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Arithmetic Geometry (hybrid meeting)

Organized by
Gerd Faltings, Bonn
Johan de Jong, New York
Peter Scholze, Bonn

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Abstract. Arithmetic geometry is at the interface between algebraic geometry and number theory, and studies schemes over the ring of integers of number fields, or their p-adic completions, and connects with representation theory, automorphic forms, Hodge theory, algebraic topology, and many other fields.

Mathematics Subject Classification (2020): 11G99.

Introduction by the Organizers

In the middle of a pandemic, the workshop *Arithmetic geometry* was attended by about half of the original invitees, a little more than 25 participants, coming from various parts of Europe. Everybody was very glad for the chance to meet and discuss mathematics. To enable remote participation via video from participants overseas, we moved all talks (except on Friday) to the afternoon. The participation via zoom turned out to work very well, but it was difficult to recreate the immersive Oberwolfach experience via video.

The talks covered a wide spectrum of topics, related to automorphic forms, Shimura varieties, Langlands correspondences, complex and p-adic Hodge theory, rational points, to foundational problems on étale cohomology and reciprocity laws encoded in sphere spectra.

In the Langlands program, concerning the relation between automorphic forms and Galois representations, it is critical to understand the cohomology of Shimura varieties. Ideally, one would like to know that all of the interesting cohomology is concentrated in the expected range of degrees; often just the middle degree. Boxer and Pilloni reported on joint work concerning the ordinary cohomology (with $\mathbb{Z}_p$-coefficients) or the cohomology of small slope (with $\mathbb{Q}_p$-coefficients), proving such results. Their proof uses the Hodge-Tate period map towards the flag variety, and is partly inspired by Kempf's proof of the classical Borel-Weil-Bott theorems on the cohomology of flag varieties.

For the study of Shimura varieties at p-adic places, one needs to find integral models. It is still an open problem to find a good characterization of those in general, but Fargues gave a related characterization of integral models of local Shimura varieties. Moreover, one wants to understand the singularities of these integral models, which leads to the study of local models of Shimura varieties. These have now been analyzed in fine detail, and in particular Richarz explained results on Cohen-Macaulayness, while Lourenço observed that in some cases local models can fail to be normal.

Concerning the local Langlands correspondence, Fargues has recently suggested a geometrization in terms of the stack Bun_G of G-bundles on the Fargues-Fontaine curve. Le Bras reported on a proof of much of Fargues' conjecture for GL_n, while Viehmann determined the topological space $|\mathrm{Bun}_G|$. Hellmann explained a closely related categorical form of the local Langlands conjecture, in terms of a fully faithful functor from the category of smooth representations of $G(\mathbb{Q}_p)$ towards coherent sheaves on the stack of L-parameters.

Of much interest is also the study of the p-adic local Langlands correspondence, which studies representations of $G(\mathbb{Q}_p)$ on p-adic vector spaces, and is important for proving automorphy (lifting) theorems following Taylor-Wiles and Breuil-Conrad-Diamond-Taylor. While the precise form this local correspondence should take remains mysterious beyond $\mathrm{GL}_2(\mathbb{Q}_p)$, one can analyze the problem via looking at global instances. New results were announced by Paškūnas, who showed in very large generality that Hodge-Tate-Sen weights match infinitesimal characters under this correspondence, and Schraën, who was able to determine the "size" of the representations, as measured by the Gelfand-Kirillov dimension, in a first interesting case beyond $\mathrm{GL}_2(\mathbb{Q}_p)$.

Like in the usual local Langlands correspondence, one expects the p-adic correspondence to be realized in the cohomology of local Shimura varieties. Motivated by this problem, Colmez announced new p-adic comparison theorems for general smooth p-adic analytic spaces; these are quite subtle as the relevant cohomology groups are infinite-dimensional, and especially (pro-)étale cohomology can be enormous.

Classical diophantine equations about rational points on curves were adressed in the talks of Edixhoven on a geometric form of the quadratic Chabauty method that was recently used to analyze rational points on modular curves, and in the talk of Habegger that proved that the number of rational points can be bounded in terms of the rank of the Jacobian. Rational points on elliptic curves, in relation to the Birch-and-Swinnerton-Dyer conjecture and p-adic variants, were discussed by Bertolini.

Česnavičius discussed variants of many classical theorems in étale cohomology in the setting of cohomology of finite flat group schemes. In particular, this includes a general continuity formula, and a statement on invariance under henselian pairs, yielding variants of Gabber's affine analogue of proper base change in étale cohomology. Gabber also generalized results in étale cohomology to new classes of algebras, related to rigid-analytic geometry. This includes new affine Lefschetz theorems, in particular proving some conjectures of Bhatt-Mathew and Hansen. On the other hand, Esnault studied the hard Lefschetz theorem in étale cohomology, and in particular the question for which classes of sheaves it holds, raising the question whether it holds for all ℓ-adic sheaves. In characteristic 0, such results follow from the work of Simpson on harmonic structures. Esnault outlined an attack to this question in terms of special loci in the space of ℓ-adic local systems, and a proof in some cases.

A central problem in Hodge theory is of course the Hodge conjecture. Starting with Cattani-Deligne-Kaplan, one tangible direction here was to study the Hodge loci, i.e. the locus where a given cohomology class is a Hodge class. The Hodge conjecture predicts that this has strong algebraicity properties. Cattani-Deligne-Kaplan proved that it is indeed an algebraic subvariety. If the ambient situation is defined over a number field, one would moreover expect that the Hodge locus is defined over a (the same) number field. A proof of this in many cases has been announced by Klingler.

Finally, Clausen gave a talk on the algebraic theory of half-integral weight modular forms, in which he made a connection between the possibility of half-integral weights and the first stable homotopy groups of sphere $\pi_1\mathbb{S} = \mathbb{Z}/2\mathbb{Z}$, and the 24-th root η of the discriminant and $\pi_3\mathbb{S} = \mathbb{Z}/24\mathbb{Z}$, by proving a relation between invertible sheaves of $\mathbb{S}_\ell^\wedge$-modules attached to the Lie algebra of an elliptic curve, and to the adelic Tate module of an elliptic curve, which is a form of a reciprocity law.

Workshop (hybrid meeting): Arithmetic Geometry

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Abstracts

On the derived category of the Iwahori-Hecke algebra

Eugen Hellmann

1. Introduction

Let F be a finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p(\!(t)\!)$ with ring of integers $\mathcal{O}_F$, uniformizer ϖ and residue field k. We denote by q the cardinality of k and fix an (algebraically closed) field C of characteristic zero containing a fixed choice of a square root $q^{1/2}$ of q. Recall that the local Langlands correspondence for $G = \mathrm{GL}_n(F)$ is a bijection

$$\left\{\begin{matrix}\text{isomorphism classes}\\ \text{of irreducible smooth}\\ \text{representations } \pi \text{ of } \mathrm{GL}_n(F)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{isomorphism classes of } n\text{-dim}\\ \text{Frobenius-semi-simple Weil-}\\ \text{Deligne representations } (\rho, N) \text{ of } F\end{matrix}\right\}$$

where on both sides the representations are on C-vector spaces. The simplest special case of this correspondence is the following bijection

$$\left\{\begin{matrix}\text{isomorphism classes}\\ \text{of those } \pi \text{ such that } \pi^I \neq 0\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{conjugacy classes of}\\ (\varphi, N) \in \mathrm{GL}_n(C) \times \mathrm{Lie}\,\mathrm{GL}_n(C)\\ \text{with } \varphi \text{ semi-simple and } N\varphi = q\varphi N\end{matrix}\right\}.$$

Here $I \subset \mathrm{GL}_n(F)$ is the choice of an Iwahori subgroup, e.g. the subgroup of those elements in $\mathrm{GL}_n(\mathcal{O}_F)$ that are upper triangular modulo ϖ. The representations on the left hand side can also be be described as those irreducible representations that are subquotients of unramified principal series representations $\iota_B^G \delta$ for some unramified character δ of a maximal split torus T, where $B \subset G$ is a Borel subgroup containing T and ι_B^G denotes (normalized) parabolic induction. Moreover, there is a third description: the set on the left is in canonical bijection with the isomorphism classes of simple modules over the Iwahori-Hecke algebra $\mathcal{H}_G = \mathrm{End}_G(\text{c-ind}_I^G \mathbf{1}_I)$.

For irreducible representations that have fixed vectors under an Iwahori subgroup this classification was extended to split reductive groups by Kazhdan-Lusztig [4]: let $\mathbb{G}$ be a split reductive group over F and write $G = \mathbb{G}(F)$ for its F-valued points and $\check{G}$ for its dual group considered as an algebraic group over C. Again we choose an Iwahori subgroup I. Then there is a surjective map

$$\left\{\begin{matrix}\text{isomorphism classes of}\\ \text{irreducible smooth}\\ G\text{-representations}\\ \pi \text{ such that } \pi^I \neq 0\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{conjugacy classes of}\\ (\varphi, N) \in \check{G}(C) \times \mathrm{Lie}\,\check{G}(C)\\ \text{with } \varphi \text{ semi-simple}\\ \text{and } \mathrm{Ad}(\varphi)N = q^{-1}N\end{matrix}\right\}.$$

The fiber of this map over a pair (φ, N) is finite and can be parametrized by certain irreducible representations of the centralizer of the pair (φ, N).

The aim of this talk is to describe a conjectural extension [2] of this parametrization to the level of (derived) categories. Similar conjectures/results are also the subject of (ongoing) work of Ben-Zvi–Chen–Helm–Nadler and of X. Zhu.

2. Formulation of a conjecture

We continue to assume that $\mathbb{G}$ is a split reductive group over F. Moreover, we fix a choice of a maximal split torus and a Borel $T \subset B \subset G = \mathbb{G}(F)$, and write $\check{T}$ respectively $\check{B}$ for the dual groups. We denote by W the Weyl group of (G, T) respectively of $(\check{G}, \check{T})$.

We write $\operatorname{Rep} G$ for the category of smooth G-representations on C-vector spaces and $\operatorname{Rep}_{[T,1]} G$ for the Bernstein block of those G-representations all of whose irreducible subquotients are also subquotients of unramified principal series representations. The category $\operatorname{Rep}_{[T,1]} G$ is canonically equivalent to the category $\mathcal{H}_G$-mod of modules over the Iwahori-Hecke algebra, for some choice of an Iwahori subgroup $I \subset G$. These categories have a $\mathfrak{Z}_G$-linear structure, where

$$\mathfrak{Z}_G = C[X_*(T)]^W = C[X^*(\check{T})]^W = \Gamma(\check{T}/W, \mathcal{O}_{\check{T}/W})$$

denotes the center of the category $\operatorname{Rep}_{[T,1]} G$ which is canonically identified with the center of $\mathcal{H}_G$.

On the other hand we consider the C-scheme $X_{\check{G}}^{\mathrm{WD}}$ of all Weil-Deligne representations

$$(\rho : W_F \to \check{G}, N \in \operatorname{Lie} \check{G}).$$

Here W_F denotes the Weil group of F. This scheme contains

$$X_{\check{G}} = \{(\varphi, N) \in \check{G} \times \operatorname{Lie} \check{G} \mid \operatorname{Ad}(\varphi) N = q^{-1} N\}$$

as a connected component. Note that $\check{G}$ acts on $X_{\check{G}}$ and we can form the stack quotient $[X_{\check{G}}/\check{G}]$. Moreover, this construction makes sense for every linear algebraic group H instead of $\check{G}$. We point out that the canonical projection to the adjoint quotient

$$X_{\check{G}} \longrightarrow \check{G} \longrightarrow \check{T}/W = \operatorname{Spec} \mathfrak{Z}_G$$

makes $\operatorname{QCoh}([X_{\check{G}}/\check{G}])$ and the derived category $\mathbf{D}^+_{\mathrm{QCoh}}([X_{\check{G}}/\check{G}])$ into $\mathfrak{Z}_G$-linear categories.

Conjecture 1. *For every (G, B, T) as above (together with the choice of a Whittaker datum ψ) there exists a fully faithful $\mathfrak{Z}_G$-linear functor*

$$R_G^\psi : \mathbf{D}^+(\operatorname{Rep}_{[T,1]} G) \longrightarrow \mathbf{D}^+_{\mathrm{QCoh}}([X_{\check{G}}/\check{G}])$$

and for each parabolic subgroup $B \subset P \subset G$ with Levi quotient M there exists a natural $\mathfrak{Z}_G$-linear isomorphism

$$\xi_P^G : R_G^\psi \circ \iota_{\overline{P}}^G \xrightarrow{\cong} R\beta_{P,*} L\alpha_P^* \circ R_M^{\psi_M},$$

where ψ_M is a Whittaker datum for M induced by ψ, $\overline{P}$ is the opposite parabolic to P, and

$$\alpha_P : [X_{\check{P}}/\check{P}] \to [X_{\check{M}}/\check{M}] \text{ respectively}$$
$$\beta_P : [X_{\check{P}}/\check{P}] \to [X_{\check{G}}/\check{G}]$$

are the morphisms of stacks induced by the canonical morphisms $\check{P} \to \check{M}$ respectively $\check{P} \to \check{G}$. These data satisfy the following properties:

- *there are compatibilities among the ξ_P^G for the various parabolic subgroups containing B.*
- *If $G = T$ is a (split) torus and $T^\circ \subset T$ is the maximal compact subgroup, then R_T is induced by the canonical identification*

$$\mathrm{Rep}_{[T,1]}T \cong C[T/T^\circ]\text{-mod} = C[X_*(T)]\text{-mod} = C[X^*(\check{T})]\text{-mod} = \mathrm{QCoh}(\check{T}),$$

where $\check{T} = X_{\check{T}}$ is equipped with the trivial $\check{T}$-action.
- *Let $(\text{c-ind}_N^G\psi)_{[T,1]}$ denote the image of $\text{c-ind}_N^G\psi$ in $\mathrm{Rep}_{[T,1]}G$. Then*

$$R_G^\psi((\text{c-ind}_N^G\psi)_{[T,1]}) \cong \mathcal{O}_{[X_{\check{G}}/\check{G}]}.$$

Remark 2. (a) There is a variant of the conjecture replacing $\mathrm{Rep}_{[T,1]}G$ by $\mathrm{Rep}\, G$ and $X_{\check{G}}$ by $X_{\check{G}}^{\mathrm{WD}}$.

(b) In the conjecture we are forced to consider derived categories instead of abelian categories, as we compare structures flat over $\mathfrak{Z}_G$ with structures that are not flat. For example $(\text{c-ind}_N^G\psi)_{[T,1]}$ is flat over $\mathfrak{Z}_G$ whereas the morphism $X_{\check{G}} \to \check{T}/W$ is not flat. In fact, at least for $\mathbb{G} = \mathrm{GL}_n$, we can identify the irreducible components of $X_{\check{G}}$ with the $\check{G}$-orbits in the nilpotent cone $\mathcal{N}_{\check{G}} \subset \mathrm{Lie}\, \check{G}$ and it is easy to see that $X_{\check{G}} \to \check{T}/W$ maps all but one of these components to proper closed subschemes of $\check{T}/W$.

(c) Even worse, spelling out the compatibility among the various ξ_P^G involves requiring that a certain base-change morphism is an isomorphism. Working with classical schemes/stacks this is not true, and we are hence forced to replace the schemes X_H for a linear algebraic group H by a *derived scheme* (defined by the equation $\mathrm{Ad}(\varphi)N = q^{-1}N$). For a reductive group like $\check{G}$ this does not change anything, but if $H \subset \check{G}$ is a parabolic subgroup we are forced to stick to the derived set-up. The main reason is that in general H does not act via finitely many orbits on $\mathrm{Lie}\, H \cap \mathcal{N}_{\check{G}}$.

3. Results

In the case $\mathbb{G} = \mathrm{GL}_n$ we can prove the following partial results. In the formulation we make use of the canonical identification $\mathrm{Rep}_{[T,1]}G \cong \mathcal{H}_G$-mod. As in the case of GL_n there is (up to isomorphism) a unique Whittaker datum ψ, we will omit ψ from the notations.

Theorem 3. *The conjecture is true for* GL_2*.*

Theorem 4. *For $\mathbb{G} = \mathrm{GL}_n$ there exists an explicit candidate $R_G = - \otimes^L_{\mathcal{H}_G} \mathcal{M}_G$ such that*

- *R_G is $\mathfrak{Z}_G$-linear.*
- *R_G satisfies compatibility with parabolic induction as in Conjecture 1, after restricting to a* regular locus $X_{\check{G}}^{\mathrm{reg}} \subset X_{\check{G}}$.
- $R_G((\text{c-ind}_N^G\psi)_{[T,1]}) \cong \mathcal{O}_{[X_{\check{G}}/\check{G}]}$.

The object $\mathcal{M}_G$ is an $\mathcal{O}_{[X_{\check{G}}/\check{G}]} \otimes_{\mathfrak{Z}_G} \mathcal{H}_G$-module given by the Iwahori invariants in the family $\mathcal{V}_G$ on $[X_{\check{G}}/\check{G}]$ interpolating the (modified) local Langlands correspondence which is suggested by the work of Emerton-Helm [1] and constructed in subsequent work of Helm [3]. More explicitly $\mathcal{V}_G$ is the unique quotient

$$(\text{c-ind}_N^G \psi)_{[T,1]} \otimes_{\mathfrak{Z}_G} \mathcal{O}_{[X_{\check{G}}/\check{G}]} \longrightarrow \mathcal{V}_G$$

such that for a generic point $\eta = (\varphi_\eta, N_\eta)$ of $X_{\check{G}}$ the fiber of $\mathcal{V}_G$ is the representation associated to (φ_η, N_η) by the local Langlands correspondence. The specializations of $\mathcal{V}_G$ to other points of $X_{\check{G}}$ can conjecturally be computed by a similar, but slightly more involved, formula. The key to compatibility with parabolic induction is the conjectural identification

$$\mathcal{M}_G \cong R\beta_{B,*}\mathcal{O}_{[X_{\check{B}}/\check{B}]}. \tag{1}$$

Obviously, this identification implies that the right hand side is concentrated in degree zero. So far we are only able to prove (1) after restriction to the regular locus $[X_{\check{G}}^{\text{reg}}/\check{G}]$ which is, by definition, the open dense subset of $[X_{\check{G}}/\check{G}]$ over which $[X_{\check{B}}/\check{B}] \to [X_{\check{G}}/\check{G}]$ is finite.

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G-bundles on the Fargues-Fontaine curve and Newton strata

Eva Viehmann

In this talk we explain current work in progress determining the topological space underlying the stack of G-bundles on the Fargues-Fontaine curve.

Fix a prime p and a finite extension F of $\mathbb{Q}_p$. Let C be an algebraically closed complete extension of $\mathbb{Q}_p$ and let $C^\flat$ be its tilt. Denote by X the Fargues-Fontaine curve for $C^\flat$. The chosen field C corresponds to a point $\infty \in X$ with $k(\infty) = C$ and $\hat{\mathcal{O}}_{X,\infty} = B_{\text{dR}}^+(C)$.

Let G be a reductive group over F. For simplicity, we assume for this abstract that G is quasi-split, and fix a maximal torus T and a Borel subgroup B containing it. Then Fargues [1] proved that the set of isomorphism classes of G-bundles on X is in bijection with Kottwitz's set $B(G)$ of Frobenius-conjugacy classes of elements $b \in G(\breve{F})$ where $\breve{F}$ denotes the completion of the maximal unramified extension of F. We denote the G-bundle corresponding to some $[b] \in B(G)$ by $\mathcal{E}_b$.

Using Kottwitz's classification [4], the elements $[b] \in B(G)$ are described by two invariants. The first is to associate with $b \in G(\breve{F})$ its image under the Kottwitz map κ_G, an element of $\pi_1(G)_\Gamma$, where Γ is the absolute Galois group of F. The second invariant is the Newton point ν_b of b, an element of $X_*(T)^\Gamma_{\mathbb{Q},\mathrm{dom}}$.

We have a partial order on $B(G)$ given by $[b] \leq [b']$ if $\kappa_G(b) = \kappa_G(b')$ and $\nu_{b'} - \nu_b$ is a non-negative rational linear combination of positive coroots. By results of Rapoport and Richartz [5], this partial order describes the specialization order among F-isocrystals with additional structure in characteristic p.

Let Bun_G denote the small v-stack of G-bundles on the Fargues-Fontaine curve. Then our main result is that the specialization order among the points of Bun_G is given by the opposite of the partial order on $B(G)$.

