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Automorphic Forms and Arithmetic (hybrid meeting)

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Abstract. The workshop was at the interface of automorphic forms and analytic number theory. The aim was to disseminate, discuss and develop important new methods and results in the analytic theory of automorphic forms, in particular on higher rank groups, as well as their arithmetic applications. This includes, for instance, the study of various aspects of L-functions such as moments, reciprocity laws, and probabilistic aspects, counting problems with automorphic forms, applications of results of algebraic geometry to automorphic forms, as well as analytic aspects of automorphic forms over function fields.

Mathematics Subject Classification (2010): 11Fxx, 11Lxx, 11Mxx, 11Nxx, 22E5x, 42Bxx, 52Cxx.

Introduction by the Organizers

First and foremost, we would like to thank the Mathematisches Forschungsinstitut Oberwolfach for making this conference possible in an equally safe and enjoyable format. The opportunity of a combined personal and digital exchange of ideas provided an excellent framework for this productive workshop. This was explicitly mentioned by many of the participants.

The workshop "Automorphic Forms and Arithmetic" was attended by 15 participants being physically present in Oberwolfach, and 36 additional participants who joined digitally, with a total of 14 female participants. We heard 22 talks featuring a selection of the latest research results in the field.

Modern analytic number theory draws from a wide array of methods ranging from algebraic geometry to real and complex analysis and ergodic theory. An overarching theme is the theory of automorphic forms where the interactions goes in both directions: families of automorphic forms and their L-functions can be investigated by means of analytic number theory, and conversely many classical questions in analytic number theory need the full power of the spectral theory of automorphic forms. The goal of the workshop was the exploration of the links between analytic number theory and automorphic forms, with a particular focus on areas where deep new ideas are emerging, such as

- moments of L-functions and equidistribution,
- automorphic forms over function fields, and
- applications of modular forms to sphere packing.

We highlight major results that were presented featuring strong methodological variety.

New and unexpected connections between moments of L-functions and equidistribution were presented in talks of Brumley, Young and Risager, focusing on quantum variance and simultaneous equidistribution of toric periods. The latter is an excellent example of the fruitful interactions of arithmetic, automorphic forms and ergodic theory: Duke's classical equidistribution theorem proves the equidistribution of toric periods within quotients of adelic quaternion algebras. It has several arithmetic incarnations, such as equidistribution of Heegner points, lattice points on the sphere and supersingular reduction of elliptic curves. In the modern framework Duke's theorem is an application of subconvexity bounds for automorphic L-functions. It is quite remarkable that the much harder set-up of simultaneous equidstribution of a diagonally embedded toric period within two copies of adelic quaternion algebras can be also approached, and even in two ways: through ergodic means (the joinings theorem of Einsiedler and Lindenstrauss) and through an analysis of fractional moments of certain L-functions.

Nelson presented his work on the cubic moment for PGL_2 over number fields that emphasizes the point of view of period integrals and a method to handle general test functions; Nunes approached the topic of reciprocity formulas using integral representations, and interesting links were noticed between parts of the two problems. Nelson's work is partly inspired by a recent breakthrough result of Petrow and Young that was reported on by Petrow – the Weyl bound for Dirichlet L-functions – which after almost 70 years improves on the Burgess bound from 1960s. The proof features a beautiful combination of analytic number theory, automorphic forms and algebraic geometry in the form of trace functions. In connection with Nelson's work, this Weyl bounds is also valid over general number fields.

Various aspects of families of automorphic forms were considered in talks of Khayutin, Milićević and Matz. Jasmin Matz, who opened the conference, discussed quantum ergodicity in the level aspect, while Milićević settled a very general version of Sarnak's density conjecture in the level aspect. The density conjecture for exceptional eigenvalues – inspired by the density conjecture for exceptional zeros of L-functions – features a flow of ideas in the other directions: deep properties of automorphic representations can be obtained from the geometry of numbers in number fields or arithmetic properties of generalized Kloosterman sums.

Khayutin presented an exciting new method to bound large values of automorphic forms, an approach which for the first time deviates from the classical Iwaniec-Sarnak technique and therefore offers a number of new features. This method is, quite surprisingly, particularly strong in the challenging squarefree level aspect; the new exponent breaks the previous 2015 record by N. Templier and goes half way towards Sarnak's purity conjecture. It is the strongest result ever obtained in any version of the sup-norm problem.

Large values from the point of view of automorphic over function fields were discussed in a talk of Will Sawin. Kowalski surveyed recent work of Sawin that defines suitable notions of "complexity" for trace functions in many variables, which have the potential to facilitate many applications of Deligne's Riemann Hypothesis to analytic number theory. The MFO workshop provided a perfect opportunity to present the powerful function field aspect of automorphic forms theory to a large audience of analytic number theorists. Kowalski's sheaf theoretic talk was complemented by a very different viewpoint on exponential sums in a talk by Lillian Pierce who discussed direct estimates in the context of harmonic analysis, yet another manifestation of the interdisciplinary nature of analytic number theory.

Important new results in classical analytic number theory were presented in talks of Harper on arithmetic insights that can be derived from the probabilistic notion of "multiplicative chaos", Merikoski on a generalization of the famous Friedlander-Iwaniec result concerning primes of the form $a^2 + b^6$ and in particular a double talk of Teräväinen and Kaisa Matomäki on higher order uniformity of the Möbius function, presenting deep and difficult breakthrough results on the non-correlation of the Möbius function with polynomial and even more general phases related to nilsequences.

Anke Pohl presented some surprising, partly experimental and still unexplained discoveries about the structure of the set of resonances on infinite-area hyperbolic surfaces, while Saha focused on the fundamental Fourier coefficients for the group GSp_4.

One of the most spectacular applications of modular forms is the sphere packing problem, in particular in dimension 8 and 24, which is intimately connected also to Fourier interpolation and the question under what conditions a function can be reconstructed from a subset of its values and a subsets of the values of its Fourier transform. The recent developments in this area were discussed in several talks: Radchenko has introduced a new surprising Fourier interpolation formula that can essentially reconstruct a function from its values at the non-trivial zeros of an L-function (for example the Riemann zeta function) and the values of its Fourier transform at the logarithms of integers. This formula is a broad generalization of the famous Riemann–Weil explicit formula. Cohn spoke about the newly discovered connection between the Cohn-Elkies linear programming bound for the sphere packings and the modular bootstrap in conformal field theory. He also presented new numerical results computing the above-mentioned bounds in high dimensions. Matthew de Courcy-Ireland gave an overview of linear programming bounds for the sphere packing and presented an approach for obtaining a "bound on the bound", while Hedenmalm highlighted the connection between the Klein-Gordon equation and Fourier interpolation.

The conference spanned a wide range of viewpoints on automorphic forms and arithmetic, emphasizing its interdisciplinary nature. Wednesday evening was devoted to a lively and interesting problem session with contributions both from people at the institute and those joining digitally from various parts of the world.

Workshop (hybrid meeting): Automorphic Forms and Arithmetic

Table of Contents

Jasmin Matz (joint with Farrell Brumley)
Quantum Ergodicity in the level aspect 1323

Paul D. Nelson
Eisenstein series and the cubic moment for PGL_2 1325

Farrell Brumley (joint with Valentin Blomer)
Simultaneous equidistribution of torus orbits and fractional moments of twisted L-functions 1328

Will Sawin
Questions on exact large values of modular forms 1330

Ilya Khayutin (joint with Paul Nelson, Raphael S. Steiner)
Fourth Moments of Modular Forms on Arithmetic Surfaces 1333

Matthew de Courcy-Ireland
Sphere packing and linear programming: a bound on the bound 1336

Danylo Radchenko (joint with Andriy Bondarenko, Kristian Seip)
Fourier interpolation from zeros of the Riemann zeta function 1338

Ramon Nunes
Spectral Reciprocity via Integral representations 1340

Adam J. Harper
Multiplicative chaos in number theory 1343

Lillian B. Pierce
On Superorthogonality 1345

Ian Petrow (joint with Matthew P. Young)
Cubic moments and a distance function on cuspidal representations 1347

Kaisa Matomäki, Joni Terävainen (joint with Maksym Radziwiłł, Terence Tao, Tamar Ziegler)
Higher order uniformity of the Möbius function I and II 1350

Anke Pohl (joint with O. Bandtlow, T. Schick, A. Weiße)
Resonances of Schottky surfaces 1352

Abhishek Saha (joint with Jesse Jääsaari, Stephen Lester)
Sign changes and large values for fundamental Fourier coefficients of Siegel cusp forms of degree 2 1355

Emmanuel Kowalski (joint with A. Forey, J. Fresán, W. Sawin)
Quantitative sheaf theory (d'après Sawin) 1357

Henry Cohn (joint with Nima Afkhami-Jeddi, Thomas Hartman, David de Laat, and Amirhossein Tajdini)
Sphere packing and the modular bootstrap 1361

Matthew P. Young (joint with Rizwanur Khan)
Moments and hybrid subconvexity for symmetric-square L-functions . . . 1362

Haakan Hedenmalm
Fourier uniqueness in even dimensions 1365

Djordje Milićević (joint with Mikołaj Frączyk, Gergely Harcos, Péter Maga)
The Density Hypothesis for Horizontal Families of Lattices 1370

Jori Merikoski
A cubic analogue of the Friedlader-Iwaniec spin along primes 1373

Morten S. Risager (joint with Asbjørn C. Nordentoft, Yiannis N. Petridis)
Small scale equidistribution of Hecke eigenforms at infinity 1375

Problem Session 1377

Abstracts

Quantum Ergodicity in the level aspect

Jasmin Matz
(joint work with Farrell Brumley)

Suppose S is a Riemannian symmetric space of non-compact type, and let G be its group of isometries. Let $\Gamma_n \leq G$, $n \in \mathbb{N}$, be a sequence of torsion-free, cocompact lattices such that the sequence of compact locally symmetric spaces $Y_n := \Gamma_n \backslash S$ Benjamini-Schramm converges to S as $n \to \infty$. In this setting, [1] showed that *limit multitplicity* holds, that is, as $n \to \infty$, the L^2-spectra of Y_n converge to the L^2-spectrum of S in a suitable sense.

A more refined question then involves the distribution of not only the spectrum, but also the corresponding eigenfunctions on the Y_n. In the situation of just one fixed closed Riemannian manifold M whose geodesic flow is ergodic, Shnirelman and others [8, 4, 9] and others showed that any orthonormal basis $\{\psi_j\}_{j\in\mathbb{N}}$ consisting of L^2-normalized Laplace eigenfunctions of M is *quantum ergodic*, so in particular, for every $a \in C(M)$,

$$\frac{1}{\#\{j : \lambda_j \leq \lambda\}} \sum_{j:\, \lambda_j \leq \lambda} \left| \langle a\psi_j, \psi_j \rangle_{L^2(M)} - \frac{1}{\mathrm{vol}(M)} \int_M a \right|^2 \longrightarrow 0$$

as $\lambda \to \infty$, whith $\lambda_j \geq 0$ denotes the Laplace eigenvalue of ψ_j.

In view of those results it make sense to ask for a type of quantum ergodicity result that looks at the behavior of the Laplace eigenfunctions in the high level aspect instead of the high energy one.

A first precise formulation of this type of question has been provided in [7] and [2], where they focus on the case of G having rank 1. In our recent preprint [3] we consider the following higher rank situation: We suppose that $S = \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$ with $d \geq 3$, and $\Gamma_n \leq \mathrm{SL}_d(\mathbb{R})$ is a sequence of torsion-free cocompact lattices. We write $Y_n = \Gamma_n \backslash S$ again. We suppose that the Γ_n are uniformly discrete, that is, the distances between any $g \in S$ and its translates γg, $\gamma \in \Gamma_n - \{1\}$ are uniformly bounded away from 0.

Before we can state our main result, we need some more notation. Let $\mathfrak{a}$ be the Lie algebra of the usual maximal split torus T of $G = \mathrm{SL}_d(\mathbb{R})$ consisting of diagonal matrices, and let W denote the Weyl group of (T, G). Then the Weyl group orbit $i\mathfrak{a}^*/W$ of the subspace $i\mathfrak{a}^*$ of the complexified dual $\mathfrak{a}^*_{\mathbb{C}}$ can be identified with the unramified tempered spectrum of G. We call a spectral parameter $\nu \in i\mathfrak{a}^*/W$ *sufficiently regular* if ν is sufficiently far away from all singular hyperplanes determined by the roots of (T, G) in $i\mathfrak{a}^*/W$.

For each n let $\{\psi_{n,j}\}_{j\in\mathbb{N}}$ denote an orthonormal basis of $L^2(Y_n)$ consisting of Laplace eigenfunctions. Let $\nu_{n,j} \in \mathfrak{a}^*_{\mathbb{C}}/W$ denote the spectral parameter of $\psi_{n,j}$. For $\nu \in i\mathfrak{a}^*/W$ and $\varrho > 0$ let $B(\nu, \varrho)$ denote the ball of radius ϱ around ν in

$i\mathfrak{a}^*/W$, and let $N_n(\nu,\varrho)$ denote the number of $j \in \mathbb{N}$ with $\nu_{n,j} \in B(\nu,\varrho)$. The main result of [3] then is:

Theorem. *Let $d \geq 3$, $S = \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}(d)$, and let $\Gamma_n \leq \mathrm{SL}_d(\mathbb{R})$ be a sequence of uniformly discrete cocompact lattices with $\mathrm{vol}(\Gamma_n\backslash S) \longrightarrow \infty$. Let $a_n \in C(\Gamma_n\backslash S)$ be a sequence of uniformly bounded functions. Then there exists $\varrho > 0$ such that for every sufficiently regular $\nu \in i\mathfrak{a}^*/W$,*

$$\frac{1}{N_n(\nu,\varrho)} \sum_{j\colon \nu_{n,j}\in B(\nu,\varrho)} \left| \langle a_n\psi_{n,j}, \psi_{n,j}\rangle_{L^2(Y_n)} - \frac{1}{\mathrm{vol}(Y_n)}\int_{Y_n} a_n \right|^2 \longrightarrow 0 \tag{1}$$

as $n \to \infty$.

Remarks:

- The analogue result for $d = 2$ had been proven in [7] under the additional assumptions that the Y_n Benjamini-Schramm converge to S as $n \to \infty$, and that the smallest non-zero Laplace eigenvalues of Y_n are uniformly bounded away from 0 for all n. We require those properties as well, but they are automatically satisfied when $d \geq 3$ because of [1] and [6].
- [2] extend the results of [7] to general rank 1 spaces. They are also able to use pseudo-differential operators of degree 0 instead of the functions a_n, and permit (sufficiently slowly) shrinking intervals instead of intervals $B(\nu,\varrho)$ of fixed length. So far, we were not able to incorporate those features into our higher rank results.

Ideas of proof

The first main step in the proof of the Theorem is to average over a kind of wave propagation operator, an idea that was already used in [7] and [2]. More precisely, we need to find a suitable family of expanding measurable sets E_t, $t \geq 0$, with $\bigcup_{t\geq 0} E_t = G$. Let $\rho_{\Gamma_n\backslash G}$ be the right regular representation on $L^2(\Gamma_n\backslash G)$, and $k_t \in C_c(G)$ the characteristic function of E_t normalized by the square-root of its measure. We then define our wave propagation operator as $U_t = \rho_{\Gamma\backslash G}(k_t) : L^2(\Gamma_n\backslash G) \longrightarrow L^2(\Gamma_n\backslash G)$. From now on we assume that $\int_{Y_n} a_n = 0$.

Our next task then is to show that we can replace $\left|\langle a_n\psi_{n,j}, \psi_{n,j}\rangle_{L^2(Y_n)}\right|^2$ by a time average: For every j with $\nu_{n,j} \in B(\nu,\varrho)$ we have

$$\left|\langle a_n\psi_{n,j}, \psi_{n,j}\rangle_{L^2(Y_n)}\right|^2 \ll_{\nu,\varrho} \left|\langle A(\tau)\psi_{n,j}, \psi_{n,j}\rangle_{L^2(Y_n)}\right|^2$$

where $A_n(\tau) = \frac{1}{\tau}\int_0^\tau U_t a_n U_t\, dt L^2(Y_n) \longrightarrow L^2(Y_n)$, and the implied constant is independent of n and the choice of functions a_n. As a consequence the left hand side of (1) is bounded by a constant multiple of the square of the Hilbert-Schmidt norm $\|A_n(\tau)\|_{HS}^2$ of $A_n(\tau)$. Proving this inequality requires a good lower bound on certain elementary spherical functions on average.

We now need to bound $\|A_n(\tau)\|_{HS}^2$ from above. More precisely, we prove that for $\tau \gg 0$ we have

$$(2) \qquad \|A_n(\tau)\|_{HS}^2 \ll \frac{\|a_n\|_{L^2(Y_n)}^2}{\tau} + \frac{e^{c\tau}}{\mathrm{InjRad}(Y_n)^{\dim S}} \mathrm{vol}((Y_n)_{\leq 2\tau + c'}) \|a_n\|_{L^\infty(Y_n)}^2$$

where $c, c' > 0$ are suitable constants, and $(Y_n)_{\leq r}$ denotes the r-thin part of Y_n. Once this inequality is established, we use the fact that the Y_n Benjamini-Schramm converge to S to find a suitable sequence of $\tau = \tau_n$ with $\tau_n \longrightarrow \infty$ such that the right hand side goes to 0 as $n \to \infty$. This finishes the proof of the Theorem once we prove a lower bound on $N_n(\nu, \varrho)$ of the right order. A key ingredient in the proof of (2) is a quantitative ergodic mean theorem due to Nevo [5]. Consequently, we need to bound the volume of the intersections $gE_t \cap E_t \subseteq G$ from above, for which our choice of E_t becomes crucial. Metric balls as used in [7, 2] do not work for us in higher rank any longer so that we need to define those expanding sets in a different way, using suitable expanding polytopes in $\mathfrak{a}$.

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Eisenstein series and the cubic moment for $\mathbf{PGL_2}$

Paul D. Nelson

We described (following the preprint [7]) how to use Eisenstein series to establish summation formulas for the cubic moment of standard L-functions on PGL_2. We deduced some applications.

The cubic moment of interest first appeared in Motohashi's formula [6] for the fourth moment of the Riemann zeta function (see also Ivić [3]):

$$\int_{t\in\mathbb{R}} |\zeta(1/2+it)|^4 h(t)\,dt = \sum_{\varphi} \frac{L(\varphi,1/2)^3}{L^*(\varphi\times\varphi,1)} \tilde{h}(r_\varphi) + (\cdots).$$

Here h is a test function, $\tilde{h}$ is an explicit integral transform, φ runs over the Hecke–Maass cusp forms on $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ of eigenvalue $1/4 + r_\varphi^2$, and $(\cdots)$ denotes the analogous contributions from holomorphic forms, Eisenstein series, plus a degenerate term.

In their 2006 ICM survey, Michel–Venkatesh [4] suggested a "one-line proof sketch" of Motohashi's formula, assuming standard facts from the theory of integral representations of L-functions. Denoting by $E_s : \Gamma\backslash\mathbb{H} \to \mathbb{C}$ the Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ with Fourier expansion

$$E_s(x+iy) = \sum_{\pm} \xi(1\pm 2s) y^{1/2\pm s} + \sum_{n\geq 1} \frac{\tau_s(n)}{\sqrt{n}} 2\cos(2\pi n x) W_s(ny),$$

their sketch consists of evaluating the regularized inner product

$$\int_0^\infty E_0^2(iy)\,d^\times y$$

via two different spectral expansions and applying standard facts from the theory of integral representations of L-functions.

