Mathematisches Forschungsinstitut Oberwolfach

Report No. 15/2020

DOI: 10.4171/OWR/2020/15

Mathematical Advances in Geophysical Fluid Dynamics (online meeting)

Organized by
Yoshikazu Giga, Tokyo
Matthias Hieber, Darmstadt
Peter Korn, Hamburg
Edriss S. Titi, Cambridge, College Station and Rehovot

17 May – 23 May 2020
as online meeting June 15 - June 19 2020

Abstract. This workshop on "Mathematical Advances in Geophysical Fluid Dynamics" was organized as an online seminar and addressed recent advances in analytical, modeling and computational studies of geophysical fluid models. Of particular interest were the contributions concerning modeling and computation of sea-ice models, well-posedness results for the primitive equations, internal waves for stratified flows and models for moist atmospheric dynamics including phase transitions.

Mathematics Subject Classification (2010): 76-XX, 86A10, 35-XX, 60Hxx.

Introduction by the Organizers

This online seminar was fostering the investigation of certain classes of geophysical models by methods stemming from analysis, computation and modeling. The complexity of fluid models taking into account geophysical factors and circumstances shows the need for accessible and reliable reduced models.

The mathematical invstigation of these reduced models involves many modern tools from analysis and computation. Of concern are in particular local and global well-posedness properties of the associated systems of equations, their rigorous justification and validation as well as the development of numerical and computational schemes for their simulations.

A particular challenge in this context is the mathematical understanding of various sea-ice models. The complexity of these coupled fluid-viscoplastic models show the need for new analytical and computational methods in this direction.

Also models for moist atmospheric dynamics including phase transitions ask for new analytical and computational tools in order to treat the Heaviside terms arising there modeling the fast time scale saturation effects.

One of the main characteristics of the online seminar was the bringing together of leading experts from diverse scientific backgrounds such as analysis, modeling, numerics and computation. The meeting ignited lively interaction and exchange of ideas, which was an inspiring experience. The presence of younger participants and gender diversity was very visible during the meeting. The workshop aimed also to encouraged younger participants to play an important role in this area of research. This was also implemented by inviting some younger speakers.

The lectures presented took 50 minutes which were followed by lively and interactive discussions for about 15 minutes. The meeting brought together a very good mixture of various communities and several leaders from different disciplines met here for the first time. The gender diversity was remarkable.

It seems that this seminar was the very first Oberwolfach online seminar. Due to the Corona crisis we needed to replace the originally planned workshop by a new format and choose this online seminar as a substitute. Of course, it is evident that such an online seminar cannot give all the privileges a classical workshop at MFO is offering to its participants.

Workshop (online meeting): Mathematical Advances in Geophysical Fluid Dynamics

Table of Contents

Sergey Danilov
Unstructured meshes in global ocean modeling 861

Carolin Mehlmann (joint with Peter Korn, Thomas Richter)
Finite elements and sea ice dynamics 862

Takahito Kashiwabara (joint with K. Furukawa, Y. Giga, M. Gries, M. Hieber, A. Hussein, M. Wrona)
Semigroup and maximal regularity approach to the primitive equations . 865

Anne-Laure Dalibard (joint with Roberta Bianchini,LaureSaint-Raymond)
Critical reflection of internal waves 867

Leslie M. Smith (joint with Sam Stechmann, Yeyu Zhang)
Fast Wave Averaging with Phase Changes: Application to Moist Atmospheric Dynamics 868

Sabine Hittmeir (joint with Rupert Klein, Jinkai Li, Edriss Titi)
Multiscale Asymptotics and Analysis for Atmospheric Flow Models with Moisture 870

Abstracts

Unstructured meshes in global ocean modeling

Sergey Danilov

Global ocean models are an important component of Earth climate models helping to study the future climate. Traditionally, such models were formulated on structured meshes. However there are factors that are difficult to take into account in this case. Geometrical factors include the shape of coastlines and the presence of narrow passages in the world ocean. Dynamical factors come from the strong decrease of the Rossby radius of deformation in high latitudes or the ocean shelf areas, which hampers the representation of mesoscale variability, generated through baroclinic instabilities and contributing to numerous exchange processes. These factors motivate the use of unstructured meshes allowing the resolution that is varied according to practical needs. Three new model development initiatives, two in Germany at the Alfred Wegener Institute (FESOM) and the Max Planck Institute for Meteorology (ICON), and one in the united States at the Los Alamos National Laboratory (MPAS-Ocean) offer such models now. Their computational efficiency approaches that of structured mesh models, they are mature enough and are already used in climate studies.

Such models are formulated on triangular (FESOM, ICON) or dual (quasi-hexagonal) meshes (MPAS). They solve the equations of ocean dynamics which include the Navier-Stokes and continuity equations, together with the conservation equations for the potential temperature and salt under traditional approximations (the Boussinesq and hydrostatic ocean in a thin layer on the sphere). The common practice in ocean modeling is the use of staggered discretizations, which place discrete velocity and pressure at different locations. The reason for that is an improved representation of pressure gradient and, hence, wave dynamics. However, numerics of staggered discretizations on triangular or dual quasi-hexagonal meshes face complications related to the geometrical structure of such meshes.

