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Non-Commutative Geometry and Cyclic Homology (hybrid meeting)

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Abstract. The workshop on "Non-Commutative Geometry and Cyclic Homology" was attended by 16 participants on site. 30 participants could not travel to Oberwolfach because of the pandemia and took advantage of the videoconference tool. This report contains the extended abstracts of the lectures both given on site and externally.

Mathematics Subject Classification (2010): 19K56, 46L80, 58B34, 58J20, 81T75, 81R60.

Introduction by the Organizers

The meeting in Oberwolfach, 28 Jun - 4 Jul 2020, was the second hybrid type workshop since the partial relaxing of the corona related restrictions in Baden-Württemberg. As the result, there were a score of problems with the workshop, both as far as the setting up the lectures and the interaction between the participants was concerned.

The decision to let the workshop proceed in the hybrid form was taken fairly late, because of the dynamics of the pandemics. Due to national travel restrictions, none of the organisers were able to participate physically. We would like to express our gratitude to both Birgit Richter and Thomas Schick, for the effort they put into making the meeting happen at all. Also, the setting up of remote lectures was initiated shortly before the beginning of the workshop, giving participants very short time to prepare. The on-line lectures require substantial amount of effort. As the result, the choice of subjects of lectures was dictated by what participants have prepared previously, in connection with web-seminars they had participated in during the last couple of months. Also, the geographic spread of the participants resulted in a relatively narrow time window - effectively three hours per day - for presentation of lectures and, possibly, related discussions.

There were interesting discussions as on the evening of Tuesday, but what was really missing was the actual physical gathering and its impromptu discussions. The feedback from the *in spe* organisers present physically in Oberwolfach, stressed that the main impact of the Oberwolfach workshops is achieved by direct interaction and discussions between the participants, be it at the break after lunch or in the evening. According to Birgit Richter and Thomas Schick, the participants present on site unanimously expressed their great satisfaction with the fact that they finally could have this kind of direct personal interaction. While the lectures are a very important element of the Oberwolfach type meeting, they do not seem able to replace the immediacy of physical presence and reactions it allows. In short, the on-line form of the workshop, while possible, does not seem to be an appropriate replacement for traditional form of Oberwolfach meetings.
The main objective of the workshop was to instigate a real interaction between the homotopy theorists and the operator algebraists, with their different approaches to the subject and applications of topological and analytic cyclic homology in all their variants. This was unfortunately made almost impossible in the hybrid type set up. But just the few lectures and interactions during the meeting related to this goal did show again the importance of this kind of event. Just for a starter, we would like to point to the lectures - on site - by Dustin Clausen on his work with Scholze and Ralf Meyer on his work with Guillermo Cortinas and Devarshi Mukherjee (Non-Archimedean analytic cyclic homology, ArXiv:1912.09366, both related to the way an appropriate version of (topological/bornological) cyclic homology relates to the cohomology of schemes in prime characteristic. While the reasons for the workshop listed below remain just as valid as in the original application, the latest developments in both TC and applications of bornological cyclic homology stress, in our minds, the need to carry out a classical type Oberwolfach workshop devoted to the interaction between the two mathematical communities.

Acknowledgement: We would like to express our gratitude to both Birgit Richter and Thomas Schick, for the effort they put into making the meeting happen at all.

Workshop (hybrid meeting): Non-Commutative Geometry and Cyclic Homology

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Abstracts

Quantized calculus and Weil positivity

Alain Connes
(joint work with Caterina Consani)

My talk surveyed the recent developments of my collaborative research program with C. Consani on the positivity of the Weil functional in number theory using the Hilbert space framework of the semi-local trace formula of [2]. This research is closely related to the main open question of the Riemann Hypothesis.

I started my talk by showing the Oberwolfach abstract of 1981 [1], *Spectral sequence and homology of currents for operator algebras*, in which cyclic cohomology, the SBI long exact sequence, and the computations for smooth manifolds and non-commutative tori were done, while the origin of the theory came from the quantized calculus whose basic equations are

$$đf := [F, f], \quad F = F^*, \ F^2 = 1, \quad \Omega^k := \{\omega = \sum f_0\, đf_1 \ldots đf_k\}$$

I then explained how the quantized calculus clarifies why the Hilbert space framework of the semi-local trace formula delivers a sum of terms as in the Riemann-Weil explicit formula [7] while the underlying geometric space, the semi-local adele class space is closely related to a product of local fields. The explanation comes from the quantum logarithmic derivative of the product of ratios of local factors which transforms a product into a sum. For the Riemann zeta function $\zeta(s)$ the explicit formula takes the following form, with $Z = \frac{1}{2} + iS$ the multi-set of non-trivial zeros of the Riemann zeta function

$$\widehat{f}(-i/2) - \sum_{s\in S} \widehat{f}(s) + \tilde{f}(i/2) = \sum_v W_v(f) \ \forall f \in C_c^\infty(\mathbb{R}_+^*) \tag{1}$$

Here one lets $\widehat{f}$ be the Fourier transform of the complex-valued function f with compact support on the group $\mathbb{R}_+^*$ whose dual is identified with $\mathbb{R}$. The terms of the sum in the right-hand side of (1) are of local nature, associated to the places v of the field $\mathbb{Q}$. Following Weil, the Riemann hypothesis (RH) is equivalent to the negativity of the right-hand side of (1) for all functions $f = g * \bar{g}^*$, $g \in C_c^\infty(\mathbb{R}_+^*)$, such that $\widehat{f}(s) = 0$ for all s in a finite set E of complex numbers with $E \supset \{\pm i/2\}$ and $E \cap S = \emptyset$. The key point is that the right-hand side of the explicit formula, when evaluated on a test function f with compact support, involves only finitely many places at a time. In the very recent paper [5] we have provided a theoretical proof for the negativity of the right-hand side of (1) in the case one restricts to test functions f with $\mathrm{Support}(f) \subset (1/2, 2)$ and whose Fourier transform vanishes at 0 and $\pm\frac{i}{2}$. In this case, the (geometric) right-hand side of the explicit formula is just the archimedean distribution $W_\mathbb{R}$ which coincides, outside $1 \in \mathbb{R}_+^*$, with a locally rational, positive function that tends to $+\infty$ as the variable tends to 1. In fact, the main result in [5] provides an operator theoretic conceptual reason beyond Weil's positivity, which is rooted in the compression of the scaling action ϑ (of $\mathbb{R}_+^*$ in the

Hilbert space $L^2(\mathbb{R})_{\mathrm{ev}}$ of square integrable even functions) on Sonin's space $S(1,1)$ of functions, which, together with their Fourier transform, vanish identically in the interval $[-1,1]$.

Theorem
Let $g \in C_c^\infty(\mathbb{R}_+^*)$ have support in the interval $[2^{-1/2}, 2^{1/2}]$ and Fourier transform vanishing at $\frac{i}{2}$ and 0. Then one has, with $W_\infty := -W_\mathbb{R}$,

$$W_\infty(g * g^*) \geq \mathrm{Tr}(\vartheta(g)\, \mathbf{S}\, \vartheta(g)^*). \tag{2}$$

An important step toward the proof of this inequality is provided by the following result which describes the functional $W_\infty(f)$ in terms of Sonin's trace and a further functional whose behavior and sign play a determining factor in this study

Theorem
Let $\mathbf{S}$ be the orthogonal projection of $L^2(\mathbb{R})_{\mathrm{ev}}$ on the closed subspace $S(1,1)$. The following functional is positive

$$\mathrm{Tr}(\vartheta(f)\mathbf{S}) = W_\infty(f) + \int f(\rho^{-1})\epsilon(\rho) d^*\rho, \ \forall f \in C_c^\infty(\mathbb{R}_+^*),$$

where $\epsilon(\rho)$ is the function of $\rho \in \mathbb{R}_+^*$, with $\epsilon(\rho^{-1}) = \epsilon(\rho)$, which is given, for $\rho \geq 1$, by

$$\epsilon(\rho) = \sum \frac{\lambda(n)}{\sqrt{1-\lambda(n)^2}} \langle \xi_n \mid \vartheta(\rho^{-1})\zeta_n \rangle.$$

where the vectors ζ_n, ξ_n are given in terms of prolate functions.

The strategy pursued in this joint research is motivated by the desire to understand the link between the analytic Hilbert space operator theoretic set-up of [2], and the geometric approach pursued in the joint work in [3, 6]. The latter unveiled a novel geometric landscape still in development for an intersection theory of divisors (on the square of the Scaling Site [6]), thus not yet in shape to handle the delicate principal values involved in the Riemann-Weil explicit formula. The contribution of [5] is to make very explicit the relation between the two approaches, thus overcoming the above problem. The connection between the operator theoretic and the geometric viewpoints is effected by the Schwartz kernels associated to operators. By implementing the additive structure of the adèles, one sees that the Schwartz kernel of the scaling operator corresponds geometrically to the divisor of the Frobenius correspondence.

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An analytic cyclic homology theory for algebras in finite characteristic

Ralf Meyer

(joint work with G. Cortiñas, J. Cuntz, G. Tamme, D. Mukherjee)

Let $\mathbb{F}_p$ be the finite field with p elements. It is difficult to define well behaved cohomology theories for $\mathbb{F}_p$-algebras. This is visible already for the affine line, that is, the algebra $\mathbb{F}_p[t]$. The usual de Rham cohomology recipe would say that the de Rham cohomology of $\mathbb{F}_p[t]$ is given by the kernel and cokernel of the differentiation map $f \mapsto f'$ on $\mathbb{F}_p[t]$. Due to the finite characteristic of $\mathbb{F}_p$, both are infinite-dimensional $\mathbb{F}_p$-vector spaces. A homotopy invariant cohomology theory should, however, be trivial for the affine line.

A well known recipe to define a better behaved theory is to lift an algebra over $\mathbb{F}_p$ to an algebra over the p-adic integers $\mathbb{Z}_p$. This is completed in a suitable way and tensored with $\mathbb{Q}_p$, and then we take de Rham cohomology. The p-adic completion still gives the wrong cohomology for the affine line. Monsky and Washnitzer [4] proposed to complete $\mathbb{Z}_p[t]$ as follows. Let $\nu\colon \mathbb{Z}_p \to \mathbb{N} \cup \{\infty\}$ be the p-adic evaluation, defined by $\nu(p^k x) = k$ if $x \in \mathbb{Z}_p \setminus p \cdot \mathbb{Z}_p$ and $\nu(0) = \infty$. The weak completion or *dagger completion* of $\mathbb{Z}_p[t]$ is defined by

$$\mathbb{Z}_p[t]^\dagger = \left\{ \sum_{n=0}^{\infty} c_n t^n \;\middle|\; \nu(c_n) \text{ grows at least linearly} \right\}.$$

A similar recipe works for polynomials in several variables. A general finitely generated algebra is written as a quotient of a polynomial algebra in order to define its dagger completion.

When we tensor the algebra $\mathbb{Z}_p[t]^\dagger$ with $\mathbb{Q}_p$, then the differentiation map becomes invertible except for the constant functions in the kernel. This is because the linear growth of $\nu(c_n)$ dominates the at most logarithmic growth of the factors that appear in the differentiation.

Monsky and Washnitzer [4] defined a cohomology theory for smooth affine varieties over $\mathbb{F}_p$ by lifting a smooth commutative algebra over $\mathbb{F}_p$ to a ("very") smooth algebra over $\mathbb{Z}_p$, dagger completing, tensoring with $\mathbb{Q}_p$, and then taking the de Rham cohomology of the resulting commutative algebra. The most difficult issue is to prove that this theory is well defined. Since this uses the smoothness of the lifting, this definition can only work for smooth varieties over $\mathbb{F}_p$. Berthelot's rigid cohomology is a theory that still works for $\mathbb{F}_p$-algebras that are not smooth. The following description of it used in [1] is based on ideas of Große-Klönne. Write an $\mathbb{F}_p$-algebra A as a quotient of a polynomial algebra $\mathbb{F}_p[x_1, \dots, x_n]$. Let I be the kernel of the resulting quotient map

$$\mathbb{Z}_p[x_1, \dots, x_n] \to \mathbb{F}_p[x_1, \dots, x_n] \to A.$$

For each $l \geq 1$, define the *tube algebra* at I^l to be

$$\sum_{j=1}^{\infty} p^{-j} I^{l \cdot j} \subseteq \mathbb{Q}_p[x_1, \dots, x_n].$$

These algebras form a projective system of finitely generated, commutative $\mathbb{Z}_p$-algebras. Now dagger complete these algebras, tensor with $\mathbb{Q}_p$, and take the de Rham complexes. The cohomology of the homotopy limit of these cochain complexes for $l \to \infty$ is isomorphic to the rigid cohomology of the affine variety defined by A.