We briefly indicated how one can develop this approach to Motohashi-type formulas rigorously, paying attention to issues of regularization. The most direct implementation of this method gives Motohashi-type formulas for rather specific weight functions. The main focus of the talk was to explain how to achieve flexible weights on the cubic moment side. Our discussion featured the diagram

$$\begin{tikzcd}
 & \mathcal{I}(0)\otimes\mathcal{I}(0) \arrow[dl, "f\mapsto h_f"'] \arrow[dr, "f\mapsto \tilde{h}_f"] & \\
\{h : (\mathrm{PGL}_2)^\wedge_{\mathrm{gen}} \to \mathbb{C}\} \arrow[rr, dashed, leftrightarrow, "?"] & & \{\tilde{h} : (\mathrm{GL}_1)^\wedge \to \mathbb{C}\}
\end{tikzcd}$$

relating elements of an induced representation of $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ (parametrizing linear combinations of products of Eisenstein series) to local weight functions, as well as the diagram

$$\begin{tikzcd}
\mathcal{I}(0)\otimes\mathcal{I}(0) \arrow[ddr, bend left, "f\mapsto \tilde{h}_f"] & \\
C_c^\infty(\mathrm{PGL}_2) \arrow[u, squiggly, "h_\phi = h_f"] \arrow[d, twoheadrightarrow, "\phi\mapsto h_\phi"'] \arrow[dr, dashed] & \\
\{h : (\mathrm{PGL}_2)^\wedge_{\mathrm{gen}} \to \mathbb{C}\} & \{\tilde{h} : (\mathrm{GL}_1)^\wedge \to \mathbb{C}\}
\end{tikzcd}$$

describing our approach for showing that every nice enough ("Paley–Wiener") test function on the cubic moment side arises from some linear combination of products of Eisenstein series.

Our main motivation for developing such an approach to Motohashi-type formulas is to introduce tools from representation theory and the theory of integral representations of L-functions into their study. The resulting method applies uniformly over number fields and in all aspects. It has some potential for generalization to higher rank groups.

We illustrated the transformation $h \mapsto \tilde{h}$ in an explicit example, showing how the two-variable exponential sums studied by Conrey–Iwaniec [2] arise.

We deduced several applications:

- Weyl-type subconvex bounds for quadratic Hecke characters over number fields, generalizing results of Conrey–Iwaniec [2].
- Improved estimates for representation numbers of ternary quadratic forms over number fields, improving upon estimates of Blomer–Harcos.
- Improved bounds for the prime geodesic theorem arithmetic 3-folds via recent work of Balog–Biró–Cherubini–Laaksonen [1].

We described how one might hope to understand recent work of Petrow–Young [8, 9] from the perspective of our method.

We indicated ongoing work with Lei Zhang that aims to generalize the basic summation formula to higher rank orthogonal groups using the (non-Gelfand) diagram

$$\begin{tikzcd} & \mathrm{SO}_{n+1} & \\ \mathrm{SO}_n \arrow[ur] & & \mathrm{SO}_{n-1} \times \mathrm{SO}_2 \arrow[ul] \\ & \mathrm{SO}_{n-1} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

One obtains in this way identities of the shape

$$\sum_{\sigma \subseteq L^2(\mathrm{SO}_{2n+1}(\mathbb{Z}) \backslash \mathrm{SO}_{2n+1}(\mathbb{R}))} h_f(\sigma) L(\sigma, 1/2)^{2n+1} = \int_{t \in \mathbb{R}^n} \tilde{h}_f(t) \mathcal{L}(t)\, dt + (\cdots).$$

These recover Motohashi's formula when $n = 1$, in which case $\mathcal{L}(t) = |\zeta(1/2+it)|^4$. When $n \geq 2$, the quantity $\mathcal{L}(t)$ is more mysterious: it is defined by an Eulerian integral, but seems unlikely to be an L-value in the traditional sense.

In summary, we gave a rigorous implementation of Michel–Venkatesh's strategy for deriving Motohashi-type formulas with flexible weights on the cubic moment side, deduced applications such as strong subconvex estimates over number fields, and indicated some directions for ongoing and future work.

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Simultaneous equidistribution of torus orbits and fractional moments of twisted *L*-functions

Farrell Brumley
(joint work with Valentin Blomer)

The classical Linnik problems are concerned with the equidistribution of periodic adelic torus orbits on the homogeneous spaces attached to inner forms of PGL_2, as the discriminant of the torus gets large. When specialized, these problems admit beautiful classical interpretations, such as the equidistribution of integer points on spheres, of Heegner points or packets of closed geodesics on the modular surface, or of supersingular reductions of CM elliptic curves. In the mid 20th century, Linnik and his school [11, 12] established the equidistribution of many of these classical variants through his ergodic method, under a congruence condition on the discriminants modulo a fixed auxiliary prime. When these methods are sufficiently quantified, the auxiliary congruence condition can be removed by appealing to the Generalized Riemann Hypothesis for the L-functions of quadratic Dirichlet characters.

More recently, the Waldspurger formula and subconvex estimates on L-functions (or, more accurately, of Fourier coefficients of half-integer weight modular forms) were used to remove these congruence conditions unconditionally, and provide effective power-savings rates. This is the fundamental work of Duke [3], dating from the late 1980's, which built on a breakthrough of Iwaniec [9]. Duke's equidistribution theorems were further generalized, and Linnik's difficult methods were understood through the lens of modern ergodic theory, in the early 2000's, thanks to the collaborative work Ellenberg, Einsiedler, Lindenstrauss, Michel, and Venkatesh [5, 6, 7, 8]. Perhaps most notably, the equidistribution of periodic adelic torus orbits on $\mathrm{PGL}_3(\mathbb{Z})\backslash\mathrm{PGL}_3(\mathbb{R})$ was established [6] by an ingenious combination of ergodic and analytic techniques.

In their 2006 ICM address [13], Michel and Venkatesh proposed a new variant of the Linnik equidistribution problems in which one considers the product of two distinct inner forms of PGL_2, along with a diagonally embedded torus of large discriminant. One can again specialize the setting to obtain interesting classical reformulations, such as the joint equidistribution of integer points on the sphere, together with the shape of the orthogonal lattice. This hybrid context has received a great deal of attention recently in the dynamics community, where, for instance, the latter problem was solved by Aka, Einsiedler, and Shapira [1], under supplementary congruence conditions modulo two fixed primes, using as critical input the joinings theorem of Einsiedler and Lindenstrauss [4]. In fact the motivating application of the Michel–Venkatesh conjecture was to the simultaneous supersingular reduction of CM elliptic curves, which has recently been shown in [2] to follow from the joinings theorem, again subject to the double Linnik condition at two distinct auxiliary primes. In contrast to the original Linnik problem, these ergodic methods have not yet been sufficiently quantified so as to allow for a removal, under the assumption of GRH, of the twin congruence conditions on the discriminants. From the perspective of effective methods in homogeneous dynamics, this seems to be a difficult problem.

In joint work with Valentin Blomer, we remove the supplementary congruence conditions in the joint equidistribution problem, conditionally on GRH, while obtaining a logarithmic rate of convergence. (We impose further technical constraints, like maximal level structure, and we test equidistribution only against the discrete automorphic spectrum.) The proof uses Waldspurger's theorem to reduce the problem to that of bounding the following fractional moment of L-functions in the family of class group twists:

$$\frac{1}{\operatorname{vol}\widetilde{\mathrm{Cl}}_E}\sum_{\chi\in\widetilde{\mathrm{Cl}}_E^\vee} L(1/2,\pi_1,\times\chi)^{1/2}L(1/2,\pi_2,\times\chi)^{1/2}.$$

Here, π_1 and π_2 are two distinct cuspidal automorphic representations of PGL_2 over $\mathbb{Q}$ of square-free level, E is a quadratic field extension of $\mathbb{Q}$ of large discriminant D, and $\widetilde{\mathrm{Cl}}_E$ is the Arakelov class group of its ring of integers. Using the pioneering method of Soundararajan [14] to estimate high moments of L-functions, and in particular its more recent use in the work of Lester–Radziwiłł [10], we show that, under GRH, the above moment decays faster than $(\log|D|)^{-\delta}$ for any $0<\delta<1/4$.

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Questions on exact large values of modular forms

Will Sawin

The sup-norm problem asks for the size of the largest value of an automorphic form on an arithmetic manifold. In investigating a function field analogue of the sup-norm problem [7], I came to a number of questions, which make sense in the classical setting as well. The goal of my talk was to present these questions.

We adapt a setup of Milićević [4], building on work of Rudnick and Sarnak [6]. Let F be an imaginary quadratic extension of $\mathbb{Q}$. Let D be a quaternion algebra over $\mathbb{Q}$. Assume D is not split at ∞ and not split over F. Let $K = K_\infty K^\infty$ be a compact subgroup of $D^\times(\mathbb{A}_F)$ with K^∞ open and

$$K_\infty = D^\times(\mathbb{R}) \mod Z(D^\times(F_\infty)).$$

Then we can take

$$M = D^\times(F)\backslash D^\times(\mathbb{A}_F)/KZ(D^\times(\mathbb{A}_F)).$$

Because

$$D^\times(F_\infty)/(K_\infty Z(D^\times(F_\infty))) = GL_2(\mathbb{C})/(U_2(\mathbb{R})\mathbb{C}^\times) = \mathbb{H}^3,$$

M is an arithmetic hyperbolic three-manifold. Because D is not split over F, M is compact.

Let f be an eigenfunction of the Laplacian on M, with eigenvalue λ, which is also an eigenform of the Hecke operators. Normalize f to have $\int_M |f(x)|^2\mu = 1$, where in this, and every subsequent integral, μ is a measure of total mass 1. Does $|f|$ take values much larger than its average value 1? Milićević showed, using the relative trace formula, that there exist infinitely many f such that the period integral

$$\int_{D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_\mathbb{Q})} f(x)d\mu(x) \tag{1}$$

is greater than a constant times $\lambda^{1/4}$ [4]. This built on earlier work of Rudnick and Sarnak [6]. It follows that for at least some

$$x \in D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_\mathbb{Q})/(K \cap D^\times(\mathbb{A}_\mathbb{Q}))Z(D^\times(\mathbb{A}_\mathbb{Q}))$$

we have $f(x)$ greater than a constant times $\lambda^{1/4}$. It turns out that this double coset space is a finite set.

I would like to know a better asymptotic formula for the values of f on this finite set. We say f is special if f arises from the quadratic base change of an automorphic form on $GL_2(\mathbb{A}_\mathbb{Q})$ with central character the quadratic character χ_F associated to $F/\mathbb{Q}$. Based on [6], we expect that for special F the integral (1) is large and for all other f this integral should vanish. It is reasonable to expect that for special f, the value at each point is close to the value of the period integral. Confirming that expectation would answer the following two questions:

Question 1. Does there exist $\delta > 0$ such that for $y \in D^\times(\mathbb{A}_Q)$, for all Hecke-Laplace eigenfunctions f, we have

$$f(y) = \int_{D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_\mathbb{Q})} f(x)d\mu(x) + O\left(\lambda^{1/4-\delta}\right)?$$

Question 2. Do we have, for $y \in D(\mathbb{A}_Q)$, for all special eigenfunctions f,

$$f(y) = (1 + o(1)) \int_{D(\mathbb{Q})\backslash D(\mathbb{A}_\mathbb{Q})} f(x)d\mu(x)?$$

To obtain an asymptotic for $f(y)$, we would also need a formula for the period integral, similar to what Lapid and Offen [3] did for GL_n instead of $D^\times$.

To answer the first question, it would suffice to estimate the period integrals

$$\int_{D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_\mathbb{Q})} f(x)g(x)d\mu(x)$$

for g a nonconstant Hecke eigenform on $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_\mathbb{Q})/(K \cap D^\times(\mathbb{A}_\mathbb{Q}))Z(D^\times(\mathbb{A}_\mathbb{Q}))$. Upper bounds for similar period integrals, in the case D nonsplit, were obtained in [5, Theorem B], using Ichino's triple product formula [2] to reduce to estimating a certain L-function. Nelson suggested a similar method could work for this problem, but only conditionally on the Ramanujan conjecture for f together with Sato-Tate-like estimates on the Fourier coefficients of f. Sufficient estimates for the L-function would also follow from the generalized Riemann hypothesis.

One can also consider questions about the behavior of f near special points $x \in D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_\mathbb{Q})$. For this, the most important invariant is the derivative of f, which is a function on $D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_\mathbb{Q})/(K^\infty \cap D^\times(\mathbb{A}_\mathbb{Q}))Z(D^\times(\mathbb{A}_\mathbb{Q}))$ that is not invariant under $K_\infty \cap D^\times(\mathbb{A}_\mathbb{Q})$ but instead transforms according to the three-dimensional representation of $(K_\infty \cap D^\times(\mathbb{A}_\mathbb{Q}))/Z(K_\infty \cap D^\times(\mathbb{A}_\mathbb{Q})) = SO(3)$. If there are no such functions, the derivative vanishes, and it is reasonable to guess that x is a local maximum, except when the eigenvalue is very small.

Question 3. If there are no nonzero functions on

$$D^\times(\mathbb{Q})\backslash D^\times(\mathbb{A}_\mathbb{Q})/(K^\infty \cap D^\times(\mathbb{A}_\mathbb{Q}))Z(D^\times(\mathbb{A}_\mathbb{Q}))$$

that transform according to the 3-dimensional spherical representation of $SO(3)$, then for all but finitely many special eigenfunctions f, is each $x \in D^\times(\mathbb{A}_\mathbb{Q})$ a local maximum?

Question 4. If there are nonzero such functions, then is each $x \in D^\times(\mathbb{A}_\mathbb{Q})$ not a local maximum for a density 1 set of special eigenfunctions f?

My final questions concern the function field setting, in the genus aspect. In this setting, we fix a quaternion algebra D over $\mathbb{F}_q(t)$, and let F be a varying quadratic extension of q. Let g be the genus of the underlying hyperelliptic curve, equivalently, let q^{2g+2} be the discriminant of F. Let K be a compact subgroup of $D^\times(\mathbb{A}_F)$. Let m be the index of $K \cap D^\times(\mathbb{A}_{\mathbb{F}_q(t)})$ in a maximal compact subgroup, which we note does not depend on the choice of maximal compact subgroup of $D^\times(\mathbb{A}_{\mathbb{F}_q(t)})$ because they all have the same Haar measure.

Let f be a complex-valued function on $D^\times(F)\backslash D^\times(\mathbb{A}_F)/KZ(D^\times(\mathbb{A}_F))$ which is an eigenfunction of the Hecke operators and such that $\int_{D^\times(F)\backslash D^\times(\mathbb{A}_F)} |f(x)|^2 d\mu(x) = 1$.

We can first ask for an analogue of the result of Milićević [4] in this setting.

Question 5. Does there exist a universal constant C such that

$$\int_{D^\times(\mathbb{F}_q(t))\backslash D^\times(\mathbb{A}_{\mathbb{F}_q(t)})} f(x)d\mu(x) = \Omega(m^{-C}q^{g/2})?$$

Question 6. Do there exist universal constants C and $\delta > 0$ such that for $y \in D^\times(\mathbb{A}_{F_q(t)})$,

$$f(y) = \int_{D^\times(\mathbb{F}_q(t))\backslash D^\times(\mathbb{A}_{\mathbb{F}_q(t)})} f(x)d\mu(x) + O\left(m^C q^{(1-\delta)g/2}\right)?$$

The implicit constants in the big O and Ω should depend on q and D but not on F or K.

I can check these hold in the case where D is split and K is the standard maximal compact subgroup (though, in that case, $D^\times(\mathbb{F}_q(t))\backslash D^\times(\mathbb{A}_{\mathbb{F}_q(t)})$ is noncompact and so the constant may depend on y) explicitly using the Fourier expansion (calculated by Drinfeld [1] in this case). Thus I suspect this is the right formula in general.

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Fourth Moments of Modular Forms on Arithmetic Surfaces

Ilya Khayutin
(joint work with Paul Nelson, Raphael S. Steiner)

In this talk we have discussed recent progress on the sup-norm problems for $\mathbf{GL}_2$. Let $\Gamma < \mathbf{SL}_2(\mathbb{R})$ be an arithmetic lattice and $\varphi\colon \Gamma\backslash\mathbf{SL}_2(\mathbb{R}) \to \mathbb{C}$ be a Hecke eigenfrom. Assume[1] $\|\varphi\|_2 = 1$. The sup-norm problem asks for the best possible bound on $\|\varphi\|_\infty$ in terms of the analytic conductor. An important variant is the problem of finding the best possible bound on $\|\varphi\restriction_\Omega\|_\infty$, where $\Omega \subset \mathbf{SL}_2(\mathbb{R})$ is a bounded open subset, in terms of the analytic conductor and Ω. In practice, we often seek a bound independently in terms of the Laplace eigenvalue, the weight or the level. The sup-norm problem is analogues to the subconvexity problem for automorphic L-functions; and indeed proving an optimal bound on the sup-norm would imply several instances of the generalized Lindelöf hypothesis.

The first breakthrough is due to Iwaniec–Sarnak [IS95]. Let $\Gamma < \mathbf{SL}_2(\mathbb{R})$ be the unit norm elements of an Eichler order in an indefinite quaternion algebra over $\mathbb{Q}$. Iwaniec and Sarnak establish the inequality

$$\|\varphi\|_\infty \ll_{\Gamma,\varepsilon} (1+\lambda_\varphi)^{5/24+\varepsilon}\|\varphi\|_2 \tag{1}$$

for a Hecke-Mass form $\varphi\colon \Gamma\backslash\mathbb{H} \to \mathbb{C}$ of Laplace eigenvalue λ_φ. The same inequality (1) with the larger exponent $1/4$ is known to hold for any Laplace eigenfunction on a compact Riemannian manifold [Sog88]. Hence (1) saves a power over the "trivial" convexity bound. Incidentally, the $5/24$ exponent has not been improved since [IS95]. Iwaniec and Sarnak have introduced the method of amplification as a major tool in studying the sup-norm problem. It has been then extensively used and developed further in [BH10, Tem10, Tem15, HT12, HT13, Sah17a, Sah17b, Sah20, HS19, Kır14, DS15, Ste17, BHM16, BHMM20, Ass17, Van97, BMi11, BMi13, BP16, BMa15, BMa16, Mar14] to establish sup-norm bounds on automorphic forms in different aspects and settings.

In this talk we have presented new results whose proof does not use amplification. Instead, the argument exploits a method to bound fourth moments of families of automorphic forms using the theta correspondence and a sharp second-moment count of integral matrices in terms of the determinant. The theta correspondence was first used by Steiner to bound sup-norms of Hecke eigenforms on the 3-sphere [Ste20], and this result motivated our study into the subject. In this talk we had several new theorems to present. Let $\Gamma < \mathbf{SL}_2(\mathbb{R})$ be the units of an Eichler order as above. Let $\mathcal{B}_m^{\mathrm{new}}$ be an orthonormal basis of Hecke eigenforms for $S_m^{\mathrm{new}}(\Gamma)$ –

[1] with respect to the probability measure on $\Gamma\backslash\mathbf{SL}_2(\mathbb{R})$.

the space of weight m holomorphic newforms attached to $\Gamma\backslash\mathbb{H}$. Denote by V the covolume of Γ.

