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Cohomology of Finite Groups: Interactions and Applications (hybrid meeting)

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Abstract. The cohomology of finite groups is an important tool in many subjects including representation theory and algebraic topology. This meeting was the fifth in a series that has emphasized the interactions of group cohomology with other areas. In spite of the Covid-19 epidemic, this hybrid meeting ran smoothly with about half the participants physically present and the other half participating via Zoom.

Mathematics Subject Classification (2010): 20Jxx, 20Cxx, 55Nxx, 57Nxx.

Introduction by the Organizers

The workshop *Cohomology of finite groups: Interactions and Applications* was held under extraordinary circumstances, due to the spread of Covid-19 and the isolation measures surrounding the pandemic. This meant that 25 of the 49 participants were present in person, with the rest participating remotely by Zoom. Three of the four organisers were physically present. The technology was up to the task, and participation was high among the remote attendees, in spite of often large time differences. Since there were a large number of North Americans taking part remotely, the majority of the talks were in the afternoon and evening, with the late morning set aside for discussions and collaborations. There was room in the schedule for everyone who submitted an abstract to give a talk. There were 20 one hour talks in total, of which 15 were given in person and 5 remotely. Of the 15 given in person, 4 were by postdocs and the rest by more seasoned researchers.

The weather was somewhat variable, but held out for the traditional Wednesday afternoon hike to St. Roman.

There were many scientific highlights of the meeting. Fusion systems and their cohomology were prominent. Nadia Mazza talked on the cohomology of pro-saturated pro-fusion systems on pro-p-groups and presented a version of the Cartan–Eilenberg stable element method in this context for the continous cohomology. Bob Oliver and Ellen Henke gave us the latest applications of fusion systems to the theory of finite simple groups. Henke's talk discussed the relationship between fusion systems and their homotopy theory, and Chermak's partial groups. The purpose was to give a smoother and more conceptual framework for their applications in Aschbacher's programme for rewriting parts of the classification of finite simple groups in terms of fusion systems. Oliver's talk was more focused on the simplicity of the fusion systems associated with the known simple groups, and whether there were exotic fusion subsystems. Radha Kessar's talk on weight conjectures for p-compact groups showed their connection to the modular representation theory of finite groups of Lie type via fusion systems, and their relations with the theory of spetses. This gave rise to extensive discussions, especially at an informal evening session organized by Tobias Barthel.

The structure of stable module categories appeared in Jon Carlson's Zoom talk, with the focus on the structure of the endomorphism ring of the trivial module after a Verdier localisation. A striking example showed how one can realise the negative part of Tate cohomology in such a construction. Dan Nakano's Zoom talk involved a noncommutative version of the tensor triangulated geometry that we know in the commutative version from Paul Balmer's work, and described how it related to various stable module categories where the tensor product is not commutative up to isomorphism.

Antoine Touzé talked about his work on functor categories with Djament and Vespa, and its application to understanding the category of polynomial functors when the coefficient ring is not a field. Jesper Grodal's talk on the group of endotrivial modules gave us an update of progress in going from finite p-groups to arbitrary finite groups, a project involving many authors, including a number of participants of the conference. He presented a new perspective, considering the stable module category of a finite group as a stable infinity category. This allows us to consider the Picard space for such category, which has the advantage of being better behaved with respect to limits, and with respect to stable module categories for p-subgroups. The group of connected components is then the Picard group. This point of view led to extensive evening discussions over a beer or two, concerning the role of infinity categories in our subject.

Staying for the moment in the realm of representation theory, Serge Bouc and Ergün Yalçın both gave Zoom talks involving permutation modules in some form. Bouc's involved biset functors and their cohomology, and is part of a long running programme for understanding extensions in this context, while Yalçın's involved the Dade group, constructed from endopermutation modules, but modified so that it works for general finite groups, not just p-groups. Closely related to this were the talks of Peter Webb, where he described what works and what doesn't work when one tries to construct transfer maps in the cohomology of categories, and that of Stefan Schwede, which was a more topological look at global Mackey functors. The latter gave a new perspective on Nakaoka's splitting result for the cohomology of the symmetric groups, by adapting a proof by Dold to this setting which then extends to orthogonal, unitary and symplectic Lie groups. This then gives applications on the regularity of Euler classes.

Markus Linckelmann talked on his joint work with Benson and Kessar on the Batalin–Vilkovisky operator in the Hochschild cohomology of a finite group, and gave a very simple construction of the corresponding operator on the ordinary cohomology of the centraliser of an element, in the centraliser decomposition of Hochschild cohomology.

Paul Balmer was going to give a Zoom talk about the role of permutation complexes in the bounded derived module category, but logistic complications got in the way, and he had to cancel. Dave Benson, as an organiser, was not expecting to talk, but gave them one at short notice on the exotic abelian symmetric tensor categories in prime characteristic that he had been developing in joint work with Etingof and Ostrik, and described a conjecture for their cohomology algebras.

Moving to the topological side, John Greenlees talked about work in progress with Benson and Stevenson, aimed at understanding the singularity category of $C^*(BG)$, the geometric cochains on the classifying space of a finite group G. This category vanishes if and only if G is a finite p-group, and otherwise it is not well understood. He showed how it is closely related to the modules for $C_*(\Omega BG^\wedge_p)$, the chains on the loops of the p-completion of BG, and described their approach when G is a finite group with a cyclic Sylow p-subgroup.

Of the four postdoc talks, three were on the topological side and one on the algebraic side of the subject. Simon Gritschacher's talk involved joint work with Adem and Gomez on topological invariants of the space of commuting pairs of elements of a compact Lie group, and in particular gave a description of π_2 of this space. Marc Stephan talked about obstructions to the topological realisation of G-equivariant chain complexes by free G-spaces motivated by a conjecture of Carlsson on free actions of elementary abelian p-groups on finite CW-complexes. Markus Hausmann's talk described the role of formal groups in the context of stable homotopy theory, focusing on Quillen's work on complex cobordism, and the analogous setting in both equivariant formal group laws and equivariant stable homotopy theory for abelian compact Lie groups, where he presented statements generalizing the classical results. The algebraic talk was by Daniel Bissinger, who told us about his work on modules of constant Jordan type with Loewy length two.

Workshop (hybrid meeting): Cohomology of Finite Groups: Interactions and Applications

Table of Contents

Abstracts

The singularity category of a group with cyclic Sylow *p*-subgroup

J.P.C. Greenlees
(joint work with D.J. Benson, G. Stevenson)

1. Context

For a compact Lie group G and a field k of characteristic p, we are interested in the cohomology ring $H^*(BG)$ (coefficients in k), but in fact a refinement shows richer and more uniform behaviour: we therefore consider the cochains $C^*(BG)$ (again with coefficients in k), or more precisely the commutative ring spectrum $C^*(BG) = \mathrm{map}(BG, k)$ with $\pi_*(C^*(BG)) = H^*(BG)$.

It is known that for finite G the ring $C^*(BG)$ is always Gorenstein ([1]), and that it is regular if and only if G is p-nilpotent. Accordingly it is natural to consider the singularity category, and we would like to define

$$\mathsf{D}_{sg}(C^*(BG)) := \mathsf{D}^b(C^*(BG))/\mathsf{D}^c(C^*(BG)).$$

It is perfectly clear how to make sense of $\mathsf{D}^c(C^*(BG))$ as the compact $C^*(BG)$-modules in $\mathsf{D}(C^*(BG))$, but it is much less clear how to define the bounded derived category $\mathsf{D}^b(C^*(BG))$.

In fact we start by choosing a faithful representation $\nu : G \to U$ in a group U with $C^*(BU)$ regular, and then say that a $C^*(BG)$-module M is *ν-finitely generated* if M is small as a $C^*(BU)$-module via restriction along $\nu^* : C^*(BU) \to C^*(BG)$.

Theorem 1 (Greenlees-Stevenson[2])**.** *The condition that a $C^*(BG)$-module M is ν-finitely generated is independent of ν.*

Choosing U so that $H^*(BU)$ has polynomial cohomology (for example $U = U(n)$ for some n) we obtain an easy characterization of finitely generated $C^*(BG)$-modules.

Corollary 2. *A $C^*(BG)$-module M is finitely generated if and only if $\pi_*(M)$ is finitely generated over $H^*(BG)$.*

We also see that the vanishing of the singularity category captures regularity as required.

Corollary 3. *The singularity category is trivial if and only if G is p-nilpotent.*

This poses the problem of calculating the singularity category in some case where it does not vanish.

2. Groups with cyclic Sylow p-subgroup

First of all, if G has cyclic Sylow p-subgroup $C = C_{p^n}$ then the maps

$$G \leftarrow N_G(C) \rightarrow N_G(C)/O_{p'}N_G(C)$$

induce an equivalence

$$BG_p^\wedge \simeq B(C \rtimes D)_p^\wedge$$

where $D = C_q$ acts faithfully on C, so $q|p-1$. This reduces us to the special case when G is a semi-direct product $G = C \rtimes D$.

2.1. **The cohomology ring.** From now on we suppose $G = C \rtimes D$ where $C = C_{p^n}$ and $D = C_q$ with $q|p-1$ and D acts faithfully on C. We write $h = p^n - (p^n - 1)/q$. To avoid G being a p-group we suppose $q \geq 2$ so $p \geq 3$.

Here we have

$$H^*(BG) = k[X] \otimes \Lambda(T) \text{ with } \deg X = -2q, \deg T = -(2q-1).$$

It is easily deduced from the known Massey products in $H^*(BC)$ that the p^n-fold Massey product is defined and (with zero indeterminacy) we have

$$\langle T, T, \ldots, T \rangle = -X^h.$$

2.2. **The Koszul dual.** The case $p^n = 3$ is a bit exceptional, but for $p^n > 3$

$$H_*(\Omega(BG_p^\wedge)) = \Lambda(\xi) \otimes k[\tau] \text{ with } \deg(\xi) = 2q-1 \, \deg(\tau) = 2q-2.$$

Theorem 4 (Benson-Greenlees [3])**.** *The h-fold Massey product is defined and (with zero indeterminacy)*

$$\langle \xi, \xi, \ldots, \xi \rangle = -\tau^{p^n}.$$

3. The singularity category

3.1. **The BGG correspondence.** The following uses the BGG correspondence of [2] to give a complete description of the singularity category.

Theorem 5. *We have equivalences*

$$\mathsf{D}_{sg}(C^*(BG)) = \mathsf{D}^b(\Omega(BG_p^\wedge))/\langle k \rangle \simeq \mathsf{D}^b(\Omega(BG_p^\wedge)[\tau^{-1}]).$$

In principle this gives a complete calculation. In particular, it immediately makes clear $(2q-2)$-periodicity and provides a functor to modules over $k[\tau, \tau^{-1}] \otimes \Lambda(\xi)$-modules. However the statement does not address the question of what the objects of the category are.

3.2. **Realizability.** An $H^*(BG)$-module M is said to be *realizable* if there is a $C^*(BG)$-module X with $\pi_*(X) \cong M$. The realization is *unique* if any two $C^*(BG)$-module realizations of M are equivalent. For example, the residue field k is uniquely realizable because $C^*(BG)$ is coconnected.

Where an $H^*(BG)$-module M is uniquely realizable, we will also write M for its realization.

3.3. **Reducing to Maximal Cohen-Macaulay modules.** If M is any finitely generated $C^*(BG)$-module, we may find a finitely generated free $H^*(BG)$-module F mapping onto $\pi_*(M)$, and then we have a short exact sequence

$$0 \to \Omega\pi_* M \to F \to \pi_* M \to 0,$$

and because $H^*(BG)$ is of Krull dimension 1, $\Omega\pi_* M$ is a maximal Cohen-Macaulay module (MCM) over $H^*(BG)$.

Of course F is a sum of suspensions of $H^*(BG)$, so we may realize it by taking $\mathbb{F}$ to be the corresponding sum of suspensions of $C^*(BG)$. We may also realize the map $F \to \pi_* M$, and the resulting cofibre sequence

$$\Omega M \to \mathbb{F} \to M$$

shows that in realizing finitely generated modules we need only consider realizing MCMs.

3.4. **Maximal Cohen-Macaulay modules.** The indecomposable MCMs over $H^*(BG) = k[X] \otimes \Lambda(T)$ are the ideals

$$M_i = (X^i, T) \text{ for } i = 0, 1, 2, 3, \ldots, \infty.$$

Theorem 6. *The modules $M_0, M_1, \ldots, M_h$ are uniquely realizable and the modules M_i for $h+1 \leq i \leq \infty$ are not realizable.*

To prove this one may check that if M_i is realizable then M_{i-1} is realizable, and that M_i is then uniquely constructed from M_{i-1} by attaching a copy of k. This uses an Adams-Eilenberg-Moore spectral sequence to make calculations, and matrix factorizations to give convenient periodic resolutions of M_i over $H^*(BG)$. This shows that the realizable indecomposable MCMs M_i are for i in an initial segment $[0, s]$ and also that they the realizable modules are uniquely realizable. Finally, one observes that M_i (if it exists) embodies the vanishing of an $(i-1)$-fold Massey power of $\xi \in [k, k]_{C^*(BG)} = H_*(\Omega(BG_p^\wedge))$.

The result now follows from Theorem 4.

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Representations and cohomology of $GL_n(R)$ with an arbitrary ring R

ANTOINE TOUZÉ
(joint work with Aurélien Djament, Christine Vespa)

Representations and cohomology of the finite groups $GL_n(\mathbb{F}_q)$ are familiar objects which are extensively studied. In particular, they are known to be closely related to the polynomial representations of $GL_n(k)$ and their cohomology. In this talk we explain similar relations when $\mathbb{F}_q$ is replaced by a more general ring R.

To give an idea of the relations we are talking about, we state two theorems. **Fix an algebraically closed field k of positive characteristic p.** Recall that a k-linear finite dimensional representation V of $GL_n(k)$ is called *polynomial* (*of degree* d) if its action morphism is given by polynomial formulas (of degree d) in the entries of $g = [g_{st}] \in GL_n(k)$. Polynomial representations yield a full abelian subcategory Pol_n of $kGL_n(k)-\mathrm{Mod}$, which is much studied in terms the Schur algebras $S(n,d)$, indeed we have an equivalence of categories [9]:

$$\mathrm{Pol}_n \simeq \bigoplus_{d\geq 0} S(n,d)-\mathrm{mod}\ .$$

The first theorem deals with simple representations. Let us recall a classical construction, which is an instance of Schur-Weyl duality. For all finite dimensional vector spaces V, and all $k\mathfrak{S}_d$-modules M, let $E_M(V)$ be the $kGL(V)$-module:

$$E_M(V) := \mathrm{Image}\left((V^{\otimes d}\otimes M)_{\mathfrak{S}_d} \xrightarrow{\mathrm{Norm}} (V^{\otimes d}\otimes M)^{\mathfrak{S}_d}\right)\ . \tag{1}$$

If S is a simple representation of $k\mathfrak{S}_d$, then either $E_S(k^n)$ is zero, or $E_S(k^n)$ is a simple polynomial representation of $GL_n(k)$. Not all simple polynomial representations of $GL_n(k)$ are of this kind: the representations obtained this way are the *p-restricted* ones. The next theorem is a variant of the celebrated tensor product theorem of Steinberg [11], the latter being usually stated for a finite field R.

Theorem 1. *Let R be a finite ring of characteristic p^r, and let $B_1,\dots,B_s$ be a set of representatives of the isomorphism classes of simple (k,R)-bimodules. A $kGL_n(R)$-module M is simple if and only if there is an isomorphism*

$$M \simeq M_1^{[B_1]}\otimes\cdots\otimes M_s^{[B_s]}$$

where each $M^{[B_i]}$ is the restriction of a p-restricted simple representation M_i along the morphism $GL_n(R)\to GL_{n\dim_k B_i}(k)$ determined by B_i.

The second theorem deals with cohomology. It should be noted that the Ext between polynomial representations look very different if they are computed in Pol_n or in $kGL_n(k)-\mathrm{Mod}$. This difference can already be seen on the behavior of Frobenius twists. Let $M^{[r]}$ denote the restriction of a polynomial representation M along the group morphism $F^r : GL_n(k)\to GL_n(k)$, such that $F^r([g_{st}]) = [g_{st}^{p^r}]$. Since F^r is an automorphism of groups, the extensions $\mathrm{Ext}^*_{kGL_n(k)}(M,N)$ are isomorphic to $\mathrm{Ext}^*_{kGL_n(k)}(M^{[r]},N^{[r]})$. On the other hand, there is no inverse for restriction along F^r in the realm of polynomial representations. In fact it is known that the induced map

$$\mathrm{Ext}^*_{\mathrm{Pol}_n}(M,N) \to \mathrm{Ext}^*_{\mathrm{Pol}_n}(M^{[r]},N^{[r]})$$

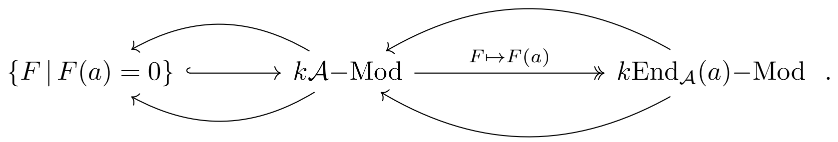
is always injective but not an isomorphism in general. The next result is a version for $GL_n(k)$ of the celebrated theorem of Cline Parshall Scott and van der Kallen [2] on the cohomology of the finite groups of Lie type (see also [1]).

Theorem 2. *Let M and N be two polynomial representations of degree d. If n is big enough with respect to d and i, there is an isomorphism:*

$$\mathrm{Ext}^i_{kGL_n(k)}(M,N) \to \underset{r}{colim}\, \mathrm{Ext}^i_{\mathrm{Pol}_n}(M^{[r]},N^{[r]}) \;.$$

We obtain theorems 1 and 2, as well as some other results in the same spirit, from the study of representations of a small additive category $\mathcal{A}$. We now explain how these two topics are connected.

Let $k\mathcal{A}-\mathrm{Mod}$ denote the abelian category of (not necessarily additive) functors $F : \mathcal{A} \to k-\mathrm{Mod}$ and natural transformations. This category is related in two ways with representations of groups. Firstly, for all objects a of $\mathcal{A}$, the value of a functor on a is naturally endowed with an action of the monoid $(\mathrm{End}_{\mathcal{A}}(a), \circ)$, and we have a recollement of abelian categories:



The theory of recollements of abelian categories asserts that evaluation on a yields a bijection between the isomorphism classes of simple functors F such that $F(a) \neq 0$ and those of the simple representations of the monoid $(\mathrm{End}_{\mathcal{A}}(a), \circ)$. Under some favorable circumstances, the simple representations of the group $(\mathrm{Aut}_{\mathcal{A}}(a), \circ)$ identify with a subset of those of $(\mathrm{End}_{\mathcal{A}}(a), \circ)$, hence we get a connection between the simple objects of $k\mathcal{A}-\mathrm{Mod}$ and the simple $k\mathrm{Aut}_{\mathcal{A}}(a)$-modules.

A second connection holds when $\mathcal{A} = \mathbf{P}_R$, the category of finitely generated projective modules over a ring R. In this case, evaluation on R^n turns functors into representations of $GL_n(R)$, and it was proved by Scorichenko [10], see also [3], that there is an isomorphism for all $i \geq 0$:

$$\mathrm{Tor}_i^{k\mathbf{P}_R}(F,G) \otimes \underset{n}{colim}\, H_i(GL_n(R),k) \simeq \underset{n}{colim}\, \mathrm{Tor}_i^{kGL_n(R)}(F(R^n),G(R^n)) \;.$$

provided F and G are polynomial functors in the sense of Eilenberg and Mac Lane [6]. If the ring R is nice, homological stabilization occurs and the colimits on both sides of the isomorphism are isomorphic to the homology of $GL_n(R)$ for n big enough with respect to i. Then the isomorphism above provides a connection between the Tor-computations in $k\mathbf{P}_R-\mathrm{Mod}$ and the homology of $GL_n(R)$ (and by dualizing we get a similar connection between Ext-computations in $k\mathbf{P}_R-\mathrm{Mod}$ and the cohomology of $GL_n(R)$).