For algebras of smooth functions on smooth manifolds, it is well known that de Rham cohomology made 2-periodic is isomorphic to periodic cyclic homology. It is shown in [1] that this is also true for the projective system of $\mathbb{Q}_p$-algebras used in the definition above. Thus rigid cohomology is expressed through the periodic cyclic homology of a certain projective system of algebras.

Periodic cyclic homology makes sense for noncommutative algebras. The result above suggests that it should be possible to extend rigid cohomology to a cohomology theory for noncommutative algebras over $\mathbb{F}_p$ that is defined by resolving the algebra by a $\mathbb{Z}_p$-algebra, then forming a tube algebra as above, and then taking a suitable analogue of the dagger completion. The appropriate dagger completion is described in [1]. It is based on analysis in complete bornological algebras over $\mathbb{Z}_p$. A bounded subset of such an algebra has a spectral radius. The bornology consisting of all subsets of $\sum_{j=1}^{\infty} p^j S^{j+1}$ for bounded subsets S is called the *linear growth bornology*. It is the smallest bornology in which every bounded subset has spectral radius at most 1. The completion of a bornological $\mathbb{Z}_p$-module is defined by p-adically completing each bounded $\mathbb{Z}_p$-submodule, then taking an inductive limit, and dividing out the closure of 0. The bornological completion in this sense of an algebra with the linear growth bornology is a noncommutative analogue of the weak completion of Monsky–Washnitzer.

A crucial result about Monsky–Washnitzer cohomology is that it does not depend on the auxiliary choice of the lifting to $\mathbb{Z}_p$. Unfortunately, there is no reason to expect that the periodic cyclic homology for noncommutative algebras over $\mathbb{Z}_p$ would have such invariance properties. Instead, we need an "analytic" variant of periodic cyclic cohomology. This theory is introduced in [2]. It is analogous to the analytic cyclic homology theory for bornological $\mathbb{C}$-algebras studied in [3]. This is closely related to the entire cyclic cohomology introduced by Connes. The point of analytic cyclic homology theories is to replace the product $\prod_{n=0}^{\infty} \Omega^n(A)$ in the bicomplex that computes periodic cyclic homology by a subcomplex, which is defined by suitable growth conditions. As in [3], the growth condition is dictated by a notion of "analytic" nilpotence. A given bornological $\mathbb{Z}_p$-algebra D has a universal extension with analytically nilpotent kernel. This is obtained from the tensor algebra extension $JD \rightarrowtail TD \twoheadrightarrow D$ by the following steps: first, we form the pro-algebra of tube algebras of TD relative to the ideals $(JD)^l$. Then we equip these with their linear growth bornologies and complete. The resulting "analytic" tensor algebra is quasi-free. Therefore, its periodic cyclic homology is computed by the X-complex, a rather small complex introduced by Cuntz and Quillen.

The analytic cyclic homology theory for algebras over $\mathbb{Z}_p$ enjoys the same nice formal properties as analytic cyclic homology for $\mathbb{C}$-algebras. Namely, it is invariant under "smooth" homotopies, satisfies excision for algebra extensions with a bounded linear section, and it is stable for generalised matrix algebras. In particular, it is Morita invariant on unital algebras. These results are proven in [2] by carrying over the proofs for the archimedean case in [3].

An important open problem is when the analytic cyclic homology theory described above is isomorphic to periodic cyclic homology. This is true for algebras that are "analytically" quasi-free. We show that this is the case for dagger completions of Leavitt path algebras and for the dagger completion of smooth commutative algebras over $\mathbb{Z}_p$ that lift affine curves over $\mathbb{F}_p$. The argument uses analytic quasi-freeness, which forces us to restrict to curves, that is, dimension 1. It is unclear to what extent this remains true for higher-dimensional smooth algebras.

In ongoing work of Mukherjee and the author, the analytic cyclic homology theory for algebras over $\mathbb{Z}_p$ defined in [3] is used to define an analytic cyclic homology theory for algebras over $\mathbb{F}_p$. We may lift such an algebra A to the free $\mathbb{Z}_p$-algebra on the underlying set of A. Then we take a tube algebra and dagger complete. The resulting algebra is already analytically quasi-free, so that its X-complex computes its analytic cyclic homology. For computations, we want to replace the free algebra by other dagger algebras that lift A. This can indeed be done. The main result in this ongoing work will show that the analytic cyclic homology of any dagger algebra that lifts A is isomorphic to the theory defined by the free algebra lifting. This independence of the lifting allows to prove that the analytic cyclic homology theory for algebras over $\mathbb{F}_p$ satisfies excision (at least for finitely generated algebras), is invariant under polynomial homotopies, and stable for matrix algebras.

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On Hopf Cyclic Cohomology and Characteristic Classes

Henri Moscovici

In the noncommutative approach to the geometry of the leaf-spaces of foliations, Hopf cyclic cohomology replaces the role played by Lie algebra cohomology in the construction of characteristic classes of foliations. The object of this talk is to reveal the intrinsic relationship between the universal characteristic classes in cyclic cohomology of étale smooth groupoids constructed out of the Hopf cyclic cohomology of the Hopf algebra $\mathcal{H}_n$ [2] and those constructed out of the differential graded (DG) Hopf algebra $\Omega^*(\mathrm{GL}_n)$ [4]. The key to the linkage is provided by Connes' map Φ relating the equivariant cohomology to cyclic cohomology [1].

As shown in [2], the periodic Hopf cyclic cohomology $HP^\bullet(\mathcal{H}_n, O_n; \mathbb{C}_\delta)$ of $\mathcal{H}_n$ (relative to O_n) is canonically isomorphic to the Gelfand-Fuks (continuous) cohomology $H_{\mathrm{c}}^\bullet\big(\mathfrak{a}_n, O_n; \mathbb{C}\big)$ of the Lie algebra $\mathfrak{a}_n$ of formal vector fields on $\mathbb{R}^n$ (relative to O_n). The latter is in turn isomorphic to the cohomology $H^\bullet\big(W(\mathfrak{gl}_n, O_n)_n\big)$ of the truncated Weil algebra (relative to O_n). On the other hand, Gorokhovsky has shown that the truncated DG Hopf cyclic cohomology $HC^\bullet(\Omega^*(\mathrm{GL}_n))_n$ is also isomorphic to $H^\bullet\big(W(\mathfrak{gl}_n, O_n)_n\big)$ [4]. Thus, indirectly, $HP^\bullet(\mathcal{H}_n, O_n; \mathbb{C}_\delta)$ and $HC^\bullet(\Omega^*(\mathrm{GL}_n))_n$ are seen to be isomorphic. In view of their close connection with the characteristic classes of foliations (cf. [2, 3, 6, 7] it is of definite interest to exhibit a more intrinsic and explicit isomorphism.

1. The Hopf algebra $\mathcal{H}_n$ is a bicrossed product of two classical types of Hopf algebras: the universal enveloping algebra of the affine Lie algebra $\mathbb{R}^n \ltimes \mathfrak{gl}_n$ and the Hopf algebra of the pro-nilpotent formal group supplementing the affine group in $\mathrm{d}\,\mathbb{R}^n$. It acts naturally on the convolution algebra $C_c^\infty(F\mathbf{\Gamma}_n)$ of the prolongation to the frame bundle of the Haefliger groupoid $\mathbf{\Gamma}_n$, giving rise to a map of cyclic complexes $\chi_* : CC^*(\mathcal{H}_n; \mathbb{C}_\delta) \to CC^*(C_c^\infty(F\mathbf{\Gamma}_n))$,

$$\chi_q^F(h^1 \otimes \ldots \otimes h^q)(a_0, a_1, \ldots, a_q) := \tau\left(a_0 h^1(a_1) \cdots h^q(a_q)\right),$$

where $\tau : C_c^\infty(F\mathbf{\Gamma}_n) \to \mathbb{C}$ is the canonical trace (integration over units $\mathbf{\Gamma}_n^0 = \mathbb{R}^n$ with respect to the canonical volume form of the frame bundle). The fact that χ_* is a chain map is automatic, since it is faithful and the cyclic object underlying the Hopf cyclic cohomology of $\mathcal{H}_n$ (with coefficients in the $\mathcal{H}_n$-module $\mathbb{C}_\delta$ associated to the canonical character $\delta \in \mathcal{H}_n^*$) is the pullback by χ_* of the standard cyclic object for the algebra $C_c^\infty(F\mathbf{\Gamma}_n)$.

Furthermore, $\mathcal{H}_n$ acts on $C_c^\infty(F\mathbf{\Gamma})$ for any *flattened* étale smooth groupoid $\mathbf{\Gamma}$ (i.e. with $\mathbf{\Gamma}^{(0)} = \bigsqcup_i U_i$, $U_i \cong \mathbb{R}^n$), giving rise to a corresponding characteristic map $\chi_*^{F\mathbf{\Gamma}} : CC^*(\mathcal{H}_n; \mathbb{C}_\delta) \to CC^*(C_c^\infty(F\mathbf{\Gamma}))$. Passing to O_n-invariants, which amounts to replacing $F\mathbf{\Gamma}$ by the quotient groupoid $P\mathbf{\Gamma} = F\mathbf{\Gamma}/O_n$, one obtains an induced characteristic map $\chi_*^{P\mathbf{\Gamma}} : CC^*(\mathcal{H}_n, \mathrm{O}_n; \mathbb{C}_\delta) \to CC^*(C_c^\infty(P\mathbf{\Gamma}))$.

The isomorphism $HP^\bullet(\mathcal{H}_n, O_n; \mathbb{C}_\delta) \cong H_{\mathrm{c}}^\bullet\big(\mathfrak{a}_n, O_n; \mathbb{C}\big)$ established in [2] is a composition of two maps. The first map (described here as in [3, 7]) is induced by the quasi-isomorphism $\mathcal{D} : C_{\mathrm{c}}^*(\mathfrak{a}_n, \mathrm{O}_n) \to C_{\mathrm{d}}^\bullet\left(\mathbf{\Gamma}_n, \Omega^*(P\mathbf{\Gamma}_n)\right)$ from the relative Lie algebra cohomology complex to the complex defining the *differentiable homology* of

$\mathbf{\Gamma}_n$ with coefficients in the sheaf of germs of currents $\Omega_*(P\mathbf{\Gamma}_n)$, formed of cochains whose values depend of finite order jets (cf. [5, IV, §4]), and is given by

$$\mathcal{D}(\omega) = \oint_{\Delta^\bullet} \hat{\sigma}^*(\tilde{\omega}) \in C^\bullet_{\mathrm{d}}\left(\mathbf{\Gamma}_n, \Omega_*(P\mathbf{\Gamma}_n)\right), \qquad \omega \in C^*_{\mathrm{c}}(\mathfrak{a}_n, \mathrm{O}_n);$$

here $\tilde{\omega}$ is the $\mathbf{\Gamma}_n$-invariant form on $P\mathbb{R}^n = J_0^\infty \mathbb{R}^n / O_n$ associated to ω, $\hat{\sigma}^*(\tilde{\omega})$ is the form on the "thick" geometric realization of the nerve of $P\mathbf{\Gamma}_n$ obtained by pullback using the standard connection on $\mathbb{R}^n$, and $\oint_{\Delta^\bullet}$ stands for integration over simplices. The second map, which has quite an elaborate expression (cf. [1, Ch. III, §2.δ]), is Connes' map $\Phi^P : C^\bullet_\nu(\mathbf{\Gamma}_n, \Omega_*(P\mathbf{\Gamma}_n)) \to CC^\bullet\left(C^\infty_c(P\mathbf{\Gamma}_n)\right)$, defined on the normalized subcomplex. It is easily seen that $\mathcal{D}(\omega) \in C^\bullet_\nu(\mathbf{\Gamma}_n, \Omega_*(P\mathbf{\Gamma}_n))$, and that $(\Phi^P \circ \mathcal{D})(\omega)$ actually belongs to the image of the characteristic map $\chi_\bullet$ induced by $\chi^F_\bullet$. Since the map $\chi_\bullet$ is faithful, this uniquely determines an element $\chi^{-1}(\Phi^P(\omega))$ in the cyclic object of $\mathcal{H}_n$ relative to O_n. The chain map thus obtained, $C^*_{\mathrm{c}}(\mathfrak{a}_n, \mathrm{O}_n) \to CC^\bullet(\mathcal{H}_n, O_n; \mathbb{C}_\delta)$ was shown in [2] to induce an isomorphism in cohomology $\sum^\oplus_{i\geq 0} H^{q-2i}_{\mathrm{c}}\big(\mathfrak{a}_n, O_n; \mathbb{C}\big) \xrightarrow{\cong} HC^q(\mathcal{H}_n, O_n; \mathbb{C}_\delta)$.

