Mathematisches Forschungsinstitut Oberwolfach

Report No. 10/2020

DOI: 10.4171/OWR/2020/10

Mini-Workshop: Superpotentials in Algebra and Geometry

Organized by
Lara Bossinger, Oaxaca
Eduardo González, Boston
Konstanze Rietsch, London
Lauren Williams, Cambridge MA

23 February – 29 February 2020

Abstract. Mirror symmetry has been at the epicenter of many mathematical discoveries in the past twenty years. It was discovered by physicists in the setting of super conformal field theories (SCFTs) associated to closed string theory, mathematically described by σ-models. These σ-models turn out in two different ways: the A-model and the B-model. Physical considerations predict that deformations of the SCFT of either σ-model should be isomorphic. Thus the mirror symmetry conjecture states that the A-model of a particular Calabi-Yau space X must be isomorphic to the B-model of its mirror $\check{X}$. Mirror symmetry has been extended beyond the Calabi-Yau setting, in particular to Fano varieties, using the so called Landau-Ginzburg models. That is a non-compact manifold equipped with a complex valued function called the *superpotential*.

In general, there is no clear recipe to construct the mirror for a given variety which demonstrates the need of joining mathematical forces from a wide range. The main aim of this Mini-Workshop was to bring together experts from the different communities (such as symplectic geometry and topology, the theory of cluster varieties, Lie theory and algebraic combinatorics) and to share the state of the art on superpotentials and explore connections between different constructions.

Mathematics Subject Classification (2020): 14J33 (primary), 13F60, 53D37, 14M25 (secondary).

Introduction by the Organizers

The first mathematical challenge in mirror symmetry is finding precise mathematical statements that reflect physicists expectations. There are various flavours of mirror symmetry statements that may be thought of as different layers. The first task in this meeting of mathematicians with diverse background was to communicate the different expectations of what properties a mirror space $\check{X}$ should have for a given variety X. Mostly, X is either a Calabi-Yau space or a Fano variety with a choice of anticanonical divisor. To overcome differences in background and field, the week started with three 3-hour lecture series.

Building on the toric case developed over two decades ago by Batyrev *et al*, Givental and Hori–Vafa (to name a few), in lectures by Pech we learned a first mirror symmetry statement:

The quantum cohomology ring of X is isomorphic to the Jacobian ring of the superpotential $W : \check{X} \to \mathbb{C}$.

Lie theoretic constructions by Rietsch *et al* successfully prove this statement for homogeneous spaces (including Grassmannians, flag varieties and quadrics, for example). Additional talks by Kim, Sherman-Bennett, Spacek and Wang revealed the most recent developements in this direction.

Another mirror symmetry program is being developed by Gross–Siebert and Gross–Hacking–Keel that can roughly be stated as follows:

Tropical points of X define a ring R of theta functions whose spectrum is $\check{X}$.

It should be noted though that in many cases the mirror space is rather a sufficiently smooth replacement of Spec(R). If X is compact the mirror is additionally endowed with a superpotential. A second lecture series by Nájera Chávez focused on the tools used in this program (like scattering diagrams, broken lines and theta functions) for the more tactile case of cluster varieties. Besides the lectures, talks by Cheung, Magee, Pomerleano built on the theory explained in the lectures.

A third mirror symmetry program was presented by Woodward in a lecture series. Given a self-transverse immersion $\phi : L \to X$ where L is a Lagrangian one can construct the Fukaya algebra $A(\phi)$. Then the A^∞-algebra structure of $A(\phi)$ is used to define the Maurer–Cartan space: the space of projective solutions of weakly bounding cochains. By construction this space is endowed with a disc potential. The quantum cohomology of X should then be recovered by the Floer cohomology of $A(\phi)$. This constructions leads us to a homological mirror symmetry statement:

The mirror to a space X with boundary D is a fibration defined by the superpotential on $\check{X}$ such that the bounded derived category of coherent sheaves on $X \setminus D$ is equivalent to the wrapped Fukaya category of $\check{X}$.

Additional related research talks were given by Keating and Lekili which showed the state of the art in the subject.

The lecture series helped overcome the potential issues posed by having an audience of such diverse mathematical background. The research talks of all participants continued in the spirit of the lectures in the sense that they were aiming to make the audience understand as much as possible while patiently answering any type of question that came up. Several discussion sessions were scheduled to allow lecturers and speakers to answer further questions arising from the talks.

Besides the above subjects, research talks and discussions revealed further connection to deformation theory, toric degenerations, Newton–Okounkov bodies, tropical geometry and convexity in tropical spaces, mutation and wall-crossing formulas, polyhedral geometry (in particular Gelfand-Tsetlin and string polytopes, reflexive polytopes and Newton polytopes), plabic graphs and cluster algebras, algebraic K-theory and higher representation theory.

Overall the conference showed how the constructions of superpotentials arising in various areas can be embedded into a broader mathematical context. We hope that this bird's eye view on the subject will particularly help the younger participants in identifying how their research fits into a wider mathematical context. The meeting was attended by 18 participants from the U.S., U.K., Mexico, Canada and Germany. Among the 18 participants, 5 were postdoctoral fellows, and three were PhD students.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Alfredo Nájera Chávez in the "Simons Visiting Professors" program at the MFO.

Mini-Workshop: Superpotentials in Algebra and Geometry

Table of Contents

Abstracts

Landau-Ginzburg potentials for Fano compactifications of cluster varieties

Alfredo Nájera Chávez

In this series of three expository lectures we discuss the construction of Landau Ginzburg potentials for Fano compactifications of cluster varieties. The main question we address is the following: given a Fano variety Y and an anticanonical divisor $D \subset Y$ such that $U = Y \setminus D$ has a cluster structure, can we use the cluster structure on U to construct a Landau-Ginzburg potential for Y? We frame the discussion within the mirror symmetry program for log Calabi-Yau varieties (see for example [1] and [2]) and focus on certain birational aspects that allow the construction of such potentials. The content presented here surveys ideas and results of [2] and [3].

Lectures 1 and 2: Theta functions. Cluster varieties are a particular class of schemes over $\mathbb{C}$, obtained by gluing a (usually infinite) collection of algebraic tori using distinguished birational maps called cluster transformations. Cluster varieties come in cluster dual pairs. So if V is a cluster variety, we let $V^\vee$ be the cluster dual cluster variety.

By construction, cluster varieties are log Calabi-Yau varieties with maximal boundary. In particular, each cluster variety V has a canonical volume form Ω (this follows from the log Calabi-Yau structure) and, using the maximal boundary property one can define the integral tropicalization of V as

$$V^{\mathrm{trop}}(\mathbb{Z}) = \{\text{divisorial discrete valuations } \nu : \mathbb{C}(V) \setminus \{0\} \to \mathbb{Z} \ : \ \nu(\Omega) < 0\} \cup \{0\}.$$

Recall that a discrete valuation ν on the field of rational functions $\mathbb{C}(V)$ is divisorial if it is of the form $\nu = \mathrm{ord}_D$, where D is a divisor on a variety V' birational to V and $\mathrm{ord}_D(f)$ is the order of vanishing of $f \in \mathbb{C}(V') \cong \mathbb{C}(V)$ along D. One can define in a natural way $V^{\mathrm{trop}}(\mathbb{R})$ and by definition $V^{\mathrm{trop}}(\mathbb{Z})$ sits inside of $V^{\mathrm{trop}}(\mathbb{R})$. As a set, the tropical space $V^{\mathrm{trop}}(\mathbb{Z})$ (resp. $V^{\mathrm{trop}}(\mathbb{R})$) can be non-canonically identified with $\mathbb{Z}^n$ (resp. $\mathbb{R}^n$) and $V^{\mathrm{trop}}(\mathbb{R})$ inherits the topology of $\mathbb{R}^n$.

A scattering diagram for V is a (possibly infinite) collection of walls in the tropical space $(V^\vee)^{\mathrm{trop}}(\mathbb{R})$. Every wall is a co-dimension 1 rational polyhedral cone in $(V^\vee)^{\mathrm{trop}}(\mathbb{R})$ decorated with a *scattering function.* The scattering function decorating a wall is a Laurent power series whose shape depends on the hyperplane containing the wall. Up to a certain notion of equivalence, there is a unique scattering diagram that codifies in a fascinating way the most valuable information related to the cluster structure of V. We refer to it as the consistent scattering diagram associated to V. The purpose of these first two lectures is to recall the definition of the consistent scattering diagram associated to V and explain how it is used to define theta functions on V. Besides the consistent scattering diagram, the most important objects used to define theta functions are *broken lines.* These are piece-wise linear rays, homeomorphic to $(-\infty, 0]$ that sit inside $(V^\vee)^{\mathrm{trop}}(\mathbb{R})$

that only bend at walls. The allowed bending for a broken line at a wall is precisely determined by the scattering function decorating the wall. Broken lines are used to associate to each $m \in (V^\vee)^{\mathrm{trop}}(\mathbb{Z})$ a (possibly infinite) sum of global functions of V. Call this expression ϑ_m. In particular, if ϑ_m is a finite sum of global functions then ϑ_m itself is a global function on V refered to as the theta function on V associated to m.

The Fock-Goncharov conjecture for V states that the algebra of regular functions on V has a canonical vector space basis parameterized by the tropical space $(V^\vee)^{\mathrm{trop}}(\mathbb{Z})$. It is known that the Fock-Goncharov conjecture in general is false, however, it is true in various interesting cases. In particular, if ϑ_m is a finite sum of global functions on V for all $m \in (V^\vee)^{\mathrm{trop}}(\mathbb{Z})$ then the Fock-Goncharov conjecture holds for V.

Lecture 3: Landau-Ginzburg potentials. Let Y be a Fano variety and $D \subset Y$ an anticanonical divisor such that $U = Y \setminus D$ has a cluster structure given by a cluster variety V. In other words, we ask that there is a birational map from V to U which is an isomorphisms outside of a set of co-dimension at least 2 (of domain and range). So in this case we have that U has the same algebra of regular functions as V and $U = \mathrm{Spec}(\Gamma(V, \mathcal{O}_V))$. In particular, U is an affine log Calabi-Yau with maximal boundary. We assume that the canonical volume form on V has a pole along every irreducible component of D so that the compactifiaction $U \subset Y$ is sensible to the cluster structure. It is expected that $U^\vee$, the mirror of U, is again an affine log Calabi-Yau variety with maximal boundary. In particular, one expects that in this case the Fock-Goncharov conjecture holds for V and moreover, that $U^\vee = \mathrm{Spec}(\Gamma(V^\vee, \mathcal{O}_{V^\vee}))$. So let us assume that Fock-Goncharov conjecture holds for V and let $D_1, \ldots, D_s$ be the irreducible components of D. By definition, the diviosorial valuation $\mathrm{ord}_{D_i} : \mathbb{C}(V) \setminus \{0\} \to \mathbb{Z}$ associated to D_i is a point of $V^{\mathrm{trop}}(\mathbb{Z})$. So there is an associated regular function $\vartheta_{D_i} : V^\vee \to \mathbb{C}$. We obtain a function $W = \vartheta_{D_1} + \cdots + \vartheta_{D_s} \in \Gamma(V^\vee, \mathcal{O}_{V^\vee})$ and thus a function $W : U^\vee \to \mathbb{C}$ (at least conjecturally). We expect that W is a Landau-Ginzburg potential on Y. In [3] the authors show that this construction recovers various superopotentials for Fano varieties arising in representation theory. In the particular case of the Grassmannian it can be shown that the potential obtained in this way is the superpotential constructed by Marsh and Rietsch in [4].