Theorem 1 (Khayutin-Steiner [KS19])**.** *Assume $m > 2$ and let $z = x + iy \in \mathbb{H}$. Then there exists $A > 0$ such that*

$$y^{2m} \sum_{f\in\mathcal{B}_m^{\mathrm{new}}} |f(z)|^4 \ll_\varepsilon V^A m^{1+\varepsilon}(1 + \mathrm{ht}_\Gamma(z)^2 m^{-1/2}) \,, \tag{2}$$

where ht_Γ *vanishes if* Γ *is co-compact and for a congruence subgroup* $\Gamma < \mathbf{SL}_2(\mathbb{Z})$

$$\mathrm{ht}_\Gamma = \min\{\mathrm{Im}(\gamma.z) \mid \gamma \in \mathbf{SL}_2(\mathbb{Z})\} \,. \tag{3}$$

This bound is best possible in terms of the power of m. As a corollary we derive the individual convexity-breaking bound $\|f\|_\infty \ll_{V,\varepsilon} m^{1/4+\varepsilon}$. This extends the results of Xia [Xia07] to the co-compact setting and improves upon Das–Sengupta [DS15] and Ramacher–Wakatsuki [RW]. A notable feature of our method is that we do not need to use any strong arithmetic information about the Hecke eigenvalues of modular forms. In particular, unlike the previously known results we do not use Deligne's bound. Instead, our method bounds the fourth moment by an expression roughly of the form

$$\sum_{1\le n\ll m} \#\left\{\xi \in \mathcal{R} \mid \det\xi = n \,, u(z, \xi.z) \ll 1/m\right\}^2 \,, \tag{4}$$

where $\mathcal{R} \subset \mathbf{M}_2(\mathbb{R})$ is an Eichler order with $\Gamma = \mathcal{R} \cap \mathbf{SL}_2(\mathbb{R})$ and $u(z,w) = |z-w|^2/(4\,\mathrm{Im}\, z\,\mathrm{Im}\, w)$. The gist of the argument is that we are able to establish an optimal upper bound for (4) in terms of m. Our main tools being geometry of numbers and several delicate applications of the divisor bound to reduce the quadratic problem into several independent linear ones.

The value of A in 1 is very far from the best possible, even though an explicit value of A can be extracted from the argument. Indeed, one would hope to prove the optimal bound $A = 1 + \varepsilon$. The covolume satisfies $V = (qD_B)^{1+o(1)}$, where q is the level of the Eichler order $\mathcal{R}$ and D_B is the reduced discriminant of the rational quaternion algebra B containing $\mathcal{R}$. Hence an optimal bound of the form $\ll_\varepsilon V^{1+\varepsilon}$ on (2) would imply rather strong new sup-norms bounds in the level aspect. It turns out that proving an optimal fourth moment bound in terms of the covolume V is a formidable challenge. The counting methods from [KS19] need be replaced with several new ideas. Very recently, together with Paul Nelson, we have been able to establish an optimal bound to the analogue of (4) in terms of V.

Theorem 2 (Kh–Nelson–Steiner '20 [KNS20])**.** *Assume $m > 2$ and let $f \in S_m^{\mathrm{new}}(\Gamma)$ be a Hecke newform of weight $m > 2$ for the lattice Γ of unit norm elements in an Eichler order* R *of* square-free level *, then*

$$\|f\|_\infty \ll_\varepsilon (mV)^{1/4+\varepsilon}\|f\|_2 \,. \tag{5}$$

Moreover, if $\varphi\colon \Gamma\backslash\mathbb{H} \to \mathbb{C}$ is a Hecke-Maass newform of eigenvalue λ_φ then

$$\|\varphi\|_\infty \ll_{\varepsilon,\lambda_\varphi} V^{1/4+\varepsilon}\|\varphi\|_2 \,. \tag{6}$$

We also establish similar results for the case of definite quaternion algebras. It is worth mentioning that in addition to several new ingredients the the proof of Theorem 2 also utilizes an idea of Blomer–Michel from [BMi11, BMi13].

I would like to conclude with an open problem, which has thus far eluded us. For simplicity we state it only in the case of $\mathbf{SL}_2(\mathbb{Z})\backslash\mathbf{SL}_2(\mathbb{R})$.

Conjecture 3. *Fix $H > 0$. Let $\mathcal{B}$ be an orthonormal basis of Hecke-Maass eigenforms of the cuspidal spectrum $L^2\left(\mathbf{SL}_2(\mathbb{Z})\backslash\mathbb{H}\right)^{\mathrm{cusp}}$. Write the Laplace eigenvalue of $\varphi \in \mathcal{B}$ as $\lambda_\varphi = 1/4 + t_\varphi^2$, then for all $T \gg 1$*

$$\sup_{z\in\mathbb{H}} \sum_{\substack{\varphi\in\mathcal{B}\\ T\leq t_\varphi\leq T+H}} |\varphi(z)|^4 \ll_{\varepsilon,H} T^{1+\varepsilon}\,. \tag{7}$$

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Sphere packing and linear programming: a bound on the bound

Matthew de Courcy-Ireland

The sphere packing problem is to arrange non-overlapping spheres of equal size so as to occupy the greatest fraction of volume in Euclidean space. The linear programming bound of Cohn-Elkies estimates the optimal density from above, and a dual construction constrains how close an approximation can be achieved by this method. We will discuss these programs and pose the question of how insights into equidistribution for modular forms in the level aspect might improve the "bound on the bound".

The method of linear programming involves the construction of an auxiliary function, which yields a bound as follows. Suppose given an integrable function $f : \mathbb{R}^d \to \mathbb{R}$ whose Fourier transform satisfies $\widehat{f} \geq 0$, while $f(x) \leq 0$ for all x satisfying $|x| \geq r$. Cohn-Elkies show that the center density in $\mathbb{R}^d$ is then bounded above by

$$\frac{f(0)}{\widehat{f}(0)} \left(\frac{r}{2}\right)^d$$

It is easy to give examples of f satisfying the constraints (a convolution square, for example), but apparently difficult to minimize this objective. Viazovska succeeded in doing so in dimension $d = 8$, where the bound exactly matches the center density of a known configuration. Her method applies also to dimension $d = 24$, where

Cohn-Kumar-Miller-Radchenko-Viazovska showed that again the bound matches a known packing. As a result, these packings are now known to be optimal.

This match between the bound and a known configuration provably occurs for $d = 1$ as well, although the optimal choices of f do not seem parallel to the constructions for $d = 8$ or 24. Numerical evidence suggests a match for $d = 2$, but a certificate f is lacking. Except for $d = 1, 2, 8, 24$, there seem to be no cases where the Cohn-Elkies bound is sharp. In practice, numerical efforts to optimize f converge quickly to values larger than the best known densities. In principle, the possibility remains that some as-yet-unexplored candidates for f yield a better bound. To rule this out, a natural approach is duality for linear programs.

The underlying duality is the Plancherel relation

$$\langle f, \mu \rangle = \langle \widehat{f}, \widehat{\mu} \rangle$$

Suppose that μ is a measure with $\mu \geq \delta_0$ and $\mu - \delta_0$ supported outside a ball $|x| < r$ (corresponding to the radius r in the Cohn-Elkies bound). Suppose moreover that the Fourier transform satisfies $\widehat{\mu} \geq c\delta_0$ for some positive real c, where δ_0 is a Dirac delta at the origin. Then the Cohn-Elkies bound is at least

$$c \left(\frac{r}{2} \right)^d$$

In other words, regardless of whether there is a packing of a certain density, the linear programming bound can be tricked by a measure imitating such a packing.

This dual linear program was studied by Torquato-Stillinger. As $d \to \infty$, their analysis of a specific choice of μ shows that the Cohn-Elkies bound for packing density (rather than center density) is at least

$$2^{-\tau d + o(d)}, \quad \tau = \frac{3 - \frac{1}{\log 2}}{2} = 0.7786\ldots$$

This is substantially higher than the best known configurations, which have density $2^{-d+o(d)}$, and substantially lower than the best known construction of f, which achieves

$$2^{-\kappa d + o(d)}, \quad \kappa = 0.5990\ldots$$

A suitable f achieving this was adapted by Cohn-Zhao from a related construction of Kabatiansky-Levenshtein on the sphere instead of Euclidean space. The true size of the Cohn-Elkies bound could be anywhere in between Torquato-Stillinger and Kabatiansky-Levenshtein. It is not clear whether optimal packings in high dimensions can achieve a better exponential rate than 2^{-d}.

The connection with the present meeting is a method, proposed by Cohn-Triantafillou, to construct suitable measures μ from modular forms. Given a holomorphic function g on the upper half-plane, let $\widetilde{g}$ be its Atkin-Lehner transform with level N. For packing in dimension d, the relevant weight is $k = d/2$ and then $\widetilde{g}$ is given by

$$\widetilde{g}(z) = (-iz)^{-k} N^{-k/2} g\left(\frac{-1}{Nz} \right)$$

Assuming that both g and $\widetilde{g}$ are invariant under $z \mapsto z+1$, and of moderate growth, then there is a Fourier pair

$$\mu = \sum_{n=0}^{\infty} a_n \delta_{\sqrt{n}} \qquad \widehat{\mu} = \left(\frac{2}{\sqrt{N}}\right)^k \sum_{n=0}^{\infty} b_n \delta_{2\sqrt{n/N}}$$

where a_n and b_n are the respective coefficients in the Fourier expansions of g and $\widetilde{g}$. In particular, one obtains such a pair whenever g is a modular form of level N. The inequalities on μ and $\widehat{\mu}$ amount to a finite-dimensional linear program, with additional constraints $a_n = 0$ for finitely many coefficients of g. The number of coefficients forced to be 0 determines what radius r can be chosen. By this approach for $d = 12$ and $d = 16$ among other examples, Cohn-Triantafillou found measures μ outperforming the packings presumed best in those dimensions.

All of the above suggests the question of whether a suitable N can be found to approximate the strongest choices of μ in the dual "bound on the bound". If so, this might help to understand whether there are other dimensions besides $d = 1, 2, 8, 24$ where the linear programming bound is sharp, or to estimate the bound as $d \to \infty$. The purpose of this talk is to survey the background above in an expository fashion, and hopefully forge connections with other large-level problems.

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Fourier interpolation from zeros of the Riemann zeta function

Danylo Radchenko

(joint work with Andriy Bondarenko, Kristian Seip)

Let $f\colon \mathbb{R} \to \mathbb{C}$ be a sufficiently nice function (at least $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$), and define the Fourier transform of f by

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx\,.$$

Consider the following question: when is it possible to reconstruct f from the restrictions $f|_A$ and $\widehat{f}|_B$, where $A, B \subset \mathbb{R}$. If such recovery is possible, (A, B) is called a uniqueness pair. Among all uniqueness pairs (A, B) of particularly interest are the ones with A and B minimal, so that there is no redundant information. Note that an obstruction to minimality of (A, B) can be manifested as a (classical or non-classical) Poisson summation formula. For some interesting results on non-classical Poisson summation formulas, which are sometimes also called crystalline measures, see [4], [5], [6]. Finally, given a uniqueness pair (A, B), one can ask whether there is an explicit formula that recovers f from $f|_A$ and $\widehat{f}|_B$.

A familiar situation when reconstruction can be done in a non-redundant way by an explicit formula is the following: if $A = \{x\colon |x| \geq 1/2\}$ and $B = \mathbb{Z}$, then the Poisson summation formula

$$f(x) = -\sum_{n\neq 0} f(x+n) + \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{2\pi i n x}$$

reconstructs f from $f|_A$, $\widehat{f}|_B$. One can also show that (A, B) is minimal in the sense that neither can "1/2" be increased, nor can any point be removed from B.

It turns out that such a reconstruction is sometimes also possible when both A and B are discrete. The main result of [7], which for simplicity we only state for even functions, is that there exists a sequence of even Schwartz functions $a_n\colon \mathbb{R} \to \mathbb{R}$ such that for every even Schwartz function $f : \mathbb{R} \to \mathbb{C}$ one has

$$f(x) = \sum_{n=0}^{\infty} f(\sqrt{n})a_n(x) + \sum_{n=0}^{\infty} \widehat{f}(\sqrt{n})\widehat{a}_n(x).$$

The functions a_n are given explicitly as an integral transform of certain weakly-holomorphic modular forms of weight 3/2, and they satisfy $a_n(\sqrt{m}) = \delta_{n,m}$ and $\widehat{a}_n(\sqrt{m}) = 0$ when $m \geq 1$. In particular, this implies that, for even functions, $A = \{\sqrt{n}\}_{n\geq 0}$, $B = \{\sqrt{n}\}_{n\geq 1}$ is a minimal uniqueness pair. A related result was proven in [2], where it was shown that a radial Schwartz function on $\mathbb{R}^d$ for certain d can be recovered from $f(\sqrt{2n})$, $f'(\sqrt{2n})$, $\widehat{f}(\sqrt{2n})$, $\widehat{f}'(\sqrt{2n})$, $n \geq 1$. The corresponding interpolation formula plays a central role in the proof of the universal optimality of the E_8 and Leech lattices.

The proof of the explicit interpolation formula from [7] relies on the structure of the set $\{\pm\sqrt{n}\}_{n\geq 0}$ in a crucial way, and, except for some small perturbations of $\sqrt{n}$, it does not seem possible to adapt it to obtain other uniqueness pairs (A, B) consisting of discrete sets. A conjectural example of such a pair is $A = \{\pm c n^{\alpha}\}_{n\geq 0}$, $B = \{\pm d n^{\beta}\}_{n\geq 0}$, where $\alpha, \beta > 0$, $\alpha + \beta = 1$, and $2cd = \alpha^{-\alpha}\beta^{-\beta}$.

In [1] we have proved a new interpolation formula with discrete A and B, with A consisting of suitably rotated nontrivial zeros of the Riemann zeta function, and $B = \{\pm\frac{\log n}{4\pi}\}_{n\geq 1}$. More precisely, let $S_\varepsilon = \{z\colon |\operatorname{Im} z| < 1/2 + \varepsilon\}$. Then there exist two sequences of rapidly decaying even entire functions $U_n(z)$, $n = 1, 2, \ldots$, and $V_{\rho,j}(z)$, $0 \leq j < m(\rho)$, with ρ ranging over the nontrivial zeros of $\zeta(s)$ with positive imaginary part, such that for every even analytic function $f\colon S_\varepsilon \to \mathbb{C}$ that satisfies

$$\sup_{|y|<1/2+\varepsilon} \int_{-\infty}^{\infty} |f(x+iy)|(1+|x|)dx < \infty$$

and every $z = x + iy$ in the strip $|y| < 1/2$ we have

$$f(z) = \sum_{n=1}^{\infty} \widehat{f}\left(\frac{\log n}{4\pi}\right) U_n(z) + \lim_{k\to\infty} \sum_{0<\gamma\le T_k} \sum_{j=0}^{m(\rho)-1} f^{(j)}\left(\frac{\rho - 1/2}{i}\right) V_{\rho,j}(z).$$

Here $T_k \to \infty$ is some increasing sequence of positive numbers that does not depend on neither f nor on z. Moreover, the corresponding uniqueness pair is minimal since $U_n(z)$ and $V_{\rho,j}(z)$ enjoy the following interpolatory properties:

$$\begin{aligned} U_n^{(j)}\left(\tfrac{\rho-1/2}{i}\right) &= 0, & \widehat{U}_n\left(\tfrac{\log n'}{4\pi}\right) &= \delta_{n,n'}, \\ V_{\rho,j}^{(j')}\left(\tfrac{\rho'-1/2}{i}\right) &= \delta_{(\rho,j),(\rho',j')}, & \widehat{V}_{\rho,j}\left(\tfrac{\log n}{4\pi}\right) &= 0. \end{aligned}$$

The proof of this interpolation formula is based on a strengthening of Knopp's abundance principle for Dirichlet series with functional equations [3], and ultimately relies on an explicit construction of weight 1/2 modular integrals for the theta group $\Gamma_\theta = \langle \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \rangle$.

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Spectral Reciprocity via Integral representations

Ramon Nunes

1. Spectral reciprocity formulae

In recent years there has been some interest in studying (automorphic) spectral reciprocity formulae. By this we mean an identity of the shape

$$\sum_{\pi\in\mathcal{F}} \mathcal{L}(\pi)\mathcal{H}(\pi) = \sum_{\pi\in\widetilde{\mathcal{F}}} \widetilde{\mathcal{L}}(\pi)\widetilde{\mathcal{H}}(\pi), \tag{1}$$

where $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are families of automorphic representations, $\mathcal{L}(\pi)$ and $\widetilde{\mathcal{L}}(\pi)$ are L-values associated to π, $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are some weight functions.

The term first appeared in this context in a paper by Blomer, Miller and Li [BML19] but such identities have been around at least since Motohashi's formula connecting the fourth moment of the Riemann zeta-function to the cubic moment of L-functions of cusp forms for GL(2) (*cf.* [Mot93]).

Besides the intrinsical beauty of such results they often have applications to non-vanishing and subconvexity for the associated L-functions. Moreover there are several works on L-function on which one can find such spectral identities hidden inside the proof.

A recent result of Blomer and Khan [BK17] shows a very interesting identity of the shape (1) where for coprime integers q and ℓ, on one side we sum over representations of GL(2) over $\mathbb{Q}$ of conductor q and the weight function involves the ℓ-th Hecke eigenvalue for these representations and on the opposite side the roles of q and ℓ are reversed.

Their proof is *classical*: it uses the Kuznetsov and Voronoi summation formulae and ultimately relies on the additive reciprocity trick

$$e\left(\frac{1}{ab}\right) = e\left(\frac{\overline{a}}{b}\right) e\left(\frac{\overline{b}}{a}\right)$$

.

In this talk we described our work [Nun20] towards a new proof of Blomer-Khan's result. We use an adelic approach and the theory of integral representations of L-functions as developped by Jacquet, Piatetski-Shapiroi and Shalika. This has the advantage of making the generalization to number fields straightforward.

2. Blomer-Khan result

Let us now describe the main result in [BK17]: Let Π be a fixed automorphic representation of GL(3) over $\mathbb{Q}$. Let q and ℓ be coprime integers. We write

$$\mathcal{M}(q,\ell;h) := \frac{1}{q} \sum_{\mathrm{cond}(\pi)=q} \frac{L(1/2,\Pi\times\pi)L(1/2,\pi)}{L(1,\mathrm{Ad},\pi)} \lambda_\pi(\ell) h(t_\pi) + (\cdots),$$

where

- t_π is the spectral parameter,
- h is a *fairly general* smooth function and
- $(\cdots)$ denotes the contribution of the Eisenstein part, the terms of lower conductor and some polar terms coming from shifting contours.

Then, Blomer an Khan have showed that

$$\mathcal{M}(q,\ell,h) = \mathcal{M}(\ell,q,\check{h}),$$

where $h \mapsto \check{h}$ is given by an explicit integral transformation.

When Π corresponds to an Eisenstein series, this has an application to subconvexity: Let π be a cuspidal automorphic representation for GL(2) oer $\mathbb{Q}$ of *squarefree* conductor, then:

$$L(1/2,\pi) \ll_\epsilon (\mathrm{cond}(\pi))^{\frac{1}{4}-\frac{1-2\vartheta}{24}+\epsilon},$$

where ϑ is an admissible exponent towards the Ramanujan conjecture (we know that $\frac{7}{64}$ is admissible and $\vartheta = 0$ corresponds to the conjecture).

3. Our result

Our setting is similar but we work over general number fields and we must make some technical assumptions along the way. Let F be a number field. Let Π be an automorphic representation of GL(3) over F and let $\mathfrak{q}$ and $\mathfrak{l}$ be unramified coprime integral ideals of F. We let

$$\mathcal{M}_0(\mathfrak{q},\mathfrak{l}) := \frac{1}{\mathrm{N}\mathfrak{q}} \sum_{\substack{\mathrm{cond}(\pi)=\mathfrak{q} \\ \pi \,\mathrm{spherical}}} \frac{\Lambda(1/2,\Pi\times\pi)\Lambda(1/2,\pi)}{\Lambda(1,\mathrm{Ad},\pi)} \lambda_\pi(\mathfrak{l}) + (\cdots),$$

where all the definitions are the natural generalizations of the ones in the Blomer-Khan scenario except that we use completed L-functions and by π spherical we mean that we are only summing over representations whose local components are spherical at every archimedean place. Moreover, unlike the previous setting we do not have the freedom of varying the weight function h. One may say that our result is specialized to a very specific weight function given by a quotient of Gamma functions accounting for the completed L functions.