2. In [4] Gorokhovsky devised a parallel Hopf algebraic construction in the differential graded (DG) context, which can be viewed as a noncommutative counterpart of the Kamber-Tondeur setup for characteristic invariants of foliations. The central object of his approach is the (Fréchet) Hopf DG algebra $(\Omega^*(\mathrm{GL}_n), d)$. Analogously to Weil algebras, the cyclic bicomplex of Hopf DG algebras (defined in [4, §3]) is naturally filtered and thus admits truncations [4, §4]. In the case of $(\Omega^*(\mathrm{GL}_n), d)$, by [4, §6], $HC^q(\Omega^*(\mathrm{GL}_n), d)_m \cong \sum^\oplus_{i\geq 0} H^{q-2i}(W(\mathfrak{gl}_n, O_n)_m)$, where $m \in \mathbb{Z}^+$ is the truncation level.

Furthermore, $(\Omega^*(\mathrm{GL}_n), d)$ acts on $\Omega^*_c(\mathbf{\Gamma}_n)$ by multiplication with the pullback by the first jet (Jacobian) map of forms in $\Omega^*(\mathrm{GL}_n)$ and, similarly to $\mathcal{H}_n$, this action gives rise to a characteristic map $\kappa_* : HC^*(\Omega^*(\mathrm{GL}_n), d)_n \to HC^*(\Omega^*_c(\mathbf{\Gamma}_n), d)$,

$$\kappa_q(\alpha^1 \otimes \alpha^2 \otimes \cdots \otimes \alpha^q)(\varpi_0, \varpi_1, \ldots, \varpi_q) = (-1)^{\sum_{i<j} \partial\alpha^i \partial\varpi_j} \tau_\wedge\big(\varpi_0\, \alpha^1(\varpi_1) \cdots \alpha^q(\varpi_q)\big)$$

where $\tau_\wedge : \Omega^*_c(\mathbf{\Gamma}_n) \to \mathbb{C}$ is the canonical graded trace (integration over the units); in this case too, the characteristic map is faithful and tautological, in the sense that it transfers to $\Omega^*_c(\mathbf{\Gamma}_n)$ the cyclic structure of the DG algebra $\Omega^*_c(\mathbf{\Gamma}_n)$.

Given a smooth étale groupoid $\mathbf{\Gamma}$, Gorokhovsky's DG analogue of the chain map $\Phi : C^\bullet_\nu(\mathbf{\Gamma}, \Omega_*(\mathbf{\Gamma})) \to CC^\bullet\left(C^\infty_c(\mathbf{\Gamma})\right)$ is the map $\Psi : C^\bullet_\nu(\mathbf{\Gamma}, \Omega_*(\mathbf{\Gamma})) \to CC^\bullet\left(\Omega^*_c(\mathbf{\Gamma})\right)$, which assigns to $\gamma \in C^\bullet_\nu(\mathbf{\Gamma}_n, \Omega_*(\mathbf{\Gamma}))$ a cochain $\Psi(\gamma) \in CC^\bullet\left(\Omega^*_c(\mathbf{\Gamma})\right)$; its expression [4, §7] resembles that of κ_* except that $\tau_\wedge$ is replaced by the current γ. Gorokhovsky made the insightful observation that by extending the Connes' notion of a cycle to DG algebras then, for any cocycle $\boldsymbol{c} = \{\gamma_{p,q}\} \in C^\bullet_\nu(\mathbf{\Gamma}, \Omega_*(\mathbf{\Gamma}))$ is a *cocycle*, $\Psi(\boldsymbol{c}) \in CC^\bullet\left(\Omega^*_c(\mathbf{\Gamma})\right)$ becomes the *character* $\mathrm{Ch}\,\mathcal{K}_{\boldsymbol{c}}$ *of a cycle* $\mathcal{K}_{\boldsymbol{c}}$ *over the DG algebra* $\Omega^*_c(\mathbf{\Gamma})$. The bi-differential bi-graded algebra underlying all such cycles is that defined by Connes in [1, Remark 15, Ch. III, §2. δ]. Furthermore, the retraction map $\mathcal{R} : CC^\bullet\left(\Omega^*_c(\mathbf{\Gamma})\right) \to CC^\bullet\left(C^\infty_c(\mathbf{\Gamma})\right)$ [4, §2] which implements the isomorphism $HP^\bullet(\Omega_c^*(\mathbf{\Gamma})) \cong HP^\bullet(C_c^\infty(\mathbf{\Gamma}))$, sends $\Psi(\gamma)$ into precisely Connes' $\Phi(\gamma)$, for any cochain $\gamma \in C_\nu^\bullet(\mathbf{\Gamma}, \Omega_*(\mathbf{\Gamma}))$. In particular, for any cocycle $\boldsymbol{c}$, $\Phi(\boldsymbol{c}) = \mathcal{R}(\mathrm{Ch}\,\mathcal{K}_{\boldsymbol{c}})$ becomes the *retracted character* of a cycle.

3. The concept that allows to bind together the two approaches sketched above is that of *continuous* (with respect to the completion topology in the formal jet series) cohomology, i.e. Haefliger's the *differentiable cohomology* of smooth étale groupoids. The relevant groups for this discussion are $H_{\mathrm{d}}^\bullet(\mathbf{\Gamma}, \Omega^*(P\mathbf{\Gamma}))$ and $H_{\mathrm{d}}^\bullet(\mathbf{\Gamma}, \Omega^*(\mathbf{\Gamma}))$, which consist of classes of characters of differentiable cycles over $\Omega_c^*(P\mathbf{\Gamma}_n)$, resp. $\Omega^*(\mathbf{\Gamma})$, and are in fact isomorphic. Indeed, the choice of a cross-section $\iota : \mathbf{\Gamma}^{(0)} \to P\mathbf{\Gamma}^{(0)}$, i.e. a metric, yields an isomorphism ι^*.

The isomorphism $\chi^{-1} \circ \Phi^P : C_{\mathrm{c}}^*(\mathfrak{a}_n, \mathrm{O}_n) \to CC^\bullet(\mathcal{H}_n, O_n; \mathbb{C}_\delta)$ from §**1** can be refined by the insertion of differentiable cohomology. Indeed, the image of the map $\mathcal{D}$ is contained in $C_{\mathrm{d},\nu}^\bullet(\mathbf{\Gamma}_n, \Omega_*(P\mathbf{\Gamma}_n))$ and $\mathcal{D} : C_{\mathrm{c}}^*(\mathfrak{a}_n, \mathrm{O}_n) \to C_{\mathrm{d},\nu}^\bullet(\mathbf{\Gamma}_n, \Omega_*(P\mathbf{\Gamma}_n))$ is a quasi-isomorphism (cf. [7]). Since $\Upsilon^P = \Phi^P \circ \mathcal{D}$ is quasi-isomorphism, so is the restriction $\Phi_{\mathrm{d}}^P = \Phi^P \mid C_{\mathrm{d},\nu}^\bullet(\mathbf{\Gamma}_n, \Omega_*(P\mathbf{\Gamma}_n))$. Moreover, the image of Φ_{d}^P is included in the image of χ. Because the latter is faithful, $\Xi = \chi^{-1} \circ \Phi_{\mathrm{d}}^P : C_{\mathrm{d},\nu}^\bullet(\mathbf{\Gamma}_n, \Omega_*(P\mathbf{\Gamma}_n)) \to CC^\bullet(\mathcal{H}_n, O_n; \mathbb{C}_\delta)$ is well-defined. Similar results hold true in the DG framework of §**2**. In particular, the map $\Upsilon = \kappa^{-1} \circ \Psi : C_{\mathrm{d},\nu}^\bullet(\mathbf{\Gamma}_n, \Omega_*(\mathbf{\Gamma}_n)) \to CC^\bullet\big(\Omega^*(\mathrm{GL}_n), d\big)_n$ is well-defined.

Given a cocyle $\boldsymbol{c} = \{\gamma_{p,q}\} \in C_{\mathrm{d},\nu}^\bullet(\mathbf{\Gamma}_n, \Omega_*(P\mathbf{\Gamma}_n))$, $\Xi(\boldsymbol{c})$ can be regarded as the *Hopf cyclic character* $\mathrm{Hch}(\mathcal{K}_{\boldsymbol{c}})$ *of the differentiable cycle* $\mathcal{K}_{\boldsymbol{c}}$ *over* $\Omega_*(P\mathbf{\Gamma}_n)$, while for a cocyle $\boldsymbol{c'} = \{\gamma'_{p,q}\} \in C_{\mathrm{d},\nu}^\bullet(\mathbf{\Gamma}_n, \Omega_*(\mathbf{\Gamma}_n))$, $\mathrm{Hch}(\boldsymbol{c'}) := \Upsilon(\boldsymbol{c'})$ is the *Hopf cyclic character of the differentiable cycle* $\mathcal{K}_{\boldsymbol{c'}}$ *over* $\Omega_*(\mathbf{\Gamma}_n)$.

The main result can now be stated as follows.

Theorem. **(i)** *Both maps* $\Xi_\bullet : H_{\mathrm{d}}^\bullet(\mathbf{\Gamma}_n, \Omega_*(P\mathbf{\Gamma}_n)) \to HP^\bullet(\mathcal{H}_n, O_n; \mathbb{C}_\delta)$ *and* $\Upsilon_\bullet : H_{\mathrm{d}}^\bullet(\mathbf{\Gamma}_n, \Omega_*(\mathbf{\Gamma}_n)) \to HP^\bullet\big(\Omega^*(\mathrm{GL}_n), d\big)_n$ *are isomorphisms.*

(ii) *The map* $\Theta : HP^\bullet(\mathcal{H}_n, O_n; \mathbb{C}_\delta) \to HP^\bullet\big(\Omega^*(\mathrm{GL}_n), d\big)_n$ *which sends the class of* $\mathrm{Hch}(\mathcal{K}_{\boldsymbol{c}})$ *to the class of* $\mathrm{Hch}(\mathcal{K}_{\iota^*(\boldsymbol{c})})$*, for any cocyle* $\boldsymbol{c} \in C_{\mathrm{d},\nu}^\bullet(\mathbf{\Gamma}_n, \Omega_*(P\mathbf{\Gamma}_n))$*, is an isomorphism.*

Concluding remarks. **(1)** The Chern-Weil theory applied in the setting of Dupont's (commutative) DG algebra of compatible De Rham forms on the nerve of $\mathbf{\Gamma}_n$, produces Chern forms and Chern-Simons forms from which one can build (cf. [7]) a Vey type basis of $H_{\mathrm{d}}^\bullet(\mathbf{\Gamma}_n, \Omega_*(P\mathbf{\Gamma}_n))$. By transport through the isomorphisms in the above theorem, the groups $H_{\mathrm{d}}^\bullet(\mathbf{\Gamma}_n, \Omega_*(\mathbf{\Gamma}_n))$, $HP^\bullet(\mathcal{H}_n, O_n; \mathbb{C}_\delta)$ and $HP^\bullet\big(\Omega^*(\mathrm{GL}_n), d\big)_n$ inherit similar bases.

(2) The isomorphism Θ implements the descent of the universal Hopf cyclic transverse characteristic classes from the Hopf algebra $\mathcal{H}_n$, acting at the level of frame bundle, to the Hopf DG algebra $\Omega^*(\mathrm{GL}_n)$, acting at the base level.

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Homology and K-theory of torsion free ample groupoids and Smale spaces

Makoto Yamashita
(joint work with Valerio Proietti)

Ample groupoids, that is, étale Hausdorff groupoids with totally disconnected topology, provide a groupoid model of dynamical systems on Cantor spaces. We relate their groupoid homology and the K-groups of crossed products.

Theorem 1. *Let G be an ample groupoid that has torsion free stabilizers and satisfies the strong Baum–Connes conjecture, and let A be a separable G-C^*-algebra which is KK^X-nuclear for $X = G^{(0)}$. Then there is a convergent spectral sequence*

$$E^2_{pq} = H_p(G, K_q(A)) \Rightarrow K_{p+q}(G \ltimes A).$$

Here, the K-group $K_q(A)$ admits a structure of unitary $C_c(G, \mathbb{Z})$-module, and $H_p(G, K_q(A))$ is the groupoid homology [1] with coefficient in the associated G-sheaf. The construction of spectral sequence is based on the triangulated categorical approach to the Baum–Connes conjecture by Meyer and Nest [3, 2]. We relate their machinery to a projective resolution of A with respect to the kernel of $\mathrm{Res}^G_X \colon \mathrm{KK}^G \to \mathrm{KK}^X$ induced by the adjoint pair of functors Res^G_X and $\mathrm{Ind}^G_X \colon \mathrm{KK}^X \to \mathrm{KK}^G$.

An interesting example is unstable equivalence relation $R^u(Y, \psi)$ of Smale spaces (Y, ψ). In this setting, there is another homology theory proposed by Putnam [4]. We show that one of the variants, H^s_*, fits into our scheme.