Theorem 1 then follows from the connection between simple $kGL_n(R)$-modules and simple objects of $k\mathbf{P}_R-\mathrm{Mod}$ explained above, together with the following:

Theorem 3. [5] *Let $F : \mathcal{A} \to k-\mathrm{Mod}$ be a polynomial functor with finite dimensional values. Then F is a simple if and only if there are pairwise non isomorphic simple additive functors $A_i : \mathcal{A} \to k-\mathrm{mod}$, some simple $k\mathfrak{S}_{d_i}$-modules S_i and an isomorphism (the functors $E_{S_i} : k-\mathrm{mod} \to k-\mathrm{mod}$ are defined by (1))*

$$F \simeq (E_{S_1} \circ A_1) \otimes \cdots \otimes (E_{S_s} \circ A_s)\ .$$

Similarly, theorem 2 follows from Scorichenko's isomorphism and a generalization of the strong comparison theorem of [7] that we now explain. Polynomial functors of interest are provided by the compositions $F \circ A$, where $A : \mathcal{A} \to k-\mathrm{mod}$ is an additive functor and $F : k-\mathrm{mod} \to k-\mathrm{mod}$ is a strict polynomial functor in the sense of Friedlander and Suslin [8], such as a symmetric power S^d, an exterior power Λ^d, a functor E_S, a Frobenius twist functor $I^{(r)}$... If F is a d-homogeneous strict polynomial functor then $F(k^n)$ is a module over the Schur algebra $S(n, d)$, and Ext and Tor in the category of strict polynomial functors can be computed in terms of Ext and Tor over Schur algebras.

Theorem 4. [4] *Let $\mathcal{A}$ be an additive $\mathbb{F}_q$-linear category, let A and B be two $\mathbb{F}_q$-linear functors from $\mathcal{A}$ to $k-\mathrm{mod}$, respectively contravariant and covariant, and let F and G be two strict polynomial functors of degree $d < q$. Assume furthermore that the Hochschild-Mitchell homology $HH_i(\mathcal{A}, A \otimes B)$ vanishes for $0 < i < e$. Then for $0 \le i < e$ there is a k-linear isomorphism*

$$\mathrm{Tor}_i^{k\mathcal{A}}(F \circ A, G \circ B) \simeq \underset{r}{colim}\, \mathrm{Tor}_i(F \circ H \circ I^{(r)}, G \circ I^{(r)})$$

where H refers to the contravariant functor $H = \mathrm{Hom}_k(-, HH_0(A \otimes B))$, and the Tor *on the right hand side are taken in the category of strict polynomial functors.*

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On the cohomology of pro-fusion systems

Nadia Mazza
(joint work with Antonio Díaz Ramos, Oihana Garaialde Ocaña, Sejong Park)

Throughout, let p denote a prime number. Fusion systems for finite groups and compact Lie groups have been defined as algebraic models for their p-completed classifying spaces, see [1]. Recently, fusion systems have been defined over pro-p groups and are called *pro-fusion systems*, see [5] (and also [3, 4, 6] for the background on profinite groups). Loosely, pro-fusion systems on pro-p groups generalise the fusion systems on finite p-groups. Recall that a fusion system on a finite p-group S is a category whose objects are the subgroups of S and the morphisms are injective group homomorphisms subject to certain axioms. A morphism of fusion systems $(f_{ij}, F_{ij})\colon \mathcal{F}_j \to \mathcal{F}_i$ consists of a group homomorphism $f_{ij}\colon S_j \to S_i$ and a functor $F_{ij}\colon \mathcal{F}_j \to \mathcal{F}_i$ such that $F_{ij}(P) = f_{ij}(P)$ for each subgroup P of S_j, and, for each morphism $\varphi\colon P \to Q$ in $\mathcal{F}_j$ we have a commutative diagram (left hand side column in S_j and right hand side in S_i)

$$\begin{array}{ccc} P & \xrightarrow{f_{ij}} & f_{ij}(P) \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle F_{ij}(\varphi)} \\ Q & \xrightarrow{f_{ij}} & f_{ij}(Q). \end{array}$$

Elaborating on this, we can define inverse systems of fusion systems on finite p-groups, and therefore pro-fusion systems. We call $\mathcal{F}$ pro-saturated if $\mathcal{F}$ is the inverse limit of saturated fusion systems. (Note that pro-saturated and saturated pro-fusion systems are distinct notions, see [5]). If $\mathcal{F}$ is a pro-fusion system on a pro-p group S, we let $\mathcal{F}^\circ$ denote the full subcategory whose objects are the open subgroups of S. We say that $\mathcal{F}$ (resp. $\mathcal{F}^\circ$) is finitely generated if there exists a finite set of morphisms X of $\mathcal{F}$ (resp. $\mathcal{F}^\circ$) such that every morphism in $\mathcal{F}$ is the composition of finitely many restrictions of morphisms in $X \cup \mathrm{Inn}(S)$.

To compute the mod-p cohomology rings of finite groups and compact Lie groups two results stand out, namely, the Cartan-Eilenberg stable elements theorem [2, XII.Theorem 10.1], and the Lyndon-Hochschild-Serre spectral sequence. In the present work, we study the corresponding tools for the continuous mod-p cohomology ring $H^*_c(\cdot;\mathbb{F}_p)$ of pro-fusion systems, where the coefficients are the trivial module $\mathbb{F}_p$. The cohomology ring of a (pro-)fusion system $\mathcal{F}$ on a (pro-)p group S is the subring $H^*_c(\mathcal{F};\mathbb{F}_p) := H^*_c(S;\mathbb{F}_p)^{\mathcal{F}}$ formed by the $\mathcal{F}$-stable elements in $H^*_c(S;\mathbb{F}_p)$. Our first main result is as follows.

Theorem 1 (Stable Elements Theorem for Pro-Fusion Systems)**.** *Let $\mathcal{F}$ be a pro-fusion system on a pro-p group S, where $\mathcal{F} = \varprojlim_{i\in I} \mathcal{F}_i$ and $S = \varprojlim_{i\in I} S_i$. Assume* *that either $\mathcal{F}$ is pro-saturated, or that $\mathcal{F}^\circ$ is finitely generated. Then there is a ring isomorphism*

$$H^*_c(S;\mathbb{F}_p)^{\mathcal{F}} \cong H^*_c(S;\mathbb{F}_p)^{\mathcal{F}^\circ} \cong \varinjlim_{i\in I} H^*(S_i;\mathbb{F}_p)^{\mathcal{F}_i}.$$

Equivalently, $H^*_c(\mathcal{F};\mathbb{F}_p) \cong H^*_c(\mathcal{F}^\circ;\mathbb{F}_p) \cong \varinjlim_{i\in I} H^*(\mathcal{F}_i;\mathbb{F}_p)$.

Let $\mathcal{F}_S(G)$ denote the pro-fusion system defined by the conjugation action of a profinite group G on a Sylow p-subgroup S. Then $\mathcal{F}_S(G)$ is pro-saturated and saturated. Moreover, $\mathcal{F}_S(G)$ is finitely generated if S is open in G; which happens in particular if G is a compact p-adic analytic group (for example if $G = \mathrm{GL}_n(\mathbb{Z}_p)$, see [3]). As a corollary, we obtain:

Corollary 2 (Stable Elements Theorem for Profinite Groups)**.** *Let G be a profinite group. Then, there is a ring isomorphism*

$$H^*_c(G;\mathbb{F}_p) \cong H^*_c(S;\mathbb{F}_p)^{\mathcal{F}_S(G)} \cong H^*_c(S;\mathbb{F}_p)^{\mathcal{F}_S(G)^\circ}.$$

Our second main result provides a version of a Lyndon-Hochschild-Serre spectral sequence that can be used to compute the continuous mod-p cohomology of pro-fusion systems. Recall that a subgroup $T \leq S$ is strongly $\mathcal{F}$-closed (in S) if $\varphi(Q) \leq T$ for all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,S)$ and for all $Q \leq T$.

Theorem 3. *Let $\mathcal{F}$ be a pro-saturated pro-fusion system on a pro-p group S and let $T \leq S$ be a strongly $\mathcal{F}$-closed subgroup. Then there exists a first quadrant cohomological spectral sequence with second page*

$$E_2^{n,m} = H^n_c(S/T; H^m_c(T))^{\mathcal{F}},$$

and which converges to $H^*_c(\mathcal{F};\mathbb{F}_p)$.

As an application, we compute the cohomology ring $H^*_c(G;\mathbb{F}_3)$, where $G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}/3$ is the 3-adic version of the finite extraspecial group of order 3^3 and exponent 3.

Theorem 4. *The continuous mod-3 cohomology of the group $G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}/3$ is*

$$H^*_c(G;\mathbb{F}_3) \cong \mathbb{F}_3[x'] \otimes \Lambda(y,y',Y,Y')/\{yy', yY, y'Y', YY', yY' - y'Y\},$$

with degrees $|y| = |y'| = 1$, $|Y| = |Y'| = |x'| = 2$.

Finally we determine the cohomology rings of the general linear groups of dimension 2 over the p-adic integers, for p odd.

Theorem 5. *We have:*

(a) *For* $p = 3$, $H^*_c(\mathrm{GL}_2(\mathbb{Z}_3);\mathbb{F}_3) \cong \mathbb{F}_3[X] \otimes \Lambda(Z_1,Z_2,Z_3)$, *with degrees* $|Z_1| = 1$, $|Z_2| = |Z_3| = 3$ *and* $|X| = 4$.

(b) *For* $p > 3$, $H^*_c(\mathrm{GL}_2(\mathbb{Z}_p);\mathbb{F}_p) \cong \Lambda(Z_1,Z_2)$ *with degrees* $|Z_1| = 1$, $|Z_2| = 3$.

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Torsion-free endotrivial modules

Jesper Grodal

(joint work with Tobias Barthel, Joshua Hunt)

The goal of my Oberwolfach talk was to explain joint work with Tobias Barthel and Joshua Hunt [BGH], in which we obtain generators for the torsion-free part of the group of endotrivial modules. Combined with my earlier work [Gro16], which described the torsion part, we are able to provide a computable model for the whole group of endotrivial modules for an arbitrary finite group. The result extends the celebrated classification for finite p–groups, due to Carlson–Thévenaz. An important new ingredient in our joint work is a systematic use of methods from higher algebra, extending the homotopy methods from my earlier work. In this extended abstract I'll explain this story in more detail, as outlined in my talk.

Let G be a finite group and k any field of characteristic p, where $p||G|$. We're interested in calculating the group of *endotrivial* modules i.e.,

$$T_k(G) = \{ \text{ iso. classes of indecomposable } kG\text{-modules } M \mid M \otimes_k M^* \cong k \oplus (\text{proj})\}$$

which, in more high-flying language, identifies with the Picard group of the stable module category StMod_{kG}. This is an abelian group under $\otimes_k$ (and discarding projectives) with unit k and inverse to M given by the dual $M^* = \mathrm{Hom}(M, k)$. It plays an important role in modular representation theory.

The group $T_k(G)$ was shown in the 1980's by Puig to be finitely generated, and hence begs the question: Which group is it? Obvious elements include 1–dimensional modules and $\Omega k = \ker(\mathrm{proj} \twoheadrightarrow k)$, the shift of the trivial module (which has inverse $\Omega^{-1}k$). These in fact often generate $T_k(G)$, but not always. Here are two small examples where they do to keep in mind:

$$T_{\mathbb{F}_7}(C_7 \times C_3) = \langle \Omega\mathbb{F}_7\rangle \times \mathrm{Hom}(C_3, \mathbb{F}_7) \cong \mathbb{Z}/2 \times \mathbb{Z}/3 \cong \mathbb{Z}/6$$

$$T_{\mathbb{F}_7}(C_7 \rtimes C_3) = \langle \Omega\mathbb{F}_7\rangle \cong \mathbb{Z}/6$$

In fundamental work from 1978 Dade showed that for A an abelian p-group, Ωk is all there is:

Theorem 1 ([Dad78a, Dad78b]). *For any finite abelian p–group A,*

$$T_k(A) \cong \langle \Omega k \rangle \cong \begin{cases} 0 & \text{if } |A| \leq 2 \\ \mathbb{Z}/2 & \text{if } A \text{ is cyclic } |A| > 2 \\ \mathbb{Z} & \text{else} \end{cases}$$

The order of Ωk encodes that k has a periodic resolution iff A is cyclic, where the periodicity is 1 or 2 as indicated. The first theorem of Carlson–Thévenaz classifies torsion in $T_k(S)$, for S a p–group:

Theorem 2 ([CT05]). *For S a finite p–group, $T_k(S)$ has torsion only if S is cyclic, generalized quaternion Q_{2^n}, $n \geq 3$, or semi-dihedral $SD_{2^n}, n \geq 4$ (the two last examples for $p = 2$).*

It has been known since the 1970s that $T_k(Q_{2^n}) \cong \mathbb{Z}/4 \oplus A$, where A is either $\mathbb{Z}/2$ or 0, depending on whether or not k has a primitive 3rd root of unity, and $T_k(SD_{2^n}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$.

The torsion in $T_k(G)$ for general finite G hence equals $T_k(G,S) = \ker(T_k(G) \xrightarrow{\text{res}} T_k(S))$, except in the few cases when $T_k(S)$ has torsion. In [Gro16], I described $T_k(G,S)$ in terms of p–local group theory, using methods from homotopy theory, and developed a number of methods to calculate it:

Theorem 3 ([Gro16]). $T_k(G,S) \xrightarrow{\cong} H^1(\mathcal{O}_p^*(G); k^\times) \cong \mathrm{Hom}(\pi_1(\mathcal{O}_p^*(G)), k^\times)$

$$\cong H^0_G(|\mathcal{C}|; H^1(-;k^\times)) \cong \lim_{[P_0 \leq \cdots \leq P_n] \in |\mathcal{C}|/G} \mathrm{Hom}(N_G(P_0 \leq \cdots \leq P_n), k^\times)$$

Here $\mathcal{O}_p^*(G)$ is the p–orbit category with objects G/P for P a non-trivial p–subgroup of G, and morphisms G–equivariant maps, and $\mathcal{C}$ is a collection of non-trivial p–subgroups such that the inclusion $\mathcal{C} \subseteq \mathcal{S}_p(G)$ induces a G–homotopy equivalence $|\mathcal{C}| \subseteq |\mathcal{S}_p(G)|$ on geometric realization. E.g., $\mathcal{C} = \mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, etc., where $\mathcal{A}_p(G)$ means non-trivial elementary abelian p–subgroups $(\mathbb{Z}/p)^r$, $r \geq 1$. The limit is taken over conjugacy classes of chains, ordered by inclusion, and H^*_G denotes Bredon equivariant cohomology.

The fundamental group of the p–orbit category $\pi_1(\mathcal{O}_p^*(G))$ turns out to be a finite p'–group, a quotient of $N_G(S)/S$ describable in terms of explicit generators and relations. Arbitrary representations of this group, not necessarily of dimension one, turn out to govern a slightly larger class of kG–modules. The last expression in Theorem 3 implies an affirmative answer to the so-called Carlson–Thévenaz conjecture, providing an algorithmic description of $T_k(G,S)$. The description also implies a number of structural results, and $T_k(G,S)$ has subsequently been calculated using these methods for all finite simple groups. The most complicated case, that of finite groups of Lie type, is established in a joint work with Carlson, Mazza, and Nakano [CGMN].

Now, for the torsion-free quotient of $T_k(G)$, Alperin establish around year 2000 a formula for the torsion-free rank of G (originally just for p–groups, but later generalized to all finite groups): If G does not contain $(\mathbb{Z}/p)^3$ as a subgroup the torsion-free rank of $T_k(G)$ equals the number of G–conjugacy classes of *maximal* rank 2 elementary abelian p–subgroups, and otherwise it is this number plus one. However it still left open what the generators were. For G a p–group, this problem was also solved by Carlson–Thévenaz in another fundamental paper.

Theorem 4 ([CT04])**.** *Let S be a finite p–group that is not cyclic, semidihedral, or generalized quaternion. Then*

$$T_k(S) \xrightarrow{\cong} \{\{n_{V_0}\} \in \lim_{[V_0 \le \cdots \le V_n] \in |\mathcal{A}_p(S)|/S} T_k(V_0) \mid \{n_{V_0}\} \text{ satisfies the “CT-conditions”}\}$$

via the restriction maps $M \mapsto M|_V \in T_k(V)$.

Here and later the assumption on S is just for simplicity. The “CT-conditions”, short for Carlson–Thévenaz-conditions, are some explicit natural restrictions, stemming from the structure of projective resolutions, which we won't repeat here. As $T_k(V) \cong \mathbb{Z}$ if V has rank at least two, and is of order 1 or 2 otherwise, the formula amounts to specifying an integer $n_V \in \mathbb{Z}$ for each component of $|\mathcal{A}_p(G)_{\geq 2}|/G$, subject to the aforementioned restrictions. This is called the *type function.*

It was speculated that the image of the restriction $T_k(G) \to T_k(S)$ might be obtained by simply replacing S–conjugacy by G–conjugacy in the formula above. The main theorem with Barthel and Hunt shows that this is *true* when $p = 2$, but is in general *false* when p is odd. We can however precisely determine the extent of the failure, which relates to how p'–elements act on p–elements, a propagation of the phenomenon we already saw in the $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$ and $\mathbb{Z}/7 \times \mathbb{Z}/3$ examples from the beginning. Anyone care to walk the Brauer tree? More precisely, we have to build into our model how the type function of Carlson–Thévenaz, as in Theorem 4, interacts with “orientations”, i.e., the 1-dimensional representations used to describe $T(G,S)$ in Theorem 3. They arise when one restricts a module M to rank 1 elementary abelian p–subgroups V, where the periodic resolutions, not just the elements in $T_k(V)$, have to match up. (In fact checking on ${}_pZ(S) \cong \mathbb{Z}/p$, the rank one subgroup of elements of order at most p in the center of S, suffices.) For a fixed kG–module M we need to consider the assignment

$$(*) \qquad \bar{V} = [V_0 \leq \cdots \leq V_n] \mapsto (n_{V_0}, \varphi_{\bar{V}})$$

where $M|_{V_0} \cong \Omega^{n_{V_0}} k$, and $\varphi_{\bar{V}} \in \mathrm{Hom}(N_G(\bar{V}), k^\times)$ is obtained by considering the action of $N_G(\bar{V}) = N_G(V_0) \cap \cdots \cap N_G(V_n)$ on the 1-dimensional Tate cohomology group $\hat{H}^{n_V}(V; M)$. The second factor captures the p'–part of the residual $N_G(\bar{V})$–action on $\Omega^{n_{V_0}} k$. When $\mathrm{rk}_p(V_0) \geq 2$ the integer n_{V_0} is uniquely defined, whereas for $\mathrm{rk}_p(V_0) = 1$ there is an ambiguity, due to the periodicity of the resolution of $M|_{V_0}$. Furthermore, when p is odd, changing n_{V_0} also changes the $N_G(\bar{V})$–action, as we already saw for $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$, and we have to adjust for this dependency. However, this is all that is needed, and our main classification theorem reads as follows:

Theorem 5 ([BGH])**.** *Let G be a finite group with Sylow p–subgroup S, which is not cyclic, semi-dihedral, or quaternionic. Then*

$$T_k(G) \xrightarrow{\cong} \{\{(n_{V_0}, \varphi_{\bar{V}})\} \in \lim_{\bar{V} \in |\mathcal{A}_p(G)|/G} F(\bar{V}) \mid \{n_{V_0}\} \textit{ satisfies the "CT-conditions"}\}$$

$$\textit{via } (*), \qquad \textit{with} \qquad F(\bar{V}) = (\mathbb{Z} \times \mathrm{Hom}(N_G(\bar{V}), k^\times))/R(V_0)\ ,$$

$$\textit{and} \qquad R(V_0) = \begin{cases} 0 & \textit{for } \mathrm{rk}_p(V_0) \geq 2 \\ \langle (m_p, \nu_{V_0}) \rangle & \textit{for } \mathrm{rk}_p(V_0) = 1 \end{cases}$$

Here ν_{V_0} is the 1–dimensional character on $N_G(\bar{V})$ obtained by restriction of $N_G(V_0) \to \mathrm{Aut}(V_0) \cong \mathbb{F}_p^\times \to k^\times$, and $m_p = 1$ if $p = 2$ and 2 otherwise.