Theorem 1 (V., in progress). *Let $[b'], [b''] \in B(G)$. Then $\mathcal{E}_{b''} \in \overline{\{\mathcal{E}_{b'}\}}$ if and only if $[b'] \leq [b'']$.*

Remark 2. In [2], Fargues and Scholze prove that if $\mathcal{E}_{b''} \in \overline{\{\mathcal{E}_{b'}\}}$, then $\kappa_G(b') = \kappa_G(b'')$. By [3], $\mathcal{E}_{b''} \in \overline{\{\mathcal{E}_{b'}\}}$ implies $\nu_{b'} \leq \nu_{b''}$, so together, these results imply one of the directions of the theorem.

The strategy of proof involves the construction of a family of G-bundles using Beauville-Laszlo uniformization. Let $\mathcal{E}_1$ be the trivial G-bundle on X. For every $x \in \mathrm{Gr}_G^{B_{dR}}(C)$ we obtain a new G-bundle $\mathcal{E}_{1,x}$ on X by gluing the trivial bundle over $X \setminus \{\infty\}$ and over $\mathrm{Spec}B^+_{dR}(C)$ using a gluing datum determined by x. However, it turns out to be very hard to determine the isomorphism class of $\mathcal{E}_{1,x}$ for a given point $x \in \mathrm{Gr}_G^{B_{dR}}(C)$. The main step of the proof is to circumvent this problem by defining for a pair $[b'] \leq [b''] \in B(G)$ two subspaces of $\mathrm{Gr}_G^{B_{dR}}$ such that the Newton stratum for $[b']$ resp. for $[b'']$ is open and dense in the respective subspace. Furthermore, these spaces are defined in such a way that we have the necessary closure relations between them.

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Higher Coleman theory

Vincent Pilloni
(joint work with George Boxer)

Let G be a split reductive group over a field k. Let B be a Borel subgroup, containing a maximal torus T. We let $X^\star(T)$ be the group of characters of T. Let $FL = B\backslash G$ be the Flag variety for G and $\pi : G \to FL$ be the projection map. Let d be the dimension of FL. Let W be the Weyl group of G and ρ be half the sum of the positive roots. We have a length function $\ell : W \to [0, d]$. Let $w_0 \in W$ be the longest element. For any $\kappa \in X^\star(T)$, we define a G-equivariant line bundle $\mathcal{L}_\kappa$ over FL. The right action of G on FL induces a left action on the cohomology groups $\mathrm{H}^i(FL, \mathcal{L}_\kappa)$. If κ is dominant, then $\mathrm{H}^0(FL, \mathcal{L}_\kappa)$ is a highest weight representation of weight κ. We introduce the dotted action $w \cdot \kappa = w(\kappa + \rho) - \rho$.

The following classical Borel-Weil-Bott theorem describes the cohomology of the sheaves $\mathcal{L}_\kappa$ over FL when the characteristic of k is 0 :

Theorem 1. *Assume that* $\mathrm{car}(k) = 0$. *Let* $\kappa \in X^\star(T)$ *then :*

(1) If there exists no $w \in W$ *such that* $w \cdot \kappa$ *is dominant then* $\mathrm{H}^i(FL, \mathcal{L}_\kappa) = 0$ *for all* i*,*

(2) If there exists $w \in W$ *such that* $w \cdot \kappa$ *is dominant, then there is a unique such* w*, and* $\mathrm{H}^i(FL, \mathcal{L}_\kappa) = 0$ *if* $\ell(w) \neq i$*, while* $\mathrm{H}^{\ell(w)}(FL, \mathcal{L}_\kappa)$ *is the highest weight* $w \cdot \kappa$ *representation.*

Following [2], section 12, one can study the cohomology of the sheaves $\mathcal{L}_\kappa$ over FL with the help of the Bruhat stratification $FL = \cup_{w \in W} B\backslash BwB$ and build a Cousin complex which computes the cohomology. Namely, for all $w \in W$, let X_w be the Schubert variety equal to the closure of $B\backslash BwB$ in FL. Consider the stratification of FL by closed subsets $FL = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_d \supseteq Z_{d+1} = \emptyset$ where $Z_i = \cup_{w, \ell(w) = d-i} X_w$. The following complex:

$$0 \to \mathrm{H}^0_{Z_0/Z_1}(FL, \mathcal{L}_\kappa) \to \mathrm{H}^1_{Z_1/Z_2}(FL, \mathcal{L}_\kappa) \to \cdots \to \mathrm{H}^d_{Z_d/Z_1}(FL, \mathcal{L}_\kappa) \to 0$$

computes $\mathrm{R}\Gamma(FL, \mathcal{L}_\kappa)$. The cohomologies $\mathrm{H}^i_{Z_i/Z_{i+1}}(FL, \mathcal{L}_\kappa)$ are by definition certain cohomology groups with support on the Bruhat cells of codimension i.

The modules appearing in the Cousin complex are infinite dimensional, but the action of the torus is very easy to determine and one can prove the following result which is valid in all characteristics.

Proposition 2. *Let* $\kappa \in X^\star(T)$ *and let* $C(\kappa) = \{w \in W,\ w(\kappa + \rho) \in X^\star(T)^+\}$. *Let* $\mathrm{R}\Gamma(FL, \mathcal{L}_\kappa)^{bw}$ *be the big weight part of* $\mathrm{R}\Gamma(FL, \mathcal{L}_\kappa)$*, which is the direct factor where the weights of* T *are* $> w \cdot \kappa$ *for all* $w \notin C(\kappa)$*. Then the cohomology* $\mathrm{R}\Gamma(FL, \mathcal{L}_\kappa)^{bw}$ *is a prefect complex of amplitude* $[\min_{w \in C(\kappa)} \ell(w), \max_{w \in C(\kappa)} \ell(w)]$.

One can show that the Cousin complex is a complex in the BGG category $\mathcal{O}$. In caracteristic 0, we derive a full proof of theorem 1 as a combination of some basic properties of the category $\mathcal{O}$ and the description of the action of the torus on the Cousin complex.

The main theme of our work is the coherent cohomology of Shimura varieties. The ideas we will employ use the close relation between Shimura varieties and flag varieties, as provided by the Hodge-Tate period map constructed in [3] and refined in [1]. We develop methods from local cohomology similar to the Grothendieck-Cousin complex of [2]. Let (G, X) be a Shimura datum. There are two opposite parabolic subgroups of G attached to (G, X), called P_μ and P_μ^{Std}. The space X embeds $G(\mathbb{R})$-equivariantly as an open subspace of $FL^{Std}_{G,\mu}(\mathbb{C}) = G/P^{Std}_\mu(\mathbb{C})$. This is the Borel embedding.

For any neat compact open subgroup $K \subseteq G(\mathbb{A}_f)$, we let $S_K(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ be the corresponding Shimura variety over $\mathbb{C}$. This is a finite disjoint union of arithmetic quotients of X.

Any representation of the Levi M_μ of P^{Std}_μ defines a G-equivariant vector bundle over $FL^{Std}_{G,\mu}$. By pull back to X and descent to $S_K(\mathbb{C})$, we obtain a functor from the category of representations of M_μ to the category of vector bundles on $S_K(\mathbb{C})$, whose essential image consists of (totally decomposed) automorphic vector bundles.We make a choice of Borel subgroup contained in P_μ, and we let T be a maximal torus contained in this Borel. We label irreducible representations of M_μ by their highest weight in $X_\star(T)^{M_\mu,+}$. For any $\kappa \in X_\star(T)^{M_\mu,+}$ we let $\mathcal{V}_\kappa$ be the corresponding vector bundle over $S_K(\mathbb{C})$.

The Shimura variety $S_K(\mathbb{C})$ has a structure of algebraic variety S_K defined over a number field E, called the reflex field. For a combinatorial choice Σ of cone decomposition, there are algebraic compactifications $S^{tor}_{K,\Sigma}$ whose boundary $D_{K,\Sigma} = S^{tor}_{K,\Sigma} \setminus S_K$ is a Cartier divisor. The vector bundles $\mathcal{V}_\kappa$ admit models over S_K and canonical extensions $\mathcal{V}_{\kappa,\Sigma}$ to $S^{tor}_{K,\Sigma}$. We denote by $\mathcal{V}_{\kappa,\Sigma}(-D_{K,\Sigma})$ the sub-canonical extension.

This paper is devoted to the study of the cohomologies of weight κ (which are independant of Σ) : $\mathrm{H}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,\Sigma})$, $\mathrm{H}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,\Sigma}(-D_{K,\Sigma}))$ as well as the interior cohomology:

$$\overline{\mathrm{H}}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,\Sigma}) = \mathrm{Im}(\mathrm{H}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,\Sigma}(-D_{K,\Sigma})) \to \mathrm{H}^i(S^{tor}_{K,\Sigma}, \mathcal{V}_{\kappa,\Sigma})).$$

We assume that (G, X) is an abelian Shimura datum and therefore S_K is (closely related) to a moduli space of abelian varieties with certain extra structures (endomorphism, polarization, level structure, Hodge tensors...).

Let p be a prime number such that $G_{\mathbb{Q}_p}$ is quasi-split. We also fix an embedding of $E \hookrightarrow \overline{\mathbb{Q}}_p$. We fix a compact open subgroup $K^p \subseteq G(\mathbb{A}^p_f)$. We now consider the following $G(\mathbb{Q}_p)$-representations arising from the cohomology of Shimura varieties:

$$\mathrm{H}^i(K^p, \kappa) = \mathrm{colim}_{K_p} \mathrm{H}^i(S^{tor}_{K^pK_p,\Sigma}, \mathcal{V}_{\kappa,\Sigma}),$$

and similarly $\mathrm{H}^i(K^p, \kappa, cusp)$ and $\overline{\mathrm{H}}^i(K_p, \kappa) = \mathrm{Im}(\mathrm{H}^i(K^p, \kappa, cusp) \to \mathrm{H}^i(K^p, \kappa))$.

We define a first direct summand as $G(\mathbb{Q}_p)$-representation $\mathrm{H}^i(K^p, \kappa)^{fs}$ of $\mathrm{H}^i(K^p, \kappa)$ that we call the finite slope part of $\mathrm{H}^i(K^p, \kappa)$. It contains all the irreducible smooth $G(\mathbb{Q}_p)$-subquotients which can be embedded in a principal series representation $\iota^{G(\mathbb{Q}_p)}_{B(\mathbb{Q}_p)}\lambda$ for a character λ of $T(\mathbb{Q}_p)$.

We define a second direct summand $\mathrm{H}^i(K^p, \kappa)^{ss}$ of $\mathrm{H}^i(K^p, \kappa)^{fs}$ that we call the small slope part of $\mathrm{H}^i(K^p, \kappa)$. This direct factor of the cohomology is the smallest which contains all irreducible subquotient smooth $G(\mathbb{Q}_p)$-representations which can be embedded in a principal series representation $\iota^{G(\mathbb{Q}_p)}_{B(\mathbb{Q}_p)}\lambda$ for a character λ of $T(\mathbb{Q}_p)$ whose p-adic valuation is small with respect to κ, in a sense that is made precise in the paper.

We adopt similar definitions for the cuspidal and interior cohomology.

Theorem 3. *For any $\kappa \in X_\star(T)^{M_\mu,+}$, any prime p such that $G(\mathbb{Q}_p)$ is quasi-split and any compact open subgroup K, let $C(\kappa)^+ = \{w \in W, w^{-1}w_{0,M}(\kappa+\rho) \in X^\star(T)^-\}$. Then $\overline{\mathrm{H}}^i(K^p,\kappa)^{ss}$ is concentrated in the range*

$$[\inf_{w\in C(\kappa)^+}\ell(w), \sup_{w\in C(\kappa)^+}\ell(w)].$$

There is a classical Archimedean result due to the combined works of Blasius-Harris-Ramakrishnan, Mirkovich, Schmid and Williams which asserts that for any compact open $K \subseteq G(\mathbb{A}_f)$, the tempered at infinity interior cohomology $\overline{\mathrm{H}}^i(S^{tor}_{K,\Sigma},\mathcal{V}_{\kappa,\Sigma})^{temp}$ is concentrated in the range $[\inf_{w\in C(\kappa)^+}\ell(w), \sup_{w\in C(\kappa)^+}\ell(w)]$.

If the weight κ is such that $\kappa+\rho$ is regular, then $C(\kappa)^+$ consists of a single element. The interior cohomology is therefore concentrated in one single degree. To any representation $\mathbb{W}$ of the group G over a $\mathbb{Q}$-vector space we can attach a local system $\mathcal{W}$ on $S_K(\mathbb{C})$. We have the Betti cohomology groups $\mathrm{H}^\star(S_K(\mathbb{C}),\mathcal{W})$, $\mathrm{H}^\star_c(S_K(\mathbb{C}),\mathcal{W})$ and the interior cohomology

$$\overline{\mathrm{H}}^\star(S_K(\mathbb{C}),\mathcal{W}) = \mathrm{Im}(\mathrm{H}^\star_c(S_K(\mathbb{C}),\mathcal{W}) \to \mathrm{H}^\star(S_K(\mathbb{C}),\mathcal{W})).$$

We can consider $\mathrm{H}^i(K^p,\mathbb{W}) = \mathrm{colim}_{K_p}\mathrm{H}^i(S_{K^pK_p}(\mathbb{C}),\mathcal{W})$, and similarly $\mathrm{H}^i_c(K^p,\mathbb{W})$, as well as $\overline{\mathrm{H}}^i(K_p,\mathbb{W}) = \mathrm{Im}(\mathrm{H}^i_c(K^p,\mathbb{W}) \to \mathrm{H}^i(K^p,\mathbb{W}))$. One can consider the finite and small slope part of these cohomology. Using Faltings's BGG spectral sequence we deduce easily:

Theorem 4. *For any prime p such that $G(\mathbb{Q}_p)$ is quasi-split, the small slope at p interior Betti cohomology $\overline{\mathrm{H}}^\star(K^p,\mathbb{W})^{ss}$ is concentrated in the middle degree* $\dim S_K(\mathbb{C})$.

In [1] Caraiani and Scholze proved a similar concentration result for the Betti cohomology of unitary Shimura varieties under a genericity condition for the action of the spherical Hecke algebra at a prime number p. Their result is much more powerful because it also applies to the cohomology with coefficients in a ℓ-torsion local system for a prime $\ell \neq p$. The three conditions of temperdness at infinity, genericity at p and small slope at p are related.

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Invariance under Henselian pairs for flat cohomology

Kęstutis Česnavičius
(joint work with Alexis Bouthier and with Peter Scholze)

The goal of this talk is to present results from [BČ20] and [ČS20] that established invariance under Henselian pairs for several cohomological functors, see Theorem 5. For context, we first explain a common utility of this invariance.

1. The role of Henselian pairs in algebraization and approximation

Definition 1. A pair (A, I) consisting of a commutative ring A and an ideal $I \subset A$ is *Henselian* if for every affine, étale A-scheme X, we have

$$X(A) \twoheadrightarrow X(A/I).$$

Example 2. If A is I-adically complete in the sense that $A \xrightarrow{\sim} \varprojlim_{n>0}(A/I^n)$, then (A, I) is Henselian. If A is merely derived I-adically complete in the sense that $A \xrightarrow{\sim} R\lim_{n>0}(A/^{\mathbb{L}}a^n)$ for $a \in I$, then (A, I) is still Henselian [ČS20, Lem. 5.6.2].

The properties of (A, I) discussed in Example 2 all depend only on the nonunital ring I. For example, by [Gab92, Prop. 1] or [SP, 09XI], the pair (A, I) is Henselian if and only if elements of $1 + I$ have multiplicative inverses and every polynomial

$$T^n(T-1) + a_nT^n + \ldots + a_1T + a_0 \quad \text{with} \quad a_n, \ldots, a_0 \in I$$

has a (necessarily unique) root in $1 + I$.

We seek to exhibit *invariant under Henselian pairs* functors F in the sense that $F(A) \xrightarrow{\sim} F(A/I)$ for Henselian pairs (A, I) such that F is defined on A and A/I.

Example 3 ([Gab94, Thm. 1])**.** For an abelian, torsion sheaf $\mathscr{F}$ on the étale site of some commutative ring A_0, on the category of A_0-algebras A, the functor

$$A \mapsto R\Gamma_{\text{ét}}(A, \mathscr{F}) \quad \text{is invariant under Henselian pairs.}$$

In Theorem 5 (b), we will establish a similar property of fppf cohomology.

The following result of Gabber shows that the functors that are invariant under Henselian pairs behave well with respect to algebraization and approximation.

Theorem 4 (Gabber (unpublished), [BČ20, Thm. 2.1.16])**.** *Let B be a topological ring that has a Henselian open nonunital subring $B' \subset B$ whose induced topology has a neighborhood base of 0 consisting of ideals of B'. If a functor*

$$F\colon B\text{-}algebras \to Sets$$

commutes with filtered direct limits and is invariant under Henselian pairs, then

$$F(B) \xrightarrow{\sim} F(\widehat{B}). \tag{1}$$

Here $\widehat{B}$ is the completion of the topological ring B. For example, we could have

(1) $B := R\{t\}[\frac{1}{t}]$ with $B' := tR\{t\}$, where R is a commutative ring and $R\{t\}$ is the Henselization of $R[t]$ along $\{t = 0\}$, so that $\widehat{B} \cong R(\!(t)\!)$; or
(2) B is a Henselian Huber ring as defined in [Hub96, Def. 3.1.2] with B' an ideal of definition in a ring of definition, so that $\widehat{B}$ is a complete Huber ring; or
(3) $B := A$ with $B' := I$ for some Henselian pair (A, I) such that B' with its coarse topology is open in B, so that $\widehat{B} \cong A/I$.

The idea of the proof of Theorem 4. We let S be a neighborhood base of 0 in B considered as a poset with the order $U \le U'$ iff $U' \subset U$, and we consider the ring

$$\mathrm{Cauchy}_S(B) := \{\text{germs of Cauchy nets } f\colon S \to B\}$$

and its ideal

$$\mathrm{Null}_S(B) := \{\text{germs of null nets } f\colon S \to B\}.$$

The nonunital ring $\mathrm{Null}_S(B)$ is Henselian because it agrees with $\mathrm{Null}_S(B')$ which, in turn, is an ideal in $\varinjlim_{U \in S} \left(\prod_{S_{\ge U}} B'\right)$. The identification

$$\widehat{B} \cong \mathrm{Cauchy}_S(B)/\,\mathrm{Null}_S(B)$$

then serves as a basic link between (1) and invariance under Henselian pairs.

This technique based on rings of Cauchy nets also leads to a reproof and a non-Noetherian generalization of the Elkik approximation theorem, see [BČ20, §2.2].

2. Flat cohomology and invariance under Henselian pairs

The following is the promised invariance under Henselian pairs for flat cohomology.

Theorem 5. *Let (A, I) be a Henselian pair.*

(a) ([BČ20, Prop. 2.1.4, Thm. 2.1.7]). *For a smooth, quasi-compact, algebraic A-stack $\mathscr{X}$ whose diagonal is quasi-affine,*

$$\mathscr{X}(A) \twoheadrightarrow \mathscr{X}(A/I).$$

In particular, for a smooth, quasi-affine A-group G,

$$H^1_{\mathrm{fppf}}(A, G) \xrightarrow{\sim} H^1_{\mathrm{fppf}}(A/I, G),$$
$$\mathrm{Ker}(H^2_{\mathrm{fppf}}(A, G) \to H^2_{\mathrm{fppf}}(A/I, G)) = \{*\},$$

where the H^1 and H^2 are interpreted in terms of torsors and gerbes.

(b) ([ČS20, Cor. 5.6.8]). *For a commutative, finite, locally free A-group G,*

$$H^i_{\mathrm{fppf}}(A, G) \xrightarrow{\sim} H^i_{\mathrm{fppf}}(A/I, G) \quad \textit{for} \quad i \ge 2.$$

(c) ([ČS20, Thm. 5.6.5]). *If A is derived I-adically complete, $I = (a_1, \ldots, a_r)$ is finitely generated, and G is as in* (b)*, then*

$$R\Gamma_{\mathrm{fppf}}(A, G) \xrightarrow{\sim} R\lim_{n>0}(R\Gamma_{\mathrm{fppf}}(A/^{\mathbb{L}}(a_1^n, \ldots, a_r^n), G)). \tag{2}$$

In particular, if A is I-adically complete, I is finitely generated, and G is as in (b)*, then we have a short exact sequence*

$$0 \to \varprojlim^1_{n>0}(G(A/I^n)) \to H^1(A, G) \to \varprojlim_{n>0}(H^1(A/I^n, G)) \to 0.$$

The continuity formula (2) continues to hold when A is merely an animated ring in the sense of [ČS20, §5.1], and this added generality is crucial for the proof.