Let $\mathcal{V}_h, \mathcal{C}_h, \mathcal{E}_h$ be the finite dimensional vector spaces related to the degrees of freedom at vertices, cells and edges of a triangular mesh, which is a set of non-overlapping triangles $\mathcal{T} = \{T_i\}$, $i = 1, \dots N_t$. In this case $\dim \mathcal{V}_h : \dim \mathcal{C}_h : \dim \mathcal{E}_h \approx 1 : 2 : 3$. If d_u and d_p are the dimensions of spaces spanned by discrete velocities and pressure, their ratio will deviate from 2, as is the case on structured quadrilateral meshes, for most of staggered finite-volume or finite-element discretizations on triangular meshes or meshes dual to them. For example, $\frac{d_u}{d_p} = \frac{2 \dim \mathcal{C}_h}{\dim \mathcal{V}_h} = 4$ for the cell–vertex (velocity–pressure) discretization of FESOM, $\frac{d_u}{d_p} = \frac{\dim \mathcal{E}_h}{\dim \mathcal{C}_h} = 3/2$ and $\frac{d_u}{d_p} = \frac{2 \dim \mathcal{E}_h}{\dim \mathcal{V}_h} = 3$ in the case of MPAS. These staggered discretizations are unbalanced, and as a consequence, maintain spurious dynamics. On the geometrical level, the occurrence of spurious dynamics can be viewed as the presence of geometrically different locations for like discrete variables within a unit cell of triangular lattice. This creates a possibility for non-trivial dynamics within the cell, which is spurious in all cases above. Spurious dynamics needs to be suppressed, and this is achieved either through specific numerical viscosity operators, or filters. Although good practical solutions are already available, this topic is still a subject of active research, aimed at designing the solutions with the least dissipation of the resolved physical dynamics.

The three discretizations above require a very similar computational efforts, for the majority of operations is performed in cycles over edges. However, they differ in details, posing a question on their effective resolution. This notion combines the geometrical resolution provided by the mesh with the accuracy of particular numerical algorithms. This is the subject of ongoing research. The set of questions mentioned here is discussed in [1], while the model description can be found in [2], [3] and [4].

References

[1] S. Danilov, *Ocean modeling on unstructured meshes*, Ocean Model. **69** (2013), 195–210.

[2] S. Danilov, D. Sidorenko, Q. Wang, T. Jung, *The Finite-volumE Sea ice – Ocean Model (FESOM2)*, Geosci. Model Dev. **10** (2017), 765–789.

[3] P. Korn, *Formulation of an unstructured grid model for global ocean dynamics*, J. Comput. Phys. **339** (2017), 525–552.

[4] T. Ringler, M. Petersen, R. Higdon, D. Jacobsen, M. Maltrud, P. W. Jones, *A multi-resolution approach to global ocean modelling*, Ocean Modell. **69** (2013), 211–232.

Finite elements and sea ice dynamics

Carolin Mehlmann

(joint work with Peter Korn, Thomas Richter)

Currently the most used approach to model sea ice dynamics in coupled climate models is the *viscous-plastic sea model* introduced by Hibler in 1979 [1]. Here, sea ice is considered as a two dimensional fluid which is located between ocean and atmosphere. Sea ice is characterized by the sea ice concentration A (the percentage of a grid cell that is covered with ice), the mean ice thickness H and the sea ice velocity v. The sea ice concentration and the sea ice thickness are advected in time by transport equations, whereas the the sea ice velocity is determined by

$$m\partial_t v = F + div(\sigma),$$

where $m = \rho H$ is the sea ice mass, F are the external forces as wind and water drag or Coriolis force and σ describes internal sea ice stresses. The ice stresses are related to the strain rate tensor $(\nabla v + \nabla v^T)$ by the viscous-plastic rheology

$$\sigma = \frac{1}{2}\zeta(v)\Big(\nabla v + \nabla v^T\Big) + \frac{3}{4}\zeta(v)\mathrm{tr}(\nabla v + \nabla v^T)I - \frac{P}{2}I,$$

where the viscosity ζ and the ice strenght P is chosen as

$$\zeta = \frac{P}{2\max(\Delta, 2\cdot 10^{-9})}, \quad P = H\exp(20(1-A)),$$
$$\Delta = \sqrt{1.25(v_{1,x}^2 + v_{2,y}^2) + 0.5(v_{1,y}+v_{2,x})^2 + 1.5(v_{1,x}v_{2,y})}.$$

For a one-dimensional reduction of the system Gray showed the ill-possdness of the viscous-plastic model [2]. The analysis has been extended by Guba and coauthors to a two dimensional case where the authors could not conclude if the model is ill or well-posed [3]. We emphasis that using a proper regularization of the viscosities

$$\zeta = \frac{P}{\sqrt{\Delta^2 + 4 \cdot 10^{-18}}},$$

as suggested by Kreyscher et al. in [4], the issues appearing in the one dimensional reduction can be circumvented and the conclusion of the ill-posedness can not be drawn.

As sea ice is part of a coupled system, we address the question how the discrete sea-ice dynamics can be formulated in a way such that the internal sea ice dynamics are captured well while the external coupling to the ocean is accomplished in a natural way. We consider this problem on triangular grids in the context of the ocean general circulation model ICON-O [5]. Here the horizontal ocean velocity is represented by its normal component placed on the edge midpoint. This staggering is equivalent to the lowest order Raviart-Thomas finite element (RT-0). The space of the Raviart-Thomas element is not rich enough to approximate the full strain rate tensor $(\nabla v + \nabla v^T)$ appearing in the sea ice rheology, see [6]. We propose to enlarge the approximation space by including the tangential velocity at the edge midpoint. This variable arrangement equals to the so called Crouzeix-Raviart finite element and allows the desired natural coupling to the underlying ocean variables on the same grid. A direct application of the Crouzeix-Raviart element to the sea-ice momentum equations leads to an unstable discretization. This instability has its origin in the discretization of the symmetric strain rate tensor in the rheology, which has a non trivial kernel in the Crouzeix-Raviart space. Thus, the element does not satisfy Korn's first inequality $\|\nabla v\|^2 \leq c\|\frac{1}{2}(\nabla v + \nabla v^T)\|^2$, see [7]. In order to circumvent this instability we follow the idea of Hansbo and Larson [8] and introduce a stabilization of the Crouzeix-Raviart element, see [9]