References

[1] M. Gross and B. Siebert, *From real affine geometry to complex geometry*, Ann. of Math. (2) **174**(3) (2011), 1301–1428.

[2] M. Gross, P. Hacking and S. Keel, *Mirror symmetry for log Calabi-Yau surfaces I*, Publ. Math. Inst. Hautes Études Sci. **122** (2015), 65–168.

[3] M. Gross, P. Hacking, S. Keel and M. Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. **31**(2) (2018), 497–608.

[4] R. Marsh and K. Rietsch, *The B-model connection and mirror symmetry for Grassmannians*, to appear in Adv. Math. (2020).

Mirror symmetry for homogeneous varieties

Clelia Pech

In this series of three talks I surveyed results by K. Rietsch and her collaborators on mirror symmetry for rational homogeneous spaces G/P. In the first talk I explained Rietsch's 2008 Lie-theoretic mirror construction [5]. In the second one, I focused on the case of Grassmannians, following the results of Marsh-Rietsch [1]. Finally, in the last lecture I sketched constructions for other homogeneous spaces such as quadrics (joint with K. Rietsch [3], and K. Rietsch and L. Williams [4]), as well as Lagrangian Grassmannians (joint with K. Rietsch [2]).

References

[1] R. Marsh and K. Rietsch, *The B-model connection and mirror symmetry for Grassmannians*, to appear in Adv. Math. (2020).

[2] C. Pech, K. Rietsch, *A Landau-Ginzburg model for Lagrangian Grassmannians, Langlands duality, and relations in quantum cohomology*, arXiv:1304.4958, (2013)

[3] C. Pech, K. Rietsch, *A comparison of Landau-Ginzburg models for odd dimensional quadrics*, Bulletin of the Institute of Mathematics Academia Sinica, Vol. 13, No. 3, 27.09.2018, p. 249-291.

[4] C. Pech, K. Rietsch, L. Williams, *On Landau-Ginzburg models for quadrics and flat sections of Dubrovin connections*. Adv. Math. **300** (2016), 275–319.

[5] K. Rietsch, *A mirror symmetric construction of* $qH_T^*(G/P)_{(q)}$, Adv. Math. **217** (2008), no. 6, 2401–2442.

Disk potentials in Lagrangian Floer theory

Chris T. Woodward

The author began the short course by describing disk potentials in immersed Lagrangian Floer theory following the work of Fukaya-Oh-Ohta-Ono [7] and Akaho-Joyce [2]. Associated to a self-transverse immersion $\phi : L \to X$ of a compact Lagrangian in a compact symplectic manifold X is a *Fukaya* A_∞ *algebra* $CF(\phi)$ additively generated over a Novikov field Λ of formal power series in a formal variable q by chains on the Lagrangian plus two copies of each self-intersection; for each $d \geq 0$ the composition maps

$$\mu_d : CF(\phi)^{\otimes d} \to CF(\phi)$$

count holomorphic disks with boundary in the Lagrangian. Via the homotopy units construction in [7, (3.3.5.2)], [4, Section 2.2] one may furnish $CF(\phi)$ with a strict unit $1_\phi \in CF(\phi)$. A *bounding cochain* (resp. *weakly bounding cochain*) is a solution $b \in CF(\phi)$ to the Maurer-Cartan equation (resp. weak Maurer-Cartan equation)

$$\mu_0(1) + \mu_1(b) + \mu_2(b, b) + \ldots = 0 \text{ (resp. } = W(b)1_\phi).$$

The space of solutions to the weak Maurer-Cartan equation is denoted $MC(\phi)$ and is equipped with a *potential function*

$$MC(\phi) \to \Lambda, \quad b \mapsto W(b).$$

For any $b \in MC(\phi)$, the operator

$$\mu_1^b : CF(\phi) \to CF(\phi), \quad c \mapsto \sum_{k_- \geq 0, k_+ \geq 0} \mu_{k_- + k_+ + 1}(\underbrace{b, \ldots, b}_{k_-}, c, \underbrace{b, \ldots, b}_{k_+})$$

squares to zero. The *Floer cohomology* of a weakly bounding cochain b is

$$HF(\phi, b) = \frac{\ker \mu_1^b}{\operatorname{im} \mu_1^b}.$$

Properties of disk potentials and Lagrangian Floer cohomology include the following:

- The homotopy type of the Fukaya algebra, and so the potential and the union of Floer cohomologies

$$W : MC(\phi) \to \Lambda, \quad HF(\phi) := \cup_{b \in MC(\phi)} HF(\phi, b)$$

  is independent of all choices up to a natural notion of *gauge equivalence* on the space $MC(\phi)$.
- Non-vanishing of the Floer cohomology for some weakly bounding cochain obstructs the Hamiltonian displaceability of the Lagrangian. That is, if there exists a Hamiltonian diffeomorphism $\psi : X \to X$ such that $\psi(\phi(L)) \cap \phi(L)$ is empty then the Floer cohomology $HF(\phi, b)$ vanishes for any $b \in MC(\phi)$. We presented several examples using the *Morse model* for Fukaya algebras in which the composition maps are defined by counting *treed disks* where the segments in the tree represent gradient trajectories of a chosen Morse function.
- Critical points of the disk potentials for Lagrangian tori give rise to non-vanishing Floer cohomology groups. Suppose for simplicity that the Morse function on L is perfect and $H^1(L) \cong CF^1(\phi) \subset MC(\phi)$. An inductive argument shows that any critical point of the pullback of W to $H^1(L)$ has the property that $HF(\phi, b)$ is non-vanishing. Fukaya-Oh-Ohta-Ono [8] have shown for Lagrangian torus orbits in toric varieties that such critical points always exist.
- There is a natural product on the Floer cohomologies which gives rise to a category, assuming the same values of the disk potential. On the chain level on obtains an A_∞ *Fukaya category* $\text{Fuk}(X)$ by taking objects to be Lagrangians as above, and morphisms to be Floer cochains. The composition maps are defined as above, but now allowing boundary conditions on multiple Lagrangians.

The Fukaya category may be related to quantum cohomology by a program of Kontsevich [6, p.18]. Let

$$HH_\bullet(\text{Fuk}(X), \text{Fuk}(X)) \quad \text{resp. } QH^\bullet(X), \quad \text{resp. } HH^\bullet(\text{Fuk}(X), \text{Fuk}(X))$$

denote the Hochschild homology resp. quantum cohomology resp. Hochschild cohomology; in the case of curved categories we define the first by taking the direct sum over values of the disk potential although we understand a treatment of

Hochschild homology of curved A_∞ categories is under development by Abouzaid-Groman-Varolgunes. There are natural *open-closed* and *closed-open maps*

$$HH_\bullet(\mathrm{Fuk}(X),\mathrm{Fuk}(X)) \xrightarrow{OC} QH^\bullet(X) \xrightarrow{CO} HH^\bullet(\mathrm{Fuk}(X),\mathrm{Fuk}(X)).$$

A criterion for the closed-open map to be an isomorphism is provided by results of Abouzaid [1] and Ganatra [5]. Given a collection $\mathcal{G}$ of Lagrangians let $\mathrm{Fuk}_{\mathcal{G}}(X)$ denote the sub Fukaya category with objects $\mathcal{G}$. Write

$$QH_{\mathcal{G}}(X) = OC(HH_\bullet(\mathrm{Fuk}_{\mathcal{G}}(X),\mathrm{Fuk}_{\mathcal{G}}(X)))$$

for the image of $HH_\bullet(\mathrm{Fuk}_{\mathcal{G}}(X),\mathrm{Fuk}_{\mathcal{G}}(X))$ under the open-closed map. We say $QH^\bullet(X)$ is *generated by* $\mathcal{G}$ iff $QH_{\mathcal{G}}(X) = QH^\bullet(X)$. In this situation, $\mathcal{G}$ split-generates $\mathrm{Fuk}(X)$. Ganatra has shown in the exact setting [5] that under these assumptions Hochschild homology and cohomology of $\mathrm{Fuk}(X)$ are isomorphic as vector spaces (after a degree shift):

$$HH_\bullet(\mathrm{Fuk}_{\mathcal{G}}(X),\mathrm{Fuk}_{\mathcal{G}}(X)) \cong HH^{\dim(X)-\bullet}(\mathrm{Fuk}_{\mathcal{G}}(X),\mathrm{Fuk}_{\mathcal{G}}(X))$$

and both isomorphic to the quantum cohomology $QH^{\dim(X)-\bullet}(X)$. Ganatra's results [5]. We see no obstruction in extending his results to the compact case using the reader's preferred perturbation scheme. We briefly explained the Cieliebak-Mohnke perturbation scheme [3] which requires that the symplectic manifold and Lagrangian submanifolds satisfy rationality hypotheses [4].

