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Real Analysis, Harmonic Analysis and Applications (hybrid meeting)

Organized by
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Abstract. The workshop hosted various research groups on topics in Real Analysis, Harmonic Analysis and Applications.

Mathematics Subject Classification (2020): 42xx.

Introduction by the Organizers

This workshop "Real Analysis, Harmonic Analyis and Applications" was planned in continuation of the triennial series at Oberwolfach that started in 1986. Because of the Covid 19 pandemic, it could not be realized in the usual format.

Instead, participants were invited to, if able to do the travel, propose small groups in the spirit of an extended "Research in Pairs" program to meet at Oberwolfach during the workshop week, advance some project, and hold informal discussions with the other groups. None of the vast majority of original participants from outside Germany were able to come. Four of the original senior participants from Germany proposed groups. Together with mostly postdocs and PhD students from their institutions, a group of 14 in-person and a small number of virtual participants was formed.

Each research group reports in the following abstracts on their work. In addition to this research work and informal discussions, each afternoon except Wednesday an online talk was given. The speakers were Alessio Martini, Alexander Volberg, David Beltran, and Joris Roos, with titles and abstracts also listed below.

The in-person participants, and in particular the younger researchers, benefited greatly from the event. All expressed gratitude that Oberwolfach had suggested to experiment with such alternative format and accommodate this ad hoc meeting. The in-person interaction was very much welcomed during the pandemic and a boost for everyone's research agenda. We thank the administration and staff of the *Mathematisches Forschungsinstitut Oberwolfach* for creating a very inspriring atmosphere in difficult times.

Workshop (hybrid meeting): Real Analysis, Harmonic Analysis and Applications

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Abstracts

Multilinear UMD properties for tuples of Banach spaces

Alex Amenta
(joint work with Zoe Nieraeth, Gennady Uraltsev)

During this week we sketched a long-term roadmap for a research program on multilinear UMD (Unconditionality of Martingale Differences) properties for tuples of Banach spaces. In the linear case, the UMD property of a Banach space is well-understood, with many equivalent characterisations coming from probability and harmonic analysis in particular. The UMD property of a Banach space X is generally necessary for the harmonic analysis of linear operators on X-valued functions, most notably since it is equivalent to the L^p-boundedness of the Hilbert transform on such functions. In the multilinear setting, one deals with a tuple of Banach spaces $(X_i)_{i=1}^M$ connected via an M-linear form $\Pi\colon \prod_{i=1}^M \to \mathbb{C}$. From this data it is possible to define X_i-valued extensions of M-linear operators which are initially defined on scalar-valued functions. For some natural multilinear operators, including the bilinear Hilbert transform, it is known that the UMD property of the individual spaces X_i is not necessary for the L^p-boundedness of vector-valued extensions; what is more important is how the spaces interact with each other through the form Π. One goal of our program is to characterise abstractly what properties of the form Π could be said to constitute a 'multilinear UMD property' (or perhaps a class of multilinear UMD properties). During our week at Oberwolfach we discussed candidate properties and implications between them, and formulated a number of conjectures which we are currently working on.

The circular maximal operator on Heisenberg radial functions

David Beltran
(joint work with Shaoming Guo, Jonathan Hickman, Andreas Seeger)

An analogue of the Stein spherical maximal function was introduced on the Heisenberg group $\mathbb{H}^n$ by Nevo and Thangavelu in 1997. Whilst the sharp L^p estimates for this object are known for $n \geq 2$ (independently obtained by Müller and Seeger and Narayanan and Thangavelu), the case of $\mathbb{H}^1$ remains open and it is currently unknown if the circular maximal function is bounded on $L^p(\mathbb{H}^1)$ for any finite p.

In this talk, we present sharp L^p estimates for this object when restricted to a class of Heisenberg radial functions. Under this assumption, the problem reduces to studying a variable-coefficient version of Bourgain's circular maximal operator on the Euclidean plane which presents a number of interesting singularities: it is associated to a non-smooth curve distribution and, furthermore, fails both the usual rotational curvature and cinematic curvature conditions.

Neumann problem for the Laplace equation with general nonlinearity

Gael Diebou Yomgne
(joint work with Lenka Slavíková)

The discussion on this project was initiated at Oberwolfach when the two authors attended the workshop "Real and Harmonic Analysis". The initial motivation for our work was related to the solvability and related questions for the Laplace equation in the positive half-space subject to an exponential nonlinearity at the boundary. This problem has direct applications in geometry (Yamabe problem with prescribed Gaussian curvature), in the modeling of corrosion of materials, etc. The available methods employed to approach such a problem require the assumption $n = 2$, a condition which emanates from the use of the Moser-Trudinger trace principle. Our idea is to remove this restriction on the dimension by introducing new techniques based on fixed points arguments and designing a new functional setting which accounts for the lack of a trace theorem. We succeeded to do so after several discussions and along the way, a new collaborator, Andrea Cianchi (University of Florence, Italy) joined the project with the suggestion of generalizing our ideas to nonlinearities with general growth properties. This is still a work in progress.