The viscous-plastic rheology introduces a strong nonlinearity to the sea ice model. Thus, solving sea ice dynamics of increasing spatial resolution is extremely difficult. There are mainly three approaches for solving the nonlinear momentum equation of the viscous-plastic sea ice model, a fixed-point method denoted as Picard solver, an inexact Newton method (JFNK) [10] and a subcycling procedure based on an elastic-viscous-plastic model approximation [11]. All methods tend to have problems on fine meshes by sharp structures in the solution. Convergence rates deteriorate such that either too many iterations are required to reach sufficient accuracy or convergence is not obtained at all. To improve robustness, globalization and acceleration approaches, which increase the area of fast convergence, are needed. We developed an implicit scheme with improved convergence properties by combining an inexact Newton method with a Picard solver. We show that the Jacobian is a positive definite matrix, guaranteeing global convergence of a properly damped Newton iteration, see [12], [13]. By comparing the performance of the new damped Newton solver to the JFNK solver, we found that we could decrease the number of time steps in which the solver can not reduce the residual to the given tolerance from 24% to 0.5% [14].

References

[1] W.D. Hibler, *A dynamic thermodynamic sea ice model*, Journal of physical oceanography **9** (1979), 815-846

[2] J. Gray, *Loss of Hyperbolicity and ill-posedness of viscous-plastic sea ice rheology in uniaxial divergent flow*, Journal of physical oceanography, **29** (1999), 2920-2929

[3] O. Guba, J. Lorenz, D. Sulsky., *On well posedness of the Viscous-plastic sea ice model*, Journal of physical oceanography, **43** (2013), 2185-2199

[4] M Kreyscher, M. Harder, P. Lemke, G. Flato, *Results of the Sea Ice Model Intercomparison Project Evaluation of sea ice rheology schemes for use in climate simulations*, Journal of Geophysical research, **105** (2000), 299-320

[5] P. Korn, *Formulation of an unstructured grid model for global ocean dynamics* Journal of Computational Physics, **339** (2017), 525-552

[6] G. Acosta, Th. Apel, R. G. Duran, A. L. Lombardi, *Error estimates for Raviart-Thomas interpolation of any order on anisotropic tetrahedra* Mathematics of Computation, **80** (2011), 141-163

[7] R. Falk. *Nonconforming Finite Element Methods for the Equations of Linear Elasticity* Mathematics of Computation, **57** (1991), 529-529,

[8] P. Hansbo and M. Larson *Discontinuous Galerkin and the Crouzeix-Raviart element: Application to elasticity*, ESAIM, **37** (2003), 63-72

[9] C. Mehlmann, P. Korn, *Sea ice on triangular grids*, arxiv.org/abs/2006.00547, (2020)

[10] J.-F. Lemieux, B. Tremblay, J. Sedlacek, P. Tupper, S. Thomas, D. Huard, J.-P. Auclair, *Improving the numerical convergence of viscous-plastic sea ice models with the Jacobian-free Newton Krylov method*, Journal of Computational Physics, **229** (2010), 2840-2852

[11] M. Kimmritz, S. Danilov, M. Losch. *On the convergence of the modified elastic-viscous-plastic method for solving the sea ice momentum equation* Journal of Computational Physics, **296** (2015), 90-100

[12] C. Mehlmann , T. Richter *A modified global Newton solver for viscous-plastic sea ice models* Ocean Modeling, **116** (2017), 96-107

[13] C. Mehlmann, T. Richter *An accelerated Newton method for nonlinear materials in structure mechanics and fluid mechanics*, Numerical Mathematics and Advanced Applications - EnuMath 2017, Springer, (2019)

[14] Efficient numerical methods to solve the viscous-plastic sea ice model at high spatial resolutions, PhD Thesis, dx.doi.org/10.25673/14011, (2019)

Semigroup and maximal regularity approach to the primitive equations

Takahito Kashiwabara
(joint work with K. Furukawa, Y. Giga, M. Gries, M. Hieber, A. Hussein, M. Wrona)

In this extended abstract, we would like to present more explanations to some of our results mentioned in the talk only for a short time.

1. $L^\infty_H L^p_z$-approach for the Dirichlet–Neumann problem

We consider the primitive equations of the following form:

$$\begin{aligned} \partial_t v - \Delta v + (u\cdot\nabla)v + \nabla_H \pi &= 0 && \text{on } \Omega\times(0,\infty),\\ \partial_z \pi &= 0 && \text{on } \Omega\times(0,\infty),\\ \mathrm{div}_H \overline{v} &= 0 && \text{on } G\times(0,\infty),\\ v(0) &= a && \text{on } \Omega, \end{aligned}$$

where $\Omega = G\times(-h,0)$ and $G=(0,1)^2$.

$$\partial_z v = 0 \text{ on } \Gamma_u\times(0,\infty),\quad \pi, v \text{ periodic on } \Gamma_l\times(0,\infty),\quad v=0 \text{ on } \Gamma_b\times(0,\infty),$$

where $\Gamma_u = G\times\{0\}$, $\Gamma_l = \partial G\times[-h,0]$ and $\Gamma_b = G\times\{-h\}$.

Our main result to this problem is as follows.