Theorem 2. *Let (Y, ψ) be an irreducible Smale space with totally disconnected stable sets. Then there is a convergent spectral sequence*

$$E^2_{pq} = E^3_{pq} = H^s_p(Y, \psi) \otimes K_q(\mathbb{C}) \Rightarrow K_{p+q}(C^*(R^u(Y, \psi))).$$

Here we look at a factor map $f\colon \Sigma \to Y$ from a shift of finite type used in the definition of $H^s_*(Y,\psi)$. Taking a good transversal $T \subset \Sigma$, we obtain an inclusion of étale groupoids $H = R^u(\Sigma,\sigma)|_T \subset G = R^u(Y,\psi)|_{f(T)}$. Then the restriction and induction for this inclusion gives the above result. In fact, we obtain $H^s_p(Y,\psi) \simeq H_p(G,\mathbb{Z})$ through a detailed analysis of the equivariant sheaves and multiple fibered product of groupoids.

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Index theorem for elliptic operators invertible at infinity

Hang Wang

(joint work with Xiaoman Chen, Hongzhi Liu, Guoliang Yu)

Let X be a complete spin manifold with metric admitting uniform positive scalar curvature outside a compact set Z. Consider the Dirac operator D on the universal covering space $\widetilde{X}$ of X. Denote by G the fundamental group of X. Let

$$\mathrm{ind}_G(D) \in K_0(C_r^*(G))$$

be the higher index of the Dirac operator D. Suppose that $g \in G$ is a nontrivial element whose conjugacy class (g) has polynomial growth. Then $C_r^*(G)$ admits a smooth subalgebra depending on g on which the delocalized trace

$$\mathrm{tr}_g : \mathbb{C}\Gamma \to \mathbb{C} \qquad \sum_{h\in G} a_h h \mapsto \sum_{h\in(g)} a_h h$$

extends continuously. The aim of the talk is to compute the pairing of the delocalized trace at g with the higher index of D, in terms of the delocalized eta invariant at infinity, denoted by $\eta_{g,\infty}(D)$:

$$\mathrm{tr}_g(\mathrm{ind}_G(D)) = -\frac{1}{2}\eta_{g,\infty}(D). \tag{1}$$

The delocalized eta invariant at infinity, introduced in [1], admits the following formula

$$\frac{1}{2}\eta_{g,\infty}(D) = \lim_{t\to 0}\int_t^\infty \mathrm{tr}_g(e^{-sD_c^-D_c^+}D_c^-[D^+,\psi_2])ds \tag{2}$$

where D_c is the invertible Dirac operator on $\widetilde{X}\backslash\widetilde{Z}$, and ψ_2 is a G-invariant cutoff function from $\widetilde{X}$ to $[0,1]$, which equals 0 on a cocompact neighbourhood of $\widetilde{Z}$, the G-Galois covering space of Z, and equals 1 outside a cocompact set far from Z. The integral on the right hand side of (2) is independent of the choice of the cutoff function ψ_2.

Let M be a manifold with boundary N, carrying a product metric near the boundary. Assume in addition that the metric on N admits positive scalar curvature. Then the delocalized eta invariant at infinity for the Dirac operator on $X := M \cup_N N \otimes [0,\infty)$ reduces to Lott's delocalized eta invariant of the Dirac operator on N, and (1) recovers a main result of [4]:

$$\mathrm{tr}_g(\mathrm{ind}_G(D)) = -\frac{1}{2}\eta_{g,\infty}(D) = -\frac{1}{2}\eta_g(D_N).$$

The main result (1)-(2) also applies to the example of a manifold with corners, recovering corresponding delocalized APS type formula. In view of the motivation from [4], the nonalgebraicity of the delocalized eta invariant at infinity is an obstruction for the Baum-Connes assembly map for G being surjective. The proof of (1)-(2) is motivated by a new way of obtaining the equivariant Atiyah-Patodi-Singer index theorem for a manifold with boundary in the joint work of the speaker with Bai-Ling Wang and Peter Hochs [2, 3].

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Bivariant hermitian K-theory

Guillermo Cortiñas

In currently ongoing work with Santiago Vega we introduce an excisive, homotopy invariant and matricially stable functor j^h from the category Alg^* of algebras with involution over a commutative ring ℓ with involution (assuming $1/2 \in \ell$) to a triangulated category kk^h, and show it is universal initial among those functors from Alg^* to triangulated categories, which are excisive, homotopy invariant and matricially stable. Weibel-style homotopy invariant hermitian K-theory is recovered as a hom in this category:

$$\hom_{kk^h}(j^h(\ell), j^h(A)[n]) = KH_n^h(A)$$

Here $[-n]$ is the n-fold suspension.

Usual Karoubi hermitian K-theory K^h (a.k.a. Grothendieck-Witt theory) maps to KH^h, and the map is an isomorphism for sufficiently regular A.

We prove a version of Karoubi's fundamental periodicity theorem ([2]) in terms of kk^h. There are an a endofunctor Λ of kk^h, which represents K-theory and natural transformations $F : 1 \to \Lambda$ and $H : \Lambda \to 1$, representing the forgetful and hyperbolic maps, as well as a change of symmetry functor ϵ. The version of Karoubi's fundamental theorem says that the polynomial homotopy fibers V of F and U of H satisfy

$$\epsilon(U(A))[1] = V(A)$$

We use kk^h to attack a long standing problem in Leavitt path algebras (purely algebraic analogues of graph, or Cuntz-Krieger C^*-algebras). It is known that two purely infinite simple CK-algebras of finite graphs have isomorphic K_0-invariants if and only if they are isomorphic. The analogue question for LPAs is wide open; in particular it is not known whether the LPA of a graph and of its Cuntz splice (which changes the sign of the determinant of the adjacency matrix but preserves K_0) are isomorphic. We classify Leavitt path algebras up to involution preserving homotopy. We show that two purely infinite simple LPAs of finite graphs with isomorphic K_0 invariants are homotopy equivalent, via a homotopy equivalence that preserves their standard involutions. This result can be seen as an improvement upon the main theorem of [1], where a similar homotopy equivalence, maybe not involution preserving, was obtained. We also consider the involution that results from composing the standard involution of a LPA with the automorphism that multiplies homogeneous elements of degree n by $(-1)^n$. We show that, equipped with this signed involution, the algebraic analogues L_n and L_{n^-} of the Cuntz algebra O_n and its Cuntz splice O_{n^-} have nonisomorphic hermitian K_0-groups.

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Liquid modules

Dustin Clausen

(joint work with Peter Scholze)

0.1. **Introduction.** I will be describing some new foundations for functional analysis. In the standard foundations, one starts with the category of topological real vector spaces, and then singles out the more relevant full subcategory of complete locally convex topological real vector spaces:

$$\{\text{complete, locally convex}\} \subset \{\text{topological real vector spaces}\}.$$

Our replacement for this inclusion will be the following:

$$\{\text{liquid } \mathbb{R}\text{-modules}\} \subset \{\text{condensed } \mathbb{R}\text{-modules}\}.$$

Actually, there is a missing parameter p in the liquid theory, but we'll get to that.

There is a natural functor from (Hausdorff) topological real vector spaces to condensed $\mathbb{R}$-modules which restricts to a functor from complete locally convex to liquid, and this functor is fully faithful on the metrizable spaces (plus many others). In particular every Frechet space can just as well be viewed as a liquid $\mathbb{R}$-module.

On the other hand, the categorical properties of the second inclusion are much better. Both condensed $\mathbb{R}$-modules and liquid $\mathbb{R}$-modules form abelian categories, and moreover of a very simple kind, being generated by a class of compact projective objects. This makes them very similar to categories of modules over a ring, which are exactly the abelian categories generated by a single compact projective object. There is also a tensor product on each of these categories which preserves colimits in each variable separately, and there are corresponding internal mapping objects.[1]

Furthermore, the full subcategory of liquid modules has extremely strong closure properties inside all condensed $\mathbb{R}$-modules: it is closed under all colimits and limits, and if M is condensed and N is liquid then the internal ext's $\underline{Ext}^i(M,N)$ are all liquid. Moreover, the inclusion functor admits a left adjoint "liquidification", which in practice functions as a kind of completion functor. All of these nice properties make the liquid theory smooth and easy to use in practice.

0.2. **Definitions.** Just as topological real vector spaces are gotten by piling algebraic structure on topological spaces, so are condensed $\mathbb{R}$-modules gotten by piling algebraic structure on our analog of topological spaces, which are called *condensed sets*. These are functors $X : \mathrm{CHaus}^{op} \to \mathrm{Sets}$ such that:

(1) For all $K, K' \in \mathrm{CHaus}$, the map $X(K \sqcup K') \to X(K) \times X(K')$ is a bijection; also $X(\emptyset) = *$.
(2) If $K' \to K$ is a surjective map in CHaus, then the map $X(K) \to X(K')$ is injective with image the subset of those elements whose two pullbacks to $X(K' \times_K K')$ agree.
(3) A technical set-theoretic condition: X is a small colimit of representable functors.[2]

The idea (the same as for Spanier's quasi-topological spaces, [9]) is that $X(K)$ stands for the set of continuous maps from K to some fictional space "X". This intuition promotes to a functor from T_1 topological spaces to condensed sets, which

[1] In general, the liquid tesnor product doesn't correspond to any of the usual tensor products on complete locally convex topological vector spaces. But on nuclear Frechet spaces it is the same as the usual tensor product (essentially unique in this case by Grothendieck's work [4]). In terms of cyclic homology, this means that the cyclic homology of things like algebras of smooth functions as naturally defined in our theory is the "correct" one, whereas the cyclic homology of, for example, C^*-algebras is pathological.

[2] Barwick and Haine, [2], have recently studied essentially the same concept, but with a different set-theoretic condition in 3 (using cut-off cardinals instead); they call that notion *pyknotic set*. At first pass (and probably at n^{th} pass as well) one should ignore the distinction.

is fully faithful on a large class of topological spaces (e.g. the metrizable ones). Encoding "topologies" via points (or K-families of points) as opposed to subsets makes for much nicer interaction with algebraic structures.

Condensed $\mathbb{R}$-modules are just condensed abelian groups together with an action of the condensed ring $\mathbb{R}$. That these form an abelian category follows from formal topos-theoretic considerations, but that it has enough compact projectives relies on a remark of Gleason's, [5], that CHaus itself "has enough projectives" given by the Stone-Cech compactifications βS of discrete sets S. It follows that the free modules $\mathbb{R}[\beta S]$ provide compact projective generators.

How to single out the liquid ones inside? It suffices to understand the liquidifications $L(\mathbb{R}[\beta S])$ of the above generators, which come equipped with natural maps $\mathbb{R}[\beta S] \to L(\mathbb{R}[\beta S])$. Indeed, then the desired axiomatics outlined in the first section will ensure that the following two properties of a condensed $\mathbb{R}$-module M are equivalent, and define when M is liquid:

(1) M is generated under colimits by modules of the form $L(\mathbb{R}[\beta S])$.
(2) For all sets S and all maps $\mathbb{R}[\beta S] \to M$, there is a unique extension to $L(\mathbb{R}[\beta S]) \to M$.

The space $\mathbb{R}[\beta S]$ can be interpreted as a certain space of measures on βS, namely the finite linear combinations of Dirac measures. Then $L(\mathbb{R}[\beta S])$ should be some larger space of measures, and the liquid condition 2 is more closely related to the classical notion of *quasi-completeness* for topological vector spaces. To check that a choice of $L(\mathbb{R}[\beta S])$'s works, meaning satisfies the strong axiomatics described in the first section, the condition one needs to check is the following: for any condensed R-module M which is the cokernel of a direct sum of copies of $L(\mathbb{R}[\beta S])$'s, we have

$$\underline{Ext}^i(L(\mathbb{R}[\beta S]), M)) \xrightarrow{\sim} \underline{Ext}^i(\mathbb{R}[\beta S], M).$$

Our solution to this problem is as follows: we fix a real number $0 < p \leq 1$, and we define $L(\mathbb{R}[\beta S]) = \mathcal{M}_{<p}(\beta S) = \cup_{c>0, q<p} \mathcal{M}_q(\beta S)_{\leq c}$, where

$$\mathcal{M}_q(\beta S)_{\leq c} = \varprojlim_{\beta S \to S_i} \mathbb{R}[S_i]_{\ell^q \leq c}.$$

Here the index is over all continuous surjective maps from βS to a finite set S_i; note that then $\beta S = \varprojlim S_i$. Also, $\mathbb{R}[S_i]_{\ell^q \leq c}$ is the compact subpsace of $\mathbb{R}[S_i]$ consisting of those points whose ℓ^q quasinorm is $\leq c$.