The map in (b) for $i = 1$ is still surjective but no longer injective: for instance,

$$H^1_{\mathrm{fppf}}(\mathbb{Z}_p, \mu_p) \not\cong 0 \quad \text{but} \quad H^1_{\mathrm{fppf}}(\mathbb{F}_p, \mu_p) \cong 0.$$

3. An overview of the proof of Theorem 5

In (a), one begins with an affine $\mathscr{X}$, for which one uses the local structure of smooth morphisms to eventually reduce to affine étale $\mathscr{X}$ (see [Gru72, I.8]) covered by Definition 1. One combines the affine case, limit arguments, and Popescu's theorem [SP, 07GC] to reduce to Noetherian, I-adically complete A. One concludes by combining the infinitesimal smoothness criterion with the formula

$$\mathscr{X}(A) \xrightarrow{\sim} \varprojlim_{n>0} \mathscr{X}(A/I^n)$$

that follows from Tannaka duality for algebraic stacks settled in [BHL17] or [HR19].

An analogous passage to the Noetherian, complete case reduces (b) to (c), except that (b) for $i = 2$ is actually an input to (c). One deduces this low degree case from (a) by combining the Bégueri sequence $0 \to G \to \mathrm{Res}_{G^*/A}(\mathbb{G}_m) \to Q \to 0$, the identification $\mathrm{Br}(R) \xrightarrow{\sim} H^2(R, \mathbb{G}_m)_{\mathrm{tors}}$ due to Gabber [Gab81, Ch. II, Thm. 1], and the definition of the Brauer group $\mathrm{Br}(-)$ in terms of PGL_n-torsors.

In (c), one assumes at the outset that A is animated and reduces to $r = 1$ with $a := a_1$ and G of p-power order for a prime p. One then uses the $i = 2$ case of (b) to establish (2) "by hand" when the appearing $H^{\geq 3}_{\mathrm{fppf}}$ all vanish. Bounds on the p-cohomological dimension of $\mathbb{F}_p$-algebras (essentially, the Artin–Schreier sequence) ensure that this includes the case when A is an $\mathbb{F}_p$-algebra. One then deduces (2) for p-Henselian A by combining deformation theory with the p-adic continuity formula [ČS20, Thm. 5.3.5], which, essentially, is the $a = p$ case of (2) and is simpler because $R\Gamma_{\text{ét}}((-)^h_{(p)}[\frac{1}{p}], G)$ satisfies p-complete arc hyperdescent [BM20, Cor. 6.17]. With this in hand, the idea is to show a-completely faithfully flat hyperdescent for the functor $A \mapsto R\Gamma_{\mathrm{fppf}}(A, G)$ on a-adically complete inputs by replacing A by its p-Henselization and using excision both for flat cohomology [ČS20, Thm. 5.4.4] and for étale cohomology [BM20, Thm. 5.4] to reduce the study of the flat cohomology of G to that of the étale cohomology of $j_!(G)$ with $j\colon \mathrm{Spec}(A[\frac{1}{p}]) \hookrightarrow \mathrm{Spec}(A)$, for which Example 3 applies. The acquired a-completely faithfully flat hyperdescent allows one to replace A by the terms A^i of a large a-completely faithfully flat hypercover, constructed so that each A^i has no nonsplit étale covers. This last property ensures that all the appearing $H^{\geq 2}_{\mathrm{fppf}}$ all vanish, to the effect that one is in the case that was already settled "by hand."

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On the Gelfand-Kirillov dimension of modulo p representations of $\mathbf{GL_2}$

Benjamin Schraen

(joint work with Christophe Breuil, Florian Herzig, Yongquan Hu, Stefano Morra)

Let p be a prime and $\mathbb{F}$ a finite field of characteristic p. Let G be a p-adic Lie group and fix $H \subset G$ a compact open uniform pro-p-subgroup. It follows from the work of Lazard that the complete group algebra $\mathbb{F}[\![H]\!] = \varprojlim_{H' \subset H} \mathbb{F}[H/H']$ is a complete noetherian local ring with maximal ideal $\mathfrak{m}_H$ whose graded ring is isomorphic to $\mathbb{F}[x_1, \dots, x_{\dim G}]$.

Let (π, V) be a smooth admissible representation of G over some $\mathbb{F}$-vector space V. The admissibility condition is equivalent to the fact that $\pi^\vee := \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ is a finitely generated $\mathbb{F}[\![H]\!]$-module. Its graded module for the $\mathfrak{m}_H$-topology is then a finitely generated $\mathrm{gr}\mathbb{F}[\![H]\!]$-module. We define the *dimension* $\dim_G \pi$ of π as the dimension of the support of $\mathrm{gr}(\pi^\vee)$ in $\mathrm{Spec}\,\mathrm{gr}(\mathbb{F}[\![H]\!])$. Equivalently there exists a positive integer C such that

$$\dim_{\mathbb{F}}(\pi^\vee/\mathfrak{m}_H^n) \sim_{n \to +\infty} \frac{C}{(\dim_G \pi)!} n^{\dim_G \pi}.$$

The integer $\dim_G \pi$ is the Gelfand-Kirillov dimension of the graded $\mathbb{F}$-vector space $\mathrm{gr}(\pi^\vee)$.

Here are some examples of representations for which we know the dimension.

- We have $\dim_G \pi = 0$ if and only if $\dim_{\mathbb{F}} V < +\infty$.
- If $P \subset G$ is a cocompact subgroup and (ψ, W) is a smooth finite dimension representation of P, we have $\dim_G(\mathrm{Ind}\,_P^G\psi) = \dim(G/P)$ where the latter is the dimension of the p-adic analytic variety G/P.
- If $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and (π, V) is absolutely irreducible, $\pi \not\simeq \chi \circ \det$ for a smooth character of $\mathbb{Q}_p^\times$, then $\dim_G \pi = 1$. This is a consequence of [1], [3] and [8].

Here is yet another example of global nature. If $G = \mathrm{GL}_2(\mathbb{Q}_p)$, and $N \geq 3$ is a prime to p integer, let

$$\widetilde{H}^1(N, \mathbb{F}) = \varinjlim_{n \to +\infty} H^1_{\text{ét}}(X(Np^n)_{\overline{\mathbb{Q}}}, \mathbb{F})$$

be the (completed) cohomology with coefficients in $\mathbb{F}$ of the tower of modular curves of tame level N. It is a smooth admissible representation of G carrying a commuting action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $\overline{r} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$ be some absolutely irreductible continuous representation such that

$$\pi(\overline{r}) := \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\overline{r}, \widetilde{H}^1(N, \mathbb{F})) \neq 0$$

(ie a modular Galois representation of tamel level N). Then we know that $\pi(\overline{r})$ is a smooth admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ and that $\dim_G \pi(\overline{r}) = 1$. This last fact is actually a consequence of the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ ([5], [2] etc.) and the local-global compatibility between the p-adic Langlands correspondence and the completed cohomology of modular curves ([6]). A consequence of Emerton's work is that $\pi(\overline{r})$ is (a finite direct sum of) the representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to the local Galois representation $\overline{r}|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)} : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{GL}_2(\mathbb{F})$.

For other groups that $\mathrm{GL}_2(\mathbb{Q}_p)$ we don't know yet if there exists an analogue of the p-adic Langlands correspondence. However looking at Galois-isotypic subspaces in the completed cohomology of Shimura curves should give us good candidates for representations of $\mathrm{GL}_2(L)$, where L is a finite extension of $\mathbb{Q}_p$, attached to 2-dimensional representations over $\mathbb{F}$ of the group $\mathrm{Gal}(\overline{L}/L)$ is some hypothetical mod p Langlands correspondence for $\mathrm{GL}_2(L)$. Our main result is the computation of the dimension of such representations.

Let F be a totally real field such that p is unramified in F. Assume for simplicity that p is inert in F. Let D be a quaternion algebra of center F which is split at places over p and at exactly one place at infinity. Let S_D be the set of ramification places of F. We fix $K^p \subset (D \otimes \mathbb{A}_F^{p;\infty})^\times$ a compact open subgroup and define

$$\widetilde{H}^1(K^p, \mathbb{F}) := \varinjlim_{K_p \subset \mathrm{GL}_2(F_p)} H^1_{\text{ét}}(\mathrm{Sh}_D(K^p K_p)_{\overline{F}}, \mathbb{F})$$

where $\mathrm{Sh}_D(K)$ denotes the Shimura curve associated to D of level K and K_p varies among open subgroups of $\mathrm{GL}_2(F_p)$. Let $\overline{r}:\mathrm{Gal}(\overline{F}/F)\to\mathrm{GL}_2(\mathbb{F})$ be some modular continuous representation, which means that

$$\pi(\overline{r}) \coloneqq \mathrm{Hom}_{\mathrm{Gal}(\overline{F}/F)}(\overline{r}, \widetilde{H}^1(K^p,\mathbb{F})) \neq 0$$

and let $S_{\overline{r}}$ be the set of ramification places of $\overline{r}$. This is a smooth admissible representation of the group $\mathrm{GL}_2(F_p)$ (recall that F_p is an unramified extension of $\mathbb{Q}_p$).

Our main result is the following:

Theorem. *We have* $\dim_{\mathrm{GL}_2(F_p)}\pi = [F_p:\mathbb{Q}_p]$ *when the following hypotheses are satisfied :*

(i) the representation $\overline{r}|_{\mathrm{Gal}(\overline{F}/F(\zeta_p))}$ *is absolutely irreducible ;*
(ii) the group K^p *is of the form* $\prod_{v\nmid p}K_v\subset\prod_{v\nmid p}(\mathcal{O}_D\otimes_{\mathcal{O}_F}\mathcal{O}_{F_v})^\times$ *and* $K_v=(\mathcal{O}_D\otimes_{\mathcal{O}_F}\mathcal{O}_{F_v})^\times$ *when* $v\notin S_D\cup S_{\overline{r}}$ *;*
(iii) if $v\in(S_D\cup S_{\overline{r}})\setminus S_p$*, the local framed deformation ring of* $\overline{r}|_{\mathrm{Gal}(\overline{F_v}/F_v)}$ *is formally smooth ;*
(iv) the local representation $\overline{r}|_{\mathrm{Gal}(\overline{F_p}/F_p)}$ *is tame and "very generic".*

Let say a word about the *very generic* condition. Let $f=[F_p:\mathbb{Q}_p]$. We say that a tame representation $\overline{\rho}:\mathrm{Gal}(\overline{F_p}/F_p)\to\mathrm{GL}_2(\mathbb{F})$ is very generic if its restriction to the inertia subgroup I_p is of one of the two following forms, up to twist by a character:

- if $\overline{\rho}$ is reducible,

$$\overline{\rho}|_{I_p}\simeq\begin{pmatrix}\omega_f^{(r_0+1)+\cdots+p^{f-1}(r_{f-1}+1)} & 0\\ 0 & 1\end{pmatrix}\text{ with } 9\le r_i\le p-12,$$

- if $\overline{\rho}$ is irreducible,

$$\overline{\rho}|_{I_p}\simeq\begin{pmatrix}\omega_{2f}^{(r_0+1)+\cdots+p^{f-1}(r_{f-1}+1)} & 0\\ 0 & \omega_{2f}^{p^f((r_0+1)+\cdots+p^{f-1}(r_{f-1}+1))}\end{pmatrix}$$
$$\text{with } 10\le r_0\le p-11,\ 9\le r_i\le p-12,\ i>0,$$

where ω_f and ω_{2f} are Serre's fundamental characters of level f and $2f$. In particular the existence of a very generic representation $\overline{\rho}$ implies that $p>19$. Howeverthe result is still expected to be true under a weaker condition as $\overline{\rho}$ generic in the sense of [4, Def. 11.7].

It is interesting to note that $[F_p:\mathbb{Q}_p]$ is the dimension of the flag variety of $\mathrm{Res}_{F_p/\mathbb{Q}_p}\mathrm{GL}_2$. Our result can be thought as an "holonomy" property for the representation $\pi(\overline{r})$. Some consequences of this property are discussed in [7].

Remark. The hypothesis p inert in F is not essential and is there only for notational convenience.

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The Sen operator and infinitesimal character

Vytautas Paškūnas

(joint work with Gabriel Dospinescu, Benjamin Schraen)

We are interested in studying Hecke eigenspaces in completed cohomology. The main result explained in the talk says that in favourable settings if the Hecke eigenspace is non-zero then the centre of the universal enveloping algebra acts on its locally analytic vectors via infinitesimal character, which can be related to the eigenvalues of the Sen operator of the corresponding Galois representation. We explain the result in somewhat abstract setting.

Let L be a finite extension of $\mathbb{Q}_p$ with the ring of integers $\mathcal{O}$, a uniformizer ϖ and the residue field k. Let $(A,\mathfrak{m})$ be a complete local noetherian $\mathcal{O}$-algebra with residue field k. Let $\mathfrak{X}^{\mathrm{rig}}$ be the rigid space associated to the formal scheme $\mathrm{Spf}\, A$ by Berthelot and let A^{rig} be its global sections. Let G be a connected reductive group over $\mathbb{Q}_p$ and let K be an open compact subgroup of $G(\mathbb{Q}_p)$. Let $\mathfrak{g}$ be the Lie algebra of G, let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let $Z(\mathfrak{g})$ be its centre. Further let M be an $A[G(\mathbb{Q}_p)]$-module, such that the induced action of $A[K]$ on M extends to the action of the completed group algebra $A[[K]]$ which makes M into a finitely generated $A[[K]]$-module. There is a canonical topology on M such that the action of $A[[K]]$ on M is continuous. Moreover, M is compact with respect to this topology. We may define a unitary Banach space representation of $G(\mathbb{Q}_p)$ by

$$\Pi(M) := \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(M, L)$$

with the topology given by the supremum norm. We may think of $\Pi(M)$ as a family of admissible unitary Banach space representations of $G(\mathbb{Q}_p)$ over the maximal spectrum $\text{m-Spec}\, A[1/p]$. Namely, for $x \in \text{m-Spec}\, A[1/p]$ we may study

$\Pi(M)[\mathfrak{m}_x]$, the subspace of $\Pi(M)$ consisting of elements killed by the maximal ideal $\mathfrak{m}_x$.

We assume that G splits over L and let $\mathrm{Irr}(G_L)$ be the set of isomorphism classes of irreducible algebraic representations of G_L. If $V \in \mathrm{Irr}(G_L)$ then we let K act on $V(L)$ on the right via $K \subset G(\mathbb{Q}_p) \subset G(L)$. We further let $M(V) := V(L) \otimes_{\mathcal{O}[[K]]} M$, where in the tensor product we consider $V(L)$ as a left $\mathcal{O}[[K]]$-module via $k \mapsto k^{-1}$. Since by assumption M is finitely generated over $A[[K]]$, $M(V)$ is a finitely generated $A[1/p]$-module and we denote the quotient of $A[1/p]$ which acts faithfully on $M(V)$ by $A_{V,M}$.

Theorem 1. *Assume we are given a homomorphism of L-algebras $\chi : Z(\mathfrak{g})_L \to A^{\mathrm{rig}}$ and M as above so that the following hold*

(1) there is an M-regular sequence $y_1, \dots, y_h \in A$ such that $M/(y_1, \dots, y_h)M$ is a finitely generated projective $\mathcal{O}[[K]]$-module;

(2) for all $V \in \mathrm{Irr}(G_L)$, the rings $A_{V,M}$ are reduced;

(3) for all $V \in \mathrm{Irr}(G_L)$ and all $x \in \text{m-Spec}\, A_{V,M}$, $Z(\mathfrak{g})_L$ acts on V via the specialization of χ at x.

Then for all $y \in \text{m-Spec}\, A[1/p]$, $Z(\mathfrak{g})_L$ acts on the locally analytic vectors in $\Pi(M)[\mathfrak{m}_y]$ via the specialisation of χ at y.

The theorem is proved using density arguments. The reader should prove the theorem when G is just the trivial group. The same commutative algebra arguments are applied to the modules $M(V)$ in the course of the proof of the theorem.

We can also associate an L-algebra homomorphism $\chi_\rho : Z(\mathfrak{g})_L \to A^{\mathrm{rig}}$, which encodes the information about the characteristic polynomial of the Sen operator, to a continuous Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to G^L(A^{\mathrm{rig}})$, where G^L is the L-group. The construction uses the theory of Sen as discussed in [1], Tannakian formalism, Chevalley's restriction theorem and Harish-Chandra homomorphism. It was pointed out by Peter Scholze during the talk that we should use the C-group instead of the L-group, so in order to apply the theorem with $\chi = \chi_\rho$ the construction of the character needs to be modified. However, in the examples below G_L is a product of GL_n's and we can get away with twisting. Informally, our result says that if we can attach Galois representations to Hecke eigenspaces in completed cohomology, so that at classical points the Hodge–Tate weights of the Galois representation match the infinitesimal character of the corresponding classical automorphic form, then this property propagates p-adically. We end the abstract with a few examples.

Example 2. $G = \mathbb{G}_{\mathbb{Q}_p}$, where $\mathbb{G}$ is a connected reductive group over $\mathbb{Q}$ with the property that the maximal $\mathbb{Q}$-split torus in the centre of $\mathbb{G}$ is a maximal $\mathbb{R}$-split torus in $\mathbb{G}$. This means that $\mathbb{G}(\mathbb{R})$ is compact modulo its centre. M is the completed 0-th homology with respect to a fixed tame level, localised at an open maximal ideal $\mathfrak{m}$ of the big Hecke algebra. A is the localisation of the big Hecke algebra at $\mathfrak{m}$. In this case $\Pi(M)$ is just the completed 0-th cohomology with p inverted and $\Pi(M)[\mathfrak{m}_y]$ is the Hecke eigenspace, see [3, §5.1] for more details. In this case M satisfies part (1) of the theorem with $h = 0$, provided the tame level is small enough, but in this generality we don't know how to attach Galois representations to automorphic forms, so we don't know how to produce χ. If $\mathbb{G}$ is a definite unitary group then we can attach Galois representations to automorphic forms and from this construct an infinitesimal character satisfying the conditions of the theorem.

Example 3. Let F be a finite extension of $\mathbb{Q}_p$ and let G be the restriction of scalars of GL_n over F to $\mathbb{Q}_p$. Then $G(\mathbb{Q}_p) = \mathrm{GL}_n(F)$ and our theorem applies to the patched module M_∞ in [2] with $A = R_\infty$, the patched deformation ring, which is an algebra over the framed deformation ring of an n-dimensional mod p representation $\bar{\rho}$ of the absolute Galois group of F, with the infinitesimal character in the theorem obtained from the Galois representation obtained from the universal framed deformation of $\bar{\rho}$ by extending scalars to R_∞. Part (1) follows from [2, Prop. 2.10], the elements $y_1, \dots, y_h$ are the patching variables, so that $S_\infty = \mathcal{O}[[y_1, \dots, y_h]]$ in the notation of [2]. The rings $A_{V,M}$ are essentially the rings $R_\infty(\sigma)[1/p]$ in the notation of [2, Lem. 4.17], which proves that they are reduced. Except that in [2] we work with $K = \mathrm{GL}_n(\mathcal{O}_F)$ and fix a type and here K is some compact open subgroup, so that our ring $A_{V,M}$ injects into a product of the rings of considered in [2, Lem. 4.17] with p inverted. Part (3) of the theorem follows from the construction of the algebraic representation π_{alg} of G out of the Hodge–Tate weights of a Galois representation in [2, §1.8]. These Hodge–Tate weights are precisely the eigenvalues of the Sen operator.

Example 4. Let D be a quaternion algebra over $\mathbb{Q}$ split at ∞. Then our theorem applies to $\Pi(M) = \widehat{H}^1(U^p, \mathcal{O})_{\mathfrak{m}} \otimes L$ (with some modifications due to the centre), the first completed cohomology group of the tower of Shimura curves associated to D, fixed tame level U^p and varying the level at p, localised at a maximal ideal corresponding to an absolutely irreducible 2-dimensional mod p Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. If the ideal $\mathfrak{m}_y$ in the theorem corresponds to a p-adic Galois representation which is not Hodge–Tate at p then we can show that the specialisation of χ at y cannot be an infinitesimal character of an irreducible finite dimensional $U(\mathfrak{sl}_2)_L$-module. This implies that if K is a compact open subgroup of $(D \otimes \mathbb{Q}_p)^\times$ then $(\widehat{H}^1(U^p, \mathcal{O})_{\mathfrak{m}} \otimes L)[\mathfrak{m}_y]$ does not have a finite dimensional K-invariant subquotient. By a different argument we can bound the Gelfand–Kirilov dimension of this Banach space representation by 1 and, putting the two ingredients together, conclude that $(\widehat{H}^1(U^p, \mathcal{O})_{\mathfrak{m}} \otimes L)[\mathfrak{m}_y]$ is of finite length as a Banach space representation of K. This is interesting because there should be a p-adic Jacquet–Langlands correspondence realised by the completed cohomology, and the result suggests that the objects on the division algebra side should be (some) admissible unitary L-Banach space representations of $(D \otimes \mathbb{Q}_p)^\times$ of finite length and of Gelfand–Kirilov dimension 1.