We ended the last lecture with a computation of quantum cohomology of the equator in the two-sphere using the open-closed technology. Let $X = S^2$ be the symplectic two-sphere with unit area and $L = S^1$ embedded via a map ϕ as the equator dividing X into two disks of equal area 2π. These two disks contribute to the disk potential which is written in coordinates $y = e^b$ as

$$W(y) = q^{1/2}(y + 1/y)$$

and has two critical points at $y = \pm 1$ giving rise to non-trivial Floer cohomology. We computed the ring structure on $HF(\phi, b)$ which in each case is a Clifford algebra. Each Clifford algebra contributes on copy of the Novikov field to the Hochschild cohomology of the Fukaya category, which we may then view as the space of functions on the critical set $\{y = \pm 1\}$, also known as the *Jacobian ring.* The is additively generated by delta functions $\delta_{y=\pm 1}$ at the critical points. The closed-open map in this case is given by

$$CO : [X] \mapsto \delta_{y=1} + \delta_{y=-1}, \quad [\mathrm{pt}] \mapsto q^{1/2}\delta_{y=1} - q^{1/2}\delta_{y=-1}.$$

In this way one obtains the relation

$$CO([\mathrm{pt}]^2) = (CO([\mathrm{pt}]))^2 = q(\delta_{y=1} + \delta_{y=-1}) = qCO([X]).$$

Since $CO([X])$ is the unit, this gives the usual description of the small quantum cohomology ring

$$QH(X) = \Lambda[[\mathrm{pt}]]/\langle [\mathrm{pt}]^2 - q\rangle.$$

References

[1] M. Abouzaid. A geometric criterion for generating the Fukaya category. *Publ. Math. Inst. Hautes Études Sci.*, (112):191–240, 2010.
[2] M. Akaho and D. Joyce. Immersed Lagrangian Floer Theory. *J. Differential Geom.* 86 (2010), no. 3, 381–500.
[3] K. Cieliebak and K. Mohnke. Symplectic hypersurfaces and transversality in Gromov-Witten theory. *J. Symplectic Geom.*, 5(3):281–356, 2007.
[4] F. Charest and C. Woodward. Floer theory and flips. To appear in *Memoirs of the A.M.S.*. http://www.arxiv.org/abs/1508.01573.
[5] S. Ganatra. Symplectic Cohomology and Duality for the Wrapped Fukaya Category. Thesis (Ph.D.) – Massachusetts Institute of Technology. ProQuest LLC, Ann Arbor, MI, 2012.
[6] M. Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 120–139, Basel, 1995. Birkhäuser.
[7] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian intersection Floer theory: anomaly and obstruction.*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2009.
[8] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. Lagrangian Floer theory on compact toric manifolds. I. *Duke Math. J.*, 151(1):23–174, 2010.

Wall-crossing for Newton-Okounkov bodies

Megumi Harada
(joint work with Laura Escobar)

This is a report on joint work with Laura Escobar.

Let X be an irreducible projective algebraic variety over $\mathbb{C}$ of dimension d. One way to study the geometry of X is to construct a toric degeneration of X, which is a flat family $\pi : \mathcal{X} \to \mathbb{A}^1$ such that $\pi^{-1}(\xi)$ for $\xi \neq 0$ is isomorphic to the original variety X and the special fiber $\pi^{-1}(0)$ over 0 is a toric variety. The philosophy is that the geometric data of X_0, which can be read off combinatorially, can encode information about the original X.

Our work concerns two general and well-known strategies for constructing toric degenerations, tropicalization and the theory of Newton-Okounkov bodies. In the tropical theory, we first choose an explicit realization $X = \mathrm{Proj}(\mathbb{C}[x_1, \ldots, x_n]/I)$ for some homogeneous ideal I. Then the tropicalization of X is

$$\tau(I) := \{w \in \mathbb{R}^n \mid \text{ the initial ideal } \mathrm{init}_w(I) \text{ of } I \text{ contains no monomials}\}$$

where the initial ideal is defined with respect to the weighting of the variables given by $w \in \mathbb{R}^n$. It is known that $\tau(I)$ is a $(d+1)$-dimensional subfan of the Gröbner fan of I. The point here is that a Gröbner degeneration of I to $\mathrm{init}_w(I)$ yields a toric degeneration if $\mathrm{init}_w(I)$ is prime and binomial. Thus, we can view $\tau(I)$ as a set of possible candidates for toric degenerations.The theory of Newton-Okounkov bodies also gives rise to toric degenerations, as follows. Let $A = \mathbb{C}[x_1, \ldots, x_n]/I$ be the homogeneous coordinate ring of X as above. Let $\nu : A \setminus \{0\} \to \mathbb{Q}^{d+1}$ be a (discrete) valuation on A (here we have equipped the target with a total order). Assume ν is full rank and has one-dimensional leaves. From this data we obtain a multiplication filtration $\mathcal{F}_\nu$ of A and an associated graded algebra $gr_\nu(A)$. The grading of $gr_\nu(A)$ is encoded by the value semigroup $S(A,\nu) := \text{image}(\nu)$. The point now is that when $S(A,\nu)$ is finitely generated, $\text{Proj}(\text{gr}_\nu(A))$ is a (possibly non-normal) toric variety. Then, by a Rees algebra construction, we obtain a toric degeneration of X to $\text{Proj}(\text{gr}_\nu(A))$.

In 2016, Kaveh and Manon showed that the tropicalization approach and the Newton-Okounkov body approach were related. To state their result we need one more piece of terminology. Recall that each cone C of $\tau(I)$ is canonically associated to an initial ideal which we denote by $\text{init}_C(I)$. Kaveh and Manon say that a cone C of $\tau(I)$ is prime if $\text{init}_C(I)$ is prime. We can now state their theorem.

Theorem. (Kaveh-Manon) *Let X, A be as above. Let C be a prime cone of $\tau(I)$ of maximal dimension (so $\dim(C) = d+1$). Pick $u_1 = (1,1,\ldots,1) \in C$ and pick rational vectors $u_2,\ldots,u_{d+1} \in C$, such that $\{u_1,\ldots,u_{d+1}\}$ is linearly independent. Let M denote the $n\times(d+1)$ matrix obtained by putting u_i^T (thought of as a row vector) as the i-th row of M. Then there exists a full-rank valuation $\nu_M : A\setminus\{0\} \to \mathbb{Q}^{d+1}$ constructed from M, with one-dimensional leaves, such that*

$$\text{gr}_\nu(A) \cong \mathbb{C}[x_1,\ldots,x_n]/\text{init}_C(I).$$

From the above theorem it also follows that $S(A,\nu_M)$ is generated by the columns of M and the Newton-Okounkov body $\Delta(A,\nu_M)$ is the convex hull of the columns of M. Note that $\Delta(A,\nu_M)$ is precisely the polytope corresponding to the special fiber $\text{Proj}(\text{gr}_\nu(A))$.

Since $\tau(I)$ has a fan structure, it is possible for two prime cones C_1, C_2 of maximal dimension to be adjacent, i.e., to share a codimension-1 facet. A natural question is to ask how the two toric degenerations on either side of the facet are related.

Theorem A. (Harada-Escobar) *In the same setting and using the same notation as above, let C_1, C_2 be two adjacent prime cones of maximal dimension in $\tau(I)$. Let $C := C_1 \cap C_2$ be the codimension-1 facet shared by C_1 and C_2. Choose $u_1 = (1,1,\ldots,1)$ and $u_2,\ldots,u_d \in C$, $u_{d+1,1} \in \text{int}(C_1)$ and $u_{d+1,2} \in \text{int}(C_2)$. Let M_1 (resp. M_2) be the matrix obtained by placing the u_i and $u_{d+1,1}$ (resp. $u_{d+1,2}$) as the rows. Let M denote the matrix obtained by taking the top d rows of either M_1 or M_2. Then under the natural projection $p : \mathbb{R}^{d+1} \to \mathbb{R}^d$ which drops the last coordinate, there exist two piecewise-linear maps F_{12} and $\mathsf{S}_{12} : \Delta(A,\nu_{M_1}) \to \Delta(A,\nu_{M_2})$ such that $p\circ\mathsf{F}_{12} = p$ and $p\circ\mathsf{S}_{12} = p$ as maps $\Delta(A,\nu_{M_1}) \to \Delta(A,\nu_M)$, where $\text{len}(\pi^{-1}(\xi)\cap\Delta(A,\nu_{M_1})) = \text{len}(\pi^{-1}(\xi)\cap\Delta(A,\nu_{M_2}))$ for all $\xi \in \Delta(A,\nu_M)$.*

We refer to the maps F_{12} ("the flip map") and S_{12} ("the shift map") as (geometric) wall-crossing maps; these can be described more explicitly, although we do not do so here. We note that our theorem also follows from the theory of complexity-1 T-varieties. In an Appendix to our paper [1], Nathan Ilten explains this point of view, and also explains the connection to the "combinatorial mutation" studied by Akhtar, Coates, Galkin, and Kasprzyk.

In the setting of Newton-Okounkov bodies and valuations, it is reasonable to consider the semigroup $S(A,\nu)$ corresponding to a valuation to be a more fundamental object than the associated Newton-Okounkov body, which is obtained from the data of the semigroup. Thus it is natural to ask whether there exists a wall-crossing formula at the level of the two semigroups in question. We now describe some first steps in this direction. When C_1, C_2 are both faces of the same maximal cone of the Gröbner fan (associated to a monomial order $<$), then the set $\mathcal{S}(< .I)$ of standard monomials associated to $<$ and I is naturally bijective with both $S(A,\nu_{M_1})$ and $S(A,\nu_{M_2})$, i.e. there exist bijections $\theta_1 : \mathcal{S}(< .I) \to S(A,\nu_{M_1})$ and $\theta_2 : \mathcal{S}(< .I) \to S(A,\nu_{M_2})$. Thus we may define an "algebraic wall-crossing map" $\Theta_{12} : S(A,\nu_{M_1}) \to S(A,\nu_{M_2})$ as $\Theta_{12} := \theta_2 \circ \theta^{-1}$. It is natural to ask how Θ_{12} is related to F_{12} and S_{12}. What we know so far is that Θ_{12} is not, in general, a restriction to $S(A,\nu_{M_1})$ of either of the (extension to the relevant cones of the) maps F_{12} or S_{12}. Moreover, Θ_{12} is not, in general, a semigroup map. (See [1] for explicit counterexamples.) However, we do have the following, in the special case of $Gr(2,n)$.

Theorem B. (Harada-Escobar) *For* $\mathrm{trop}(Gr(2,n))$ *(i.e.* $\tau(I_{2,n})$ *for* $I_{2,n}$ *the ideal for the usual Plücker embedding) the algebraic wall-crossing map* Θ_{12} *is the restriction of the flip map* F_{12}*.*

References

[1] M. Harada and L. Escobar, Wall-crossing for Newton-Okounkov bodies and the tropical Grassmannian. ArXiv: 1912.04809.