The abstract theory of L^p spaces for outer measures

Marco Fraccaroli

The theory of L^p spaces for outer measures, or outer L^p spaces, was developed in [1] in the context of time-frequency analysis. It provides a framework to encode the boundedness of multilinear operators satisfying a certain group of symmetries in a two-step programme. The programme consists of a version of Hölder's inequality for outer L^p spaces and embedding theorems between classical and outer L^p spaces via wave packet decompositions.

The outer L^p spaces can be thought of as a generalization of the product of classical L^p spaces. Given two finite sets with strictly positive measures (X, ω_X), (Y, ω_Y), and two exponents $p, q \in [1, \infty)$, we define the norm of a function f on the Cartesian product $X \times Y$ by

$$\begin{aligned}
\|f\|_{L^\infty(X,\omega_X;L^q(Y,\omega_Y))} &= \sup_{x\in X}(\sum_{y\in Y}\omega_Y(y)|f(x,y)|^q)^{\frac{1}{q}} \\
&= \sup_{x\in X}\omega_X(x)^{-\frac{1}{q}}(\sum_{y\in Y}\omega(x,y)|f(x,y)|^q)^{\frac{1}{q}}, \\
\|f\|_{L^p(X,\omega_X;L^q(Y,\omega_Y))} &= \|\|f(x,\cdot)\|_{L^q(Y,\omega_Y)}\|_{L^p(X,\omega_X)} \\
&= (\sum_{x\in X}\omega_X(x)(\sum_{y\in Y}\omega_Y(y)|f(x,y)|^q)^{\frac{p}{q}})^{\frac{1}{p}},
\end{aligned}$$

where $\omega = \omega_X \otimes \omega_Y$ is the induced measure on $X \times Y$, with the obvious extensions for $q = \infty$. We observe that the measure ω_X on X induces an outer measure μ on

$X \times Y$, namely a monotone subadditive function on every subset of X attaining the value zero on the empty set. In particular, for every $A \subseteq X \times Y$, we define

$$\mu(A) = \omega_X(\pi_X(A)),$$

where π_X is the projection on the first component.

The following quantities reproduce the norms defined in the previous display, namely

$$\|f\|_{L^\infty_\mu(\ell^q_\omega)} = \sup_{B \subseteq X \times Y} \mu(B)^{-\frac{1}{q}} \|f 1_B\|_{L^q(X \times Y, \omega)},$$
$$\|f\|_{L^p_\mu(\ell^q_\omega)} = (\int_0^\infty p \lambda^{p-1} \inf\{\mu(A) \colon A \subseteq X \times Y, \|f 1_{A^c}\|_{L^\infty_\mu(\ell^q_\omega)} \leq \lambda\} \,\mathrm{d}\lambda)^{\frac{1}{p}},$$

with the obvious extensions for $q = \infty$. As a matter of fact, under reasonable assumptions, we can extend these definitions to the case of an arbitrary set Z together with an outer measure μ and a measure ω.

These quantities define the outer L^p spaces. It is easy to prove that they are quasi-norms, although in the particular Cartesian product setting they are actually norms. In [2], we addressed the equivalence of the outer L^p quasi-norms to norms in the general setting recalled above. In particular, we proved a quasi-triangle inequality independent of the number of summands and uniform in the finite setting (Z, μ, ω) in the range of exponents $p, q \in (1, \infty)$.

During the stay at MFO for the workshop, through fruitful discussion with Christoph Thiele, I began the investigation of the equivalence of outer L^p quasi-norms to norms in the case of iterated outer L^p spaces. These are the spaces defined recursively on a set Z together with a measure ω, and n outer measures $\mu_1, \dots, \mu_n$ in analogy with the n-fold product of classical L^p spaces. Already in the case of two outer measures μ, ν, the question becomes more difficult, due to the interplay of the subadditivities of the two outer measures and the role of the exponents p, q, r. In upcoming work, we provide conditions on μ, ν under which a quasi-triangle inequality independent of the number of summands and uniform in the finite setting (Z, μ, ν, ω) can hold in the range of exponents $p, q, r \in (1, \infty)$. Moreover, we exhibit counterexamples to the uniformity in the finite setting (Z, μ, ν, ω) without additional conditions at least in a certain range of exponents $p, q, r \in (1, \infty)$.