Note that the Carlson–Thévenaz conditions are only conditions on S. Since the extra compatibility condition is trivial for $p = 2$ we get that, for S as above,

$$T_k(G) \cong \{\{n_V\} \in \prod_{V \in \pi_0(|\mathcal{A}_p(G)_{\geq 2}|/G)} \mathbb{Z} \mid \{n_V\} \text{ satisfies the "CT-conditions" }\} \times T_k(G,S).$$

Contrary to the expectation before our work, the image of

$$T_k(G) \to \{\{n_V\} \in \prod_{V \in \pi_0(|\mathcal{A}_p(G)_{\geq 2}|/G)} \mathbb{Z} \mid \{n_V\} \text{ satisfies the "CT-conditions"}\}$$

may be a proper subgroup when p is odd. E.g., for $G = \mathfrak{S}_{p^2}$, p odd, it has index $p-1$ and for $G = \mathrm{PSL}_3(\mathbb{F}_p)$, $3|p-1$, it has index 3. The cokernel will however always be annihilated by $p-1$ as one sees from our model. Furthermore $T_k(\mathrm{PSL}_3(\mathbb{F}_p)) \hookrightarrow T_k(\mathrm{SL}_3(\mathbb{F}_p))$ has index 3 when $3|p-1$, with both groups torsion-free, despite that $\mathrm{SL}_3(\mathbb{F}_p) \to \mathrm{PSL}_3(\mathbb{F}_p)$ induces an isomorphism on p–fusion; this also implies that torsion-free generators for $T_k(\mathrm{PSL}_3(\mathbb{F}_p))$ cannot be chosen to lie in the principal block. These statements provide counterexamples to conjectures of Carlson–Mazza–Thévenaz.

The way we prove Theorem 5 is by extending the identification of my earlier theorem, Theorem 3, "to the right"; we keep the notation from that theorem.

Theorem 6 ([BGH])**.** *There are exact sequences*

$$0 \to H^1(\mathcal{O}_p^*(G); k^\times) \to T_k(G) \xrightarrow{\mathrm{res}} \lim_{\mathcal{O}_p^*(G)^{\mathrm{op}}} T_k(-) \xrightarrow{\alpha} H^2(\mathcal{O}_p^*(G); k^\times)$$

and

$$0 \to H^0_G(|\mathcal{C}|; H^1(-; k^\times)) \to T_k(G) \xrightarrow{\mathrm{res}} \lim_{\mathcal{O}_p^*(G)^{\mathrm{op}}} T_k(-) \xrightarrow{\beta} H^1_G(|\mathcal{C}|; H^1(-; k^\times)).$$

The maps α and β can be described explicitly, and provide the additional relations of Theorem 5. The key ingredient in proving these theorems is systematic use of higher algebra. In particular, a theorem of Mathew provides a decomposition of StMod_{kG} as an ∞–category, in terms of StMod_{kP} for p–subgroups P. This allows us to get an obstruction theory whose obstructions we then identify and calculate explicitly, leading to Theorems 6 and 5.

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On the space of commuting pairs in a Lie group

Simon Gritschacher
(joint work with Alejandro Adem and José Manuel Gómez)

Let G be a simply–connected and simple compact Lie group. The space of ordered n-tuples of commuting elements in G,

$$\{(g_1, \dots, g_n) \in G^n \mid g_i g_j = g_j g_i\} \subseteq G^n ,$$

is naturally identified with the space $\mathrm{Hom}(\mathbb{Z}^n, G)$ of group homomorphisms $\mathbb{Z}^n \to G$. It has the structure of a real algebraic variety, usually with complicated singularities. The study of varieties of commuting elements in algebraic groups has a long history [6]. The interest in the topology of spaces of commuting elements with a view towards their higher homotopical invariants is more recent [1].

In the talk I presented results, obtained in joint work with Alejandro Adem and José Manuel Gómez, concerning the second homotopy group of the space of commuting pairs $\mathrm{Hom}(\mathbb{Z}^2, G)$. The topology of $\mathrm{Hom}(\mathbb{Z}^2, G)$ is intricate, and little is known about the homology even in simple cases such as $SU(3)$. The representation space $\mathrm{Rep}(\mathbb{Z}^2, G) := \mathrm{Hom}(\mathbb{Z}^2, G)/G$, obtained by factoring out the adjoint action of G, describes the coarse moduli space of semistable $G_{\mathbb{C}}$-bundles over an elliptic curve of genus one, which is known to have the structure of a weighted projective space $\mathbb{CP}(n_0^\vee, \dots, n_r^\vee)$ [4]. The weights are the coroot integers of G, that is, the coefficients in the expansion of the coroot dual to the highest root in terms of the simple coroots.

The main result presented in the talk was

Theorem. *Let G be a simply–connected simple compact Lie group of rank r. Then*

$$\pi_2(\mathrm{Hom}(\mathbb{Z}^2, G)) \cong \mathbb{Z} \text{ and } \pi_2(\mathrm{Rep}(\mathbb{Z}^2, G)) \cong \mathbb{Z} ,$$

and on these groups the quotient map

$$\mathrm{Hom}(\mathbb{Z}^2, G) \to \mathrm{Rep}(\mathbb{Z}^2, G)$$

induces multiplication by the Dynkin index $D = \mathrm{lcm}(n_0^\vee, \dots, n_r^\vee)$.

The Dynkin index is 1 for A_n $(n \geq 1)$ and C_n $(n \geq 2)$, 2 for B_n $(n \geq 3)$, D_n $(n \geq 4)$ and G_2, 6 for E_6 and F_4, 12 for E_7, and 60 for E_8.

It is not too difficult to see – by what has been known already – that the group $\pi_2(\mathrm{Hom}(\mathbb{Z}^2, G))$ has rank one; the difficult part is to prove the rather surprising fact that this group is torsionfree. The result should also be compared with the fundamental fact from Lie group theory that $\pi_2(G) = 0$ and $\pi_3(G) \cong \mathbb{Z}$. The latter group is represented by the homomorphism $S^3 \cong SU(2) \to G$ corresponding to the highest root of G. We show that the same homomorphism induces an isomorphism $\pi_2(\mathrm{Hom}(\mathbb{Z}^2, SU(2))) \cong \pi_2(\mathrm{Hom}(\mathbb{Z}^2, G))$.

By the results of [5], and by standard arguments with the universal cover, our calculations apply to a wider class of groups.

Corollary. *Let G be a connected real or complex reductive algebraic group. Then*

$$\pi_2(\mathrm{Hom}(\mathbb{Z}^2, G)) \cong \mathbb{Z}^s ,$$

where s is the number of simple factors in $\mathfrak{g}$.

The theorem applies to the study of families of flat bundles over the 2-torus. A homomorphism $\phi\colon \mathbb{Z}^2 \to G$ is associated with a continuous map $B\phi\colon B\mathbb{Z}^2 \to BG$ which classifies a flat principal G-bundle over the 2-torus $B\mathbb{Z}^2$ with holonomy ϕ. (This bundle is necessarily trivial, since BG is 3-connected). The assignment $\phi \mapsto B\phi$ defines a continuous map from $\mathrm{Hom}(\mathbb{Z}^2, G)$ to the based mapping space $\mathrm{map}_*(B\mathbb{Z}^2, BG)$. By reduction to the case $G = SU(2)$ we obtain

Corollary. *Let G be a simply–connected simple compact Lie group. Then the map*

$$\mathrm{Hom}(\mathbb{Z}^2, G) \to \mathrm{map}_*(B\mathbb{Z}^2, BG)$$

induces an isomorphism on π_i for all $i \leq 2$.

Loosely speaking, the corollary implies that every principal G-bundle over $S^2 \times (S^1)^2$ is induced by an S^2-family of holonomies $\mathbb{Z}^2 \to G$, unique up to homotopy.

The proof of the theorem is based on the observation that

$$\pi_2(\mathrm{Hom}(\mathbb{Z}^2, G)) \cong H_2(EG \times_G \mathrm{Hom}(\mathbb{Z}^2, G); \mathbb{Z}) .$$

On the right is the Borel homology of the G-space $\mathrm{Hom}(\mathbb{Z}^2, G)$ with the adjoint action, which can be calculated via the Atiyah-Hirzebruch spectral sequence

$$E^2_{s,t} = H^G_s(\mathrm{Hom}(\mathbb{Z}^2, G); \mathcal{H}_t) \Longrightarrow H_{s+t}(EG \times_G \mathrm{Hom}(\mathbb{Z}^2, G); \mathbb{Z}) .$$

On the E^2-page we have the Bredon homology of $\mathrm{Hom}(\mathbb{Z}^2, G)$, where $\mathcal{H}_t$ means the coefficient system $G/K \mapsto H_t(BK; \mathbb{Z})$.

The groups $E^2_{s,0}$ are the integral homology groups of $\mathrm{Rep}(\mathbb{Z}^2, G)$, which are the homology groups of a weighted projective space and hence known by [3]. A direct argument shows furthermore that $E^2_{0,2} = 0$.

The calculation of the groups $E^2_{s,1}$ requires a careful analysis of centralisers of commuting pairs in G along the lines of [2]. The key idea is to study, for each prime p, the space

$$X_G(p) := \{(x,y) \in \mathrm{Hom}(\mathbb{Z}^2, G) \mid \pi_0(Z(x,y)) \otimes \mathbb{Z}_{(p)} \neq 0\}\,.$$

We prove a decomposition

$$H^G_s(\mathrm{Hom}(\mathbb{Z}^2, G); \mathcal{H}_1) \cong \bigoplus_{p \in \mathcal{P}} H^G_s(X_G(p); \mathcal{H}_1 \otimes \mathbb{Z}_{(p)})\,,$$

where $\mathcal{P}$ is the set of torsion primes of G. We show that the restriction of the coefficient system $\mathcal{H}_1 \otimes \mathbb{Z}_{(p)}$ to $X_G(p)$ is constant at $\mathbb{Z}/p$ in all cases except when $p = 2$ and $G = E_7$ or $G = E_8$. In these cases the argument must be adapted, but otherwise the calculation is reduced to the calculation of the mod-p homology of $X_G(p)/G$. We then proceed to show that $X_G(p)/G$ is homotopy equivalent to a weighted projective space, of which the homology is known.

In the range considered, the analysis of the differentials in the spectral sequence is straightforward, and no non-trivial extensions occur.

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Weight conjectures for p-compact groups and spetses

Radha Kessar

(joint work with Gunter Malle, Jason Semeraro)

We describe some of the results of the paper [4] which reveal new connections between the local-global conjectures of modular representation theory of finite groups, the theory of fusion systems, the theory of ℓ-compact groups, and the representation theory of finite groups of Lie type. The local-global conjectures considered are the Alperin weight conjecture and Robinson's ordinary weight conjecture. Here, we will concentrate mainly on the Alperin weight conjecture, briefly touching upon the Robinson ordinary weight conjecture at the end.

If k is an algebraically closed field of characteristic p, G is a finite group, B_0 is the principal block of kG, and $\mathcal{F}_p(G)$ is the fusion system of G (on some Sylow

p-subgroup of G), then Alperin's Weight Conjecture (AWC) predicts the equality

$$|\ell(B_0)| = \mathbf{w}(\mathcal{F}_p(G))$$

where, for a saturated fusion system $\mathcal{F}$,

$$\mathbf{w}(\mathcal{F}) := \sum_{P \in \mathcal{F}^{cr}/\mathcal{F}} z(k\mathrm{Out}_{\mathcal{F}}(P)),$$

and the sum runs over $\mathcal{F}$-conjugacy class representatives of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups (see [2, IV, Prop. 5.46]). Here, for a finite group H, $z(kH)$ denotes the number of isomorphism classes of projective simple kH modules and $\ell(B_0)$ denotes the number of isomorphism classes of simple B_0-modules.

Let q be a prime power prime to p, $\mathbf{G}$ a connected reductive linear algebraic group and F a Frobenius morphism on $\mathbf{G}$ with respect to an $\mathbb{F}_q$-structure and let e be the order of q-modulo p. By results of Geck-Hiss, Broué-Malle-Michel and Cabanes-Enguehard (see [4, Prop. 4.1]), if p is very good for $\mathbf{G}$, and B_0 is the principal block of $k\mathbf{G}^F$, then $\ell(B_0)$ equals $|\mathrm{Irr}(W_e)|$, where W_e is a certain complex reflection subquotient of the Weyl group W of $\mathbf{G}^F$ described via Lehrer-Springer theory. Thus AWC for B_0 is equivalent to the assertion

$$\mathbf{w}(\mathcal{F}_p(\mathbf{G}^F)) = |\mathrm{Irr}(W_e)|\,. \tag{1}$$

Let X be a simply connected p-compact group with associated p-adic reflection group W and let τ be a self-equivalence of X whose class in the outer automorphism group of X is of p'-order. If p is odd, then Broto and Møller have shown that the space of homotopy fixed points under $\tau\psi^q$, where ψ^q is an unstable Adams operator, is the classifying space of an p-local finite group (which they call a p-local Chevalley group), and in particular the triple (X, q, τ) determines a saturated fusion system $\mathcal{F}({}^{\tau}X(q))$ on a finite p-group [3, Thm. A]. Moreover, the structure of the underlying p-group is controlled by a certain Springer-Leher subquotient W_e of the Weyl group W of X ([4, Theorem 3.6]). If W is rational and $\mathbf{G}^F$ is a group of Lie type associated to W as above, results of Friedlander–Mislin and Quillen imply that for suitable X and τ, $\mathcal{F}({}^{\tau}X(q))$ is the fusion system of $\mathbf{G}^F$ on a Sylow p-subgroup of $\mathbf{G}^F$ and the equality (1) for the group $\mathbf{G}^F$ is equivalent to the equality

$$\mathbf{w}(\mathcal{F}({}^{\tau}X(q))) = |\mathrm{Irr}(W_e)|.$$

If W is not rational, then in infinitely many cases, the fusion system $\mathcal{F}({}^{\tau}X(q))$ is exotic, that is, there is no finite group G such that $\mathcal{F}({}^{\tau}X(q)) = \mathcal{F}_p(G)$. Our main result is that the above equality extends to the non-rational case.

Theorem 1. [4, Theorem 1] *Let $p > 2$, X a simply connected ℓ-compact group, q a prime power prime to ℓ and τ an automorphism of X whose image in the outer automorphism group of X has finite order prime to p. If ℓ is very good for X, then*

$$\mathbf{w}(\mathcal{F}({}^{\tau}X(q))) = |\mathrm{Irr}(W_e)|,$$

where e is the order of q modulo ℓ.

The above result may be viewed as saying that a version of AWC holds for Chevalley p-local finite groups. As a byproduct, we recover the validity of AWC for the principal ℓ-blocks of simply connected groups of Lie type, in particular the previously unknown cases of types E_6, E_7 and E_8, for all very good primes $\ell > 2$. The proof of the Theorem is on a case by case basis, the bulk being devoted to the case of the generalised Grassmanians. This case is handled by developing an equivariant version of the Alperin–Fong proof of AWC for finite general linear groups [1], combined with a result of Ruiz [5] identifying the relevant fusion systems as subsystems of fusion systems of finite general groups.

In the second half of [4], combining the theory of spetses, Lusztig's Jordan decomposition of characters of finite groups of Lie type, and the structure of centralisers of p-elements in p-compact groups, we associate to the triple (X, τ, q) as above a finite set which plays the role of the set of ordinary character degrees in the principal block of a finite group of Lie type. We formulate (and prove some cases of) a version of Robinson's ordinary weight conjecture in this context.

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Simplicity of fusion systems of finite simple groups

Bob Oliver

(joint work with Albert Ruiz)

For a prime p and a finite group G, the *fusion system* of G over a Sylow p-subgroup S of G is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S, and whose morphisms are those homomorphisms between subgroups induced by conjugation by elements of G. The fusion system of G thus encodes the conjugacy relations among its p-subgroups and p-elements (its "p-local structure"). Motivated by connections with modular representation theory, Lluis Puig in the 1990s defined the concept of abstract fusion systems (first in unpublished notes and later published in [Pg]): a (saturated) fusion system $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, and whose morphisms are injective homomorphisms satisfying certain axioms motivated by properties of finite groups such as the Sylow theorems (see [AKO, Definitions I.2.1–2]).

Normal fusion subsystems and simple fusion systems are defined by analogy to those of finite groups. One special case of this is that where the fusion system of a subgroup $Q \trianglelefteq S$ is normal in a fusion system $\mathcal{F}$ over S: in this case, $\mathcal{F}_Q(Q) \trianglelefteq \mathcal{F}$ (usually written $Q \trianglelefteq \mathcal{F}$) if and only if each morphism φ in $\mathcal{F}$ extends to a morphism between subgroups containing Q and sending Q to itself.

The fusion system of a finite simple group G need not in general be simple. For example, if a Sylow p-subgroup $S \leq G$ is abelian, then by a theorem of Burnside [Bu, § 123], $\mathcal{F}_S(G) = \mathcal{F}_S(N_G(S))$, and hence $S \trianglelefteq \mathcal{F}_S(G)$ (i.e., $\mathcal{F}_S(S) \trianglelefteq \mathcal{F}_S(G)$).

Our main result, described below, was to determine for exactly which known finite simple groups G and which primes p the p-fusion system of G is simple. This question had been studied and answered in most cases by Aschbacher in Chapter 16 of his memoir [A2], but a few cases (all involving groups of Lie type in cross characteristic) were left open, and it was on those cases that we focused in our work. We also corrected two errors found among Aschbacher's conclusions. One problem of particular interest was that of for which G and p, the p-fusion system of G contains a normal subsystem of index prime to p that is exotic, and we were able to describe that situation quite precisely.

Our detailed group-by-group results are summarized in the following theorem, which depends on the classification of finite simple groups. In the theorem, by analogy with the notation used for finite groups, $O^{p'}(\mathcal{F})$ denotes the smallest normal fusion subsystem of index prime to p (see [AKO, Theorem I.7.7]). Also, a fusion system is *realizable* if it is isomorphic to the fusion system of some finite group, and is *exotic* otherwise.