Theorem 1 ([1]). (i) Let $p\in(3,\infty)$ and $a\in X_{\bar\sigma}$, where $X = C([0,1]^2; L^p(-h,0))\cap L^p(\Omega)$ and $X_{\bar\sigma} = \{v\in X : \mathrm{div}_H\,\overline{v}=0\}$. Then there exists a unique solution of the primitive equations such that

$$v\in C([0,\infty);X_{\bar\sigma}),\quad t^{1/2}\nabla v\in L^\infty((0,\infty);X),\quad \limsup_{t\to0+} t^{1/2}\|\nabla v(t)\|_{L^\infty_H L^p_z(\Omega)} = 0.$$

(ii) For $p\in(3,\infty)$, there exists a constant $C_0>0$ such that if $a=a_1+a_2$ with $a_1\in X_{\bar\sigma}$ and $a_2\in L^\infty_H L^p_z(\Omega)^2$ ($\mathrm{div}_H\,\overline{a_2}=0$) with $\|a_2\|_{L^\infty_H L^p_z(\Omega)}\le C_0$, then there exists a unique solution to the primitive equations such that

$$v\in C([0,\infty);L^p(\Omega))\cap L^\infty((0,T);L^\infty_H L^p_z(\Omega))^2$$

$$t^{1/2}\nabla v\in L^\infty((0,\infty);X),\quad \limsup_{t\to0+} t^{1/2}\|\nabla v\|_{L^\infty_H L^p_z(\Omega)}\le C\|a_2\|_{L^\infty_H L^p_z},$$

We emphasize that the solutions constructed in the above theorem are global in time and strong (even classical after $t>0$). The important ingredient needed to prove this theorem is several semigroup estimates involving the hydrostatic Stokes operator A given by

$$Av = \Delta v + \frac{1}{h}\nabla_H(-\Delta_H)^{-1}\mathrm{div}_H\Big(\partial_z v|_{z=-h}\Big).$$

Note that A differs from the Laplace operator unlike the case of the Neumann boundary conditions, which makes the analysis more involved.

2. Convergence of anisotropic Navier–Stokes equations to PEs

We consider the following anisotropic Navier–Stokes equations:

$$
\left\{
\begin{array}{rcl}
\partial_t v_\varepsilon + u_\varepsilon \cdot \nabla v_\varepsilon - \Delta v_\varepsilon + \nabla_H p_\varepsilon = & 0 & \text{in } (0,T)\times\Omega, \\
\varepsilon^2(\partial_t w_\varepsilon + u_\varepsilon \cdot \nabla w_\varepsilon - \Delta w_\varepsilon) + \partial_z p_\varepsilon = & 0 & \text{in } (0,T)\times\Omega, \\
\operatorname{div} u_\varepsilon = & 0 & \text{in } (0,T)\times\Omega,
\end{array}
\right.
\tag{1}
$$

where $\Omega = \mathbb{T}^3$ (more precisely, the problem with the Neumann boundary conditions can be reduced to periodic boundary conditions by adopting suitable even/odd extensions). Making $\varepsilon = 0$ in (1), we formally obtain the primitive equations. The following theorem justifies this intuition rigorously, establishing the rate of convergence $O(\varepsilon)$ as well.

The main strategy is to consider not the anisotropic Navier–Stokes equations themselves but the difference equations, namely, the PDEs obtained by subtracting the primitive equations from (1). The unknowns for these equations are denoted by $(V_\epsilon, W_\epsilon, P_\epsilon)$. This strategy enables us to construct a solution to the difference solution based on the theory of global solutions with small data (the smallness is provided by that of ε), rather than the theory of local solutions with large data. The main tool to solve a linearized problem is the theory of maximal regularity.

Theorem 2 ([2]). Let $T > 0$, $q \in \left(\frac{4}{3}, \infty\right)$, and $p \geq \max\left\{\frac{q}{q-1}; \frac{2q}{3q-4}\right\}$. Let $u = (v,w) \in \mathbb{E}_1(T) = H^{1,p}(0,T; L^q(\Omega)) \cap L^p(0,T; H^{2,q}(\Omega))$ be the solution to the primitive equations. Then for all $0 < \varepsilon \leq {}^{\exists}\varepsilon_0$ there exists a unique solution $(V_\varepsilon, W_\varepsilon, P_\varepsilon)$ of the difference equations, mentioned above, such that

$$\|(V_\varepsilon, \varepsilon W_\varepsilon)\|_{\mathbb{E}_1(T)} + \|\nabla_H P_\varepsilon\|_{L^p(0,T;L^q(\Omega))} + \|\varepsilon^{-1}\partial_z P_\varepsilon\|_{L^p(0,T;L^q(\Omega))} \leq C\varepsilon.$$

Here the constant $C > 0$ depends only on p, q, T, u.

As an immediate corollary, we obtain the global strong solution of (1) by simply considering $(v_\varepsilon, w_\varepsilon, p_\varepsilon) := (v + V_\varepsilon, w + W_\varepsilon, p + P_\varepsilon)$.

It is possible to extend the above result to the case of Dirichlet boundary conditions. In this case, however, it becomes much more difficult to establish maximal regularity, which must be uniform in ε, for a linearized problem (i.e. anisotropic Stokes equations). Moreover, it is not trivial to ensure that $w \in \mathbb{E}_1(T)$ and we assume more regularity of the initial data. For more details, see our preprint [3].

References

[1] Y. Giga, M. Gries, M. Hieber, A. Hussein, and T. Kashiwabara, *The hydrostatic Stokes semigroup and well-posedness of the primitive equations on spaces of bounded functions*, J. Funct. Anal. **279**, 108561, 2020.

[2] K. Furukawa, Y. Giga, M. Hieber, A. Hussein, T. Kashiwabara, and M. Wrona, *Rigorous justification of the hydrostatic approximation for the primitive equations by scaled Navier–Stokes equations*, to appear in Nonlinearity.

[3] K. Furukawa, Y. Giga, and T. Kashiwabara, *The hydrostatic approximation for the primitive equations by the scaled Navier–Stokes equations under the no-slip boundary condition*, submitted, arXiv:2006.02300

Critical reflection of internal waves

Anne-Laure Dalibard
(joint work with Roberta Bianchini,LaureSaint-Raymond)

Internal waves describe the (linear) response of an incompressible stably stratified fluid to small perturbations. The inclination of their group velocity with respect to the vertical is completely determined by their frequency: indeed, the dispersion relation is

$$\omega = \pm N \frac{|k_1|}{|k|},$$

where ω is the time frequency of an internal wave, $k = (k_1, k_3)$ its wavenumber, and N is the Brunt-Väisälä frequency, which is a parameter of the problem. Therefore the reflection on a sloping boundary cannot follow Descartes' laws, and it is expected to be singular if the slope has the same inclination as the group velocity.