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Perturbed Hardy spaces for Fourier integral operators

Dorothee Frey
(joint work with Yonas Mesfun)

Hardy spaces for Fourier integral operators were initially considered by Smith and developed further only recently by Hassell, Portal and Rozendaal [2, 3]. Those spaces can serve as an effective tool in the study of hyperbolic equations such as the wave equation $\partial_t^2 u = Lu$ in L^p, where L is a uniformly elliptic operator with fairly rough coefficients. Compared to L^p spaces, the spaces have the major advantage that they are invariant under a large class of Fourier integral operators. The recent work [1] gives a new approach to fixed-time L^p estimates for wave equations based on Hardy spaces. For fixed $t \in \mathbb{R}$, it yields boundedness of the wave propagator $e^{it\sqrt{L}}$ from $W^{s(p),p}$ to L^p with some sharp parameter $s(p)$. However, in [1], the operator L is assumed to have a specific algebraic structure. In our ongoing work, we aim to relax the structural assumptions on L such that the optimal fixed time L^p estimate for the wave propagator still holds. We have obtained first perturbation results in this direction, with more general results being work in progress.

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Maximal excess charge of atoms, regularity theory for fractional Laplacians

Dirk Hundertmark
(joint work with Nikolaos Pattakos and Marvin Raimund Schulz)

According to an ancient Chinese curse “May you live in interesting times”, we certainly do live in one.

We had just started the first 4 months of living in the global Covid-19-Pandemic, when we attended the workshop at Oberwolfach in June 2020. The week at Oberwolfach gave us the serenity to be able to focus on research, due to the peaceful and intellectually highly stimulating atmosphere.

We were lucky to be able to attend in person, thanks to the efforts of the organizers and, in particular, the staff at Oberwolfach. Since attendees from abroad weren’t able to travel, their talks were broadcasted live. This worked out surprisingly well: discussions during and after the talks did not differ much from regular talks.

Besides the talks, the possibility of being at Oberwolfach during June 2020 also had a deep impact on our own research: The workshop offered plenty of time and, as usual in Oberwolfach, an exciting environment for research. We used part of this time to focus on an ongoing project, which at the time still was in its infancy: The open question of the maximal excess charge of atoms, see for example [1, Problem 10D], [2], [3] and [4]. We had an idea on how one could make progress in this longstanding problem at the foundations of quantum mechanics, but we had not made much progress before coming to Oberwolfach. During the week at Oberwolfach this changed. Based on the progress we made at Oberwolfach we drastically improved and generalized the method in [3]. In addition, we developed ideas for an alternative proof of the asymptotic neutrality of heavy ions in Oberwolfach. Our proof is not based on compactness arguments but direct and constructive.

Asymptotic neutrality of large atoms was first proven in [5] using a highly non–constructive contradiction argument based on compactness and Choquet's Theorem. In addition, our proof is not only constructive, but, in contrast to [5], we can also include magnetic fields, as long as they are not too strong. We are writing up the manuscript at the moment.

In addition to the work sketched above, we also started another line of research at Oberwolfach: Regularity theory for fractional Laplacians. So far, this has been dealt with with the approach of Caffarelli and Silvestre, mapping the problem onto one for the usual Laplacian, via an harmonic extension to upper half spaces. We want to avoid this extension and aims on directly ealing with fractional Laplacians, which are highly non local beasts. As a first step we were able to develop a direct approach to sub- and super-solution estimates for fractional Laplacians in Oberwolfach, which avoids the Caffarelli–Silvestre method and allows us to easily include semi-relativistic operators of the form $(P^2 + c^4m^2)^{1/2} - c^2m$, where c is the speed of light and m is the mass of the particle. This type of operators are relevant for quantum mechanics.

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A Fourier integral operator approach to the sub-Riemannian wave equation

Alessio Martini
(joint work with Detlef Müller, Sebastiano Nicolussi Golo)

Let $\mathcal{L}$ be a sub-Laplacian on a sub-Riemannian manifold of dimension n. We show in [3] that the ranges of validity of spectral multiplier estimates of Mihlin–Hörmander type and wave propagator estimates of Miyachi–Peral type for $\mathcal{L}$ cannot be wider than the corresponding ranges for the Laplace operator on $\mathbb{R}^n$ — despite the possible lack of ellipticity of $\mathcal{L}$.