Theorem A ([OR, Theorems A and 4.8])**.** *Fix a prime p and a known finite simple group G such that $p \mid |G|$. Let S be a Sylow p-subgroup of G, and set $\mathcal{F} = \mathcal{F}_S(G)$. Then one of the following holds: either*

(a) $S \trianglelefteq \mathcal{F}$*; or*
(b) $p = 3$ *and* $G \cong G_2(q)$ *for some* $q \equiv \pm 1$ *(mod 9), in which case* $|O_3(\mathcal{F})| = 3$*, and* $O^{3'}(\mathcal{F})$ *has index 2 in* $\mathcal{F}$ *and is realized by* $SL_3(q)$ *(if* $q \equiv 1$ *(mod 9)) or* $SU_3(q)$ *(if* $q \equiv -1$*); or*
(c) $p \geq 5$*,* G *is one of the classical groups* $PSL_n^{\pm}(q)$*,* $PSp_{2n}(q)$*,* $\varOmega_{2n+1}(q)$*, or* $P\varOmega_{2n+2}^{\pm}(q)$ *where* $n \geq 2$ *and* $q \not\equiv 0, \pm 1$ *(mod p), in which case* $O^{p'}(\mathcal{F})$ *is simple and exotic; or*
(d) $O^{p'}(\mathcal{F})$ *is simple, and it is realized by a known finite simple group* G^**.*

Moreover, in case (c)*, there is a subsystem* $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ *of index at most 2 in* $\mathcal{F}$ *with the property that for each (saturated) fusion system* $\mathcal{E}$ *over* S *such that* $O^{p'}(\mathcal{E}) = O^{p'}(\mathcal{F})$*,* $\mathcal{E}$ *is realizable if and only if it contains* $\mathcal{F}_0$*. In case* (d)*, if* $p = 2$*, then* $\mathcal{F}$ *is always simple (i.e.,* $O^{p'}(\mathcal{F}) = \mathcal{F}$*).*

Note that case (d) includes all cases where the fusion system is simple. Note also that when $p = 2$, there are only two possibilities: either $S \trianglelefteq \mathcal{F}$ (case (a)), or $\mathcal{F}$ is simple (case (d)).

Here are a few examples of pairs (G,p) where these four cases occur:

(a) By Burnside's theorem, if S is abelian, then $S \trianglelefteq \mathcal{F}$, and we are in case (a). But we also have $S \trianglelefteq \mathcal{F}$ in a few other cases, including groups of Lie type in defining characteristic p and of Lie rank 1 (e.g., $(Sz(2^{2n+1}),2)$ and $(PSU_3(p^n),p)$), and a few sporadic groups such as J_3 when $p=3$.

(b) As stated in Theorem A, if $p=3$ and $G \cong G_2(q)$ for $q \equiv \pm 1 \pmod 9$, then $\mathcal{F}$ contains a normal subsystem of index 2 (the fusion system of $SL_3(q)$ or $SU_3(q)$), and also contains a normal subsystem of order p. By comparison, if $G \cong G_2(q)$ for $q \equiv \pm 2, \pm 4 \pmod 9$, then $S \trianglelefteq \mathcal{F}$.

(c) The smallest example where case (c) occurs is the group $G = SL_{20}(2)$ for $p=5$. Here, G contains the group $\Gamma L_5(2^4)$ (the extension of $GL_5(\mathbb{F}_{16})$ by its field automorphisms), and $\mathcal{F}$ contains an exotic normal subsystem of index 4 containing the fusion system of $GL_5(2^4)$ (first shown by Ruiz [Ru]).

(d) When G is an alternating group A_n, and $n \geq 8$ (if $p=2$) or $n \geq p^2$ (if p is odd), then $\mathcal{F} = \mathcal{F}_S(G)$ is simple whenever $n \equiv 0,1 \pmod p$ (in particular, whenever $p=2$). If $n \equiv k \pmod p$ where $2 \leq k < p$, then $\mathcal{F}$ is isomorphic to the fusion system of Σ_n and also that of Σ_{n-k}, and contains the fusion system of A_{n-k} as a normal subsystem of index 2.

When G is a sporadic group and $S \ntrianglelefteq \mathcal{F}$, then $\mathcal{F}$ is simple in most cases. But there are a few pairs (G,p), such as $(M_{24},3)$ or $(He,3)$, for which $\mathcal{F}$ is almost simple, and contains a normal simple realizable subsystem of index 2 (in these two cases, $O^{p'}(\mathcal{F})$ is the fusion system of M_{12}).

Theorem A, and other results in our paper [OR], were originally motivated by our work (still being written up) with Carles Broto and Jesper Møller, proving tameness of all realizable fusion systems (see [AKO, § III.6.1] for more on tameness, including the definition). While trying to determine automorphism groups of certain realizable fusion systems, we found that it is first necessary to understand more precisely the normal fusion subsystems of fusion systems of simple groups. Independently of that, some of the results in [OR] were used in recent work of Radha Kessar, Gunter Malle, and Jason Semeraro [KMS] to calculate weights (in the context of the Alperin weight conjecture in modular representation theory) attached to exotic fusions arising from homotopy fixed points of p-compact groups.

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Noncommutative Tensor Triangular Geometry

Daniel K. Nakano
(joint work with Kent B. Vashaw, Milen T. Yakimov)

Given a symmetric monoidal tensor triangulated category, Balmer [1] initiated the study of tensor triangular geometry by viewing the category like a commutative ring via its tensor structure. In his seminal work, he defined a categorical spectrum and related this notion to the important concept of support datum. Recently, the authors [4] developed a general noncommutative version of Balmer's theory that deals with an arbitrary monoidal triangulated category **K** (MΔC for short). Noncommutative versions of Balmer's theory were sought after before, however, for various reasons, a general noncommutative version of tensor triangular geometry had not been fully realized. There are many important MΔCs, and one motivating example is the stable module categories of finite dimensional Hopf algebras (which are in general not cocommutative).

Prior attempts in the noncommutative setting focused on the fact that the conditions on the objects in **K** do not satisfy the usual axioms for commutative support data from [1]. These considerations attempted to mimic the treatment of completely prime ideals in a noncommutative ring. In general there are too few completely prime ideals so a new definition of prime ideal was introduced in terms of tensoring thick ideals of **K**, and not to use object-wise tensoring. With this new definition, completely prime ideals are prime ideals.

For arbitrary noncommutative rings, the prime spectrum of a ring is very hard to describe as a topological space (e.g., the spectra of universal enveloping algebras of Lie algebras and quantum groups). However, in the categorical setting, the authors [4], were successful in developing strategies for computing the Balmer spectrum Spc(**K**) that are as useful as the commutative counterparts. Furthermore, the set of right ideals of a noncommutative ring are rarely classifiable with the exceptions of very few rings. Surprisingly, in the categorical setting, we provide effective methods to classify the thick right ideals of an MΔC.

In this talk, I will show how to construct a general noncommutative version of Balmer's tensor triangular geometry that is applicable for arbitrary monoidal triangulated categories (MΔC). Insights from noncommutative ring theory is used to obtain a framework for prime, semiprime, and completely prime (thick) ideals of an MΔC, **K**, and then to associate to **K** a topological space–the Balmer spectrum Spc(**K**).

We develop a general framework for (noncommutative) support data, coming in three different flavors, and show that Spc(**K**) is a universal terminal object for the first two notions (support and weak support). The first two types of support data are then used in a theorem that gives a method for the explicit classification of the thick (two-sided) ideals and the Balmer spectrum of an MΔC. The third type (quasi support) is used in another theorem that provides a method for the explicit classification of the thick right ideals of **K**, which in turn can be applied to classify the thick two-sided ideals and Spc(**K**). Applications will be given for quantum groups and non-cocommutative finite-dimensional Hopf algebras studied by Benson and Witherspoon [2].

The problem of whether the cohomological support map of a finite dimensional Hopf algebra has the tensor product property has attracted a lot of attention following the earlier developments on representations of finite group schemes. Many authors have focussed on concrete situations where positive and negative results have been obtained by direct arguments.

At the end of the talk I will demonstrate that it is natural to study questions involving the tensor product property in the broader setting of a monoidal triangulated category. Using [5], we give an intrinsic characterization by proving that the tensor product property for the universal support datum is equivalent to complete primeness of the categorical spectrum. From these results one obtains information for other support data, including the cohomological one. Two theorems are proved giving complete primeness and non-complete primeness in certain general settings.

As an illustration of these methods, we solve a recent conjecture of Negron and Pevtsova [6] on the tensor product property for the cohomological support maps for the small quantum Borel algebras for all complex simple Lie algebras.

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Transfer in the homology and cohomology of categories

Peter Webb

Let $R\mathcal{C}$ denote the category algebra of the finite category $\mathcal{C}$ over the commutative ring R. The *cohomology ring* of $\mathcal{C}$ over R is $H^*(\mathcal{C}, R) := \mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, \underline{R}) \cong H^*(|\mathcal{C}|; R)$, where $\underline{R}$ is the constant functor on $\mathcal{C}$ and $|\mathcal{C}|$ is the nerve of $\mathcal{C}$. Similarly, the *homology* of $\mathcal{C}$ over R is $H_*(\mathcal{C}, R) := \mathrm{Tor}_*^{R\mathcal{C}}(\underline{R}, \underline{R}) \cong H_*(|\mathcal{C}|; R)$. The isomorphisms indicate both an algebraic and a topological definition of these groups, and are explained in [8].

These definitions extend the definition of group homology and cohomology to arbitrary small categories, although we are interested only in finite categories here. Because every finite CW-complex can be realized, up to homotopy, as the nerve of a finite category, the cohomology of finite categories includes the cohomology of such spaces, for instance, and there are many special cases of interest.

Group cohomology comes with several operations that relate the cohomology of a group to that of subgroups and quotients; namely restriction, corestriction or transfer, conjugation and inflation. We ask how similar operations can be formulated for the cohomology of categories. For some of these operations the answer is straightforward. If we are given a functor $F : \mathcal{C} \to \mathcal{D}$, the induced map $|F| : |\mathcal{C}| \to |\mathcal{D}|$ gives a contravariant map $F^* : H^*(|\mathcal{D}|) \to H^*(|\mathcal{C}|)$, and this construction gives maps that generalize restriction, inflation and conjugation. What about the transfer map? In what circumstances can this map be defined? What properties should it satisfy? For instance, for finite groups these maps satisfy relations that make cohomology a Mackey functor, and cohomology with trivial coefficients both a global Mackey functor and an inflation functor, in the terminology of [7].

The approach we take is to use the formalism of biset functors for categories, extending the theory developed for groups that is documented in [4]. Biset functors for categories were introduced in [9], and we summarize the construction. First, for groups, if G and H are finite groups, a (G,H)-*biset* is a set ${}_G\Omega_H$ with a left action of G and a right action of H, so that these actions commute. This is the same as a set with a left action of $G \times H^{\mathrm{op}}$ and, if we regard G and H as categories $\mathcal{G}$ and $\mathcal{H}$ with a single object and all morphisms invertible, it is the same as a functor $\mathcal{G} \times \mathcal{H}^{\mathrm{op}} \to \mathrm{Set}$. A *biset* for categories $\mathcal{C}, \mathcal{D}$ is thus defined to be a functor $\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathrm{Set}$.

It turns out that such bisets for categories have already appeared in the literature, and are called *distributors* or *profunctors*. They were introduced in [5] and [1], and an exposition is given in [2]. In particular, it is shown that there is a product operation on distributors, associative up to natural isomorphism, giving rise to a distributor bicategory where the objects are categories, the morphisms are distributors and the 2-morphisms are natural transformations of distributors.

For our purposes we change this definition in two important ways: we linearize, and we impose a relation. We define $A_R(\mathcal{C}, \mathcal{D})$ to be the free R-module with the finite $(\mathcal{C}, \mathcal{D})$-bisets as basis with the following relation imposed: $\Omega \sqcup \Psi = \Omega + \Psi$. Here the disjoint union $\Omega \sqcup \Psi$ of bisets Ω, Ψ is the biset whose value at the object (x, y) of $\mathcal{C} \times \mathcal{D}^{\mathrm{op}}$ is $\Omega(x, y) \sqcup \Psi(x, y)$. Now $A_R(\mathcal{C}, \mathcal{D})$ is the free R-module with basis the isomorphism types of $(\mathcal{C}, \mathcal{D})$-bisets that are indecomposable with respect to $\sqcup$. We define the *biset category* $\mathbb{B}$ over R to have finite categories as objects, and $\mathrm{Hom}_{\mathbb{B}}(\mathcal{D}, \mathcal{C}) = A_R(\mathcal{C}, \mathcal{D})$, with composition given by the product of bisets. A *biset functor* is an R-linear functor $M : \mathbb{B} \to R\text{-mod}$.

The category of biset functors for categories has a number of properties that extend those of biset functors for groups, but the aspect that interests us here is that of defining $H^*(\mathcal{C}, R)$ and $H_*(\mathcal{C}, R)$ as biset functors in a suitable sense.

We cannot define them as functors on $\mathbb{B}$, because in the special case of groups, group cohomology does not possess the deflation map enjoyed by biset functors in general. In the case of groups, we get round this by using only bisets for which the group action on one side is free. We now show how to extend this condition to categories.

Employing an abuse of terminology, we will say that a functor $\Theta : \mathcal{C} \to \mathrm{Set}$ is *representable* if $\Theta \cong \bigsqcup_i \mathrm{Hom}(x_i, -)$, for some objects $x_i \in \mathcal{C}$. Usually this term is reserved for the situation where there is only one object x_i. We say that a $(\mathcal{C}, \mathcal{D})$-biset Ω is *representable on the right* if $\bigsqcup_{x \in \mathcal{C}} \Omega(x, -)$ is representable, with a similar definition for *representable on the left.*

Theorem 1. *If ${}_{\mathcal{C}}\Omega_{\mathcal{D}}$ and ${}_{\mathcal{D}}\Psi_{\mathcal{E}}$ are bisets that are representable on the left (or right), then so is the biset product ${}_{\mathcal{C}}\Omega \circ \Psi_{\mathcal{E}}$.*

Let $\mathbb{B}^{1,\mathrm{all}}$ be the subcategory of $\mathbb{B}$ obtained by using only bisets that are representable on the left, and $\mathbb{B}^{\mathrm{all},1}$ be the subcategory using bisets free on the right. Our main theorem is the following. It is our solution to question of constructing a transfer map.

Theorem 2. *Let R be a field. Then $\mathcal{C} \mapsto H^*(\mathcal{C}, R)$ has the structure of a biset functor on the category $\mathbb{B}^{all,1}$, and $\mathcal{C} \mapsto H_*(\mathcal{C}, R)$ has the structure of a biset functor on $\mathbb{B}^{1,all}$.*

When the categories are groups our construction provides the usual notion of restriction, transfer and inflation as the functorial effect of cohomology on the bisets ${}_HG_G$, ${}_GG_H$, and ${}_GQ_Q$, where H is a subgroup of G and Q is a quotient of G. The proof of this result puts together a result on Hochschild homology and an analogue for homology of the result of Xu [10] that the cohomology of a category is naturally a summand of its Hochschild cohomology. The theorem on Hochschild cohomology quotes independent work of Bouc and Keller and can be expressed as follows.

Theorem 3 (Bouc [3], Keller [6])**.** *Hochschild homology $\mathcal{C} \mapsto HH_*(R\mathcal{C})$ has the structure of a functor on $\mathbb{B}^{1,all}$.*

In [10], Xu showed that there are canonical maps

$$H^*(\mathcal{C}, R) \to HH^*(R\mathcal{C}) \to H^*(\mathcal{C}, R)$$

with composite the identity, providing a canonical decomposition $HH^*(R\mathcal{C}) = H^*(\mathcal{C}, R) \oplus Y$ for some summand Y. We show that the same is true for homology. For groups this splitting is well known. For categories, the argument goes via use of the *factorization category* $F(\mathcal{C})$ of Quillen.

We combine Xu's decomposition with the construction of Bouc and Keller in Hochschild homology, getting a definition of homology $H_*(\mathcal{C}, R)$ as a biset functor. So far, the argument does not require R to be a field. When R is a field, $H_*(\mathcal{C}, R)$ and $H^*(\mathcal{C}, R)$ are the homology and cohomology of a space (the nerve), and so one is the dual of the other. Transporting the result for homology to its dual, we obtain a dependence of cohomology as a biset functor.

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Splittings of global Mackey functors and regularity of equivariant Euler classes

Stefan Schwede

In 1962, Dold [3] published an elegant proof of Nakaoka's splitting [8] of the cohomology of symmetric groups. Dold's proof only uses formal aspects of group cohomology; his argument is a lot more general and also provides a splitting of the values of global Mackey functors at symmetric groups. The relevant kind of global Mackey functor has been studied under different names, for example as *inflation functors* [14] or *global* $(\emptyset, \infty)$*-Mackey functors* [6]. These objects are special cases of *biset functors* [1], and they are equipped with restrictions, transfers and inflations (but possibly no deflations).

Dold's arguments from [3] also prove the following result.

Theorem. *For every inflation functor* F *and every* $n \geq 1$*, the restriction homomorphism*

$$\mathrm{res}^{\Sigma_n}_{\Sigma_{n-1}} \ : \ F(\Sigma_n) \ \to \ F(\Sigma_{n-1})$$

is a naturally split epimorphism.

I would not be surprised if this splitting were published somewhere in the algebraic literature on the subject; however, I am not aware of a reference. Dold's proof proceeds as follows. We write

$$F(\Sigma; k) \ = \ \ker(\mathrm{res}^{\Sigma_k}_{\Sigma_{k-1}} : F(\Sigma_k) \to F(\Sigma_{k-1}))$$

for the kernel of the restriction homomorphism. For $k = 0$, we interpret this as $F(\Sigma; 0) = F(e)$, the value of F at the trivial group; the group $F(\Sigma; 1)$ is actually trivial. For $0 \leq k \leq n$, we write

$$p^*_{k,n-k} \ : \ F(\Sigma_k) \ \to \ F(\Sigma_k \times \Sigma_{n-k})$$

for the inflation homomorphism associated to the projection to the first factor, and we write

$$\mathrm{tr}_{k,n-k} \ : \ F(\Sigma_k \times \Sigma_{n-k}) \ \to \ F(\Sigma_n)$$

for the transfer along the inclusion of the block subgroup $\Sigma_k \times \Sigma_{n-k}$ inside Σ_n. We define a natural homomorphism

$$\psi_{k,n} \ : \ F(\Sigma; k) \ \to \ F(\Sigma_n)$$

as the following composite

$$F(\Sigma; k) \xrightarrow{\text{inclusion}} F(\Sigma_k) \xrightarrow{p^*_{k,n-k}} F(\Sigma_k \times \Sigma_{n-k}) \xrightarrow{\mathrm{tr}_{k,n-k}} F(\Sigma_n) \ .$$

For example, $\psi_{0,n}$ is inflation along the unique homomorphism $\Sigma_n \to \Sigma_0$, and $\psi_{n,n}$ is the inclusion $F(\Sigma; n) \to F(\Sigma_n)$. The key observation is that these maps satisfy the relation

$$\mathrm{res}^{\Sigma_n}_{\Sigma_{n-1}} \circ \psi_{k,n} \ = \ \psi_{k,n-1} \ ,$$

an almost immediate consequence of the double coset formula for the subgroups Σ_{n-1} and $\Sigma_k \times \Sigma_{n-k}$ of Σ_n. Induction on n then shows that the map

$$\sum_{k=0}^{n} \psi_{k,n} \ : \ \bigoplus_{k=0}^{n} F(\Sigma; k) \ \to \ F(\Sigma_n)$$

is an isomorphism and restriction from Σ_n to Σ_{n-1} is a naturally split epimorphism.

Equivariant homotopy theory provides a more general kind of global Mackey functor with values at all compact Lie groups. My main result is an analog of Dold's splitting for the values of these global Mackey functors at orthogonal, unitary and symplectic groups. As a consequence of these splittings, certain long exact sequences of equivariant homotopy groups decompose into short exact sequences. This in turn implies that the Euler class of the tautological $U(n)$-representation in homotopical equivariant bordism is a non zero-divisor.