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Density of Arithmetic Representations

Hélène Esnault

(joint work with Moritz Kerz)

Let X be a smooth projective variety of dimension d defined over an algebraically closed field k. Let ℓ be a prime number not equal to the characteristic of k. Let $\eta \in H^2(X, \mathbb{Z}_\ell)$ be the Chern class of a polarization of X. Let $\mathcal{F}$ be a $\bar{\mathbb{Q}}_\ell$-local system. If $\mathcal{F}$ is *semi-simple* one predicts the *Hard Lefschetz property* (HL) to be true, that is

$$\cup^i \eta : H^{d-i}(X, \mathcal{F}) \to H^{d+i}(X, \mathcal{F})$$

to be an isomorphism. Indeed, for $k = \mathbb{C}$, by the comparison isomorphism between ℓ-adic and de Rham cohomology, (HL) is true for trivial coefficients by harmonic theory (Hodge). By work of Simpson $\mathcal{F}$ is semi-simple if and only if $\mathcal{F}$ carries a harmonic metric ([7]). If char. $k = p > 0$, one easily reduces (HL) to the situation where X is defined over a finite field, so $k = \bar{\mathbb{F}}_p$. If $\mathcal{F}$ itself descends to a finite field, i.e. if $\mathcal{F}$ is *arithmetic*, (HL) is proved by in [3] and [1] based on Deligne's theory of weights, and by the Langlands correspondence ([6]).

In absence of weights, we propose in [4] and [5] the following strategy. Let X be a smooth connected scheme of finite type over $\bar{\mathbb{F}}_p$. Fix a semi-simple continuous residual representation $\bar{\rho} : \pi_1^{\text{ét}}(X) \to GL_r(\mathbb{F})$, where $\mathbb{F}$ is a finite extension of $\mathbb{F}_\ell$ and $\pi_1^{\text{ét}}(X)$ is the geometric fundamental group based at a geometric point. Consider the set $\mathcal{S}_{\bar{\rho}}$ of isomorphism classes of semi-simple $\bar{\mathbb{Q}}_\ell$-local systems of a given rank r with semi-simplified residual representation isomorphic to $\bar{\rho}$. By [2], $\mathcal{S}_{\bar{\rho}}$ is the set of $\bar{\mathbb{Q}}_\ell$-points of a formal scheme defined over $W(\mathbb{F})$, which is noetherian and Jacobson. Assuming the Frobenius Φ leaves $\bar{\rho}$ invariant, which is true after replacing the field $\mathbb{F}_q$ of definition of X by $\mathbb{F}_{q^m}$ for some natural number $m > 0$, Φ acts on $\mathcal{S}_{\bar{\rho}}$ as a homeomorphism. We define the set of *arithmetic points* $\mathcal{A}_{\bar{\rho}} \subset \mathcal{S}_{\bar{\rho}}$ as the set of points stabilized by some power of Φ, that is the set of isomorphism classes of semi-simple $\bar{\mathbb{Q}}_\ell$-local systems which are defined over $\mathbb{F}_{q^m}$ for some positive natural number m.

Conjecture 1. *(see [5]) Let $Z \subset \mathcal{S}_{\bar{\rho}}$ be Zariski closed, stabilized by Φ^m for some positive natural number m. Then $\mathcal{A}_{\bar{\rho}} \cap Z$ is Zariski dense.*

We prove the following.

Proposition 2. *1) Conjecture 1 in rank r implies (HL) in rank r (*[4]*,* [5]*).*
2) Conjecture 1 on $\mathbb{P}^1 \setminus \{0,1,\infty\}$ for $\bar{\rho}$ tame implies Conjecture 1 in general ([5]*).*

Theorem 3. *The conjecture is true in the following cases.*
1) For $r = 1$ when $H^1(X, \mathbb{Q}_\ell)$ is pure of weights $\neq 0$, e.g. when X is proper or a torus ([4]*);*
2) For any r for $Z = \mathcal{S}_{\bar{\rho}}$ and $\ell > 2$ when X is a curve ([5]*);*
3) For $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$, $r = 2$ and $\bar{\rho}$ tame ([5]*).*

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Triple product p-adic L-functions, endoscopy and rational points on ellliptic curves

Massimo Bertolini
(joint work with Marco A. Seveso, Rodolfo Venerucci)

Let $E/\mathbb{Q}$ be an elliptic curve over the rational numbers, which throughout this abstract is assumed to be semistable for simplicity. Let $\varrho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_n(\bar{\mathbb{Q}})$ be an n-dimensional Artin representation, and let $L(E, \varrho, s)$ denote the L-series of E twisted by ϱ. The equivariant Birch and Swinnerton-Dyer conjecture states that the vanishing of $L(E, \varrho, s)$ at $s = 1$ implies the existence of a non-trivial point in the ϱ-isotypic component $E(\bar{\mathbb{Q}})^\varrho$ of $E(\bar{\mathbb{Q}})$. This talk considers representations ϱ of the form $\varrho_1 \otimes \varrho_2$, where ϱ_i is a 2-dimensional *odd* Artin representation and the *self-duality* assumption $\det(\varrho_1) = \det(\varrho_2)^{-1}$ is satisfied. In this setting, the modularity theorems of Wiles, Taylor-Wiles and Khare-Wintenberger guarantee the existence of modular forms f, g and h of respective weights 2, 1 and 1, such that $L(E, s) = L(f, s)$, $L(\varrho_1, s) = L(g, s)$ and $L(\varrho_2, s) = L(h, s)$. To fix ideas say that – in addition to f – g and h are also *cuspidal*. Note that $L(E, \varrho, s)$ is equal to the triple product L-function $L(f \otimes g \otimes h, s)$, and admits an analytic continuation to the whole of $\mathbb{C}$ as well as a functional equation relative to $s \longmapsto 2 - s$. The previous considerations motivate the next assumption:

Assumption 1. $L(f \otimes g \otimes h, 1) = 0$.

The above setting encompasses the following

Classical setting. Let K be a quadratic imaginary field and let η_g, η_h be ray class characters of K, which do not arise from Dirichlet characters and have inverse central characters. Set $\varrho_1 = \mathrm{Ind}_K^{\mathbb{Q}}(\eta_g)$ and $\varrho_2 = \mathrm{Ind}_K^{\mathbb{Q}}(\eta_h)$. These Artin representations arise from weight one theta-series g and h respectively. Note that

$$\varrho = \varrho_1 \otimes \varrho_2 = \mathrm{Ind}_K^{\mathbb{Q}}(\varphi) \oplus \mathrm{Ind}_K^{\mathbb{Q}}(\psi)$$

for $\varphi = \eta_g \eta_h$ and $\psi = \eta_g \eta_h^c$, where η_h^c is the Hecke character obtained by composing the complex conjugation on fractional ideals of K with η_h. It follows that φ and ψ are ring class characters of K, for which the factorisations

$$L(E, \varrho, s) = L(E/K, \varphi, s) \cdot L(E/K, \psi, s), \qquad E(\bar{\mathbb{Q}})^{\varrho} = E(\bar{\mathbb{Q}})^{\varphi} \oplus E(\bar{\mathbb{Q}})^{\psi}$$

hold. Assumption 1 implies, say, that $L(E/K, \varphi, 1) = 0$. When $L(E/K, \varphi, s)$ has a simple zero at $s = 1$, the Gross-Zagier formula guarantees the existence of a non-trivial *Heegner point* in $E(\bar{\mathbb{Q}})^{\varphi}$ and hence also in $E(\bar{\mathbb{Q}})^{\varrho}$. A construction of a non-trivial rational point in $E(\bar{\mathbb{Q}})^{\varphi}$ is not known in general. Let K_φ be the ring class field extension of K cut out by φ and let p be an ordinary prime for E. When $L(E/K, \varphi, s)$ vanishes to odd order at $s = 1$, the family of Heegner points defined over the anticyclotomic p-adic tower above K_φ can be descended to produce a non-trivial class in a suitable p-adic Selmer group $S_p(E/\bar{\mathbb{Q}})^{\varphi}$ containing $E(\bar{\mathbb{Q}})^{\varphi}$. When $L(E/K, \varphi, s)$ vanishes to even order at $s = 1$ the existence of a non-trivial class in $S_p(E/\bar{\mathbb{Q}})^{\varphi}$ follows from (a suitable version of) the anticyclotomic Main Conjecture for E over the ring class field K_φ.

Remarks 2.
1) The construction of Selmer classes alluded to above points to the method of p-adic deformations, which will be further pursued in the rest of this abstract.
2) The proof of the anticyclotomic Main Conjecture is heavily based on the Jacquet-Langlands correspondence. This can be viewed as an instance of the use of endoscopy in the arithmetic study of elliptic curves.

Returning to the general setting of a triple of cuspidal eigenforms (f, g, h) of weight $(2, 1, 1)$, let $(\underline{f}, \underline{g}, \underline{h})$ be the triple of Coleman p-adic families interpolating (f, g, h). Write (f_k, g_ℓ, h_m) for the specialisation of $(\underline{f}, \underline{g}, \underline{h})$ at a classical triple of weights (k, ℓ, m). Let Σ^f be the region of classical triples (k, ℓ, m) for which $k \geq \ell + m$ and define Σ^g, Σ^h similarly. Moreover, let Σ^{bal} be the complement of $\Sigma^f \cup \Sigma^g \cup \Sigma^h$ in the region of classical triples. The generic sign $\epsilon(k, \ell, m)$ of the functional equation for the triple-product L-function $L(f_k \otimes g_\ell \otimes h_m, s)$ is constant on the *balanced* region Σ^{bal} and on the union $\Sigma^f \cup \Sigma^g \cup \Sigma^h$ of the 3 *unbalanced* regions, and changes when moving from the balanced region to any of the unbalanced ones. One is thus led to define the *indefinite case*, in which $\epsilon(k, \ell, m)$ is -1 on Σ^{bal} and $+1$ on its complement, and the *definite case*, in which the signs are opposite to those of the indefinite case. (This terminology is motivated by the fact that the relevant special values are described in terms of modular forms arising from indefinite and definite quaternion algebras, respectively.)

Indefinite case. Choose Σ^f as the region of classical interpolation. One may define a p-adic L-function $L_p(\underline{f},\underline{g},\underline{h})(k,\ell,m)$, which interpolates the square-root of the algebraic part of the central critical values $L(f_k\otimes g_\ell\otimes h_m,(k+\ell+m)/2)$. See [8] for the precise definition. The point $(2,1,1)$ belongs to Σ^f and therefore the behaviour of $L_p(\underline{f},\underline{g},\underline{h})$ at $(2,1,1)$ is expected to be related to a p-adic regulator by a p-adic Bloch-Kato conjecture, generalising the Birch and Swinnerton-Dyer conjecture. This line of research is undertaken in [1]; see also [6] for a special case. In a restricted setting, one obtains the following explicit formula.

Theorem 3. *Assume that (f,g,h) belongs to the classical setting above. Let p be multiplicative prime for E which is inert in K. Under the additional assumptions of* [2] *(to which we refer for details) the second partial derivative with respect to k of $L_p(\underline{f},\underline{g},\underline{h})$ at $(2,1,1)$ is equal to $\log_p(P^\varphi)\cdot\log_p(P^\psi)$, where P^φ resp. P^ψ is a Heegner point in $E(\bar{\mathbb{Q}})^\varphi$ resp. $E(\bar{\mathbb{Q}})^\psi$ and $\log_p$ denotes the formal group logarithm on E.*

Remarks 4.
1) When K is replaced by a *real* quadratic field and P^φ, P^ψ are replaced by so-called Stark-Heegner points, a similar formula to that of Theorem 3 is obtained in [2] and [7].
2) By an explicit reciprocity law in the sense of Perrin-Riou, $L_p(\underline{f},\underline{g},\underline{h})$ is realised as the big logarithm of the p-adic Abel-Jacobi image of a family of diagonal cycles defined in the "geometric region" Σ^{bal}. In particular it follows that in the Heegner or Stark-Heegner case $\log_p(P^\varphi)\cdot\log_p(P^\psi)$ arises as a p-adic limit of diagonal classes.
3) The starting point in the proof of the reciprocity law mentioned above is a formula of Harris-Kudla, which for $(k,\ell,m)\in\Sigma^f$ relates the central critical value $L(f_k\otimes g_\ell\otimes h_m,(k+\ell+m)/2)$ to the quantity $(f_k,\delta(g_\ell)\cdot h_m)$, where $(\ ,\ )$ is the Petersson inner product on weight k forms and δ is a suitable differential operator. This expression lends itself to p-adic interpolation, and admits a geometric interpretation in terms of diagonal cycles at points in Σ^{bal}.

Definite case. Since the sign $\epsilon(2,1,1)$ is equal to -1, this setting is somewhat more natural for the task of constructing rational points on E. The p-adic L-function $L_p(\underline{f},\underline{g},\underline{h})$ arises from the region of classical interpolation Σ^{bal} as in [8]. Since the point $(2,1,1)$ does not belong to Σ^{bal}, the behaviour of this p-adic L-function is not expected to be governed by a direct p-adic variant of the Birch and Swinnerton-Dyer conjecture. Assume here for simplicity that ϱ is irreducible.

Conjecture 5 ((See [3]))**.** *There is a canonical multiple $L_p^{\mathrm{can}}(\underline{f},\underline{g},\underline{h})$ of $L_p(\underline{f},\underline{g},\underline{h})$, whose value at $(2,1,1)$ is equal to the formal group logarithm of a rational point P^ϱ in $E(\bar{\mathbb{Q}})^\varrho$. Moreover, P^ϱ is non-trivial if and only if $L(E,\varrho,1)$ has a simple zero at $s=1$.*

The above conjecture has been verified in various instances of the classical setting, building on the main theorem of [5]. In the general setting, the following result of [4] provides evidence for the conjecture.

Theorem 6. *Assume that* $L_p(\underline{f}, \underline{g}, \underline{h})$ *does not vanish at* $(2, 1, 1)$*. Then* $S_p(E/\bar{\mathbb{Q}})^{\varrho}$ *is non-zero.*

Remark 7. The opening gambit in the proof of Theorem 6 is the p-adic interpolation of the following formula of Böcherer-Furusawa-Schulze-Pillot. For (k, ℓ, m) and (k', ℓ, m) in Σ^{bal} the product

$$L(f_k \otimes g_\ell \otimes h_m, (k + \ell + m)/2) \cdot L(f_{k'} \otimes g_\ell \otimes h_m, (k' + \ell + m)/2)$$

is related to a Petersson product $(\delta(\vartheta(f_k, f_{k'})_{|\mathcal{H}\times\mathcal{H}}), g_\ell, h_m)$, where $\vartheta(f_k, f_{k'})$ is a Yoshida (endoscopic) lift of the pair $(f_k, f_{k'})$ to the genus 2 Siegel half space $\mathcal{H}_2$, $|\mathcal{H} \times \mathcal{H}$ denotes restriction to two copies of the usual upper half plane $\mathcal{H}$ diagonally embedded in $\mathcal{H}_2$, and δ is an appropriate differential operator. This opens the way to a geometric interpretation of $L_p(\underline{f}, \underline{g}, \underline{h})(2, 1, 1)$ as a p-adic limit of classes arising in the cohomology of the Siegel threefold attached to GSp_4.

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Fields of definitions of Hodge loci

Bruno Klingler

The purpose of this talk was to explain the results on fields of definition of Hodge loci obtained in [KOU].

Let $(\mathbb{V}, \mathcal{V}, F^\bullet, \nabla)$ be a variation of $\mathbb{Z}$-Hodge structure ($\mathbb{Z}$VHS) on a smooth complex irreducible algebraic variety S. Recall this means the following: $\mathbb{V}$ is a finite rank $\mathbb{Z}_{S^{\mathrm{an}}}$-local system on the complex manifold S^{an} associated to S; and the holomorphic module with integrable connection $(\mathcal{V}^{\mathrm{an}} := \mathbb{V} \otimes_{\mathbb{Z}_{S^{\mathrm{an}}}} \mathcal{O}_{S^{\mathrm{an}}}, \nabla^{\mathrm{an}})$ on S^{an} associated to $\mathbb{V}$ by the Riemann-Hilbert correspondence is endowed with a decreasing filtration $(F^\bullet)^{\rm an}$ of $D_{S^{\rm an}}$-modules such that for each $s \in S^{\rm an}$ the filtration $F^\bullet_s$ on $\mathbb{V}_s$ is the Hodge filtration of a pure Hodge structure. Following Deligne [De70, Theor.5.9] there exists a unique *algebraic* module with regular integrable connection $(\mathcal{V}, \nabla)$ whose analytification is $(\mathcal{V}^{\rm an}, \nabla^{\rm an})$. We moreover assume $\mathbb{V}$, and all $\mathbb{Z}$VHSs in this paper, to be polarizable. In this case there exists a unique filtration $F^\bullet$ on the D_S-module $(\mathcal{V}, \nabla)$ whose analytification provides $(F^\bullet)^{\rm an}$, see [Sc73, (4.13)].

A typical example of such a $\mathbb{Z}$VHS, referred to as "the geometric case", is $(\mathbb{V} := R^{2k} f^{\rm an}_* \mathbb{Z}(k), \mathcal{V} := R^{2k} f_* \Omega^\bullet_{X/S}, F^\bullet, \nabla)$ associated to a smooth projective morphism of smooth irreducible complex quasi-projective varieties $f : X \to S$. In this case the Hodge filtration $F^\bullet$ is induced by the stupid filtration on the algebraic De Rham complex $\Omega^\bullet_{X/S}$ and ∇ is the Gauß-Manin connection.

From now on we abbreviate the $\mathbb{Z}$VHS $(\mathbb{V}, \mathcal{V}, F^\bullet, \nabla)$ simply by $\mathbb{V}$. Let $\mathbb{V}^\otimes$ be the infinite direct sum of $\mathbb{Z}$VHS $\bigoplus_{a,b \in \mathbb{N}} \mathbb{V}^{\otimes a} \otimes (\mathbb{V}^\vee)^{\otimes b}$, where $\mathbb{V}^\vee$ denotes the $\mathbb{Z}$VHS dual to $\mathbb{V}$. The (tensorial) Hodge locus $\mathrm{HL}(S, \mathbb{V}^\otimes)$ is the subset of points $s \in S^{\rm an}$ for which the Hodge structure $\mathbb{V}_s$ admits more Hodge tensors than the very general fiber $\mathbb{V}_{s'}$. Following Deligne [De72, 7.5] this is a meager subset of $S^{\rm an}$. While a priori $\mathrm{HL}(S, \mathbb{V}^\otimes)$ has no nice geometric feature, in the geometric case the Hodge conjecture easily implies that $\mathrm{HL}(S, \mathbb{V}^\otimes)$ is a countable union of closed irreducible algebraic subvarieties of S, see [Weil79]. Remarkably, Cattani, Deligne and Kaplan [CDK95] (see also [BKT18] for a simplified proof) proved unconditionnally that for any $\mathbb{Z}$VHS $\mathbb{V}$ on S the Hodge locus $\mathrm{HL}(S, \mathbb{V}^\otimes)$ is a countable union of irreducible algebraic subvarieties of S, called the strict special subvarieties of S for $\mathbb{V}$ (or sometimes "the irreducible components of the Hodge locus $\mathrm{HL}(S, \mathbb{V}^\otimes)$ "). A special subvariety of dimension zero is called a special point. We refer to [Voi13] for a survey on Hodge loci.

Let us now turn to fields of definitions of special subvarieties. In the geometric case, let us suppose that the morphism $f : X \to S$ is defined over a number field K. In that case the filtered algebraic D_S-module $(\mathcal{V}, F^\bullet, \nabla)$ is also defined over K. Again, the Hodge conjecture is easily seen to imply that each special subvariety Y of S for $\mathbb{V}^\otimes$ is defined over a finite extension of K and that each of the finitely many $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-conjugates of Y is a special subvariety of S for $\mathbb{V}$. In fact this follows from the weaker conjecture that Hodge classes are absolute Hodge, see [ChSc14, 3.5].