Cluster duality for Lagrangian and Orthogonal Grassmannians

Charles Wang

Let $\mathbb{X} = \mathrm{LGr}(n,2n)$ be the variety of n-dimensional Lagrangian subspaces of $\mathbb{C}^{2n}$, with respect to the symplectic form $\omega_{ij} = (-1)^j \delta_{i,2n+1-j}$, embedded as a subvariety of $\mathrm{Gr}(n,2n)$ in its Plücker embedding. $\mathbb{X}$ has dimension $N = \binom{n+1}{2}$, and a distinguished anticanonical divisor $D_{ac} = D_0 + \cdots + D_n$ consisting of the $n+1$ hyperplanes $D_i = \{p_{\mu_i} = 0\}$, where μ_i is the $n \times i$ rectangle. We view $\mathbb{X}$ as an $\mathcal{X}$-cluster variety, and use the seeds of the associated cluster structure to obtain *network charts* $(\mathbb{C}^*)^N \hookrightarrow \mathbb{X}$. Then the data of a seed and an ample divisor of the form $D = \sum_{i=0}^n r_i D_i$ determines a Newton-Okounkov body for $\mathbb{X}$ as follows: the choice of seed determines a discrete valuation val on $\mathbb{C}(\mathbb{X}) \setminus \{0\}$, and we apply this to $L_{rD} = H^0(\mathbb{X}, \mathcal{O}(rD))$ for $r \in \mathbb{Z}_{>0}$ to get:

$$\overline{\mathrm{conv}\left(\bigcup_{r=1}^{\infty} \frac{1}{r}\mathrm{val}(L_{rD})\right)}.$$

We consider the Landau-Ginzburg model $(\check{\mathbb{X}}^\circ, W)$ for $\mathbb{X}$ studied in [1]. Here, $\check{\mathbb{X}}^\circ$ is the complement of a particular anticanonical divisor in a Langlands dual

$\check{\mathbb{X}} = \mathrm{OG}(n, 2n+1)$, and $W : \check{\mathbb{X}}^\circ \to \mathbb{C}$ is a regular function called the *superpotential*. Recall that the number of terms of W should be the index of X, and we keep track of this by writing $W = \sum_{i=0}^n W_i$. As before, we want to associate a polytope to a choice of seed and divisor. In this setting, we view $\check{\mathbb{X}}$ as an $\mathcal{A}$-cluster variety, and use the seeds of the associated cluster structure to obtain *cluster charts* $(\mathbb{C}^*)^N \hookrightarrow \check{\mathbb{X}}^\circ$. We obtain a polytope from a choice of seed and divisor as follows. First, the restriction of each W_i to the cluster chart corresponding to the seed gives a Laurent polynomial expression for W_i in the coordinates of that chart. Interpret each Laurent polynomial expression as an inequality by "tropicalizing" (replace multiplication with addition, and addition with min), denoted by $\mathrm{Trop}(W_i)$. Finally, define the *superpotential polytope* by the inequalities $\mathrm{Trop}(W_i) + r_i \geq 0$. We can now state our main theorem:

Fix an ample divisor in $\mathbb{X}$ of the form $D = \sum_{i=0}^n r_i D_i$, and a seed Σ in the cluster structure for $\mathbb{X}$. Then the associated Newton-Okounkov body associated to this choice of seed and divisor is a rational polytope, and is equal to the superpotential polytope for the same choice of seed and divisor.

The proof follows the general strategy of [2], where the analogous statements are established for Grassmannians $\mathrm{Gr}(k, n)$.

References

[1] C. Pech, K. Rietsch, *A Landau-Ginzburg model for Lagrangian Grassmannians, Langlands duality, and relations in quantum cohomology*, arXiv:1304.4958, (2013)

[2] K. Rietsch, L. Williams, *Newton-Okounkov Bodies, Cluster Duality, and Mirror Symmetry for Grassmannians*, Duke Math. J. **168** (2019), 3437–3527.

Symplectic geometry and algebraic K-theory

Yanki Lekili

(joint work with T. Dyckerhoff and G. Jasso)

This is joint work with T. Dyckerhoff and G. Jasso.

We show that the perfect derived categories of Iyama's d-dimensional Auslander algebras of type $\mathbb{A}$ are equivalent to the partially wrapped Fukaya categories of the d-fold symmetric product of the 2-dimensional unit disk with finitely many stops on its boundary. Furthermore, we observe that Koszul duality provides an equivalence between the partially wrapped Fukaya categories associated to the d-fold symmetric product of the disk and those of its $(n-d)$-fold symmetric product; this observation leads to a symplectic proof of a theorem of Beckert concerning the derived Morita equivalence between the corresponding higher Auslander algebras of type $\mathbb{A}$.

As a byproduct of our results, we deduce that the partially wrapped Fukaya categories associated to the d-fold symmetric product of the disk organise into a paracyclic object equivalent to the d-dimensional Waldhausen $\S_\bullet$-construction, a simplicial space whose geometric realisation provides the d-fold delooping of the connective algebraic K-theory space of the ring of coefficients.

Let n and d be natural numbers and consider the poset

$$\left\{ {\mathbf{n} \atop d} \right\} = \{I \in \mathbb{N}^d : 1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n\}$$

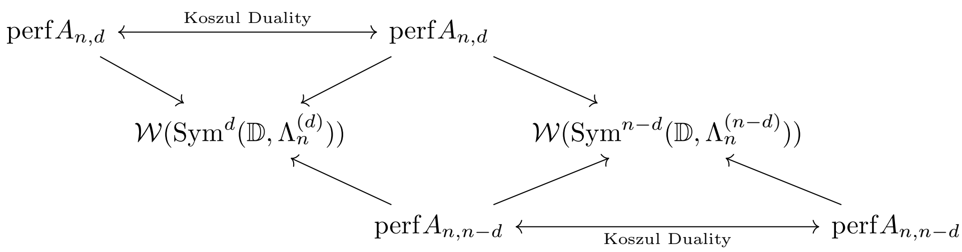
Further, introduce the subset $\left\{ {\mathbf{n} \atop d} \right\}^\flat =\subseteq \left\{ {\mathbf{n} \atop d} \right\}$ consisting of those $I \in \left\{ {\mathbf{n} \atop d} \right\}$ such that there exists an index $1 \leq a < d$ with $i_a = i_{a+1}$.

$$A_{n,d} := \Big(\bigoplus_{I \leq J} \mathbf{k} f_{JI}\Big) \Big/ \langle f_{KK} \,|\, K \in \left\{ {\mathbf{n} \atop d} \right\}^\flat \rangle$$

equipped with the multiplication law

$$f_{KJ'} \cdot f_{JI} = \begin{cases} f_{KI} & \text{if } J = J', \\ 0 & \text{otherwise.} \end{cases}$$

Theorem *Let $n \geq d \geq 1$. There is a commutative diagram of quasi-equivalences of triangulated A_∞-categories*



where $\mathcal{W}(\mathrm{Sym}^d(\mathbb{D}, \Lambda_n^{(d)}))$ stands for the partially wrapped Fukaya category.

References

[1] T. Dyckerhoff, G. Jasso, and Y. Lekili, *The symplectic geometry of higher Auslander algebras: Symmetric products of disks*, arXiv:1911.11719 (2019).

Laurent polynomial potentials for cominuscule homogeneous spaces

Peter Spacek

We present the results from [5], where a local Laurent polynomial expression is given for Rietsch's Lie-theoretic Landau-Ginzburg models for projective homogeneous spaces G/P [4].

Let $B_\pm^\vee$, $U_\pm^\vee$ be opposite Borel and unipotent subgroups of the group $G^\vee$ Langlands dual to G, denote by W and W^P the Weyl groups associated to the Dynkin diagrams of G and P respectively and write $T^P = (T^\vee)^{W_P}$ for the subset of the maximal torus $T^\vee$ of $G^\vee$ that is invariant under W_P. Let $w_0 \in W$ and $w_P \in W_P$ be the maximal elements, write w^P for the minimal coset representative of $w_0 W_P$ and fix a reduced expression $w^P = s_{r_1} \cdots s_{r_d}$. Take Chevalley generators $\{e_i^\vee, f_i^\vee, h_i^\vee\}$ for $\mathfrak{g}^\vee$ and represent $s_i \in W$ by $\bar{s}_i = \exp(-e_i^\vee)\exp(f_i^\vee)\exp(-e_i^\vee)$ in $G^\vee$.

Recall that the Lie-theoretic mirror model is given by the pair $(\mathcal{Z}^\vee, \mathcal{W})$ consisting of the subset $\mathcal{Z}^\vee = B_-^\vee \bar{w}_0^{-1} \cap U_+^\vee T^P \bar{w}_P U_-^\vee$ inside $G^\vee$ and the potential $\mathcal{W} : \mathcal{Z}^\vee \to \mathbb{C} : u_+ t \bar{w}_P u_- \mapsto \sum_i (e_i^\vee)^*(u_+^{-1}) + \sum_i (f_i^\vee)^*(u_-)$.

In [3], a rational expression is found for this model applied to the case that $G/P = \mathrm{LG}(n, 2n)$. As an intermediate result, the authors show that on the algebraic torus $\mathcal{Z}^\circ = \{u_+ t \bar{w}_P u_- \in \mathcal{Z}^\vee \mid u_- = \exp(a_d f_{r_d}) \cdots \exp(a_1 f_{r_1})\}$, $\mathcal{W}$ can be written as

$$\mathcal{W}|_{\mathcal{Z}^\circ} = a_1 + a_2 + \ldots + a_d \pm q \frac{\mathrm{Coef}(u_-^{-1}\, \bar{w}_P \bar{s}_k \cdot v^+, v^-)}{\mathrm{Coef}(u_-^{-1} \cdot v^+, v^-)}, \tag{1}$$

where $d = \dim(G/P)$, v^+ is a choice of highest weight vector in the spin representation of $\mathfrak{g}^\vee = \mathfrak{so}(2n)$, $v^- = \bar{w}_0 \cdot v^+$ is the associated lowest weight vector, and $\mathrm{Coef}(v, v^-)$ is the constant $\lambda \in \mathbb{C}$ such that $\pi(v) = \lambda v^-$ for π the projection to the lowest weight space.

It turns out that the arguments used to get to this result hold for any cominuscule homogeneous space G/P. (Indeed, we believe that it holds for any G/P with P maximal.) Recall that every (conjugacy class of a) parabolic subgroup is associated to a selection of fundamental weights $\{\omega_i\}$. We call G/P cominuscule when P is associated to a single fundamental weight ω_k that is cominuscule. (Recall that ω_k is cominuscule when the associated fundamental coweight is minuscule, i.e. it satisfies $2\frac{(\omega_k^\vee, \alpha^\vee)}{(\alpha^\vee, \alpha^\vee)} \in \{-1, 0, +1\}$ for any coroot $\alpha^\vee$.) For cominuscule G/P, equation (1) holds with v^+ a choice of highest weight vector in the fundamental weight representation $V(\omega_k^\vee)$ of $\mathfrak{g}^\vee$.