Where did this come from? The naive initial attempt was to take $L(\mathbb{R}[\beta S])$ to be the analogously defined $\mathcal{M}_1(\beta S)$, which is essentially the usual space of Radon measures on βS showing up in the notion of quasi-completeness[3], but this turned out to fail the above Ext condition even for $M = \mathbb{R}$ and $i = 1$: there are exotic Ext's on the left hand side, variants of Ribe's construction [7] based on the entropy functional. This is fixed by taking the union of $q < p \leq 1$ as above, in accordance

[3]The topology is not the Banach topology, but the "Smith space" topology, [1], for which the closed unit ball is compact. Smith spaces are actually better building blocks than Banach spaces because every Banach space is a filtered colimit of Smith spaces and filtered colimits are homologically nice, whereas to build Smith spaces from Banach spaces you need inverse limits which are homologically complicated.

with Kalton's theorem, [6], that any extension of p-Banach spaces is q-Banach for all $q < p$. We thus find that to obtain good categories of "complete" $\mathbb{R}$-modules it is necessary to take non-locally convex spaces as basic building blocks.

A word about the proof, an account of which (plus more information on the above) can be found in my collaborator Peter Scholze's lecture notes [8]. It is long and difficult, but there is one crucial shift which is worth mentioning. To prove this statement about $\mathbb{R}$ we need to deform to a ring of (overconvengent) arithmetic Laurent series $\mathbb{Z}((T))_{>r}$ which recovers $\mathbb{R}$ after modding out by some regular element. The reason it's easier to prove the statement for this other ring is that $\mathbb{Z}((T))_{>r}$ is an increasing union of profinite sets, which are zero-dimensional and therefore easy to control even after taking something like infinite products.

In particular, we find that the real numbers are not isolated, but rather sit in a natural one-parameter family of theories parametrized by $0 < p \leq 1$. The relevant category of modules, that of the liquid modules, varies as you vary p, so this is quite a non-trivial deformation. One can access the corresponding formal deformation as a formal completion of the ring $\mathbb{Z}((T))_{>r}$. This itself carries natural notions of liquid modules very much related to Ribe's extension and the entropy functional. Such structure has been to some extent previsaged by Connes and Consani when they defined their archimedean analog of B_{dR}, based also on the entropy functional, [3]. What we have provided is the natural theory of modules which expresses the essential structure and non-triviality of such deformations.

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Slant products on the Higson–Roe exact sequence

Christopher Wulff

(joint work with Alexander Engel and Rudolf Zeidler)

This talk is on the recent paper with the same title [5]. Let G, H be countable groups acting properly and isometrically on proper metric spaces X, Y, respectively. We have to assume that X, Y satisfy some kind of bounded geometry assumption, which will not be specified in this abstract.

The Higson–Roe exact sequence associated to the G-space X is a long exact sequence

$$\dots \to \mathrm{S}^G_p(X) \to \mathrm{K}^G_p(X) \xrightarrow{\mu} \mathrm{K}_p(\mathrm{C}^*_G X) \to \mathrm{K}^G_{p-1}(X) \to \dots$$

which relates the equivariant K-homology $\mathrm{K}^G_*(X)$, the K-theory of the equivariant Roe algebra $\mathrm{K}_*(\mathrm{C}^*_G X)$ and the analytic structure group $\mathrm{S}^G_*(X)$ [6]. Its importance comes from index theory: If X is a complete Riemannian manifold, then equivariant elliptic operators over it have fundamental classes $[D] \in \mathrm{K}^G_*(X)$ and indices $\mathrm{ind}(D) \coloneqq \mu([D]) \in \mathrm{K}_*(\mathrm{C}^*_G X)$ which are the images of the fundamental classes under the assembly map μ. If the operator is invertible, then its index vanishes and in this case the invertibility can be used to construct an element in the structure group, a so called secondary index invariant, which is mapped to the fundamental group and can thus be seen as a reason for the vanishing of the index. Both index and secondary index invariants are important obstructions to topological properties. For example, if X carries a G-invariant spin-structure, then the index of the spin-Dirac operator is an obstruction to the existence of metrics of uniformly positive scalar curvature, and metrics g of uniformly positive scalar curvature give rise to the ρ-invariants $\rho(X, g) \in \mathrm{S}^G_*(X)$ which distinguish these metrics up to concordance.

One of the challenges surrounding the Higson–Roe exact sequence is to construct maps out of it into something more computable. Chern characters into cyclic homology were constructed in [1, 7], see also the talks by Xie and Zenobi at this workshop.

The main technical innovation of our work is the construction of slant products from the Higson–Roe exact sequence of the product space $X \times Y$ into the arguably simpler Higson–Roe exact sequence of X alone:

$$\begin{tikzcd}
\mathrm{S}^{G\times H}_*(X\times Y) \arrow[r] \arrow[d, "/\theta"] & \mathrm{K}^{G\times H}_*(X\times Y) \arrow[r, "\mu"] \arrow[d, "/\mu^*(\theta)"] & \mathrm{K}_*(\mathrm{C}^*_{G\times H}(X\times Y)) \arrow[r] \arrow[d, "/\theta"] & \mathrm{S}^{G\times H}_{*-1}(X\times Y) \arrow[d, "/\theta"] \\
\mathrm{S}^G_{*-q}(X) \arrow[r] & \mathrm{K}^G_{*-q}(X) \arrow[r, "\mu"] & \mathrm{K}_{*-q}(\mathrm{C}^*_G X) \arrow[r] & \mathrm{K}^G_{*-1-q}(X)
\end{tikzcd}$$

Here, $\theta \in \mathrm{K}_{1-q}(\mathfrak{c}^{\mathrm{red}}Y \rtimes H)$, where $\mathfrak{c}^{\mathrm{red}}Y$ denotes the stable Higson corona of Y, and $\mu^*\colon \mathrm{K}_{1-q}(\mathfrak{c}^{\mathrm{red}}Y \rtimes H) \to \mathrm{K}_{-q}(\mathrm{C}_0 Y \rtimes H) =: \mathrm{K}^q_H(Y)$ denotes a version of the co-assembly map [2, 3, 4].

The important feature of the slant products is that they can be used to show (rational) split injectivity of well known exterior products. Given $z \in \mathrm{K}^H_q(Y)$, these exterior products are natural transformations

$$\begin{tikzcd}
\mathrm{S}^G_*(X) \arrow[r] \arrow[d, "\times z"] & \mathrm{K}^G_*(X) \arrow[r, "\mu"] \arrow[d, "\times z"] & \mathrm{K}_*(\mathrm{C}^*_G X) \arrow[r] \arrow[d, "\times \mu(z)"] & \mathrm{S}^G_{*-1}(X) \arrow[d, "\times z"] \\
\mathrm{S}^{G\times H}_{*+q}(X\times Y) \arrow[r] & \mathrm{K}^{G\times H}_{*+q}(X\times Y) \arrow[r, "\mu"] & \mathrm{K}_{*+q}(\mathrm{C}^*_{G\times H}(X\times Y)) \arrow[r] & \mathrm{S}^{G\times H}_{*-1+q}(X\times Y)
\end{tikzcd}$$

with the property that composing them from the right with the slant products yields exactly multiplication with pairings $\langle\theta,\mu(z)\rangle = \langle\mu^*(\theta),z\rangle \in \mathbb{Z}$. Thus, if a θ exists for which these pairings are ± 1 ($\neq 0$), then the exterior products with z are (rationally) split injective. This allowed us to use the index theory on X to derive numerous interesting implications for metrics of uniformly positive curvature on the product manifold $X \times Y$.

One must also ask the question of how our slant products compare with already existing slant products. As it turns out, the slant product $\mathrm{K}_p(X\times Y)\otimes \mathrm{K}^q(Y) \to \mathrm{K}_{p-q}(X)$ in the non-equivariant case $G = H = 1$ agrees with the slant product obtained as the composition product

$$\mathrm{E}_p(\mathrm{C}_0(X\times Y),\mathbb{C})\otimes \mathrm{E}_{-q}(\mathbb{C},\mathrm{C}_0(Y)) \to \mathrm{E}_{p-q}(\mathrm{C}_0(X),\mathbb{C})\,, \quad x\otimes y \mapsto x\circ(\mathrm{id}_X\otimes y)$$

in E-theory. Unfortunately, it cannot be generalized directly to the non-equivariant case by simply replacing the three E-theory groups by their $G\times H$-, H- and G-equivariant versions, most importantly because the products cannot get rid of the H-invariance.

This gives rise to the following question, which could be of interest to other participants of this particular conference as well: Given a G-C*-algebra A, an H-C*-algebra B and a group homomorphism $\alpha\colon G\to H$, are there "α-equivariant" E-theory groups $\mathrm{E}^\alpha_*(A,B)$ which generalize the usual equivariant E-theory groups in the sense that for $G=H$, $\alpha=\mathrm{id}_G$ we have

$$\mathrm{E}^{\mathrm{id}}_*(A,B)\cong \mathrm{E}^G_*(A,B)$$

and which include the equivariant K-homology and K-theory of spaces as the special cases with trivial group homomorphisms

$$\mathrm{E}^{1\to H}_{-*}(\mathbb{C},\mathrm{C}_0(Y))\cong \mathrm{K}^*_H(Y) \quad\text{and}\quad \mathrm{E}^{G\to 1}_*(\mathrm{C}_0(X),\mathbb{C})\cong \mathrm{K}^G_*(X) \quad ?$$

Of course, straightforward adaptions of the well-known properties of bivariant K-theory should hold, so in particular there should be composition and exterior products of the form

$$\begin{aligned}
&\mathrm{E}^\alpha_p(A,B)\otimes \mathrm{E}^\beta_q(B,C)\to \mathrm{E}^{\beta\circ\alpha}_{p+q}(A,C)\\
&\mathrm{E}^\alpha_p(A,B)\otimes \mathrm{E}^\gamma_q(C,D)\to \mathrm{E}^{\alpha\times\gamma}_{p+q}(A\otimes C,B\otimes D)\,.
\end{aligned}$$

Such a theory could also explain our equivariant slant products from an E-theoretic viewpoint. The same question can be asked about equivariant KK-theory.

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Stability of Loday constructions

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(joint work with Ayelet Lindenstrauss)

For a commutative ring spectrum R and a commutative R-algebra spectrum A the Loday construction $\mathcal{L}^R_X(A)$ for a finite simplicial set X generalizes the concept of topological Hochschild homology of A which corresponds to the case where X is the circle, S^1, and R is the sphere spectrum S. Of particular interest is the case $X = T^n = (S^1)^n$, an n-torus, as $\mathcal{L}^S_{T^n}(A)$ is the target of an iterated trace map from the n-fold iterated algebraic K-theory of A. The homotopy groups of $\mathcal{L}^S_{T^n}(A)$ are in general difficult to calculate.

If we assume that R and A are cofibrant, then the homotopy type of $\mathcal{L}^R_X(A)$ only depends on the homotopy type of X. In several classes of examples it actually only depends on the homotopy type of ΣX. In this case one says that $R \to A$ is stable. For such $R \to A$ one can for instance determine the homotopy type of $\mathcal{L}^R_{T^n}(A)$ in terms of $\mathcal{L}^R_{S^k}(A)$ for $1 \leq k \leq n$ and the homotopy groups of the latter are known in many examples, such as when R is the sphere spectrum and A is the Eilenberg-MacLane spectrum $H\mathbb{F}_p$ for any prime p [2].

In the talk we present several different notions of stability together with their structural properties and we discuss examples an non-examples of stability.

A strong notion of stability is the following: Let $R \to A$ be a cofibration of commutative S-algebras with R cofibrant. We call $R \to A$ *multiplicatively stable* if for every pair of pointed simplicial sets X and Y an equivalence $\Sigma X \simeq \Sigma Y$ implies that $\mathcal{L}^R_X(A) \simeq \mathcal{L}^R_Y(A)$ as augmented commutative A-algebras. There are also linear variants of stability.

An easy stability result says that for any augmented commutative R-algebra spectrum A, $A \to R$ and $R \to \mathcal{L}^R_{\Sigma X}(A; R) \to R$ are multiplicatively stable.

Dundas and Tenti show [3] that the 2-torus is a witness for the fact that $H\mathbb{Q}[t]/t^2$ is *not* stable and in [4] we show that $H\mathbb{Q} \to \mathbb{Q}[t]/t^m$ is not multiplicatively stable for all $m \geq 2$ by using the m-torus as a witness.

In [4] we also show that for any commutative Hopf algebra spectrum $\mathcal{H}$ and every equivalence $\Sigma(X_+) \simeq \Sigma(Y_+)$ in the infinity category of pointed spaces $\mathcal{S}_*$, there is an equivalence $\mathcal{L}_X(\mathcal{H}) \simeq \mathcal{L}_Y(\mathcal{H})$. This generalizes a result by Berest, Ramadoss, Yeung for commutative Hopf algebras over a field [1].

Other concrete examples are that $HR \to HR/(a_1, \dots, a_n)$ is multiplicatively stable if R is a commutative ring and $(a_1, \dots, a_n)$ is a regular sequence and if $R \to A$ is a cofibration of commutative S-algebras with R cofibrant, then $A \to \mathcal{L}^R_{\Sigma X}(A)$ is multiplicatively stable for all $X \in sSets_*$ [5].