In this talk, following the paper [1], we prove that in this critical geometry the weakly viscous and weakly nonlinear wave equations have actually a solution which is well approximated by the sum of the incident wave packet, a reflected second harmonic and some boundary layer terms. This result confirms a prediction by Dauxois and Young, and provides precise estimates on the time of validity of this approximation.

More precisely, the talk is devoted to the analysis of the Boussinesq system

$$\begin{aligned} \partial_t v + \delta(v \cdot \nabla)v + \nabla P + be_3 &= \nu \Delta v, \\ \partial_t b + \delta(v \cdot \nabla)b - N^2 v_3 &= \kappa \Delta b, \\ \nabla \cdot v &= 0, \end{aligned} \tag{1}$$

in the domain $\Omega = \{(x_1, x_3) \in \mathbb{R}^2,\ x_1 \sin\gamma - x_3 \cos\gamma > 0\}$. We are interested in the limit $\delta, \kappa, \nu \ll 1$, and in the case when an incident wave packet arrives on the sloping boundary $\partial\Omega$ with a time frequency $\omega = N \sin\beta$ with $\beta \simeq \gamma$ (critical regime).

We prove that in the linear case ($\delta = 0$) we can build an almost exact solution of equation (1) consisting of the incident wave packet and of a reflected boundary layer. This boundary layer actually consists of a superposition of several boundary layers of different sizes: in particular, when $\beta - \gamma = O(\nu^{1/3})$ and $\kappa \sim \nu$, there is one boundary layer of size $\nu^{1/2}$ and one of size $\nu^{1/3}$.

Once the linear theory is well-understood, one can study (1) in a weakly nonlinear setting and construct the first nonlinear correction to the linear solution. We then exhibit a second harmonic, which is not confined to a boundary layer, and whose size is much smaller than the one of the original solution. The final result is the following;

Theorem. *Consider the Boussinesq equations* (1) *in the scaling* $\nu \simeq \kappa$, $\beta - \gamma = O(\nu^{1/3})$, *with boundary conditions*

$$v_{|\partial\Omega} = 0, \quad \partial_n b_{|\partial\Omega} = 0$$

Then there exists a vector field

$$\mathcal{W}_{app} := (v_{app}, b_{app}) = \mathcal{W}_{inc} + \mathcal{W}_{BL} + \mathcal{W}_{II}^1 + \mathcal{W}_{corr},$$

where $\mathcal{W}_{inc}$, $\mathcal{W}_{BL}$, $\mathcal{W}_{II}^1$, $\mathcal{W}_{corr}$ *are respectively an incident wave packet, a boundary layer, a second harmonic wave packet and an additional correction term, which is an approximate solution, in the sense that*

$$\begin{aligned}\partial_t v_{app} + \delta(v_{app}\cdot\nabla)v_{app} + \nabla P_{app} + b_{app}e_3 &= \nu\Delta v_{app} + O(\delta\nu^{1/3}),\\ \partial_t b_{app} + \delta(v_{app}\cdot\nabla)b_{app} - N^2 v_{3,app} &= \kappa\Delta b_{app} + O(\delta\nu^{1/3}),\\ \nabla\cdot v_{app} &= 0,\end{aligned}$$

where the remainders $O(\delta\nu^{1/3})$ *have to be understood in the sense of the* $L^2(\Omega)$ *norm, and endowed with the boundary conditions*

$$v_{app|\partial\Omega} = 0, \quad \partial_n b_{app|\partial\Omega} = 0\,.$$

Furthermore, denoting by $\mathcal{W}$ *the unique weak solution to the Cauchy problem associated with system* (1) *with initial data*

$$\mathcal{W}_0 = \mathcal{W}_{app}(t=0),$$

we have the following stability estimate:

$$\|(\mathcal{W}_{app} - \mathcal{W})(t)\|_{L^2(\Omega)} \le \delta\nu^{1/3}\exp((\delta\nu^{-1/3}+1)t). \tag{2}$$

References

[1] Roberta Bianchini, Anne-Laure Dalibard, Laure Saint-Raymond, *Near-critical reflection of internal waves*, to appear in Analysis&PDE, 2020.

Fast Wave Averaging with Phase Changes: Application to Moist Atmospheric Dynamics

Leslie M. Smith

(joint work with Sam Stechmann, Yeyu Zhang)

We develop a fast-wave averaging framework for a moist Boussinesq system with phase changes between water vapor and liquid water. The goal is to determine the influence of fast, propagating waves on the slowly varying component of the system, in the limit of asymptotically large rotation and stratification. For purely saturated flow with rainfall but no phase changes, it is straightforward to prove that the slow modes evolve independently in a distinguished limit with $\varepsilon \to 0$, as in the dry dynamis [1]. When phase changes are present, the fast timescale is not associated with simple Fourier waves because the buoyancy changes its functional form across phase boundaries, thus complicating the analysis.

