{G}(S)$,*
- *$\mathcal{G}(S)$ has no witnesses which are annuli (see below for definition).*

Then $\mathcal{G}(S)$ is a hierarchically hyperbolic space.

In particular, we have certain subsurfaces of S called **witnesses** for $\mathcal{G}(S)$, which are important in the hierarchically hyperbolic structure on $\mathcal{G}(S)$. A witness is a subsurface X of S which has the property that every vertex of $\mathcal{G}(S)$ (which is a multicurve) intersects X; see Figure 2 for an example. In a sense X is a subsurface which "sees" every vertex of $\mathcal{G}(S)$.

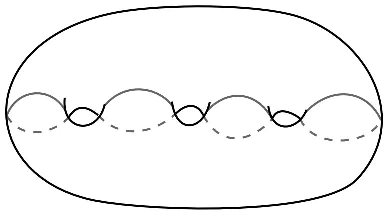


FIGURE 2. Let S be the genus 3 surface and $\mathcal{G}(S)$ be the separating curve graph, with a vertex for every curve that cuts S into two pieces. Then the two complementary components of the multicurve shown are both witnesses for $\mathcal{G}(S)$.

The witnesses are those subsurfaces that we use for the subsurface projections; for $\mathrm{MCG}(S)$ all subsurfaces are witnesses but for the graphs of multicurves we consider the sets of witnesses are smaller.

Suppose that $\mathcal{G}(S)$ is one of the graphs of multicurves for which Theorem 1 applies. The way that the hierarchically hyperbolic structure on $\mathcal{G}(S)$ works implies that pairs of disjoint witnesses give rise to **product regions** in $\mathcal{G}(S)$. These product regions are, in particular, obstructions to $\mathcal{G}(S)$ being Gromov hyperbolic. Results of Behrstock–Hagen–Sisto [4] and Bowditch [5] combine with Theorem 1 to give a converse to this.

Corollary 2. *Let $\mathcal{G}(S)$ be a graph of multicurves satisfying the hypotheses of Theorem 1. Then $\mathcal{G}(S)$ is Gromov hyperbolic if and only if there exists no pair of disjoint witnesses for $\mathcal{G}(S)$.*

A weakening of Gromov hyperbolicity is **relative hyperbolicity**, where roughly speaking we allow a space to have non-hyperbolic regions, but stipulate that these should be sufficiently isolated within the space. In recent work, Jacob Russell gives a sufficient condition for a hierarchically hyperbolic space to be relatively hyperbolic in terms of the hierarchically hyperbolic structure (in this setting, the set of witnesses), and applies this to certain graphs of multicurves [10]. Combined with Corollary 2 above, this led to a partial classification of (relative) hyperbolicity for the separating curve graph, which we completed in joint work [11]. The separating curve graph is the full subgraph of the curve graph spanned by curves which cut the surface into two connected components.

Theorem 3 (with J. Russell)**.** *Let $S = S_{g,b}$, with $g \geq 3$. Then the separating curve graph is:*

- *Gromov hyperbolic if $b \geq 3$,*
- *relatively hyperbolic if $b = 0$ or $b = 2$,*
- *neither hyperbolic nor relatively hyperbolic if $b = 1$.*

We prove the last part of the theorem by showing that the separating curve graph of a surface with one boundary component is a **thick metric space** (of order at most two). This is a concept introduced by Behrstock, Druţu and Mosher which is an obstruction to relative hyperbolicity [1].

We are working on completing the classification for all graphs of multicurves for which Theorem 1 applies.

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Surgery obstructions from character varieties

Raphael Zentner

(joint work with Steven Sivek)

If a closed 3-manifold can be obtained by surgery on a knot in S^3 then its fundamental group is normally generated by one element. Every 3-manifold can be obtained by Dehn surgery on a link in S^3 by a Theorem of Lickorish-Wallace. It isn't an easy task to show that a 3-manifold which satisfies this condition on its fundamental group cannot be obtained by surgery on a knot in S^3. The first obstructions (due to Auckly) have used instanton gauge theory [1], and in particular Donaldson's diagonalisation theorem [4]. Later obstructions have been found using Heegaard Floer homology by Hom, Karakurt, and Lidman, [8], or using the Casson-Walker invariant by Boyer and Lines [2].

We present an infinite family of graph 3-manifolds, satisfying the fundamental group constraint, cannot be obtained by surgery on a knot in S^3. Our methods use classical 3-manifold topology methods together with results of Gordon-Luecke [7] on toroidal surgery on knots with complete hyperbolic complement, together with our classification results on $SU(2)$-cyclic Seifert fibered manifolds from [13] and our classification of $SU(2)$-cyclic surgeries on iterated torus knots, building on previous work in [12]. This gives a new method of obstructing 3-manifolds from being surgery on a knot in S^3 that is remarkable for the absence of gauge theory in the set of tools that is being used.

To be more precise, our graph 3-manifolds in question are of the form

$$Y(T_{a,b}, T_{c,d}) := \big(S^3 \setminus N(T_{a,b})\big) \cup_{T^2} \big(S^3 \setminus N(T_{c,d})\big),$$

in which we glue a meridian of one torus knot exterior to a Seifert fiber of the other and vice versa. Here, $N(T_{a,b})$ denotes a tubular neighborhood of the torus knot $T_{a,b}$. These manifolds have been considered by Motegi [10] who has shown that these manifolds have only representations of their fundamental group in $SU(2)$ with cyclic image (or short they are $SU(2)$-cyclic). They contain an incompressible torus, namely the one along which we glue the two knot exteriors.

Some infinite classes of these graph manifolds appear as half-integral surgeries on a class of knots introduced by Eudave-Muñoz [5], as has been shown by Ni and Zhang in [11].

Our results include the following sample application. We say that a set $T \subset \mathbb{N}$ has density zero if $\lim_{n\to\infty} \frac{|T\cap\{1,2,\ldots,n\}|}{n} = 0$.

Theorem 1. *There is a set $S \subset \mathbb{N}$ of density zero such that if $T_{a,b}$ is a nontrivial torus knot with $a, b > 2$ and $ab \notin S$, then $Y(T_{a,b}, T_{-a,b})$ is not Dehn surgery on any knot in S^3.*

Our proof follows the following lines: By some linking form computations we rule out the possibility of such graph manifolds being the result of integral surgery on a knot in S^3, for a a set of values of density 1. The numbers are chosen so that these do not arise as the half-integral surgeries on a Eudave-Muñoz knot.

By Thurston's geometrisation theorem for Haken manifolds, it is enough to check the cases whether the manifolds in question may arise as non-integral surgeries on a hyperbolic knot, a torus knot, or a satellite knot. No surgery on a torus knot contains an incompressible surface, however, so this case cannot produce a Motegi manifold. Gordon and Luecke have shown that if a surgery on a hyperbolic knot contains an incompressible torus, then the surgery is integral or half-integral, and if it is non-integral, then it arises as one of the half-integral surgeries on the Eudave-Muñoz knots. So we are left to deal with satellite knots. This uses classical 3-manifold topology, and in particular results due to Culler-Gordon-Luecke-Shalen [3], of Gordon [6], of Miyazaki and Motegi [9], and some new result that we prove which classifies $SU(2)$-cyclic surgeries on iterated torus knots. One consequence of our classification result is that none of these contain an incompressible torus. The combination of these results rules out the possibility that one of these Motegi manifolds arises as non-integral surgery on a satellite knot.

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The Berge conjecture for tunnel number one knots

Tao Li, Tali Pinsky

(joint work with Yoav Moriah)

In these two talks, we give an overview of a recent proof of the Berge conjecture for tunnel number one knots. A knot is called *tunnel number one* if the knot complement has a Heegaard splitting of genus two. Let K be such a knot. If K admits a non-trivial lens space surgery, the Heegaard surface of genus 2 is a Heegaard surface for the lens space as well, hence stabilized. Thus, there exists a planar surface P in the compression body (punctured by K) and a disk D in the handlebody that intersect at a single point, and (P, D) extends to a stabilizing pair for the splitting of the lens space given by Σ. John Berge observed [1], that if one can find such a (P, D) pair with P an annulus, then a Dehn surgery on K yields a lens space. Such a knot is said to be *doubly primitive* as it is a promitive curve in both handlebodies of the Heegaard splitting. Berge compiled a list of twelve families of doubly primitive knots $K \subset S^3$ including known cases of torus and satellite knots.

Conjecture 1 (The Berge Conjecture)**.** *Let $K \subset S^3$ be a non-trivial knot which has Dehn surgery resulting in a lens space. Then K is doubly primitive.*

As the notion of doubly primitive knots is not limited to S^3, one can ask whether a conjecture equivalent to the Berge Conjecture holds for other manifolds with Heegaard genus at most 2. The answer to this is known to be false if M is the Poincaré homology sphere [3] and is false for lens spaces in general [4]. The next conjecture, which was made by J. Greene (see [7, Conjecture 1.8]) and by Baker, Buck and Lecuona (see [4, Conjecture 1.1]), says that the answer is expected to be true if M is $S^2 \times S^1$.

Conjecture 2 (Berge Conjecture for $S^2 \times S^1$)**.** *If K is a knot in $S^2 \times S^1$ which admits a non-trivial lens space Dehn surgery, then K is doubly primitive.*

In the talks we sketch the proof of the following theorem, proving both conjectures in the tunnel number one case:

Theorem 1. *Let $K \subset M$ be a tunnel number one knot, where M is either $S^3, S^2 \times S^1$ or $(S^2 \times S^1)\#L(r,s)$, (where $L(r,s)$ is any lens space). If a non-trivial Dehn surgery on K yields a lens space, then K is doubly primitive.*

The general method of proof is by starting with a genus two Heegaard surface Σ and a (P,D) pair as above. One may choose a second meridian E for the handlebody so that it does not intersect P or D. Next, let C be the unique compressing disk in the compression body. Moreover, we choose a meridian A punctured by K exactly once as a second meridian for the handlebody from the trivial Dehn filling on the compression body. We study the Heegaard diagram given by these disks.

We assume minimality of the choices of P, D, and A. Furthermore, we assume P is incompressible, by else compressing it. The proof heavily relies on a theorem proven by Homma, Ochiai and Takahashi in [8], saying that any genus two Heegaard diagram for S^3 either is standard or contains a wave. If $M = (S^1 \times S^2)\#L(p,q)$, we use an analogous theorem proven by Negami and Okita in [9].

These theorems allow us to co-orient ∂D and ∂E in $\Sigma \setminus \partial P$, as conflicting orientations between the meridians on the two sides immediately prevent the existence of waves, contradicting these theorems. We view $\Sigma \setminus \partial P$ as a union of two annuli and two rectangles, containing ∂D and ∂E except the intersection point $P \cap D$. The curves ∂D and ∂E in $\Sigma \setminus \partial P$ are disjoint, and thus there are at most two families of parallel such arcs that can pass through each annulus.

Putting a convenient product structure on $\Sigma \setminus \partial P$ and collapsing the vertical direction, we obtain a traintrack carrying ∂D. The proof then is completed, case by case, according to the four possible configurations for this traintrack, corresponding to the number of different families of parallel arcs in the two annuli of $\Sigma \setminus \partial P$.

In each of these cases, the proof terminates at one of the following conclusions:

(1) A contradiction to the Theorems of Homma-Ochiai-Takahashi or Negami-Okita, or
(2) A contradiction to the minimality assumption, or
(3) The knot K is a Berge-Gabai knot [2, 6] and hence doubly primitive, or
(4) There exists a different genus 2 Heegaard splitting for $M \setminus K$, in which K is doubly primitive with respect to the new splitting.

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Developments in 4-manifolds

Mark Powell

I was asked to give an overview of the current state of the art in 4-manifold topology, and to describe some recent advances. We will consider some of the major open problems on 4-manifolds, discuss what classification results are known, and then talk about recent interest in studying symmetries of 4-manifolds.

Fundamental problems

I want to start by drawing the following stark contrast in our knowledge. The following five statements are true and known in the topological category with locally flat embeddings for every n, but are unknown and open in the smooth category for $n = 4$.

(1) The Poincaré conjecture that $M^n \simeq S^n$ implies $M \cong S^n$.
(2) The Schoenflies problem, that for every embedding $f \colon S^{n-1} \hookrightarrow S^n$, $f(S^{n-1})$ is unknotted.
(3) The unknot problem, that for every embedding $K \colon S^{n-2} \hookrightarrow S^n$ with $S^n \setminus K(S^{n-2}) \simeq S^1$ is unknotted.

(4) Let $F\colon D^n \xrightarrow{\cong} D^n$ be an equivalence with $F|_{\partial D^n} = \mathrm{Id}$. Then F is isotopic to Id.
(5) If $M^n \simeq T^n$ then $M \cong T^n$.

Let me briefly mention the status of the statements for other n in the smooth and PL categories. They are also relatively well understood. The statements all hold for $n \leq 3$ in the smooth and PL categories. For smooth manifolds, statements (1), (4), and (5) are generally false in dimensions $n \geq 5$, for example due to the existence of exotic spheres. Statements (1) and (4) do hold in the PL category for $n \geq 5$, but (5) does not. Statements (2) and (3) also hold in the smooth and PL categories for $n \geq 5$.

What do we know about 4-manifolds?