We show [5] that stability satisfies certain inheritance properties: If $f\colon A \to B$ is multiplicatively stable, then so is $C \wedge_R f\colon C \wedge_R A \to C \wedge_R B$. Multiplicative stability is closed under pushouts: If $R \to B$ and $R \to C$ are multiplicatively stable, then so is $R \to B \wedge_R C$.

Multiplicative stability is also closed under forming Loday constructions: If $R \to A$ is multiplicatively stable, then so is $R \to \mathcal{L}^R_Z(A)$ for any Z. If $S \to A$ and $S \to B$ are cofibrations of commutative S-algebras and if A and B are multiplicatively stable, then for connected X and Y with $\Sigma X \simeq \Sigma Y$, there is an equivalence

$$\mathcal{L}^S_X(A \times B) \simeq \mathcal{L}^S_Y(A \times B)$$

of commutative S-algebras.

Beware, however, that stability is not transitive: If $R \to A$ and $A \to B$ satisfy stability then this does *not* imply that $R \to B$ is stable. A concrete example is $\mathbb{Q} \to \mathbb{Q}[t]$ and $\mathbb{Q}[t] \to \mathbb{Q}[t]/t^m$.

Dundas and Tenti [3] show that for $k \to A$ smooth, the map $Hk \to HA$ is stable. We develop an adequate generalization of this phenomenon for ring spectra [5]. We show that for every simplicial set X there is a weak equivalence of commutative R-algebras

$$\mathcal{L}^R_X(\mathbb{P}_R(M)) \simeq \mathbb{P}_R(X_+ \wedge M),$$

in particular, if $\Sigma X \simeq \Sigma Y$, then $\mathcal{L}^R_X(\mathbb{P}_R(M)) \simeq \mathcal{L}^R_Y(\mathbb{P}_R(M))$ as commutative R-algebra spectra. Here, $\mathbb{P}_R(M)$ is the free commutative R-algebra spectrum generated by an R-module spectrum M.

For ring spectra there are several non-equivalent notions of étale maps. Let $R \to A \to B$ be a sequence of cofibrations of commutative S-algebras with R cofibrant. Then this sequence *satisfies étale descent* if for all connected X the canonical map

$$\mathcal{L}^R_X(A) \wedge_A B \to \mathcal{L}^R_X(B)$$

is an equivalence.

We call a map of cofibrant S-algebras $\varphi\colon R \to A$ *really smooth* if it can be factored as $R \xrightarrow{i_R} \mathbb{P}_R(M) \xrightarrow{f} A$ where i_R is the canonical inclusion, M is an R-module, and $R \xrightarrow{i_R} \mathbb{P}_R(M) \xrightarrow{f} A$ satisfies étale descent.

We establish, for instance, that periodic complex topological K-theory, KU, is stable and we deduce with the Galois descent property of $KO \to KU$ that periodic real topological K-theory, KO, is also stable.

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Higher index theorem for proper action of Lie groups

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(joint work with Peter Hochs, Xiang Tang)

Let G be a connected real reductive group and H be its Cartan subgroup. For any smooth function f on G with compact support and regular element $h \in H^{\mathrm{reg}}$, the orbital integral

$$\int_{G/H} f(ghg^{-1})dg$$

defines a trace on the convolution algebra $C_c^\infty(G)$, which extends to Harish-Chandra's Schwartz algebra $\mathcal{C}(G)$. Orbit integrals have been proved to be essential tools in both representation theory and number theory. We study orbital integrals as tools to probe the tempered dual, and exam their relationship to the Connes-Kasparov isomorphism.

In the first part, we introduce a construction of higher orbital integrals in the direction of higher cyclic cocycles on the Harish-Chandra Schwartz algebra of G. Using the Fourier transform or Harish-Chandra's Plancherel formula, the higher orbital integrals can be expressed as integrals on the tempered dual of G. In general, the higher orbital integrals are very difficult to compute. However, the pairing between the higher orbit integrals and the K-theory of the reduced group C^*-algebra are easier to compute and can be used to detect some information about the Plancherel measure of tempered representations of G.

In the second part, we consider a Riemannian manifold M together with a proper, cocompact, isometric G-action. Let D be a G-equivariant, elliptic differential operator on M. By the assembly map in the Baum-Connes conjecture, D has an equivariant index in the K-theory of the reduced group C^*-algebra $C_r^*(G)$. We prove a topological index formula for the pairings of the higher orbital integrals with the equivariant index of D, which generalizes Connes-Moscovici's L^2-index theorem to the non-equal rank case, and the Atiyah-Bott fixed point theorem to the non-compact Lie group action case.

The Novikov conjecture for geometrically discrete groups of diffeomorphisms

Jianchao Wu
(joint work with Sherry Gong and Guoliang Yu)

The Novikov conjecture is a central problem in differential topology. It states that the higher signatures of closed oriented smooth manifolds are invariant under orientation preserving homotopy equivalences. Noncommutative geometry provides a potent approach to tackle this conjecture. In particular, Connes [2] proved a very striking theorem that the Novikov conjecture holds for higher signatures associated to Gelfand-Fuchs classes of groups of diffeomorphisms. Connes's proof is a technical tour de force and uses the full power of noncommutative geometry.

Using C^*-algebraic and K-theoretic tools, we [3] prove that *the (rational strong) Novikov conjecture holds for geometrically discrete subgroups of the group of volume preserving diffeomorphisms of any closed smooth manifold* N. More precisely, given a density ω on N, which we regard as a measure on N, we let $\mathrm{Diff}(N,\omega)$ denote the group of diffeomorphisms on N that fix ω. A countable subgroup Γ of $\mathrm{Diff}(N,\omega)$ is said to be a *geometrically discrete* subgroup if the integral

$$\int_N (\log(\|D\varphi\|))^2 d\omega$$

goes to infinity when $\varphi \to \infty$ in Γ. Here $D\varphi$ is the Jacobian of a diffeomorphism φ, and the norm $\|\cdot\|$ denotes the operator norm, computed using an arbitrarily fixed Riemannian metric on N. Intuitively speaking, this function measures how much a diffeomorphism φ deviates from an isometry in an L^2-sense. Observe that when the subgroup Γ actually fixes a Riemannian metric, the above integral is bounded on Γ. This suggests that geometrically discrete subgroups of $\mathrm{Diff}(N,\omega)$ are, in a sense, conceptual antitheses to subgroups of isometries. We remark that since the group of isometries of a closed Riemannian manifold is a Lie group, all its countable subgroups satisfy the rational strong Novikov conjecture by [5]. Thus our theorem gives hope for a unified approach to prove the rational strong Novikov conjecture for all countable subgroups of $\mathrm{Diff}(N,\omega)$.

The crucial geometric property of these groups Γ that we exploit is the fact that they admit isometric and proper actions on a type of infinite-dimensional symmetric space of nonpositive curvature called the *space of L^2-Riemannian metrics*, which is defined as the completion of the space of all bounded Borel maps from N to the symmetric space $X := \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$ with regard to the following metric:

$$d(\xi,\eta) = \left(\int_{y\in N} (d_X(\xi(y),\eta(y)))^2 \, d\omega(y)\right)^{\frac{1}{2}} \text{ for two such maps } \xi \text{ and } \eta \,,$$

where d_X is the standard Riemannian metric on X. Observe that this symmetric space parametrizes all inner products on $\mathbb{R}^n$ with a fixed volume form. Thus Riemannian metrics on N that induce ω correspond to the smooth sections of an X-bundle over N. Upon taking a Borel trivialization of this bundle, these smooth sections are embedded into the space of all bounded Borel maps from N to X, and thus also into $L^2(N, \omega, X)$, with a dense image. This explains the terminology "L^2-Riemannian metrics".

In fact, in [3], we prove a more general statement that *the rational strong Novikov conjecture holds for any discrete group admitting an isometric and proper action on an admissible Hilbert-Hadamard space. Admissible Hilbert-Hadamard spaces* are a type of (possibly infinite-dimensional) non-positively curved (i.e., CAT(0)) metric spaces that include Hilbert spaces, complete simply connected Riemannian-Hilbertian manifolds with non-positive sectional curvature, and certain infinite-dimensional symmetric spaces such as the space of L^2-Riemannian metrics introduced above. Hence our result partially extends earlier ones on the rational strong Novikov conjecture for groups admitting isometric and proper actions on Hadamard manifolds ([6]) and on Hilbert spaces ([4]).

A key ingredient in our proof is the construction of a C^*-algebra associated to a Hilbert-Hadamard space, which generalizes a construction of Higson and Kasparov [4] for a Hilbert space and is analogous to the one constructed by Kasparov and Yu [7] for Banach spaces with property (H). This algebra comes with a natural dual Dirac (or Bott) element, but the construction of a Dirac element remains elusive. As a result, our proof deviates from the standard Dirac-dual-Dirac method: We develop a novel deformation technique that "trivializes" the Γ-action at the KK-theoretic level. This deformation technique is only accessible in the framework of infinite-dimensional spaces. In addition, we make use of the newly developed KK-theory of real coefficients ([1]) to go beyond the case of torsion-free groups.

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Connes-Chern characters for higher rho invariants and higher eta invariants

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(joint work with Xiaoman Chen, Jinmin Wang and Guoliang Yu)

Higher index theory is a far-reaching generalization of the classic Fredholm index theory by taking into consideration of the symmetries of the underlying space. Let X be a complete Riemannian manifold of dimension n with a discrete group G acting on it properly and cocompactly by isometries. Each G-equivariant elliptic differential operator D on X gives rise to a higher index class $\mathrm{Ind}_G(D)$ in the K-group $K_n(C_r^*(G))$ of the reduced group C^*-algebra $C_r^*(G)$. This higher index is an obstruction to the invertibility of D. The higher index theory plays a fundamental role in the studies of many problems in geometry and topology such as the Novikov conjecture, the Baum-Connes conjecture and the Gromov-Lawson-Rosenberg conjecture. Higher index classes are invariant under homotopy and often referred to as primary invariants.

When the higher index class of an operator is trivial and given a specific trivialization, a secondary index theoretic invariant naturally arises. One such example is the associated Dirac operator on the universal covering $\widetilde{M}$ of a closed spin manifold M equipped with a positive scalar curvature metric. It follows from the Lichnerowicz formula that the Dirac operator on $\widetilde{M}$ is invertible. In this case, there is a natural C^*-algebraic secondary invariant introduced by Higson and Roe in [3, 4, 5, 8], called the higher rho invariant, which lies in $K_n(C^*_{L,0}(\widetilde{M})^\Gamma)$, where Γ is the fundamental group $\pi_1(M)$ of M and $C^*_{L,0}(\widetilde{M})^\Gamma$ is a certain geometric C^*-algebra. This higher rho invariant is an obstruction to the inverse of the Dirac operator being local and has important applications to geometry and topology.

Despite its importance, the higher rho invariant is difficult to compute in general. Connes' cyclic cohomology theory provides a powerful method to compute the higher rho invariant. Roughly speaking, this is done through a pairing between the Connes-Chern character of a higher rho invariant and the cyclic cohomology of the relevant group algebra. In the case of higher rho invariants arisen from invertible[1] operators on manifolds, such a pairing coincides with Lott's higher eta invariants [6].

Let us first recall the definitions of some geometric C^*-algebras.

(1) $C^*(\widetilde{M})^\Gamma$ denotes the C^*-closure of all Γ-invariant locally compact and finite propagation operators on $L^2(\widetilde{M})$.
(2) $C^*_L(\widetilde{M})^\Gamma$ is the C^*-closure of uniformly continuous maps
$$\varphi : [0,\infty) \to C^*(\widetilde{M})^\Gamma$$
with the propagation of $\varphi(t)$ goes to zero, as $t \to \infty$.
(3) $C^*_{L,0}(\widetilde{M})^\Gamma$ is the kernel of the map $C^*_L(\widetilde{M})^\Gamma \to C^*(\widetilde{M})^\Gamma$.

[1]Here "invertible" means being invertible on the universal cover of the manifold.

Note that $C^*(\widetilde{M})^\Gamma \cong C^*_r(\Gamma) \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators. Also, we have $K_i(C^*_L(\widetilde{M})^\Gamma) \cong K_i(M)$, where $K_i(M)$ is the K-homology of M.

Here is an example of higher rho invariants that naturally arise from invertible operators. For simplicity, we shall only discuss the odd dimensional case. The even dimensional case is completely similar. Suppose M is an odd-dimensional closed spin manifold equipped with a positive scalar curvature metric. Let D be the associated Dirac operator. The higher rho invariant of D (with respect to the given metric) is defined to be

$$\rho(\widetilde{D}) = e^{2\pi i \frac{F_t+1}{2}} \in K_1(C^*_{L,0}(\widetilde{M})^\Gamma)$$

where $F_t = \mathrm{Ch}i(\widetilde{D}/t)$ for some normalizing function $\mathrm{Ch}i$, where normalizing function $\mathrm{Ch}i\colon \mathbb{R} \to [-1,1]$ is an odd continuous function such that $\mathrm{Ch}i(\lambda) \to \pm 1$, as $\lambda \to \pm\infty$. For example, one can choose

$$\mathrm{Ch}i(\lambda) = \frac{2}{\sqrt{\pi}} \int_0^\lambda e^{-s^2} ds.$$

The Connes-Chern character for a secondary invariant, i.e., an element $[u]$ in $K_1(C^*_{L,0}(\widetilde{M})^\Gamma)$, is defined as follows.