Let us say that a general $\mathbb{Z}$VHS $\mathbb{V}$ is *defined over a number field* $K \subset \mathbb{C}$ if S, $\mathcal{V}$, $F^\bullet$ and ∇ are defined over K: $S = S_K \otimes_K \mathbb{C}$, $\mathcal{V} = \mathcal{V}_K \otimes_K \mathbb{C}$, $F^\bullet \mathcal{V} = (F^\bullet_K \mathcal{V}_K) \otimes_K \mathbb{C}$ and $\nabla = \nabla_K \otimes_K \mathbb{C}$ with the obvious compatibilities. Following Simpson [Si90, "Standard conjecture" p.372], such a $\mathbb{Z}$VHS defined over a number field ought to be motivic: there should exist a $\overline{\mathbb{Q}}$-Zariski-open subset $U \subset S$ such that the restriction of $\mathbb{V}$ to U is a direct factor of a geometric $\mathbb{Z}$VHS on U. In particular Simpson's "standard conjecture" and the remark above concerning the geometric case implies:

Conjecture 1. *Any special subvariety associated to a $\mathbb{Z}$VHS defined over K is defined over a finite extension L of K and its finitely many* $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$*-conjugates are still special subvarieties of S for $\mathbb{V}$.*

For simplicity of notations, we will refer to this situation by saying that $\mathbb{V}$ is defined over $\overline{\mathbb{Q}}$, that special subvarieties are defined over $\overline{\mathbb{Q}}$ and that their Galois conjugates are special subvarieties.

Let us mention the few results in the direction of Conjecture 1 we are aware of:

(1) In [Voi07, Theor. 0.6 (ii)] (see also [Voi13, Theor. 7.8]) Voisin proves that if $f : X \to S$ is defined over $\mathbb{Q}$ and if the special subvariety $Y \subset S$ defined by a Hodge class $\alpha \in H^{2k}(X_0, \mathbb{Z}(k))$ satisfies that any locally constant Hodge substructure $L \subset H^{2k}(X_y, \mathbb{Z}(k))$, $y \in Y^{\mathrm{an}}$, is purely of type $(0, 0)$ then Y is defined over $\overline{\mathbb{Q}}$ and its $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-translates are still special subvarieties of S for $\mathbb{V}^{\otimes}$.

(2) In [SaSc16] Saito and Schnell prove that for any $\mathbb{Z}$VHS defined over $\overline{\mathbb{Q}}$ a special subvariety is defined over $\overline{\mathbb{Q}}$ if (and only if) it contains a single $\overline{\mathbb{Q}}$-point of $S_{\overline{\mathbb{Q}}}$; this generalizes the well-known fact that the special subvarieties of Shimura varieties are defined over $\overline{\mathbb{Q}}$ (as any special subvariety of a Shimura variety contains a CM-point, and CM-points are defined over $\overline{\mathbb{Q}}$).

Notice that both Voisin's and Saito-Schnell's criteria are difficult to check in practice as one usually knows very little about the geometry of a special variety Y: in Voisin's case one would need to control the Hodge structure on the cohomology of a smooth compactification of $X_{|Y}$; in Saito-Schnell's case, there is no natural candidate for points over $\overline{\mathbb{Q}}$.

In [KOU] we provide a geometric criterion for a special subvariety of a $\mathbb{Z}$VHS $\mathbb{V}$ defined over $\overline{\mathbb{Q}}$ to satisfy Conjecture 1. Let us first recall the notion of algebraic monodromy group. Let S be a smooth irreducible complex algebraic variety and $\mathbb{V}$ a local system on S^{an}. Given an irreducible closed subvariety $Y \subset S$, a natural invariant attached to Y and $\mathbb{V}$ is the algebraic monodromy group $\mathbf{H}_Y$ of Y for $\mathbb{V}$: the connected component of the Tannaka group of the category $\langle \mathbb{V}_{|Y^{\mathrm{nor}}} \rangle^{\otimes}_{\mathbb{Q}\mathrm{Loc}}$ of local systems on (the normalisation of) Y tensorially generated by the restriction of $\mathbb{V}$ and its dual; equivalently the connected component of the Zariski-closure of the monodromy of the local system $\mathbb{V}_{|Y^{\mathrm{nor}}}$.

Definition 2. Let S be a smooth irreducible complex algebraic variety and $\mathbb{V}$ a local system on S^{an}. Let $Y \subset S$ be an irreducible closed subvariety. We say that Y is *weakly non-factor* for $\mathbb{V}$ if it is not contained in a closed irreducible $Z \subset S$ such that $\mathbf{H}_Y$ is a strict normal subgroup of $\mathbf{H}_Z$.

Our main result in this paper is the following:

Theorem 3. *Let $\mathbb{V}$ be a polarized variation of $\mathbb{Z}$-Hodge structure on a smooth quasi-projective variety S. Suppose that $\mathbb{V}$ is defined over $\overline{\mathbb{Q}}$. Then:*

(1) any special subvariety of S for $\mathbb{V}$ which is weakly non-factor is defined over $\overline{\mathbb{Q}}$;
(2) its Galois-translates are special subvarieties of S for $\mathbb{V}$.

As an explicit corollary we obtain:

Corollary 4. *Let $\mathbb{V}$ be a polarized variation of $\mathbb{Z}$-Hodge structure on a smooth quasi-projective variety S. Suppose that $\mathbb{V}$ is defined over $\overline{\mathbb{Q}}$ and that its adjoint generic Mumford-Tate group $\mathbf{G}_S^{\mathrm{ad}}$ is simple. Then any strict special subvariety $Y \subset S$ for $\mathbb{V}$, which is positive dimensional for $\mathbb{V}$ and maximal for these properties, is defined over $\overline{\mathbb{Q}}$, and its Galois-translates are special subvarieties of S for $\mathbb{V}$.*

Theorem 3 also enables to reduce the full Conjecture 1 to the case of points:

Corollary 5. *Special subvarieties for $\mathbb{Z}$VHSs defined over $\overline{\mathbb{Q}}$ are defined over $\overline{\mathbb{Q}}$ if and only if it holds true for special points. Similarly their Galois-translates are special subvarieties if and only if it holds true for special points.*

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Geometric quadratic Chabauty

Bas Edixhoven

(joint work with Guido Lido)

This is about joint work with Guido Lido, which started in 2018. It was announced in Oberwolfach Report 34/2018. The aim of this work is to give a relatively simple geometric description of the cohomological quadratic Chabauty method developed by Balakrishnan, Besser, Dogra, Müller, Tuitman and Vonk. This approach gives a method that probably (but not yet provably) determines if a given list of rational points on a given curve of genus g at least two Picard number ρ and Mordell Weil rank r satisfying $r < g+\rho-1$ is complete. It avoids all p-adic (non-abelian) Hodge theory, and works in terms of $\mathbb{Z}/p^2$-valued points of the curve, its jacobian and its Poincaré bundle.

By now there are good references for it: the preprint [1], 4 lectures by Edixhoven at the Arizona Winter School [2], 1 lecture in the Number Theory Web Seminar [3].

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`https://sites.google.com/view/ntwebseminar/previous-talks?authuser=0`

Cohomology of p-adic analytic spaces

Pierre Colmez

(joint work with Wiesława Nizioł)

Let K be a field complete for v_p (discrete), and let C be the completion of its algebraic closure, $\breve{C} = W(k_C)[\frac{1}{p}]$, where k_C is the residue field of C. Let G_K be the absolute Galois group of K.

Theorem 1. *Let Y/K be a smooth, geometrically connected, dagger analytic space. If $i \leq r$, we have a bicartesian diagram of G_K-modules:*

$$\begin{array}{ccc} H^i_{\text{proet}}(Y_C, \mathbb{Q}_p(r)) & \longrightarrow & (\mathbf{B}^+_{\text{st}} \otimes H^i_{\text{HK}}(Y))^{N=0,\varphi=p^r} \\ \downarrow & & \downarrow \\ H^i(\text{Fil}^r(\mathbf{B}^+_{\text{dR}} \otimes \text{R}\Gamma_{\text{dR}})) & \longrightarrow & \mathbf{B}^+_{\text{dR}} \otimes H^i_{\text{dR}}(Y) \end{array}$$

Note:

- Dagger analytic spaces include analytifications of algebraic varieties, overconvergent affinoids, étale coverings of Drinfeld's symmetric spaces or, more generally, analytic spaces with no boundary.

- The theorem gives a description of the proétale cohomology in terms of de Rham data which is more amenable to computations. In the realm of algebraic varieties (and with étale cohomology in place of proétale), theorems of this type have a long history, starting with the formulation of the conjectures $C_{\rm cris}$, $C_{\rm st}$, $C_{\rm dR}$ of Fontaine (refining the conjecture of Tate on the existence of a Hodge-like decomposition for étale cohomology), and their ensuing proofs by Fontaine-Messing, Kato, Hyodo-Kato, Faltings, Tsuji, Nizioł, Yamashita, Scholze, etc., the most general result being that of Beilinson [1] with no asumption on the existence of good models nor, even, smoothness.
- In the theorem, the Hyodo-Kato cohomology group $H^i_{\rm HK}(Y)$ is a $\breve{C}$-module with a frobenius φ, a monodromy operator N, a (pro)smooth action of G_K, and an isomorphism $\iota_{\rm HK}: C\otimes_{\breve{C}} H^i_{\rm HK}(Y)\xrightarrow{\sim} H^i_{\rm dR}(Y)$. The definition [4, 5] of $H^i_{\rm HK}(Y)$ and $\iota_{\rm HK}$ is a big part of the theorem; it is adapted from Beilinson's and uses the alterations of Hartl and Temkin to produce good models (locally).
- In the case of algebraic varieties (or proper analytic ones), all cohomology groups in the diagram are finite dimensional (as in [3]) and the kernels of the horizontal arrows are 0. This is not the case for a general analytic variety and the tensor products are (derived) completed tensor products. Even if $H^i_{\rm dR}(Y)$ is finite dimensional, $H^i({\rm Fil}^r(\mathbf{B}^+_{\rm dR}\otimes {\rm R}\Gamma_{\rm dR}))$ surjects onto $C\otimes\Omega^r(Y)^{d=0}$ and hence can be huge (and so is $H^i_{\rm proet}(Y_C,\mathbb{Q}_p(r))$).
- In the Stein case or for an overconvergent affinoid, the horizontal arrows are surjective and their kernels are isomorphic to $(C\otimes\Omega^{r-1}(Y))/{\rm Ker}\, d$ (as in [2]).
- The general case reduces to the quasi-compact case, and then uses an induction on the number of affinoids needed to cover the space. This induction uses fine properties of the category of BC's (BC stands for Banach-Colmez).
- Using results of Fontaine of the type ${\rm Hom}_{G_K}(\mathbf{B}^+_{\rm dR}/t^N,\mathbf{B}_{\rm dR})=0$, one can recover $H^i_{\rm HK}(Y)$ and $H^i_{\rm dR}(Y)$ from $H^i_{\rm proet}(Y_C,\mathbb{Q}_p)$. For example, we have

$$H^i_{\rm dR}(Y)^* = {\rm Hom}_{G_K}(H^i_{\rm proet}(Y_C,\mathbb{Q}_p),\mathbf{B}_{\rm dR}).$$

- If we don't assume Y to be defined over K, one can still prove the above results but this requires (in progress) to promote the diagram to a diagram of BC's (slightly generalized since the spaces involved do not satisfy the finiteness conditions required in the definition of BC's).

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On Uniform Bounds for the Number of Rational Points on a Curve

Philipp Habegger
(joint work with Vesselin Dimitrov and Ziyang Gao)

The following theorem is due to Faltings and is also known as the Mordell Conjecture.

Theorem 1 ([5]). *Let F be a number field and C a smooth, geometrically irreducible projective curve of genus $g \geq 2$ defined over F. Then the set $C(F)$ of F-rational points of C is finite.*

In the following we abbreviate smooth, geometrically irreducible projective curve by *smooth curve.*

Faltings's Theorem leads to the following questions for a smooth curve C of genus $g \geq 2$ defined over a number field F:

- Suppose C is presented as the solution set in some projective space of finitely many homogeneous polynomial equations. Does there exist an algorithm that produces a complete list of points in $C(F)$?
- Can one bound the cardinality $\#C(F)$ from above in terms of suitable and known invariants of C?

As of today, the answer to the first question is unknown. The second question can be interpreted in many ways as the nature of the invariants is not specified. But the answer is yes for many, and possibly all reasonable, interpretations. For example, shortly after Faltings's work, Parshin [12] presented a first upper bound for $\#C(F)$.

Later, Vojta [13] gave a new proof of the Mordell Conjecture based on ideas from diophantine approximations. This proof led to many developments towards the second question. Indeed, Vojta gave an upper bound for $\#C(F)$ in his original paper [13, Theorem 6.1]. We cite here a later result of Rémond. Let $\overline{F}$ be an algebraic closure of F and let $\mathrm{Jac}(C)$ denote the Jacobian of C. Recall that $\mathrm{Jac}(C)(F)$ is a finitely generated abelian group by the Mordell–Weil Theorem.

Theorem 2 ([10]). *There exists $c(\mathrm{Jac}(C)_{\overline{F}}) \geq 1$ which depends only on $\mathrm{Jac}(C)_{\overline{F}}$ with*

$$\#C(F) \leq c(\mathrm{Jac}(C)_{\overline{F}})^{1+\mathrm{rk}(\mathrm{Jac}(C)(F))}.$$

We also mention earlier work of Silverman [11] on uniformity on the number of rational points on twists of C.

In my talk, I presented joint work with Dimitrov and Gao on an estimate in the spirit of Theorem 2.

Theorem 3 ([3])**.** *There exists $c([F:\mathbb{Q}],g)\geq 1$ which depends only on the degree $[F:\mathbb{Q}]$ and $g\geq 2$ such that if C is a smooth curve of genus g defined over F, then*

$$\#C(F)\leq c([F:\mathbb{Q}],g)^{1+\mathrm{rk}(\mathrm{Jac}(C)(F))}.$$

The main feature is that $c([F:\mathbb{Q}],g)$ does not depend on the moduli of C. Our result answers affirmatively a question of Mazur [9], see also the earlier mention [8].

In earlier work, David and Philippon [1] obtained a completely explicit estimate of the same quality when the curve is immersed in a power of an elliptic curve. Here, the moduli of the elliptic curve does not appear in the bound. Later, Dimitrov, Gao, and myself obtained [4] a precursor to Theorem 3 where we considered C arising in a 1-parameter family of smooth curves.

Let us indicate some of the ingredients of the proof of Theorem 3.

Our approach follows the strategy laid out by Vojta [13] in his proof of the Mordell Conjecture.

Let C be as in the theorem and let us make the (harmless) assumption that $C(F)$ is non-empty. Then the Abel–Jacobi map based at a fixed F-rational point of C defines a closed immersion $C\to\mathrm{Jac}(C)$. The Jacobian $\mathrm{Jac}(C)$ comes equipped with a principal polarization to which we can attach the Néron–Tate height $\hat{h}\colon\mathrm{Jac}(C)(\overline{F})\to[0,\infty)$. It satisfies the following properties. First, the function $\hat{h}$ differs from a suitable Weil height by a bounded function on $\mathrm{Jac}(C)(\overline{F})$. By the Northcott Theorem, a subset of $\mathrm{Jac}(C)(F)$ on which the Néron–Tate height is bounded from above is finite. Second, $\hat{h}$ is a quadratic form, *i.e.*, $\langle P,Q\rangle=(\hat{h}(P+Q)-\hat{h}(P)-\hat{(}Q))/2$ is a bilinear form where $P,Q\in\mathrm{Jac}(C)(\overline{F})$. Third, $\hat{h}$ vanishes precisely on the set of points of finite order of $\mathrm{Jac}(C)(\overline{F})$. We can view $\hat{h}$ as the square of an Euclidean norm $\|\cdot\|$ on the $\mathbb{R}$-vector space $\mathrm{Jac}(C)(\overline{F})\otimes\mathbb{R}$. Observe that the subspace $\mathrm{Jac}(C)(F)\otimes\mathbb{R}$ is finite dimensional by the Mordell–Weil Theorem.

At the core of Vojta's approach is his deep height inequality. An important consequence is that there exist constants $c_1(C)\in(0,\infty)$ and $c_2(C)\in(0,1)$ with the following property. For each $y\in\mathrm{Jac}(C)(F)\otimes\mathbb{R}$ with $y\neq 0$ the truncated cone

$$\left\{x\in\mathrm{Jac}(C)(F)\otimes\mathbb{R}:\|x\|^2\geq c_1(C)\text{ and }\langle x,y\rangle\geq(1-c_2(C))\|x\|\|y\|\right\}$$

can afford at most finitely many points from $C(F)$. The finite dimensional vector space $\mathrm{Jac}(C)(F)\otimes\mathbb{R}$ is covered by a finite number of cones of given angle. So at most finitely many points in $C(F)$ have Néron–Tate height at least $c_1(C)$; these are called the *large points*. The set of *small points*, those points of Néron–Tate height less than $c_1(C)$, is finite by the Northcott Theorem.

To obtain an upper bound for $\#C(F)$ we need more information on $c_1(C)$ and $c_2(C)$. It turns out that $c_2(C)$ is absolute. A result of Mumford together with a ball packing argument implies that the number of large points is bounded from above solely in terms of $\mathrm{rk}(\mathrm{Jac}(C)(F))$; the dependency in the rank is exponential as in Theorems 2 and 3.

The value of $c_1(C)$, roughly speaking, depends on the height of the point in a suitable moduli space that corresponds to C, see work of de Diego [2]. So we cannot hope that a straightforward application of the Northcott Theorem leads to the desired uniformity. Rather we prove a height inequality which suggests that points on C cannot become too close to one another with respect to $\|\cdot\|$.

To formulate this inequality we let $\mathbb{M}_g$ denote the fine moduli space of smooth curves of genus g with suitable level structure. This is an irreducible quasi-projective variety on which we can fix a Weil height. We can realize C as a fiber of the universal family $\mathcal{M}_g$ above $\mathbb{M}_g$, after a finite field extension of F. We write $h([C])$ for the Weil height of the $\overline{F}$-point on $\mathbb{M}_g$ below C.

Let $m = 3g-2$. Our height inequality amounts to the following statement. There exists a Zariski closed and proper subset Z in the $(m+1)$-fold fibered power $\mathcal{M}_g^{[m+1]} = \mathcal{M}_g \times_{\mathbb{M}_g} \cdots \times_{\mathbb{M}_g} \mathcal{M}_g$ and constants $c_3 = c_3(g) > 0, c_4 = c_4(g) \geq 0$ with the following property. If each entry of $(P_0, P_1, \ldots, P_m) \in (\mathcal{M}_g^{[m+1]} \smallsetminus Z)(\overline{F})$ lies on the fiber C of $\mathcal{M}_g \to \mathbb{M}_g$, then

$$\max_{1\leq j\leq m} \|P_j - P_0\|^2 \geq c_3 h([C]) - c_4. \tag{1}$$

This inequality together with other tools such as a ball packing argument and a careful analysis of $c_1(C)$, which happens to be linear in $h([C])$, ultimately leads to an upper bound on the number of small points and Theorem 3. To prove (1) we require a functional transcendence result of Gao [6] in the spirit of the Ax–Schanuel Theorem. An earlier height inequality for a one-parameter family of abelian varieties, used in [4], was proved by Gao and myself in [7].

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On the ordinary part of coherent cohomology of Hilbert modular varieties

George Boxer
(joint work with Vincent Pilloni)

1. Motivation: the case of the modular curve

Let p be a prime. We first recall the notion of the ordinary part in the sense of Hida [5]. If M is a finitely generated $\mathbb{Z}_p$-module and T is an endomorphism of M, then there is a unique T-stable decomposition $M = eM \oplus (1-e)M$ so that T is an isomorphism on eM and T is topologically nilpotent on $(1-e)M$. The idempotent e is called the ordinary projector associated to T, and eM is called the ordinary part of M for T.

Now let $\mathfrak{X}/\mathbb{Z}_p$ be the (compactified) modular curve of level $\Gamma_1(N)$ for $N \geq 3$ prime to p. Let $\omega/\mathfrak{X}$ be the modular line bundle, defined as $e^*\Omega^1_{\mathcal{E}/\mathfrak{X}}$ where e is the identity section of the universal generalized elliptic curve $\mathcal{E} \to \mathfrak{X}$. For an integer $k \in \mathbb{Z}$, the line bundle ω^k is the sheaf of modular forms of weight k. Let $X = \mathfrak{X}_{\mathbb{F}_p}$ be the special fiber, and let $X^{\mathrm{ord}} \subset X$ be the ordinary locus.

Then we recall the following "Hida control theorem" proved in [1]. The first case that $k \geq 3$ is a classical theorem of Hida [5], which is itself a reinterpretation of a theorem of Jochnowitz [6].

Theorem 1. *There is a normalized Hecke operator T_p acting on $\mathrm{R}\Gamma(X, \omega^k)$, $\mathrm{R}\Gamma(X^{\mathrm{ord}}, \omega^k)$, and $\mathrm{R}\Gamma_c(X^{\mathrm{ord}}, \omega^k)$ for all $k \in \mathbb{Z}$, and we have isomorphisms*

$$e\mathrm{R}\Gamma(X, \omega^k) \simeq e\mathrm{R}\Gamma(X^{\mathrm{ord}}, \omega^k)$$

for $k \geq 3$, and

$$e\mathrm{R}\Gamma(X, \omega^k) \simeq e\mathrm{R}\Gamma_c(X^{\mathrm{ord}}, \omega^k)$$

for $k \leq -1$.