The moist model may be written in non-dimensional form as

$$\frac{D\mathbf{u}}{Dt} + \varepsilon^{-1}\hat{z}\times\mathbf{u} + \varepsilon^{-1}\nabla\phi = \varepsilon_1^{-1} b\,\hat{z}, \quad \nabla\cdot\mathbf{u} = 0, \tag{1}$$

$$\frac{D\theta_e}{Dt} + \varepsilon_1^{-1} w = 0, \quad \frac{Dq_t}{Dt} - \varepsilon_2^{-1} w - \frac{\partial q_r}{\partial z} = 0, \tag{2}$$

where $\mathbf{u}(\mathbf{x}, t)$ is velocity, $\phi(\mathbf{x}, t)$ is effective pressure and $b(\mathbf{x}, t)$ is buoyancy. The anomalous contributions to the other thermodynamic variables are temperature $\theta(\mathbf{x}, t)$, equivalent potential temperature $\theta_e(\mathbf{x}, t) = \theta(\mathbf{x}, t) + q_v(\mathbf{x}, t)$, total water mixing ratio $q_t(\mathbf{x}, t) = q_v(\mathbf{x}, t) + q_r(\mathbf{x}, t)$, and water vapor (liquid water) mixing ratio $q_v(\mathbf{x}, t)$ $(q_r(\mathbf{x}, t))$. The three small parameters $\varepsilon, \varepsilon_1, \varepsilon_2$ incorporate the constraints of rapid rotation and strong stable stratification, typical of the mid-latitude atmosphere at synoptic scales. The distinguished limit in terms of $\varepsilon \to 0$ is described in [2], and here we set all $O(1)$ constants equal to unity for ease of the calculations. The buoyancy may be written

$$b = H_u b_u + H_s b_s, \tag{3}$$

where $b_u = [\theta_e + (\varepsilon - 1)q_t]$ and $b_s = [\theta_e - \varepsilon q_t]$, and phase boundaries are indicated by the Heaviside operators H_u, H_s defined as $H_u = 1$ for $q_t < 0$, $H_u = 0$ for $q_t \geq 0$, $H_s = 1 - H_u$. The background thermodynamic states $\tilde{\theta}_e(z)$ and $\tilde{q}_t(z)$ are linear functions of altitude z, and for convenience we set $\tilde{q}_r = 0, \tilde{q}_t = \tilde{q}_v = q_{vs}(z)$, where $q_{vs}(z)$ is the saturation profile. In abstract form, one may write (1)-(3) as

$$\frac{\partial \mathbf{v}}{\partial t} + \varepsilon^{-1} \Big[H_u(\mathbf{v}) L_u(\mathbf{v}) + H_s(\mathbf{v}) L_s(\mathbf{v}) \Big] + B(\mathbf{v}, \mathbf{v}) = 0, \quad \mathbf{v}(\mathbf{x}, 0) = \bar{\mathbf{v}}(\mathbf{x}) \tag{4}$$

where $\mathbf{v}$ is the state vector, L_u and L_s are linear, B is bilinear and the initial state $\bar{\mathbf{v}}(\mathbf{x})$ is unbalanced (contains fast waves). Since phase boundaries are determined by the full state vector, H_u, H_s *depend on fast waves*, and the term $[H_u L_u + H_s L_s]$ incorporating the effects of rotation and stratification is *nonlinear*.

We here presume existence and uniqueness of $\mathbf{v}^{\varepsilon}(\mathbf{x}, t)$ for each value of ε, where $\mathbf{v}^{\varepsilon}(\mathbf{x}, t)$ is determined by the full nonlinear dynamics (4). The goal is then to decompose $\mathbf{v}^{\varepsilon}(\mathbf{x}, t)$ into fast and slow components, and to discover if (and how) these components are coupled. Since $\mathbf{v}^{\varepsilon}(\mathbf{x}, t)$ is known, then $H_u(\mathbf{v}^{\varepsilon}) = H_u(\mathbf{x}, t)$ and $H_s(\mathbf{v}^{\varepsilon}) = H_s(\mathbf{x}, t)$ are also known. With this perspective in mind, we apply multi-scale analysis to the linearized version of (4)

$$\frac{\partial \mathbf{v}}{\partial t} + \varepsilon^{-1} \Big[H_u(\mathbf{x}, t) L_u(\mathbf{v}) + H_s(\mathbf{x}, t) L_s(\mathbf{v}) \Big] + B(\mathbf{v}, \mathbf{v}) = 0, \quad \mathbf{v}(\mathbf{x}, 0) = \bar{\mathbf{v}}(\mathbf{x}), \tag{5}$$

with given functions $H_u(\mathbf{x}, t), H_s(\mathbf{x}, t)$. The solution is expanded as

$$\mathbf{v}^{\varepsilon}(\mathbf{x}, t, \tau) = \mathbf{v}^{0}(\mathbf{x}, t, \tau)|_{\tau = t/\varepsilon} + \varepsilon \mathbf{v}^{1}(\mathbf{x}, t, \tau)|_{\tau = t/\varepsilon} + \cdots \tag{6}$$

where, upon collecting terms order-by-order, one may find explicit expressions for $\mathbf{v}^{0}, \mathbf{v}^{1}$ in terms of $\bar{\mathbf{v}}$. The fast-wave-averaging equation arises as the condition to suppress secular growth in $\mathbf{v}^{1}$. After projection of the fast-wave averaging equation onto slow and fast modes, then the coupling terms must be evaluated.

Slow modes are solutions that do not change in time as $\varepsilon \to 0$, and hence they satisfy $[H_u(\mathbf{x}, t) L_u + H_s(\mathbf{x}, t) L_s]\mathbf{v} = 0$. One slow mode is the equivalent potential vorticity PV_e, which is a moist analogue of the dry potential vorticity PV. An additional slow mode $M = q_v + \theta_e$ arises from including water [2]. Even though Fourier analysis is not available as in the dry case, coupling terms may be analyzed using structural information about the linear and bilinear operators. For example, in the absence of rainfall, the fast-wave averaging equation for M is given by

$$\frac{\partial M}{\partial t} = -\langle \mathbf{u}_{(M,PV_e)} \rangle \cdot \nabla M - \langle \mathbf{u}_{(Wave)} \rangle \cdot \nabla M \tag{7}$$