Definition 1. For $u \in K_1(C^*_{L,0}(\widetilde{M})^\Gamma)$, we define its Connes-Chern character to be

$$\mathrm{Tch}(u) := \sum_{k=0}^{\infty} (-1)^k \frac{(k-1)!}{\pi i} \Big(\sum_{j=0}^{k} \int_0^\infty \mathrm{tr}((u \otimes u^{-1})^{\otimes j} \otimes \dot{u}u^{-1} \otimes (u \otimes u^{-1})^{\otimes k-j}) dt \Big).$$

If u is the higher rho invariant $\rho(\widetilde{D})$ arising from an invertible elliptic operator $\widetilde{D}$, then the above formula can be identified with Lott's higher eta invariants, cf. [2].

Now suppose $\mathcal{A}$ is a locally m-convex Fréchet smooth dense subalgebra of $C^*_r(\Gamma)$. Let $CC_{*,\langle 1\rangle}(\mathbb{C}\Gamma)$ be the $\langle 1\rangle$-summand of cyclic chain complex $CC_*(\mathbb{C}\Gamma)$ of the group algebra $\mathbb{C}\Gamma$ and $CC_{*,\langle 1\rangle}(\mathcal{A})$ be the closure of $CC_{*,\langle 1\rangle}(\mathbb{C}\Gamma)$ in $CC_*(\mathcal{A})$. We have the short exact sequence of cyclic chain complexes

$$0 \to CC_{*,\langle 1\rangle}(\mathcal{A}) \to CC_*(\mathcal{A}) \to CC_*(\mathcal{A})/CC_{*,\langle 1\rangle}(\mathcal{A}) \to 0.$$

We define $HP_{e/o}(\mathcal{A})_{\langle 1\rangle}$ (resp. $HP_{e/o}(\mathcal{A})_{\mathrm{del}}$) to be the even/odd homology of the cyclic chain complex $CC_{*,\langle 1\rangle}(\mathcal{A})$ (resp. $CC_*(\mathcal{A})/CC_{*,\langle 1\rangle}(\mathcal{A})$). Then the above Connes-Chern character fits into the following commutative diagram:

$$\begin{tikzcd} K_0(M) \arrow[r] \arrow[d, "\mathrm{Ch}_{\mathrm{loc}}"] & K_0(C^*_r(\Gamma)) \arrow[r] \arrow[d, "\mathrm{Ch}"] & K_1(C^*_{L,0}(\widetilde{M})^\Gamma) \arrow[d, "\mathrm{Tch}"] \\ HP_e(\mathcal{A})_{\langle 1\rangle} \arrow[r] & HP_e(\mathcal{A}) \arrow[r] & HP_e(\mathcal{A})_{\mathrm{del}} \end{tikzcd} \tag{1}$$

where $\mathrm{Ch}_{\mathrm{loc}}$ is the (local) Connes-Chern character for K-homology classes and Ch is the Connes-Chern character for K-theory classes. More precisely, we have the following theorem.

Theorem 2 ([2]). *Let $\mathcal{A}$ be a Fréchet locally m-convex smooth dense subalgebra of $C_r^*(\Gamma)$. Then the Connes-Chern character map*

$$\mathrm{Tch}\colon K_{o/e}(C^*_{L,0}(\widetilde{M})^\Gamma) \to HP_{e/o}(\mathcal{A})_{\mathrm{del}}$$

given by the above formula is well-defined and makes the diagram (1) *commute.*

Despite our explicit formula for the Connes-Chern character map Tch of secondary invariants, the homology group $HP_*(\mathcal{A})$ is rather difficult to compute in general. In order to compute the Connes-Chern character, we shall pair it with $HP_*(\mathbb{C}\Gamma)$ to obtain numerical invariants, where $HP_*(\mathbb{C}\Gamma)$ is much better understood. Indeed, by a theorem of Burghelea [1], we have

$$HP_*(\mathbb{C}\Gamma) \cong \bigoplus_{\langle g\rangle} HP_*(\mathbb{C}\Gamma)_{\langle g\rangle}$$

summing over all conjugacy classes of Γ. Moreover, for a finite order element g, $HP_*(\mathbb{C}\Gamma)_{\langle g\rangle} \cong H_*(N_g)$, where Z_g is the centralizer group of g and $N_g = Z_g/\{g\}$ is the quotient of Z_g by the cyclic subgroup generated by g. In particular, we have

$$HP_{e/o}(\mathbb{C}\Gamma)_{\langle 1\rangle} \cong H_{e/o}(\Gamma)$$

and

$$HP_{e/o}(\mathbb{C}\Gamma)_{\mathrm{del}} \cong \bigoplus_{\langle g\rangle\neq\langle 1\rangle} HP_{e/o}(\mathbb{C}\Gamma)_{\langle g\rangle}.$$

where $H_{e/o}(\Gamma)$ is the even/odd group homology of Γ. This naturally brings us to the following question.

Question 3. When does a cyclic cocycle of $\mathbb{C}\Gamma$ extends to a cyclic cocycle of $\mathcal{A}$?

The above question is wide open for general groups. For word hyperbolic groups, we have the following theorem.

Theorem 4 ([2]). *If Γ is a word hyperbolic group, then every element in $HC^n(\mathbb{C}\Gamma)$ has a representative of polynomial growth,for all $n \geq 0$. Furthermore, when $n \neq 1$, every element in $HC^n(\mathbb{C}\Gamma)$ has a bounded representative.*

As a consequence, we have the following corollary.

Corollary 5 ([2]). If Γ is hyperbolic and $\mathcal{A}$ is the Puschnigg smooth dense subalgebra of $C_r^*(\Gamma)$, then any cyclic cocycle of $\mathbb{C}\Gamma$ extends to a cyclic cocycle of $\mathcal{A}$.

To summarize, we have the following theorem.

Theorem 6 ([2]). *Let M be a closed manifold whose fundamental group Γ is word hyperbolic. Suppose $\langle h\rangle$ is a non-trivial conjugacy class of Γ. Then the paring between delocalized Connes-Chern characters and delocalized cyclic cohomology*

$$K_i(C^*_{L,0}(\widetilde{M})^\Gamma) \otimes HP^{2k+1-i}(\mathbb{C}\Gamma)_{\langle h\rangle} \to \mathbb{C}$$

given by $(u,\varphi) \mapsto \langle \mathrm{Tch}(u), \varphi\rangle$ *is well-defined. In particular, if* D *is an elliptic operator on* M *such that the lift* $\widetilde{D}$ *of* D *to the universal cover* $\widetilde{M}$ *of* M *is invertible, then Lott's higher eta invariant*

$$\eta_\varphi(\widetilde{D}) := -\langle \mathrm{Tch}(\rho(\widetilde{D})), \varphi\rangle$$

converges absolutely, where $\rho(\widetilde{D})$ *is the higher rho invariant of* $\widetilde{D}$.

So far, we have seen that the existence of a smooth dense subalgebra $\mathbb{C}\Gamma \subset \mathcal{A} \subset C_r^*(\Gamma)$ and the extendability of a cyclic cocycle φ of $\mathbb{C}\Gamma$ to $\mathcal{A}$ together provide a sufficient condition for the convergence of the pairing

$$\eta_\varphi(\widetilde{D}) := -\langle \mathrm{Tch}(\rho(\widetilde{D})), \varphi\rangle.$$

This naturally brings us to the following question.

Question 7. What happens in the general? Does this pairing always converge? Are there other geometric situations where the convergence holds?

In the rest of this report, we shall discuss some results that partially answer the above question. Let us first discuss the degree-zero cyclic cocycle case. For a given $h \in \Gamma$ with $h \neq 1$, let $\langle h\rangle$ be its conjugacy class in Γ. The following trace

$$\mathrm{tr}_{\langle h\rangle}\colon \mathbb{C}\Gamma \to \mathbb{C}, \quad \sum_\beta c_\beta \beta \mapsto \sum_{\beta\in\langle h\rangle} c_\beta$$

defines a degree-zero cyclic cocycle of Γ. In this case, the pairing between the Connes-Chern character of the higher rho invariant $\rho(\widetilde{D})$ and $\mathrm{tr}_{\langle h\rangle}$ takes the following form:

$$\langle \mathrm{Tch}(\rho(\widetilde{D})), \mathrm{tr}_{\langle h\rangle}\rangle = \frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{tr}_{\langle \alpha\rangle}(\widetilde{D}e^{-t^2\widetilde{D}^2})dt$$

where we have chosen the normalizing function $\mathrm{Ch}i$ to be

$$\mathrm{Ch}i(\lambda) = \frac{2}{\sqrt{\pi}}\int_0^\lambda e^{-s^2}ds$$

in the definition of $\rho(\widetilde{D}) = \exp(2\pi i\frac{\mathrm{Ch}i(\widetilde{D}/t)+1}{2})$. We have the following theorem.

Theorem 8 ([2])**.** *If* $\langle h\rangle \neq \langle 1\rangle$ *and the spectral gap of* $\widetilde{D}$ *at zero is sufficiently large, then the pairing*

$$\langle \mathrm{Tch}(\rho(\widetilde{D})), \mathrm{tr}_{\langle h\rangle}\rangle$$

absolutely converges.

Let us make precise of what "sufficiently large spectral gap" means. Fix a finite generating set S of Γ and a fundamental domain $\mathcal{F}$ for the action of Γ on $\widetilde{M}$. Let ℓ be the corresponding word length function on Γ determined by S. The spectral gap of $\widetilde{D}$ at zero is said to be sufficiently large if its spectral gap is larger than the constant

$$\frac{2K_{\langle h\rangle}\cdot c_D}{C_{\langle h\rangle}},$$

where

(1) $K_{\langle h\rangle}$ is the smallest number nonnegative number such that
$$\#\{\beta \in \langle\alpha\rangle \colon \ell(\beta) \leq n\} \leq A \cdot e^{K_{\langle h\rangle}\cdot n};$$
(2) $c_D = \sup\big\{\|\sigma_D(x,v)\| \colon x \in M, v \in T_x^*M, \|v\| = 1\big\}$;
(3) and
$$C_{\langle h\rangle} = \liminf_{\substack{g\in\langle h\rangle\\ \ell(g)\to\infty}} \Big(\inf_{x\in\mathcal{F}} \frac{\mathrm{dist}(x,gx)}{\ell(g)}\Big).$$

Note that if $\langle h\rangle$ has subexponential growth, then any spectral gap of $\widetilde{D}$ at zero is automatically sufficiently large, hence the following immediate corollary.

Corollary 9 ([2])**.** If $\langle h\rangle \neq \langle 1\rangle$ has subexponential growth and $\widetilde{D}$ has a spectral gap at zero, then the pairing
$$\langle \mathrm{Tch}(\rho(\widetilde{D})), \mathrm{tr}_{\langle h\rangle}\rangle$$
absolutely converges.

Now for higher degree cyclic cocycles, we have the following analogue.

Theorem 10 ([2])**.** *Given $\langle h\rangle \neq \langle 1\rangle$, if $\varphi \in HP^e(\mathbb{C}\Gamma)_{\langle h\rangle}$ is a cyclic cocycle of at most exponential growth and the spectral gap of $\widetilde{D}$ at zero is sufficiently large, then the pairing*
$$\langle \mathrm{Tch}(\rho(\widetilde{D})), \varphi\rangle$$
absolutely converges.

Here we say the spectral gap of $\widetilde{D}$ is sufficiently large if it is larger than the following constant:
$$\sigma_\varphi =\colon \frac{2(K_G + K_\varphi)\cdot c_D}{C_\Gamma},$$
where

(1) K_Γ is the exponential growth rate of the group Γ and K_φ is the exponential growth rate of the cyclic cocycle φ;
(2) $c_D = \sup\big\{\|\sigma_D(x,v)\| \colon x \in M, v \in T_x^*M, \|v\| = 1\big\}$;
(3) and
$$C_\Gamma = \liminf_{\ell(g)\to\infty} \Big(\inf_{x\in\mathcal{F}} \frac{\mathrm{dist}(x,gx)}{\ell(g)}\Big).$$

We would like to point out that Piazza, Schick, and Zenobi gave alternative proofs for some of the results mentioned in this report, cf. [7].