More specifically, let $\chi \in C_c^\infty((0,\infty))$ be any nontrivial cutoff, and denote by $L^q_s(\mathbb{R})$ the L^q Sobolev space on $\mathbb{R}$ of fractional order s. Then we prove that, if, for some $s \geq 0$ and $p \in (1,\infty)$, the estimate

$$\|F(\mathcal{L})\|_{p\to p} \lesssim \sup_{t\geq 0} \|F(t\cdot)\chi\|_{L^\infty_s}$$

holds for all smooth $F : \mathbb{R} \to \mathbb{C}$, then $s \geq n|1/p - 1/2|$. Moreover, we prove that, if, for some $\alpha \geq 0$ and $p \in [1,\infty]$, the estimate

$$\|\chi(t\sqrt{\mathcal{L}}/\lambda)\cos(t\sqrt{\mathcal{L}})\|_{p\to p} \lesssim \lambda^\alpha$$

holds for all small t and large λ, then $\alpha \geq (n-1)|1/p - 1/2|$. In the case $\mathcal{L}$ is elliptic, these results could be easily obtained by transplantation [2], since the "local model" for $\mathcal{L}$ (obtained by freezing the coefficients at some point and considering the principal part) is just the Euclidean Laplacian. The novelty of the results in [3] is that they also apply to non-elliptic sub-Laplacians $\mathcal{L}$ on sub-Riemannian manifolds of arbitrary step, where ellipticity is replaced by a form of Hörmander's bracket generating condition.

The proof hinges on a Fourier integral representation for the wave propagator associated with $\mathcal{L}$ and subtle nondegeneracy properties of the sub-Riemannian geodesic flow [1].

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Instability Mechanisms in Inverse Problems

Angkana Rüland
(joint work with Herbert Koch, Mikko Salo)

The question of stability is a central aspect of the study of inverse problems, in particular, since these problems are often notoriously ill-posed. While instability mechanisms had been introduced for the Calderon problem in seminal work due to Mandache [1] which had later been generalized by Di Cristo-Rondi [2], these mechanisms only applied to specific geometric set-ups with high symmetry. In a joint long-term project [3] with Herbert Koch and Mikko Salo (U. Jyväskylä), we thus investigated robust mechanisms for quantitative ill-posedness in inverse problems, relying on (analytic) smoothing, iterative low but global smoothing and microlocal smoothing of the forward map. During the stay in Oberwolfach, Herbert Koch and I discussed methods of formulating our microlocal setting and the associated results more elegantly by Weyl type asymptotics for microlocalization operators.

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A triangular Hilbert transform with curvature

Joris Roos
(joint work with Polona Durcik, Michael Christ)

This talk is about a recent joint work with Michael Christ and Polona Durcik on a variant of the triangular Hilbert transform involving curvature. The proof relies on a key trilinear smoothing inequality, and on bounds for an anisotropic variant of the twisted paraproduct. Our results unify various previously known results such as bounds for a bilinear Hilbert transform with curvature and a maximally modulated singular integral of Stein-Wainger type, and Bourgain's non-linear Roth theorem in the reals.

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Bilinear multipliers for convex sets

Olli Saari
(joint work with Christoph Thiele)

Our project focuses on bilinear Fourier multipliers. Given a bounded measurable function m in the plane, we define the associated bilinear Fourier multiplier operator acting on a pair of one-dimensional Schwartz functions (f, g) by

$$B_m(f,g)(x) := \iint_{\mathbb{R}^2} m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i(\xi+\eta)x}\, d\xi d\eta.$$

We are interested in bounds $L^{p_1} \times L^{p_2} \to L^{p_3}$ where (p_1, p_2, p_3) is a Hölder triple (in local L^2 range), and we specialize to the case where m is an indicator function.

When m is a characteristic function of a half-plane, we encounter the well-known family of bilinear Hilbert transforms. The state of the art covers their theory and some other sets such as the disc and certain suitably tame bounded sets with smooth boundary. Our program is to understand the case of a general convex set. The paradigmatic approach to the problem consists in three subproblems: the paraproduct bound, the tree selection algorithm and a synthesis part. During the hybrid workshop, we advanced our understanding of the key phenomena. We have formulated and proved partial results for the paraproduct part of the program, [1].

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A solution of a problem of Enflo

Alexander Volberg
(joint work with Paata Ivanishvili, Ramon van Handel)

A nonlinear analogue of the Rademacher type of a Banach space was introduced in classical work of Enflo. The key feature of Enflo type is that its definition uses only the metric structure of the Banach space, while the definition of Rademacher type relies on its linear structure. We prove that Rademacher type and Enflo type coincide, settling a long-standing open problem in Banach space theory. The proof is based on a novel dimension-free analogue of Pisier's inequality on the discrete cube, which, in its turn, is based on a certain formula that we used before in improving the constants in scalar Poincaré inequality on Hamming cube. I will also show several extensions of Pisier's inequality (originally considered by Hytönen and Naor) with ultimate assumptions on a Banach space structure. In the second part of the presentation I will talk about singular integrals on Hamming cube and about two approaches: Lust-Piquard's approach via non-commutative random variables and another approach using Bellman function technique.

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Reporter: Christoph Thiele

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