We may now state our theorem

Theorem. *Let the notation be as above and suppose that* Π *is an everywhere unramified cuspidal automorphic representation of* GL(3) *over* F*. Then*

$$\mathcal{M}_0(\mathfrak{q},\mathfrak{l}) = \mathcal{M}_0(\mathfrak{l},\mathfrak{q}).$$

3.1. **Application to non-vanishing.** As an application, We may deduce a non-vanishing result which is similar in spirit to a result of Khan (*cf.* [Kha12, Theorem 1.2]):

Corollary. *For prime ideals* $\mathfrak{p}$ *with sufficiently large norm, there is at least one automorphic representation* π *of conductor* $\mathfrak{p}$*, unramified for every archimedean place and such that* $\Lambda(\frac{1}{2},\Pi\times\pi)$ *and* $\Lambda(\frac{1}{2},\pi)$ *are both non-zero.*

3.2. **Loose ends.** There are still some unresolved issues that keep us from claiming a complete generalization of Blomer-Khan's result to general number fields.

The first one fixed automorphic automorphic representation Π we work with needs to be taken cuspidal in order for our arguments to hold. This is is unfortunate since when Π corresponds to a minimal parabolic Eisenstein series, the

L-function $L(s, \Pi \times \pi)$ simplifies and as in [BK17], this may lead to a strong sub-convexity estimate. This issue is likely to be resolved by using the technique of renormalizations of integrals. We hope to look into that in the future.

Another drawback of our result as it stands is that in order to complete our arguments we have no freedom to choose weight functions h. For applications it would be useful to allow for more general weight functions as well as understanding the integral transform involved in the transformation $h \mapsto \check{h}$ in the representation-theoretic viewpoint. A very similar question to this one was addressed by Paul Nelson in [Nel19]. We hope that his ideas can be carried over to our context.

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Multiplicative chaos in number theory

Adam J. Harper

I gave a survey style talk describing a probabilistic object called *multiplicative chaos*, and some of its emerging connections with analytic number theory.

Multiplicative chaos was first studied by Kahane in 1985 [2]. Informally, the idea is to construct a random measure (i.e. a random weighting) on a set $\mathcal{H}$ by integrating test functions against the exponential of some collection of random variables $(X(h))_{h \in \mathcal{H}}$ defined on that set. Thus for a test function g, we can look at

$$\int_{\mathcal{H}} g(h) e^{\gamma X(h)} dh,$$

where $\gamma > 0$ is a real parameter. For number theoretic purposes, it generally suffices to consider the simple case where $\mathcal{H}$ is the interval $[-1/2, 1/2]$ (say), and the test function $g(h) \equiv 1$.

One needs to make assumptions on $X(h)$ in order for this construction to be interesting. It turns out one gets something very interesting if the $X(h)$ are Gaussian random variables (or close to this); they each have mean zero $\mathbb{E}X(h) = 0$, and the same (or similar) finite non-zero variance $\mathbb{E}X(h)^2$; and the covariance $\mathbb{E}X(h)X(h')$ (i.e. the dependence between $X(h)$ and $X(h')$) decays *logarithmically* as $|h - h'|$ increases.

To make the connection with number theory, we consider the situation of having a family of functions $F_j(s)$, for $j \in \mathcal{J}, s \in \mathbb{C}$, that each have an Euler product structure (either exact or approximate) as well as some orthogonality/independence between the contribution from different primes, when we vary over $j \in \mathcal{J}$. Then in place of $\int_{\mathcal{H}} g(h)e^{\gamma X(h)}dh$, we look at

$$\int g(h)|F_j(1/2+ih)|^{\gamma}dh$$

as $j \in \mathcal{J}$ varies (giving our "randomness"). Now if $F_j(s)$ has an (approximate) Euler product structure, then $\log|F_j(s)| = \mathrm{Re}\log F_j(s)$ is (approximately) a sum over primes. Furthermore, if the contributions from different primes are orthogonal/independent as j varies, we can expect $\log|F_j(s)|$ to behave like a sum of independent contributions. In many situations, this means that $\log|F_j(1/2+ih)|$ will behave roughly like Gaussians with mean zero and comparable variances. The desired logarithmic covariance structure emerges because there is a *multiscale structure* in an Euler product: $p^{ih} = e^{ih\log p}$ varies on an h-scale roughly $1/\log p$, so contributions from small primes remain correlated over large h intervals, whilst contributions from larger primes decorrelate more quickly.

I described various specific applications of these ideas (by myself and others), including to random Euler products; random multiplicative functions; shifts and "typical large values" of the Riemann zeta function; averages of character sums; and pseudomoments of the Riemann zeta function. Here are a couple of sample theorems:

Theorem 1. *Uniformly for all large T and all $0 \le q \le 1$, we have*

$$\frac{1}{T}\int_T^{2T}\left(\int_{-1/2}^{1/2}|\zeta(1/2+it+ih)|^2dh\right)^q dt \ll \left(\frac{\log T}{1+(1-q)\sqrt{\log\log T}}\right)^q.$$

Theorem 2. *Let r be a large prime. Then uniformly for all $1 \le x \le r$ and $0 \le q \le 1$, if we set $L := \min\{x, r/x\}$ we have*

$$\frac{1}{r-2}\sum_{\chi\neq\chi_0 \ mod\, r}|\sum_{n\le x}\chi(n)|^{2q} \ll \left(\frac{x}{1+(1-q)\sqrt{\log\log 10L}}\right)^q.$$

Theorem 1 is from my paper [1], whilst Theorem 2 is my work in preparation, improving a previously announced weaker bound

$$\left(\frac{x}{1+(1-q)\sqrt{\min\{\log\log 10L, \log\log\log r\}}}\right)^q.$$

I conjecture that both bounds are sharp.

The key feature of the theorems, which is characteristic of the application of multiplicative chaos ideas, is the $\sqrt{\log\log}$ term in the denominators. In contrast, a simple application of Hölder's inequality to compare with the $q = 1$ case would produce weaker bounds $\log^q T$ and x^q.

The $\sqrt{\log\log}$ terms reflect the standard deviation of the Euler product corresponding to the object under study (so a product roughly over primes $\le T$ in

Theorem 1, and over primes $\leq x$ in Theorem 2). The proofs are inspired by the multiplicative chaos literature. The most fundamental idea is that, when studying distributional properties (including low moments) of $\int_{\mathcal{H}} g(h)e^{\gamma X(h)}dh$ or its number theoretic avatars, one can restrict to the case where all the partial Euler products involved obey certain size bounds. This restriction can be imposed because it holds with very high probability (i.e. for very many $T \leq t \leq 2T$ or χ respectively), but it reduces the size of the averages computed by a quantity reflecting the number of partial products involved, namely the appropriate standard deviation.

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On Superorthogonality

Lillian B. Pierce

Let $\{f_n\}_n$ be a sequence of functions associated to a function f. We study two types of inequalities.

The direct inequality:

$$\|\sum_n f_n\|_{L^p} \leq c_p\|(\sum_n |f_n|^2)^{1/2}\|_{L^p}.$$

The converse inequality:

$$\|(\sum_n |f_n|^2)^{1/2}\|_{L^p} \leq c_p{}'\|f\|_{L^p}.$$

A fundamental reason such inequalities are of interest in harmonic analysis is the following: given an operator with a suitable decomposition

$$T = \sum_n T_n,$$

upon setting $f_n = T_n(f)$, if both estimates were true, they would imply that

$$\|Tf\|_{L^p} \leq c_p{}'c_p\|f\|_{L^p}.$$

Superorthogonality can be used to prove one or both of these inequalities. Superorthogonality is the property that for any tuple of functions $f_{n_1}, \ldots, f_{n_{2r}}$ from the given sequence $\{f_n\}_n$,

$$\int f_{n_1}\bar{f}_{n_2}\cdots f_{n_{2r-1}}\bar{f}_{n_{2r}}dx = 0 \tag{1}$$

as long as an appropriate condition is satisfied by the tuple of indices $(n_1, \ldots, n_{2r})$.

We show that the framework of superorthogonality, and associated direct and converse inequalities, unites a wide variety of topics in harmonic analysis and number theory. In particular, this perspective gives clean proofs of central results relating to Khintchine's inequality, Walsh-Paley series, discrete operators, decoupling, counting solutions to systems of Diophantine equations, multicorrelation of trace functions, and the Burgess bound for short character sums.

We exhibit three main types of superorthogonality.

Type I: Type I superorthogonality is the case in which (1) holds if the tuple $(n_1, \dots, n_{2r})$ has the property that some value n_j appears an odd number of times. Any collection of functions with Type I superorthogonality satisfies a direct inequality. Type I superorthogonality classically appeared in Khintchine's inequality for the Rademacher functions, which can be viewed as both a direct and a converse inequality. A refinement of Type I superorthogonality underpins the recent work [GGPRY19], a philosophical converse to the proof of the Vinogradov Mean Value Theorem via decoupling. This notion of superorthogonality shows that counts for the number of diagonal solutions and near-solutions to a system of Diophantine equations can imply a direct inequality for a square function; this in turn implies a decoupling inequality for the extension operator associated to the corresponding curve.

Type II: Type II superorthogonality is the case in which (1) holds if the tuple $(n_1, \dots, n_{2r})$ has the property that some value n_j appears precisely once. Any collection of functions with Type II superorthogonality satisfies a direct inequality. Any sequence $\{f_n\}_n$ in which $f_1, f_2, \dots, f_n, \dots$ are mutually independent random variables, and each has mean zero (in the sense that $\int f_n dx = 0$), satisfies the Type II condition. Type II superorthogonality is also widely used in recent work in the area of discrete analogues in harmonic analysis.

Type III: Type III superorthogonality is the case in which (1) holds if the tuple $(n_1, \dots, n_{2r})$ has the property that some value n_j appears precisely once and is strictly greater than all other values in the tuple. This type of superorthogonality occurred in Paley's work on Walsh-Paley series.

Quasi-superorthogonality: Fourth, we introduce quasi-superorthogonality: we no longer assume that (1) vanishes, but instead that it exhibits quantitative cancellation. Now instead of a direct inequality, we obtain a variant that also includes an "off-diagonal" term on the right-hand side. Such inequalities are nevertheless very useful.

In fact, we observe that a deep application of ℓ-adic cohomology and the Riemann Hypothesis over finite fields proves that Type I quasi-superorthogonality holds for sequences of "trace functions"; this is a statement of multicorrelation of trace functions proved in [FKM15]. Hence an approximate direct inequality holds for such functions. Moreover, the source of quasi-superorthogonality of trace functions is a consequence of "exact" superorthogonality in the sense of (1) for a different set of functions, combined with the Riemann Hypothesis over finite fields; this is an observation of Emmanuel Kowalski.

As an application, we give a complete proof of the Burgess bound [Bur57] from the perspective of quasi-superorthogonality and an approximate direct inequality for square functions. As remarked in [GM10], "While the original argument [of Burgess] is easily followed line-by-line, it seems hard to comprehend the larger sense of it, because several technical difficulties are being dealt with at the same time that the main idea is unfolding." The new perspective of superorthogonality, combined with ideas introduced by modern proofs in [GM10, HB12], shows that the Burgess method is not an isolated anomaly, but fits into a unified class of proofs. This also clarifies which aspects of the method have no slack, and which might be possible points for improvisation.

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Cubic moments and a distance function on cuspidal representations

Ian Petrow

(joint work with Matthew P. Young)

Let $\mathcal{H}_{it}(m,\psi)$ denote the set of GL_2 cuspidal automorphic representations over $\mathbf{Q}$ of conductor m, central character ψ, and unramified at infinity with spectral parameter it. Generalizing work of Conrey and Iwaniec [CI00], in [PY20, PY19] we proved the following.

Theorem 1. *There exists a $B > 2$ such that for all primitive χ modulo q not quadratic and $\varepsilon > 0$ we have*

$$\sum_{|t_j|\le T}\sum_{m|q}\sum_{\pi\in\mathcal{H}_{it_j}(m,\chi^2)} L(1/2,\pi\otimes\overline{\chi})^3 + \int_{-T}^{T}|L(1/2+it,\chi)|^6 dt \ll_\varepsilon T^B q^{1+\varepsilon}. \tag{1}$$

Theorem 2. *For all primitive χ modulo q, $\delta,\varepsilon > 0$, and $T \gg q^\delta$ we have*

$$\sum_{T\le t_j<T+1}\sum_{m|q}\sum_{\pi\in\mathcal{H}_{it_j}(m,\chi^2)} L(1/2,\pi\otimes\overline{\chi})^3 + \int_{T}^{T+1}|L(1/2+it,\chi)|^6 dt \ll_{\delta,\varepsilon} T^{1+\varepsilon} q^{1+\varepsilon}. \tag{2}$$

Note that $\pi\otimes\overline{\chi}$ has trivial central character and conductor exactly q^2 [JL70, Prop. 3.8(iii)]. By deep results of Guo [Guo96] we have $L(1/2,\pi\otimes\overline{\chi}) \ge 0$ and thus by positivity the Weyl-strength subconvexity bound: for any primitive Dirichlet character χ modulo q and $\varepsilon > 0$

$$L(1/2+it,\chi) \ll_\varepsilon (q(1+|t|))^{1/6+\varepsilon}. \tag{3}$$

The crucial idea in the above is the shape of the family of automorphic forms over which we prove a Lindelöf-on-average upper bound for the 3rd moment. To that end, we now make some further comments on families of automorphic forms which are useful in analytic number theory.

Fix a degree $n \geq 1$. Let $\mathcal{A}(\mathrm{GL}_n)$ denote the space of automorphic forms on GL_n, and let $C(\pi)$ denote the analytic conductor of $\pi \in \mathcal{A}(\mathrm{GL}_n)$. The function $a : \mathcal{A}(\mathrm{GL}_n) \times \mathcal{A}(\mathrm{GL}_n) \to \mathbf{R}$ defined by

$$\exp(a(\pi_1, \pi_2)) = \frac{C(\pi_1 \otimes \overline{\pi_2})}{C(\pi_1 \times \overline{\pi_1})^{1/2} C(\pi_2 \otimes \overline{\pi_2})^{1/2}}$$

provides a reasonable notion of the distance between π_1 and π_2. We call it the Rankin-Selberg distance between π_1 and π_2.

Many families of automorphic forms in the literature can be seen to particularly small diameter with respect to the Rankin-Selberg distance (or, the exponentiation thereof). For example, consider the family $\mathcal{F}_\chi = \bigcup_{t_j \ll 1} \{\pi \otimes \overline{\chi} : \pi \in \mathcal{H}_{it_j}(m, \chi^2),\, m \mid q\}$, as in Theorem 1. The diameter with respect to a of $\mathcal{F}_\chi$ remains bounded as $q \to \infty$, so this family is particularly close-knit. Indeed, when $\chi^2 \neq 1$ the local component at $p < \infty$ of any member of $\mathcal{F}_\chi$ is isomorphic to the principal series $\chi_p \boxplus \overline{\chi}_p$ for some character χ_p of $\mathbf{Q}_p^\times$ coinciding with χ on $\mathbf{Z}_p^\times$.

Similarly, another example of a close-knit family is given by specifying for each $p < \infty$ a supercuspidal representation σ_p of $\mathrm{GL}_2(\mathbf{Q}_p)$ with trivial central character, and letting $\mathcal{F}_{\sigma_p} = \bigcup_{t_j \ll 1} \{\pi : \pi \in \mathcal{H}_{it_j}(p^{f(\sigma_p)}, 1),\, \pi_p \simeq \sigma_p\}$. Such supercuspidal representations may be constructed from a quadratic extension $E/\mathbf{Q}_p$ and a character θ of $E^\times$ satisfying $\theta|_{\mathbf{Q}_p^\times} = 1$ by compact induction theory. Again, we have that the diameter of $\mathcal{F}_{\sigma_p}$ with respect to a remains bounded as $p \to \infty$. Studying the cubic moment of L-functions over the family $\mathcal{F}_{\sigma_p}$ is joint work in progress with Y. Hu.

The Rankin-Selberg distance also arises in a natural way in unpublished work of Michel and Nelson on hybrid subconvexity for Rankin-Selberg L-functions (private communication).

At the level of local representations, we can be more precise. Recall that a pseudometric on X is a function $d : X \times X \to \mathbf{R}$ satisfying the axioms of a metric, except possibly the fact that $d(x,y) = 0$ implies that $x = y$. Let F be a non-archimedean local field of finite residue characteristic and $f(\pi)$ denote the conductor of a generic irreducible representation π of $\mathrm{GL}_n(F)$.

Definition 3. The function d defined by

$$d(\pi_1, \pi_2) = f(\pi_1 \otimes \overline{\pi_2}) - \frac{1}{2} f(\pi_1 \otimes \overline{\pi_1}) - \frac{1}{2} f(\pi_2 \otimes \overline{\pi_2})$$

is called the local Rankin-Selberg distance between generic irreducible representations π_1 and π_2 of $\mathrm{GL}_n(F)$.

Proposition 4. *The local Rankin-Selberg distance defines a pseudometric on the set of irreducible supercuspidal representations of* $\mathrm{GL}_n(F)$.

The proposition follows quickly from the following important theorem of Bushnell-Henniart [BH17]. The statement of their theorem below is quoted from [Lap20, Thm. 1], wherein a simpler proof of this theorem is given.

Theorem 5 (Bushnell-Henniart)**.** *Let π_i be irreducible supercuspidal representations of* $\mathrm{GL}_{n_i}(F)$, $i = 1, 2, 3$. *Then*

$$\frac{f(\pi_1 \otimes \overline{\pi_3})}{n_1 n_3} \leq \max\left(\frac{f(\pi_1 \otimes \overline{\pi_2})}{n_1 n_2}, \frac{f(\pi_2 \otimes \overline{\pi_3})}{n_2 n_3}\right). \tag{4}$$

Consequently,

$$\frac{f(\pi_1 \otimes \overline{\pi_2})}{n_1 n_2} \geq \max\left(\frac{f(\pi_1 \otimes \overline{\pi_1})}{n_1^2}, \frac{f(\pi_2 \otimes \overline{\pi_2})}{n_2^2}\right). \tag{5}$$

Proof of Proposition 4. The only unclear point is the triangle inequality, that is $d(\pi_1, \pi_3) \leq d(\pi_1, \pi_2) + d(\pi_2, \pi_3)$. By (5) we have

$$f(\pi_2 \otimes \overline{\pi_2}) \leq \max\left(f(\pi_2 \otimes \overline{\pi_2}), f(\pi_3 \otimes \overline{\pi_3})\right) \leq f(\pi_2 \otimes \overline{\pi_3}) \tag{6}$$

for any irreducible supercuspidal π_2, π_3.

Next, suppose that π_1, π_2, π_3 are such that $f(\pi_1 \otimes \overline{\pi_2}) \geq f(\pi_2 \otimes \overline{\pi_3})$. Then we have by (4)

$$f(\pi_1 \otimes \overline{\pi_3}) \leq \max\left(f(\pi_1 \otimes \overline{\pi_2}), f(\pi_2 \otimes \overline{\pi_3})\right) = f(\pi_1 \otimes \overline{\pi_2}). \tag{7}$$

Putting together (6) and (7) we get

$$f(\pi_1 \otimes \overline{\pi_3}) + f(\pi_2 \otimes \overline{\pi_2}) \leq f(\pi_1 \otimes \overline{\pi_2}) + f(\pi_2 \otimes \overline{\pi_3}),$$

which after subtracting

$$\frac{1}{2} f(\pi_1 \otimes \pi_1) + f(\pi_2 \otimes \overline{\pi_2}) + \frac{1}{2} f(\pi_3 \otimes \overline{\pi_3})$$

from both sides yields the desired inequality. The case where $f(\pi_1 \otimes \overline{\pi_2}) \leq f(\pi_2 \otimes \overline{\pi_3})$ follows similarly by applying (6) with π_1, π_2 replacing π_2, π_3. ☐

Natural questions regarding the local Rankin-Selberg pseudometric present themselves. For instance, let $B_r(\pi)$ be the ball of radius r surrounding π inside the set of irreducible supercuspidal representations of $\mathrm{GL}_n(F)$. What is the Plancherel measure of $B_r(\pi)$?