A *global functor* in the sense of [11, Definition 4.2.2] is an additive functor from the global Burnside category of [11, Construction 4.2.1] to the category of abelian groups. In more explicit terms, a global functor specifies values on all compact Lie groups, restriction homomorphisms along continuous group homomorphisms, and transfers along inclusions of closed subgroups; this data has to satisfy a short list of explicit relations that can be found after Theorem 4.2.6 of [11]. The data of a global functor is equivalent to that of a *functor with regular Mackey structure* in the sense of Symonds [12, §3, p.177]. When restricted to finite groups, we obtain an inflation functor. My main result is now as follows.

Theorem. For every global functor F, and every $n \geq 1$, the restriction homomorphisms

$$\begin{aligned} \mathrm{res}^{O(n)}_{O(n-1)} &: F(O(n)) \to F(O(n-1)) \ , \\ \mathrm{res}^{U(n)}_{U(n-1)} &: F(U(n)) \to F(U(n-1)) \qquad \text{and} \\ \mathrm{res}^{Sp(n)}_{Sp(n-1)} &: F(Sp(n)) \to F(Sp(n-1)) \end{aligned}$$

are naturally split epimorphism.

The families of alternating groups, special orthogonal groups and special unitary groups have the same kind of structure as the symmetric, orthogonal, unitary and symplectic groups; so one might wonder about the existence of splittings for the values of global functors at A_n, $SO(n)$ and $SU(n)$. However, the restriction homomorphisms between adjacent groups in these families do *not* split naturally, except in some low-dimensional cases and for half of the special orthogonal groups.

The strategy of proof of the splitting theorem is the same as in Dold's argument presented above; the proof of the key relation

$$\mathrm{res}^{O(n)}_{O(n-1)} \circ \psi_{k,n} \ = \ \psi_{k,n-1}$$

now involves an instance of the double coset formula for the subgroups $O(n-1)$ and $O(k) \times O(n-k)$ of $O(n)$, and here things become a little more subtle. Every global functor satisfies a generalization of the double coset formula in the context of compact Lie groups, see [5, IV §6] or [11, Theorem 3.4.9]. In this generality, the double coset space is typically not discrete, and the statement of the double coset formula is substantially more involved than for finite groups. The double coset space comes with a stratification by locally closed subspaces that are manifolds (typically not compact). The summands in the double coset formula are indexed by the path components of these *orbit type manifolds*, and they have integer coefficients given by internal Euler characteristics (i.e., Euler characteristics based on compactly supported cohomology).

In the case relevant to us, the double coset space $O(n-1)\backslash O(n)/O(k)\times O(n-k)$ is a closed interval with orbit type stratification as the two endpoints and the interior; the double coset formula thus has three terms, one of which has coefficient -1, the internal Euler characteristic of the open interval.

The splitting theorem for global functors has several applications to equivariant stable homotopy theory. Indeed, for every global equivariant spectrum X, i.e., an object of the global stable homotopy category [11, Section 4], and every integer m, the collection of m-th equivariant stable homotopy groups $\pi_m^G(X)$ naturally forms a global functor as G varies over all compact Lie groups. Moreover, the preferred t-structure on the global stable homotopy category shows that every global functor arises in this way, see [11, Theorem 4.4.9]. The splittings thus show that for every global equivariant spectrum X, the restriction homomorphism $\mathrm{res}^{O(n)}_{O(n-1)} : \pi_*^{O(n)}(X) \to \pi_*^{O(n-1)}(X)$ is a naturally split epimorphism. And the analogous statements hold for unitary and symplectic groups.

The splitting results imply regularity properties for certain equivariant Euler classes of the global Thom spectrum **MU** defined in [11, Example 6.1.53]. For every compact Lie group G, the underlying G-homotopy type of **MU** is that of tom Dieck's homotopical equivariant bordism [13]. These equivariant spectra have received a lot of attention; for example, for abelian compact Lie groups, the equivariant cohomology theory represented by **MU** is the universal complex-oriented equivariant cohomology theory [2, Theorem 1.2], and the equivariant homotopy groups of **MU** carry the universal global group law [4, Theorem C].

Since the global theory **MU** is complex-oriented, every unitary representation W of a compact Lie group G has an equivariant Euler class $e_{G,W} \in \mathbf{MU}_G^{2n}$, where $n = \dim_{\mathbb{C}}(W)$. The Euler class of the tautological $U(n)$-representation ν_n on $\mathbb{C}^n$ and the restriction homomorphism from $U(n)$ to $U(n-1)$ feature in a well-known long exact sequence of equivariant homotopy groups. Because **MU** is global spectrum, our splitting theorem implies that the restriction from $U(n)$ to $U(n-1)$ is surjective, and the long exact sequence decomposes into short exact sequences, leading to the following results.

Corollary. For every compact Lie group G, every character $\chi : G \to U(1)$ and every $n \geq 1$, the Euler class of the $(G \times U(n))$-representation $\chi \otimes \nu_n$ is a non zero-divisor in the graded-commutative ring $\mathbf{MU}^*_{G\times U(n)}$.

Corollary. For all $k_1, \dots, k_m \geq 1$, the Euler class of the tautological representation of the group $U(k_1) \times \cdots \times U(k_m)$ is a non zero-divisor in the graded ring $\mathbf{MU}^*_{U(k_1)\times\cdots\times U(k_m)}$.

Another consequence of our splittings for global functors are stable splitting of global classifying spaces of orthogonal, unitary and symplectic groups. In the model of [11], unstable global homotopy types are represented by *orthogonal spaces*, continuous functors to spaces from the category of finite-dimensional inner product spaces and linear isometric embeddings. An important example is the *global classifying space* $B_{\mathrm{gl}}G$ of a compact Lie group G, see [11, Definition 1.1.27]. The unstable global homotopy type of $B_{\mathrm{gl}}G$ 'globally represents' principal G-bundles over equivariant spaces, see [11, Proposition 1.1.30]. In particular, the underlying non-equivariant homotopy type of $B_{\mathrm{gl}}G$ is a classifying space for the Lie group G.

The *global stable homotopy category* $\mathcal{GH}$ is the localization of the category of orthogonal spectra at the class of global equivalences [11, Definition 4.1.3]. The global stable homotopy category is a compactly generated tensor triangulated category, see [11, Section 4.4]. The 0-th G-equivariant homotopy group functor $\pi_0^G : \mathcal{GH} \to$ (abelian groups) is represented by the unreduced suspension spectrum of $B_{\mathrm{gl}}G$, compare [11, Theorem 4.4.3]. The natural splittings of $O(n)$-, $U(n)$- and $Sp(n)$-equivariant stable homotopy groups thus correspond to splittings of the representing objects. The upshot is:

Corollary. For every $n \geq 0$, there are sum decompositions in the global stable homotopy category

$$\Sigma^\infty_+ B_{\mathrm{gl}} O(n) \cong \bigvee_{k=0,\dots,n} \Sigma^\infty B_{\mathrm{gl}}(O(k), O(k-1)) \ ,$$

$$\Sigma^\infty_+ B_{\mathrm{gl}} U(n) \cong \bigvee_{k=0,\dots,n} \Sigma^\infty B_{\mathrm{gl}}(U(k), U(k-1)) \qquad \text{and}$$

$$\Sigma^\infty_+ B_{\mathrm{gl}} Sp(n) \cong \bigvee_{k=0,\dots,n} \Sigma^\infty B_{\mathrm{gl}}(Sp(k), Sp(k-1)) \ .$$

Here $B_{\mathrm{gl}}(O(k), O(k-1))$ denotes the unreduced mapping cone of the morphism $B_{\mathrm{gl}}O(k-1) \to B_{\mathrm{gl}}O(k)$ induced by the embedding $O(k-1) \to O(k)$; it is globally equivalent to the global Thom space of the global vector bundle associated with the tautological $O(k)$-representation on $\mathbb{R}^k$. And similarly for the unitary and symplectic groups.

If we apply the forgetful functor from the global stable homotopy category to the non-equivariant stable homotopy category, the above global splittings specialize to the classical stable splittings due to Snaith [9, Theorem 4.2], [10, Theorem 2.2] and Mitchell-Priddy [7, Theorem 4.1]. If G is a compact Lie group, we can apply the forgetful functor [11, Theorem 4.5.23]

$$U_G \ : \ \mathcal{GH} \ \to \ G\text{-}\mathcal{SH}$$

from the global to the genuine G-equivariant stable homotopy category. This forgetful functor turns the global splittings of $B_{\mathrm{gl}}O(n)$, $B_{\mathrm{gl}}U(n)$ and $B_{\mathrm{gl}}Sp(n)$ into G-equivariant stable splittings of the classifying G-spaces for G-equivariant real, complex and quaternionic vector bundles. These equivariant splittings appear to be new.

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Realizability of equivariant chain complexes

Marc Stephan
(joint work with Henrik Rüping)

A classical conjecture in the theory of transformation groups by Carlsson [Car86] states that the sum of the mod-p Betti numbers of a finite, free $(\mathbb{Z}/p)^n$-CW complex $X \neq \emptyset$ is at least 2^n. An algebraic version [Car86, Conjecture II.2] (see also [AS95]) of this conjecture predicted the same bound for the total dimension of the homology of any finite, nonacyclic, free $\mathbb{F}_p(\mathbb{Z}/p)^n$-chain complex. In [IW18] Iyengar and Walker constructed counterexamples to this algebraic conjecture and asked if they can be realized topologically in order to produce counterexamples to Carlsson's conjecture. In this talk, I explained that these counterexamples can not be realized topologically.

The counterexamples are for $n \geq 8$ and odd primes p. They arise as the mapping cone

$$\mathrm{Cone}(w\colon \Sigma^2 K_* \to K_*)$$

of multiplication with a certain cycle w of degree 2 of the Koszul complex K_* of the group ring $\mathbb{F}_p(\mathbb{Z}/p)^n$. For arbitrary $n \geq 1$, prime p and cycles $w \in K_*$, we classified which mapping cones $\mathrm{Cone}(w\colon \Sigma^r K_* \to K_*)$ can be realized topologically, where $r = \deg(w)$.

Theorem 1 ([RS19, Theorem A]). *There is a free $(\mathbb{Z}/p)^n$-space whose singular chain complex is quasi-isomorphic to* $\mathrm{Cone}(w\colon \Sigma^r K_* \to K_*)$ *as $\mathbb{F}_p(\mathbb{Z}/p)^n$-chain complexes if and only if w is a boundary or the degree of w is at most* 1.

I focused on establishing nonrealizability, explaining the difficulty, a general approach adapted from Carlsson's solution [Car81] to the Steenrod problem [Las65, Problem 51] that does not provide obstructions, and a new approach.

If an equivariant chain complex C_* can be realized topologically as the singular chain complex of an equivariant space, then its cohomology will have a multiplicative structure. A challenge in establishing nonrealizability lies in not knowing how this potential multiplicative structure looks like. Group cohomology helps to get a handle on the multiplicative structure in the cohomology of the coinvariants of C_*. For any finite group G, if a free $\mathbb{F}_pG$-chain complex C_* is quasi-isomorphic to the singular chain complex $C_*(X;\mathbb{F}_p)$ of a free G-space X, then $H^*(C_*/G)$ becomes a $H^*(BG;\mathbb{F}_p)$-module. This action by the group cohomology can be calculated algebraically from C_*, just as the multiplication in group cohomology can be defined algebraically from any free resolution of the coefficient field. Moreover, the annihilator ideal in $H^*(BG;\mathbb{F}_p)$ of the action must be closed under Steenrod operations and thus can provide obstructions to realizability. In general, this approach does not provide obstructions for realizing $\mathrm{Cone}(w\colon \Sigma^r K_* \to K_*)$.

The new approach is to use the interplay between group cohomology and the multiplicative structure in the following spectral sequence. Compatibility with the Steenrod operations gets replaced by compatibility with the differentials.

Theorem 2 ([RS19, Theorem B])**.** *Let $G = (\mathbb{Z}/p)^n$. For any free G-space X, there is a multiplicative spectral sequence with E_1-page*

$$E_1^{*,q} \cong H^q(X/G;\mathbb{F}_p) \otimes \frac{\mathbb{F}_p[y_1,\dots,y_n]}{(y_1^p,\dots,y_n^p)}$$

converging to $H^q(X;\mathbb{F}_p)$. The differential d_1 is determined by

$$d_1(x\otimes y_j) = (-1)^{|x|} x \cup a_j,$$

where the a_j's form an explicit basis of $H^1(BG;\mathbb{F}_p)$.

It can be calculated algebraically from the associated cochain complex $C^*(X;\mathbb{F}_p)$ by filtration with powers of the augmentation ideal I^{L-k}, where L is the nilpotency index of I minus one. For the mapping cones $\mathrm{Cone}(w\colon \Sigma^r K_* \to K_*)$, I explained how to use the potential $H^1(BG;\mathbb{F}_p)$-action to produce contradictions to the Leibniz rule, establishing the nonrealizability part of Theorem 1.

I concluded remarking that the spectral sequence from Theorem 2 extends to any finite p-group G and asked for a group theoretic interpretation of the d_1-differential in general.

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Dade groups for finite groups and dimension functions

Ergün Yalçin
(joint work with Matthew Gelvin)

Let G be a finite group with Sylow p-subgroup S and let k be an algebraically closed field of characteristic $p > 0$. We assume that all kG-modules are finitely generated. A kG-module M is called an *endo-permutation module* if $\mathrm{End}_k(M) \cong M^* \otimes_k M$ is a permutation kG-module, i.e., if it has a G-invariant basis.

When $G = S$ is a p-group, a group of endo-permutation modules is defined as follows: An endo-permutation kS-module M is said to be *capped* if it has a summand with vertex S. Two capped endo-permutation modules M and N are said to be *equivalent* if $M \oplus N$ is an endo-permutation module, or equivalently if $M^* \otimes N$ is a permutation module. The *Dade group* $D(S)$ of a p-group S is defined to be the group whose elements are the equivalence classes of capped endo-permutation kS-modules and whose group operation is induced by tensor product, i.e., $[M] + [N] := [M \otimes N]$. The Dade group of a p-group has been studied by Bouc, Thévenaz, Carlson, Mazza, and many others. A complete description of $D(S)$ in terms of the genetic sections of the group S is given by Bouc in [2].

When G is a finite group, the situation is more complicated since a transitive permutation kG-module need not be indecomposable. A kG-module M is a *p-permutation module* if it is a summand of a permutation kG-module. A kG-module M is called an *endo-p-permutation module* if $\mathrm{End}(M) \cong M^* \otimes M$ is a p-permutation module. Endo-p-permutation modules were studied by Urfer [6] using an equivalence relation on the sources of endo-p-permutations modules, and by Lassueur [5] by considering the class of strongly capped endo-p-permutation modules. As a generalization of Lassueur's definition of strongly capped endo-p-permutation modules (see [5, Prop 5.2]), we define a notion of a Dade kG-module.

Definition 1. A kG-module M is a *Dade kG-module* if there is an integer $n \geq 0$ such that

$$\mathrm{End}(M) \cong k^n \oplus W$$

for some p-permutation module W, all of whose indecomposable summands have vertices that are non-Sylow p-subgroups of G.

A Dade module is *capped* if it has a Sylow-vertex component, or equivalently if $n \geq 1$ in the above decomposition. We show that a Dade kG-module has a unique cap up to isomorphism: If U and V are two Sylow-vertex components of a Dade module M, then $U \cong V$. We declare that two capped Dade modules M and N are *equivalent* if $M \oplus N$ is a Dade module.

Theorem 2. *Let $D(G)$ denote the set of equivalence classes of capped Dade kG-modules under the equivalence relation defined above. Then the operation $[M] + [N] := [M \otimes N]$ defines an abelian group structure on $D(G)$. The group $D(G)$ defined this way is isomorphic to the Dade group defined by Lassueur in* [5].

One important source of Dade modules is the kG-modules defined as relative syzygies. Given a G-set X, the kernel of the augmentation map $\varepsilon : kX \to k$ is called the *relative syzygy of* X, and is denoted by $\Delta(X)$. We show that if X is a finite G-set such that $X^S = \emptyset$, then $\Delta(X)$ is a capped Dade kG-module. For such a G-set, we define $\Omega_X := [\Delta(X)]$ in $D(G)$. We extend this definition to all G-sets by declaring $\Omega_X = 0$ whenever $X^S \neq \emptyset$.

Definition 3. The subgroup of $D(G)$ generated by the elements Ω_X as X ranges over all G-sets is called the *Dade group generated by relative syzygies*, and is denoted by $D^{\Omega}(G)$.

For a finite group G and a fixed prime p, we denote by $\mathcal{F}_p$ the family of all p-subgroups in G. A function $f : \mathcal{F}_p \to \mathbb{Z}$ that is constant on the G-conjugacy classes of subgroups in $\mathcal{F}_p$ is called a *superclass function.* The set of superclass functions defined on $\mathcal{F}_p$ forms a group under addition, denoted by $C(G, p)$. For each G-set X, there is a superclass function ω_X defined by

$$\omega_X(P) = \begin{cases} 1 & \text{if} \quad X^P \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

for every p-subgroup $P \leq G$. One of our main results is the following:

Theorem 4. *There is a well-defined surjective group homomorphism*

$$\Psi_G : C(G, p) \to D^{\Omega}(G)$$

that sends ω_X *to* Ω_X *for every* G*-set* X*.*

We call the homomorphism Ψ_G the *Bouc homomorphism* for G since it is a generalization of the homomorphism defined by Bouc [1] for p-groups.

When $G = S$ is a p-group, the kernel of the Bouc homomorphism is completely determined. Let $C(S)$ denote the group of all superclass functions $f : \mathrm{Sub}(S) \to \mathbb{Z}$, and $C_b(S)$ denote the subgroup of $C(S)$ formed by superclass functions satisfying Borel-Smith conditions. It is known that $C_b(S)$ is also equal to the image of the dimension homomorphism $\mathrm{Dim} : R_{\mathbb{R}}(S) \to C(S)$ from the real representation ring $R_{\mathbb{R}}(G)$ to $C(S)$ which sends a real G-representation V to the function $\mathrm{Dim}(V)$ defined by

$$\mathrm{Dim}(V)(P) = \dim_{\mathbb{R}}(V^P)$$

for all $P \leq S$. Bouc and Yalçın [3] showed that the kernel of Ψ_S is equal to $C_b(S)$, and hence equals the image of the homomorphism $\mathrm{Dim} : R_{\mathbb{R}}(S) \to C(S)$.

A real G-representation V is called *k-oriented* if the $N_G(P)/P$-action on the reduced homology group $\widetilde{H}_*(S(V)^P, k) \cong k$ is trivial for every $P \in \mathcal{F}_p$. The set of isomorphism classes of k-orientable real representations forms a group under direct sum, which we denote by $R^+_{\mathbb{R}}(G, k)$. There is a group homomorphism

$$\mathrm{Dim} : R^+_{\mathbb{R}}(G, k) \to C(G, p)$$

that takes a representation V to its dimension function $\mathrm{Dim}(V)$. We prove the following:

Theorem 5. *The image of the dimension function* $\mathrm{Dim} : R^+_{\mathbb{R}}(G, k) \to C(G, p)$ *lies in the kernel of Bouc homomorphism* $\Psi_G : C(G, p) \to D^{\Omega}(G)$*.*

As in the p-group case, it is possible to characterize the image of the dimension homomorphism $\mathrm{Dim} : R^+_{\mathbb{R}}(G, k) \to C(G, p)$ by a set of conditions defined on certain subquotients of G. Using this characterization we conclude that when $p = 2$, the image of the homomorphism Dim is equal to $C_b(G, 2)$, the group of superclass functions in $C(G, 2)$ satisfying the Borel-Smith conditions when restricted to a Sylow 2-subgroup. This gives the following corollary of Theorem 5.