Thanks to the work of Freedman and Quinn, surgery theory allows one to classify 4-manifolds for certain fundamental groups. I will briefly describe the known classifications. These exhibit cases where the topology of 4-manifolds corresponds closely to the algebra of intersection forms. In each case, by a classification I mean that there is a collection of algebraic-topological invariants of a 4-manifold in the relevant class, and these invariants coincide if and only if the associated 4-manifolds are homeomorphic. The intersection form on the middle homology and the Kirby-Siebenmann invariant always appear.

(1) Freedman and Quinn classified closed, simply connected 4-manifolds [FQ90, Chapter 10].
(2) Freedman and Quinn classified closed, orientable 4-manifolds with fundamental group $\mathbb{Z}$ [FQ90, Chapter 10].
(3) Wang classified closed, nonorientable 4-manifolds with fundamental group $\mathbb{Z}$ [Wan95].
(4) Hambleton and Kreck classified closed 4-manifolds with finite cyclic fundamental groups [HK88].
(5) Hambleton, Kreck and Teichner classified closed nonorientable 4-manifolds with fundamental group $\mathbb{Z}/2$ [HKT94].
(6) Hambleton, Kreck and Teichner classified closed orientable 4-manifolds with fundamental group $\mathbb{Z} \ltimes \mathbb{Z}[1/2]$ [HKT09].
(7) Freedman and Quinn classified closed aspherical 4-manifolds with good fundamental groups for which the high dimensional Borel conjecture is known [FQ90].
(8) Brookman, Davis, and Kahn classified 4-manifolds homotopy equivalent to the connected sum of two copies of projective space $\mathbb{RP}^4 \# \mathbb{RP}^4$ [BDK07].
(9) Boyer classified simply connected compact 4-manifolds with a fixed 3-manifold as the boundary [Boy86].
(10) There are no known classifications for 4-manifolds with boundary that have nontrivial fundamental group. The only groups that admit a surjection to $\mathbb{Z}/2$ for which nonorientable 4-manifolds are classified are $\mathbb{Z}$ and $\mathbb{Z}/2$.

In joint work with A. Conway and D. Crowley, we are working on a classification of compact 4-manifolds with fundamental group $\mathbb{Z}$ and nonempty boundary. There is a corresponding and equally interesting discussion of results on embedded surfaces in 4-manifolds, omitted for space reasons. All of these classifications rely on Freedman's disc embedding theorem. With Behrens, Kalmár, Kim and Ray, I am an editor of a new book, written in a collaborative project with 20 authors, that gives a new and complete proof of the disc embedding theorem.

One of the big themes of 4-manifold topology is the ubiquity of exoticness, for example 4-manifolds that are homeomorphic but not diffeomorphic, diffeomorphisms that are topologically but not smoothly isotopic, and locally flat embeddings that are not smoothable. There is a wealth of fascinating complication in smooth 4-manifolds, which makes any kind of classification scheme for smooth structures hard to approach.

Stable classification

One way to obtain classification results for 4-manifolds in terms of algebraic topology is to consider the stable classification.

Two closed 4-manifolds M and N are said to be *stably diffeomorphic* if there are natural numbers m and n such that $M\#^m S^2 \times S^2$ and $N\#^n S^2 \times S^2$ are diffeomorphic.

Kreck reduced this question to bordism: for example two spin 4-manifolds with fundamental group π are stably diffeomorphic if and only if they represent, for some choices of spin structure and maps to π, bordant elements of $\Omega_4^{Spin}(B\pi)$. This enables one to reduce the question to algebraic topological invariants again. Hambleton-Kreck-Teichner [HKT09] studied the case of 2-dimensional groups. With Kasprowski, Land, and Teichner [KLPT17], I studied the case of 4-manifolds with 3-dimensional groups. In a forthcoming paper with Kasprowski and Teichner, we study spin 4-manifolds with abelian fundamental groups.

Another paper with Kasprowski and Teichner [KPT18] studied the analogous question where one stabilises with copies of $\mathbb{CP}^2$ instead of $S^2 \times S^2$.

Diffeomorphisms of 4-manifolds

A lot of recent activity has been on symmetries of 4-manifolds. Despite our lack of knowledge in the smooth category, with regards to the five questions I started with, one can still obtain information on the symmetries and families of such symmetries. The following is a useful guiding question.

Question 1. *For a fixed 4-manifold M, what are the homotopy types of* $\mathrm{Homeo}_\partial(M)$ *and* $\mathrm{Diff}_\partial(M)$*?*

These are the spaces, in fact topological groups, of homeomorphisms and diffeomorphisms of M that restrict to the identity on the boundary, equipped with the compact-open and Whitney topologies respectively.

There are two reasons to study the homotopy type. First, it can be easier than trying to study these purely as groups. Second, the homotopy type contains information on the classification of fibre bundles with fibre M.

In dimension 2, every connected component of the space of diffeomorphisms of a compact surface is contractible, so one is left with studying the mapping class group. In dimension 3, there is a strong understanding of diffeomorphism spaces, thanks in particular to Thurston, Hatcher, and Perelman. In dimension 4, our knowledge is somewhat limited, but there has been some exciting progress recently.

Theorem 2 (Watanabe, 2018). $\pi_k(\mathrm{Diff}_\partial(D^4)) \neq 0$ *for* $k = 1, 4, 8$.

Watanabe's theorem [Wat18] implies in particular that the 4-dimensional generalised Smale conjecture does not hold, that is $\mathrm{Diff}(S^4)$ is not homotopy equivalent to $O(5)$. Note that it was already known, and is not too hard to see, that $\mathrm{Diff}(S^n) \simeq \mathrm{Diff}_\partial(D^n) \times O(n+1)$.

The idea of Watanbe's proof is to construct bundles $D^4 \to E \to S^{k+1}$, using a 4-dimensional version of the Goussarov-Habiro clasper surgery. He then evaluates Kontsevich configuration space integrals on these bundles, using parametrised Morse theory. In 3-dimensions that analogous integrals give invariants of diffeomorphism classes of 3-manifolds, whereas in dimension 4 these are invariants of *families*.

Here are some more selected results on diffeomorphism and homeomorphism spaces of 4-manifolds. I find it striking that there are interesting results of a similar flavour coming from such different techniques. We already mentioned that Watanabe used configuration space integral characteristic classes.

Let Diff_0 denote the subset of diffeomorphisms that are proper homotopy equivalent to the identity.

Theorem 3 (Gabai, 2017). $\pi_0(\mathrm{Diff}_0(S^2 \times D^2)/\,\mathrm{Diff}_0(D^4)) = 0$.

Gabai's proof [Gab17] uses intricate geometric constructions to show that homotopy implies isotopy for certain embedded 2-spheres in 4-manifolds.

Theorem 4 (Budney-Gabai, 2019). $\pi_0(\mathrm{Diff}_\partial(S^1 \times D^3)/\,\mathrm{Diff}_\partial(D^4)) \neq 0$.

Budney-Gabai [BG19] use the Goodwillie-Klein-Weiss embedding calculus.

Theorem 5 (Baraglia-Konno, 2019). $\pi_1(\mathrm{Homeo}(K3)/\!/\,\mathrm{Diff}(K3)) \neq 0$.

Here $\mathrm{Homeo}(K3)/\!/\,\mathrm{Diff}(K3)$ denotes the homotopy quotient, which is homotopy equivalent to the homotopy fibre of the forgetful map $\mathrm{BDiff}(K3) \to \mathrm{BHomeo}(K3)$. Baraglia and Konno [BK19] use family Seiberg-Witten theory.

Theorem 6 (Galatius-Randal-Williams, 2014). *The limit homology*

$$\mathrm{colim}_{n\to\infty} H_k(\mathrm{BDiff}_{\frac{1}{2}\partial}(\natural^n S^2 \times S^2 \setminus \mathring{D}^4); \mathbb{Q})$$

is generated by a collection of well-understood characteristic classes, the generalised Miller-Morita-Mumford κ*-classes.*

In particular, this homology is computed. Galatius and Randal-Williams [GRW14] apply a notion of parametrised surgery to the Galatius-Madsen-Tillman-Weiss theorem on the homotopy type of the cobordism category. It is unknown whether, for a fixed k, there is an n such that the homology equals the limit homology.

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Circular Heegaard splittings and the additivity of the Morse-Novikov number

Kenneth L. Baker

Given an oriented link L in S^3, the *Morse-Novikov number* of L is the count $\mathcal{MN}(L)$ of the minimum number of critical points among regular Morse functions $f\colon S^3 - L \to S^1$, see [VPR01] and [Paj06, Definition 14.6.2]. Pajitnov attributes to M. Boileau and C. Weber the question of whether if the Morse-Novikov number is additive on the connected sum of oriented knots, see the beginning of [Paj10, Section 5] and the end of [Paj06, Section 14.6.2]. We show that it is.

Theorem 1. *The Morse-Novikov number is additive: If $K = K_a \# K_b$ is a connected sum of two oriented knots K_a and K_b in S^3, then*

$$\mathcal{MN}(K) = \mathcal{MN}(K_a) + \mathcal{MN}(K_b).$$

Instead of working with circle-valued Morse functions directly, we use the handle-theoretic interpretation of the Morse-Novikov number presented in [God06, §3] and [GP05, §2] that enables the use of techniques from the theory of Heegaard splittings. This approach is rooted in Goda's work on handle numbers of sutured manifolds and Seifert surfaces [God92, God93] and Manjarrez-Gutierrez's work on circular generalized Heegaard splittings and circular thin decompositions [MG09, MG13], and it uses a key proposition of Manjarrez-Gutierrez & Eudave-Munoz about connected sums [EMnMG12].

On our way to Theorem 1, we establish the following:

Lemma 2. *The handle number of a knot is realized by an incompressible Seifert surface.*

Furthermore, our proof of Theorem 1 also applies in the context of cabling.

Theorem 3. *The Morse-Novikov number is unchanged under cabling: If $K_{p,q}$ is the (p, q)–cable of a knot K for coprime integers p and q, then*

$$\mathcal{MN}(K_{p,q}) = \mathcal{MN}(K).$$

The driving observation for the proofs is that a related count called the *handle index* is (a) unchanged by weak reductions and amalgamations and (b) equivalent to the handle number of a generalized Heegaard splitting when the compression bodies have no handlebody components. Figure 1 schmatically conveys the proof of Lemma 2.

While we state our results for knots in S^3, Lemma 2, Theorem 1, and Theorem 3 all can be immediately generalized to null-homologous knots in rational homology spheres. With a little more attention they should also generalize to rationally null-homologous knots in other orientable 3–manifolds.

A few questions arise.

Question 4. *Is the handle number of a knot always realized by a minimal genus Seifert surface?*

Goda shows this is so for all small crossing knots [God93]. However he also shows that there are knots with other minimal genus Seifert surfaces that do not realize the handle number of the knot.

One may also wonder whether Theorem 1 might be proven more simply, albeit possibly more indirectly, by expressing the Morse-Novikov number in terms of other established knot invariants.

Question 5. *Can the Morse-Novikov number of a knot be expressed in terms of other established knot invariants?*

Question 6. *What knot invariants detect fibered knots and are additive under connected sum?*

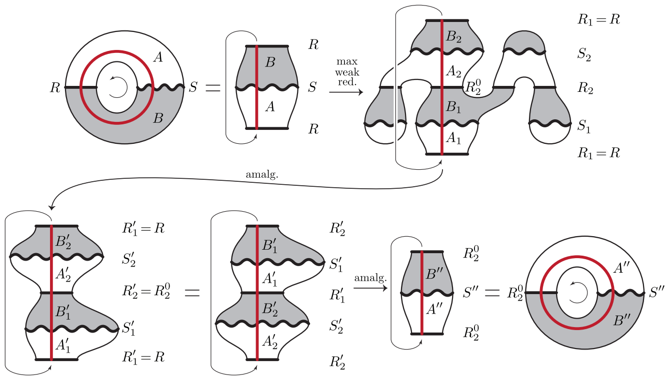

FIGURE 1. A maximal weak reduction followed by amalgamations transforms a circular Heegaard splitting (M, R, S) with R compressible into a circular Heegaard splitting (M, R_2^0, S'') with R_2^0 incompressible and the same handle number. The central loops/arcs represent the toroidal/annular sutures and the vertical boundary of the compression bodies.

We note that the log of the rank of the knot Floer Homology of a knot K in the highest non-zero grading, $LR(K) = \log \mathrm{rkHFK(K, g(K))}$, is both additive on connected sums [OS04, Theorem 7.1] and equals zero precisely for fibered knots [Ghi08, Ni07]. However, LR is distinct from MN; neither is a function of the other.

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Taut foliations and fundamental group left orders in Heegaard genus 2

Sarah Rasmussen

In addition to making predictions about Heegaard Floer homology, the L-space conjecture [1, 3] implies that any closed oriented irreducible 3-manifold M admits a cooriented taut foliation if and only if its fundamental group $\pi_1(M)$ admits a left multiplication-invariant order. Dunfield has found particularly strong numerical evidence for the "if" direction of this implication [2], but until now, the only known examples in which one could associate a taut foliation on M to a left order on $\pi_1(M)$ were those in which the left order was originally obtained from the taut foliation in question.

The purpose of this communication is to describe a new construction by which, under certain hypotheses, I can build a cooriented taut folation on M from a left order on $\pi_1(M)$. Unlike all taut foliation constructions on hyperbolic manifolds in the literature, this novel method makes no recourse to branched surfaces. It is particularly successful for M of Heegaard genus 2, but it is hoped that higher Heegaard genus versions of this method will prove fruitful as well. In addition to slightly stronger but more technical results, I can prove the following.