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Higher order uniformity of the Möbius function I and II

Kaisa Matomäki, Joni Teräväinen
(joint work with Maksym Radziwiłł, Terence Tao, Tamar Ziegler)

Let λ denote the Liouville function. It is well known that the fact that

$$\sum_{n\leq x}\lambda(n)=o(x)$$

is equivalent to the prime number theorem, whereas the claim

$$\sum_{n\leq x}\lambda(n)=O_{\varepsilon}(x^{1/2+\varepsilon})\quad\text{for every } \varepsilon>0$$

is equivalent to the Riemann Hypothesis.

Concerning higher order correlations, Chowla has conjectured that, for any distinct $h_1,\ldots,h_k$ one has

$$\frac{1}{x}\sum_{n\leq x}\lambda(n+h_1)\cdots\lambda(n+h_k)=o(1).$$

In light of the above equivalences it is natural to see this as an analogue of the notoriously hard prime k-tuple conjecture.

Tao [5] has shown that the logarithmically averaged version of Chowla's conjecture is equivalent to the logarithmically averaged higher order Fourier uniformity conjecture. We discuss our recent progress [4] on the latter conjecture as well as applications of our result.

Before stating the full higher order uniformity conjecture, let us state the special case of polynomial phases.

Conjecture 1. *Let $k\geq 1$ and $H=H(X)\to\infty$ as $X\to\infty$. Then*

$$\sup_{\substack{\deg(P)=k\\ P(x)\in\mathbb{R}[x]}}\Big|\sum_{x\leq n\leq x+H}\mu(n)e(P(n))\Big|=o(H)$$

for almost all $x\in[X,2X]$.

Towards this, we show in [4] the following.

Theorem 1. *Let $k\geq 1$ and $H\geq X^{\varepsilon}$ ($\varepsilon>0$). Then*

$$\sup_{\substack{\deg(P)=k\\ P(x)\in\mathbb{R}[x]}}\Big|\sum_{x\leq n\leq x+H}\mu(n)e(P(n))\Big|=o(H)$$

for almost all $x\in[X,2X]$.

Actually we obtain an analogous result for any "non-pretentious" multiplicative function taking values in the unit disc. Furthermore, in this case of polynomial phases we managed to obtain the result in a wider range $H \geq \exp((\log X)^{5/8+\varepsilon})$.

Our main result extends this to non-correlation with higher order nilsequences. These are the characters of the higher order Fourier analysis that contain e.g. polynomial phases as well as "bracket polynomial" phases such as $e(\alpha n\lfloor\beta n\rfloor)$. Let us now state the full Fourier uniformity conjecture from [5].

Conjecture 2. *Let G/Γ be a nilmanifold of degree $k \geq 1$ and let $H = H(X) \to \infty$ as $X \to \infty$. Then for any fixed Lipschitz map $F : G/\Gamma \to \mathbb{C}$,*

$$\sup_{g\in \mathrm{Poly}(\mathbb{Z}\to G)} \Big| \sum_{x\leq n\leq x+H} \mu(n)F(g(n)\Gamma)\Big| = o(H)$$

for almost all $x \in [X, 2X]$.

Our main theorem takes the following step towards this conjecture.

Theorem 2. *Let G/Γ be a nilmanifold of degree $k \geq 1$ and let $H \geq X^{\varepsilon}$. Then for any fixed Lipschitz map $F : G/\Gamma \to \mathbb{C}$,*

$$\sup_{g\in \mathrm{Poly}(\mathbb{Z}\to G)} \Big| \sum_{x\leq n\leq x+H} \mu(n)F(g(n)\Gamma)\Big| = o(H)$$

for almost all $x \in [X, 2X]$.

Thus for example

$$\sup_{\alpha,\beta\in\mathbb{R}} \Big| \sum_{x\leq n\leq x+H} \mu(n)e(\alpha n\lfloor\beta n\rfloor)\Big| = o(H), \quad H = X^{\varepsilon}$$

for almost all $x \in [X, 2X]$.

By the inverse theorem of the Gowers norms [1], we can state this equivalently in terms of the local Gowers norm $\|f\|_{U^k[N][x,x+H]} := \|f1_{[x,x+H]}\|_{U^k[N]}/\|1_{[x,x+H]}\|_{U^k[N]}$.

Theorem 3. *For any $k \geq 1$,*

$$\|\mu\|_{U^k[x,x+x^{\varepsilon}]} = o(1)$$

for almost all $x \in [X, 2X]$.

We note that the $k = 0$ case of this was proved earlier in [2] (for arbitrarily short intervals), and the $k = 1$ case was recently established in [3].

We have two applications of our results. The first concerns the number of sign patterns of the Liouville function — let

$$s(k) = \{v \in \{-1, +1\}^k : \ (\lambda(n+1), \ldots, \lambda(n+k)) = v \text{ for some } n\}$$

be the number of sign patters of length k in λ.

Theorem 4 (Superpolynomial word complexity for the Liouville sequence)**.** *$s(k) \geq ck^A$ for every fixed A and some $c = c(A)$.*

The second application is a new averaged version of Chowla's conjecture.

Theorem 5 (Chowla's conjecture with a short one-variable average)**.** *Let $k \geq 1$. Then*

$$\frac{1}{x^{\varepsilon}} \sum_{h \leq x^{\varepsilon}} \left| \frac{1}{x} \sum_{n \leq x} \lambda(n)\lambda(n+h)\cdots\lambda(n+kh) \right| = o(1).$$

We also get a result for much more general polynomial patterns.

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Resonances of Schottky surfaces

Anke Pohl

(joint work with O. Bandtlow, T. Schick, A. Weiße)

The investigation of L^2-Laplace eigenvalues and eigenfunctions for hyperbolic surfaces of *finite area* is a classical and exciting topic at the intersection of number theory, harmonic analysis and mathematical physics. In stark contrast, for geometrically finite hyperbolic surfaces of *infinite area*, the discrete L^2-spectrum is finite. A natural replacement are the resonances of the considered hyperbolic surface, which are the poles of the meromorphically continued resolvent

$$R(s) = (\Delta - s(1-s))^{-1}$$

of the hyperbolic Laplacian Δ. These spectral entities also play an important role in number theory and various other fields, and many fascinating results about them have already been found; the generalization of Selberg's 3/16-theorem by Bourgain, Gamburd and Sarnak [4] is a well-known example. However, an enormous amount of the properties of such resonances, also some very elementary ones, is still undiscovered. Prominent open questions include the existence of a Weyl law, the fractal Weyl law conjecture by Lu, Sridhar and Zworski [6], and the essential spectral gap conjecture by Jakobson and Naud [5].

A few years ago, by means of numerical experiments, Borthwick [2] noticed for some classes of Schottky surfaces (hyperbolic surfaces of infinite area without cusps and conical singularities) that their sets of resonances exhibit unexpected and nice patterns, which are not yet fully understood. He used the method of *periodic orbit expansion*, which is well-suited for investigations of resonances with positive real part and of Schottky surfaces with large funnel widths and Euler characteristic near -1.

We discussed an alternative method, termed *domain-refined Lagrange–Chebychev approximation*, which has some advantages over the method of period orbit expansion. Figure 1 displays a part of the resonance set of a so-called funneled torus Schottky surface, calculated with this method.

Observation ([1])**.** *The method of domain-refined Lagrange–Chebychev approximation allows us to calculate resonances also for Schottky surfaces with smaller Euler characteristic or small funnel widths as well as resonances with negative real part. This method is efficient and does not require any specific properties (e.g., additional symmetries) of the Schottky surfaces.*

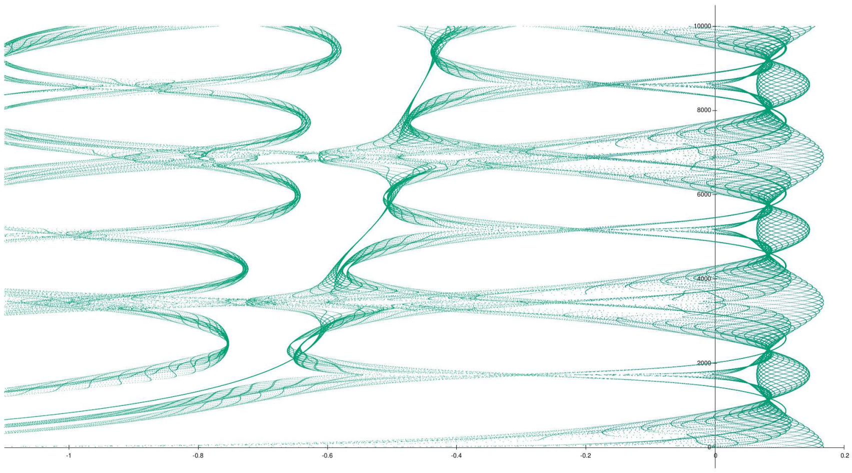

FIGURE 1. Resonances for a Schotty surface

The methods of periodic orbit expansion and of domain-refined Lagrange–Chebychev approximation have the same starting point as both take advantage of the interpretation of resonances as zeros of the Selberg zeta function and of a transfer-operator-based representation of this zeta function. For any Schottky surface X, the Selberg zeta function Z_X is given by the Euler product

$$Z_X(s) = \prod_{\ell \in L_X} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell}\right) \qquad \text{for } \operatorname{Re} s \gg 1, \tag{1}$$

and its holomorphic continuation to all of $\mathbb{C}$. Here, the multiset L_X in the first product of (1) refers to the primitive geodesic length spectrum of X. There exists a family of transfer operators $(\mathcal{L}_{X,s})_{s\in\mathbb{C}}$ for X, which derives from a discretization of the geodesic flow on X and whose Fredholm determinant equals the Selberg zeta function of X:

$$Z_X(s) = \det(1 - \mathcal{L}_{X,s})\,. \tag{2}$$

For the method of periodic orbit expansion one infers from (2) a series expansion

$$Z_X(s) = \sum_{n=0}^{\infty} d_n(s)\,,$$

whose coefficients $(d_n(s))_{n\in\mathbb{N}_0}$ are defined and calculated recursively in terms of the traces $(\mathrm{Tr}\mathcal{L}^m_{X,s})_{m\in\mathbb{N}}$ of the transfer operator $\mathcal{L}_{X,s}$. The zeros of the truncated series approximate the zeros of Z_X, and hence the resonances of X.

For the method of domain-refined Lagrange–Chebychev approximation we note that the transfer operator $\mathcal{L}_{X,s}$ has an integral kernel. Thus,

$$(\mathcal{L}_{X,s}f)w = \int_\Omega K_s(z,w)f(z)\,dz\,,$$

where Ω is a finite union of certain open subsets of $\mathbb{C}$, the map f belongs to a well-chosen function space, and the integral kernel K_s has a rather simple structure. We use the Gauss–Chebychev quadrature rule to approximate K_s or, equivalently, Lagrange–Chebychev interpolation for the functions f. Then the transfer operator $\mathcal{L}_s$ gets approximated by a finite matrix, say M_s, and hence the Selberg zeta function $Z_X(s) = \det(1-\mathcal{L}_{X,s})$ is approximated by $D(s) := \det(1-M_s)$. The zeros of D serve as an approximation of the zeros of Z_X, and in turn of the resonances of X.

The method of periodic orbit expansion allowed us to discover that the resonance set exhibits astonishing structures in the positive half-plane, as shown by Borthwick's seminal work [2] and subsequent investigations (we refer to [3, 1] for extensive references). With the method of domain-refined Lagrange–Chebychev approximation we see that these structures not just extend to the negative half-plane but show new patterns there.

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Sign changes and large values for fundamental Fourier coefficients of Siegel cusp forms of degree 2

Abhishek Saha
(joint work with Jesse Jääsaari, Stephen Lester)

Let Φ belong to a family of automorphic forms whose members have "Fourier expansions" of the form

$$\Phi(z) = \sum_{n \in \mathcal{S}} \Phi_n e_n(z)$$

where $\mathcal{S}$ is the indexing set for the Fourier coefficients and e_n are some functions. Let $\mathcal{D}$ be an *interesting* subset of $\mathcal{S}$. We are interested in situations where the following implication is true:

$$\Phi_n = 0 \; \forall \; n \in \mathcal{D} \Rightarrow \Phi = 0. \tag{1}$$

Equivalently: $\Phi \neq 0 \Rightarrow$ there exists $n \in \mathcal{D}$ such that $\Phi_n \neq 0$. In other words, *the subset of Fourier coefficients from $\mathcal{D}$ determine the automorphic form Φ.* In practice, this situation is most interesting for eigenforms Φ for which the Fourier coefficients are *not* multiplicative. Given a situation where it is possible to prove (1), one may ask the following refined questions:

- Are there many sign changes among $\{\Phi_n\}_{n \in \mathcal{D}}$?
- How large are the non-zero $\{\Phi_n\}_{n \in \mathcal{D}}$

In this talk, I considered this question for Φ a Siegel cusp form of degree 2. Recall that the Siegel upper-half space of degree 2 is defined by

$$\mathbb{H}_2 = \{Z \in M_2(\mathbb{C}) \mid Z = Z^t, \; \mathrm{Im}(Z) \text{ is positive definite}\}.$$

For any positive integer N, define

$$\Gamma_0^{(2)}(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}) \mid C \equiv 0 \pmod{N} \right\}. \tag{2}$$

Let $S_k^{(2)}(N)$ denote the space of holomorphic functions F on $\mathbb{H}_2$ which satisfy the relation

$$F(\gamma Z) = \det(J(\gamma, Z))^k F(Z) \tag{3}$$

for $\gamma \in \Gamma_0^{(2)}(N)$, $Z \in \mathbb{H}_2$, and vanish at all the cusps. Any $F \in S_k^{(2)}(N)$ has a Fourier expansion

$$F(Z) = \sum_{T \in \mathcal{S}} a(F, T) e(\mathrm{Tr}(TZ)),$$

where

$$\mathcal{S} = \{T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} : a, b, c \in \mathbb{Z}, \; \mathrm{disc}(T) := b^2 - 4ac < 0\}.$$

In fact, because $\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \in \Gamma_0^{(2)}(N)$ for all $A \in \mathrm{SL}_2(\mathbb{Z})$, we have, using (3), that

$$a(F, ATA^t) = a(F, T) \tag{4}$$

for all $A \in \mathrm{SL}_2(\mathbb{Z})$, thus showing that $a(F,T)$ only depends on the class of T in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{S}$, where the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{S}$ is $A \cdot T = ATA^t$.

Our interest is in the natural subset $\mathcal{S}_{\mathrm{fund}}$ of $\mathcal{S}$ consisting of the matrices whose discriminant is a fundamental discriminant. These are the "basic building blocks", in the sense that one cannot use the theory of Hecke operators to relate the Fourier coefficients at these matrices to those at simpler matrices. In this situation, it was proved some years ago that (1) holds. More precisely,

Theorem 1 (Saha, 2013; Saha–Schmidt, 2013)**.** *Let $k > 2$ and N be a squarefree integer. Moreover, if $N > 1$, assume that k is even. Let $0 \neq F \in S_k^{(2)}(N)$ belong to the orthogonal complement of the oldspace. Then, one has the lower bound*

$$|\{0 < |d| < X,\ d \textit{ squarefree },\ a(F,S) \neq 0 \textit{ for some } S \textit{ with } d = \mathrm{disc}(S)\}| \gg_F X^{5/8}.$$

In particular, there are infinitely many distinct $S \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{S}_{fund}$ such that $a(F,S) \neq 0$.

The above theorem had important consequences for the existence of unramified Bessel models and the non-vanishing of central L-values, as explained in [1] and [2]. In ongoing joint work with Jesse Jaasaari and Steve Lester, we can now answer the remaining two questions posed at the beginning, while simultaneously improving the exponent in Theorem 1.

Theorem 2 (Jääsaari–Lester–S, 2020)**.** *Let F be as in Theorem 1. Then for all large X there exist $r_X \gg_\epsilon X^{1-\epsilon}$ distinct elements $\{S_i\}_{1 \le i \le r_X}$ in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{S}_{fund}$ with $\mathrm{disc}(S_1) < \mathrm{disc}(S_2) < \ldots < \mathrm{disc}(S_{r_X}) < X$ and $a(F,S_i)a(F,S_{i+1}) < 0$.*

Theorem 3 (Jääsaari–Lester–S, 2020)**.** *Let F be as in Theorem 1. Then for all large X there exist $r_X \gg_\epsilon X^{1-\epsilon}$ distinct elements $\{S_i\}_{1 \le i \le r_X}$ in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{S}_{fund}$ with $\mathrm{disc}(S_i) \asymp X$ and*

$$|a(F,S_i)| \gg_F |\mathrm{disc}(S_i)|^{\frac{k}{2}-\frac{3}{4}} \exp\left(\frac{1}{82}\frac{\sqrt{\log|\mathrm{disc}(S_i)|}}{\sqrt{\log\log|\mathrm{disc}(S_i)|}}\right)$$

The proofs of the above theorems rely on transferring to the setup of classical half-integral weight forms, via the Fourier–Jacobi expansion. Once that is done, we extend (to the setup of half-integral weight forms with level that are not necessarily Hecke eigenforms) methods previously developed by Lester–Radziwiłł and Gun–Kohnen–Soundararajan resepctively.

A key interest of the above theorems come from a remarkable connection to the refined Gan–Gross–Prasad conjecture. Let $d < 0$ be a fundamental discriminant and put $K = \mathbb{Q}(\sqrt{d})$. Let Cl_K denote the ideal class group of K. It is a fact going back to Gauss that the $\mathrm{SL}_2(\mathbb{Z})-$equivalence classes of binary quadratic forms of discriminant d are in natural bijective correspondence with the elements of Cl_K. In view of (4), for any $c \in \mathrm{Cl}_K$ the notation $a(F,c)$ makes sense.

For $F \in S_k^{(2)}(N)$, an imaginary quadratic field K with discriminant equal to d, and a character Λ of the finite group Cl_K, we define

$$R(F,K,\Lambda) = \sum_{c\in \mathrm{Cl}_K} a(F,c)\Lambda^{-1}(c). \tag{5}$$

Then for a non-lift newform F, a consequence of the refined Gan–Gross–Prasad conjecture is the following identity proved in [3]:

$$\frac{|R(F,K,\Lambda)|^2}{\langle f,f\rangle} = \frac{2^{4k-4}\ \pi^{2k+1}}{(2k-2)!}\ w(K)^2\ |d|^{k-1}\ \frac{L(1/2,F\times \mathcal{AI}(\Lambda^{-1}))}{L(1,F,\mathrm{Ad})}\prod_{p|N} c_p \tag{6}$$

where $w(K)$ is the number of roots of unity and c_p are some local constants. This tells us that the Fourier coefficients $a(F,S)$ for $S\in\mathcal{S}$ are mysterious objects which may be viewed as *unipotent periods* whose weighted averages are *Bessel periods* whose absolute squares are essentially central L-values of degree 8 L-functions. The above theorems help shed light on these underlying periods and their statistical properties.

Remark 4. *Using Theorem 3 and* (6)*, it is not hard to give a* lower bound *for* $L(1/2,F\times\mathcal{AI}(\Lambda^{-1}))$ *for infinitely many* Λ *(i.e., an* Ω*-result). In ongoing work, we are studying the fractional moments of* $L(1/2,F\times\mathcal{AI}(\Lambda^{-1}))$ *to go in the other direction and prove an* upper bound *for* $a(F,S)$.