where are $\mathbf{u}_{(M,PV_e)}$ is the component of $\mathbf{u}$ found from (M, PV_e) inversion after setting the wave complement to zero, and $\mathbf{u}_{(Wave)} = \mathbf{u} - \mathbf{u}_{(M,PV_e)}$. The notation $\langle\ \rangle$ denotes a time average over $O(1)$ times as the fast time scale $\tau \to \infty$. Thus, the limiting equations show that phase boundaries can induce coupling between the slow modes and a time average of the fast motions. Time averages over the wave complement are non-zero, such as the term $\langle \mathbf{u}_{(Wave)} \rangle$, because buoyancy frequencies are different in saturated and unsaturated regions. Notice also that $\mathbf{u}_{(M,PV_e)}$ has a fast component, since (M, PV_e) inversion involves H_u, H_s depending on τ. The PV_e-equation is similar, but contains more coupling terms, as well as a term $\langle \partial_z \mathbf{u}_{(M,PV_e)} \cdot \nabla \theta_{e(M,PV_e)} \rangle$ which is identically zero in saturated regions and 'turns on' in unsaturated regions, with no analogue single-phase dynamics [3].

References

[1] A.J. Majda & P. Embid, *Averaging over fast gravity waves for geophysical flows with unbalanced initial data*, Theoretical and Computational Fluid Dynamics **11** (1998), 155–169.

[2] L.M. Smith & S.N. Stechmann, *Precipitating quasigeostrophic equations and potential vorticity inversion with phase changes*, Journal of the Atmospheric Sciences **74** 2017, 3285-3303.

[3] Y. Zhang, L.M. Smith & S.N. Stechmann, *Fast Wave Averaging with Phase Changes: Application to Moist Atmospheric Dynamics*, submitted.

Multiscale Asymptotics and Analysis for Atmospheric Flow Models with Moisture

Sabine Hittmeir

(joint work with Rupert Klein, Jinkai Li, Edriss Titi)

Model reductions in meteorology by scale analysis are inevitable and therefore have a long history in meteorology. The key technique for a systematic study of complex processes involving the interaction of phenomena on different length and time scales is multiple scales asymptotics. Due to their major contribution to the energy transport of particular interest are hot towers, which are large cumulonimbus clouds that live on small horizontal scales. In comparison to existing studies we not only incorporate moisture into the model via balance equations for water vapor, cloud water and rain, but also refine the thermodynamics by taking into account the different gas constants and heat capacities for water in contrast to dry air. This refined setting is demonstrated to be essential by leading to different force balances.

These deep convective clouds furthermore constitute the building blocks of larger scale convective storms, which we study in a next step by incorporating the setting of organised convection into the multiscale approach. This requires systematic averaging procedures, allowing to quantify the modulation of the larger scale flow by the moisture processes in the small scale regions. This work is joint work with R. Klein.

While the just described multiscale asymptotics are purely formal, in collaboration with R. Klein, J. Li and E. Titi, we also proceed further in the rigorous analysis of the atmospheric flow models with moisture and phase transitions. We study the global existence and uniqueness of solutions for the moisture balance equations coupled to the thermodynamic equation building the basis for the above expansions. In a first step we assume the flow field to be given and then extend the analysis to the moisture model coupled to the primitive equations.

References

[1] S. Hittmeir, R. Klein, J. Li, E. Titi. *Global well-posedness for the primitive equations coupled to nonlinear moisture dynamics with phase changes,* Nonlinearity **33**(7) (2020), 3206–3236.

[2] S. Hittmeir, R. Klein. *Asymptotics for moist deep convection I: Refined scalings and self-sustaining updrafts.* Theoretical and Computational Fluid Dynamics, **32** (2018), 137–164.

[3] S. Hittmeir, R. Klein, J. Li, E. Titi. *Global well-posedness for passively transported nonlinear moisture dynamics with phase changes.* Nonlinearity **30** (2017), 3676–3718.

Reporter: Matthias Hieber

Participants

Dr. Bilal Al Taki
Beijing International Center for
Mathematical Research (BICMR)
Beijing University
Quan 23-1
No. 5 Yiheyuan Road, Haidian District
Beijing 100871
CHINA

Prof. Dr. Claude Bardos
31, Avenue Trudaine
75009 Paris Cedex
FRANCE

Prof. Dr. Yann Brenier
D M A
École Normale Supérieure
45, rue d'Ulm
75230 Paris Cedex 05
FRANCE

Ph.D. Jorge Cardona
Fachbereich Mathematik (Analysis)
Technische Universität Darmstadt
Schloßgartenstrasse 7
64289 Darmstadt
GERMANY

Prof. Dr. Peter Constantin
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Dr. Michele Coti Zelati
Department of Mathematics
Imperial College London
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Sergey Danilov
Department of Climate System
Alfred-Wegener-Institute
Jacobs University
Bussestrasse 24
27570 Bremerhaven
GERMANY

Prof. Dr. Robert Denk
Fachbereich Mathematik und Statistik
Universität Konstanz
Fach 193, F 522
Postfach 5560
78434 Konstanz
GERMANY

Dr. Karoline Disser
Fachbereich Mathematik
Technische Universität Darmstadt
Schlossgartenstrasse 7
64289 Darmstadt
GERMANY

Prof. Dr. Eduard Feireisl
Institute of Mathematics of the AS CR
Charles University
Žitná 25
115 67 Praha 1
CZECH REPUBLIC

Prof. Daniel Feltham
Department of Meteorology
University of Reading
Office 2L42-Met
Whiteknights
P.O. Box 243
Reading, Berkshire RG6 6AH
UNITED KINGDOM

Prof. Dr. Giovanni Paolo Galdi
Department of Mechanical Engineering
and Materials Science
University of Pittsburgh
607 Benedum Hall
3700 O'Hara Street
Pittsburgh, PA 15261
UNITED STATES