Theorem 2. *Suppose M is a closed oriented irreducible 3-manifold, equipped with a left order $>_{\mathrm{L}}$ on $\pi_1(M)$. If a minimal intersection Heegaard diagram $\mathcal{H}$ for M is of genus 2 and its associated fundamental group presentation has no subwords that are trivial in $\pi_1(M)$, then one can use $\mathcal{H}$ and $>_{\mathrm{L}}$ to build a cooriented taut foliation on M with continuous tangent distribution, additionally certifying that M is a non-L-space.*

The Heegaard-diagrammatic tools developed in this construction also provide new methods for analysing fundamental group left orders, and this holds for arbitrary Heegaard genus.

A *left order* on a group G is a strict total order $>_{\mathrm{L}}$ on G such that $g_1 >_{\mathrm{L}} g_2$ if and only if $hg_1 >_{\mathrm{L}} hg_2$, for any $g_1, g_2, h \in G$. If G admits a left order $>_{\mathrm{L}}$ and is also countable, one can associate a *dynamically realised real line action* $\rho : G \to \mathrm{Homeo}_+\mathbb{R}$ to $>_{\mathrm{L}}$, such that $\rho(g)(0) >_{\mathrm{L}} \rho(h)(0)$ if and only if $g >_{\mathrm{L}} h$. Such ρ is unique up to semiconjugacy and is an important ingredient in our construction.

Adopting the convention that all foliations are cooriented, a foliation $\mathcal{F}$ on a closed oriented 3-manifold M is called *taut* if it is of codimension 1 and any leaf of $\mathcal{F}$ intersects a *closed transversal*, i.e., a simple closed curve which has only transverse intersections with leaves of $\mathcal{F}$.

One key ingredient in our construction is the classical notion of the *complete transversely foliated bundle*. Given oriented manifolds B and F with B closed, and a representation $\rho : \pi_1(B) \to \mathrm{Homeo}_+F$, there is a unique (up to conjugacy) complete transversely foliated bundle $\pi : E_\rho \to B$ with foliation $\mathcal{F}_\rho$ and holonomy representation ρ, constructed by taking

$$E_\rho := (\widetilde{B} \times F)/(x,t) \sim (x \cdot g, \rho(g^{-1})(t)) \text{ for all } g \in \pi_1(B), \tag{1}$$

$$\mathcal{F}_\rho := \textstyle\coprod_{t \in F} \widetilde{B} \times \{t\}/\sim \ , \tag{2}$$

where $\widetilde{B}$ is the universal cover of B.

Suppose that $F = \mathbb{R}$, $B = M$ is a closed, oriented, irreducible 3-manifold, and $\pi_1(M)$ admits a left order $<_{\mathrm{L}}$, to which we associate a dynamically-realised real line action $\rho' : \pi_1(M) \to \mathrm{Homeo}_+\mathbb{R}$. Since any co-oriented $\mathbb{R}$-bundle is necessarily trivial, the complete transversely foliated bundle $E_{\rho'}$ is homeomorphic to $M \times \mathbb{R}$. If $E_{\rho'}$ admits an embedding $f : M \to E_{\rho'}$ transverse to $\mathcal{F}_{\rho'}$, then the induced foliation $\mathcal{F}_{\rho'}|_{f(M)}$ on M is of codimension 1 and has no compact leaves, which one can show implies tautness. Unfortunately, it is not known what dynamical or group theoretic properties, if any, ρ' must satisfy in order for such f to exist, nor is it known how to construct such f if such properties are satisfied.

We therefore pursue the following modification of the above approach. Suppose $M = U_\alpha \cup_\Sigma U_\beta$ is a Heegaard splitting for M, for some closed oriented Heegaard surface Σ of genus g. Let $\iota : \Sigma \hookrightarrow M$ be the associated embedding of Σ in M, and pull the above $\rho' : \pi_1(M) \to \mathrm{Homeo}_+\mathbb{R}$ back to the representation

$$\rho := \rho' \circ \iota_* : \pi_1(\Sigma) \to \mathrm{Homeo}_+\mathbb{R}. \tag{3}$$

The complete transversely foliated $\mathbb{R}$-bundle $\pi : E_\rho \to \Sigma$ with holonomy representation ρ and associated transverse foliation $\mathcal{F}_\rho$ is now a foliated 3-manifold, with $E_\rho \cong \Sigma \times \mathbb{R}$. Although $\Sigma \times \mathbb{R}$ is not the same manifold as M, it does occur as the normal bundle to Σ in M, and it is from this point of view that we use $\mathcal{F}_\rho$ to build a taut foliation on M.

Our construction adopts the following basic outline:

(1) construct certain simple, generically boundary-transverse, foliations on the compression bodies U_α and U_β;
(2) use the transverse foliation $\mathcal{F}_\rho$ on E_ρ to interpolate between these two compression-body foliations;
(3) perform surgeries or isotopies to cancel or smooth out singularities created by gluing these compression body foliations to the transverse foliation.

To accomplish this, we choose certain global sections $s_\alpha, s_\beta : \Sigma \hookrightarrow E_\rho$ which intersect each other in at most finitely many points, and we decompose E_ρ along these sections into three components,

$$E^\alpha \amalg E^0 \amalg E^\beta := E_\rho \setminus (s_\alpha(\Sigma) \cup s_\beta(\Sigma)), \text{ with } E_\rho = \overline{E^\alpha} \cup_{s_\alpha(\Sigma)} \overline{E^0} \cup_{s_\beta(\Sigma)} \overline{E^\beta}, \tag{4}$$

such that E^α is below E^0 and E^0 is below E^β, with respect to the natural ordering on $\mathbb{R}$ in $\Sigma \times \mathbb{R} \cong E_\rho$. We further demand that the induced singular foliation $\mathcal{F}_{s_\alpha} := \mathcal{F}_\rho|_{s_\alpha(\Sigma)}$ on $s_\alpha(\Sigma) = \partial U_\alpha$ has only isolated singularities, and extends to a singular foliation, say $\mathcal{F}_\alpha$, on U_α, which restricts to a (nonsingular) foliation by disks on the interior of U_α. We impose an analogous condition for β. Then since

$$M \cong U_\alpha \cup_{s_\alpha(\Sigma)} \overline{E^0} \cup_{s_\beta(\Sigma)} U_\beta, \tag{5}$$

we can glue together the foliations $\mathcal{F}_\alpha$ on U_α, $\mathcal{F}_0 := \mathcal{F}_\rho|_{\overline{E^0}}$ on $\overline{E^0}$, and $\mathcal{F}_\beta$ on U_β.

If $s_\alpha(\Sigma) \cap s_\beta(\Sigma) = \emptyset$, then the resulting glued up foliation $\mathcal{F}' = \mathcal{F}_\alpha \cup \mathcal{F}_0 \cup \mathcal{F}_\beta$ will necessarily contain singularities along some of the singular points of $\mathcal{F}_{s_\alpha}$ and $\mathcal{F}_{s_\beta}$, and we can attempt to surger $\mathcal{F}'$ to cancel pairs of singularities. In practice, we instead move these cancelling pairs on top of each other during our initial selection of s_α and s_β, so that all potential singularities of $\mathcal{F}'$ lie in the finite set $s_\alpha(\Sigma) \cap s_\beta(\Sigma)$. For suitable choices of s_α and s_β, this makes $\mathcal{F}'$ nonsingular as a topological foliation. It is then straightforward to isotop $\mathcal{F}'$ to a foliation $\mathcal{F}$ with continuous tangent distribution, and we show that such $\mathcal{F}$ is taut.

The primary technical challenge of this construction is to find global sections $s_\alpha, s_\beta : \Sigma \to E_\rho$ that satisfy the needed conditions. To construct s_α, we first build an explicit model of $\mathcal{F}_\rho$ on $\Sigma \times \mathbb{R}$ by combining suspension foliations along annuli whose cores are boundaries of compressing disks for U_α. We then take $s_\alpha(\Sigma) = \pi_\alpha^{-1}(0) \subset E_\rho$, for $\pi_\alpha : E_\rho \to \mathbb{R}$ the horizontal projection corresponding to the identification $E_\rho \to \Sigma \times \mathbb{R}$ determined by our foliation model. Since the section s_β is constructed similarly, the height comparison of these two sections is governed by a type of decorated Heegaard diagram whose underlying structure is an ordinary Heegaard diagram.

Given a pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, x_0)$ for the Heegaard splitting $M = U_\alpha \cup_\Sigma U_\beta$, we associate an element of $\pi_1(M, x_0)$ to each component $c \in \pi_0(\Sigma \setminus \coprod_{i=1}^{g} \alpha_i \cup \beta_i)$ corresponding to the based oriented knot associated to x_0 and a point in c. Studying the left (and dual right) order of these group elements informs the adaptations needed to build our more intricate decorated Heegaard diagram. For example, there are basepoint-invariant notions of minimal or maximal components of $\Sigma \setminus \coprod_{i=1}^{g} \alpha_i \cup \beta_i$ with respect to the fundamental group right order.

This Heegaard diagrammatic perspective is also a useful tool in its own right for analysing fundamental group left (and right) orders. In the case of genus 2, this analysis simplifies considerably, enabling us to prove the earlier-stated theorem.

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Quasimorphisms on homeomorphism groups

Richard Webb

(joint work with Jonathan Bowden, Sebastian Hensel)

Let us write $\mathrm{Diff}(M)$ for the group of orientation-preserving, compactly-supported diffeomorphisms $M \to M$ of an orientable manifold M. This is a topological group and may have infinitely many connected components. The component containing the identity is the group $\mathrm{Diff}_0(M)$ which consists of those diffeomorphisms $M \to M$ that are isotopic to the identity.

The *support* of a diffeomorphism is the closure of the set of points that are not fixed. A *disk-supported* map is a diffeomorphism whose support lies within an open disk of M. The fragmentation lemma states that the disk-supported maps generate $\mathrm{Diff}_0(M)$. With respect to this generating set the word length or *fragmentation norm* $\mathrm{frag}(f)$ of $f \in \mathrm{Diff}_0(M)$ is the minimum length of a product of disk-supported maps which equals f, and by convention, frag of the identity is zero. It is straightforward to see that the fragmentation norm endows $\mathrm{Diff}_0(M)$ with a metric which is both left and right invariant, or equivalently, the fragmentation norm is a conjugation-invariant norm. The same applies for homeomorphisms and $\mathrm{Homeo}_0(M)$. In fact, a classic trick shows that any disk-supported homeomorphism is a commutator of disk-supported homeomorphisms. This shows that $\mathrm{Homeo}_0(M)$ is a perfect group, and that the commutator length satisfies $\text{cl}(f) \leq \text{frag}(f)$. Commutator length is another example of a conjugation-invariant norm. But this classic trick does not seem to work for diffeomorphisms.

In their insightful paper Burago–Ivanov–Polterovich [BIP08] showed that for *any* conjugation-invariant norm $\rho\colon \text{Diff}_0(M) \to \mathbb{R}$ we have that there exists $C > 0$ such that $\rho(f) \leq C\text{frag}(f)$. They also show that one can take $C = 2$ when $\rho = \text{cl}$. This uses the fact that $\text{Diff}_0(M)$ is perfect, which is a deep theorem of Thurston. This is an interesting relationship between an algebraically-defined function and a topologically-defined function ... but these functions are possibly bounded!

Burago–Ivanov–Polterovich then go on to show that the fragmentation norm (and hence all conjugation-invariant norms) on $\text{Diff}_0(M)$ are bounded whenever M is what they call a *portable* manifold, this includes for example any open disk, open handlebody, or $N \times (0, 1)$ e.g. an open annulus. They also prove the theorem when M is the n-sphere ($n \geq 1$), or indeed when M is any closed, orientable 3-manifold. Tsuboi [Tsu08, Tsu12] extended this result to all dimensions at least 5 and for 4-manifolds without 2-handles in a handlebody decomposition. This leaves open the general case of closed, orientable manifolds of dimension 2 or dimension 4: is cl bounded on $\text{Diff}_0(M)$ i.e. is it uniformly perfect? We prove that in dimension 2 a new phenomenon occurs [BHW].

Theorem 1. *Let $S = S_g$ be the closed, orientable surface of genus $g \geq 1$ then* $\text{Homeo}_0(S)$ *and* $\text{Diff}_0(S)$ *are not uniformly perfect and* frag *is unbounded.*

The purpose of this talk is to explain how we prove this theorem. We find unbounded quasimorphisms on our groups. The theory of quasimorphisms is an important tool for showing certain groups are not uniformly perfect. Bestvina–Fujiwara [BF02] introduced a criterion, which we shall call (BF), that enables one to construct "many" unbounded quasimorphisms on a group. The criterion requires an action by isometries on a hyperbolic space with two "independent" loxodromic elements. Using their criterion they showed that any subgroup of the mapping class group is either virtually abelian or satisfies (BF).

We would like to apply (BF) for surface homeomorphism groups. Fortunately there already is a hyperbolic space associated to a surface S. Masur and Minsky [MM99] showed that the curve graph $\mathcal{C}(S)$ is hyperbolic. Unfortunately $\text{Homeo}_0(S)$ and $\text{Diff}_0(S)$ act trivially on $\mathcal{C}(S)$ because the vertices are isotopy classes of simple closed curves. We instead study $\mathcal{C}^\dagger(S)$ where vertices correspond to simple closed curves (not their isotopy classes!) and edges connect two vertices if the curves have empty intersection. We explain how to prove the following, which in turn proves Theorem 1 [BHW].