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Quantitative sheaf theory (d'après Sawin)

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(joint work with A. Forey, J. Fresán, W. Sawin)

Deligne's Riemann Hypothesis is now a fundamental tool of analytic number theory; in the context of automorphic forms, it appears prominently through the Ramanujan–Petersson conjecture for classical holomorphic modular forms, but also in a number of significant works, such as the estimate of Conrey and Iwaniec for the cubic moment of special values of Hecke L-functions.

From the very beginning, a basic challenge has been to bound the "number of roots" after applying the Riemann Hypothesis. Indeed, an exponential sum might be expressed (by the Grothendieck–Lefschetz trace formula) as a sum of Weil numbers with square-root cancellation

$$\sum_{x\in\mathbf{F}_p^n} e\Big(\frac{f(x)}{p}\Big) = \sum_{i=1}^{N_p}\alpha_i, \qquad |\alpha_i| = p^{n/2},$$

but we need to bound N_p to get a non-trivial result (if we had $N_p = p^{2n}$, then the resulting direct estimate would be trivial).

The formalism of algebraic geometry ("étale cohomology") does not immediately imply such bounds in general.

For this particular case, bounds for N_p are due to Bombieri, Adolphson–Sperber and especially Katz, but this is not sufficient for many other situations. Indeed, the problem becomes even worse when we use the Riemann Hypothesis in more complicated situations where knowing the integer N_p is not sufficient.

For instance, we might want to estimate

$$\sum_{\substack{x\in\mathbf{F}_p^n\\ x=g(y)}} e\Big(\frac{f(x)}{p}\Big)$$

for some polynomials $g = (g_1, \dots, g_n)$ in m variables.

Or

$$\sum_{0\leq x_i\leq X} e\Big(\frac{f(x)}{p}\Big)\lambda_1(x_1)\cdots\lambda_n(x_n)$$

for some other interesting arithmetic functions λ_i.

This problem was particularly evident in the papers of Fouvry, Kowalski Michel (such as [2]), where we consider general one-variable trace functions and analytic expressions like

$$\sum_{n\leq X} \lambda_f(n)t(n)$$

for some modular form f and some trace function t modulo a prime q. We defined (on Feb. 28, 2012) a "complexity" invariant c that turns out to give a good theory for one-variable sums, in the sense that in analytic estimates, the only dependency on t is through c. However, many natural problems involve two (or more) variables.

Let $n \geq 1$ be an integer. Let ψ (resp. χ) be a character of $\mathbf{F}$ (resp. of $\mathbf{F}^\times$). Put $\chi(0) = 0$ if χ is non-trivial, and otherwise $\chi(0) = 1$.

The following are trace functions on $\mathbf{A}^n$:

- (AS) For any polynomial $f \in \mathbf{F}[x_1, \dots, x_n]$, the function $t_1(x) = \psi(f(x))$.
- (K) For any polynomial $f \in \mathbf{F}[x_1, \dots, x_n]$, the function $t_2(x) = \chi(f(x))$.
- (FC) For any n-tuple of polynomials $g = (g_1, \dots, g_n)$ in m variables, the function

$$t_4(x) = |\{y \in \mathbf{F}^m \mid g(y) = x\}|.$$

- (TT) The constant functions $|\mathbf{F}|^{1/2}$ and $|\mathbf{F}|^{-1/2}$.

Moreover, given trace functions t_1 and t_2 in n variables:

- (DS) The functions $t_1 + t_2$, $t_1 - t_2$ are trace functions in n variables.
- (TP) The function t_1t_2 is a trace function in n variables.
- (PB) Given $g = (g_1, \dots, g_n)$ with $g_i \in \mathbf{F}[x_1, \dots, x_m]$, the function $t_1 \circ g$ is a trace function in m variables.

(DI) Given $h = (h_1, \dots, h_m)$ with $h_i \in \mathbf{F}[x_1, \dots, x_n]$, the function

$$t_3(y) = \sum_{h(x)=y} t_1(x)$$

is a trace function in m variables.

(D) The complex conjugate $\overline{t}_1$ is a trace function.

Application. This formalism allows us to defined already the Fourier transform. Denote $x \cdot y = x_1y_1 + \cdots + x_ny_n$.

Corollary. Let t be a trace function in n variables. The Fourier transform

$$\widehat{t}(y) = \frac{1}{|\mathbf{F}|^{n/2}} \sum_{x \in \mathbf{F}^n} t(x)\psi(x \cdot y)$$

is a trace function in n variables.

Main theorem. To each underlying geometric object of a trace function, Sawin associates an integer $c(\mathcal{F})$. (This is, roughly speaking, the maximum of the "number of roots"/sum of Betti numbers for the restrictions of $\mathcal{F}$ to "generic" affine subspaces of all dimensions $\leq n$.)

To each tuple $g = (g_1, \dots, g_m)$ of polynomials in n variables (giving a morphism $\mathbf{A}^n \to \mathbf{A}^m$) he also associates an integer $c(g)$. (This has a similar definition, but can be bounded from above explicitly in terms of the number and degrees of the polynomials g_i).

These measure the *complexity* of the trace function, or of the morphism. In all operations, the complexity "after" is bounded in terms of the complexity "before" – this is a form of *continuity.* Moreover, in most cases, the complexity can increase at most linearly. Also, crucially, the complexity controls the "number of roots" and other analytic invariants of the trace functions.

For instance we have:

$$c(\psi(f)) \ll c(f), \quad c(\chi(f)) \ll c(f), \quad c(|\mathbf{F}|^{1/2}t) = c(t)$$
$$c(t_1 \pm t_2) \ll c(t_1) + c(t_2), \quad c(t_1t_2) \ll c(t_1)c(t_2), \quad c(\bar{t}) \ll c(t)$$

as well as $c(t(g(y))) \ll c(g)c(t(x))$ for $g = (g_1, \dots, g_n)$ with $g_i \in \mathbf{F}[x_1, \dots, x_m]$ and $c(h_!t) \ll c(h)c(t)$ given $h = (h_1, \dots, h_m)$ with $h_i \in \mathbf{F}[x_1, \dots, x_n]$, where

$$h_!t(y) = \sum_{f(x)=y} t(x).$$

And given a trace function t associated to the complex $\mathcal{F}$, we have:

- The "total number of roots" of $\mathcal{F}$ is $\leq c(\mathcal{F})$ (this means that the L-function of $\mathcal{F}$ (constructed using extensions of $\mathbf{F}$) can be written as f_1/f_2 for polynomials f_1 and f_2 with $\deg(f_1) + \deg(f_2) \leq c(\mathcal{F})$).
- Under suitable conditions (e.g., for two geometrically irreducible perverse sheaves) the Riemann Hypothesis becomes

$$\Big| \frac{1}{|\mathbf{F}|^n} \sum_{x \in \mathbf{F}^n} t_1(x)\overline{t_2(x)} - (\text{main term}) \Big| \ll c(\mathcal{F}_1)c(\mathcal{F}_2)|\mathbf{F}|^{-1/2}.$$

- For one variable trace functions, we have

$$c_{\mathrm{fkm}}(\mathcal{F}) \leq c(\mathcal{F}) \leq 3c_{\mathrm{fkm}}(\mathcal{F})^2.$$

For instance, it follows that

$$c(\widehat{t}) \ll c(t)$$

for the Fourier transform, where the implied constant depends only on n.

Application 1 (equidistribution along primes)
Combining the formalism of complexity with Deligne's Riemann Hypothesis, we can for instance prove the following equidistribution result, which answers a question of Katz:

Theorem. Let $n \geq 1$ and $e \geq 1$ be integers. Let $P(n,e)$ be the set of polynomials of degree e in n variables. of Deligne type[1] For $f \in P(n,e)(\mathbf{F}_p)$, let

$$S(f;p) = \frac{1}{p^{n/2}} \sum_{x \in \mathbf{F}_p^n} e\Big(\frac{f(x)}{p}\Big).$$

The families $(S(f;p))_{f \in P(n,e)(\mathbf{F}_p)}$ become equidistributed as $p \to +\infty$ with respect to the measure which is the image under the trace of the probability Haar measure on $\mathrm{U}_{(e-1)^n}(\mathbf{C})$.

Application 2 (tannakian equidistribution)
In work in progress with Forey and Fres'an [1], we generalize Katz's work on Mellin transforms to other groups (e.g. to exponential sums parameterized by tuples $(\chi_1, \ldots, \chi_n)$ of multiplicative characters, or by pairs (χ, ψ) of multiplicative and additive characters).

For instance, we can get "vertical" equidistribution statements for

$$S(\chi, \psi; \mathbf{F}) = \frac{1}{|\mathbf{F}|^{1/2}} \sum_{x \in \mathbf{F}} \chi(x)\psi(x)t(x)$$

for suitable trace functions t.

We can also obtain applications to things like the variance of arithmetic functions for twists of higher-degree L-functions over $\mathbf{F}[u]$ (generalizing work of Hall, Keating and Roditty–Gershon).

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[1] I.e., the homogeneous part of degree e defines a smooth hypersurface.

Sphere packing and the modular bootstrap

Henry Cohn
(joint work with Nima Afkhami-Jeddi, Thomas Hartman, David de Laat, and Amirhossein Tajdini)

In 2019, Hartman, Mazáč, and Rastelli [5] discovered a remarkable connection between linear programming bounds for sphere packing [3] and the modular bootstrap in conformal field theory. The modular bootstrap is a special case of the conformal bootstrap program, which seeks to understand constraints on the space of possible conformal field theories (CFTs) via self-consistency relations. Specifically, the modular bootstrap looks at constraints arising from the modular invariance of the torus partition function. This connection puts the solution of the sphere packing problem in 8 and 24 dimensions [7, 4] into a broader context.

In particular, the spinless modular bootstrap for two-dimensional CFTs with conformal algebra $U(1)^c \times U(1)^c$ turns out to be identical to the linear programming bound for sphere packing in $\mathbb{R}^{2c}$, while the spinning modular bootstrap amounts to a new bound for Narain lattices (Euclidean lattices that are even unimodular lattices under a split inner product).

In [1, 2], we study both of these bounds in more detail. Highlights include the following two conjectures:

Conjecture 1. The linear programming upper bound for the sphere packing density in $\mathbb{R}^n$ is $2^{-(\lambda+o(1))n}$ as $n \to \infty$, with $\lambda \approx 0.6044$.

For comparison, the best bound currently known has $\lambda \approx 0.5990$, which has not been improved since Kabatyanskii and Levenshtein's work [6] in 1978. Conjecture 1 remains unproved, but numerical evidence indicates that this exponential improvement is possible.

We also conjecture that the spinning modular bootstrap is sharp for $U(1)^2 \times U(1)^2$, where it matches the $SU(3)_1$ Wess-Zumino-Witten model. That is the content of the following conjecture, stated in terms of the Fourier transform

$$\widehat{f}(y) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x,y \rangle}\, dx$$

of a function $f\colon \mathbb{R}^d \to \mathbb{R}$.

Conjecture 2. There exists a Schwartz function $f\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ (not identically zero) such that $f(0,0) = 0$, $\widehat{f} = -f$, and $f(x,y) \geq 0$ whenever $|x|^2 - |y|^2 \in 2\mathbb{Z}$ and $|x|^2 + |y|^2 \geq 4/3$.

It is natural to expect that this function should be connected with automorphic forms, along the lines of Viazovska's integral transform [7], but it is unclear how. Unlike what happens in previous bounds [7, 4], this function will not be radial (instead, $f(x,y)$ can be assumed to depend on just $|x|$ and $|y|$, but not $|x|^2 + |y|^2$), and the inequalities are imposed only given the integrality constraint $|x|^2 - |y|^2 \in 2\mathbb{Z}$.

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Moments and hybrid subconvexity for symmetric-square L-functions

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(joint work with Rizwanur Khan)

Background. The widely studied subconvexity problem for automorphic L-functions is completely resolved for degree ≤ 2. For uniform bounds, over arbitrary number fields, this is due to Michel and Venkatesh [MV]; for superior quality bounds in various special cases, this is due to many authors, of which a small sample is [JM, BH, Bo, BHKM, PY]. The next frontier is degree 3, but here the subconvexity problem remains a great challenge, save for a few spectacular successes. The first breakthrough is due to Xiaoqing Li [Li], who established subconvexity for $L(f, 1/2+it)$ on the critical line (t-aspect), where f is a fixed self-dual Hecke-Maass cusp form for $SL_3(\mathbb{Z})$. This result was generalized by Munshi [M1], by a very different method, to forms f that are not necessarily self-dual. Munshi [M2] also established subconvexity for twists $L(f \times \chi, 1/2)$ in the p-aspect, where χ is a primitive Dirichlet character of prime modulus p. Subconvexity in the spectral aspect of f itself is much harder, and even more so when f is self-dual due to a conductor-dropping phenomenon. Blomer and Buttcane [BB] and Kumar, Mallesham, and Singh [KMS] have established subconvexity for $L(1/2, f)$ in the spectral aspect of f in many cases, but excluding the self-dual forms.

A self-dual GL_3 Hecke-Maass cusp form is known to be a symmetric-square lift from GL_2 [S]. Let u_j be a Hecke-Maass cusp form for the full modular group $SL_2(\mathbb{Z})$, with Laplace eigenvalue $1/4 + t_j^2$. It is an outstanding open problem to prove subconvexity for the associated symmetric-square L-function $L(\mathrm{sym}^2 u_j, 1/2)$ in the t_j-aspect. Such a bound would represent major progress in the problem of obtaining a power-saving rate of decay in the Quantum Unique Ergodicity problem

[IS]. A related problem is that of establishing the Lindelöf-on-average bound

$$\sum_{T\le t_j\le T+\Delta} |L(\mathrm{sym}^2 u_j, 1/2+it)|^2 \ll \Delta T^{1+\epsilon} \tag{1}$$

for Δ as small as possible. Such an estimate is interesting in its own right, and also yields by positivity a bound for each L-value in the sum. At the central point ($t=0$), if (1) can be established for $\Delta = T^{\epsilon}$, it would give the convexity bound for $L(\mathrm{sym}^2 u_j, 1/2)$; the hope would then be to insert an amplifier in order to prove subconvexity. While a second moment bound which implies convexity at the central point is known in the level aspect by the work of Iwaniec and Michel [IM], in the spectral aspect the problem is much more difficult. The best known result until now for (1) was $\Delta = T^{1/3+\epsilon}$ by Lam [La]. (Lam's work actually involves symmetric-square L-functions attached to holomorphic Hecke eigenforms, but his method should apply equally well to Hecke-Maass forms.) Other works involving moments of symmetric square L-functions include [Bl, K, J, KD, BF, Ba, N].

Main results. One of the main results of this paper is an approximate version of the subconvexity bound for $L(\mathrm{sym}^2 u_j, 1/2)$. Namely, we establish subconvexity for $L(\mathrm{sym}^2 u_j, 1/2+it)$ for t small, but not too small, compared to $2t_j$. This hybrid bound (stated precisely below) seems to be the first subconvexity bound for symmetric-square L-functions in which the dominant aspect is the spectral parameter t_j. For comparison, note that bookkeeping the proofs of Li [Li] or Munshi [M1] would yield hybrid subconvexity bounds for t_j (very) small compared to t. Our method also yields a hybrid subconvexity bound for $L(\mathrm{sym}^2 u_j, 1/2+it)$ when t is larger (but not too much larger) than $2t_j$, but for simplicity we refrain from making precise statements. We do not prove anything when t is close to $2t_j$, for in this case the analytic conductor of the L-function drops. In fact it is then the same size as the analytic conductor at $t=0$, where the subconvexity problem is the hardest.

Our approach is to establish a sharp estimate for the second moment as in (1), which is strong enough to yield subconvexity in certain ranges. Let $0<\delta<2$ be fixed, and let $U, T, \Delta > 1$ be such that

$$\frac{T^{3/2+\delta}}{\Delta^{3/2}} \le U \le (2-\delta)T. \tag{2}$$

We have

$$\sum_{T<t_j<T+\Delta} |L(\mathrm{sym}^2 u_j, 1/2+iU)|^2 \ll \Delta T^{1+\epsilon}. \tag{3}$$

Let $0<\delta<2$ be fixed. For $|t_j|^{6/7+\delta} \le U \le (2-\delta)|t_j|$, we have the hybrid subconvexity bound

$$L(\mathrm{sym}^2 u_j, 1/2+iU) \ll |t_j|^{1+\epsilon} U^{-1/3}. \tag{4}$$

Proof. The bound follows by taking $\Delta = T^{1+\delta}U^{-2/3}$ in (3) with δ chosen small enough. When $U \ge T^{6/7+\delta}$, this bound is subconvex. □

For the central values we do not get subconvexity but we are able to improve the state of the art for the second moment. This is the other main result: we establish a Lindelöf-on-average estimate for the second moment with Δ as small as $T^{1/5+\epsilon}$. For $\Delta \geq T^{1/5+\varepsilon}$ and $U \ll T^{\varepsilon}$ we have

$$\sum_{T<t_j<T+\Delta} |L(\mathrm{sym}^2 u_j, 1/2 + iU)|^2 \ll \Delta T^{1+\varepsilon}. \tag{5}$$

It is a standing challenge to prove a Lindelöf-on-average bound in (5) with $\Delta = 1$.

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Fourier uniqueness in even dimensions

Haakan Hedenmalm

1. Fourier uniqueness

Fourier uniqueness questions have appeared in various forms associated with the uncertainty principle. We work with the Fourier transform in $\mathbb{R}^D$, $D = 1, 2, 3, \ldots$:

$$\hat{f}(y) = \int_{\mathbb{R}^D} \mathrm{e}^{-\mathrm{i}2\pi\langle x,y\rangle} f(x)\,\mathrm{diffvol}_D(x),$$

where $\mathrm{diffvol}_D$ is volume measure in $\mathbb{R}^D$.

The uniqueness problem. Suppose f is a test function in $\mathbb{R}^D$, and that $A, B \subset \mathbb{R}^D$ are closed subsets. If

$$f|_A = 0 \text{ and } \hat{f}|_B = 0 \implies f = 0$$

we say that (A, B) is a *Fourier uniqueness pair*.

Remark 1.1.

(1) The class of test functions may play a role. Naturally, we would like to have as wide a class as possible.
(2) If instead of the commutative topological group $\langle \mathbb{R}^D, +\rangle$ we consider a non-commutative group, the Fourier transform gets replaced by representation theory. Similar uniqueness problems may be formulated then.

In work with Montes-Rodriguez, the related concept of Heisenberg uniqueness pairs was introduced. It concerns the Fourier transforms of finite Borel complex measures $\mu \in \mathrm{Meas}(\mathbb{R}^2)$:

$$\hat{\mu}(y) = \int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}2\pi\langle x,y\rangle}\,\mathrm{d}\mu(x).$$

We let $\mathrm{Meas}_{\mathrm{ac}}(\Gamma) \subset \mathrm{Meas}(\mathbb{R}^2)$ denote the subset of measures supported on a rectifiable curve Γ, which are absolutely continuous with respect to arc length.

Definition 1.2 (Heisenberg uniqueness pairs)**.** Let $\Gamma \subset \mathbb{R}^2$ be a rectifiable curve, and $\Lambda \subset \mathbb{R}^2$ a closed subset. Then (Γ, Λ) is a Heisenberg uniqueness pair if for every $\mu \in \mathrm{Meas}_{\mathrm{ac}}(\Gamma)$ we have that

$$\hat{\mu}|_\Lambda = 0 \implies \mu = 0.$$

Remark 1.3. The concept has various generalizations to higher dimensions.