Now, a lot is known about the general structure of $V(\omega_k^\vee)$ for $\omega_k^\vee$ minuscule. Using the presentation given in [2], we showed that (1) can be rewritten as

$$\mathcal{W}|_{\mathcal{Z}^\circ} = a_1 + a_2 + \ldots + a_d + q \frac{\sum_{i \in \mathcal{I}} a_{i_1} \cdots a_{i_{\ell'}}}{a_1 \cdots a_d}, \tag{2}$$

where $\mathcal{I} = \{(i_1, \ldots, i_{\ell'}) \mid s_{r_{i_1}} \cdots s_{r_{i_{\ell'}}} = w'\}$ is the set of subexpressions of w' in the fixed reduced expression $s_{r_1} \cdots s_{r_d}$ of w^P. Here $w' = w^P (w'')^{-1}$ for w'' the minimal coset representative of $w_P s_k W_P$. This is Theorem 5.7 of [5].

We also mentioned that the set of subexpressions of w' in $w^P = s_{r_1} \cdots s_{r_d}$ can be obtained combinatorially using the quiver associated by [1] to w^P. Each subexpression of w' corresponds to a set of vertices of this quiver, and in [5] we show how to obtain all subexpressions from the quiver. The corresponding result is Corollary 8.12 in [5].

References

[1] P.E. Chaput, L. Manivel and N. Perrin, *Quantum cohomology of minuscule homogeneous spaces*, Transform. Groups **13** (2008), no. 1, 47-89.

[2] R.M. Green, *Representations of Lie algebras arising from polytopes*, Int. Electron. J. Algebra **4** (2008), 27-52.

[3] C.M.A. Pech and K. Rietsch, *A Landau-Ginzburg model for Lagrangian Grassmannians, Langlands duality and relations in quantum cohomology*, arXiv (2013), math.AG/1304.4958.

[4] K. Rietsch, *A mirror symmetric construction of $qH_T^*(G/P)_{(q)}$*, Adv. Math. **217** (2008), no. 6, 2401–2442.
[5] P. Spacek, *Laurent polynomial Landau-Ginzburg models for cominuscule homogeneous spaces*, arXiv (2019), math.AG/1912.09122.

Convexity in tropical spaces and compactifications of cluster varieties

Timothy Magee
(joint work with Man-Wai Cheung, Alfredo Nájera Chávez)

Earlier in the week Alfredo Nájera Chávez gave a series of lectures on *cluster varieties*, *scattering diagrams*, *broken lines*, and *theta functions* following [3]. This talk relies upon the material presented in his lecture series.

The construction of projective toric varieties via polytopes is fundamental in toric geometry. By drawing a convex rational polytope P, we can define a graded ring where the r^{th} homogeneous component has a basis parametrized by the integral points of $r \cdot P$. Taking Proj of the resulting graded ring, we obtain a projective toric variety. We describe an analogous construction for cluster varieties, where the cocharacter lattice N of a torus T is replaced by the integral tropicalization $U^{\text{trop}}(\mathbb{Z})$ of a cluster variety U, and characters of the dual torus $T^\vee$ are replaced by theta functions on the dual cluster variety $U^\vee$. The goal then is to say which subsets S of $U^{\text{trop}}(\mathbb{R})$ define graded rings through the same dilation procedure, and in turn projective compactifications of $U^\vee$. Since $U^{\text{trop}}(\mathbb{R})$ has only a *piecewise* linear structure, there is *a priori* no notion of convexity in it. However, broken lines in $U^{\text{trop}}(\mathbb{R})$ are a natural alternative to lines in $N \otimes \mathbb{R}$. We define:

Definition. A closed set $S \subset U^{\text{trop}}(\mathbb{R})$ is *broken line convex* if for every pair of rational points x, y in S, every broken line segment connecting x and y is entirely contained in S.

Our main result is that this notion does for cluster varieties precisely what convexity does for toric varieties.

Theorem. *A closed set $S \subset U^{\text{trop}}(\mathbb{R})$ defines a graded ring whose r^{th} homogeneous component is the span of the theta functions associated to integral points of $r \cdot S$ if and only if S is broken line convex.*

We propose two exciting applications of this concept. First, it provides an intrinsic version of a Newton-Okounkov body in the cluster setting. There is a natural "valuation" ν in the cluster world, which sends a theta function ϑ_p on $U^\vee$ to the point p in $U^{\text{trop}}(\mathbb{Z})$ that indexes it. Moreover there is a natural partial order coming from scattering functions which can be used to extend ν to the section ring of a polarized projective compactification of $U^\vee$. Using this valuation ν, we obtain an intrinsic version of a Newton-Okounkov body by replacing "convex hull" in the usual definition with "broken line convex hull". That is, given $(Y, \mathcal{L})$ a polarized projective compactification of $U^\vee$, we define

$$\Delta_\nu^{\mathrm{BL}}(\mathcal{L}) := \overline{\mathrm{conv}_{\mathrm{BL}}\left(\bigcup_{r\geq 1} \frac{1}{r}\nu\left(\Gamma(Y, \mathcal{L}^{\otimes r})\right)\right)}, \tag{1}$$

where $\mathrm{conv}_{\mathrm{BL}}$ denotes the broken line convex hull. Notably, if Y is a Grassmannian and $\mathcal{L}$ is the pullback of $\mathcal{O}(1)$ under the Plücker embedding, $\Delta_\nu^{\mathrm{BL}}(\mathcal{L})$ is simply the broken line convex hull of the valuations of Plücker coordinates. This is closely related to results of [4].

Next, it gives a candidate for a cluster version of a classic toric mirror symmetry construction due to Batyrev.[1] It is easy to write down the Landau-Ginzburg mirror of a Fano toric variety. Each divisorial component of the toric anticanonical divisor is naturally identified with a character on the dual torus, and the potential is simply the sum of these characters. Next, generic sections of the anticanonical bundle cut out (mildly singular) Calabi-Yau hypersurfaces in our toric Fano. Meanwhile, level sets of the potential on the dual torus are *almost* Calabi-Yau. The hang-up is that they are non-compact. But there is a simple fix to this– if we take the Newton polytope of the potential, we obtain the anticanonical polytope of another toric Fano. The potential is itself a section of this anticanonical bundle and defines (mildly singular) Calabi-Yau hypersurfaces in precisely the same way as before. Batyrev proposed that the families of Calabi-Yau hypersurfaces in the pair of toric Fanos were mirror to one another. There is a natural way for this picture to generalize to the cluster world. Let (X, D) be a Fano compactification of a cluster variety U, with D anticanonical. Generic anticanonical sections still give mildly singular Calabi-Yau hypersurfaces in X. Meanwhile, divisorial components of D now correspond to points in $U^{\mathrm{trop}}(\mathbb{Z})$, and so to theta functions on the dual $U^\vee$. The potential W is once again the sum of these functions. As before, it *almost* cuts out Calabi-Yau hypersurfaces in $U^\vee$, but they will fail to be compact. We can compactify $U^\vee$ (and in turn the hypersurfaces) by essentially the same method. We define the *theta function analogue of the Newton polytope of* W as

$$\mathrm{Newt}_\vartheta(W) := \mathrm{conv}_{\mathrm{BL}}\left(p \in U^{\mathrm{trop}}(\mathbb{Z}) \middle| \, \vartheta_p \text{ is a non-zero summand of } W\right). \tag{2}$$

By construction, this defines a polarized projective compactification Y of $U^\vee$, with W a section of the line bundle in question. Our hope is that just like the toric case, the generic hypersurfaces in X and Y described in this way will be mirror Calabi-Yau varieties.

References

[1] V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Alg. Geom. (1994), 493–535.

[2] M.-W. Cheung, T. Magee, and A. Nájera Chávez, *Compactifications of cluster varieties and convexity*, arXiv:1912.13052 [math.AG] (2019).

[3] M. Gross, P. Hacking, S. Keel, and M. Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. **31**(2) (2018), 497–608.

[4] K. Rietsch and L. Williams, *Newton-Okounkov bodies, cluster duality, and mirror symmetry for Grassmannians*, Duke Math. J. **168**(18) (2019), 3437–3527.

Many cluster structures on Schubert varieties in the Grassmannian

Melissa Sherman-Bennett
(joint work with Chris Fraser)

Let $\mathrm{Gr}_{k,n}$ denote the (complex) Grassmannian, considered as a projective variety with respect to the Plücker embedding. We denote Plücker coordinates by P_J, where J is a k-element subset of $\{1, \ldots, n\}$. Let $\widehat{\mathrm{Gr}}_{k,n}$ denote the affine cone over the Grassmannian with respect to this embedding. In [4], Scott showed that the coordinate ring of $\widehat{\mathrm{Gr}}_{k,n}$ is a *cluster algebra*, and seeds for this cluster algebra come from Postnikov's plabic graphs for $\mathrm{Gr}_{k,n}$.

In [5], we showed an analogous result for the *open Schubert varieties* X_I° in the Grassmannian; that is, the coordinate ring of $\widehat{X}_I^\circ$ is a cluster algebra, and seeds for this cluster algebra are given by Postnikov's plabic graphs for X_I°.

There are a few key differences between the cluster structure on the Grassmannian and the cluster structure on open Schubert varieties. For the Grassmannian, the following hold.

- **1A:** Every Plücker coordinate P_J is a cluster variable or a frozen variable.
- **2A:** Every seed consisting entirely of Plücker coordinates comes from a plabic graph for $\mathrm{Gr}_{k,n}$ [3].
- **3A:** For G a plabic graph for $\mathrm{Gr}_{k,n}$, let $\Sigma_T(G)$ and $\Sigma_S(G)$ denote the seeds obtained from target and source labellings, respectively. Then $\Sigma_T(G)$ and $\Sigma_S(G)$ are mutation equivalent.

In constrast, for X_I°, we have the following.

- **1B:** There are many Plücker coordinates which are not identically zero on X_I°, but are neither frozen variables nor cluster variables.
- **2B:** There are seeds consisting of variables $P_J \cdot L$, where L is a Laurent monomial in frozen variables (which are Plücker coordinates), which do not come from a plabic graph for X_I°.
- **3B:** For G a plabic graph for X_I°, the seeds $\Sigma_T(G)$ and $\Sigma_S(G)$ are never mutation equivalent, as they have different frozen variables.

In particular, a plabic graph G for X_I° gives rise to two different cluster structures on the coordinate ring for $\widehat{X}_I^\circ$ (the "source" structure and the "target" structure), which have different cluster variables and, a priori, different cluster monomials. In [2], we establish that these two cluster structures are not too different.

Theorem. [2] *Let G be a reduced plabic graph for X_I°. Then $\Sigma_T(G)$ and $\Sigma_S(G)$ are in the same seed orbit pattern, as introduced in* [1]. *In particular, there is a seed Σ mutation equivalent to $\Sigma_T(G)$ such that the mutable part of the quiver of Σ is the same as the mutable part of the quiver of $\Sigma_S(G)$, and the cluster variables of Σ are, up to multiplication by Laurent monomials in frozen variables, equal to the cluster variables of $\Sigma_S(G)$.*

As an immediate corollary of this theorem, we obtain that the cluster monomials of the source structure and the target structure coincide.