Theorem 2. *Let $S = S_g$ be the closed, orientable surface of genus $g \geq 1$. Then $\mathcal{C}^\dagger(S)$ is hyperbolic and the action of* $\text{Homeo}_0(S)$ *on $\mathcal{C}^\dagger(S)$ satisfies* (BF).

It would be interesting to know whether this is the first example of a simple group that acts on a hyperbolic space satisfying (BF). Of course there are simple groups that act on hyperbolic spaces for example $PSL(2, \mathbb{R})$ but this group is uniformly perfect so cannot satisfy (BF).

Theorem 2 applies also to the smooth category. In fact we are able to define our quasimorphisms on the entire diffeomorphism group such that it is unbounded on each component.

These are the first known examples where $\mathrm{Diff}_0(M)$ is not uniformly perfect and M is a closed or open manifold. We believe our methods will determine precisely which orientable surfaces of finite type have this property. However little is known in dimension 4.

Question 3. *Are there* 4*-dimensional closed, orientable, smooth manifolds* M *with* $\mathrm{Diff}_0(M)$ *not uniformly perfect?*

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Symmetries and the Character Variety

Kate Petersen

(joint work with Jay Leach)

For a finite volume hyperbolic 3-manifold, M, the $\mathrm{SL}_2(\mathbb{C})$ character variety of M is

$$X(M) = \{\chi_\rho \mid \rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})\}$$

where the character $\chi_\rho : \pi_1(M) \to \mathbb{C}$ is defined by $\chi_\rho(\gamma) = \mathrm{trace}(\rho(\gamma))$ for all $\gamma \in \pi_1(M)$. The character variety is a complex algebraic set with defining coefficients in $\mathbb{Q}$, and its isomorphism type is independent of the presentation of $\pi_1(M)$. An irreducible component of $X(M)$ is called a canonical component and written $X_0(M)$ if it contains the character of a discrete and faithful representation. The complex dimension of $X_0(M)$ equals the number of cusps of M [7].

We call an orientation preserving homeomorphism $\sigma : M \to M$ a symmetry of M. A symmetry σ naturally acts on $\pi_1(M)$ by sending a loop class to its image under σ; we write σ_* for this action. Any symmetry σ of M acts on $X(M)$ as well. This action is given by $\hat{\sigma} : X(M) \to X(M)$ where $\hat{\sigma}(\chi_\rho) = \chi_{\sigma_{\rho \circ \sigma_*}}$. That is, for $\gamma \in \pi_1(M)$,

$$\hat{\sigma}(\chi_\rho)(\gamma) = \chi_{\sigma_{\rho \circ \sigma_*}}(\gamma) = \mathrm{trace}(\rho(\sigma_*(\gamma))).$$

This action $\hat{\sigma}$ is trivial on $X(M)$ if σ fixes unoriented free homotopy classes of loops in M. In this case, $\rho(\gamma)$ is sent to a matrix that is conjugate to $\rho(\gamma)^{\pm 1}$ under the induced action, and therefore has the same trace. However, this is not a sufficient condition for $\hat{\sigma}$ to act trivially on $X(M)$ [1].

If M is a knot complement in S^3, then σ extends to a symmetry of any Dehn filling of M. It follows that $\hat{\sigma}$ fixes a discrete and faithful representation, and all representations associated to hyperbolic Dehn filling of $\pi_1(M)$. The corresponding characters form a Zariski dense set in the curve $X_0(M)$, and therefore $\hat{\sigma}$ fixes $X_0(M)$ point-wise.

Culler and Shalen [2] showed that if $\pi_1(M)$ acts non-trivially and without inversions on a tree T, then there is an essential surface in M dual to this action. Moreover, they demonstrated how to construct such actions on trees from $X(M)$, and specifically from ideal points of $X(M)$.

Our main result shows that symmetries affect the detection of surfaces on a canonical component. We state the consequence for boundary slopes, where G is the symmetry group of M. We call a boundary slope symmetric if it is the slope of a surface preserved by the symmetry.

Theorem 1. *Let M be a finite volume, orientable, hyperbolic 3-manifold with a single cusp, such that the orbifold quotient M/G has a flexible cusp. Any boundary slope detected on $X_0(M)$ is a symmetric slope.*

One key ingredient in our proof is the following extension of Culler and Shalen's theory to orbifolds.

Proposition 2. *Let Q be a compact, orientable, irreducible 3-orbifold. If $\pi_1^{orb}(Q)$ acts non-trivially and without inversions on a tree T, then there exists an essential 2-suborbifold F in Q dual to this action.*

The association between $Q = M/G$ and M is made through Long and Reid's result [4] which implies that when Q has a flexible cusp, then $X_0(M)$ and $X_0(Q)$ are birationally equivalent.

As an application, we consider the symmetric double twist knots K_n, which in two-bridge notation are associated to the rational number $2n/(4n^2 - 1)$. Let M_n be the complement of K_n in S^3. These knots have a symmetry corresponding to turning the four-plat upside down which acts non-trivially on $X(M_n)$. In fact [5],

$$X(M_n) = X_{red}(M_n) \cup X_0(M_n) \cup X_1(M_n)$$

where $X_{red}(M_n)$ is the component of $X(M_n)$ containing characters of reducible representations. By work of Hatcher and Thurston [3] the boundary slopes of M_n are 0, $-8n + 2$ and $-4n$. The slopes 0 and $-8n + 2$ are symmetric slopes of M_n whereas $-4n$ is not symmetric. All three slopes are detected by ideal points of $X(M_n)$ by work of Ohtsuki [6]. However, as a consequence of Theorem 1 we have the following.

Corollary 3. *The ideal points on the canonical component $X_0(K_n)$ detect only the slopes 0 and $-8n + 2$, and not $-4n$.*

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Dehn fillings on knot groups

Kimihiko Motegi

(joint work with Tetsuya Ito, Masakazu Teragaito)

Let K be a nontrivial knot in S^3 with its exterior $E(K)$. We denote the knot group $\pi_1(E(K))$ by $G(K)$. By the loop theorem the inclusion map $i : \partial E(K) \to E(K)$ induces a monomorphism $i_* : \pi_1(\partial E(K)) \to G(K)$, thence we have a peripheral subgroup $P(K) = i_*(\pi_1(\partial E(K))) \subset G(K)$. A *slope element* in $G(K)$ is a primitive element γ in $P(K) \cong \mathbb{Z} \oplus \mathbb{Z}$, which is represented by an oriented simple closed curve in $\partial E(K)$. Denote by $\langle\langle \gamma \rangle\rangle$ the normal closure of γ in $G(K)$. Using the standard meridian-longitude pair (μ, λ) of K, each slope element γ is expressed as $\mu^p\lambda^q$ for some relatively prime integers p, q. As usual we use the term *slope* to mean the isotopy class of an unoriented simple closed curve in $\partial E(K)$. Two slope elements γ and its inverse γ^{-1} represent the same slope which is identified with $r = p/q \in \mathbb{Q} \cup \{\infty\}$. Since $\langle\langle \gamma \rangle\rangle = \langle\langle \gamma^{-1} \rangle\rangle$, it is convenient to denote them by $\langle\langle r \rangle\rangle$. Thus each slope defines the normal subgroup $\langle\langle r \rangle\rangle \subset G(K)$, which will be referred to as the *normal closure of the slope* r for simplicity.

A normal closure $\langle\langle r \rangle\rangle$ of a slope r of K naturally arises via Dehn filling on $E(K)$. Denote by $K(r)$ the 3-manifold obtained by r–Dehn filling of $E(K)$. Then we have the following short exact sequence which relates $G(K)$, $\langle\langle r \rangle\rangle$ and $\pi_1(K(r))$.

$$1 \to \langle\langle r \rangle\rangle \to G(K) \xrightarrow{p_r} G(K)/\langle\langle r \rangle\rangle = \pi_1(K(r)) \to 1.$$

In the talk we propose to study Dehn fillings from a group theoretic viewpoint.

For a given nontrivial element $g \in G(K)$, how many Dehn fillings trivialize g? To make precise define the function

$$\mathcal{D} : G(K) - \{1\} \to \mathbb{Z}_{\geq 0} \cup \{\infty\} \quad \text{by}$$

$$\mathcal{D}(g) = \sharp\{r \in \mathbb{Q} \mid p_r(g) = 1 \text{ in } \pi_1(K(r))\} = \sharp\{r \in \mathbb{Q} \mid g \in \langle\langle r \rangle\rangle\}.$$

If g is nontrivial and there are infinitely many such slopes, then $\mathcal{D}(g)$ is defined to be ∞. In this context Property P [4] says that $\langle\langle r \rangle\rangle = \langle\langle \infty \rangle\rangle = G(K)$ if and only if $r = \infty$, and $\mathcal{D}(\mu) = 0$ for a meridian μ of K.

For simplicity, in what follows, we focus on hyperbolic knots in S^3. Then we have the following results [1, 2]:

(1) $\langle\langle r_1 \rangle\rangle = \langle\langle r_2 \rangle\rangle$ if and only if $r_1 = r_2$.

(2) For any infinite family of slopes $\{r_1, r_2, \dots\}$, the intersection $\langle\langle r_1 \rangle\rangle \cap \langle\langle r_2 \rangle\rangle \cap \cdots = \{1\}$. This implies that $\mathcal{D}(g) < \infty$ for any nontrivial element $g \in G(K)$.

(3) For any finite family of slopes $\{r_1, \dots, r_n\}$, the intersection $\langle\langle r_1 \rangle\rangle \cap \cdots \cap \langle\langle r_n \rangle\rangle$ is an infinite subgroup. More precisely $\langle\langle r_1 \rangle\rangle \cap \cdots \cap \langle\langle r_n \rangle\rangle$ is finitely generated if and only if all the r_i are finite surgery slopes. This implies that for a given $N > 0$, there are infinitely many elements $g \in G(K)$ such that $\mathcal{D}(g) \geq N$.

We would like to ask:

Question. Which value is a "generic" for $\mathcal{D}(g)$ over $g \in G(K)$?

In spite of (3), it might be reasonable to expect that "most" elements $g \in G(K)$ satisfies $\mathcal{D}(g) = 0$, i.e. g never becomes trivial by any nontrivial Dehn filling of $E(K)$. On the other hand, except for a meridian, and a pseudo-meridian, i.e. a nontrivial element which is not conjugate to a meridian, but its normal closure coincides with $G(K)$, we have no explicit examples of elements $g \in G(K)$ with $\mathcal{D}(g) = 0$. Recall that a meridian or a pseudo-meridian is a homological generator of $H_1(E(K)) \cong \mathbb{Z}$.

As the first step, we prove the following results.

Theorem 1. *Let K be a hyperbolic knot in S^3. Then there exist infinitely many, mutually non-conjugate elements $g \in [G(K), G(K)]$ such that $\mathcal{D}(g) = 0$ if and only if K has no cyclic surgeries.*

Theorem 2. *Let K be a hyperbolic knot in S^3. There exist infinitely many, mutually non-conjugate elements $g \in G(K)$ which represent a generator of $H_1(E(K))$ and $\mathcal{D}(g) = 0$.*

Furthermore, if we require that K has no finite surgeries, we have:

Theorem 3. *Let K be a hyperbolic knot without finite surgeries. Then in each homology class $\alpha \in H_1(E(K))$, there are infinitely many, mutually non-conjugate elements $g \in G(K)$ such that g represents α and $\mathcal{D}(g) = 0$.*

Following Theorems 1 and 3, we have infinitely many, mutually non-conjugate elements $g \in G(K)$ which are not pseudo-meridians, but $\mathcal{D}(g) = 0$. On the other hand, nontrivial elements $g \in G(K)$ given in Theorem 2 may be pseudo-meridians.

Question. Let K be a hyperbolic knot and $g \in G(K)$ a nontrivial element which represents a generator of $H_1(E(K))$ and satisfies $\mathcal{D}(g) = 0$. Then is g a pseudo-meridian?

Obviously if K has a finite surgery slope s, there does not exist a subgroup H such that $H \cap (\bigcup_{r\in\mathbb{Q}}\langle\langle r\rangle\rangle) = \{1\}$, because that for any element $g \in G(K) - \bigcup_{r\in\mathbb{Q}}\langle\langle r\rangle\rangle$ we have $g^n \in \langle\langle s\rangle\rangle$ for some integer $n > 0$.

Our results immediately imply that if K is a hyperbolic knot which admits neither finite surgeries nor reducing surgeries, then there are infinitely many cyclic subgroups C_i such that

$$C_i \cap (\bigcup_{r\in\mathbb{Q}}\langle\langle r\rangle\rangle) = \{1\}.$$

We would like to close the abstract with the following question.

Question. Let K be a hyperbolic knot which admits no finite surgery. Then does there exist a subgroup $H \subset G(K)$ which is a rank two free group and satisfies

$$H \cap (\bigcup_{r\in\mathbb{Q}}\langle\langle r\rangle\rangle) = \{1\}?$$

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Special covers of alternating links

David Futer

(joint work with Edgar A. Bering IV)

The spectacular resolution of Thurston's virtual conjectures by Agol [2] and Wise [9] shines a bright light on the power of cube complexes in studying 3–manifolds and their fundamental groups. In particular, Agol [1] and Przytycki–Wise [8] proved that every non-positively curved 3–manifold M is virtually fibered by first proving M is *cubulated* (homotopy equivalent to a non-positively curved cube complex X) and then proving M is *virtually special* (X has a finite cover whose hyperplanes are free of several pathologies). Ever since the appearance of these results, mathematicians have asked for a quantitative version: What is the degree of a fibered cover of M? What is the degree of a cover $\widehat{M}$ with $b_1(\widehat{M}) \geq 2$? Or $b_1(\widehat{M}) \geq 10$? For instance, see Agol [3, Question 11.4].