Corollary 6. *There is a short exact sequence of abelian groups*

$$0 \longrightarrow C_b(G,2) \xrightarrow{\ j\ } C(G,2) \xrightarrow{\ \Psi_G\ } D^{\Omega}(G) \longrightarrow 0$$

where the first map is the inclusion and the second map is the Bouc homomorphism for G.

The proof of Theorem 5 relies on topological methods. A *Moore G-space over k relative to the family $\mathcal{F}_p$* is a G-CW-complex X such that for every $P \in \mathcal{F}_p$, the fixed-point subspace X^P is a Moore space over k, i.e., the reduced homology $\widetilde{H}_*(X^P; k)$ is nonzero only in a single dimension. We show that the reduced homology of an n-dimensional Moore G-space X whose point-stabilizers are all non-Sylow p-subgroups is a capped Dade kG-module. Moreover, if X_i denotes the G-set of i-cells of X, then the equality

$$[\widetilde{H}_n(X;k)] = \sum_{i=1}^{n} \Omega_{X_i}$$

holds in $D^{\Omega}(G)$. This is analogous to [7, Thm 1.2] which proves the same statement for Moore S-spaces when S is a p-group.

We then consider Moore G-spaces whose isotropy subgroups are arbitrary, including the possibility that X^S is nonempty. We prove a similar theorem in this case, and apply it to a k-orientable real representation sphere $X = S(V)$ to obtain that $\Psi_G(\mathrm{Dim}(X)) = 0$ in $D^{\Omega}(G)$. This gives the conclusion of Theorem 5.

It is an open question whether the equality $C_{ba^+}(G,p) = \ker \Psi_G$ also holds when p is odd. This is related to the question whether $D^{\Omega}(G)$ has a well-defined biset functor structure, which is also open.

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Varying the object sets of linking systems

Ellen Henke

Saturated fusion systems play a role in local finite group theory, in the modular representation theory of finite groups and in parts of homotopy theory. The main examples are the categories $\mathcal{F}_S(G)$, where G is a finite group, S a Sylow p-subgroup of S, the objects of $\mathcal{F}_S(G)$ are all subgroups of S, and the morphisms are the conjugation maps by elements of G. In the same setting, there is also a category $\mathcal{L}_S^c(G)$ defined which is called the centric linking system and is closely related to $\mathcal{F}_S(G)$. As shown by Broto, Levi and Oliver [3], we have $BG_p^\wedge \simeq |\mathcal{L}_S^c(G)|_p^\wedge$ where $(\cdot)_p^\wedge$ denotes the Bousfield-Kan p-completion functor.

In general, a saturated fusion system is a category $\mathcal{F}$ together with a p-group S such that the objects of $\mathcal{F}$ are all subgroups of S and the morphisms in $\mathcal{F}$ are injective group homomorphisms subject to certain axioms. If $\mathcal{F}$ is a saturated fusion system over S, then the following are defined somewhat analogously to the corresponding concepts in finite group theory:

(1) A normalizer $N_\mathcal{F}(P)$ and a centralizer $C_\mathcal{F}(P)$ for $P \leq S$. If P is suitably chosen in its $\mathcal{F}$-isomorphism class, then $N_\mathcal{F}(P)$ and $C_\mathcal{F}(P)$ are saturated.
(2) Normal subsystems.
(3) Subnormal subsystems; here $\mathcal{E}$ is subnormal in $\mathcal{F}$ if there exists a "subnormal series" $\mathcal{E} = \mathcal{E}_0 \trianglelefteq \mathcal{E}_1 \trianglelefteq \cdots \trianglelefteq \mathcal{E}_k = \mathcal{F}$ of subsystems of $\mathcal{F}$.
(4) Components of $\mathcal{F}$; a component is a subnormal subsystem of $\mathcal{F}$ which is (in a certain sense) a perfect central extension of a simple fusion system.
(5) A normal subsystem $F^*(\mathcal{F})$ called the generalized Fitting subsystem of $\mathcal{F}$.

The precise notion of a normal subsystems as well as the definitions mentioned in (4) and (5) are due to Aschbacher [1, 2] and play an important role in his program to revisit the classification of finite simple groups using fusion systems.

Broto, Levi and Oliver [4] built the foundation for the homotopy theory of fusion systems by introducing centric linking systems. The longstanding conjecture that there is a unique centric linking system associated to each saturated fusion system was first shown by Chermak [6] and subsequently by Oliver [12] using an important idea from Chermak's proof. Both proofs use originally the classification of finite simple groups, but work of Glauberman–Lynd [9] removes the dependence of Oliver's proof on the classification. This leads also to a classification-free proof of the Martino–Priddy conjecture, which was originally shown by Oliver.

Linking systems of a more general kind were defined and studied in [5], [11] and [10]. If $\mathcal{F}$ is a saturated fusion system over a p-group S, the object set of a centric linking system associated to $\mathcal{F}$ is the set $\mathcal{F}^c$ of *centric subgroups* of S, whereas the object set of a linking system in the definition of Oliver [11] is a subset of the set $\mathcal{F}^q$ of *quasicentric subgroups*. We will use our definition of a linking system [10], which is the most general one currently in the literature. The object set of a linking system in this definition is a subset of the set $\mathcal{F}^s$ of *subcentric subgroups*. The existence and uniqueness of centric linking systems implies that there is a unique *quasicentric linking system* and a unique *subcentric linking system* associated to $\mathcal{F}$ meaning linking systems whose object sets are the sets $\mathcal{F}^q$ and $\mathcal{F}^s$ respectively; see [5] and [10]. More generally, given any set Δ of subcentric subgroups of S, there is a unique linking system with object set Δ associated to $\mathcal{F}$ provided Δ fulfills some conditions which are necessary for such a linking system to exist. Every linking system is a full subcategory of a subcentric linking system. Moreover, the homotopy type of the nerve of a linking system depends not on the choice of the object set; see [10]. If t is any of the symbols c, q or s (standing for the centric, quasicentric and subcentric subgroups of a fusion system), then a further interesting choice for the object set of a linking system is the set

$$\mathcal{F}^{\delta(t)} := \{P \leq S \colon (P \cap F^*(\mathcal{F}))O_p(\mathcal{F}) \in \mathcal{F}^t\}.$$

We introduced this set in unpublished notes generalizing ideas of Chermak [7]. His set $\delta(\mathcal{F})$ turns out to be equal to $\mathcal{F}^{\delta(s)}$.

In the talk we outline the advantages and disadvantages of different object sets of linking systems. The underlying problem is that there is no meaningful notion of morphisms of linking systems. We ask however the more modest question how linking systems associated to the saturated subsystems mentioned in (1),(2),(3) (and thus to the subsystems mentioned in (4) and (5)) can be seen inside of a linking system associated to $\mathcal{F}$. The case (2) is here particularly important as a basis for formulating an extension theory of fusion systems. We moreover believe that understanding these concepts on the level of linking systems will lead to simplifications in Aschbacher's program. Already now our results enable us (partly in joint work with Chermak) to revisit and significantly extend the theory of fusion systems. Our new algebraic concepts allow us moreover to formulate some conjectures about maps between p-completed nerves of linking systems.

To summarize now the answer to the above question, it turns out that a centric, quasicentric or subcentric linking system associated to one of the saturated subsystems mentioned in (1) can be seen inside of a centric, quasicentric or subcentric linking system associated to $\mathcal{F}$. However, if $\mathcal{E}$ is a normal subsystem of $\mathcal{F}$, then a centric or quasicentric linking system associated to $\mathcal{E}$ can only in special cases be seen inside of a centric or quasicentric linking system for $\mathcal{F}$. In contrast, the subcentric linking system associated to $\mathcal{E}$ can be seen inside of the subcentric linking system associated to $\mathcal{F}$; the same holds if we replace "subcentric" by "t-regular" where t stands again for either of the symbols c, q or s. As a consequence, a subcentric or t-regular linking system associated to a subnormal subsystem of $\mathcal{F}$ can be seen inside of a linking system associated to $\mathcal{F}$ which is subcentric or t-regular respectively. However, in the case of subcentric linking systems it is not clear that different subnormal series lead to the same embedding. Thus, when looking at subnormal subsystems it is good to work with t-regular linking systems. This becomes even more evident if one considers localities rather than linking systems.

The concept of a locality was introduced by Chermak [6] in the context of his proof of the existence and uniqueness of centric linking systems. A locality is a "partial group" $\mathcal{L}$ together with a "Sylow p-subgroup" S, and a set Δ of "objects". Every linking system corresponds to a locality with the same object set which we then call a linking locality. Similarly we will talk about centric, quasicentric, subcentric and t-regular linking localities. Because of the group-like structure of a locality, there are natural notions of "partial normal subgroups" and thus of "partial subnormal subgroups".

Together with Chermak we proved that, for any linking locality $\mathcal{L}$ associated to $\mathcal{F}$, there is a one-to-one correspondence between the partial normal subgroups of $\mathcal{L}$ and the normal subsystems of $\mathcal{F}$. If $\mathcal{L}$ is a subcentric linking locality, then a subcentric linking locality associated to a normal subsystem $\mathcal{E}$ can be seen inside of the partial normal subgroup $\mathcal{N}$ corresponding to $\mathcal{E}$. If $\mathcal{L}$ is instead t-regular, then even $\mathcal{N}$ itself is a t-regular locality. As a consequence, there is a one-to-one correspondence between the subnormal subsystems of the fusion systems and the partial subnormal subgroups of an associated t-regular locality. This induces a similar correspondence for components.

In Aschbacher's program one needs to consider components of fusion systems as well as the saturated normalizers and centralizers of p-subgroups from (1). It seems however that there is no systematic way to see the t-regular linking localities associated to these normalizers and centralizers inside of the t-regular linking locality associated to $\mathcal{F}$. We propose therefore to work in this context with the subcentric linking locality associated to $\mathcal{F}$ using results about components of such localities which we proved together with Valentina Grazian.

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Constant Jordan type modules of Loewy length 2

Daniel Bissinger

In the modular representation theory of finite groups one direction of research has been the study of modules via their restrictions to algebraic families of subalgebras of type $k[T]/(T^p)$, where $p = \mathrm{char}(k) > 0$. One example are rank varieties, defined by Carlson [5], for modules of elementary abelian p-groups.

More recently, modules of constant Jordan type and modules with the equal images property were introduced by Carlson, Friedlander and Pevtsova [6] for finite group schemes over k. They have been studied subsequently especially in the context of elementary abelian p-groups [1, 7]: Let $r \geq 2$ and $E_r \cong (\mathbb{Z}/(p))^r$ be an elementary abelian p-group of rank r and V be a k-complement of $\mathrm{Rad}^2(kE_r)$ in $\mathrm{Rad}(kE_r)$. A module $M \in \mathrm{mod}\, kE_r$ has constant Jordan type, provided the Jordan canonical form of the nilpotent operator $x_M \colon M \to M, m \mapsto x \cdot m$ is independent of $x \in V \setminus \{0\}$. We let $\mathrm{CJT}(r)$ be the full subcategory of all modules of constant Jordan type and denote for $M \in \mathrm{CJT}(r)$ its Jordan type by $\mathrm{Jt}(M) \in \mathbb{N}_0^p$, i.e. for each $i \in \{1, \ldots, p\}$ and each $x \in V \setminus \{0\}$ the Jordan canonical form of x_M has exactly $\mathrm{Jt}(M)_i$ blocks of size i.

When investigating the category $\mathrm{CJT}(r)$, the rank variety is not a suitable tool as it only distinguishes projective and non-projective modules in $\mathrm{CJT}(r)$. A more promising approach is to look at the map $\Phi\colon \mathrm{CJT}(r) \mapsto \mathbb{N}_0^p, M \mapsto \mathrm{Jt}(M)$ to distinguish modules and get information about $\mathrm{CJT}(r)$. One of the main objectives today in studying modules of constant Jordan type is to determine the image of Φ, in other words to determine which tuples $(a_1, \ldots, a_p) \in \mathbb{N}_0^p$ can be realized as the Jordan type of a module in $\mathrm{CJT}(r)$.

Among other things, this is motivated by a connection between kE_r-modules of constant Jordan type and vector bundles on $\mathbb{P}^{r-1}$. The connection has been established by Benson and Pevtsova in [2] by means of functors $\mathcal{F}_i$ for $1 \leq i \leq p$ that assign to each finite dimensional kE_r-module a coherent sheaf $\mathcal{F}_i(M)$ on the projective space $\mathbb{P}^{r-1}$. Building on the foundational work by Friedlander-Pevtsova [8], they proved that a module M has constant Jordan type if and only if $\mathcal{F}_i(M)$ is an algebraic vector bundle for all $i \in \{1, \ldots, p\}$ and that in this case the rank of $\mathcal{F}_i(M)$ is given by the number $\mathrm{Jt}(M)_i$. Hence one might hope that a classification of indecomposable objects in $\mathrm{CJT}(r)$ and their Jordan types may lead to new indecomposable vector bundles on $\mathbb{P}^{r-1}$.

Unfortunately, such a classification is deemed hopeless as $\mathrm{CJT}(r)$ is of wild representation type for $(p, r) \neq (2, 2)$. One is therefore led to consider smaller subcategories that might be easier to handle.

We investigate the full category $\mathrm{CJT}_2(r)$ of modules of constant Jordan type and Loewy length ≤ 2. Although these categories remain wild for $r \geq 3$, considerations in $\mathrm{CJT}_2(r)$ have compared to the general case the advantage, that modules of Loewy length ≤ 2 can be understood as representations for the Kronecker quiver Γ_r with r arrows. The indecomposable representations in the category of finite dimensional representations $\mathrm{rep}(\Gamma_r)$ fall into three classes: there are the preprojective ones, the preinjective ones and the regular ones. The preprojective and preinjective representations are well understood, whereas very little is known about the regular representations for $r \geq 3$.

We call an arbitrary non-zero representation X regular, provided every indecomposable direct summand of X is regular. The building blocks of the full subcategory $\operatorname{reg}(\Gamma_r) \subseteq \operatorname{rep}(\Gamma_r)$ of regular representations are the so-called elementary representations:

Definition. A non-zero regular representation E is called elementary, provided there is no short exact sequence

$$0 \to X \to E \to Y \to 0$$

with X and Y non-zero and regular.

By definition each regular representation X possesses a filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$$

such that each filtration factor X_i/X_{i-1} is elementary.

By considering the generic Jordan type for elementary representations and combining our findings with Kac's Theorem [9], we give a complete answer to the realization problem for $\operatorname{CJT}_2(r)$. By considering $\Phi(M)$ as an element in $\mathbb{N}_0^2$ (as $\Phi(M)_i = 0$ for all $M \in \operatorname{CJT}_2(r)$ and $i > 2$) this result may summarized as follows:

Theorem 1. ([4, 4.3]) Let $r \geq 2$ and denote by $\operatorname{ind}\operatorname{CJT}_2(r)$ the class of all indecomposable modules in $\operatorname{CJT}_2(r)$, then

$$\Phi(\operatorname{ind}\operatorname{CJT}_2(r)) = \{(a_1, a_2) \in \mathbb{N}^2 \mid r - 1 \leq a_1, q_{\Gamma_r}(a_2, a_1 + a_2) \leq 1\} \cup \{(1, 0)\},$$

where

$$q_{\Gamma_r} \colon \mathbb{Z}^2 \to \mathbb{Z}, x \mapsto x_1^2 + x_2^2 - r x_1 x_2$$

denotes the Tits quadratic form.

Moreover, we explain why an indecomposable module $M \in \operatorname{mod} kE_r$ of Loewy length ≤ 2 that satisfies

$$q_{\Gamma_r}((\dim_k M/\operatorname{Rad}_{kE_r}(M), \dim_k \operatorname{Rad}_{kE_r}(M))) + \dim_k M - 2\dim_k \operatorname{Rad}_{kE_r}(M) \geq 1$$

has the equal images property and therefore is an element in $\operatorname{CJT}_2(r)$. This result naturally leads to the following definition.

Definition. An element $\delta \in \mathbb{N}_0^2$ has the equal images property, provided

1. there exists an indecomposable kE_r-module N of Loewy length ≤ 2 such that $\delta = (\dim_k N/\operatorname{Rad}_{kE_r}(N), \dim_k \operatorname{Rad}_{kE_r}(N))$, and
2. every such indecomposable module has the equal images property.

We consider all dimension vectors $\delta \in \mathbb{N}_0^2$ with the equal images property. Using the universal covering of $\pi \colon C_r \to \Gamma_r$ and Kac's Theorem, these dimension vectors can be characterized by the Tits form as follows:

Theorem 2. ([3, 4.10, 5.1]) Let $\delta \in \mathbb{N}_0^2$. The following statements are equivalent.

(i) The vector δ has the equal images property.

(ii) $q_{\Gamma_r}(\delta) \leq 1$ and $q_{\Gamma_r}(\delta) + \delta_1 - \delta_2 \geq 1$.

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Some exotic tensor categories in prime characteristic

David Benson

This talk was about joint work with Pavel Etingof and Victor Ostrik. Let k be an algebraically closed field, and $\mathcal{C}$ a rigid abelian symmetric tensor category over k. We shall also assume that $\mathcal{C}$ has a generator, under the operations of direct sum, tensor product, and taking subobjects and quotient objects.

We say that an object X in $\mathcal{C}$ has *moderate growth* if for some Schur functor S^λ we have $S^\lambda X = 0$. The tensor nth power functor has a filtration by Schur functors, and the number of terms in the filtration grows worse than exponentially with n. It follows that if $X^{\otimes n}$ has finite length that grows at most exponentially with n then X has moderate growth. The category $\mathcal{C}$ is said to be of moderate growth if every object in $\mathcal{C}$ is.

Theorem 1 (Deligne [4])**.** *If k has characteristic zero and $\mathcal{C}$ has moderate growth then there exists a symmetric tensor functor from $\mathcal{C}$ to the category $\mathsf{SVec}(k)$ of finite dimensional super vector spaces over k, that doesn't kill any non-zero object.*

A functor of this nature to a smaller and better understood category is called a *fibre functor*. So the theorem says that in characteristic zero, with moderate growth, there is always a fibre functor to $\mathsf{SVec}(k)$. This category has as its objects the $\mathbb{Z}/2$-graded finite dimensional vector spaces $V = V_0 \oplus V_1$, with symmetric braiding $V \otimes W \to W \otimes V$ given by $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$.

As a consequence of this theorem, the Tannakian point of view (see for example Deligne [3]) implies that $\mathcal{C}$ is equivalent to the category of finite dimensional modules for an affine supergroup scheme G (plus a bit more structure).

In characteristic p, the same is no longer true. More targets are necessary for the fibre functors. We say that $\mathcal{C}$ is *finite* if it is equivalent as an abelian category to the category of finite dimensional modules for a finite dimensional algebra. This algebra is automatically self injective.

Theorem 2 (Ostrik [5])**.** *If k has characteristic p and $\mathcal{C}$ is finite and semisimple then there is a fibre functor from $\mathcal{C}$ to Ver_p.*

Here, Ver_p is the Verlinde category, namely the semisimplification of $\mathsf{rep}(\mathbb{Z}/p)$. To semisimplify means to quotient out all maps $f\colon X \to Y$ with the property that for all $g\colon Y \to X$ we have $\mathsf{Tr}(gf) = 0$. There are $p-1$ objects $S_1, \ldots, S_{p-1}$ in Ver_p, and the tensor product is determined by the statement that $S_2 \otimes S_i \cong S_{i-1} \oplus S_{i+1}$, with the convention that S_0 and S_p are taken to be zero. Note that for p odd, Ver_p decomposes as a Deligne tensor product $\mathsf{Ver}_p^+ \boxtimes \mathsf{SVec}(k)$, where Ver_p^+ has simples $S_1, S_3, \ldots, S_{p-2}$. So for example we have $\mathsf{Ver}_2 = \mathsf{Vec}(k)$ and $\mathsf{Ver}_3 = \mathsf{SVec}(k)$.