Dr. Mi-Ho Giga
Graduate School of Mathematical
Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Yoshikazu Giga
Graduate School of Mathematical
Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Kenneth M. Golden
Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City, UT 84112-0090
UNITED STATES

Prof. Dr. Patrick Heimbach
Oden Institute for Computational
Engineering and Sciences (CRIOS)
The University of Texas at Austin
1 University Station C 0200
201 E. 24th Street
P.O. Box 4.232
Austin, TX 78712-1229
UNITED STATES

Prof. Dr. Matthias Hieber
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstrasse 7
64289 Darmstadt
GERMANY

Prof. Dr. Michael Hinze
Mathematisches Institut
Campus Koblenz
Universität Koblenz-Landau
Universitätsstrasse 1
56070 Koblenz
GERMANY

Dr. Amru Hussein
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstrasse 7
64289 Darmstadt
GERMANY

Prof. Dr. Slim Ibrahim
Department of Mathematics and
Statistics
University of Victoria
P.O. Box 3060
Victoria BC V8W 3R4
CANADA

Dr. Takahito Kashiwabara
Graduate School of Mathematical
Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Boualem Khouider
Department of Mathematics and
Statistics
University of Victoria
P.O. Box 3060
Victoria BC V8W 3R4
CANADA

Prof. Dr. Rupert Klein
Fachbereich Mathematik und Informatik
Freie Universität Berlin
Arnimallee 6
14195 Berlin
GERMANY

Prof. Dr. Peter Korn
Max Planck Institut für Meteorologie
Universität Hamburg
Bundesstrasse 55
20146 Hamburg
GERMANY

Prof. Dr. Hideo Kozono
Department of Mathematics
School of Fundamental Science and Engineering
Waseda University
1-104 Totsukamachi, Shinyuku-ku
Tokyo 169-8050
JAPAN

Prof. Dr. Igor Kukavica
Department of Mathematics
University of Southern California, Dornsife
KAP 104
3620 S. Vermont Avenue
Los Angeles, CA 90089-2532
UNITED STATES

Prof. Jinkai Li
South China Research Center for Applied
Mathematics and Interdisciplinary Studies
South China Normal University
No. 55, Zhongshan West Avenue
Guangzhou 510631
CHINA

Quyuan Lin
Department of Mathematics
Texas A & M University
Blocker 630 A
College Station, TX 77843-3368
UNITED STATES

Dr. Xin Liu
Institut für Mathematik
Freie Universität Berlin
Arnimallee 9
14195 Berlin
GERMANY

Dr. Evelyn Lunasin
United States Naval Academy
Mathematics Department
Stop 9E, Chauvenet Hall
572C Holloway Road
Annapolis, MD 21402
UNITED STATES

Prof. Dr. Yasunori Maekawa
Department of Mathematics
Graduate School of Science
Kyoto University
Kitashirakawa, Oiwake-cho, Sakyo-ku
Kyoto 606-8502
JAPAN

Dr. Carolin Mehlmann
Max-Planck-Institut für Meteorologie
Universität Hamburg
Bundesstrasse 53
20146 Hamburg
GERMANY

Prof. Dr. Alexander Mielke
Weierstraß-Institut für
Angewandte Analysis und Stochastik
Mohrenstrasse 39
10117 Berlin
GERMANY

Prof. Dr. Stefano Modena
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstrasse 7
64289 Darmstadt
GERMANY

Prof. Dr. Camilla Nobili
Department Mathematik
Universität Hamburg
Bundesstrasse 55
20146 Hamburg
GERMANY

Prof. Dr. Helena J. Nussenzveig-Lopes
Instituto de Matemática
Universidade Federal do Rio de Janeiro
Ilha do Fundao
Caixa Postal 68530
Rio de Janeiro, RJ 21941-909
BRAZIL

Prof. Dr. Jürgen Saal
Mathematisches Institut
Heinrich-Heine-Universität Düsseldorf
Gebäude 25.13
Universitätsstrasse 1
40225 Düsseldorf
GERMANY

Prof. Dr. Senjo Shimizu
Graduate School of Human and Environmental Studies
Course of Mathematical Science
Kyoto University
Yoshida-nihonmatsu-cho, Sakyo-ku
Kyoto 606-8501
JAPAN

Prof. Dr. Leslie M. Smith
Department of Mathematics
University of Wisconsin, Madison
480 Lincoln Drive
Madison WI 53706-1388
UNITED STATES

Prof. Dr. Samuel N. Stechmann
Department of Mathematics
University of Wisconsin, Madison
480 Lincoln Drive
Madison, WI 53706-1388
UNITED STATES

Dr. Ryo Takada
Department of Mathematical Sciences
Kyushu University
744 Motooka Nishi-ku
Fukuoka 819-0395
JAPAN

Prof. Dr. Roger Temam
Institute for Scientific Computing and Applied Mathematics
Indiana University at Bloomington
Rawles Hall
831 East 3rd Street
Bloomington, IN 47405-7106
UNITED STATES

Dr. Marita Thomas
Weierstraß-Institut für Angewandte Analysis und Stochastik
Mohrenstrasse 39
10117 Berlin
GERMANY

Prof. Dr. Edriss S. Titi
Department of Applied Mathematics and Theoretical Physics
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WA
UNITED KINGDOM

Dr. Patrick Tolksdorf
Institut für Mathematik
Johannes Gutenberg-Universität Mainz
Geb.: 2413
Staudingerweg 9
55128 Mainz
GERMANY

Prof. Dr. Joe Tribbia
University Corporation for
Atmospheric Research (UCAR)
P.O. Box 3000
Boulder, CO 80307
UNITED STATES

Prof. Dr. Tsuyoshi Yoneda
Department of Mathematics
Graduate School of Mathematical
Sciences
The University of Tokyo
3-8-1 Komaba, Meguro
Tokyo 153-8914
JAPAN