In the proof, we provide a sequence of seeds in the same seed orbit pattern as $\Sigma_T(G)$, all of which come from *generalized plabic graphs*, which are plabic graphs whose boundary labels have been permuted. Each of these generalized plabic graphs gives rise to a seed of the kind mentioned in **2B**, so our proof also sheds light on these well-behaved seeds for X_I° which do not come from usual plabic graphs.

References

[1] C. Fraser, *Quasi-homomorphisms of cluster algebras.* Adv. in Appl. Math. 81 (2016), 40–77.

[2] C. Fraser and M. Sherman-Bennett. *Many cluster structures on Schubert varieties in the Grassmannian*, in preparation.

[3] S. Oh, A. Postnikov, D. Speyer, *Weak separation and plabic graphs.* Proc. Lond. Math. Soc. (3) 110 (2015), no. 3, 721–754.

[4] J. Scott, *Grassmannians and cluster algebras.* Proc. London Math. Soc. (3) 92 (2006), no. 2, 345–380.

[5] K. Serhiyenko, M. Sherman-Bennett, L. Williams, *Proc. Lond. Math. Soc.* (6) 119 (2019) 1694-1744.

Homological mirror symmetry for log Calabi-Yau surfaces

Ailsa Keating

(joint work with Paul Hacking)

Let (Y, D) be a log Calabi–Yau surface with maximal boundary: Y is a smooth rational projective surface over $\mathbb{C}$, and $D \in |-K_Y|$ a singular nodal curve. We study homological mirror symmetry for (Y, D). We focus on the case where (Y, D) has a distinguished complex structure within its deformation class, which will be mirror to an exact symplectic manifold. Given such a pair (Y, D), we construct a four-dimensional Weinstein domain M and a Lefschetz fibration $w : M \to \mathbb{C}$, with fibre Σ, such that:

- Σ is a k–punctured elliptic curve, where k is the number of irreducible components of D; there is a quasi-equivalence $D^\pi \mathcal{F}(\Sigma) \simeq \mathrm{Perf}(D)$, due to Lekili–Polishchuk [5], where $\mathcal{F}(\Sigma)$ is the Fukaya category of Σ, with objects compact Lagrangian branes;
- $D^b \mathcal{F}^{\to}(w) \simeq D^b\mathrm{Coh}(Y)$, where $\mathcal{F}^{\to}(w)$ is the directed Fukaya category of w;
- $D^b \mathcal{W}(M) \simeq D^b\mathrm{Coh}(Y \backslash D)$, where $\mathcal{W}(M)$ is the wrapped Fukaya category of M.

Gross, Hacking and Keel implemented part of the Gross–Siebert mirror symmetry program to construct a mirror family to (Y, D) as the spectrum of an algebra with canonical basis the theta functions associated to (Y, D) [2]. While their mirror family is in general only formal, its fibre is expected to be the total space of an almost-toric fibration, the integral affine base of which already appears explicitly in [2]. We prove that our mirror space M is Weinstein deformation equivalent to the total space of this almost-toric fibration; in particular, we get an explicit description of M as the result of attaching Weinstein 2-handles to $D^*(T^2)$. One consequence is that the cluster structures of [6] are precisely the same as the ones in [3].

Our construction uses the following fact: possibly after blowing up Y at nodes of D to get a log Calabi–Yau pair $(\tilde{Y},\tilde{D})$, there exists a smooth toric pair $(\bar{Y},\bar{D})$ and a birational map $(\tilde{Y},\tilde{D}) \to (\bar{Y},\bar{D})$ given by blowing up interior points of components of $\bar{D}$; varying the blow-up locus within the interior of each component varies the complex structure; for the distinguished one, we blow up only the -1 points of a fixed torus action. Say $\bar{D}$ has irreducible components $\bar{D}_1 + \ldots + \bar{D}_k$, and that we perform m_i blow ups on $\bar{D}_i$ to get $(\tilde{D},\tilde{D})$. Let V_0 be a fixed longitude of Σ, and $W_i, i = 1,\ldots,k$ the natural collection of cyclically symmetric meridiens; let $d_{ij} = (\bar{D}_1 + \ldots + \bar{D}_j) \cdot \bar{D}_i$, and $V_j = \prod_i \tau_{W_i}^{d_{ij}} V_0$. Then one choice of distinguished collection of vanishing cycles for the fibration mirror to $(\tilde{Y},\tilde{D})$ is $(W_{11},\ldots,W_{1m_1},\ldots,W_{k1},\ldots,W_{km_k},V_0,\ldots,V_{k-1})$, where each W_{ij} is a copy of W_i. Our proof of mirror symmetry relies on localisation results of Abouzaid–Seidel or Ganatra–Pardon–Shende [1], building on extensive earlier work of Seidel on Fukaya categories associated to Lefschetz fibrations.

References

[1] S. Ganatra, J. Pardon, and V. Shende, *Sectorial descent for wrapped Fukaya categories*, arXiv:1809.03427v2.

[2] M. Gross, P. Hacking, and S. Keel, *Mirror symmetry for log Calabi-Yau surfaces I*, Pub. IHES, 122, 2015, 65–168.

[3] M. Gross, P. Hacking, and S. Keel, *Birational geometry of cluster algebras*, Algebraic geometry 2, 2015, 137–175.

[4] A. Keating, *Homological mirror symmetry for hypersurface cusp singularities*, Selecta Math. (N. S.), 24, 2018, 1411-1452.

[5] Y. Lekili and A. Polishchuk, *Arithmetic mirror symmetry for genus 1 curves with n marked points*, Selecta Math. (N. S.), 23, 2017, 1851–1907.

[6] V. Shende and D. Treumann, David and H. Williams, *On the combinatorics of exact Lagrangian surfaces*, arXiv:1603.07449.

Mirror symmetry and categorical birational geometry

Daniel Pomerleano

A **maximally degenerate Calabi-Yau pair** $(M,\mathbf{D})$ is a smooth projective variety M together with an snc anticanonical divisor $\mathbf{D}$ which has a zero dimensional stratum. To such a pair $(M,\mathbf{D})$, Gross and Siebert [7] have proposed an **intrinsic mirror construction**, where the mirror partner is given by taking $\mathrm{Spec}(A)$ of an explicit commutative ring of **theta functions**, A. The algebra A comes with a **canonical basis** which is indexed by integral points $B_{\mathbb{Z}}(M,\mathbf{D})$ of a certain affine manifold with singularities $B(M,\mathbf{D})$. In this canonical basis $(\theta_{\mathbf{v}})_{\mathbf{v}\in B_{\mathbb{Z}}(M,\mathbf{D})}$, the structure constants of the multiplication are defined by certain punctured log

Gromov-Witten invariants. It is expected that this construction generalizes previously existing constructions in the theory of cluster varieties [5, 6].

While the construction of [7] is very general and elegant, it remains an open question to formulate and prove a mirror symmetry conjecture which applies to all such $(M, \mathbf{D})$. The chief difficulty in doing this is that the variety $\mathrm{Spec}(A)$ is typically mildly singular outside of special cases (see page 15 of [7]). Moreover, there exist examples arising in toric Calabi-Yau mirror symmetry [2] which show that in dimension > 3, the singularities of $\mathrm{Spec}(A)$ are not even crepant meaning that these singularities cannot be resolved inside of traditional algebraic geometry. The goal of this talk is to describe a precise mirror conjecture (Conjecture 2) and describe progress [4, 10] towards proving it. Our approach makes use of **categorical crepant resolutions** in the sense of [3, 8].

In order to simplify our discussion, we restrict to positive pairs $(M, \mathbf{D})$ — those pairs for which $\mathbf{D}$ supports an ample line bundle $\mathcal{L}$. Equipping M with a Kahler form for $\mathcal{L}$, the complement $X = M \setminus \mathbf{D}$ becomes a Weinstein manifold to which we can attach various symplectic invariants, the most important for us being the symplectic cohomology $SH^*(X)$ [11] and the wrapped Fukaya category $\mathcal{W}(X)$ [1]. It is expected that the first of these invariants allows us to give a symplectic interpretation of the algebra A (c.f. page 5 of [7]).

Conjecture 1. *For any maximally degenerate (positive) Calabi-Yau pair* $(M, \mathbf{D})$*, there is a ring isomorphism* $SH^0(X) \cong A$.

An additive version of this conjecture was recently established for many maximally degenerate Calabi-Yau pairs in [4]. A key ingredient in [4] is the construction, due to McLean [9], of a "nice" Liouville manifold $\bar{X}$ which is deformation equivalent to X and such that the Reeb flow along the boundary of $\bar{X}$ can be explicitly described and come in pseudo Morse-Bott families $\mathcal{F}_{\mathbf{v}}$ for $\mathbf{v} \in B_{\mathbb{Z}}(M, \mathbf{D})$.

Definition. A *weak section* is an exact, conical Lagrangian $L \subset X$ which is diffeomorphic to $\mathbb{R}^n$ such that:

- For every pseudo Morse-Bott family $\mathcal{F}_{\mathbf{v}}$ of Reeb orbits on $\partial\bar{X}$ (for the nice Reeb flow from [9]), a contractible submanifold $\mathcal{G}_{\mathbf{v}}$ of these are also chords of L.
- L has no other Reeb chords.

The important thing about weak sections for us is that an easy argument shows that the closed-open map

$$\mathcal{CO} : SH^0(X) \cong WF^0(L, L) \tag{1}$$

is an isomorphism. It follows that there is a fully faithful functor

$$\pi^* : \mathrm{Perf}(\mathrm{Spec}(SH^0(X))) \to \mathcal{W}(X)$$

which sends the structure sheaf to L. Assuming a certain finiteness of wrapped Floer groups which will be established in [10], the Lagrangian L also gives rise to a natural functor:

$$\pi_* : \mathcal{W}(X) \to \mathrm{D^b Coh}(\mathrm{Spec}(SH^0(X)), \quad L' \to WF^*(L, L')$$

Conjecture 2. *Let $(M, \mathbf{D})$ be a maximally degenerate, (positive) Calabi-Yau pair which admits a weak section. Then the data π_*, π^* give $\mathcal{W}(X)$ the structure of a categorical crepant resolution of $SH^0(X)$ in the sense of Kuznetsov* [8].