If Γ is the hyperbola $x_1x_2+1=0$, and $\mu \in \mathrm{Meas}_{\mathrm{ac}}(\Gamma)$, then $\hat{\mu}$ solves the Klein-Gordon equation

$$\partial_{y_1}\partial_{y_2}\hat{\mu} = 4\pi^2\hat{\mu},$$

which is a hyperbolic PDE for which we expect no small uniqueness sets. The set Λ considered was a lattice-cross,

$$\Lambda = (\alpha\mathbb{Z}\times\{0\}) \cup (\{0\}\times(\{0\}\times\beta\mathbb{Z}),$$

for positive reals α,β.

Theorem 1.4. (Γ,Λ) *is a Heisenberg uniqueness pair if and only if* $\alpha\beta \le \frac{1}{4}$.

So we find small uniqueness sets along the characteristic directions for the PDE. An alternative formulation is that the functions

$$\mathrm{e}^{\mathrm{i}2\pi\alpha m t},\ \mathrm{e}^{-\mathrm{i}2\pi\beta/t}, \qquad m,n\in\mathbb{Z},$$

form a weak-star complete system in $L^\infty(\mathbb{R})$ if and only if $\alpha\beta\le\frac{1}{4}$. Equivalently:

Theorem 1.5 (Alternative formulation)**.** *If* $0<\gamma\le 1$, $\phi\in L^1(\mathbb{R})$, *and*

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi m t}\phi(t)\mathrm{d}t = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\gamma\pi n/t}\phi(t)\mathrm{d}t = 0,$$

for all $m,n\in\mathbb{Z}$, *then* $\phi=0$. *If* $\gamma>1$ *the conclusion fails.*

The proof relies on dynamical properties of Gauss-type maps. In particular, the critical parameter case $\alpha\beta=\frac{1}{4}$ involves the Gauss-type map $t\mapsto -1/t \bmod 2\mathbb{Z}$ on the interval $[-1,1]$, which has a weakly repelling fixed point at ±1 and the absolutely continuous ergodic invariant measure has infinite mass. This is in contrast with the usual Gauss map $t\mapsto 1/t \bmod \mathbb{Z}$ on $[0,1]$.

Of particular interest is the critical density case $\alpha\beta=\frac{1}{4}$, which corresponds to $\gamma=1$ in the alternative formulation. The following expansion appears to be natural:

$$f(t) = \sum_{n\in\mathbb{Z}} a_n\mathrm{e}^{\mathrm{i}\pi n t} + b_n\mathrm{e}^{-\mathrm{i}\pi n/t},$$

which is analogous the Fourier series expansion of 2-periodic functions. Would such an expansion be unique? May we represent all functions that are e.g. bounded? We would like the coefficients a_n to be the usual Fourier coefficients if f is 2-periodic, and $b_n=0$ then. On the other hand,

$$f(-1/t) = \sum_{n\in\mathbb{Z}} b_n\mathrm{e}^{\mathrm{i}\pi n t} + a_n\mathrm{e}^{-\mathrm{i}\pi n/t},$$

which means that b_n are the Fourier coefficients of $f(-1/t)$ if that function is 2-periodic, and $a_n=0$ then. This problem is basically solved in joint work with Bakan, Radchenko, Viazovska (see also [BRS]).

A strong version of ergodicity which can be obtained for the Gauss-type maps we work with gives the following. Let $H^1_+(\mathbb{R})$ denote the subspace of $L^1(\mathbb{R})$ whose Poisson extension to the upper half-space $\mathbb{H}$ are holomorphic.

Theorem 1.6. *If* $\phi \in L^1(\mathbb{R})$

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi m t}\phi(t)\mathrm{d}t = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\pi n/t}\phi(t)\mathrm{d}t = 0,$$

for all $m, n = 0, 1, 2, 3, \ldots$, *then* $\phi \in H^1_+(\mathbb{R})$.

This theorem is stronger than the other, as $H^1_+(\mathbb{R}) \cap H^1_-(\mathbb{R}) = \{0\}$. The formulation is for critical density $\gamma = 1$, but the result generalizes to all γ with $0 < \gamma \leq 1$.

Informally, the Fourier transform of a Gaussian is another Gaussian. Gaussians are not necessarily radial, but to keep things simple we restrict to *radial complex Gaussians*

$$G_\tau(x) := \mathrm{e}^{\mathrm{i}\pi\tau|x|^2}, \qquad x \in \mathbb{R}^D,$$

which decay nicely for $\tau \in \mathbb{H}$. The Fourier transform of G_τ is then given by

$$\hat{G}_\tau(y) = \left(\frac{\tau}{\mathrm{i}}\right)^{-D/2} \mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}, \qquad y \in \mathbb{R}^D.$$

If $\tau \in \bar{\mathbb{H}} = \mathbb{H} \cup \mathbb{R}$, the function G_τ is at least bounded, and as such it has a Fourier transform in the sense of tempered distributions. The Fourier transform equals the bounded function $\hat{G}_\tau$, understood as a tempered distribution. Note that if D is odd, the square root is of τ/i is the principal branch.

If $\phi \in L^1(\mathbb{R})$, we may consider the related function

$$\mathbf{G}\phi(x) := \int_{\mathbb{R}} G_\tau(x)\phi(\tau)\mathrm{d}\tau = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi\tau|x|^2}\phi(\tau)\mathrm{d}\tau, \qquad x \in \mathbb{R}^D.$$

We might call this the Gaussian transform of ϕ, which is a radial function. We note that

$$\mathbf{G}\phi(x) = \hat{\phi}_1(-\tfrac{1}{2}|x|^2),$$

where the subscript 1 corresponds to taking the Fourier transform on $\mathbb{R}^1$. This means that $\mathbf{G}\phi$ is a pretty general radial function, and that only the values of $\hat{\phi}_1$ on $\mathbb{R}_-$ matter. The Fourier transform of the radial function $\mathbf{G}\phi$ is also radial, and given by

$$\widehat{\mathbf{G}\phi}(y) = \int_{\mathbb{R}} \hat{G}_\tau(y)\phi(\tau)\mathrm{d}\tau = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}\phi(\tau)(\tau/\mathrm{i})^{-D/2}\mathrm{d}\tau, \qquad y \in \mathbb{R}^D.$$

Suppose Φ is a test function in $\mathbb{R}^D$, and suppose it is radial. Then its Fourier transform $\hat{\Phi}$ is radial too, and both may be thought of as functions on $\mathbb{R}_+$. A natural Fourier uniqueness problem in this context is the following.

Problem 1.7 (Radial)**.** Suppose $\Phi(x) = \hat{\Phi}(y) = 0$ for all $x, y \in \mathbb{R}^D$ with $|x|^2, |y|^2 \in \mathbb{Z}$. Does it follow that $\Phi = 0$ everywhere?

Here, part of the problem is to find the correct class of test functions for which the conclusion $\Phi = 0$ is valid. We might also leave the context of radiality.

Problem 1.8 (Nonradial)**.** Suppose Φ is a test function in $\mathbb{R}^D$ which need not be radial. If $\Phi(x) = \hat{\Phi}(y) = 0$ for all $x, y \in \mathbb{R}^D$ with $|x|^2, |y|^2 \in \mathbb{Z}$, does it follow that $\Phi = 0$ everywhere?

The condition is that of vanishing on spherical shells. It has been investigated by Stoller [Sto]. What about other surfaces?

We focus on $D = 4$, and consider radial $\Phi = \mathbf{G}\phi$. Then if $|x|^2 = m \in \mathbb{Z}$ and $|y|^2 = n \in \mathbb{Z}$, and if $\Phi(x) = \hat{\Phi}(y) = 0$ for such $x, y \in \mathbb{R}^4$, we get the following conditions on ϕ:

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi m\tau}\phi(\tau)\mathrm{d}\tau = 0, \qquad m = 1, 2, 3, \ldots,$$

and

$$\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\pi n/\tau}\phi(\tau)\tau^{-2}\mathrm{d}\tau = 0, \qquad n = 1, 2, 3, \ldots.$$

Here, we initially assumed that $\phi \in L^1(\mathbb{R})$, but it turns out to be better to assume instead that $\phi \in C_0(\mathbb{R})$ and $\phi' \in L^1(\mathbb{R})$. Then the Fourier transform of ϕ remains well-defined, so that $\Phi = \mathbf{G}\phi$ is well-defined as well.

Next, by integration by parts,

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi m\tau}\phi'(\tau)\mathrm{d}\tau = 0, \qquad m = 0, 1, 2, \ldots,$$

and

$$\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\pi n/\tau}\phi'(\tau)\mathrm{d}\tau = 0, \qquad n = 1, 2, \ldots.$$

The case $m = 0$ is special, that the integral of ϕ' vanishes is a consequence of $\phi \in C_0(\mathbb{R})$. For the second set of equation, we are lucky that

$$\partial_\tau \mathrm{e}^{-\mathrm{i}\pi n/\tau} = \mathrm{i}\pi n\tau^{-2}\mathrm{e}^{-\mathrm{i}\pi n/\tau}.$$

Now we are in the setting of the strengthened ergodicity theorem. Since $\phi' \in L^1(\mathbb{R})$ is assumed, we conclude from the theorem that $\phi' \in H^1_+(\mathbb{R})$. Since by integration by parts

$$\Phi(x) = \mathbf{G}\phi(x) = \frac{\mathrm{i}}{\pi|x|^2}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi|x|^2\tau}\phi'(\tau)\mathrm{d}\tau, \qquad x \in \mathbb{R}^4,$$

it follows from a classical characterization of $H^1_+(\mathbb{R})$ that $\Phi = \mathbf{G}\phi = 0$ identically on $\mathbb{R}^4 \setminus \{0\}$.

We consider $D = 2d$, where $d \geq 2$ is an integer. Is there a counterpart of our integration by parts trick? We start with ϕ as before and form the radial function $\Phi = \mathbf{G}\phi$. It turns out to be natural to assume that $\phi^{(j)} \in C_0(\mathbb{R})$ for $j = 0, \ldots, d-2$, whereas $\phi^{(d-1)} \in L^1(\mathbb{R})$. It is given to us that ϕ meets

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\pi m\tau}\phi(\tau)\mathrm{d}\tau = 0, \qquad m = 1, 2, 3, \ldots,$$

and

$$\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\pi n/\tau}\phi(\tau)\tau^{-d}\mathrm{d}\tau = 0, \qquad n = 1, 2, 3, \ldots.$$

These integrals need not be absolutely convergent, but instead make sense in terms of integration by parts.

By iterated integration by parts, we find that

$$\Phi(x)=\mathbf{G}\phi(x)=\left(\frac{\mathrm{i}}{\pi|x|^2}\right)^{d-1}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\pi|x|^2\tau}\phi^{(d-1)}(\tau)\mathrm{d}\tau,\qquad x\in\mathbb{R}^{2d}\setminus\{0\}.$$

If we introduce the radial function

$$H_{\tau,d}(x):=\frac{\mathrm{e}^{\mathrm{i}\pi|x|^2\tau}}{|x|^{2d-2}},\qquad x\in\mathbb{R}^{2d}\setminus\{0\},$$

this means that

$$\Phi(x)=\mathbf{G}\phi(x)=(\mathrm{i}/\pi)^{d-1}\int_{\mathbb{R}}H_{\tau,d}(x)\phi^{(d-1)}(\tau)\mathrm{d}\tau,\qquad x\in\mathbb{R}^{2d}\setminus\{0\}.$$

As a radial tempered distribution, $H_{\tau,d}$ has a Fourier transform:

$$\hat{H}_{\tau,d}(y)=\frac{(\mathrm{i}\tau)^{d-2}}{|y|^{2d-2}}\left\{\sum_{j=0}^{d-2}\frac{1}{j!}(-\mathrm{i}\pi|y|^2/\tau)^j-\mathrm{e}^{-\mathrm{i}\pi|y|^2/\tau}\right\}.$$

Since $H_{\tau,d}$ solves the differential equation

$$\partial_\tau H_{\tau,d}(x)=\mathrm{i}\pi|x|^2H_{\tau,d}(x),$$

it follows that $\hat{H}_{\tau,d}$ solves the Schrödinger equation

$$\mathrm{i}\partial_\tau\hat{H}_{\tau,d}=\frac{1}{4\pi}\Delta\hat{H}_{\tau,d},$$

with initial datum for $\tau=0$ given by

$$\hat{H}_{0,d}(y)=\frac{\pi^{d-2}}{(d-2)!}|y|^{-2}.$$

By inspection, for fixed $y\neq0$, $\tau\mapsto H_{\tau,d}(y)$ defines a bounded holomorphic function in $\mathbb{H}$, and in particular, a function in $L^\infty(\mathbb{R})$.

It remains to show that if $\psi:=\phi^{(d-1)}\in L^1(\mathbb{R})$, then the conditions

$$\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\pi m\tau}\psi(\tau)\mathrm{d}\tau=0,\qquad m=0,1,2,\dots,$$

and

$$\int_{\mathbb{R}}(\mathrm{i}\tau)^{d-2}\left\{\sum_{j=0}^{d-2}\frac{1}{j!}(-\mathrm{i}\pi n/\tau)^j-\mathrm{e}^{-\mathrm{i}\pi n/\tau}\right\}\psi(\tau)\mathrm{d}\tau=0,$$

for $n=1,2,3,\dots$ imply that $\psi\in H^1_+(\mathbb{R})$. If this is attained, then the assertion that $\Phi=0$ on $\mathbb{R}^{2d}\setminus\{0\}$ follows from the known characterization of the Hardy space. In turn, this boils down to a kind of dynamical problem. We have a weighted transfer operator (transfer operators are often called Perron-Frobenius operators), and we need to show that it does not have 1 as an eigenvalue in a slightly bigger space than $L^1[-1,1]$. This bigger space contains the restrictions of Hilbert transforms of L^1 functions as well.

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The Density Hypothesis for Horizontal Families of Lattices

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(joint work with Mikołaj Frączyk, Gergely Harcos, Péter Maga)

Selberg's celebrated Eigenvalue Conjecture states that all nonzero Laplacian eigenvalues on congruence quotients of the upper half-plane are at least $1/4$, or equivalently that the archimedean constituents of non-trivial automorphic representations occurring in the spherical discrete L^2-spectrum for congruence subgroups are tempered. In its absence, for analytic applications in a *family* of automorphic forms, the non-tempered spectrum can often be satisfactorily handled if the exceptions in the family are known to be *sparse and not too bad*, in the following sense resembling the classical density theorems of prime number theory.

Let G be a semisimple Lie group without compact factors. To every irreducible unitary representation $\pi \in \widehat{G}$, we may associate the (extended) real number $p(\pi) \in [2, \infty]$ such that $p(\pi) = 2$ if and only if π is tempered and $p(\pi) = \infty$ if and only if $\pi = \mathbf{1}$ is the trivial representation; specifically,

$$p(\pi) := \inf \left\{ 2 \le p \le \infty : \pi \text{ has a nonzero matrix coefficient in } L^p(G) \right\}.$$

Following [7, 8], we say that a family $\mathcal{F}$ of lattices $\Gamma \subseteq G$ satisfies the *density hypothesis* if, for every bounded subset $\mathcal{B} \subseteq \widehat{G}$,

$$\sum_{\pi \in \mathcal{B}} \mathrm{m}(\pi, \Gamma) \ll_{\mathcal{B},\epsilon} \mathrm{vol}(\Gamma \backslash G)^{2/p(\mathcal{B})+\epsilon} \tag{1}$$

holds for every $\epsilon > 0$ uniformly for all $\Gamma \in \mathcal{F}$, where $\mathrm{m}(\pi, \Gamma)$ is the multiplicity with which π appears in the discrete spectrum of $\Gamma \backslash G$, and $p(\mathcal{B}) = \inf_{\pi \in \mathcal{B}} p(\pi)$. Originally formulated and proved in $\mathrm{SL}_2(\mathbb{R})$ as the density conjecture for principal congruence subgroups $\Gamma(N)$ of a fixed lattice and $|\mathcal{B}| = 1$ by Sarnak–Xue [8], the density hypothesis has been proved for various families of congruence sublattices of a *fixed* lattice Γ in $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{C})$, and more recently in $\mathrm{GL}(3)$, cohomological representations on $\mathrm{U}(n,1)$, and by Blomer [2] in $\mathrm{GL}(n)$.

1. **Connections and applications.** A family $\mathcal{F}$ of lattices in G intersecting the center of G in the same subgroup Θ has the *limit multiplicity property* if

$$\mu_\Gamma := \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\pi\in\widehat{G/\Theta}} \mathrm{m}(\pi,\Gamma)\delta_\pi \quad \longrightarrow \quad \mu_{\widehat{G/\Theta}} \quad (\mathrm{vol}(\Gamma\backslash G)\to\infty)$$

weakly-* to the Plancherel measure on $\widehat{G/\Theta}$. Originally discovered by DeGeorge and Wallach [4], the limit multiplicity property implies a small-o version of (1) and may also be thought of as the lattice (in particular, level) counterpart of Weyl's law. Previously studied only for families of congruence subgroups of a fixed arithmetic lattice, the limit multiplicity property was recently proved by Abért et al. [1] and Frączyk [5] for all cocompact torsion-free arithmetic lattices whose trace field is of bounded degree over $\mathbb{Q}$ (indeed for any family of lattices suitably uniformly away from $\{1\}$ in a simple group of higher rank), as well as for all cocompact torsion-free congruence lattices of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$; this motivated our Theorem 1.

Following [7], say that a family of lattices $\mathcal{F}$ satisfies the *spherical density hypothesis* if there exists an $L>0$ such that for every bounded subset $\mathcal{B}\subset\widehat{G}_{\mathrm{sph}}$ of the spherical unitary dual, we have uniformly for all $\Gamma\in\mathcal{F}$

$$\sum_{\pi\in\mathcal{B}} \mathrm{m}(\pi,\Gamma) \ll_\epsilon \mathrm{vol}(\Gamma\backslash G)^{2/p(\mathcal{B})+\epsilon}(1+\|\mathcal{B}\|)^L,$$

where $\|\mathcal{B}\| = \sup_{\pi\in\mathcal{B}}\Omega(\pi) < \infty$ and Ω is the Casimir operator. Golubev and Kamber [7] prove that the this spherical density hypothesis for principal congruence subgroups $\Gamma(\mathfrak{n})$ along with a uniform spectral gap implies the optimal lifting property that almost all elements of $\Gamma(\mathfrak{n})\backslash\Gamma$ can be lifted to elements of Γ lying in a ball of volume roughly $\mathrm{vol}(\Gamma(\mathfrak{n})\backslash G)$. Density theorems also have implications for quantum computing, and for sharp-cutoff Weyl laws as in [3].

2. **Main result.** In the presented paper, we prove the spherical density hypothesis for wide families of arithmetic lattices in the Lie group $G=\mathrm{SL}_2(\mathbb{R})^a\times\mathrm{SL}_2(\mathbb{C})^b$.