To explain the notation let $SS \subset X$ be the supersingular divisor. Then

$$H^i(X^{\mathrm{ord}}, \omega^k) = \varinjlim_n H^i(X, \omega^k(nSS))$$

and

$$H^i_c(X^{\mathrm{ord}}, \omega^k) = \varprojlim_n H^i(X, \omega^k(-nSS)).$$

This notion of coherent cohomology with compact support is due to Hartshorne [4].

This "control theorem" also implies a vanishing theorem. Indeed by the affiness of X^{ord}, for all $k \in \mathbb{Z}$ we have that $\mathrm{R}\Gamma(X^{\mathrm{ord}}, \omega^k)$ is concentrated in degree 0 and $\mathrm{R}\Gamma_c(X^{\mathrm{ord}}, \omega^k)$ is concentrated in degree 1. The resulting vanishing theorem for

$e\mathrm{R}\Gamma(X, \omega^k)$ is not particularly interesting, because one can prove easily using Riemann-Roch that $H^1(X, \omega^k) = 0$ for $k \geq 3$ (even without applying the ordinary projector.) Nonetheless we believe this method of proving vanishing results can be generalized to other Shimura varieties.

2. The case of Hilbert modular varieties

From now on let $F/\mathbb{Q}$ be a totally real number field in which the prime p is totally inert (we assume this for simplicity, one can also treat the case that p is unramified in F.) Let $d = [F : \mathbb{Q}]$.

Let $\mathfrak{X}/\mathbb{Z}_{p^d}$ be a (toroidally compactified) Hilbert modular variety of some neat level prime to p. It is a Shimura variety (of abelian type) for the group $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$, and (away from the boundary) it can also be viewed as a coarse moduli space of abelian schemes A/S with an action of $\mathcal{O}_F$ making $\mathrm{Lie}_{A/S}$ into a free $\mathcal{O}_S \otimes \mathcal{O}_F$-module of rank 1. $\mathfrak{X}/\mathbb{Z}_{p^d}$ is smooth of relative dimension d. We write $D \subset \mathfrak{X}$ for the boundary divisor and $X = \mathfrak{X}_{\mathbb{F}_{p^d}}$ for the special fiber.

By a weight we mean a tuple of integers $\kappa = ((k_\tau)_{\tau \in \mathrm{Hom}(F, \mathbb{Q}_{p^d})}; k)$ where the k_τ and k all have the same parity. Associated to κ is a modular line bundle $\omega^\kappa/\mathfrak{X}$ (which we remark is independent of k.) We say that κ is strictly regular if $k_\tau \neq 0, 1$, or 2 for all $\tau \in \mathrm{Hom}(F, \mathbb{Q}_{p^d})$. We then write

$$I(\kappa) = \{\tau \in \mathrm{Hom}(F, \mathbb{Q}_{p^d}) \mid k_\tau < 1\}.$$

Fakhruddin and Pilloni [2] have constructed a Hecke operator T_p acting on $\mathrm{R}\Gamma(\mathfrak{X}, \omega^\kappa)$ and $\mathrm{R}\Gamma(\mathfrak{X}, \omega^\kappa(-D))$ for any weight κ. We would like to study the ordinary part for this Hecke operator.

Our first result is a vanishing theorem for this ordinary cohomology in strictly regular weight.

Theorem 2. *Let κ be a strictly regular weight and let $I = I(\kappa)$. Then:*

- *When $\#I = 0$, $e\mathrm{R}\Gamma(\mathfrak{X}, \omega^\kappa(-D))$ is concentrated in degree 0.*
- *When $0 < \#I < [F : \mathbb{Q}]$, $e\mathrm{R}\Gamma(\mathfrak{X}, \omega^\kappa) = e\mathrm{R}\Gamma(\mathfrak{X}, \omega^\kappa(-D))$ is concentrated in degree $\#I$.*
- *When $\#I = d$, $e\mathrm{R}\Gamma(\mathfrak{X}, \omega^\kappa)$ is concentrated in degree d.*

It can easily be seen that unlike the case of the modular curve, when $d > 1$ this theorem is false without first passing to ordinary parts. For example this follows from the existence of Hilbert modular forms in weights κ where some of the k_τ are negative, like the partial Hasse invariants introduced by Goren [3]. However one can expect this stronger vanishing after placing a stronger restriction on the weight κ, as for example is proved in the work of Lan and Suh [7].

We deduce this vanishing theorem from the following control theorem.

Theorem 3. *Let κ be a strictly regular weight and let $I = I(\kappa)$. Then there is an isomorphism*

$$e\mathrm{R}\Gamma(X, \omega^\kappa) \simeq \begin{cases} e\mathrm{R}\Gamma(C_I, \omega^{\kappa+\eta_I})[-\#I] & \text{if} \#I \leq \frac{d}{2} \\ e\mathrm{R}\Gamma_c(C_I, \omega^{\kappa+\eta_I})[\#I - d] & \text{if} \#I \geq \frac{d}{2} \end{cases}$$

In this theorem $C_I \subset X$ is an Oort leaf. It is the reduced, locally closed, subscheme for which $A[p^\infty]$ is geometrically isomorphic to $\mathrm{LT}_I \times \mathrm{LT}_{I^c}$, where $I^c = \mathrm{Hom}(F, \mathbb{Q}_{p^d}) - I$, and where the generalized Lubin-Tate group $\mathrm{LT}_I/\overline{\mathbb{F}}_p$ is the unique up to isomorphism p-divisible group of height d with $\mathcal{O}_F$-action such that

$$\mathrm{Lie}_{\mathrm{LT}_I} \simeq \bigoplus_{\tau \in I} \overline{\mathbb{F}}_p(\tau)$$

where $\overline{\mathbb{F}}_p(\tau)$ denotes $\overline{\mathbb{F}}_p$ with the $\mathcal{O}_F$ action via τ. The Oort leaf C_I is smooth and affine (except in the case that $I = \emptyset$ or $\mathrm{Hom}(F, \mathbb{Q}_{p^d})$, in which case $C_I = X^{\mathrm{ord}}$ is not affine because of the cusps) and it has dimension $|\#I - \#I^c|$. The weight shift η_I appearing in the theorem has a simple combinatorial description which we do not recall.

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On the irreducible case of Fargues' conjecture for GL_n

Arthur-César le Bras

(joint work with Johannes Anschütz)

1. Statement of the main result

Let p be a prime, and let $n \geq 1$. Let Perf be the category of perfectoid spaces over $\overline{\mathbb{F}}_p$. Set

$$\mathrm{Bun}_n$$

to be the small v-stack sending $S \in \mathrm{Perf}$ to the groupoid of vector bundles of rank n on the Fargues–Fontaine curve $X_{\mathrm{FF},S}$ relative to S (and the local field $\mathbb{Q}_p$[1]). There is a map $\mathrm{GL}_n(\breve{\mathbb{Q}}_p) \to \mathrm{Bun}_n(\overline{\mathbb{F}}_p)$, $b \mapsto \mathcal{E}_b$ inducing a bijection

$$B(\mathrm{GL}_n) := \mathrm{GL}_n(\breve{\mathbb{Q}}_p)/\sigma - \text{conjugacy} \cong |\mathrm{Bun}_n|.$$

[1] In all the talk, we fix the base local field to be $\mathbb{Q}_p$, for simplicity.

Moreover, the bundle $\mathcal{E}_b$ is semistable if and only if b basic. The (open) semistable locus in Bun_n has the description

$$\coprod_{d\in\mathbb{Z}}[*/\underline{G_b(\mathbb{Q}_p)}] \cong \coprod_{b\in B(\mathrm{GL}_n)\text{ basic}} \mathrm{Bun}_n^b = \mathrm{Bun}_n^{\mathrm{sst}} \overset{j}{\hookrightarrow} \mathrm{Bun}_n$$

with $\deg(\mathcal{E}_b) = d$, where Bun_n^b is the substack parametrizing vector bundles which are fiberwise on S isomorphic to $\mathcal{E}_b$, G_b is the σ-centralizer of b (an inner form of GL_n) and $\underline{G_b(\mathbb{Q}_p)}$ is the sheaf on Perf associated to the topological group $G_b(\mathbb{Q}_p)$.

Let also $\mathrm{Div}^1 = \mathrm{Spd}(\breve{\mathbb{Q}}_p)/\varphi^{\mathbb{Z}}$ be the moduli space of "relative Cartier effective divisors of X_{FF} of degree 1". For each prime ℓ, the category of finite dimensional continuous $\overline{\mathbb{Q}}_\ell$-representations of $W_{\mathbb{Q}_p}$ is equivalent to the category of finite rank $\overline{\mathbb{Q}}_\ell$-local systems on Div^1.

For applications to representation theory, one needs to speak about ℓ-adic sheaves on such geometric objects, for $\ell \neq p$. This is subtle (especially when one wants to deal with $\overline{\mathbb{Q}}_\ell$-coefficients), but is part of the work in progress of Fargues-Scholze, [3], which we will use here as a black box. For a small v-stack Y, they can define a certain full subcategory

$$D_{\mathrm{lis}}(Y, \overline{\mathbb{Q}}_\ell) \subseteq D(Y_v, \overline{\mathbb{Q}}_\ell)$$

A key property is that $D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ admits an infinite semi-orthogonal decomposition by the categories $D_{\mathrm{lis}}(\mathrm{Bun}_n^b, \overline{\mathbb{Q}}_\ell), b \in B(\mathrm{GL}_n)$, and that for each $b \in B(\mathrm{GL}_n)$, there are equivalences

$$\begin{array}{ccccc} D(\mathrm{Rep}^\infty_{\overline{\mathbb{Q}}_\ell} G_b(\mathbb{Q}_p)) & \cong & D_{\mathrm{lis}}([*/\underline{G_b(\mathbb{Q}_p)}], \overline{\mathbb{Q}}_\ell) & \cong & D_{\mathrm{lis}}(\mathrm{Bun}_n^b, \overline{\mathbb{Q}}_\ell) \\ \pi & & \mapsto & & \mathcal{F}_\pi \end{array}$$

with $\mathrm{Rep}^\infty_{\overline{\mathbb{Q}}_\ell} G_b(\mathbb{Q}_p)$ the category of *smooth* $\overline{\mathbb{Q}}_\ell$-representations of $G_b(\mathbb{Q}_p)$.

Here is the main result (in progress) of this talk, which is a special case of Fargues' conjecture (for the group GL_n, and in the irreducible, instead of indecomposable, case).

Theorem 1. *For each irreducible ℓ-adic $W_{\mathbb{Q}_p}$-representation E of dimension n, there exists an object $\mathrm{Aut}_E \in D_{lis}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ such that*

(1) Aut_E is a Hecke eigensheaf with eigenvalue $\underline{E}$.
(2) Aut_E is cuspidal, i.e. killed by the geometric constant term functors attached to all proper parabolics of GL_n (in particular, $\mathrm{Aut}_E \cong j_! j^* \mathrm{Aut}_E$).
(3) Aut_E is concentrated in degree 0 and irreducible on each connected component of Bun_n.
(4) For $b \in B(\mathrm{GL}_n)$ basic,

$$j_b^* \mathrm{Aut}_E \in D_{lis}(\mathrm{Bun}_n^b, \overline{\mathbb{Q}}_\ell) \cong D(\mathrm{Rep}^\infty_{\overline{\mathbb{Q}}_\ell} G_b(\mathbb{Q}_p))$$

is the (Jacquet-)Langlands correspondent $\mathrm{LL}_b(E)$ of E.

Remark 2. Our proof uses the existence of the local (Jacquet)-Langlands correspondence and its realization in the ℓ-adic cohomology of the Lubin-Tate tower. The local (Jacquet-)Langlands correspondence is anyway invoked already in the formulation of Fargues' conjecture, but it would be nice to obtain, starting from E, a more direct geometric proof (not relying on local Langlands) of the existence of a non-zero object in $D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ satisfying properties (1)-(3) of the theorem. This is the subject of work (very much) in progress, inspired by [4].

2. Around the spectral action

As the local (Jacquet-)Langlands correspondence for GL_n is known, properties (2) and (4) in Theorem 1 force

$$\mathrm{Aut}_E \cong \bigoplus_{b\in B(\mathrm{GL}_n)\text{ basic}} j_{b,!}(\mathcal{F}_{\mathrm{LL}_b(E)}).$$

This gives a possible definition of Aut_E, but it seems difficult to use it to check property (1) (and also (2)) in the theorem. We therefore take a somehow opposite road, and define a candidate for Aut_E which *by design* satisfies property (1) – the work is then in proving that it also satisfies all the other properties. This definition is inspired by ideas of Beilinson, Drinfeld and Gaitsgory, coming from the geometric Langlands program.

To explain them, let us first state a very strong conjecture, which is an analog of the main conjecture of [1]. Let $X_{\hat{G}}$ be the Artin stack of n-dimensional ℓ-adic representations of $W_{\mathbb{Q}_p}$, i.e., of homomorphisms $W_{\mathbb{Q}_p} \to \hat{G}(\overline{\mathbb{Q}}_\ell)$, taken up to conjugacy.

Conjecture 3 (Fargues–Scholze[2]). *There exists an equivalence*

$$\mathbb{L}\colon D^b(\mathcal{C}oh(X_{\hat{G}})) \xrightarrow{\simeq} D_{lis}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega.$$

Here $D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)^\omega$ denotes the category of compact objects in $D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$. The equivalence $\mathbb{L}$ is expected to satisfy the following conditions:

(1) $\mathbb{L}$ is equivariant for the action of $\mathrm{Rep}(\hat{G})$ on both sides, which is defined as follows. Choose a completed algebraic closure C of $\mathbb{Q}_p$ (which we already did when considering $W_{\mathbb{Q}_p}$), i.e., a point of Div^1. The usual Hecke functor, combined with pullback along $\mathrm{Spd}(C^\flat) \to \mathrm{Div}^1$ and the crucial invariance

$$D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \cong D_{\mathrm{lis}}(\mathrm{Bun}_n \times \mathrm{Spd}(C^\flat), \overline{\mathbb{Q}}_\ell),$$

provide an action $T\colon \mathrm{Rep}(\hat{G}) \times D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \to D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$. For the left hand side, note that we have a morphism

$$f\colon X_{\hat{G}} \to [\mathrm{Spec}(\overline{\mathbb{Q}}_\ell)/\hat{G}]$$

and an induced monoidal functor $f^*\colon \mathrm{Rep}(\hat{G}) \to \mathrm{Perf}(X_{\hat{G}})$. Thus, $V \in \mathrm{Rep}(\hat{G})$ acts simply by tensoring with the vector bundle $f^*(V)$.

[2]See also the report by Hellmann in this volume for related ideas and results.

(2) For E an irreducible ℓ-adic representation of $W_{\mathbb{Q}_p}$, the object of $D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ attached to E by Fargues' conjecture is

$$\mathbb{L}(k(E)),$$

with $k(E)$ the *regular* representation of the center $\mathbb{G}_m \cong \hat{Z} \subseteq \hat{G}$ at the closed substack $[\mathrm{Spec}(\overline{\mathbb{Q}}_\ell)/\hat{Z}] \subseteq X_{\hat{G}}$ determined by E.

(3) $\mathbb{L}(\mathcal{O}_{X_{\hat{G}}}) \cong \mathcal{W}_\psi$, where $\mathcal{W}_\psi := j_{1,!}(\mathcal{F}_{\mathrm{cInd}_{N(\mathbb{Q}_p)}^{\mathrm{GL}_n(\mathbb{Q}_p)}\psi})$, for some generic character $\psi\colon N(\mathbb{Q}_p) \to \overline{\mathbb{Q}}_\ell^\times$, with $N \subseteq \mathrm{GL}_n$ the standard unipotent subgroup of strictly upper triangular matrices.

If we assume Conjecture 3, we see that an action

$$\mathrm{Perf}(X_{\hat{G}}) \times D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell) \to D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell),\ (W, \mathcal{F}) \mapsto W * \mathcal{F},$$

called "the spectral action", of the category $\mathrm{Perf}(X_{\hat{G}})$ of perfect complexes on $X_{\hat{G}}$ on $D_{\mathrm{lis}}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ is expected to exist. Of course, the action of $\mathrm{Perf}(X_{\hat{G}})$ should extend the action of $\mathrm{Rep}(\hat{G})$, $\mathbb{L}$ should be linear for the actions of $\mathrm{Perf}(X_{\hat{G}})$ on both sides.

A nice consequence of the results of Fargues-Scholze is that one can construct this spectral action, despite the fact that Conjecture 3 is currently out of reach, by combining the existence and properties of the geometric Hecke action together with some general categorical machinery, developed in recent work of Gaitsgory-Kazhdan-Rozenblyum-Varshavsky, [5].

Theorem 4 (Fargues–Scholze, [3])**.** *The spectral action of the category* $\mathrm{Perf}(X_{\hat{G}})$ *on* $D_{lis}(\mathrm{Bun}_n, \overline{\mathbb{Q}}_\ell)$ *exists.*

3. Some ideas on the proof of the main result

Assume momentarily Conjecture 3. From the equality $\mathbb{L}(\mathcal{O}_{X_{\hat{G}}}) \cong \mathcal{W}_\psi$, one derives the formula

$$\mathbb{L}(\mathcal{V}) \cong \mathcal{V} * \mathcal{W}_\psi$$

for $\mathcal{V} \in \mathrm{Perf}(X_{\hat{G}})$. In particular, let E be an *irreducible* ℓ-adic representation of $W_{\mathbb{Q}_p}$ of rank n. Then $k(E) \in \mathrm{IndPerf}(X_{\hat{G}})$, and we can apply to it the previous considerations and note that the right hand-side now makes sense *without* assuming the conjecture, by Theorem 4. Therefore, we *define*

$$\mathrm{Aut}_E := k(E) * \mathcal{W}_\psi$$

as a candidate for Fargues' sheaf associated to E.

The fact that Aut_E satisfies property (1) in Theorem 1 is now easy. Much more work is required to verify properties (2)-(4). (Note that it is not even clear a priori that Aut_E is non-zero!) To deal with them, the main idea is to consider the spectral action of another object of $\mathrm{Perf}(X_{\hat{G}})$ than $k(E)$. Namely, we consider the *first averaging functor*

$$\mathrm{Av}^1_{E^\vee,n} := R\Gamma(W_{\mathbb{Q}_p}, f^* V_{\mathrm{st}} \otimes E^\vee) * (-).$$

Since

$$R\Gamma(W_{\mathbb{Q}_p}, f^*V_{\mathrm{st}} \otimes E^\vee) \cong k(E)_1 \oplus k(E(1))_1[-1] \in \mathrm{Perf}(X_{\hat{G}})$$

(where $k(E)_1$ corresponds to the 1-dimensional representation $x \mapsto x$ of $\hat{Z} \cong \mathbb{G}_m$, and similarly for $E(1)$), understanding the properties of $\mathrm{Av}^1_{E^\vee,n}$ gives some information about the action of $k(E)$. It is possible, due to the concrete description of this functor: explicitly,

$$\mathrm{Av}^1_{E^\vee,n}(\mathcal{F}) = \overrightarrow{h}_!(\overleftarrow{h}^*(\mathcal{F}) \otimes \alpha^* E^\vee)[n-1]$$

where

$$\mathrm{Mod}^1_n := \{\mathcal{E} \hookrightarrow \mathcal{E}' \text{ fiberwise injective}, \deg(\mathcal{E}') = \deg(\mathcal{E}) + 1\},$$

the map $\overleftarrow{h} : \mathrm{Mod}^1_n \to \mathrm{Bun}_n$, resp. $\overrightarrow{h} : \mathrm{Mod}^1_n \to \mathrm{Bun}_n$, remembers $\mathcal{E}$, resp. $\mathcal{E}'$, and the map $\alpha : \mathrm{Mod}^1_n \to \mathrm{Div}^1$ remembers the support of the cokernel of the modification. This allows us to check that

$$\mathrm{Av}^1_{E^\vee,n}(\mathcal{W}_\psi) \cong j_{b,!}\mathcal{F}_{\mathrm{LL}_b(E)} \oplus j_{b,!}\mathcal{F}_{\mathrm{LL}_b(E(1))}[-1],$$

where $b \in B(\mathrm{GL}_n)$ is the class corresponding to $\mathcal{O}(1/n)$. This identification is an important step towards the proof of properties (3)-(4).