Quantifying virtual fibering appears to be very hard. Quantifying virtual specialness or the growth of betti numbers is more tractable, at least in particular families of examples. For instance, given an alternating link diagram $D = D(K)$, a classical construction introduced by Dehn [7] produces a square complex X_D that is homotopy equivalent to $S^3 \setminus K$. This complex has two vertices (one on each side of the projection plane), one edge through each region of $D(K)$, and one square for each crossing of $D(K)$. See Figure 1 for an example.

In recent joint work [4], Bering and I explicitly construct a special cover of X_D, with control on the covering degree.

Theorem 1. *Let $D = D(K)$ be a prime, alternating link diagram with c crossings. Then the Dehn complex X_D has a special cover $\widehat{X}_D$ of degree at most $12(c-1)!$.*

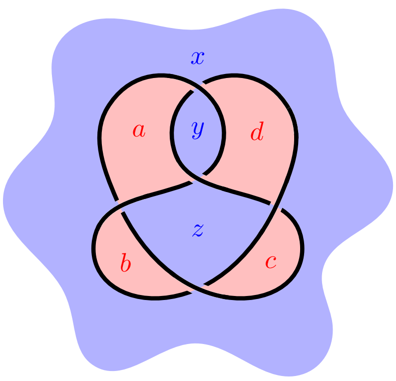
Beyond bounding the degree of the cover $\widehat{X}_D$, we actually *construct* the cover. As a consequence, we are able to gain structural understanding of surfaces in small-degree covers of the link complement $S^3 \setminus K$. For instance, we get the following effective version of a theorem of Cooper, Long, and Reid [5].

Corollary 2. *Let $D = D(K)$ be a prime, alternating link diagram with $c \geq 3$ crossings. Then $S^3 \setminus K$ has a cover $\widehat{M}$ of degree at most $12(c-1)!$, which contains four disjoint, orientable surfaces whose union does not separate $\widehat{M}$. Recording intersections of a loop with these surfaces yields a surjection $\pi_1(\widehat{M}) \to F_4$.*

Given that the fundamental group of the cover $\widehat{X}_D$ embeds into a right-angled Coxeter group with a simple description, we also learn something about linear representations of the original knot group.

Corollary 3. *Let $D = D(K)$ be a prime, alternating link diagram with c crossings. Then $S^3 \setminus K$ embeds into $SL(m, \mathbb{Z})$, where $m \leq 288((c-1)!)^2$.*

Finally, we obtain a quantification of residual finiteness.



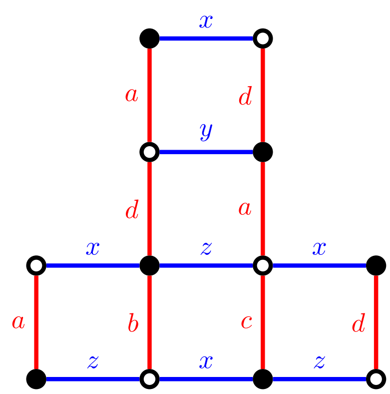


FIGURE 1. Left: an alternating diagram $D = D(K)$. Right: the Dehn complex X_D corresponding to D. Labels on the edges of X_D correspond to regions of $D(K)$. Edges with the same label are identified. The hyperplanes of this complex correspond to the checkerboard surfaces shown on the left.

Corollary 4. *Let $D = D(K)$ be a prime, alternating link diagram with c crossings. Let $\sigma \subset S^3 \setminus K$ be a closed curve that intersects the checkerboard surfaces of $D(K)$ a total of n times. Then there is a cover $M_\sigma \to S^3 \setminus K$ of degree at most $(n+1) \cdot 12(c-1)!$ such that σ does not lift to M_σ.*

Our next aim in this investigation is to strengthen Corollary 4 to quantify the separability of non-trivial finitely generated subgroups. That is: given a subgroup $H \subset \pi_1(S^3 \setminus K)$ and an element $\sigma \notin H$, can one bound the index of a subgroup $G \subset \pi_1(S^3 \setminus K)$ such that $H \subset G$ but $\sigma \notin G$? Geometrically, this is equivalent to bounding the degree of a cover of $M = S^3 \setminus K$ where some immersion of a compact manifold lifts to be an embedding. So far, we can quantify the separability of Abelian subgroups.

It is reasonable to ask whether the bound of Theorem 1 is anywhere close to optimal. We can give lower bounds on the degree of a special cover *of the Dehn complex*, but we cannot rule out the possibility that some small-degree cover of $M = S^3 \setminus K$ is homotopy equivalent to some other, seemingly unrelated, cube complex. This prompts the following question.

Problem 5. *Find an algorithm that takes as input a triangulated (hyperbolic?) 3–manifold M and decides whether M is homotopy equivalent to a special cube complex. Equivalently, the algorithm should decide whether $\pi_1(M)$ embeds into a right-angled Artin group.*

In dimension 2, this problem was solved by Crisp and Wiest [6]. In dimension 3, normal surface theory should be helpful. For instance, it is not hard to show that any cubulation of a closed, hyperbolic 3–manifold M can be expressed using immersed normal surfaces in a 1–vertex triangulation of M. What is less clear is how to go in the opposite direction and certify that a collection of immersed normal surfaces provides a wall-space for a proper and cocompact cubulation. Still harder is the question of how far one needs to search for a special cubulation via normal surfaces before deciding that it does not exist.

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A geometric approach to the embedding calculus

Danica Kosanović

The embedding calculus of Goodwillie and Weiss (see [1] for details and references) is a certain homotopy theoretic technique for studying spaces of embeddings. It gives a new language in which many of the results of Whitney, Haefliger, Dax and others can be recast, but also provides powerful new tools and results.

The embedding calculus

The idea is to use the fact that any manifold Q has a simple local description to approximate the space[1] $Emb(Q, M)$ of embeddings of Q into M. This is done via spaces $T_n(Q, M) := T_n Emb(Q, M)$ that fit into a Taylor tower[2] $\cdots \to T_n(Q, M) \to T_{n-1}(Q, M) \to \cdots \to T_1(Q, M)$. The definition of $T_n(Q, M)$ is as a certain *homotopy limit*, which extrapolates from the data of n disjointly embedded disks in Q. Moreover, there are natural *evaluation maps* $ev_n \colon Emb(Q, M) \to T_n(Q, M)$ approximating the domain 'better' as n increases.

As an example, $T_1(Q, M)$ is the coarsest approximation and is equivalent to the space of immersions $Imm(Q, M)$. Since immersions are given by a local condition (on their derivative), the space $Imm(Q, M)$ is the pullback of the diagram $Imm(V_1, M) \to Imm(V_1 \cap V_2, M) \leftarrow Imm(V_2, M)$ for any open cover $Q = V_1 \cup V_2$. Hirsch-Smale theory then shows that *this pullback is also a homotopy pullback*, and precisely this property characterises the *linear* approximation $T_1(Q, M)$.

The more precise way in which ev_n are increasingly better approximations is expressed in the following fundamental theorem of Goodwillie and Klein.

Theorem 1 ([2]). *The map ev_n is $((3 - m) + (n + 1)(m - q - 2))$-connected (isomorphism on homotopy groups below this degree and surjective in it) if*

$$(\dim Q, \dim M) = (q, m) \neq (1, 3).$$

Hence, in codimension at least 3 *the connectivity increases with* n, so the induced map from $Emb(Q, M)$ to the limit of the tower $\lim_n T_n$ is a weak homotopy equivalence. However, even when the codimension is smaller, the tower remains a useful object to study, since the spaces T_n are amenable to the tools of homotopy theory (in analogy to how immersions adhere to the h-principle) and so $\pi_i ev_n$ gives invariants of $\pi_i Emb(Q, M)$, for $i \geq 0$.

[1] We equip the set of smooth embeddings $Emb(Q, M)$ with the Whitney C^∞ topology.
[2] This name is not without reason: many constructions here are in a formal analogy to the Taylor expansion of a function.

Knots in 3-manifolds

One particularly appealing instance of embedding calculus is exactly when Theorem 1 does not apply, namely, the *space of long knots* in M, a 3-manifold with nonempty boundary. More precisely, let

$$Knots(M) := Emb_\partial(I, M) := \{f\colon I \hookrightarrow M \mid f \equiv \mathrm{U} \text{ near } \partial I\},$$

where $I = [0, 1]$ and $\mathrm{U}\colon I \hookrightarrow M$ is some fixed proper embedding. The set of path components $\mathbb{K}(M) := \pi_0(Knots(M))$ consists of isotopy classes of long knots in M which agree with U near thier boundary.

The formula above for the connectivity of ev_n predicts that in this case – for which their techniques were unsuccessful – all ev_n are 0-connected, i.e. surjective on π_0. In my thesis I *confirm this prediction*:

Theorem 2 ([3]). *$\pi_0 ev_n\colon \mathbb{K}(M) \to \pi_0 T_n(I, M)$ is surjective for all $n \geq 1$.*

The geometric calculus

As a corollary, we also prove a part of the following conjecture by Budney, Conant, Scannell and Sinha [4]:

Conjecture 3. *$\pi_0 ev_n\colon \mathbb{K}(I^3) \to \pi_0 T_n(I, I^3)$ is a universal additive Vassiliev invariant of type $\leq n-1$ over $\mathbb{Z}$.*

Here U is the standard unknot and the operation of stacking makes $\mathbb{K}(I^3)$ into an abelian monoid (which is isomorphic to the more commonly used $\pi_0(Emb(S^1, \mathbb{R}^3))$ under connected sum). Equivalently, the conjecture says that $\pi_0 ev_n$ is a homomorphism of monoids (this was proven in [5]), that it is *surjective* and that its kernel consists of knots which are *n-equivalent to the unknot*.

The n-equivalence relation comes from the geometric viewpoint on Vassiliev theory, introduced independently by Gusarov and Habiro [6, 7] in terms of *claspers* and extended by Conant and Teichner [8] in terms of *gropes*. Two knots are n-equivalent $K \sim_n K'$ if there is a sequence of clasper surgeries or (simple capped) grope cobordisms in I^3 of degree n leading from K to K'.

For $n = 1$ this is just a sequence of crossing changes. An example of a grope cobordism of degree $n = 2$ is depicted in Figure 1; the grope removes J_0 from the unknot U and replaces it by the long blue arc (the remaining part of the boundary of the yellow 'punctured torus' $T^2 \setminus int(D^2)$) to give the trefoil.

In general, a grope cobordism of degree n is a certain geometric object built from $n-1$ copies of a punctured torus, by inductively attaching its boundary to a simple closed curve generating the first homology of the object constructed so far; finally, n copies of D^2, called *caps*, are attached at the top. For the induction step one follows the shape of a rooted planar tree with n leaves.

Habiro proved that $\mathbb{K}(I^3)/\sim_n$ is an abelian group and that the projection $\nu_n\colon \mathbb{K}(I^3) \to \mathbb{K}(I^3)/\sim_n$ is a universal additive Vassiliev invariant of type $\leq n-1$. Hence, Conjecture 3 asserts that the evaluation map factors through:

$$\pi_0\overline{ev}_n\colon \quad \mathbb{K}(I^3)/\sim_n \to \pi_0 T_n(I^3) \tag{6}$$

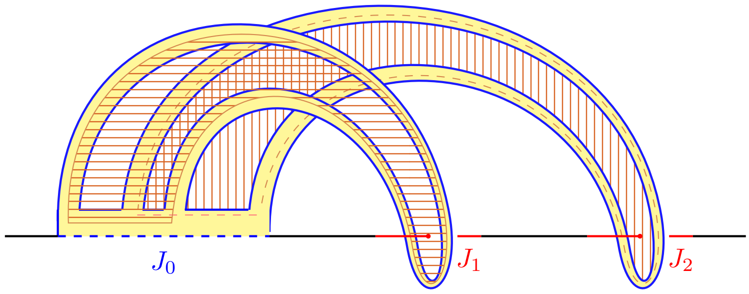
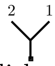


FIGURE 1. A grope cobordism with the underlying tree 

from the unknot to the trefoil. Its caps are the two dashed disks.

and that this map is an isomorphism. The existence of the factorisation was shown in [5], and *our Theorem 2 proves the surjectivity.*

We actually prove a more general result implying this. A large part of [3] is about properties of the *punctured knots model* for the Taylor tower, which has not been extensively studied in the literature, but serves best for geometric purposes. As a consequence of those more general results, and also recent work of Pedro Boavida de Brito and Geoffroy Horel [9], we also obtain the following.

Corollary 3. *The morphism* (6) *is an isomorphism after tensoring both sides*

(1) with $\mathbb{Q}$ *for all* $n \geq 1$*;*
(2) with p*-adic integers* $\mathbb{Z}_p$ *for* $n \leq p+2$.

In other words, we confirm Conjecture 3 rationally and, moreover, show that the Kontsevich integral factors through the embedding calculus tower.