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Higher ϱ-numbers and delocalized cocycles

Vito Felice Zenobi
(joint work with Paolo Piazza, Thomas Schick)

1. Long exact sequences in K-theory

Let M be a smooth compact manifold with fundamental group Γ. Let us denote by $\widetilde{M}$ the universal covering of M. Then we can associate to it the following long exact sequence of groups, called the *analytic surgery exact sequence* of Higson and Roe,

$$\cdots \longrightarrow K_{*+1}(C_r^*\Gamma) \longrightarrow \mathrm{S}_*^\Gamma(\widetilde{M}) \longrightarrow K_*^\Gamma(\widetilde{M}) \longrightarrow K_*(C_r^*\Gamma) \longrightarrow \cdots$$

where $K_*(C_r^*\Gamma)$ is the K-theory of the reduced group C*-algebra, $K_*^\Gamma(\widetilde{M})$ is the Γ-equivariant K-homology of $\widetilde{M}$ and $\mathrm{S}_*^\Gamma(\widetilde{M})$ is the so-called Γ-equivariant *structure group* of $\widetilde{M}$.

Among all the realizations of the Higson-Roe exact sequence we will consider the one obtained as the long exact sequence in K-theory induced by the following short exact sequence of C*-algebras

$$0 \to C_r^*(\widetilde{M} \times_\Gamma \widetilde{M}) \otimes C_0(0,1) \to \mathcal{C}(C(M) \to \Psi_\Gamma^0(\widetilde{M})) \to \mathcal{C}(C(M) \to C(S^*M)) \to 0$$

where $\Psi_\Gamma^0(\widetilde{M})$ and $C_r^*(\widetilde{M} \times_\Gamma \widetilde{M})$ are the C*-closure of the 0-order and the smoothing Γ-equivariant pseudodifferential operators on $\widetilde{M}$, respectively; $C(S^*M)$ is the algebra of continuous functions on the cosphere bundle of M; finally $\mathcal{C}$ denotes the mapping cone C*-algebra. See [5] for the details.

2. Chern characters to non-commutative de Rham homology

Let us now consider a dense and holomorphically closed Frèchet subalgebra $\mathcal{A}\Gamma$ of $C_r^*\Gamma$. Let $\widehat{\Omega}_*(\mathcal{A}\Gamma)$ be the universal differential graded algebra of $\mathcal{A}\Gamma$. Moreover denote by $\widehat{\Omega}_*^e(\mathcal{A}\Gamma)$ the localized part of $\widehat{\Omega}_*(\mathcal{A}\Gamma)$, namely the subcomplex generated by those forms $g_0dg_1\dots dg_n$ such that $g_0g_1\dots g_n = e$. Then the short exact sequence of complexes

$$0 \longrightarrow \widehat{\Omega}_*^e(\mathcal{A}\Gamma) \longrightarrow \widehat{\Omega}_*(\mathcal{A}\Gamma) \longrightarrow \widehat{\Omega}_*^{del}(\mathcal{A}\Gamma) \longrightarrow 0$$

induces the following long exact sequence of homology groups

$$\cdots \longrightarrow H_*^e(\mathcal{A}\Gamma) \longrightarrow H_*(\mathcal{A}\Gamma) \longrightarrow H_*^{del}(\mathcal{A}\Gamma) \longrightarrow \cdots$$

By using relative Chern characters in non-commutative de Rham homology, we are able to prove the following theorem.

Theorem A. *There exists the following commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & K_{*-1}(C_r^*\Gamma) \arrow[r] \arrow[d, "Ch_\Gamma"] & \mathrm{S}_*^\Gamma(\widetilde{M}) \arrow[r] \arrow[d, "Ch_\Gamma^{del}"] & K_*^\Gamma(\widetilde{M}) \arrow[r] \arrow[d, "Ch_\Gamma^e"] & \cdots \\
\cdots \arrow[r] & H_{[*-1]}(\mathcal{A}\Gamma) \arrow[r] & H_{[*-1]}^{del}(\mathcal{A}\Gamma) \arrow[r, "\delta"] & H_{[*]}^e(\mathcal{A}\Gamma) \arrow[r] & \cdots
\end{tikzcd}$$

where $H_{[*]}$ denotes the direct sum $\bigoplus_{n\in\mathbb{N}} H_{2n+*}$ for $* = 0, 1$.

3. Pairings with delocalized cocycles

Thanks to the Burghelea's Theorem we have that the cyclic cohomology group of the group ring $\mathbb{C}\Gamma$ is given by the following product

$$HC^*(\mathbb{C}\Gamma) \cong \prod_{\langle x\rangle\in\langle\Gamma\rangle} HC^*(\mathbb{C}\Gamma;\langle x\rangle)$$

where $\langle\Gamma\rangle$ is the set of the conjugancy classes of Γ. Observe that $H_{[*]}^{del}(\mathcal{A}\Gamma)$ is a subgroup of the cyclic homology $HC_{[*]}(\mathcal{A}\Gamma)$. Therefore, thanks to a key result of Puschnigg (see [4]), we can prove the following theorem.

Theorem B. *If Γ is Gromov-hyperbolic, then the following pairing*

$$\mathrm{S}_*^\Gamma(\widetilde{M}) \times HC^k(\mathbb{C}\Gamma;\langle x\rangle) \to \mathbb{C},$$

given by $(\varrho,\tau) \mapsto \langle Ch_\Gamma^{del}(\varrho),\tau\rangle$, is well-defined for all $\langle x\rangle \neq \langle e\rangle$ in $\langle\Gamma\rangle$.

If $u\colon M \to B\Gamma$ is the classifying map of the universal covering $\widetilde{M}$, let us consider the singular cohomology relative group $H^*(M \xrightarrow{u} B\Gamma)$. After realizing this cohomology group in terms of delocalized cocycles à la Alexander-Spanier, we obtain the following result.

Theorem C. *1) There exists a morphism of relative cohomology groups*

$$\chi\colon H^*(M \xrightarrow{u} B\Gamma) \to HC^*(\Psi^0_{\Gamma,c}(\widetilde{M}), C^\infty(M))$$

where $\Psi^0_{\Gamma,c}(\widetilde{M})$ is the algebra of Γ-compactly supported equivariant ΨDOs on $\widetilde{M}$ and $C^\infty(M)$ is the subalgebra of smooth functions on M.

2) If Γ is Gromov-hyperbolic, then the following pairing

$$\mathrm{S}^\Gamma_*(\widetilde{M}) \times H^*(M \xrightarrow{u} B\Gamma) \to \mathbb{C},$$

given by $(\varrho, \lambda) \mapsto \langle \varrho, \chi_\lambda \rangle$, is well-defined.

Recall that so far we used the realization of $\mathrm{S}^\Gamma_*(\widetilde{M})$ as K-group of the mapping cone associated to the pair $(\Psi^0_\Gamma(\widetilde{M}), C(M))$.

4. Geometric applications

Let g be a Riemannian metric on M with positive scalar curvature. In [2] Piazza and Schick associated to g an element $\varrho(g) \in \mathrm{S}^\Gamma_*(\widetilde{M})$. This K-theory element is well-defined on concordance classes of metrics with positive scalar curvature (shortly *psc*). Then the previous pairings allows to associate *higher ϱ-numbers* to psc metrics. This numbers will be used to study the moduli space of concordance classes of psc metric, namely the quotient of the set of concordance classes under the action of Diffeo(M), the diffeomorphisms group of M.

Some of the results of this work where independently treated in [1].

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Cyclic homology and group actions on manifolds

Raphaël Ponge

There is a huge amount of work on the cyclic homology of crossed-product algebras (see, e.g., [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 18, 19, 20, 21, 28]). However, at the notable exception of the characteristic map of Connes [9, 10] from the early 80s, we don't have explicit chain maps that produce isomorphisms at the level of homology and provide us with geometric constructions of cyclic cycles in the case of group actions on manifolds or varieties.

This talk reported on the construction of explicit quasi-isomorphisms for crossed products associated with actions of discrete groups [22, 23, 24, 27]. In the algebraic setting the results hold for unital algebras over rings containing $\mathbb{Q}$. They extend *mutatis mutandis* to continuous actions on locally convex algebras. Along the way we recover and clarify various earlier results (in the sense that we obtain explicit chain maps that yield quasi-isomorphisms). In particular, we recover the spectral sequences of Feigin-Tsygan [13] and Getzler-Jones [14], and derive an additional spectral sequence. In the case of the localization at infinite order conjugacy classes we obtain a module structure over group homology. This result goes back to Nistor [20], but we are able to implement the action at the chain level by means of an explicit coproduct for the paracyclic category (i.e., by using a biparayclic version of the Alexander-Whitney map).

In the case of group actions on manifolds we have an explicit description of cyclic homology and periodic cyclic homology. In the finite order case, the results are expressed in terms of what the so-called 'mixed equivariant homology', which interpolates group homology and de Rham cohomology. This is actually the natural receptacle for a cap product of group homology with equivariant cohomology. As a result taking cap products of group cycles with equivariant characteristic classes naturally gives to a geometric construction of cyclic cycles. For the periodic cyclic homology we recover earlier results of Connes [9, 10] and Brylinski-Nistor [8] via a Poincaré duality argument. For the non-periodic cyclic homology the results seem to be new. In the infinite order case, we fix and simplify the misidentification of cyclic homology by Crainic [11]. In the case of finite group actions we also recover earlier results of Baum-Connes [2] for proper actions. In the case of group actions on smooth varieties we obtain the exact analogues of the results for group actions on manifolds. In particular, in the special case of finite group actions on smooth varieties we recover recent results of Brodzki-Dave-Nistor [6] via the construction of explicit quasi-isomorphism. (Actions of infinite groups are not dealt with in [6].)

The approach consists in several intermediate steps that are put together. Roughly speaking we devise a simple machinery that produces quasi-isomorphisms for the localizations at finite order and infinite order conjugacy classes once we have quasi-isomorphisms for twisted cyclic homology of the sole algebra. Combining this with twisted versions of the Hochshild-Kostant-Rosenberg (due to Brylinski-Nistor [8] and Brodzki-Dave-Nistor [6]) leads us to the aforementioned results for group actions on manifolds and smooth varieties.

Several of the ideas were contained in earlier works. However, an important issue is to find the right homological setting to work with. In the case of localization at finite order cases the setting of the cylindrical complexes of Getzler-Jones [14] is especially relevant. However, we need to go beyond cylindrical complexes to deal with the infinite order case. To this end we introduce the notions of para-S-module and triangular S-module. The former is a "para" version of the notion of an S-module introduced by Jones-Kassel [15]. The latter is some kind of bivariant combination of the former and parachain complexes. It allows us to consider tensor products of para-S-modules and parachain complexes; this precisely what we need in order to deal with the infinite order case.

The main difference between a para-S-module and an S-module is the removal of the condition $d \circ d = 0$ for the differential. As a result we don't obtain chain complexes in the usual sense (although we may get a chain complexe by taking a suitable tensor product). This does not allow us to speak about quasi-isomorphisms. Nevertheless, we still can speak about chain homotopy equivalences. As a result this forces us to derive chain homotopy equivalences, and so the quasi-isomorphisms that we get actually are chain homotopy equivalences. A very convenient tool for constructing chain homotopy equivalences is provided by a suitable generalization of Brown's perturbation lemma to para-S-modules [25]. This can be seen as a para-version of the results of Kassel [16]. As an application, building on earlier work of Bauval [3] and Khalkhali-Rangipour [17], we obtain a version of Eilenberg-Zilber Theorem for bi-paracyclic modules [26]. This is stated in terms of chain homotopy equivalences at the level of unnormalized chains, and so we actually get a bi-paracyclic version of Dold-Puppe Theorem. In particular, this leads us to a natural notion of cup product for paracyclic comodules.

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Rigidity and characteristic classes for cyclic homology over *p*-adic numbers

Boris Tsygan

In the 80s Goodwillie [1] proved two theorems about cyclic homology over rational numbers $\mathbb{Q}$. First, periodic cyclic homology is rigid, i. e., it does not change if one factorises our algebra by a nilpotent ideal. Second, relative K theory of a nilpotent ideal is isomorphic to relative cyclic homology. Closely related to the first statement is Getzler's construction of a Gauss-Manin connection on the periodic cyclic complex of a family of algebras. (See [2])

The first theorem is obviously false over integers (consider for example an algebra with zero multiplication). What it implies, though, is that, over the rationals, any two algebra structures on the same space that are the same modulo an ideal with a certain divided power property have isomorphic (completed) periodic cyclic complexes. We also construct a regulator map from algebraic K theory of such an ideal to (completed) cyclic homology (a version of a recent work of Beilinson [3]), as well as a version of Getzler's Gauss Manin connection.

In particular, if an algebra over $\mathbb{Z}/p$ ($p > 2$) admits a lifting to an algebra over p-adic numbers, then the completed periodic cyclic complex of the lifting is independent of the choice of a lifting. Furthermore, this complex can be defined even if a lifting does not exist. This construction is closely related to a recent work of Petrov, Vaintrob, and Vologodsky [4].

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