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On the Dedekind eta function

Dustin Clausen

The results in this talk were motivated by a talk of Deligne's, [3]. Let $\mathcal{M}$ denote the moduli stack of elliptic curves; we'll need to restrict to $\mathbb{Q}$-schemes at some point so I'll take it over $\mathbb{Q}$. There is an important line bundle on $\mathcal{M}$, the *Hodge bundle* ω, defined as the dual to the relative Lie algebra of the universal elliptic curve. Modular forms of weight k (of level one with poles at ∞) are sections of $\omega^{\otimes k}$. Now, the basic classical facts I want to draw attention to are the following.

(1) There is a canonical trivialization $\omega^{\otimes 12} \simeq \mathcal{O}$, corresponding to the modular form of weight 12 known as Ramanjuan's Δ-function, whose q-expansion is $\Delta(q) = q \cdot \prod_{n\geq 1}(1-q^n)^{24}$.

(2) Over the scheme $\mathcal{M}[24]$ parametrizing elliptic curves with level 24-structure, there is a square root $\omega^{\otimes 1/2}$ of ω and a section $\eta : \omega^{\otimes 1/2} \simeq \mathcal{O}$, the *Dedekind eta function*, whose 24^{th} power is (the pullback of) Δ.

The thing to note here is that the q-expansion of Δ shows that the 24^{th} root of Δ, as a function on the upper half-plane, has a nice q-expansion with integral coefficients (neglecting the leading $q^{1/24}$). But even more is true by (2): η is itself a modular form of some level, though of *half-integral weight*, just like the theta series. In the background to this fact about η is a more general puzzle which Deligne highlighted in his talk: why is there a reasonable theory of modular forms of half-integral weight, but not, say, 1/3-integral weight?

The answer is that only for 1/2-integral weight modular forms is there a notion of Hecke operators (developed by Shimura, [4]). But then why is this? First, recall why ordinary modular forms admit Hecke operators: it is because the Lie algebra of an elliptic curve (over a $\mathbb{Q}$-scheme) is an isogeny invariant. Thus, when you form a Hecke correspondence, the line bundle on the top is pulled back from either side, letting one compose pullback and transfer to get an operator. In adelic terms, the explanation is that the moduli stack $\mathcal{M}[\infty]$ of elliptic curves with full level structure carries a $\mathrm{GL}_2(\mathbb{A}_f)$-action, and (the pullback of) ω is canonically equivariant for this action.

Now, according to Deligne, the reason why 1/2-integral weight modular forms have Hecke operators is that *the μ_2-gerbe of square roots of the Lie algebra of an elliptic curve up to isogeny is purely determined by its $\mathbb{A}_f$-Tate module*. More precisely, every lisse $\mathbb{A}_f$-sheaf canonically determines a μ_2-gerbe, such that if the lisse sheaf comes from an elliptic curve up to isogeny then this agrees with the μ_2-gerbe of square roots of the Lie algebra. (Moreover, this is no longer true if you replace μ_2 by μ_n for $n > 2$.)

Indeed, this implies that on $\mathcal{M}[\infty]$ there is a canonical square root $\omega^{\otimes 1/2}$ of ω which is equivariant with respect to the action of $\widetilde{\mathrm{GL}_2(\mathbb{A}_f)}$, a canonical "metaplectic" μ_2-central extension of $\mathrm{GL}_2(\mathbb{A}_f)$. In concrete terms, this central extension resolves the sign ambiguitites which arise in this theory of Hecke operators. This whole story is some version of the usual story of automorphic forms on metaplectic groups, but living natively in algebraic geometry.

To construct the required μ_2-gerbe associated to a lisse $\mathbb{A}_f$-sheaf is not so difficult (in one way or another it corresponds to the product of the Hilbert symbols at the finite primes p), but proving that it agrees with the μ_2-gerbe of square roots of ω in presence of an elliptic curve is tricky. Deligne proceeds in three steps:

(1) He does it over $\mathbb{C}$ where it becomes a purely topological question which reduces to Hilbert's form of the quadratic reciprocity law.
(2) Then he does it explicitly over $\mathbb{Z}((T)) \otimes \mathbb{Q}$, using Gauss sums.
(3) Then he argues that the general case follows from (1) and (2) by some sort of gluing principle.

We will give an approach which directly proves the result over $\mathcal{M}$, bypassing (1) and (2). In particular, the quadratic reciprocity law becomes a corollary rather than an ingredient. (We obtain essentially the proof of quadratic reciprocity given in [2], which was the other inspiration for this work.) On the other hand I have the impression that Deligne knows how to make the whole story work over $\mathbb{Z}$, whereas for what I'm about to say we do have to work over $\mathbb{Q}$.

Actually, we'll prove something quite a bit more general. Note that the μ_2-gerbe of square roots of a line bundle $\mathcal{L}$ over a base X is classified by the etale cohomology class in $H^2(X;\mu_2)$ given by the mod 2 reduction of the first Chern class $c_1(\mathcal{L}) \in H^2(X;\mathbb{Z}_2(1))$. Then we have the following:

Theorem. *Let ℓ be a prime and $n \geq 1$. There are functorial characteristic classes for lisse $\mathbb{A}_f$-sheaves valued in $H^{2n(\ell-1)}(-;\mathbb{Z}/\ell\mathbb{Z})$, such that for an abelian variety up to isogeny over a $\mathbb{Q}$-scheme, the value of these characteristic classes on the Tate module agrees with the (mod ℓ) reduction of the $n(\ell-1)^{st}$ Chern class of the vector bundle given by the relative Lie algebra.*

Note that the (mod ℓ) reduction of the Chern class a priori lives in etale cohomology with values in $\mathbb{Z}/\ell\mathbb{Z}(n(\ell-1))$, but the $(\ell-1)^{st}$ Tate twist of $\mathbb{Z}/\ell\mathbb{Z}$ is canonically trivial so this matches up.

Why is it only when the degree is a multiple of $2(\ell-1)$ that the (mod ℓ) reduction of the Chern class "transfers over" to the Tate module? The answer is in a result proved by Thom (at least for $\ell = 2$) in the topological context, [5]: in exactly those degrees, the (mod ℓ) reduction of the Chern classes of a vector bundle only depend on the associated stable spherical fibration. Indeed, the deeper statement underlying our above theorem is that there is an etale stable spherical fibration associated to any lisse $\mathbb{A}_f$-sheaf which, applied to Tate modules, agrees with the stable spherical fibration associated to the Lie algebra of an abelian variety up to isogeny. To prove this one realizes these spherical fibrations as appropriate dualizing objects in a theory of etale sheaves of spectra; then one sees that the dualizing object for an abelian variety only depends on its "homotopy type", controlled by the Tate module. This is an analog of a result of Atiyah, [1] in the topological context. To get the isogenies to play along requires some more nuanced set-up, but it's not so hard in the end.

So much for the general theory of half-integer weight modular forms. But what about η? I have to admit I don't have a fully satisfactory story. But one aspect is an analog to the above theorem for Euler classes instead of Chern classes:

Theorem. *Let ℓ be a prime. There is a characteristic class for rank n lisse $\mathbb{Z}_\ell$-sheaves $\mathcal{T}$ over $\mathbb{Z}[1/\ell]$-schemes X valued in etale cohomology $H^n(X;\Lambda^n\mathcal{T})$, such that if $\mathcal{T}$ is the $\mathbb{Z}_\ell$-Tate module of an abelian variety over X, then this class agrees with the ℓ-adic Euler class of the relative Lie algebra.*

Again we can note that the Euler class of a vector bundle of rank d (which agrees with its top Chern class c_d) lives in $H^{2d}(X;\mathbb{Z}_\ell(d))$, but both the degrees and the coefficients match up to make the statement of the above theorem well-defined (this time for slightly less trivial reasons).

Restricted to elliptic curves, this shows that the full first etale Chern class $c_1(\omega) \in H^2(\mathcal{M}; \widehat{\mathbb{Z}}(1))$ is completely determined by the $\widehat{\mathbb{Z}}$-Tate module of the universal elliptic curve. If we take for granted that the class of ω in $\mathrm{Pic}(\mathcal{M})$ is torsion (as can be proved without exhibiting Δ), it follows that the question of the order of ω in $\mathrm{Pic}(\mathcal{M})$ being 12, as well as the problem of producing η as a modular form — that is, the two classical facts we started the lecture with — can be decided purely in the world of characeteristic classes for lisse $\mathbb{Z}_\ell$-sheaves of rank 2, and in fact can be produced by direct computation of the continuous cohomology of $\mathrm{GL}_2(\mathbb{Z}_\ell)$.

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Finiteness and duality for étale cohomology over certain generalizations of Tate algebras

Ofer Gabber

I discussed properties of étale cohomology of certain schemes over a rank 1 valuation ring V with residue field k, $K = \mathrm{Frac}(V)$ and value group Γ s.t. $\Gamma/\ell\Gamma$ is finite (of dimension r over $\mathbb{F}_\ell$), where ℓ is a prime invertible in V and the coefficient ring is $\Lambda = \mathbb{Z}/\ell^n$ $(n > 0)$. Let $\eta = \mathrm{Spec}(K)$ (resp. $s = \mathrm{Spec}(k)$) be the generic (resp. closed) point of $S = \mathrm{Spec}\, V$, $j : \eta \hookrightarrow S$. Then $K_S = \tau^s_{\leq r-1} Rj_*\Lambda$ is a dualizing complex on S.

Slicing lemma: Let $S = \mathrm{Spec}\, A$ be a topologically nœtherian affine scheme, $S' = \mathrm{Spec}\, A(t)$ where $A(t)$ is the localization of $A[t]$ w.r.t. the multiplicative set of fiberwise nonzero polynomials, $\pi : \mathbb{P}^1_S \to S$. If $K \in D^+(\mathbb{P}^1_S, \Lambda)$, where Λ is a nœtherian torsion ring and the étale topology is understood, $i \in \mathbb{Z}$, and $\mathcal{H}^i K_{|S'}$, $R^i\pi_* K$, $R^{i-1}\pi_* K$ are constructible, then $\mathcal{H}^i K$ is constructible.

This (and variants) gives alternate proofs of results of [3], and more generally that in our context Rf_* preserves D^b_c for a morphism between finite type V-schemes. For $a : X \to S$ (separated) of finite type let $K_X = Ra^! K_S$; extending [3] 4.3, one shows that it is a dualizing complex. The case $r = 0$ is analogous to manifolds with boundary in topology.

Let us consider the modified dimension function $\delta(s) = 0$, $\delta(\eta) = r$ on S, extended to schemes of finite type over S as in [6] 2.1. Similarly to [6] 2.4 one can prove the "affine Lefschetz" theorem stating that for $f : X \to Y$ an affine morphism between S-schemes of finite type, $\delta(R^q f_* F) \leq \delta(F) - q$ for F a sheaf of Λ-modules on X. One deduces that if V is an absolutely integrally closed valuation ring (of any rank) and ℓ a prime invertible on V and X an affine V-scheme of finite type, then $cd_\ell X$ is the maximum of the dimension of fibers of $X \to \operatorname{Spec} V$. When V is complete, using the slicing lemma, and techniques of partial algebraization and comparison to the completion one can prove a finiteness result for schemes of finite type over a restricted formal series ring $V\{x_1, \ldots, x_n\}$ (finiteness for Rj_* when j is the inclusion of the generic fiber) reproving results of Huber and Berkovich, see [2] and references therein. By a method similar to that of [5] one can reprove [1] 7.3 and prove the statement expected in [1] 7.4. More generally in our case "affine Lefschetz" holds for schemes of finite type over $V\{x_1, \ldots, x_n\}$. To prove this the idea is to use a weak local uniformization result as in [7] XV to approximate the studied situation by a finite type situation with the same combinatorics. This can be done for a certain generalization of the affinoid algebras used in classical rigid geometry. From now on V is complete without restriction on Γ. Recall that a special case of [4] 7.4.1 gives that if A is a flat V-algebra complete and separated for the a-adic topology (a being a pseudo-uniformizer) s.t. the strict henselizations of A/aA are strict henselizations of finite type V-algebras and $A \underset{V}{\otimes} k$ is nœtherian, then A is a-adically t.u. adhesive, in particular A_K is nœtherian. We say that such an A is tempered if there is an étale faithfully flat $A/aA \to A'$ and a weakly étale $B \to A'$, B flat of finite presentation over V/a. [This condition is independent of the choice of a.] In the above situation if B is a quotient of a smooth V-algebra C then we have a formal liftings $A/a^n \to A'_n \leftarrow C$ and $C \to A'$ lifts uniquely to a formally étale $C \to A''$ (A' being the quotient of A'' by an ideal of definition), with $\operatorname{Spf} A''$ playing the role of a smooth ambient space for $\operatorname{Spf}(\varprojlim A'_n)$.

By a small extension of V we mean the normalization of V in a complete valued L/K which is a finite extension when $\operatorname{char}(K) = 0$ and a purely inseparable extension of finite height of a finite extension when $\operatorname{char}(K) > 0$.

For a tempered V-algebra A we can extend Kiehl's excellence result [8] and show that A_K is excellent. This comes with a dimension function and a Jacobian criterion, and in particular one gets that morphisms like $A[X] \to A\{X\}$ and $A \to \widehat{A}_x^h$ ($x \in \operatorname{Spec}(A \underset{V}{\otimes} k)$) are flat, and regular on generic fibers, and then using Popescu's theorem and some work they are filtered colimits of smooth algebras. This gives an approximation technique similar to the one used in [7] VII to prove weak local uniformization for quasi-excellent schemes.

We say that a morphism of log schemes is locally plurinodel if étale locally it is a finite composition of morphisms which are locally pullbacks of morphisms defined by the monoid map $0 \to \mathbb{N}$ or $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$. We equip S with the standard log structure coming from the log ring $V - \{0\} \to V$. If $(X, M) \xrightarrow{\varphi} (S, N)$ is a morphism of log schemes s.t. X is of finite type over a tempered V-algebra, we say that φ is of plurinodel type along the special fiber if étale locally along the special fiber (X, M) admits a formally a-adically weakly étale strict morphism to a locally plurinodel $T \to (S, N)$. One can omit "along the special fiber" by adding a suitable condition. In this case the log structure M is determined by the locus of triviality as in the case of log regular schemes.

Local uniformization: If A is a tempered V-algebra, X/A V-flat of finite type, $Z \subset X$ nowhere dense closed subset, then there is a v-covering $X' = \coprod_{i \in I} X'_i \to X$ and a nowhere dense $Z' \subset X'$ containing the preimage of Z, s.t. I is finite and $\forall\, i \in I$, X'_i is of finite type, generically finite and maximally dominating over $X \underset{A}{\otimes} \big(A \underset{V}{\widehat{\otimes}} V_i\big)$ for small V_i/V, and (X'_i, Z'_i) is of plurinodel type over V_i.

Using this one can mimic some results of [7].

In particular if $\Gamma/\ell\Gamma$ is finite we have that Rf_* preserves constructibility for a morphism of finite type A-schemes, and one can construct canonical dualizing complexes on finite type A-schemes using the local ambient spaces mentioned above with globalization using vanishing of negative Ext's.

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Cohen-Macaulayness of local models

Timo Richarz

(joint work with Thomas J. Haines, partly with João N. P. Lourenço)

A prototypical example of a Shimura variety is the $\mathbb{Q}$-space of isomorphism classes of g-dimensional principally polarized abelian varieties with level structure. If the level at a prime p is parahoric, one can construct an integral model of this space over $\mathbb{Z}_p$ by considering isogeny chains of abelian schemes having the same shape as the lattice chain which determines the parahoric subgroup of $\mathrm{GSp}_{2g}(\mathbb{Q}_p)$. Typically the resulting schemes have bad reduction. The local models serve as a tool to understand the singularities arising in the reduction modulo p. They are projective schemes over $\mathbb{Z}_p$ defined in terms of linear algebra -thus are easier to handle- and are étale locally isomorphic to the integral model for the Shimura variety. Here I report on some recent results about the geometry of local models for general groups, see [5] and [6].

For expository purposes we restrict ourselves to the following set-up: We fix a triple $(G, \{\mu\}, \mathcal{G}_{\mathbf{f}})$ where G is a reductive $\mathbb{Q}_p$-group, $\{\mu\}$ a minuscule conjugacy class of geometric cocharacters defined over a finite extension $E/\mathbb{Q}_p$, and $\mathcal{G}_{\mathbf{f}}$ is a parahoric $\mathbb{Z}_p$-group scheme with generic fiber G. We assume that $G \cong \mathrm{Res}_{F/\mathbb{Q}_p}(G_1)$ where $F/\mathbb{Q}_p$ is a finite (possibly wildly ramified) extension, and G_1 is a tamely ramified reductive F-group. Pappas-Zhu [11] and Levin [9] attach to the triple $(G, \{\mu\}, \mathcal{G}_{\mathbf{f}})$ the *Pappas-Zhu local model*

$$\mathbb{M} \;=\; \mathbb{M}(G, \{\mu\}, \mathcal{G}_{\mathbf{f}}),$$

which is a flat, projective $\mathcal{O}_E$-scheme equipped with a left action of $\mathcal{G} \otimes \mathcal{O}_E$. Recall that its construction requires certain auxiliary choices, but that the PZ local model depends, up to equivariant isomorphism, only on the data $(G, \{\mu\}, \mathcal{G}_{\mathbf{f}})$, see [7, Thm. 2.7] and [9, Rmk. 4.2.5]. We also note that He, Pappas and Rapoport [7, Def. 2.10] define the *local model* to be the PZ local model of a suitable z-extension of $(G, \{\mu\}, \mathcal{G}_{\mathbf{f}})$, which by [5, Cor. 2.3] coincides with the (weak) normalization of $\mathbb{M}$. The advantage of this point of view is that the so defined local model is characterized by Scholze's conjecture, see [7, Cor. 2.17] and Lourenço's report in the volume for results in this direction.

In this report, we work with the PZ local model $\mathbb{M}$. The generic fiber $\mathbb{M} \otimes E$ is the variety of type $\{\mu\}$ parabolic subgroups in G. The special fiber is equidimensional, but not irreducible in general, and is equipped with a closed embedding into the partial affine flag variety

$$\mathbb{M} \otimes \bar{\mathbb{F}}_p \;\hookrightarrow\; \mathrm{Fl}_{G^\flat, \mathbf{f}^\flat}.$$

The pair $(G^\flat, \mathbf{f}^\flat)$ is an equal characteristic analogue over the local function field $\bar{\mathbb{F}}_p(\!(t)\!)$ of the pair $(G, \mathbf{f})$. Under this embedding the reduced locus of $\mathbb{M} \otimes \bar{\mathbb{F}}_p$ identifies by [4, Thm. 5.14], with no restriction on p, with the admissible locus

$$\mathcal{A}(G, \{\mu\}) \;\subset\; \mathrm{Fl}_{G^\flat, \mathbf{f}^\flat},$$

which is the union of the affine Schubert varieties indexed by the $\{\mu\}$-admissible set of Kottwitz-Rapoport. The following theorem proves results on the geometry of PZ local models under weaker hypotheses than the hypothesis $p \nmid |\pi_1(G_{\mathrm{der}})|$ in [11, Thm. 9.3] and [9, Thm. 4.3.2]. We recover as special cases the results of [11] in this direction and those of [3], which treats unramified groups and facets $\mathbf{f}$ whose closure contains a special vertex[1].

Theorem 1. *Let $(G, \{\mu\}, \mathcal{G}_{\mathbf{f}})$ be as above, and assume that all Schubert varieties inside $\mathcal{A}(G, \{\mu\})$ are normal.*

[1] But we note that [3] includes $p = 2$.

(1) *The special fiber* $\mathbb{M} \otimes \bar{\mathbb{F}}_p$ *is geometrically reduced. More precisely, as closed subschemes of* $\mathrm{Fl}_{G^\flat, \mathbf{f}^\flat}$*, one has*

$$\mathbb{M} \otimes \bar{\mathbb{F}}_p \; = \; \mathcal{A}(G, \{\mu\}).$$

Further, each irreducible component of $\mathbb{M} \otimes \bar{\mathbb{F}}_p$ *is normal, Frobenius split and has only rational singularities.*

(2) *The PZ local model* $\mathbb{M}$ *is normal and, if* $p > 2$*, also Cohen-Macaulay with dualizing sheaf given by the double dual of the top differentials*

$$\omega_{\mathbb{M}} \; = \; \left(\Omega^d_{\mathbb{M}/\mathcal{O}_E}\right)^{*,*},$$

where d *is the Krull dimension of the generic fiber.*

Furthermore, all Schubert varieties inside $\mathcal{A}(G, \{\mu\})$ *are known to be normal*

- *if* $p \nmid |\pi_1(G_{\mathrm{der}})|$*, see* [10, Thm. 8.4]*;*
- *if* $\bar{\mu} \in X_*(T)_I$ *is minuscule for the échelonnage root system, and the facet* $\mathbf{f}$ *contains a very special vertex, for example, if* G *is unramified,* $\{\mu\}$ *is minuscule and* $\mathcal{G}_{\mathbf{f}}(\mathbb{Z}_p)$ *is an Iwahori subgroup, see* [6].