For non-semisimple $\mathcal{C}$ the theorem is no longer true. For example, in characteristic two, we have a category $\mathcal{C}_1$ whose objects are pairs (V, d) with V a finite dimensional vector space and d an endomorphism satisfying $d^2 = 0$. We put the usual tensor product on this, so that $d(v \otimes w) = d(v) \otimes w + v \otimes d(w)$, but the symmetric braiding is given by $x \otimes y \mapsto y \otimes x + dy \otimes dx$. This category does not fibre over $\mathsf{Vec}(k)$. The category $\mathcal{C}_1$ appears in algebraic topology as the target for Morava K-theory in characteristic two.

This turns out to be part of an infinite sequence of examples that are *incompressible* in the sense that they do not fibre over anything smaller. We call these categories Ver_{p^n} by analogy with the non-symmetric versions in characteristic zero. The following theorem was proved with Etingof [1] in characteristic two, and with Etingof and Ostrik [2] in general.

Theorem 3. *There are incompressible finite rigid abelian symmetric tensor categories $\mathsf{Ver}_{p^n} \supseteq \mathsf{Ver}_{p^n}^+$ in characteristic p with the following properties.*

(1) For $p = 2$ we have $\mathsf{Vec} = \mathsf{Ver}_2 \subseteq \mathsf{Ver}_{2^2}^+ \subseteq \mathsf{Ver}_{2^2} \subseteq \mathsf{Ver}_{2^3}^+ \subseteq \cdots$

(2) For p odd we have $\mathsf{Ver}_{p^n} = \mathsf{Ver}_{p^n}^+ \boxtimes \mathsf{SVec}(k)$, and

$$\begin{array}{ccccccc}
\mathsf{SVec}(k) & \hookrightarrow & \mathsf{Ver}_p & \hookrightarrow & \mathsf{Ver}_{p^2} & \hookrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathsf{Vec}(k) & \hookrightarrow & \mathsf{Ver}_p^+ & \hookrightarrow & \mathsf{Ver}_{p^2}^+ & \hookrightarrow & \cdots
\end{array}$$

(3) For $n \geq 2$, Ver_{p^n} is not semisimple.

(4) Ver_{p^n} has $p^{n-1}(p-1)$ isomorphism classes of simple modules.

(5) The Grothendieck ring of $\mathsf{Ver}_{p^n}^+$ is isomorphic to $\mathbb{Z}[2\cos(\pi/p^n)]$.

It is plausible that this is a complete list, and that every finite rigid abelian symmetric tensor category in characteristic p fibres over the union Ver_{p^∞}.

The construction of the categories Ver_{p^n} uses tilting modules for $SL(2, k)$, and is described in detail in [2]. Here is a brief sketch. Let T_i be the ith tilting module, numbered so that $T_0 = k$ and T_1 is the natural two dimensional module. Then $T_{p^n-1} = \mathsf{St}_n$, the nth Steinberg module of dimension p^n. The category Ver_{p^n}, as an abelian category, is representations of the endomorphism ring $E = \mathsf{End}(T)$ where $T = \bigoplus_{i=p^{n-1}-1}^{p^n-2} T_i$. Furthermore, the symmetric tensor structure also comes through projective resolutions from $SL(2,k)$. The point here is that projective E-modules correspond to modules in $\mathsf{Add}(T)$, so we know the symmetric tensor structure on projective resolutions. The crucial property is that this tensor product is exact, so that when we tensor together two projective resolutions of E-modules, we get a projective resolution of some E-module, and that gives the candidate for the tensor product.

We have a conjecture for the Ext ring of the tensor identity in these categories, and we plan to write a paper about this in the near future.

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The endomorphism ring of the trivial module

Jon F. Carlson

Let k be a field of characteristic $p > 0$, algebraically closed, and let G be a finite group. Assume that kG-modules are finitely generated unless otherwise indicated. We work in the stable category $\mathrm{stmod}(kG)$ that has all finitely generated kG-modules as objects and morphisms (for M and N objects) given by $\underline{\mathrm{Hom}}_{kG}(M,N) = \mathrm{Hom}_{kG}(M,N)/\,\mathrm{PHom}_{kG}(M,N)$ where PHom means homomorphisms that factor through projective modules.

The support variety $V_G(M)$ of a kG-module M is the closed set of all primes in the projectivized prime ideal spectrum of $\mathrm{H}^*(G,k)$ that contain the annihilator of $\mathrm{Ext}^*_{kG}(M,M)$. A subcategory $\mathcal{M}$ in $\mathrm{stmod}(kG)$ is thick provided it is triangulated and closed under taking direct summand. It is a thick tensor ideal if in addition it has the property that if M is in $\mathcal{M}$ and N is any finitely generated kG-module, then $M \otimes N$ is in $\mathcal{M}$. We know [2] that if $\mathcal{M}$ is a thick tensor ideal in $\mathrm{stmod}(kG)$, then there exists a collection $\mathcal{V}$ of subvarieties of $V_G(k)$ which is closed under specializations and finite unions such that $\mathcal{M}$ is the subcategory of all kG-modules M with $V_G(M) \in \mathcal{V}$.

If $\mathcal{M}$ is a thick subcategory of a triangulated category $\mathcal{C}$, then the Verdier localization of $\mathcal{C}$ at $\mathcal{M}$ is the category whose objects are the same as those of $\mathcal{C}$ and whose morphisms are obtained by inverting a morphism if the third object in the triangle of that morphism is in the subcategory $\mathcal{M}$. Thus, a morphism from L to N in the localized category has the form

$$L \xrightarrow{\gamma} M \xleftarrow{\theta} N$$

where the third object in the triangle of the map θ is in $\mathcal{M}$. So in the localized category, $\theta^{-1}\gamma$ is a morphism.

Jeremy Rickard has shown in [3] that that associated to a thick tensor ideal $\mathcal{M}$ there is a triangle of idempotent modules

$$\dots \longrightarrow \mathcal{E}_{\mathcal{M}} \xrightarrow{\mu} k \longrightarrow \mathcal{F}_{\mathcal{M}} \xrightarrow{\nu} \dots$$

This is a triangle in the stable category of all kG-modules. This has the property that any map from an object M in $\mathcal{M}$ to k factors through μ, In turn this implies that for any map $\theta : k \to N$ such that the third object in the triangle of that map is in $\mathcal{M}$, there is a map $\phi : M \to \mathcal{F}_{\mathcal{M}}$ such that $\phi\theta = \nu$. Indeed, it can be shown, in the case that set $\mathcal{V}$ is the collection of all subvarieties of a fixed closed set, that $\mathcal{F}$ is a homotopy colimit of a sequence

$$k \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots \qquad \dots \subseteq \mathcal{F}$$

of finitely generated modules such that for any such θ as above, and for n large enough, there is a map $\phi' : M \to F_n$ with $\phi'\theta$ equal to the inclusion of k in F_n. Thus any endomorphism of the trivial module in the Verdier localization can be taken to have the form $\zeta^{-1}\gamma$ where for some n, ζ and γ map k to F_n. The third object in the triangle of ζ is in $\mathcal{M}$.

We consider finite groups of the form $G = H \times C$ where C is a cyclic group of order 2. Let z be a generator for C, and $Z = z + 1$, so that $kC \cong k[Z]/(Z^2)$ and $kG \cong kH \otimes k[Z]/(Z^2)$. Let $V = \operatorname{res}^*_{G,C}(V_C(k))$, the image under the map induced by restriction to C. Let $\mathcal{M} = \mathcal{M}_V$, the thick tensor ideal of all modules M with $V_G(M) \subseteq V$, and let $\mathcal{C}$ be the localization of $\operatorname{stmod}(kG)$ at $\mathcal{M}$.

Suppose that

$$\dots \longrightarrow P_2 \xrightarrow{\partial} P_1 \xrightarrow{\partial} P_0 \xrightarrow{\varepsilon} k \longrightarrow 0$$

is a minmal projective kH-resolution of k. Let $\mathcal{E}$ be the module whose restriction to H is the infinite direct sum $P_0 \oplus P_1 \oplus P_2 \oplus \dots$, with the element Z acting by the boundary map ∂. Because $\partial^2 = Z^2 = 0$, the module is well defined. In addition, the augmentation provides a map $\mu : \mathcal{E} \to k$.

Similarly, let $\mathcal{F}$ be the kG-module which when restricted to H is the direct sum $k \oplus P_0 \oplus P_1 \oplus P_2 \oplus \dots$. Here the action of Z is by the boundary homomorphism in addition to the relation that $Zm = \varepsilon(m)$ for $m \in P_0$. We have a natural map $\nu : k \to \mathcal{F}$ that sends k to the first direct factor. The critical result is that we have a triangle

$$\dots \longrightarrow \mathcal{E} \xrightarrow{\mu} k \xrightarrow{\nu} \mathcal{F} \longrightarrow \dots$$

and it is the canonical triangle for the thick tensor ideal $\mathcal{M}$. That is, $\mathcal{E} \cong \mathcal{E}_{\mathcal{M}}$ and $\mathcal{F} \cong \mathcal{F}_{\mathcal{M}}$. These are idempotent modules and the universal properties are satisfied. Proving this fact requires mostly computing the support varieties of the modules.

The structure for the idempotent modules leads us to the main theorem.

Theorem: In the described situation we have an isomorphism of rings

$$\mathrm{End}_{\mathcal{C}}(k) \quad \cong \quad \hat{\mathrm{H}}^{\leq 0}(H,k),$$

the ring of negative Tate cohomology. This says that for any finite group H the negative cohomology ring of H is realized as the endomorphism of the trivial module in a Verdier localization of $\mathrm{stmod}(kG)$ for an extension G of H.

The point is that $\mathrm{Hom}_{kH}(k,P_n) \cong \hat{\mathrm{H}}^{-n-1}(H,k)$. So the equation in the theorem is correct as vector spaces. We need only worry about compositions. But it can be shown that the composition is really chain maps on the projective resolution and this is the definition of the cup product in negative cohomology [1].

An immediate corollary of the theorem is that if H is an elementary abelian group of rank 2 or more, then $\mathrm{End}_{\mathcal{C}}(k)$ is an infinitely generated local ring whose radical has square zero [1]. A similar thing should be true in other cases where H has 2-rank at least two.

We note that all of the above is generalized to odd characteristics $p > 2$, and it is also proved in the case that G and H are finite group schemes. The assumption that G and H are groups or group schemes guarantees us that the cohomology is finitely generated. But in the end, while we need the algebras to be cocommutative Hopf algebras, it makes little difference what the coalgebra structure is.

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Equivariant formal groups and bordism rings

Markus Hausmann

1. Introduction: Quillen's theorem

In [Qui69], Quillen proved that the complex bordism ring MU_* is isomorphic to the Lazard ring carrying the universal formal group law. This theorem provides a bridge from the algebraic theory of formal group laws to the stable homotopy category, which since then has proved to be one of the main organizational principles for studying the latter.

One example for this principle is the thick subcategory theorem [HS98] due to Hopkins–Smith, which can be rephrased as the computation of the Balmer spectrum [Bal05] of the category of finite spectra. The generalized homology theory $MU_*(-)$ gives rise to a support theory on the category of finite spectra by sending a finite spectrum X to the support of $MU_*(X)$. Here, $MU_*(X)$ is viewed as a comodule over the Hopf algebroid (MU_*, MU_*MU) classifying formal group laws and their strict isomorphisms, and the support is taken in the invariant prime ideals of MU_*. The thick subcategory theorem can then be phrased as saying that this support theory is the universal one, inducing a homeomorphism between the the Zariski spectrum of invariant prime ideals of the Lazard ring and the Balmer spectrum of finite spectra.

2. An equivariant version

This talk concerned an equivariant version of Quillen's theorem over abelian compact Lie groups A. The relevant equivariant generalization of MU is the homotopical A-equivariant bordism spectra MU_A introduced by tom Dieck [tD70]. The notion of an A-equivariant formal group law was first defined by Cole–Greenlees-Kriz [CGK00] (building on Cole's thesis [Col96]), as a certain quintuple

$$(k, R, \Delta, \theta, y(\epsilon))$$

consisting of a commutative ground ring k, a complete commutative topological k-algebra R, a comultiplication $\Delta\colon R \to R\hat{\otimes}_k R$, an augmentation $\theta\colon R \to k^{A^*}$ and a coordinate $y(\epsilon) \in R$, satisfying various conditions. Here, $A^* = \mathrm{Hom}(A, \mathbb{T})$ denotes the dual group. The augmentation $\theta\colon R \to k^{A^*}$ is equivalent to an A^*-action on R that is compatible with the coproduct. Non-equivariantly, the conditions this data needs to satisfy imply that R is a power series ring over k on the coordinate $y(\epsilon)$, but in the equivariant case R is generally more complicated.

The authors of [CGK00] also showed:

(1) There exists a universal A-equivariant formal group law, defined over an A-equivariant Lazard ring L_A.
(2) The coefficients of any A-equivariant complex oriented cohomology theory E carry an A-equivariant formal group law with $R = E^*(\mathbb{C}P(\mathcal{U}_A))$ the value of E at the base space of the universal A-equivariant line bundle.

Since MU_A is canonically complex oriented, one obtains a map

$$\varphi_A\colon L_A \to (MU_A)_*$$

and the A-equivariant version of Quillen's theorem amounts to showing that this map is an isomorphism. First progress was made by Greenlees [Gre00, Gre01] who showed that for finite A the map φ_A is surjective, and that the kernel consists of Euler-torsion and Euler-divisible elements. In [HW18], Hanke–Wiemeler proved that φ_{C_2} is an isomorphism, bulding on an explicit presentation of $(MU_{C_2})_*$ given by Strickland [Str01]. In [Hau19] we proved the general case:

Theorem 1 (H.). *The map*

$$\varphi_A\colon L_A \to (MU_A)_*$$

is an isomorphism for every abelian compact Lie group A.

2.1. **Balmer spectra.** Given this theorem, one may hope that as in the non-equivariant case the theory of A-equivariant formal group laws gives a good approximation to the category of A-spectra. In joint work with Lennart Meier, we show that at least with respect to the Balmer spectrum of finite A-spectra this is indeed the case, in a similar way to the non-equivariant thick subcategory theorem sketched in the introduction.

The Balmer spectrum of the category Sp_A^c of finite A-spectra for abelian A was determined in the three papers by Balmer–Sanders [BS17], Barthel–H.–Naumann–Nikolaus–Noel–Stapleton [BHN+19] and Barthel–Greenlees–H. [BGH20]. The main tool in all three papers are the geometric fixed point functors for closed subgroups of A, to pull back information from the category of non-equivariant finite spectra. The following theorem shows that the result again allows an algebro-geometric interpretation in terms of equivariant formal groups:

Theorem 2 (H.–Meier)**.** *The assignment*

$$\begin{aligned} Sp_A^c &\to \mathrm{Spec}^{inv}(L_A) \\ X &\mapsto \mathrm{supp}((MU_A)_*X) \end{aligned}$$

is the universal support theory on Sp_A^c, i.e., it induces a homeomorphism

$$\mathrm{Spec}^{inv}(L_A) \cong \mathrm{Spc}(Sp_A^c)$$

from the Zariski spectrum of invariant prime ideals of L_A (with respect to strict isomorphisms of A-equivariant formal group laws) and the Balmer spectrum of the category of finite A-spectra.

3. Idea of proof and global version

The main tool in the proof of Theorem 1 is the global equivariant structure, which helps us to obtain a better understanding of equivariant Lazard rings. The various Lazard rings assemble to a functor

$$\begin{aligned} L_{gl}\colon (\text{ab. compact Lie groups})^{op} &\to \text{comm. rings} \\ A &\mapsto L_A, \end{aligned}$$

and the universal Euler class $e \in L_{\mathbb{T}}$ forms a distinguished element at the circle group $\mathbb{T}$. The main observation used in the proof is that the pair (L_{gl}, e) allows a global characterization that looks quite different from the universal properties of the individual values L_A:

Theorem 3 (H.)**.** *(1) For every torus A and split surjective character $V\colon A \to \mathbb{T}$ the sequence*

$$0 \to L_A \xrightarrow{e_V \cdot} L_A \xrightarrow{\mathrm{res}^A_{\ker(V)}} L_{\ker(V)} \to 0$$

is exact. Here, the first map is given by multiplication with the Euler class e_V (the pullback of $e \in L_{\mathbb{T}}$ along V), and the second map is restricting to the kernel of V.

(2) The pair (L_{gl}, e) is initial under functors with this property.

We call a contravariant functor X from abelian compact Lie groups to commutative rings equipped with an element at the circle group satisfying the analog of the exact sequences above a 'global group law'. Hence, the pair (L_{gl}, e) is the initial global group law. The motivation for this definition comes from global homotopy theory [Sch18]: Given a complex oriented ring spectrum E, the sequences

$$0 \to E_A \xrightarrow{e_V \cdot} E_A \to E_{\ker(V) \to 0}$$

are (split) exact as a consequence of the A-equivariant cofiber sequence

$$A/\ker(V)_+ \to S^0 \to S^V,$$

the complex orientation and the global functoriality. In particular this holds for global complex bordism **MU** (see [Sch18, Example 6.1.53]), which restricts to MU_A at a fixed A.

The reason the characterization in Theorem 3 is useful is that it tells us something about the regularity of Euler classes in Lazard rings, which was the major issue in previous approaches to proving the equivariant Quillen theorem. In particular, Theorem 3 directly implies that when A is a torus and $V \in A^*$ is split, then $e_V \in L_A$ is a regular element.

The main remaining work in showing that $L_A \to (MU_A)_*$ is an isomorphism is to show that (L_{gl}, e) in fact has even stronger regularity properties: The Euler classes of all non-trivial characters for tori are regular elements; more generally, given a linearly independent tuple $V_1, \ldots, V_n$ of characters over a torus A, the sequence $(e_{V_1}, \ldots, e_{V_n})$ of Euler classes in the Lazard ring L_A is regular. These properties are not shared by the coefficients of complex oriented global theories in general, but do hold for **MU** as a consequence of the theorem (due to Löffler [Löf73] and Comezaña [Com96]) that for abelian A the homotopy groups $(MU_A)_*$ are concentrated in even degrees.

Remark 1. *Finally we remark that there is a similar story for A-equivariant real (unoriented) bordism MO_A, see* [Hau19, Section 4]. *Here, the family of abelian compact Lie groups needs to be replaced by the family of elementary abelian* 2*-groups, and the formal groups that arise are all* 2*-torsion.*

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Extensions of simple biset functors

Serge Bouc

1. Introduction

Let R be a commutative ring (with identity element). The *biset category* $R\mathcal{C}$ over R is defined as follows:

- The objects of $R\mathcal{C}$ are the finite groups.
- For finite groups G and H, the set $\mathrm{Hom}_{R\mathcal{C}}(G,H)$ is the R-linear extension $RB(H,G) = R\otimes_{\mathbb{Z}} B(H,G)$ of the double Burnside group $B(H,G)$, i.e. the Grothendieck group of the category of finite (H,G)-bisets, for relations given by disjoint union decomposition.
- The composition of morphisms in $R\mathcal{C}$ is the linear extension of the usual composition (also called tensor product) of bisets: if V is a (K,H)-biset and U is an (H,G)-biset, then $V\circ U = V\times_H U = (V\times U)/H$, where H acts on the right the cartesian product $V\times U$ by $(v,u)\cdot h = (vh, h^{-1}u)$.
- The identity morphism of the group G in $R\mathcal{C}$ is (the class of) the set G, viewed as a (G,G)-biset by left and right multiplication.