Fix $c\geq 0$. For any number field k with $[k:\mathbb{Q}]=a+2b+c$ and any division quaternion algebra A/k such that $A\otimes_{\mathbb{Q}}\mathbb{R}\simeq \mathrm{M}_2(\mathbb{R})^a\times\mathrm{M}_2(\mathbb{C})^b\times\mathbb{H}^c$ (with $\mathbb{H}$ being the Hamilton quaternions), the algebraic group $\mathbf{G}=\mathrm{SL}_1(A)$ satisfies $\mathbf{G}(\mathbb{A})\simeq G\times\mathrm{SU}_2^c\times\mathbf{G}(\mathbb{A}_f)$, and every open compact subgroup $U\subset\mathbf{G}(\mathbb{A}_f)$ gives rise to an arithmetic subgroup $\Gamma_U\leq G$. In particular, for every integral ideal $\mathfrak{n}\subseteq\mathfrak{o}_k$ and every map $\kappa:\{\mathfrak{p}\mid\mathfrak{n}\}\to\{0,1\}$, denoting $\mathfrak{n}_\mathfrak{p}=\mathfrak{n}\mathfrak{o}_\mathfrak{p}$ and by $K_0(\mathfrak{n}_\mathfrak{p})$ and $K_1(\mathfrak{n}_\mathfrak{p})$ the groups of matrices in $\mathrm{SL}_2(\mathfrak{o}_\mathfrak{p})$ congruent to upper-triangular and scalar matrices modulo $\mathfrak{n}_\mathfrak{p}$, respectively, we have the open compact subgroup

$$K_\kappa(\mathfrak{n}) = \prod_{\mathfrak{p}\in\mathrm{ram}_f(A)} \mathrm{SL}_1(A_\mathfrak{p}) \times \prod_{\mathfrak{p}\mid\mathfrak{n}} K_{\kappa(\mathfrak{p})}(\mathfrak{n}_\mathfrak{p}) \times \prod_{\substack{\mathfrak{p}\notin\mathrm{ram}_f(A)\\ \mathfrak{p}\nmid\mathfrak{n}}} \mathrm{SL}_2(\mathfrak{o}_\mathfrak{p})$$

and the corresponding lattice $\Gamma_\kappa(\mathfrak{n}) = \Gamma_{K_\kappa(\mathfrak{n})}$. For given $a, b, c \geq 0$, we let $\mathcal{F}_{a,b,c}$ be the family of all congruence lattices $\Gamma_\kappa(\mathfrak{n}) \leq \mathrm{SL}_2(\mathbb{R})^a \times \mathrm{SL}_2(\mathbb{C})^b$, over all number fields k, all quaternion algebras A, and all $\mathfrak{n} \subset \mathfrak{o}$ and $\kappa : \{\mathfrak{p} \mid \mathfrak{n}\} \to \{0, 1\}$ as above.

Let S_∞^G be the set of the $a+b$ infinite places of k where A splits. The spherical unitary dual $\widehat{G}_{\mathrm{sph}}$ is parametrized as $\pi_{\boldsymbol{s}} \simeq \bigotimes_{v \in S_\infty^G} \pi_{s_v}^{k_v}$, where $\pi_{s_v}^{k_v}$ is the spherical principal series representation over $k_v \in \{\mathbb{R}, \mathbb{C}\}$ of normalized Casimir eigenvalue $1/4 - s_v^2$, which is tempered if and only if $s_v \in i\mathbb{R}$. For any $S \subseteq S_\infty^G$ and any $\boldsymbol{\sigma} = (\sigma_v) \in [0, \frac{1}{2}]^S$, $\boldsymbol{T} = (T_v) \in \mathbb{R}^{S_\infty^G \setminus S}$, we introduce the bounded subset of $\widehat{G}_{\mathrm{sph}}$,

$$\mathcal{B}(\boldsymbol{\sigma}, \boldsymbol{T}) := \left\{ \pi_{\boldsymbol{s}} \ : \ \boldsymbol{s} \in \prod_{v \in S} [\sigma_v, 1/2] \times \prod_{v \in S_\infty^G \setminus S} i[T_v - 1, T_v + 1] \right\},$$

and the quantity in the spirit of analytic conductor

$$\mathcal{C}(\Gamma, \boldsymbol{T}) := \mathrm{vol}(\Gamma \backslash G) \prod_{v \in S_\infty^G \setminus S} (1 + |T_v|)^{\rho_v},$$

where $\rho_v = [k_v : \mathbb{Q}]$. In particular, $p(\mathcal{B}(\boldsymbol{\sigma}, \boldsymbol{T})) = p(\boldsymbol{\sigma})$ satisfies $2/p(\boldsymbol{\sigma}) = 1 - 2\max_{v \in S} |\sigma_v|$. Our main result is as follows.

Theorem 1. *For every $a, b, c \geq 0$, the family $\mathcal{F}_{a,b,c}$ of congruence lattices in $G = \mathrm{SL}_2(\mathbb{R})^a \times \mathrm{SL}_2(\mathbb{C})^b$ satisfies the spherical density hypothesis. More precisely, for every $\Gamma \in \mathcal{F}_{a,b,c}$, $S \subseteq S_\infty^G$, $\boldsymbol{\sigma} \in [0, 1/2]^S$, and $\boldsymbol{T} \in \mathbb{R}^{S_\infty^G \setminus S}$, we have for any $\epsilon > 0$*

$$\sum_{\pi \in \mathcal{B}(\boldsymbol{\sigma}, \boldsymbol{T})} \mathrm{m}(\pi, \Gamma) \ll_{\epsilon, a, b, c} \mathcal{C}(\Gamma, \boldsymbol{T})^{2/p(\boldsymbol{\sigma}) + \epsilon}. \tag{2}$$

Already for $S = S_\infty^G$ (or for fixed $\boldsymbol{T} \in \mathbb{R}^{S_\infty^G \setminus S}$), Theorem 1 extends for the first time the results of [8] to families of non-commensurable lattices. Theorem 1 allows for groups G of arbitrary rank $a + b$ and a wider variety of congruence subgroups $\Gamma_\kappa(\mathfrak{n})$, even when considering subgroups of a fixed lattice. Moreover, it holds uniformly over all lattices in $\mathcal{F}_{a,b,c}$ and all possible pairs (k, A), with no dependence on a particular fixed ambient lattice, and addresses for the first time the (natural) dependence on the tempered components of π. In the fully degenerate case $S = \emptyset$, Theorem 1 recovers up to $\mathrm{vol}(\Gamma \backslash G)^\epsilon$ the local bound from Weyl's law, for the first time uniformly across all lattices in $\mathcal{F}_{a,b,c}$.

Our proof of Theorem 1 proceeds by the Arthur–Selberg trace formula, with a test function chosen so as to emphasize contributions of representations occurring on the left-hand side of (2). In the sum over semisimple conjugacy classes on its geometric side, we estimate the volumes of adelic quotients building on the work of Borel, Ono, and Ullmo–Yafaev, the orbital integrals using various integral transforms and counting in Bruhat–Tits trees, and the number of contributing conjugacy classes using the geometry of numbers input from [6].

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A cubic analogue of the Friedlader-Iwaniec spin along primes

Jori Merikoski

In 1998 Friedlander and Iwaniec [1] famously proved that there are infinitely many prime numbers of the form a^2+b^4. Remarkable about this result is the fact that this is a very sparse sequence, that is, the number of integers of the form a^2+b^4 up to x is of the order $x^{3/4}$. More precisely, if $\Lambda(n)$ denotes the von Mangoldt function, then we have the following.

Theorem 1. (Friedlander-Iwaniec, 1998). *There are infinitely many primes of the form a^2+b^4. In fact, for a certain constant $C>0$ we have*

$$\sum_{a^2+b^4\leq x} \Lambda(a^2+b^4) \sim Cx^{3/4}.$$

One of the key ingredients in the argument is their proof that the so-called spin (s/r) is equidistributed along Gaussian primes $\pi = r+is$.

Theorem 2. (Friedlander-Iwaniec, 1998). *We have*

$$\sum_{\substack{r^2+s^2=p\leq x\\ r\ odd}} \left(\frac{s}{r}\right) \ll x^{1-1/77},$$

where $(\frac{s}{r})$ denotes the Jacobi symbol.

It is then reasonable to ask if their argument could be extended to show that there are infinitely many prime numbers of the form a^2+b^6. However, there are two large obstacles to this. Firstly, this sequence is even sparser than the previous, and it turns out that due to this the Friedlander-Iwaniec method breaks down at certain point.

The second issue is a structural one. More precisely, the proof of Theorem 2 relies heavily on the Law of quadratic reciprocity. With the sequence a^2+b^6 we end up with cubic residues, which unfortunately do not satisfy a nice reciprocity law on $\mathbb{Z}$.

The second problem can be rectified if we lift the whole set-up from $\mathbb{Z}$ to the Eisenstein integers $\mathbb{Z}[\zeta_3]$, where $\zeta_n := e^{2\pi i/n}$ denotes a primitive nth root of unity. In this setting we can then prove a cubic analogue of Theorem 2.

To state our main result, recall that it is possible to define for all coprime $r, s \in \mathbb{Z}[\zeta_3]$ the cubic residue character $\left[\frac{s}{r}\right]_3$ (analogously to the Jacobi symbol), which takes values on cube roots of unity. Say that $r \in \mathbb{Z}[\zeta_3]$ is primary if $r \equiv \pm 1 \pmod 3$. Then crucial to us is the law of cubic reciprocity, which states that for any primary $r, s \in \mathbb{Z}[\zeta_3]$ we have $\left[\frac{s}{r}\right]_3 = \left[\frac{r}{s}\right]_3$ (cf. [2, Chapter VIII.5] for background on reciprocity laws).

Analogous to the Gaussian integers, in this set-up we have $\mathbb{Z}[\zeta_3, i] = \mathbb{Z}[\zeta_{12}]$, the ring of integers of the twelfth cyclotomic field $\mathbb{Q}[\zeta_{12}]$. A new feature is that the unit group of $\mathbb{Z}[\zeta_{12}]$ is infinite, which means that some care is needed in formulating the cubic analogue of Theorem 2. Say that $z = r + is \in \mathbb{Z}[\zeta_{12}]$ is primary if $z \equiv \pm 1 \pmod 3$. It turns out that for any ideal $\mathfrak{a} \subseteq \mathbb{Z}[\zeta_{12}]$ coprime to 3 there is a primary generator $r + is$, $r, s \in \mathbb{Z}[\zeta_3]$ of $\mathfrak{a}$, and we can define the *cubic spin*

$$[\mathfrak{a}]_3 = [(r + is)]_3 := \left[\frac{s}{r}\right]_3.$$

We show that this is independent of the choice of the primary generator $r + is$. Our main theorem is then the following (the manuscript is still work-in-progress).

Theorem 3. (Merikoski, 2020). *We have*

$$\sum_{\substack{\mathfrak{p} \subseteq \mathbb{Z}[\zeta_{12}] \\ N_{\mathbb{Q}(\zeta_{12})}\mathfrak{p} \leq x}} [\mathfrak{p}]_3 \ll x^{1-1/143}$$

This is related to primes of the form $\alpha^2 + \beta^6 \in \mathbb{Z}[\zeta_3]$ similarly as Theorem 2 is related to primes of the form $a^2 + b^4$ in the work of Friedlander and Iwaniec.

To state an open problem, we note that the proof of relies mainly on cubic reciprocity. Thus, it seems plausible that the result could be greatly generalized as follows. If an algebraic number field K contains a primitive mth root of unity, then we can define the mth power residue character on K which satisfies a reciprocity law (by Hilbert reciprocity, for instance).

Given a quadratic extension L/K we can then define a spin at elements of $\mathcal{O}_L$ (probably some assumptions are required, the simplest case would be $K = \mathbb{Q}(\zeta_m)$ with m odd prime and $L = K(i)$). Hopefully the argument could be generalized to obtain equidistribution of this spin along principal prime ideals of $\mathcal{O}_L$.

To elaborate on this, let

$$\lambda_3(n) := \sum_{N_{\mathbb{Q}(\zeta_{12})}\mathfrak{a} = n} [\mathfrak{a}]_3.$$

Corollary 4.

$$\sum_{n\le x} \Lambda(n)\lambda_3(n) \ll x^{1-1/143}.$$

Similar to this Friedlander and Iwaniec use the Jacobi symbol to define "$\lambda_2(n)$" and write that this might be "related to the Fourier coefficients of some kind of Metaplectic Eisenstein series or a cusp form" [1, Section 23]. Obtaining such a description should be helpful for understanding the generalization.

Open problem. Define $\lambda_m(n; L/K)$ in general (K, L, m as in the above) and find an automorphic interpretation. Show equidistribution along primes.

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Small scale equidistribution of Hecke eigenforms at infinity

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(joint work with Asbjørn C. Nordentoft, Yiannis N. Petridis)

Let H_k be a Hecke basis of holomorphic cusp forms f of weight k for the full modular group Γ normalized to have Petersson norm 1. We investigate the limiting behaviour of

$$\mu_f(\psi) := \int_M \psi(z) y^k |f(z)|^2 d\mu(z)$$

on a suitable set of testfunctions ψ depending possibly on k. Here $M = \Gamma\backslash\mathbb{H}$. Holowinsky and Soundararajan [1] famously proved that for a dense set of fixed testfunctions in $L^2(M)$ we have

$$\mu_f(\psi) \to \nu(\psi) := \frac{3}{\pi}\int_M \psi(z) d\mu(z), \text{ as } k \to \infty.$$

We investigate what happens if we allow the support of ψ to shrink as $k \to \infty$ in the following way: Let $B_H := \{z \in M | \operatorname{Im}(z) > H\}$, $B = B_1$ and let furthermore $C_0^\infty(M, B_H)$ denote the set of smooth functions on M that are decaying rapidly in the cusp, supported in B_H, with compactly supported zero Fourier coefficient. We then consider the following operator

$$M_f(\psi)(z) = \psi(x + iy/H)$$

mapping $C_0^\infty(M, B)$ to $C_0^\infty(M, B_H)$. Note that M_f 'pushes' ψ up towards the cusp. We consider $H = (k-1)^\theta$ for $\theta > 0$. For this operator we want to investigate to what extend

$$\mu_f(M_{(k-1)^\theta}\psi) = \nu(M_{(k-1)^\theta}\psi) + o\left(\int_M |M_{(k-1)^\theta}\psi| d\mu\right), \text{ as } k \to \infty \tag{1}$$

i.e. to what extend the measures $y^k|f(z)|^2 d\mu(z)$ equidistribute on the shrinking sets $B_{(k-1)^\theta}$. We expect this to be the case all the way down to the Planck scale which is at $\theta = 1$. We first show that below the Planck scale we do not have equidistribution.

Proposition 1. *If $\theta \geq 1$ there exist $\psi \in C_0^\infty(M, B)$ such that*

$$\mu_f(M_{(k-1)^\theta}\psi) = o(\nu(M_{(k-1)^\theta}\psi)) \text{ as } k \to \infty.$$

We then move on to show that below the Planck scale, i.e. for $\theta < 1$ we have equidistribution on average over $f \in H_k$ and k of size K.

Theorem 1. *If $0 < \theta < 1$ then for $\psi \in C_0^\infty(M, B)$, and $u \in C_c^\infty(\mathbb{R}_+)$ we have*

$$\sum_{2|k} u\left(\frac{k-1}{K}\right) \sum_{f \in H_k} \left|\mu_f(M_{(k-1)^\theta}\psi) - \nu(M_{(k-1)^\theta}\psi)\right|^2 \ll_{u,\psi,\theta} K^{2-2\theta-\min(1/5,1-\theta)+\varepsilon},$$

as $K \to \infty$.

Note that this implies that on a density one subset we have equidistribution in this type of shrinking set $B_{(k-1)^\theta}$.

We next show that if we restrict to $C_{0,0}^\infty(M, B) = \{\psi \in C_0^\infty(M, B) | \int_M \psi d\mu = 0\}$ then we can compute the variance. Let $L(sym^2 f, s)$ denote the symmetric square L-function related to f.

Theorem 2. *Let $0 < \theta < 1$. There exist $\delta_\theta > 0$ and a bilinear Hermitian form $B_\theta : C_{0,0}^\infty(M, B) \times C_{0,0}^\infty(M, B) \to \mathbb{C}$ such that*

$$\sum_{2|k} u\left(\frac{k-1}{K}\right) \sum_{f \in H_k} L(sym^2 f, 1) \left|\mu_f(M_{(k-1)^\theta}\psi) - \nu(M_{(k-1)^\theta}\psi)\right|^2 = B_\theta(\psi, \psi) K^{1-\theta} + O_{\psi,\theta}(K^{1-\theta-\delta_\theta}).$$

as $K \to \infty$.

This is related to strong results by Luo and Sarnak [4] who proved the analogous statement for $\theta = 0$. In their case they showed that the bilinear form is diagonalized by Maass cusp forms. They then showed that the corresponding eigenvalues equals the central values of the L-function of the Maass cusp form ϕ times a universal constant. Such statements does not make sense in our case as Maass cusp forms are not supported in B. But by a continuity argument one may extend B_θ which includes $1_B\phi$ and we may prove that $B_\theta(1_B\phi, 1_B\phi) \geq 0$. We can also show that $B_\theta(1_B\phi, 1_B\phi)$ is given by a Dirichlet series expression. Using this we may show that if ϕ is an even Hecke–Maass cusp form with eigenvalue $s_\phi(1 - s_\phi)$ and Hecke eigenvalues $\lambda_\phi(n)$, then

$$\sum_{m,n \geq 1} \frac{\tau_1((m,n))\lambda_\phi(m)\lambda_\phi(n)}{(mn)^{1/2}} \int_{\max(m,n)}^\infty |K_{s_\phi - 1/2}(2\pi y)|^2 \frac{dy}{y} \geq 0. \tag{2}$$

Here τ_1 is the sum of divisors function and $K_s(y)$ is the K-bessel function. It seems hard to see that this series expression is non-negative without going through Theorem 2.

Finally we discuss the explicit form of B_θ. The decomposition

$$C^\infty_{0,0}(M,B) = C^\infty_{\mathrm{cusp}}(M,B) \oplus C^\infty_{\mathrm{Eis}}(M,B), \tag{3}$$

into the cuspidal and the Eisenstein part is orthogonal with respect to B_θ for all $0 < \theta < 1$. Furthermore we observe 3 different regimes in the sense that B_θ is independent of θ on each of the three intervals $0 < \theta < 1/2$, $\theta = 1/2$ and $1/2 < \theta < 1$. This shows that there is a transition phenomena occurring half way to the Planck scale. This is related to transition phenomena observed in [2] [3].

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Problem Session

(1) (Farrell Brumley) Prove quantum ergodicity for Maass forms on locally symmetric spaces of non-compact type, e.g., $\Gamma\backslash \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$.

(2) (Farrell Brumley, passing along a question of Wee Teck Gan) Periods of automorphic forms GL_3 along the diagonal torus are related via an exceptional theta correspondence to Fourier coefficients of automorphic forms on G_2. Given a fixed Hecke–Maass form on GL_3, show that there is at least one non-vanishing closed Cartan period.

(3) (Philippe Michel) Let φ be a Maass form. Let $D > 0$ be a large fundamental discriminant. Find a geodesic γ of discriminant D so that

$$\int_\gamma \varphi \neq 0,$$

assuming $h(D)$ is "huge". Equivalently, show that there is a class group character χ of $\mathbb{Q}(\sqrt{D})$ so that $L(\varphi \otimes \theta_\chi, 1/2) \neq 0$.

The issue is that harmonic analytic methods most readily give such a conclusion for some unramified Hecke character χ "close to" a class group character.

(4) (Merikoski) There is a problem stated in Merikoski's talk.

(5) (Sawin) There were problems advertised in Sawin's talk.

(6) (Kowalski) For a prime p and an interval I of size $|I| = p^{1/2-\gamma}$ with $\gamma > 0$ fixed, improve upon the trivial bound, which is $|I|^3/p^2$, for

$$\frac{1}{p-1}\sum_{a\in\mathbb{F}_p^\times}\left|\frac{1}{\sqrt{p}}\sum_{x\in I} e(\frac{ax+\bar{x}}{p})\right|$$

by a small power of p. This has applications to Kloosterman paths as in the paper of Kowalski–Sawin and is probably hard.

(7) (Harcos) Prove a sup-norm bound on $\mathrm{GL}_n(\mathbb{R})$ for automorphic forms of general K-type using the Fourier–Whittaker decomposition (near the cusps) complemented by the pretrace formula. Motivated by papers of Blomer–Harcos–Milićević and Humphries. The point is not to get anything optimal; the point is just to get some exponent that is reasonable.

A complementary question is if one can produce lower bound in the cusp in the style of Brumley–Templier?

(8) (Michel) Prove that the zeros of newforms in $S_2(N)$ have $o(N)$ multiplicity after pushforward to $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

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