Generalisations of the geometric calculus

In current work in progress with Peter Teichner [10] we investigate to what extend the geometric calculus (using either claspers or gropes) extends to other dimensions and codimensions. This was inspired by the results of Watanabe [11] who uses such constructions to obtain non-trivial classes in homotopy groups of $\mathrm{Diff}(D^4, \partial)$. See also the recent preprint [12] giving non-trivial classes in the fundamental group of the embedding space of circles in a certain 4-manifold.

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From veering triangulations to link spaces and back again

Henry Segerman

(joint work with Saul Schleimer)

A *veering triangulation* is an ideal triangulation of a cusped oriented three-manifold together with some combinatorial data: the triangulation admits a taut angle structure, and there is a colouring of the edges of the triangulation satisfying the following condition: The taut angle structure assigns angles of π to a pair of opposite edges in each tetrahedron, and angle zero to all other edges. For each tetrahedron, the four edges with angle zero are coloured either red or blue, as in Figure 1. The π angle edges do not have colours specified. When the tetrahedra are glued together to form the triangulation, the specified colours must agree at each edge of the triangulation.

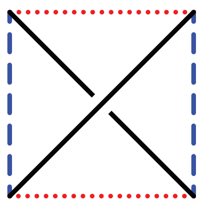

Figure 1. A veering tetrahedron. The zero angles are at the four sides of the square, and the π angles are the diagonals. The zero angle edges of a tetrahedron are coloured either red (dotted) or blue (dashed).

Agol [1] introduced veering triangulations, and showed that for any pseudo-Anosov surface bundle the result of drilling out singular fibers of the stable and unstable foliations admits a veering triangulation canonically associated to the pseudo-Anosov monodromy. Veering triangulations generated by Agol's construction are always layered – Agol asks whether or not non-layered examples exist, and what such would mean.

Hodgson, Rubinstein, Tillmann and I [4] found the first non-layered examples by computer search. In recent work, Giannopolous, Schleimer and I [2] generate the census of all transverse veering triangulations with up to 16 tetrahedra, of which there are 87,047. The first non-layered examples have five tetrahedra, and as the number of tetrahedra increase, the proportion of non-layered veering triangulations also seems to increase – we conjecture that non-layered veering triangulations dominate.

Guéritaud [3] gave an alternative construction of (layered) veering triangulations from pseudo-Anosov surface bundles, by associating maximal rectangles in the singular Sol structure with tetrahedra of the triangulation. In unpublished work, Guéritaud and Agol generalised this construction to any closed manifold equipped with a pseudo-Anosov flow without perfect fits. In work in progress, Schleimer and I build the reverse map, thus showing that veering triangulations are the correct combinatorialisation of pseudo-Anosov flows without perfect fits. As a first step, we construct the *link space* for a given veering triangulation. This is a copy of $\mathbb{R}^2$, equipped with a transverse pair of foliations, from which the Agol-Guéritaud construction recovers the veering triangulation. The link space is analogous to Fenley's orbit space for a pseudo-Anosov flow.

The first major step in building the link space is to construct a nested sequence of *continents* in the universal cover $\widetilde{\mathcal{T}}$ of the triangulation $\mathcal{T}$. These are taut polyhedra made from a finite collection of taut tetrahedra. The sequence exhausts $\widetilde{\mathcal{T}}$. This continental exhaustion implies that $\widetilde{\mathcal{T}}$ is layered, even when $\mathcal{T}$ is not. It also implies that there is a canonical circular ordering of the cusps of $\widetilde{\mathcal{T}}$. This construction requires the veering structure: there are examples of taut angle structures which admit uncountably many different circular orders on the cusps.

Agol's layered construction uses splitting sequences of train tracks, drawn on the two-skeleton of $\mathcal{T}$. These give us a hint for how to proceed. We split these train tracks through the layers of our layering of $\widetilde{\mathcal{T}}$ to generate stable and unstable laminations in the *veering circle* – the completion of the canonical circular ordering on the cusps.

The *link space* is (suppressing some details), the set of pairs of linked leaves, one leaf from each lamination. The laminations become transverse foliations in the link space. It turns out that the set of points of the link space for which both corresponding leaves link a given edge of $\widetilde{\mathcal{T}}$ is a rectangle in the link space. Similarly, each face and each tetrahedron has a corresponding rectangle. We show that a rectangle in the link space is maximal if and only if it corresponds to a tetrahedron rectangle. Following Guéritaud's construction, this allows us to recover the combinatorics of the veering triangulation from the link space.

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Embedding spaces of Hopf links

Rachael Boyd
(joint work with Corey Bregman)

Let us start with a brief history of this topic: The homotopy type of embedding spaces of knots is well understood, due in the most part to work of Hatcher and Budney. In particular Hatcher showed that each connected component of the embedding space of *long knots* is a $K(\pi, 1)$ [8], and Hatcher and McCullough showed that furthermore it is a finite dimensional CW-complex [9]. Hatcher computed the homotopy type of the component corresponding to the unknot, a torus knot, or a hyperbolic knot [8]. Following this Budney gave a recursive formula to compute the homotopy type of any connected component of the space of embeddings of long knots, using the little disks operad and JSJ decomposition of the knot complement [2]. To finish the program, Budney and Cohen provided a formula which gives the relationship between the homotopy type of the long knot component to the homotopy type of the connected component of the corresponding embedding of S^1 in S^3, given by taking one point compactification [3].

Our work focuses on the homotopy type of embedding spaces of links.

Definition 1. Let $\mathrm{SH}_{m,n}$ be the connected component of

$$\mathrm{Emb}(\bigsqcup_{2m+n} S^1, \mathbb{R}^3)/\,\mathrm{Diff}(\bigsqcup_{2m+n} S^1)$$

containing the split union of m Hopf links and n unknots.

Working modulo the diffeomorphism group gives that each component is unoriented and unparametrised, and the components of link are unordered. The notation used here is motivated as follows: the letters in $\mathrm{SH}_{m,n}$ stand for Smooth *H-trivial* link, and in $\mathrm{RH}_{m,n}$ for Round *H-trivial.* The term H-trivial comes from [4], and describes that the link is trivial except for potential Hopf link split sublinks. Since the number of Hopf link or unlink components corresponds different connected components of the embedding space, we use the indices (m, n) to identify a component.

Definition 2. A *round unknot* is an unparametrised embedding $\phi \in \mathrm{SH}_{0,1}$ for which the image of ϕ is a Euclidean circle in some plane $\mathbb{R}^2 \subset \mathbb{R}^3$. A *round Hopf link* is an unparametrised embedding $\psi \in \mathrm{SH}_{1,0}$ such that the image is a Hopf link for which each link component is round unknot. We do not require the radius of the components to be equal or for the disks that they span to intersect at a particular angle; we only require the two round components to link exactly once.

Define $\mathrm{RH}_{m,n}$ to be the subspace of $\mathrm{SH}_{m,n}$ consisting of those embeddings whose images are split links with split sublinks m round Hopf links and n round unknots.

Recall the space $\mathrm{SH}_{0,n}$ is the embedding space of the n-component unlink in $\mathbb{R}^3$ studied by Brendle and Hatcher [1]. They prove that the inclusion map

$$\mathrm{RH}_{0,n} \hookrightarrow \mathrm{SH}_{0,n}$$

is a homotopy equivalence. Our work is a generalisation of this, considering the space $\mathrm{SH}_{m,n}$.

Previously, work on these embedding spaces has focused on computation of the fundamental group, or *motion group* of the link in $\mathbb{R}^3$. Goldsmith [6] showed that $\pi_1(\mathrm{SH}_{1,0}) \cong Q_8$, as a special example of her work on motion groups of torus links, and Damiani and Kamada [4] computed presentations for $\pi_1(\mathrm{RH}_{1,0})$ (again this is isomorphic to Q_8) and $\pi_1(\mathrm{RH}_{1,1})$.

Our result is the first step in a program to study the full homotopy type of embedding spaces of links.

Theorem 3 (B. - Bregman 2020)**.** *The inclusion map* $\mathrm{RH}_{m,n} \hookrightarrow \mathrm{SH}_{m,n}$ *is a homotopy equivalence.*

In particular, this Theorem shows that $\mathrm{SH}_{m,n}$ has the homotopy type of a finite dimensional CW complex, and that it is not a $K(\pi, 1)$, due to the presence of torsion in the fundamental group.

In future work we hope to compute this homotopy type up to extensions in fibration sequences. We believe our proof can be extended to embedding spaces of 'Hopf trees'- these are split links in which each split sublink corresponds to a tree graph in the following manner. Given a graph, place an unknot at each vertex, and link two of these with linking number one if an edge exists between the vertices. Then a link configuration corresponds to a disjoint union of trees, and the fundamental group of the link complement in $\mathbb{R}^3$ is given by the right angled Artin group (RAAG) associated this disjoint union of tree graphs. We conjecture that the motion group of these spaces is isomorphic to the 'symmetric automorphism group' of this right angled Artin group. Note here that the only RAAGs that appear as fundamental groups of 3-manifolds are those corresponding to disjoin unions of tree and triangle graphs, by work of Droms [5].

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The Trace Embedding Lemma

Lisa Piccirillo

There is a rich interplay between the fields of knot theory and 3- and 4-manifold topology. In this talk, I'll discuss the trace embedding lemma, an observation first made by Fox and Milnor in 1957, which has on numerous occasions since provided outsized contributions to both fields by translating an intractible problem in one field to a tractible one in the other.

I will highlight three applications of the lemma:

(1) $\mathbb{R}^4$ admits multiple smooth structures
(2) The Conway knot is not slice
(3) There are PL 4-manifolds homotopy equivalent to S^2 which do not admit any PL embedding of S^2 inducing the homotopy equivalence.

1. The trace embedding Lemma

Definition 3. For an integer n and a knot K in S^3, the *knot trace* $X(K)$ is the 4-manifold obtained by attaching an 0-framed 2-handle to the 4-ball along K.

Definition 4. A knot K is *slice* if it bounds a smooth D^2 embedded in B^4. K is *topologically slice* if it bounds a locally flat D^2 embedded in B^4.

Lemma 1 ([FM66]). *K is slice if and only if $X_0(K)$ embeds smoothly in S^4.*

2. Exotica

Motivating Problem 1. *What is the simplest closed compact orientable 4-manifold which admits multiple smooth structures?*

Theorem 4 ([AP08]). *$\mathbb{C}P^2 \#_2 \overline{\mathbb{C}P^2}$ admits multiple smooth structures.*

Open Problem 1. *Does there exist any closed orientable compact definite 4-manifold with multiple smooth structures?*

Theorem 5. *$\mathbb{R}^4$ admits multiple smooth structures.*

The sketch that follows is folklore; it seems to have originated in Berkeley in the late 1970s.

Proof. Let K be a knot which is topologically slice but not (smoothly) slice. A version of Lemma 1 implies that there is a homeomorphism f embedding $X_0(K)$ in $\mathbb{R}^4$. Define X to be $\mathbb{R}^4 \smallsetminus \nu(f(X_0(K)))$, where ν denotes an open tubular neighborhood. Observe that X is a non-compact topological 4-manifold with boundary.

By work of Freedman-Quinn [FQ90] X admits a smooth structure, we'll call this smooth manifold X_s. Now define $W := X_s \cup X_0(K)$. Since $X_0(K)$ is smooth by construction, work of Moise [Moi52] implies W has a smooth structure. One can write down a homeomorphism $\mathbb{R}^4$ to W by forgetting the smooth structure on X. Observe that $X_0(K)$ embeds smoothly in W. Since K is not (smoothly) slice, Lemma 1 implies that W cannot be diffeomorphic to $\mathbb{R}^4$. □

It is natural to ask whether this argument can be used to produce compact exotic 4-manifolds. If one tries to replace $\mathbb{R}^4$ with S^4 and run the above argument verbatum, the trouble arises when trying to put a smooth structure on $S^4 \setminus \nu(f(X(K)))$.

Fact 1. *If K can be obstructed from being smoothly slice by an obstruction coming from*

- *classical abelian or metabelian concordance invariants*
- *Donaldson's theorem, Furuta's 10/8 theorem*
- *Heegaard Floer homology or gauge theory*
- *adjunction-type inequalities*

then K is not slice in any integer homology B^4.

Thus, if K is known not to be smoothly slice via an obstruction coming from any of these theories, $S^4 \setminus \nu(f(X(K)))$ does not admit any smooth structure.

Open Problem 2. *Does $S^4 \setminus \nu(f(X(K)))$ admit a smooth structure for K the Conway knot?*

Open Problem 3. *Build a smooth X^4 with the integer homology type of S^4 such that there exists some non-slice knot K such that $X(K)$ embeds smoothly in X.*

Open Problem 4. *Find any pair of closed oriented simply connected smooth 4-manifolds X and W such that X is homotopy equivalent to W and such that there exists some K so that $X(K)$ embeds smoothly in X but not in W.*

3. The Conway knot is not slice

The field of knot concordance is concerned with the study of knots up to sliceness, and concordance theorists have developed a rich suite of sliceness obstructions over the years. However, none of the obstructions are perfect, and there is an 11 crossing knot, the Conway knot, which sits in the intersection of the blind spots of all known sliceness obstructions.

Theorem 6 ([Pic20])**.** *The Conway knot is not slice*

Sketch. Build a knot J such that $X(J)$ is diffeomorphic to $X(Conway)$. Lemma 1 implies that the Conway knot is slice if and only if J is. Show (using Rasmussen's s invariant) that J is not slice. □

There remain some important open sliceness problems in concordance, for example

Open Problem 5. *Is the positive Whitehead double of the left handed trefoil or the* $(2, 1)$ *cable of the figure eight knot slice?*

The main technical hurdle of this technique is constructing a knot J with $X(J)$ diffeomorphic to $X(K)$ for a given K. For some knots, such as the unknot, both trefoils, and the figure eight knot, it is known that no such J exists.