The category $R\mathcal{C}$ is R-linear. A *biset functor* over R is an R-linear functor from $R\mathcal{C}$ to the category $R\text{-}\mathsf{Mod}$ of all R-modules. Biset functors, together with natural transformations of functors, form an R-linear abelian category $\mathcal{F}_R$.

2. Extensions

2.1. **Simple biset functors.** The simple biset functors $S_{H,W}$ over R are parametrized by pairs (H, W), where H is a finite group, and W is a simple $R\mathrm{Out}(H)$-module. If S is a simple biset functor, then $S \cong S_{H,W}$ if and only if H has minimal order such that $S(H) \neq \{0\}$, and $W = S(H)$.

For a finite group G, we denote by $\Phi(G)$ its Frattini subgroup (the intersection of all the maximal proper subgroups of G), and by $\mathrm{Soc}(G)$ its socle (the subgroup generated by all the minimal non trivial normal subgroups of G).

Definition 1. *Let H and G be finite groups. We say that G is a* Frattini faithful quotient *of H if there exists a surjective group homomorphism $s : H \twoheadrightarrow G$ such that* $\mathrm{Ker}\, s \cap \Phi(H) = \mathbf{1}$.

Theorem 2. *Let $S_{G,V}$ and $S_{H,W}$ be simple biset functors over a field $\mathbb{F}$ of characteristic 0.*

(1) If $\mathrm{Ext}^1_{\mathcal{F}_\mathbb{F}}(S_{G,V}, S_{H,W}) \neq \{0\}$ then either G is a proper Frattini faithful quotient of H, or H is a proper Frattini faithful quotient of G.
(2) If $\mathrm{Soc}(G) \leq \Phi(G)$ and $\mathrm{Soc}(H) \leq \Phi(H)$, then $\mathrm{Ext}^1_{\mathcal{F}_\mathbb{F}}(S_{G,V}, S_{H,W}) = \{0\}$.

2.2. **Nilpotent groups.**

Definition 3. *([4] Section 6) A finite nilpotent group G is called* atoric *if it cannot be split as a direct product $E \times K$, where E is a non-trivial elementary abelian group.*

One can show easily that a finite nilpotent group G is atoric if and only if $\mathrm{Soc}(G) \leq \Phi(G)$. Moreover each finite nilpotent group has a well defined largest atoric quotient $G^@$, isomorphic to G/N, where N is any normal subgroup of G maximal such that $N \cap \Phi(G) = \mathbf{1}$. If G and H are nilpotent finite groups, then $G^@ \cong H^@$ if and only if G and H have a common Frattini faithful quotient. Moreover:

Theorem 4. *Let $S_{G,V}$ and $S_{H,W}$ be simple biset functors over a field $\mathbb{F}$ of characteristic 0. If G and H are nilpotent, and if $\mathrm{Ext}^1_{\mathcal{F}_\mathbb{F}}(S_{G,V}, S_{H,W}) \neq \{0\}$, then $G^@ \cong H^@$.*

2.3. ***B*-groups.**

Definition 5. *For a normal subgroup N of a finite group G, let*

$$m_{G,N} = \frac{1}{|G|} \sum_{\substack{X \leq G \\ XN = G}} |X| \mu(X, G),$$

where μ is the Möbius function of the poset of subgroups of G.

*The group G is a B-*group *([1] Section 7.2.3 or [3] Definition 5.4.6) if $m_{G,N} = 0$ for any non-trivial normal subgroup N of G.*

The group G is a minimal *B-group if G is a non-trivial B-group, and any quotient B-group of G is either trivial or isomorphic to G.*

Recall ([3] Theorem 5.4.11) that any finite group G has a largest quotient B-group $\beta(G)$, well defined up to isomorphism. Let $\mathbb{F}$ be a field of characteristic 0. If H is a B-group, then the dimension of $S_{H,\mathbb{F}}(G)$ is equal to the number of conjugacy classes of subgroups K of G such that $\beta(K) \cong H$. Also recall ([3] Proposition 4.4.8) that the simple biset functor $S_{\mathbf{1},\mathbb{F}}$ is isomorphic to the functor $\mathbb{F}R_{\mathbb{Q}}$.

Theorem 6. *Let $S_{H,W}$ be a simple biset functor over a field $\mathbb{F}$ of characteristic 0. The following are equivalent:*

(1) $\mathrm{Ext}^1_{\mathcal{F}_{\mathbb{F}}}(S_{\mathbf{1},\mathbb{F}}, S_{H,W}) \neq \{0\}$.
(2) $\mathrm{Ext}^1_{\mathcal{F}_{\mathbb{F}}}(S_{H,W}, S_{\mathbf{1},\mathbb{F}}) \neq \{0\}$.
(3) The group H is a minimal B-group, and $W = \mathbb{F}$.

Moreover if this holds, then $\mathrm{Ext}^1_{\mathcal{F}_{\mathbb{F}}}(S_{\mathbf{1},\mathbb{F}}, S_{H,W}) \cong \mathbb{F} \cong \mathrm{Ext}^1_{\mathcal{F}_{\mathbb{F}}}(S_{H,W}, S_{\mathbf{1},\mathbb{F}})$.

A finite group G is a minimal B-group if and only if:

- Either G is non-solvable, and then G contains a normal subgroup $N \cong S^k$, where S is a non-abelian simple group and $k \geq 1$, the centralizer $C_G(N)$ is trivial, the quotient G/N is cyclic and acts transitively on $\{1, \dots, k\}$,
- Or G is solvable, and then either $G \cong C_p \times C_p$, where p is a prime number, or $G \cong \mathbb{F}_q \rtimes \langle\lambda\rangle$, where $\mathbb{F}_q$ is a finite field of cardinality q, and λ is a primitive element of $\mathbb{F}_q$ (with $\lambda \neq 1$ if q is prime).

2.4. **Groups of odd order.** Recall ([2]) that the assignment sending a finite group G to the group $B^\times(G)$ of units of its Burnside ring is a biset functor over $\mathbb{F}_2$. By an observation of tom Dieck ([6] Section 1.5), based on a theorem of Dress ([5]), Feit-Thompson's theorem is equivalent to the assertion that $B^\times(G) = \{\pm 1\}$ (i.e. $\dim_{\mathbb{F}_2} B^\times(G) = 1$) when G has odd order. This is a good motivation for studying biset functors over $\mathbb{F}_2$, and for restricting them to groups of odd order, throughout the rest of this section.

Definition 7. *For a normal subgroup N of a finite group G of odd order, let*

$$\overline{m}_{G,N} = \sum_{\substack{X \leq G \\ XN = G}} \mu(X, G)$$

*be the image of $m_{G,N}$ in $\mathbb{F}_2$. The group G is a B_2-*group *if $\overline{m}_{G,N} = 0_{\mathbb{F}_2}$ for any $\mathbf{1} \neq N \trianglelefteq G$. The group G is a* minimal *B_2-group if G is a non-trivial B_2-group, and any quotient B_2-group of G is either trivial or isomorphic to G.*

One can show that any finite group of odd order has a largest quotient B_2-group $\beta_2(G)$, well defined up to isomorphism ([3], Remark 5.4.12). Moreover, Feit-Thompson's theorem has the following consequence:

Theorem 8. *Let G be a group of odd order.*

(1) $\beta_2(G) \cong G/\Phi(G)$. In particular G is a B_2-group if and only if $\Phi(G) = \mathbf{1}$.
(2) A simple functor $S_{H,W}$ over $\mathbb{F}_2$ is a subquotient of $\mathbb{F}_2B$ if and only if $\Phi(H) = \mathbf{1}$ and $W = \mathbb{F}_2$.
(3) If H has odd order and $\Phi(H) = \mathbf{1}$, then $\dim_{\mathbb{F}_2} S_{H,\mathbb{F}_2}(G)$ is equal to the number of conjugacy classes of subgroups K of G such that $K/\Phi(K) \cong H$.

Theorem 9. *Let $S_{H,W}$ be a simple biset over $\mathbb{F}_2$ (where $|H|$ is odd). The following are equivalent:*

(1) $\mathrm{Ext}^1_{\mathcal{F}_{\mathbb{F}_2}}(S_{\mathbf{1},\mathbb{F}_2}, S_{H,W}) \neq \{0\}$.
(2) $\mathrm{Ext}^1_{\mathcal{F}_{\mathbb{F}_2}}(S_{H,W}, S_{\mathbf{1},\mathbb{F}_2}) \neq \{0\}$.
(3) The group H is a minimal B_2-group - that is H has odd prime order, and $W = \mathbb{F}_2$.

Moreover if this holds, then $\mathrm{Ext}^1_{\mathcal{F}_{\mathbb{F}_2}}(S_{\mathbf{1},\mathbb{F}_2}, S_{H,W}) \cong \mathbb{F}_2 \cong \mathrm{Ext}^1_{\mathcal{F}_{\mathbb{F}_2}}(S_{H,W}, S_{\mathbf{1},\mathbb{F}_2})$.

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On degree −1 operators in cohomology

Markus Linckelmann
(joint work with Dave Benson, Radha Kessar)

The purpose of this talk is to describe the content of the paper [2]. We describe an algebraic recipe for calculating the components of the BV operator Δ on the Hochschild cohomology of a finite group algebra with respect to the centraliser decomposition. We use this to investigate general properties of Δ and to make some computations for particular classes of finite groups.

1. Degree −1 operators on $\mathrm{Ext}^*_A(U, V)$

We start with an elementary construction principle for degree −1 operators on $\mathrm{Ext}^*_A(U, V)$ determined by a central element in an algebra A which annihilates both modules U and V. Let k be a commutative ring.

Theorem 1.1. *Let A be a k-algebra, let $z \in Z(A)$, and let U, V be A-modules. Suppose that z annihilates both U and V. Let $P = (P_n)_{n\geq 0}$ together with a surjective A-homomorphism $\pi\colon P_0 \to U$ be a projective resolution of U, with differential $\delta = (\delta_n\colon P_n \to P_{n-1})_{n\geq 1}$. For notational convenience, set $P_i = 0$ for $i < 0$ and $\delta_i = 0$ for $i \leq 0$. Then the following hold.*

(i) *There is a graded A-homomorphism $s\colon P \to P$ of degree 1 such that the chain endomorphism $\delta \circ s + s \circ \delta$ of P is equal to multiplication by z on P.*

(ii) *The graded k-linear map*

$$s^\vee = \mathrm{Hom}_A(s, V) \colon \mathrm{Hom}_A(P, V) \to \mathrm{Hom}_A(P[1], V)$$

sending $f \in \mathrm{Hom}_A(P_n, V)$ to $f \circ s \in \mathrm{Hom}_A(P_{n-1}, V)$ for all $n \geq 0$ is a homomorphism of cochain complexes. In particular, $s^\vee$ induces a graded k-linear map of degree -1

$$D_z^A = H^*(s^\vee) \colon \mathrm{Ext}_A^*(U, V) \to \mathrm{Ext}_A^{*-1}(U, V)$$

(iii) *The graded map D_z^A is independent of the choice of the projective resolution P and of the choice of the the homotopy s satisfying (i). In particular, we have $D_0^A = 0$.*

Example 1.2. Let G be a finite group and $g \in Z(G)$. Then $g - 1 \in Z(kG)$ annihilates the trivial kG-module k, and hence induces, by the above Theorem, a degree -1 operator

$$D_{g-1} : H^*(G, k) \to H^{*-1}(G, k)$$

In degrees 1 and 2, this takes the following form. Let $\zeta \in H^1(G, k) = \mathrm{Hom}(G, k)$. Then $D_{g-1}(\zeta) = \zeta(g) \in k = H^0(G, k)$. Let $\eta \in H^2(G, k)$. Then η is represented by a central group extension

$$1 \longrightarrow k \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1$$

where here k is the additive group structure on k. For $h \in G$ denote by $\hat{h}$ an inverse image of h in $\hat{G}$. Then $D_{g-1}(\eta)$ is the group homomorphism $G \to k$ determined by

$$D_{g-1}(\eta)(h) = [\hat{h}, \hat{g}] \in k = \ker(\hat{G} \to G) \ .$$

2. Background on Hochschild cohomology

Let A be a k-algebra such that A is projective as a k-module. By classical results of Gerstenhaber, the Hochschild cohomology

$$HH^*(A) = \mathrm{Ext}^*_{A \otimes_k A^{\mathrm{op}}}(A, A)$$

of A is graded-commutative with respect to the cup product and carries a graded Lie algebra structure of degree -1, called the Gerstenhaber bracket. In particular, $HH^1(A)$ is a Lie algebra, canonically isomorphic to the quotient $\mathrm{Der}(A)/\mathrm{IDer}(A)$ of the Lie algebra $\mathrm{Der}(A)$ of k-linear derivations on A by the Lie ideal $\mathrm{IDer}(A)$ of inner derivations.

If $A = kG$ for some finite group G, then $HH^*(kG)$ admits a degree -1 operator

$$\Delta : HH^*(kG) \to HH^{*-1}(kG)$$

called the *Batalin-Vilkovisky operator*, or BV operator for short. The Gerstenhaber bracket measures the failure of Δ to be a graded derivation on $HH^*(kG)$; that is, we have

$$[x, y] = (-1)^{|x|}\Delta(x, y) - (-1)^{|x|}\Delta(x)y - x\Delta(y)$$

for homogeneous elements x, y in $HH^*(kG)$. This formula implies in particular that the cup product and the BV-operator determine the Gerstenhaber bracket.

The Batalin-Vilkovisky formalism emerged in the early 1980s in theoretical physics, giving rise to a rich literature. In the 2000s the BV-formalism appeared in the context of symmetric algebras in papers of Tradler [15] and Menichi [10], and has subsequently been used by many authors for calculations in $HH^*(kG)$, including Ivanov, Volkov, Menichi, Witherspoon, Zhou, and others.

The BV-operator Δ in the Hochschild cohomology of finite group algebras can be constructed as the dual of the Connes B-operator, obtained from the Connes exact sequence relating Hochschild and cyclic (co-)homology

$$\longrightarrow HH_{n+1}(kG) \xrightarrow{I} HC_{n+1}(kG) \xrightarrow{S} HC_n(kG) \xrightarrow{B} HH_{n+1}(kG) \longrightarrow$$

Thus $B \circ I$ is a degree 1 operator on Hochschild homology of kG. Duality yields the degree -1 operator Δ on Hochschild cohomology of kG. More background material for this section may be found in Benson [1, §§2.11–2.15], Loday [9, Chapter 7], Burghelea [3], and Karoubi and Villamayor [6], for instance.

3. The centraliser decomposition of $HH^*(kG)$

Let G be a finite group. We have a canonical graded k-linear identification

$$HH^*(kG) = \oplus_{g \in G/\sim} H^*(C_G(g), k) ,$$

where the notation $g \in G/\sim$ means that g runs over a set of representatives of the conjugacy classes in G. Both sides are graded algebras, but this identification is not an equality of graded algebras. Siegel and Witherspoon described in [14] the cup product on the left side in terms of the decomposition on the right side.

As a consequence of the work of Burghelea [3], the Connes exact sequence respects the centraliser decomposition. It follows that the BV operator $\Delta : HH^*(kG) \to HH^{*-1}(kG)$ preserves the centraliser decomposition; that is we have

$$\Delta = \oplus_{g \in G/\sim} \Delta_g ,$$

where $\Delta_g : H^*(C_G(g), k) \to H^{*-1}(C_G(g), k)$.

In order to calculate the components Δ_g of the BV operator, note first that $g \in Z(C_G(g))$. It is easy to see that in order to calculate Δ_g, we may therefore assume that $g \in Z(G)$. The following result relates Δ_g to the construction principle of degree -1 operators described above.

Theorem 3.1. *Let $g \in Z(G)$. Then $g - 1 \in Z(kG)$ annihilates the trivial kG-module, and we have*

$$\Delta_g = D_{g-1} : H^*(G, k) \to H^{*-1}(G, k) .$$

It follows from the general properties of the construction of degree -1 operators above that Δ_g is a derivation on $H^*(G, k)$, and that Δ_g commutes with restriction, transfer, Bockstein and Steenrod operations.

4. Applications to the Lie algebra $HH^1(kG)$

Let G be a finite group and let k be a field of prime characteristic p. Then $HH^1(kG)$ is a restricted Lie algebra; that is, in addition to the Lie algebra structure, there is p-power map on $HH^1(kG)$, induced by mapping a derivation f on kG to the p-fold composition $f \circ f \circ \cdots \circ f$ (p times) of f.

Theorem 4.1 (Jacobson-Witt, 1940s)**.** *Let P be an elementary abelian p-group of order at least* 3*. Then $HH^1(kP) = \mathrm{Der}(kP)$ is a simple Lie algebra.*

These Lie algebras, called Jacobson-Witt Lie algebras, were the first non classical finite-dimensional simple Lie algebras in prime characterstic. The combined efforts of many authors have culminated in a classification of finite-dimensional simple Lie algebras in characteristic $p > 3$. See [11] and [12] for more details.

Theorem 4.2 (Fleischmann-Janiszczak-Lempken [5])**.** *Suppose that p divides the group order of G. Then $HH^1(kG)$ is nonzero.*

Perhaps surprisingly, the proof of this result requires the classification of finite simple groups via the centraliser decomposition. The nonvanishing of $HH^1(kG)$ is equivalent to the nonvanishing of $\mathrm{Hom}(C_G(g), k)$ for some $g \in G$. Since the additive group k has exponent p, this is in turn equivalent to the group theoretic condition $O^p(C_G(g)) < C_G(g)$. Using the classification of finite simple groups, it is verified in [5] that every finite group of order divisible by p has an element satisfying this condition.

It is not known which simple Lie algebras can occur as $HH^1(kG)$, or more generally, as $HH^1(B)$ for B a block of kG. The only simple Lie algebras presently known to arise in this way are the Jacobson-Witt Lie algebras. For blocks with one simple module, the following result shows that no other simple Lie algebras arise in this way.

Theorem 4.3 (Linckelmann-Rubio [7])**.** *Let B be a block of kG. Suppose that B has a unique isomorphism class of simple modules. The following are equivalent.*

(1) $HH^1(B)$ is a simple Lie algebra.
(2) $HH^1(B)$ is a Jacobson-Witt Lie algebra.
(3) B is Morita equivalent to kP for some elementary abelian p-group of order at least 3*.*

By standard block theory, statement (3) is equivalent to the statement that B is a nilpotent block with an elementary abelian defect group of order at least 3. It is not known whether $HH^1(B)$ is nonzero for any block B with a nontrivial defect group. There are a number of recent criteria for the solvability of $HH^1(B)$.

Theorem 4.4 (Eisele-Raedschelders [4], Rubio-Schroll-Solotar [13])**.** *Suppose that $p = 2$ and that B is a tame block not Morita equivalent to the Klein four group algebra. Then $HH^1(B)$ is a solvable Lie algebra.*

Theorem 4.5 (Linckelmann-Rubio [8], Rubio-Schroll-Solotar [13])**.** *Let A be a finite-dimensional split algebra over a field. Suppose that the quiver of A is a* *simple graph (that is, no loops or double arrows). Then $HH^1(A)$ is a solvable Lie algebra.*

We add to this some criteria when a finite p-group algebra yields a solvable Lie algebra (in contrast to elementary abelian groups).

Theorem 4.6. *Let P be an extraspecial finite p-group. Then $HH^1(kP)$ is a solvable Lie algebra.*

The proof combines the Siegel-Witherspoon formula and the above description of the components of the BV operator.

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