In view of [8] the corresponding integral models of Shimura varieties with parahoric level structure are normal and Cohen-Macaulay as well. The reader is referred to [1] for some applications of the Cohen-Macaulay property to the coherent cohomology of Shimura varieties.

Theorem 1 (1) gives new cases of normal PZ local models with reduced special fiber. The proof follows the original argument of Pappas-Zhu, using as a key input the Coherence Conjecture proved by Zhu [12]. This is justified by our assumption on the normality of Schubert varieties inside $\mathcal{A}(G, \{\mu\})$. For (2), the normality of $\mathbb{M}$ is an immediate consequence of (1) by Serre's criterion. The Cohen-Macaulayness of $\mathbb{M}$ is deduced from a homological algebra result combined with the well-known theorem of Zhu [12, Thm. 6.5]; here the restriction $p > 2$ enters. In particular, our method avoids using any finer geometric structure of the admissible locus $\mathcal{A}(G, \{\mu\})$ as for example in [2, §4.5.1] or [3].

Furthermore, our assumption on the normality of Schubert varieties inside $\mathcal{A}(G, \{\mu\})$ is necessary for $\mathbb{M}$ being well-behaved, see [5, Rmk. 2.4] and [6]:

Proposition 2. *If a single Schubert variety inside* $\mathcal{A}(G, \{\mu\})$ *is not normal, then* $\mathbb{M}$ *is not normal, not Cohen-Macaulay and its special fiber is not reduced.*

An example is given by the Weil restriction of scalars $G = \mathrm{Res}_{F/\mathbb{Q}_2}(\mathrm{PGL}_2)$ along a totally ramified quadratic extension $F/\mathbb{Q}_2$, the unique minuscule class $\{\mu\}$ and the special vertex $\mathbf{f}$ corresponding to the standard lattice $\mathcal{O}_F^2$. In this case, the admissible locus $\mathcal{A}(G, \{\mu\})$ is the quasi-minuscule Schubert variety in the affine Grassmannian for PGL_2 over $\bar{\mathbb{F}}_2$. The completed local ring at the singular point is the $\bar{\mathbb{F}}_2$-algebra

$$\bar{\mathbb{F}}_2[\![x, y, v, w]\!]/(vw + x^2y^2, v^2 + x^3y, w^2 + xy^3, xw + yv).$$

This is a surface singularity which is geometrically unibranch, but neither weakly normal, nor Cohen-Macaulay, nor Frobenius split. The existence of non-normal

Schubert varieties was first observed by Lourenço and came as a total surprise to the author, see [6]:

Theorem 3. *Assume that G as above is absolutely almost simple and semisimple. If $p \mid |\pi_1(G)|$, then there are only finitely many Schubert varieties in the partial affine flag variety $\mathrm{Fl}_{G^\flat,\mathbf{f}^\flat}$ over $\bar{\mathbb{F}}_p$ which are normal. The non-normal Schubert varieties are geometrically unibranch, but neither weakly normal, nor Cohen-Macaulay, nor Frobenius split.*

In fact, we give in [6] an effective criterion to determine which Schubert varieties are normal. Interestingly, the Schubert varieties inside the admissible locus $\mathcal{A}(G,\{\mu\})$ are still normal if $\bar{\mu} \in X_*(T)_I$ is minuscule for the échelonnage root system and $\mathbf{f}$ contains a special vertex in its closure. Note that in the example of PGL_2 above $\bar{\mu}$ is only quasi-minuscule. The classification of normal Schubert varieties when $p \mid |\pi_1(G)|$ seems to be a challenging problem which is closely related to the combinatorics of integral Demazure modules in the Kac-Moody setting.

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On the geometry of mixed characteristic affine Grassmannians

João Lourenço

Affine Grassmannians are certain infinite-dimensional spaces of central importance in arithmetic geometry. Given a smooth affine group scheme G over a base S, Gr_G parametrizes G-torsors over the "formal disk" trivialized away from the origin. Here the formal disk can either be a power series ring $R[[t]]$ or a relative de Rham period ring $B^+_{\mathrm{dR}}(R)$ with R a perfectoid Tate ring. In our talk we described some progress in the study of their geometry in mixed characteristic.

1. de Rham periods

Here we assume familiarity with the v-topology and v-sheaves as in [16]. An adic space X naturally gives rise to a v-sheaf $X^\diamondsuit$ over the untilt sheaf $\mathrm{Spd}(\mathbb{Z}_p)$, which can be regarded as the topologization of X:

Theorem 1 ([16], [10]). *The functor $X \mapsto X^\diamondsuit$ from topologically finite type $\mathbb{Z}_p$-adic spaces to v-sheaves over $\mathrm{Spd}(\mathbb{Z}_p)$ factors over the absolute weak normalization functor via a fully faithful embedding.*

Let G be a connected reductive group over $\mathbb{Q}_p$. The affine Grassmannian Gr_G over $\mathrm{Spd}(\mathbb{Q}_p)$ is an ind-proper ind-spatial diamond. Its closed L^+G-orbits $\mathrm{Gr}_{G,\leq\mu}$, called Schubert varieties, are defined over an extension $E/\mathbb{Q}_p$ and indexed by geometric conjugacy classes μ of cocharacters of G.

Given a smooth connected affine model $\mathcal{G}$ of G, we may define the affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$. There is the fundamental result going back to [15] in the power series setting:

Theorem 2 ([16], [9]). *$\mathrm{Gr}_{\mathcal{G}}$ is ind-proper over $\mathrm{Spd}(\mathbb{Z}_p)$ iff $\mathcal{G}$ is parahoric.*

The special fiber of $\mathrm{Gr}_{\mathcal{G}}$ is the v-sheaf attached to the Witt vector affine Grassmannian introduced in [20] and proved to be representable by a perfect ind-scheme in [2]. This also admits a stratification parametrized by a certain quotient of the Iwahori-Weyl group.

Integrally we want to look at the closure $\mathrm{Gr}_{\mathcal{G},\leq\mu}$ defined over $\mathrm{Spec}(\mathcal{O}_E)$ of $\mathrm{Gr}_{\mathcal{G},\leq\mu}$ inside $\mathrm{Gr}_{\mathcal{G}}$. Despite some surprising subtleties, we show in [1] that $\mathrm{Gr}_{\mathcal{G},\leq\mu}$ is stable under $L^+\mathcal{G}$ and has topologically dense generic fiber. More importantly we are working towards:

Theorem 3 ([1], in progress). *The special fiber of $\mathrm{Gr}_{\mathcal{G},\leq\mu}$ coincides with the μ-admissible locus.*

The μ-admissible locus is given as the union of certain Schubert varieties corresponding to coweight representatives of μ. It will rely on computing nearby cycles and the geometric Satake equivalence of [5].

Now we would like to discuss representability questions, which is not possible in the non-minuscule case by Banach-Colmez theory.

Conjecture 4 ([16]). *If μ is minuscule, $\mathrm{Gr}_{\mathcal{G},\mu}$ is representable by a unique normal flat projective $\mathcal{O}_E$-scheme $\mathbb{M}_{\mathcal{G},\mu}$ with reduced special fiber.*

We can give the following partial evidence to the conjecture.

Theorem 5 ([8], [10]). *For all pairs of abelian type, $\mathrm{Gr}_{\mathcal{G},\mu}$ is representable by a unique weakly normal flat projective $\mathcal{O}_E$-scheme. Its special fiber is reduced if $p > 2$ or if $p = 2$ and the quadratic splitting extensions of the absolutely simple factors of $G^{\mathrm{ad}}_{\breve{\mathbb{Q}}_p}$ is given by a cyclic Eisenstein polynomial.*

Abelian type means here that a generic Hodge embedding can be found up to central modification. Reducedness of the special fiber will follow from Thm. 9. We have some ideas for proving this away from type E (thus mixing $D^{\mathbb{H}}$ and $D^{\mathbb{R}}$), which would involve revisiting [2].

2. Power series

To study phenomena in mixed-characteristic, we need "parahoric group schemes" over $\mathcal{O}[t]$ with $\mathcal{O}$ a mixed characteristic Dedekind domain. These were constructed by [13] in tame cases and [7] for restriction of scalars of those. Generalizing them for quasi-split groups is the main input in getting new local models with reduced special fiber when $p = 2$.

Let G be a quasi-split connected reductive group over $\mathbb{Q}(\zeta_e, t)$ split over $\mathbb{Q}(\zeta_e, t^{1/e})$ with e a natural number. Fix moreover a quasi-pinning consisting of a maximal split torus S and coherent realizations of all root groups U_a, cf. [18, Appendix]. Exactly as in [18], we extend G to a certain smooth connected group scheme $\underline{G}$ over $\mathbb{Z}[\zeta_e, t^{\pm}]$. It turns out $\underline{G} \otimes \bar{\mathbb{F}}_p(t)$ is at most quasi-reductive for $p \mid e$ and in small characteristics $e = 2, 3$ recovers the basic exotic and non-reduced groups pseudo-reductive groups in the sense of [3]. So we need:

Theorem 6 ([17], [9]). *Bruhat-Tits theory is available for quasi-reductive groups.*

Then we identify the appartement combinatorics of $(\underline{G}, \underline{S})$ over various $k((t)))$ and construct parahoric models $\underline{\mathcal{G}}$ over $\mathbb{Z}[\zeta_e, t]$ by means of birational group law techniques. Using the rotation action and a lemma from [14] on affine hulls, we get:

Theorem 7 ([13], [8], [10]). *$\underline{\mathcal{G}}$ is a smooth affine connected $\mathbb{Z}[\zeta_e, t]$-group.*

The affine Grassmannian $\mathrm{Gr}_{\underline{\mathcal{G}}}$ is an ind-projective ind-scheme. We can prove the following on its Schubert varieties:

Theorem 8 ([4], [12], [8], [10], [6]). *If $G^{\mathrm{sc}} = G^{\mathrm{der}}$, then $\mathrm{Gr}_{\underline{\mathcal{G}}, \leq \mu}$ is geometrically normal over $\mathbb{Z}[\zeta_e]$. Otherwise it is almost never the case.*

In the simply connected absolutely simple case, we recover the affine flag varieties of [11]. The proof requires calculating distributions for odd unirary groups. The existence of non-normal Schubert varieties was explained in Richarz's talk.

If we replace the power series disk by a curve with a moving point, we get the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}^{\mathrm{BD}}_{\underline{\mathcal{G}}, \mathbb{Z}[\zeta_e, t]/\mathbb{Z}[\zeta_e]}$, an ind-scheme over $\mathbb{Z}[\zeta_e, t]$. We have the following far-reaching generalization of the coherence theorem of Zhu:

Theorem 9 ([19], [13], [8], [10], [6]). *If $G^{\mathrm{sc}} = G^{\mathrm{der}}$, then the global Schubert variety $\mathrm{Gr}^{\mathrm{BD}}_{\underline{\mathcal{G}}, \leq \mu}$ is geometrically reduced over the base and its fibers are given by the μ-admissible locus. This is usually false when $G^{\mathrm{sc}} \neq G^{\mathrm{der}}$.*

Combined with the methods of [7], this yields the reducedness part of Thm. 5 by comparing minuscule local models of GL_n.

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Minimality of integral models of Rapoport-Zink spaces

Laurent Fargues

My talk dealt with the main result of my article "Groupes Rigides Analytiques p-divisivles II". Let $\mathcal{M}$ be the Rapoport-Zink moduli space of deformations by quasi-isogenies of a fixed p-divisible group over $\overline{\mathbb{F}}_p$. This is a locally formally of finite type formal scheme over $\mathrm{Spf}(W(\overline{\mathbb{F}}_p))$. This is moreover formally smooth (but not topologically smooth in general since this is not a p-adic formal scheme in general). The problem is how to recover the integral model $\mathcal{M}$ from its generic fiber $\mathcal{M}_\eta$ as a rigid analytic space?

For this let $\omega := \omega_{H^{\text{univ}}}$ where H^{univ} is the universal deformation, a p-divisible group over $\mathcal{M}$. This is a vector bundle over $\mathcal{M}$. Let $\mathrm{sp} : \mathcal{M}_\eta \to \mathcal{M}$ be the specialization morphism, and $\mathcal{O}^+_{\mathcal{M}_\eta}$ the sheaf of bounded by 1 holomorphic functions on $\mathcal{M}_\eta$. Consider $\omega^+ := \mathrm{sp}^{-1}\omega \otimes_{\mathrm{sp}^{-1}\mathcal{O}_{\mathcal{M}}} \mathcal{O}^+_{\mathcal{M}_\eta}$, a locally free $\mathcal{O}^+_{\mathcal{M}_\eta}$-module.

The main result of my talk is then the following

Theorem 1. *Let $\mathcal{X}$ be a formally smooth locally formally of finite type formal scheme over* $\mathrm{Spf}(W(\overline{\mathbb{F}}_p))$. *Then a morphism $f : \mathcal{X}_\eta \to \mathcal{M}_\eta$ extends to a morphism $\mathcal{X} \to \mathcal{M}$ if and only if the locally free $\mathcal{O}^+_{\mathcal{X}_\eta}$-module $f^*\omega^+$ extends to a vector bundle on $\mathcal{X}$.*

In particular the couple $(\mathcal{M}_\eta, \omega^+)$ determines $\mathcal{M}$ uniquely.

The proof relies on a notion of families of rigid analytic p-divisible groups. More precisely, let S be a rigid analytic space. Let $\mathcal{C}_S$ be the category of rigid analytic commuative groups G over S such that

(1) $G \to S$ together with its unit section is a locally trivial fibration in pointed (by its zero section) balls $\mathring{\mathbb{B}}^d_S \to S$
(2) $\times p : G \to G$ is finite surjective
(3) $\times p : G \to G$ is "topologically nilpotent"

For such a G one can define $(\mathrm{Lie}\, G)^+$ as a locally free $\mathcal{O}^+_S$-module, with $(\mathrm{Lie}\, G)^+\left[\frac{1}{p}\right] = \mathrm{Lie}\, G$. Here is the main theorem we use to prove the preceding.

Theorem 2. *Let $\mathcal{S} = \widehat{\mathbb{A}}^n_{\mathcal{O}_K}$ be the p-adic affine space for some $K|\mathbb{Q}_p$ complete, with generic fibre $S = \mathbb{B}^b_K$. Then the generic fiber functor induces an equivalence*

$$\{\textit{formal p-divisible groups over } \mathcal{S}\} \xrightarrow{\sim} \{G \in \mathcal{C}_S \mid (\mathit{Lie}\, G)^+ \textit{ extends to a vector bundle on } \mathcal{S}\}.$$

This is deduced from Bartenwerfer's theorem $H^1(\widehat{\mathbb{A}}^n_{\mathcal{O}_K}, \mathcal{O}^+) = 0$.

Reporter: Lucas Mann

Participants

Prof. Dr. Joseph Ayoub
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
8057 Zürich
SWITZERLAND

Prof. Dr. Massimo Bertolini
Fakultät für Mathematik
Universität Duisburg-Essen
Mathematikcarrée
Thea-Leymann-Straße 9
45127 Essen
GERMANY

Professor Bhargav Bhatt
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor MI, 48109-1043
UNITED STATES

Prof. Dr. Yuri Bilu
IMB
Université de Bordeaux
351, cours de la Libération
33405 Talence
FRANCE

Dr. George Boxer
Unité de Mathématiques Pures et Appliquées
ENS de Lyon
46, allée d'Italie
69364 Lyon Cedex 07
FRANCE

Dr. Ana Caraiani
Imperial College London
Department of Mathematics
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Dr. Kestutis Česnavičius
Institut de Mathématiques d'Orsay
Université Paris Sud (Paris XI)
Bât. 307
91405 Orsay Cedex
FRANCE

Dr. Dustin Clausen
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Pierre Colmez
Institut de Mathématiques de Jussieu
CNRS, Case 247, Theorie des Nombres
Université de Paris VI
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Johan de Jong
Department of Mathematics
Columbia University
2990 Broadway
New York NY 10027
UNITED STATES

Prof. Dr. Bas Edixhoven
Mathematisch Instituut
Universiteit Leiden
Postbus 9512
2300 RA Leiden
NETHERLANDS

Prof. Dr. Dr. h.c.mult. Hélène Esnault
Fachbereich Mathematik und Informatik
Freie Universität Berlin
Arnimallee 3
14195 Berlin
GERMANY

Prof. Dr. Gerd Faltings
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Laurent Fargues
Institut de Mathématiques de Jussieu
Université de Paris VI
CNRS, Case 247
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Ofer Gabber
I.H.E.S.
Le Bois-Marie
Université Paris-Saclay
35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

Prof. Dr. Toby Gee
Department of Mathematics
Imperial College London
Huxley Building
180, Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Alexander Goncharov
Department of Mathematics
Yale University
Box 208 283
New Haven, CT 06520
UNITED STATES

Prof. Dr. Philipp Habegger
Department für Mathematik und Informatik
Universität Basel
Spiegelgasse 1
4051 Basel
SWITZERLAND

Prof. Dr. Michael Harris
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Prof. Dr. Eugen Hellmann
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Prof. Dr. Wei Ho
Department of Mathematics
University of Michigan
East Hall
530 Church Street
Ann Arbor, MI 48109-1109
UNITED STATES

Prof. Dr. Adrian Iovita
Department of Mathematics & Statistics
Concordia University
7141 Sherbrooke St. W.
Montréal QC H4B 1R6
CANADA

Prof. Dr. Moritz Kerz
Fakultät für Mathematik
Universität Regensburg
Universitätsstrasse 31
93053 Regensburg
GERMANY

Prof. Dr. Mark Kisin
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Prof. Dr. Bruno Klingler
Fachbereich Mathematik
Humboldt Universität Berlin
Unter den Linden 6
10099 Berlin
GERMANY

Dr. Jaclyn Lang
Département de Mathématiques
Université Paris 13
Institut Galilee
99, Ave. Jean-Baptiste Clement
93430 Villetaneuse Cedex
FRANCE

Dr. Arthur-César Le Bras
C.N.R.S.
Université Paris XIII
99, Avenue Jean-Baptiste Clement
93430 Villetaneuse Cedex
FRANCE

Dr. Shizhang Li
East Hall 1854
Department of Mathematics
University of Michigan
530 Church Street
48109 Ann Arbor, MI 48109-1043
UNITED STATES

Prof. Dr. Max Lieblich
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle, WA 98195-4350
UNITED STATES

Prof. Dr. Christian Liedtke
Zentrum Mathematik - M11
Technische Universität München
Boltzmannstrasse 3
85748 Garching bei München
GERMANY

Dr. Daniel Litt
Department of Mathematics
University of Georgia
Athens GA 30602
UNITED STATES

Lucas Mann
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Akhil Mathew
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Prof. Dr. Sophie Morel
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Dr. Matthew Morrow
Institut de Mathématiques de Jussieu
Case 247
Sorbonne Université
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Wieslawa Niziol
Département de Mathématiques, UMPA
ÉNS de Lyon
46, Allée d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Martin Olsson
Department of Mathematics
University of California, Berkeley
970 Evans Hall
Berkeley CA 94720-3840
UNITED STATES

Prof. Dr. Vytautas Paškunas
Fakultät für Mathematik
Universität Duisburg-Essen
Thea-Leymann-Strasse 9
45127 Essen
GERMANY

João Nuno Pereira Lourenço
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Vincent Pilloni
Département de Mathématiques
École Normale Supérieure de Lyon
46, Allée d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Michael Rapoport
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Emanuel Reinecke
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Prof. Dr. Timo Richarz
Fachbereich Mathematik
Technische Universität Darmstadt
Schloßgartenstrasse 7
64289 Darmstadt
GERMANY

Dr. Daniil Rudenko
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Prof. Dr. Peter Scholze
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Benjamin Schraen
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Bâtiment 307
91405 Orsay Cedex
FRANCE

Dr. Georg Tamme
Fakultät für Mathematik
Universität Regensburg
Universitätsstrasse 31
93053 Regensburg
GERMANY

Prof. Dr. Eva Viehmann
Zentrum Mathematik - M 11
Technische Universität München
Boltzmannstrasse 3
85748 Garching bei München
GERMANY

Prof. Dr. Kirsten G. Wickelgren
Department of Mathematics
Duke University
Room 117, Physics
120 Science Drive
Durham, NC 27708-0320
UNITED STATES

Mingjia Zhang
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Wei Zhang
Department of Mathematics
MIT
77 Massachusetts Avenue
Cambridge MA 02139
UNITED STATES