Open Problem 6. *For which* K *does there exist a distinct knot* J *with* $\partial(X(K)) \cong \partial(X(J))$*? What about a distinct* J *so that* $\partial(X(K))$ *and* $\partial(X(J))$ *are just integer homology cobordant? Give an algorithm for producing such a* J*.*

4. PL representatives of homotopy equivalences

Motivating Problem 2. *Given manifolds* X^n *and* M^m *which are homotopy equivalent, does there exist an embedding* $\phi : M \to X$ *inducing the homotopy equivalence?*

This classical problem is interesting in many dimensions, categories of manifolds and embeddings, and even relaxed from manifolds to spaces. In the case of simply connected compact orientable PL manifolds where ϕ is required to be a PL embedding, much is known:

Theorem 7 (Combined work of Browder, [Bro68], Casson, Haefliger [Hae68], Sullivan, and Wall [Wal70])**.** *When* n *is not 4 and* $m < n$ *is not 2, such a* ϕ *always exists.*

When $n = 4$ and $m = 2$ the problem appears on Kirby's list and was resolved only recently:

Theorem 8 (Proven by Levine-Lidman [LL19], later proof in [HP19])**.** *There exists* X^4 *homotopy equivalent to* S^2 *such that no such* ϕ *exists.*

The following lemma plays a key role in both proofs:

Lemma 2. *A smooth 4-manifold* X^4 *homotopy equivalent to* S^2 *admits such a* ϕ *if and only if there is some knot* J *in* S^3 *such that* $X(J)$ *embeds smoothly in* $X_0(J)$ *inducing a homotopy equivalence.*

To prove the theorem, Levine and Lidman show that for their candidate examples X, no such J exists. Because they have to obstruct trace embedding for *every* knot J they are forced to use a difficult sophisticated obstruction (sets of Heegaard Floer d-invariants), and their example X is quite contrived in order to be able to compute the invariants. The trace embedding lemma allows a drastic simplification; build a candidate example X which embeds in S^4. Then the trace embedding lemma implies that if X admits a trace embedding, the knot J must be slice. It is much easier to show that many candidate manifolds X do not admit a slice trace embedding (for example with the adjuction inequality for Stein domains).

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Kleinian groups generated by two parabolic transformations

Makoto Sakuma

(joint work with Shunsuke Aimi, Hirotaka Akiyoshi, Gaven Martin, Donghi Lee, Ken'ichi Ohshika, Shunsuke Sakai, John Parker, Han Yoshida)

In 2002, Agol [2] announced the following classification theorem of non-free Kleinian groups generated by two parabolic transformations, which generalises the results of Adams [1].

Theorem 1. *A non-free Kleinian group* Γ *is generated by two non-commuting parabolic elements if and only if one of the following holds.*

(1) Γ *is conjugate to the hyperbolic* 2*-bridge link group,* $G(r)$*, for some rational number* $r = q/p$*, where* p *and* q *are coprime integers such that* $q \not\equiv \pm 1 \pmod{p}$*.*

(2) Γ *is conjugate to the Heckoid group,* $G(r; n)$*, for some* $r \in \mathbb{Q}$ *and some* $n \in \frac{1}{2}\mathbb{N}_{\geq 3}$*.*

Here $G(r)$ is the Kleinian group that uniformises the complement $S^3 - K(r)$ of the hyperbolic 2-bridge link $K(r)$, and $G(r; n)$ is the Kleinian group that uniformises the Heckoid orbifold $\mathcal{S}(r; n)$, which is isomorphic to one of the orbifolds in Figure 1 (see [4, Definition 3.4 and the paragraph preceding it] for the precise definition).

Agol [2] also announced the following classification of parabolic generating pairs of the groups in Theorem 1, which also refines and extends Adams' results in [1] (see Figure 1).

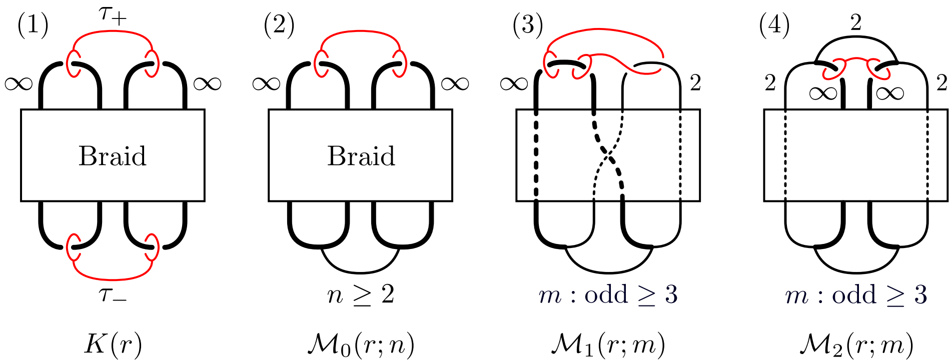

FIGURE 1. The black graphs illustrate weighted graphs representing 2-bridge links and Heckoid orbifolds, where the thick edges with weight ∞ correspond to parabolic loci and thin edges with integral weights represent the singular set. The red thin graphs represent the parabolic generating pairs of the hyperbolic 2-bridge link groups and the Heckoid groups.

Theorem 2. (1) *If Γ is a hyperbolic 2-bridge link group, then it has precisely two parabolic generating pairs, namely the upper and lower meridian pairs, up to equivalence.*

(2) *If Γ is a Heckoid group, then it has a unique parabolic generating pair, up to equivalence.*

Here, by a *parabolic generating pair* of a Kleinian group Γ, we mean an unordered pair $\{\alpha, \beta\}$ of parabolic transformations α and β that generate Γ. Two parabolic generating pairs $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$ are said to be *equivalent* if $\{\alpha', \beta'\}$ is equal to $\{\alpha^{\epsilon_1}, \beta^{\epsilon_2}\}$ for some $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ up to simultaneous conjugacy.

In the joint paper [4] with Hirotaka Akiyoshi, Ken'ichi Ohshika, John Parker and Han Yoshida, we gave a full proof to Theorem 1, and in the joint paper [3] with Shunsuke Aimi, Donghi Lee and Shunsuke Sakai, we gave an alternative proof to Theorem 2.

In the first part of my talk, I explained an outline of the proof Theorem 1, by pointing out the following facts behind the proof.

(1) The orbifold surgeries on hyperbolic 2-bridge link complements and Heckoid orbifolds, that replace the index ∞ with 2, produce spherical dihedral orbifolds $\mathcal{O}(r; d_+, d_-)$ with $d_+ = 1$ or 2 in Figure 2.

(2) Geometric orbifolds with dihedral fundamental groups can be classified easily (see [4, Theorem 4.1]). In particular, spherical dihedral orbifold is isomorphic to the orbifold $\mathcal{O}(r; d_+, d_-)$ in Figure 2.

The proof of Theorem 1 depends on two deep theorems, the orbifold theorem established by Boileau-Leeb-Porti and Cooper-Hodgson-Kerckhoff, and the relative version of the tameness theorem for orbifolds (see [4, Theorem 5.1]) which is a

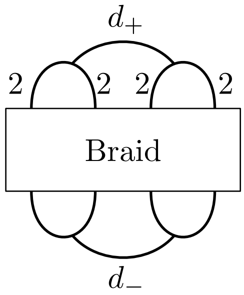

FIGURE 2. The spherical dihedral orbifold $\mathcal{O}(r; d_+, d_-)$, where the singular set consists of the 2-bridge link $K(r)$ together with the upper/lower tunnels.

variant of the tameness theorem established by Agol, Calegari-Gabai, Soma, and Bowditch.

Next, we explain a conjecture extending Theorems 1 and 2, which I proposed in the third part of my talk. The conjecture is based on the fact that two-parabolic-generator groups are commensurable with the images of certain special type-preserving representations of the fundamental group of the once-puncture torus (see [5, Lemma 5.3.2]).

Let T, S and $\mathcal{O}$, respectively, be the once-punctured torus, the 4-times punctured sphere, and the $(2,2,2,\infty)$-orbifold (i.e., the 2-orbifold with underlying space a punctured sphere and with three cone points of indices 2). They have $\mathbb{R}^2 - \mathbb{Z}^2$ as the common covering space. To be precise, let Λ and $\tilde{\Lambda}$, respectively, be the groups of transformations on $\mathbb{R}^2 - \mathbb{Z}^2$ generated by π-rotations about points in $\mathbb{Z}^2$ and $(\frac{1}{2}\mathbb{Z})^2$. Then $T = (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$, $S = (\mathbb{R}^2 - \mathbb{Z}^2)/\Lambda$ and $\mathcal{O} = (\mathbb{R}^2 - \mathbb{Z}^2)/\tilde{\Lambda}$. In particular, there is a $\mathbb{Z}_2$-covering $T \to \mathcal{O}$ and a $(\mathbb{Z}_2)^2$-covering $S \to \mathcal{O}$: the pair of these coverings is called the Fricke diagram and each of T, S and $\mathcal{O}$ is called a Fricke surface. The fundamental groups $\pi_1(T)$ and $\pi_1(S)$ are identified with the normal subgroups of the orbifold fundamental group $\pi_1(\mathcal{O})$ of index 2 and 4, respectively.

The isotopy classes of essential simple loops in a Fricke surface are in one-to-one correspondence with $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{1/0\}$: A representative of the isotopy class corresponding to $r \in \hat{\mathbb{Q}}$ is the projection of a line in $\mathbb{R}^2 - \mathbb{Z}^2$. The element $r \in \hat{\mathbb{Q}}$ associated to a loop is called its *slope*. We denote an essential loop of slope r in T or $\mathcal{O}$ (resp. S) by β_r (resp. α_r). Then, after an isotopy, the restriction of the projection $T \to \mathcal{O}$ to β_r $(\subset T)$ gives a homeomorphism from β_r $(\subset T)$ to β_r $(\subset \mathcal{O})$, while the restriction of the projection $S \to \mathcal{O}$ to α_r gives a two-fold covering from α_r $(\subset S)$ to β_r $(\subset \mathcal{O})$. By regarding α_r and β_r as (conjugacy classes of) elements of $\pi_1(\mathcal{O})$, we have $\alpha_r = \beta_r^2$.

Now recall that the 2-bridge link $(S^3, K(r))$ of slope r is obtained as the sum of the rational tangles of slope ∞ and r. This implies that the link group $G(r) \cong$

$\pi_1(S^3 - K(r))$ is isomorphic to the quotient group $\pi_1(S)/\langle\langle \alpha_\infty,\ \alpha_r \rangle\rangle$. The preceding fact implies that $G(r)$ is an index 4 normal subgroup of $\pi_1(\mathcal{O})/\langle\langle \beta_\infty^2,\ \beta_r^2 \rangle\rangle$. Thus $G(r)$ is commensurable with the quotient group $\pi_1(T)/\langle\langle \beta_\infty^2,\ \beta_r^2 \rangle\rangle$. Similarly, it turns out that Heckoid group $G(r;n)$ with $n = \frac{m}{2} \in \frac{1}{2}\mathbb{N}_{\geq 3}$ is commensurable with the quotient group $\pi_1(T)/\langle\langle \beta_\infty^2,\ \beta_r^m \rangle\rangle$. The natural embeddings of $G(r)$ and $G(r;n)$ into $\mathrm{PSL}(2,\mathbb{C})$ determine type-preserving representations of $\pi_1(T)$ to $\mathrm{PSL}(2,\mathbb{C})$, which have discrete images (and nonfaithful). Here a representation $\rho : \pi_1(T) \to \mathrm{PSL}(2,\mathbb{C})$ is said to be *type-preserving* if it is irreducible and maps peripheral elements to parabolic transformations.

This observation together with Theorems 1 and 2 naturally leads us to the following conjecture.

Conjecture 3. *If a nonfaithful type-preserving representation $\rho : \pi_1(T) \to \mathrm{PSL}(2,\mathbb{C})$ is discrete, then its kernel is equal to $\langle\langle \beta_r^m \rangle\rangle$ for some $r \in \hat{\mathbb{Q}}$ and $m \geq 2$ or $\langle\langle \beta_{r_1}^{m_1},\ \beta_{r_2}^{m_2} \rangle\rangle$ for some distinct $r_1, r_2 \in \hat{\mathbb{Q}}$ and $m_1, m_2 \geq 2$. In particular, the image of ρ is isomorphic to $\pi_1(T)/\langle\langle \beta_\infty^m \rangle\rangle$ for some $m \geq 2$ or $\pi_1(T)/\langle\langle \beta_\infty^2,\ \beta_r^m \rangle\rangle$. for some $r \in \hat{\mathbb{Q}}$ and $m \geq 2$.*

It is also natural to conjecture that similar results hold for type-preserving $\mathrm{PSL}(2,\mathbb{C})$-representations of the orbifold fundamental group of the 2-orbifold $T(n)$ with underlying space the closed torus and with a single cone point of index $n \geq 2$.