The main thing needed to establish Conjecture 2 is to show that $\mathcal{W}(X)$ is *relatively* Calabi-Yau over $SH^0(X)$, meaning that the identity is a relative Serre functor over $\mathrm{D^b Coh}(\mathrm{Spec}(SH^0(X))$.

References

[1] M. Abouzaid and Paul Seidel, *An open string analogue of Viterbo functoriality*, Geometry & Topology, Vol. 14, 2010, no. 2., 627–718.
[2] M. Abouzaid, D. Auroux, L. Katzarkov, *Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces*, Pub. Math. IHES, Volume 123 (2016), 199–282.
[3] A. Bondal, D. Orlov, *Derived categories of coherent sheaves*, Proceedings of the International Congress of Mathematicians (Beijing, 2002), Vol. II, Higher Ed. Press, Beijing 2002, 47–56.
[4] S. Ganatra and D.Pomerleano, *Symplectic cohomology rings of affine varieties in the topological limit*, to appear in GAFA.
[5] M. Gross, P. Hacking, S. Keel, *Mirror symmetry for log Calabi-Yau surfaces I*, Publ. Math. Inst. Hautes Études Sci. 122 (2015), 65–168.
[6] M. Gross, P. Hacking, S. Keel, and M. Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. 31 (2018), no. 2, 497–608.
[7] M. Gross and B. Siebert, *Intrinsic Mirror Symmetry*, 2019, https://arxiv.org/abs/1909.07649.
[8] A. Kuznetsov, *Lefschetz decompositions and categorical resolutions of singularities*, Selecta Math, New ser. (2008), 13:661.
[9] M. McLean, *The growth rate of symplectic homology and affine varieties*, GAFA,Volume 22, 2012, No. 2, pp. 369–442.
[10] D. Pomerleano, *Intrinsic Mirror Symmetry and the Bondal-Orlov conjecture*, in preparation.
[11] P. Seidel, *A biased view of symplectic cohomology*, Current Developments in Mathematics Volume 2006, 2008, pp. 211–253.

Marsh–Rietsch's mirrors from disc counting

Yoosik Kim
(joint work with Hansol Hong and Siu-Cheong Lau)

Strominger–Yau–Zaslow (SYZ) conjecture has been provided a geometric way of understanding mirror symmetry of a dual pair of Calabi–Yau manifolds. The conjecture can be generalized into a class of Fano varieties by considering a mirror Landau–Ginzburg (LG) model. Putting a Fano manifold X on one side, a mirror manifold $\check{X}$ together with a holomorphic function (which is often called a superpotential) should serve as a mirror of X.

In principle, the SYZ approach can produce $\check{X}$ by dualizing a (special) Lagrangian torus fibration on the complement of an anti-canonical divisor of X in [1]. Namely, the moduli of flat unitary connections on Lagrangian torus fibers can be taken as $\check{X}$. In the case when the Lagrangian torus fibers are weakly unobstructed, Fukaya–Oh–Ohta–Ono's disc potential [4] is defined on $\check{X}$ and plays a role of a superpotential of the mirror LG model.

In the Fano toric manifolds [3], this disc counting bounded by Lagrangian toric fibers successfully leads to a LG mirror consisting of an algebraic torus and a Laurent polynomial. It agrees with the well-known Givental–Hori–Vafa's mirror. In the case of toric manifolds, the disc potential has been defined and computed beyond the Fano setting, see [5, 14] for instance.

The closed mirror symmetry of homogeneous varieties has been extensively explored in depth and from various perspectives. Restricting to Grassmannians, Marsh–Rietsch (MR) [9] described the Lie theoretical mirror of Rietsch [12] in terms of the dual Grassmannian and the dual Plücker coordinates. Their mirror can compute the quantum cohomology ring for instance, while the LG mirror of the algebraic torus cannot generally.

The natural question one can ask is how to produce the MR's mirrors via the SYZ mirror symmetry and the disc counting. Any partial flag variety admits a Gelfand–Cetlin toric degeneration [6]. A Lagrangian torus fibration that degenerates into the toric integrable system produces a mirror chart serving as an initial chart by Floer theory, see [10].

One may consider other toric degenerations to obtain other Lagrangian torus fibrations by using the construction in [7] for instance. In principle, their mirrors of certain degenerations give rise to cluster charts. To relate such Lagrangian torus fibrations, one produces a Lagrangian torus fibration with *singular* fiber(s) interpolating a suitable pair of fibrations. A cluster mutation is then recovered by a wall-crossing formula, which a singular fiber causes. In the case of Grassmannian of two planes, Nohara and Ueda [11] partially recovered the MR's mirrors from their generalized Gelfand–Cetlin systems and proved that the wall-crossings coincide with the cluster transformations.

To relate Lagrangian tori in two different chambers from Floer theory, one needs to classify holomorphic strips bounded by two Lagrangians. They lead to a quasi-isomorphism between two Lagrangian branes. Moreover, it systematically glues the mirror charts, the disc potentials [13], and moreover the localized mirror functors [2]. The glued functor is from a Fukaya (sub)category to the category of matrix factorizations, which provides a foundational step toward homological mirror symmetry.

One important point is that the glued cluster charts cannot cover in general the MR's mirror, the complement of the anti-canonical divisor defined by the frozen variables in the dual Grassmannian. Homological mirror symmetry at least in this case implies that the critical loci should correspond to certain Lagrangians (with deformations). In our picture, the singular Lagrangians will be responsible for filling up the missing parts.

To analyze the singular Lagrangians is difficult. It is partly because it bounds constant discs with insertions of immersed sectors. The moduli of such discs has a larger dimension than the expected dimension so that a virtual perturbation is necessary to deal with the situation. Turning on the abstract perturbation makes a direct computation almost impossible. The quasi-isomorphisms among the singular Lagrangian and smooth Lagrangian tori allow us to compute the disc potential of the singular Lagrangian.

As a starting point, we take a Lagrangian fibration which degenerates into a toric moment map. Let us choose the mirror chart of the chosen fibration as the initial seed. Consider a Lagrangian torus fibration with interpolating other Lagrangian fibrations (adjacent to Lagrangian fibrations), admitting singular fibers. Singular Lagrangians produce mirror charts which can be computed as above. We glue them with the initial cluster chart to compactify it partially. We believe this procedure systematically produces a partially compactified mirror.

In the case of Grassmannian of two planes in [8], we glue local mirror charts to obtain the MR's mirrors including the strata that cannot be covered by the cluster charts. Namely, the glued mirror of the initial chart, the cluster charts adjacent to the initial chart, and the mirror charts from certain singular Lagrangians recovers the desired mirror.

References

[1] D. Auroux, *Mirror symmetry and T-duality in the complement of an anticanonical divisor*, J. Gökova Geom. Topol. GGT **1** (2007), 51–91.

[2] C.-H. Cho, H. Hong, and S.-C. Lau, *Gluing localized mirror functors*, preprint, arXiv:1810.02045.

[3] C.-H. Cho and Y.-G. Oh, *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*, Asian J. Math. **10** (2006), no. 4, 773–814.

[4] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian intersection Floer theory: anomaly and obstruction. Part I and II*, AMS/IP Studies in Advanced Mathematics, vol. 46, American Mathematical Society, Providence, RI, 2009.

[5] ———, *Lagrangian Floer theory on compact toric manifolds. I*, Duke Math. J. **151** (2010), no. 1, 23–174.

[6] N. Gonciulea and V. Lakshmibai, *Degenerations of flag and Schubert varieties to toric varieties*, Transform. Groups **1** (1996), no. 3, 215–248.

[7] M. Harada and K. Kaveh, *Integrable systems, toric degenerations and Okounkov bodies*, Invent. Math. **202** (2015), no. 3, 927–985.

[8] H. Hong, Y. Kim, and S.-C. Lau, *Immersed two-spheres and SYZ with application to Grassmannians*, preprint, arXiv:1805.11738.

[9] R. Marsh and K. Rietsch, *The B-model connection and mirror symmetry for Grassmannians*, preprint, arXiv:1307.10853.

[10] T. Nishinou, Y. Nohara, and K. Ueda, *Toric degenerations of Gelfand-Cetlin systems and potential functions*, Adv. Math. **224** (2010), no. 2, 648–706.

[11] Y. Nohara and K. Ueda, *Potential functions on Grassmannians of planes and cluster transformations*, preprint, arXiv:1711.04456.

[12] K. Rietsch, *A mirror symmetric construction of $qH^*_T(G/P)_{(q)}$*, Adv. Math. **217** (2008), no. 6, 2401–2442.

[13] P. Seidel, *Lectures on categorical dynamics and symplectic topology*, preprint, http://math.mit.edu/seidel/937/lecture-notes.pdf.

[14] C. Woodward, *Gauged Floer theory of toric moment fibers*, Geom. Funct. Anal. **21** (2011), no. 3, 680–749.

An example of family Floer mirror

Man-Wai Mandy Cheung
(joint work with Yu-Shen Lin)

Fukaya [1] initiated the family Floer mirrors so as to understand mirror symmetry. Later, Abouzaid showed that the family Floer functors induce the homological mirror symmetry. While the idea of the construction is laid out, there are very few explicit examples are known. The challenge comes from the difficulty of controlling which Lagrangians bound the holomorphic discs. On the other hand, Gross, Hacking, and Keel [2] gave the construction of the mirror family to log Calabi-Yau surfaces by considering the canonical basis of the vector spaces with algebraic structures. Together with Yu-Shen Lin, we have checked the two constructions coincides (analytically) for the case of del Pezzo surface of degree 5. This gives an explicit example of family Floer mirror, verifies the construction by Gross, Hacking, and Keel, and shows the mirror symmetry for the cluster varieties of type A_2.

The mirror construction in [2] and [3] begins with a pair (Y, D), where Y is a smooth rational projective surface over an algebraically closed field $\mathbb{K}$ of characteristic 0, and D is a anti-canonical divisor as a chain of projective lines. The tropicalization of (Y, D) is a pair (B, Σ), where B is an integral linear manifold with singularities, and Σ is a decomposition of B into cones. The pair (B, Σ) can be constructed by associating each node $p_{i,i+1}$ of D a rank two lattice with basis v_i, v_{i+1}. Take $\sigma_{i,i+1} \subset M_{i,i+1} \otimes \mathbb{R}$ to be the cone generated by v_i, v_{i+1}. The cones $\sigma_{i,i+1}$ and $\sigma_{i-1,i}$ will be glued along the ray $\rho_i = \mathbb{R}_{\geq 0} v_i$ to obtain a piecewise linear manifold B homeomorphic to $\mathbb{R}^2$ and $\Sigma = \{\sigma_{i,i+1}\} \cup \{\rho_i\} \cup \{0\}$. Then one can associate $B \setminus \{0\}$ with an integral affine structure.