Finally, I explain a conjectural picture on the space of Kleinian groups generated by two parabolic transformations, which was presented in the second part of my talk. To describe the conjecture, recall that the *Riley slice* $\mathcal{R}$ is the subspace of $\mathbb{C}^* = \mathbb{C} - \{0\}$ consisting of the non-zero complex numbers ω such that the following marked group G_ω has the domain of discontinuity whose quotient is homeomorphic to the four-times punctured sphere:

$$G_\omega = \langle A,\ B_\omega \rangle \quad \text{with } A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B_\omega = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}$$

It has been studied by Keen and Series [6] by using pleating rays, and further by Komori and Series [7]. Ohshika and Miyachi [9] proved that the closure $\bar{\mathcal{R}}$ of $\mathcal{R}$ in $\mathbb{C}^*$ is equal to the space of the complex parameters ω such that G_ω is a rank 2 free Kleinian groups and that the boundary (frontier) $\partial\mathcal{R}$ of $\mathcal{R}$ in $\mathbb{C}^*$ is a Jordan curve. In the joint work [5] with Akiyoshi, Wada and Yamashita, we studied extensions of the rational pleating rays to the outside of the Riley slice, and announced that they correspond to continuous families of certain hyperbolic cone manifolds and that their endpoints correspond to hyperbolic 2-bridge link complements (or the 2-dimensional hyperbolic base orbifolds of the Seifert fibered structures of non-hyperbolic 2-bridge links). See [5, Figure 0.2b]. Assuming this announcement, Theorems 1 and 2 imply that all non-free discrete groups are located on the extended rational pleating rays.

Through discussion with Gaven Martin and John Parker, and through collaboration with Hirotaka Akiyoshi, the following conjecture arose.

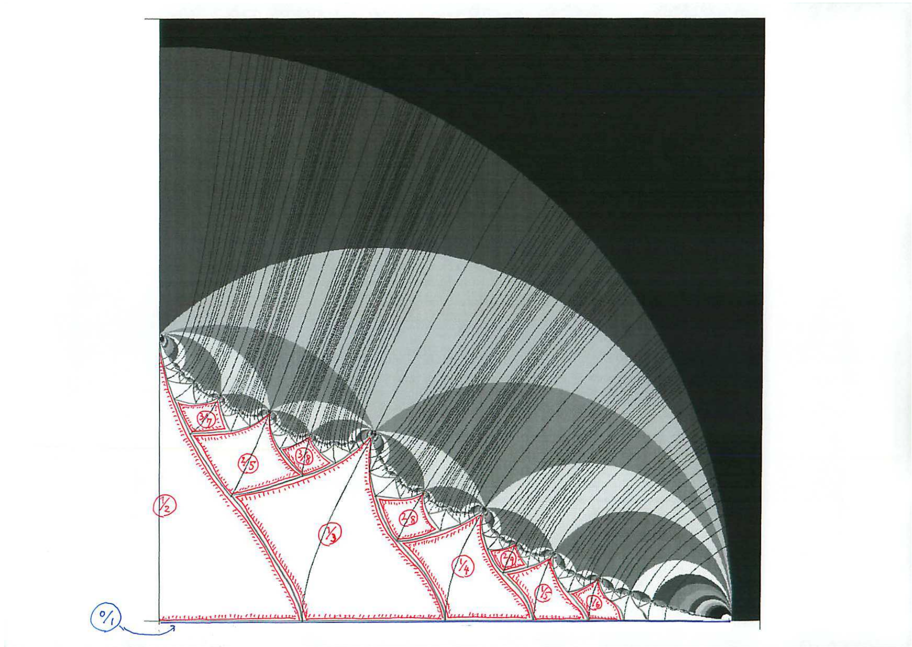

FIGURE 3. Conjectural tiling of the complement of $\bar{\mathcal{R}}$

Conjecture 4. *The complement of* $\bar{\mathcal{R}}$ *admits a natural tiling* $\{\mathcal{R}^c(r)\}_{r\in\mathbb{Q}/2\mathbb{Z}}$*, as shown in Figure 3, where each tile* $\mathcal{R}^c(r)$ *is characterized as follows.*

Consider the link $K(r^*)\cup\alpha_r$*, where (i)* r^* *is the Farey neighbour of* r *such that if* r *has the continued fraction expansion* $[a_1,\cdots,a_n]$ *then* $r^*=[a_1\cdots,a_{n-1}]$*, and (ii)* α_r *is a simple loop of slope* r *on the* 2*-bridge sphere of the* 2*-bridge link* $K(r^*)$*. Consider hyperbolic Dehn fillings of* $S^3-(K(r^*)\cup\alpha_r)$ *which keeps the cusps around* $K(r^*)$ *complete, and consider the subspace of the hyperbolic Dehn filling space spanned by the line segments joining the points* ∞*,* $(\pm1,1)$*,* $(0,1)$*,* $(\mp1,1/2)$ *cyclically in this order, where the sign* $\pm$ *is determined by* r*. Then the tile* $\mathcal{R}^c(r)$ *corresponds to the restrictions of the holonomy representations of the (possibly incomplete) hyperbolic manifolds corresponding to the points in the above subspace to the subgroup determined by the upper meridian pair of* $K(r^*)$*. In particular, each tile* $\mathcal{R}^c(r)$ *is a "quadrangle" which contains the extended pleating ray of slope* r *as a diagonal.*

Thus, each point in the complement of $\bar{\mathcal{R}}$ *can be regarded as a subgroup of the holonomy group of a certain (generically incomplete) hyperbolic manifold.*

We note that Martin [8] identified the exterior of $\mathcal{R}$ as the Julia set of a certain semigroup of polynomials and proved a "supergroup density theorem" for groups in the exterior of $\mathcal{R}$.

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Reporter: José Pedro Quintanilha

Participants

Prof. Dr. Kenneth L. Baker
Department of Mathematics
University of Miami
1365 Memorial Drive
Coral Gables, FL 33146-4250
UNITED STATES

Prof. Dr. Michel Boileau
Institut de Mathématiques de Marseille
(I2M)
Centre de Mathématiques et
Informatique (CMI)
Technopôle Château-Gombert
39, rue F. Joliot Curie
13453 Marseille Cedex
FRANCE

Dr. Rachael Jane Boyd
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Steven Boyer
Département de Mathématiques
Université du Québec à Montréal
Succursale Centre-ville
P.O. Box 8888
Montréal, Québec H3C 3P8
CANADA

Prof. Benjamin Burton
School of Mathematics and Physics
The University of Queensland
Building 67, Room 547
Brisbane, QLD 4072
AUSTRALIA

Dr. Michelle Chu
Department of Mathematics, Statistics
and Computer Science
University of Illinois at Chicago
SEO 616
1200 West Harrison Street
Chicago, IL 60607
UNITED STATES

Dr. Anthony Conway
Department of Mathematical Sciences
Durham University
Science Laboratories, CM 123
South Road
Durham DH1 3LE
UNITED KINGDOM

Prof. Dr. Nathan M. Dunfield
Department of Mathematics, MC-382
University of Illinois at
Urbana-Champaign
273 Altgeld Hall
1409 West Green Street
Urbana IL 61801-2975
UNITED STATES

Prof. Dr. Stefan Friedl
Fakultät für Mathematik
Universität Regensburg
Raum M 118
93040 Regensburg
GERMANY

Dr. David Futer
Department of Mathematics
Temple University
Wachman Hall, office 1026
Philadelphia PA 19122
UNITED STATES

Prof. Dr. David Gabai
Department of Mathematics
Princeton University
Fine Hall 502
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Prof. Dr. Cameron M. Gordon
Department of Mathematics
The University of Texas at Austin
PMA 12.112
2515 Speedway
Austin, TX 78712
UNITED STATES

Dr. François Guéritaud
Laboratoire Painlevé: UMR-CNRS 8524
UFR de Mathématiques
Université de Lille
Bât. M3, Bureau 230
59655 Villeneuve d'Ascq - Cédex
FRANCE

Prof. Dr. Shelly L. Harvey
Department of Mathematics
Rice University
MS 136
Herman Brown 446
Houston TX 77251-1892
UNITED STATES

Dr. Marc Kegel
Institut für Mathematik
Humboldt-Universität zu Berlin
Rm. 318 (Haus 1)
Rudower Chaussee 25
12489 Berlin
GERMANY

Dr. Autumn Kent
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison, WI 53706-1388
UNITED STATES

Dr. Eiko Kin
Department of Mathematics
Graduate School of Science
Osaka University
Machimakeyama-cho 1-1, Toyonaka
Osaka 560-0043
JAPAN

Dr. Danica Kosanović
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Dr. Markus Land
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 København Ø
DENMARK

Dr. Ana G. Lecuona
School of Mathematics and Statistics
University of Glasgow
Room 438
University Place
Glasgow G12 8QQ
UNITED KINGDOM

Dr. Lukas Lewark
Fakultät für Mathematik
Universität Regensburg
Universitätsstrasse 31
93053 Regensburg
GERMANY

Prof. Dr. Tao Li
Department of Mathematics
Boston College
140 Commonwealth Avenue
Chestnut Hill, MA 02467-3806
UNITED STATES

Prof. Dr. Joseph Maher
Department of Mathematics
The City University of New York
College of Staten Island
1S-215
2800 Victory Boulevard
Staten Island, NY 10314
UNITED STATES

Prof. Dr. Gordana Matic
Department of Mathematics
University of Georgia
Rm. 503 Boyd GSRC
Athens, GA 30602
UNITED STATES

Prof. Yair Minsky
Department of Mathematics
Yale University
10 Hillhouse Avenue
P.O. Box 208283
New Haven, CT 06520
UNITED STATES

Prof. Dr. Yoav Moriah
Department of Mathematics
TECHNION - Israel Institute of Technology
Haifa 3200003
ISRAEL

Dr. Kimihiko Motegi
Department of Mathematics
College of Humanities and Sciences
Nihon University
3-25-40 Sakurajosui, Setagaya-ku
Tokyo 156-8550
JAPAN

Dr. Danielle O'Donnol
Department of Mathematics
Marymount University
Caruthers Hall
2807 North Glebe Road
Arlington, Virginia 22207
UNITED STATES

Dr. Brendan Owens
School of Mathematics and Statistics
University of Glasgow
University Place
Glasgow G12 8QQ
UNITED KINGDOM

Anna Parlak
Mathematics Institute
University of Warwick
Zeeman Building
Coventry CV4 7AL
UNITED KINGDOM

Prof. Priyam Sanjay Patel
Department of Mathematics
University of Utah
155 South 1400 East, JWB 233
Salt Lake City, UT 84112
UNITED STATES

Prof. Dr. Kathleen Petersen
Department of Mathematics
Florida State University
600 W College Avenue
Tallahassee FL 32306-4510
UNITED STATES

Dr. Lisa M. Piccirillo
Department of Mathematics
Brandeis University
MS 050
415 South Street
Waltham, MA 02453
UNITED STATES

Dr. Tali Pinsky
Department of Mathematics
TECHNION - Israel Institute of Technology
Haifa 3200003
ISRAEL

Dr. Mark A. Powell
Department of Mathematical Sciences
Durham University
South Road
Durham DH1 3LE
UNITED KINGDOM

José Pedro Quintanilha
Fakultät für Mathematik
Universität Regensburg
M 302
93040 Regensburg
GERMANY

Dr. Sarah Rasmussen
DPMMS
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Arunima Ray
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Alan W. Reid
Department of Mathematics
Rice University
MS 136
Houston TX 77005-1892
UNITED STATES

Prof. Dr. Rachel Roberts
Department of Mathematics
Washington University in St. Louis
Campus Box 1146
One Brookings Drive
St. Louis MO 63105
UNITED STATES

Prof. Dr. Makoto Sakuma
Department of Mathematics
Graduate School of Science
Hiroshima University
1-3-1 Kagamiyama, Higashi
Hiroshima 739-8526
JAPAN

Dr. Saul Schleimer
Mathematics Institute
University of Warwick
Zeeman Building, B2.14
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Jennifer C. Schultens
Department of Mathematics
University of California, Davis
MSB 3145
One Shields Avenue
Davis, CA 95616
UNITED STATES

Dr. Henry Segerman
Department of Mathematics
Oklahoma State University
504 Mathematical Sciences
Stillwater OK 74078
UNITED STATES

Oliver Singh
Department of Mathematical Sciences
Durham University
Mountjoy Centre, CM 208
Stockton Road
Durham DH1 3LE
UNITED KINGDOM

Prof. Dr. András I. Stipsicz
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Reáltanoda utca 13-15
1053 Budapest
HUNGARY

Prof. Dr. Abigail A. Thompson
Department of Mathematics
University of California, Davis
1224 Sequoia Place
Davis, CA 95616-8633
UNITED STATES

Dr. Richard Webb
Department of Mathematics
The University of Manchester
Oxford Road
Manchester M13 9PL
UNITED KINGDOM

Dr. Anastasiia Tsvietkova
Okinawa Institute of Science and Technology
Graduate University
1919-1 Tancha, Onna-son, Kunigami-gun
Okinawa 904-0495
JAPAN

Dr. Mehdi Yazdi
Mathematical Institute
Oxford University
Andrew Wiles Building
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Prof. Dr. Jeremy Van Horn-Morris
Department of Mathematical Sciences
University of Arkansas
420 SCEN
525 Old Main
Fayetteville, AR 72701
UNITED STATES

PD Dr. Raphael Zentner
Fakultät für Mathematik
Universität Regensburg
Rm. M 122
Universitätsstrasse 31
93040 Regensburg
GERMANY

Dr. Katie Vokes
I.H.E.S.
Bureau: 1N4
35 route de Chartres
91440 Bures-sur-Yvette
FRANCE