Now consider Y the del Pezzo surface of degree 5 and D the anti-canonical cycle of five (-1)-curves. Then B can be viewed as five cones in $\mathbb{R}^2$ together with a monodromy. The next step is to associate wall functions to the rays in Σ. Roughly speaking, the coefficients of the functions are given by some Gromov-Witten invariants for counting of $\mathbb{A}^1$ curves. The pair (B, Σ), together with the wall functions, is a consistent scattering diagram. One can then construct the family of mirrors according to the scattering diagram structure. For the sake of our family Floer mirror calculation, we would consider a particular mirror in the family. We associate each cone with a torus and glue the tori by the wall crossing functions. The theta functions are then introduced to give the compactification of the space. The constructed mirror would be the spectrum of the set of theta functions with the algebraic structure.

On the symplectic side, Lin and I consider the rational elliptic surface Y' with singular configuration II^*II. Let $X' = Y'$ minus the fibre of type II^*. The hyperKähler rotation of X' gives us the special Lagrangian fibration $X \to B = \mathbb{R}^2$. Then we associate each point $u \in B$ with a Maurer-Cartan space $\mathcal{M}(L_u)$, where L_u is the Lagrangian fibre of u. In our case, the Maurer-Cartan spaces are $H^1(L_u, \Lambda_+)$, where Λ is the Novikov field. The family Floer mirror is morally the disjoint unions of the Maurer-Cartan spaces quotient by isomorphisms. In this case, Lin [4] has shown that the isomorphisms behave as wall crossings. Particularly, the coefficients of the wall crossing functions are open Gromov-Witten invariants [4]. To obtain the wall and chamber structure as in [2], we take the BPS rays as our walls, where the BPS rays are defined to be the support of loci with non-trivial open Gromov-Witten invariants. Combining the BPS rays, and the automorphisms of the Lagrangian fibre as the wall crossing functions, would be resulted as a scattering diagram. This scattering diagram is consistent and can be identified as the one described in the last paragraph. It is then natural to look for the gluing of tori. Instead of having algebraic tori, the Novikov field leads to analytic tori. Theta functions are defined similarly and one can see that the set of theta functions shares the same algebraic structure as in the Gross, Hacking, Keel [2] mirror which gives the del Pezzo surface of degree 5.

References

[1] K. Fukaya, *Floer homology for families-a progress report*, Contemporary Mathematics, **309**, 33–68.

[2] M. Gross, P. Hacking, S. Keel, *Mirror symmetry for log Calabi-Yau surfaces I*, Publications mathématiques de l'IHÉS, **122(1)** (2015), 65–168.

[3] M. Gross, P. Hacking, S. Keel, B. Siebert, *The mirror of the cubic surface*, arXiv preprint, arXiv:1910.08427 (2019).

[4] Y.S. Lin, *On the tropical discs counting on elliptic K3 surfaces with general singular fibres*, Transactions of the American Mathematical Society, **373(2)** (2020), 1385-1405.

Laurent polynomial superpotentials and deformations of toric varieties

Andrea Petracci

(joint work with Alessio Corti, Matej Filip, Paul Hacking)

In this talk I have tried to explain the following principle which is somehow contained in [4].

Slogan. *The choice of a 'special' Laurent polynomial f selects a way to deform the (possibly singular) toric variety associated to the face fan of the Newton polytope of f. Moreover f is mirror to the general fibre of this deformation.*

This should hold both

- in the *affine case*: here the Newton polytope of f is a lattice polytope F in an affine lattice and we consider the toric affine variety U_F associated to the cone over F put at height 1; and
- in the *Fano case*: here the Newton polytope of f is a reflexive polytope P in a lattice and we consider the toric Fano variety X_P associated to the face fan of P.

We also expect a local-to-global phenomenon: since X_P is the obtained by gluing the affine charts U_F's where F runs among the facets of P, the choice of a 'special' Laurent polynomial f on P will induce by restriction 'special' Laurent polynomials

$f|_F$ on each facet $F < P$, consequently deformations of the U_F's; the deformation of X_P induced by f should be obtained by gluing these deformations of the U_F's.

Of course one of the first issues is to identify a/the class of 'special' Laurent polynomials — in general, these are polynomials which can be mutated via cluster transformations.

I have started my talk by considering an example due to Altmann [2] (see also [8]): here P is a 3-dimensional pyramid over a lattice hexagon F and X_P is the projective cone over the anticanonical embedding of del Pezzo surface of degree 6. One can see that X_P deforms to two different smooth Fano 3-folds, namely $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and the $(1,1)$-divisor in $\mathbb{P}^2 \times \mathbb{P}^2$. These two smoothings of X_P are mirror to two different Laurent polynomials supported on P, which can be constructed from the two different Minkowski decomposition of the hexagon F.

Then I have stated the following theorem which generalises the example above to many toric Fano 3-folds with Gorenstein singularities. The following theorem builds on Altmann's construction of deformations of toric affine varieties [1].

Theorem. (Corti–Hacking–P. [7]) *Let P be a reflexive polytope of dimension 3 and let X_P be the toric Fano 3-fold associated to the face fan of P. For each facet F of P choose a Minkowski decomposition of F into A-triangles (i.e. either unit segments or lattice triangles without interior lattice points and at least two unit edges). If a technical condition (which is necessary by* [9]*) is satisfied, then we construct a smoothing of X_P.*

It is possible to compute the Hodge numbers of the smoothing [3] and consequently understand which smooth Fano 3-folds one obtains from this theorem.

From the choice of the Minkowski decompositions of the facets of P one immediately constructs a Laurent polynomial f with Newton polytope P. Thanks to [5] one can verify that f is mirror to the constructed smoothing of X_P.

The theorem above does not say anything when there is a facet which is not Minkowski-decomposable into A-triangles. Therefore it is crucial to understand deformations of all toric Gorenstein 3-fold singularities.

We have defined a new class of Laurent polynomials with integer coefficients. Roughly speaking, a Laurent polynomial is called *0-mutable* if it becomes a monomial via a finite sequence of factorisations and mutations. We expect that Altmann's construction of deformations of toric affine varieties generalises to the following conjecture.

Conjecture. (Corti–Filip–P. [6]) *Let F be a lattice polytope of dimension 3 and let U_F be the Gorenstein toric affine 3-fold associated to the cone over F put at height 1. Then the smoothing components of the miniversal deformation of U_F are in one-to-one correspondence with the 0-mutable Laurent polynomials with Newton polytope F.*

References

[1] K. Altmann, *Minkowski sums and homogeneous deformations of toric varieties*, Tohoku Math. J. (2) **47** (1995), no. 2, 151–184.

[2] K. Altmann, *The versal deformation of an isolated toric Gorenstein singularity*, Invent. Math. **128** (1997), no. 3, 443–479.

[3] T. Coates, A. Corti, G. da Silva Jr, *On the Topology of Fano Smoothings*, arXiv:1912.04383.

[4] T. Coates, A. Corti, S. Galkin, V. Golyshev, A. Kasprzyk, *Mirror symmetry and Fano manifolds*. In: European Congress of Mathematics, 285-300, Eur. Math. Soc., Zürich, 2013.

[5] T. Coates, A. Corti, S. Galkin, A. Kasprzyk, *Quantum periods for 3-dimensional Fano manifolds*, Geom. Topol. 20 (2016), no. 1, 103–256.

[6] A. Corti, M. Filip, A. Petracci, *Smoothing Gorenstein toric affine 3-folds, 0-mutable polynomials, and mirror symmetry*, in preparation.

[7] A. Corti, P. Hacking, A. Petracci, *Smoothing toric Fano 3-folds*, in preparation.

[8] A. Petracci, *An Example of Mirror Symmetry for Fano Threefolds*. In: E. Colombo, B. Fantechi, P. Frediani, D. Iacono, R. Pardini (eds), Birational Geometry and Moduli Spaces. Springer INdAM Series, vol 39. Springer, Cham (2020).

[9] A. Petracci, *Some examples of non-smoothable Gorenstein Fano toric threefolds*, to appear on Math. Z., doi.org/10.1007/s00209-019-02369-8.

Reporter: Melissa Sherman-Bennett

Participants

Dr. Lara Bossinger
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Centro Historico
Antonio de Léon # 2, altos
Oaxaca de Juárez, CP. 68000
MEXICO

Dr. Man-Wai Mandy Cheung
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Prof. Dr. Eduardo Gonzalez
Department of Mathematics
University of Massachusetts at Boston
100 William T. Morrissey Boulevard
Boston, MA 02125-3393
UNITED STATES

Prof. Dr. Megumi Harada
Department of Mathematics and Statistics
McMaster University
1280 Main Street West
Hamilton ON L8S 4K1
CANADA

Dr. Ailsa Keating
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WA
UNITED KINGDOM

Dr. Yoosik Kim
Center of Mathematical Sciences and Applications
Harvard University
20 Garden Street
Cambridge, MA 02138
UNITED STATES

Dr. Yanki Lekili
Department of Mathematics
King's College London
Strand
London WC2R 2LS
UNITED KINGDOM

Dr. Timothy Magee
School of Mathematics and Statistics
The University of Birmingham
Edgbaston
Birmingham B15 2TT
UNITED KINGDOM

Dr. Alfredo Nájera Chávez
Instituto de Matemáticas
Universidad Nacional Autonoma de Mexico
Centro Historico, Office 4
León # 2, altos
68000 Oaxaca de Juárez
MEXICO

Dr. Clélia Pech
School of Mathematics, Statistics and Actuarial Science
University of Kent
Sibson Building
Parkwood Road
Canterbury, Kent CT2 7FS
UNITED KINGDOM

Dr. Andrea Petracci
Institut für Mathematik
Freie Universität Berlin
Raum 115
Arnimallee 3
14195 Berlin
GERMANY

Prof. Dr. Daniel Pomerleano
Department of Mathematics and
Computer Science
University of Massachusetts, Boston
100 William T. Morrissey Boulevard
Boston, MA 02125-3393
UNITED STATES

Prof. Dr. Konstanze Rietsch
Department of Mathematics
King's College London
Strand
London WC2R 2LS
UNITED KINGDOM

Melissa Sherman-Bennett
Department of Mathematics
Harvard University
SC Rm. 425b
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Peter Spacek
School of Mathematics, Statistics and
Actuarial Science
University of Kent
Sibson Building, Rm. 123
Parkwood Road
Canterbury, Kent CT2 7FS
UNITED KINGDOM

Charles M. Wang
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Prof. Dr. Lauren K. Williams
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Prof. Dr. Christopher Woodward
Department of Mathematics
Rutgers University
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
UNITED STATES