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Lefschetz Properties in Algebra, Geometry and Combinatorics (hybrid meeting)

Organized by
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Abstract. The themes of the workshop are the Weak Lefschetz Property – WLP – and the Strong Lefschetz Property – SLP. The name of these properties, referring to Artinian algebras, is motivated by the Lefschetz theory for projective manifolds, initiated by S. Lefschetz, and well established by the late 1950's. In fact, Lefschetz properties of Artinian algebras are algebraic generalizations of the Hard Lefschetz property of the cohomology ring of a smooth projective complex variety. The investigation of the Lefschetz properties of Artinian algebras was started in the mid 1980's and nowadays is a very active area of research.

Although there were limited developments on this topic in the 20th century, in the last years this topic has attracted increasing attention from mathematicians of different areas, such as commutative algebra, algebraic geometry, combinatorics, algebraic topology and representation theory. One of the main features of the WLP and the SLP is their ubiquity and the quite surprising and still not completely understood relations with other themes, including linear configurations, interpolation problems, vector bundle theory, plane partitions, splines, d-webs, differential geometry, coding theory, digital image processing, physics and the theory of statistical designs, etc. among others.

Mathematics Subject Classification (2010): 05E40, 05E45, 06A07, 06A11, 13A02, 13A50, 13D02, 13E10, 13F55, 14F99, 14J60, 14L30, 14M05, 14M10, 14M15, and 17B10.

Introduction by the Organizers

The workshop *Lefschetz Properties in Algebra, Geometry and Combinatorics*, organised by Martina Juhnke-Kubitzke (Osnabrück), Juan Migliore (Notre Dame), Rosa María Miró-Roig (Barcelona), Justyna Szpond (Cracow) was attended by 11 present and 26 remote participants from Europe, Asia and both South and North America. There was a diversity in experience level ranging from early postdocs to established, internationally recognized professors. Thanks to this diversity we were able to achieve considerable progress on topics highlighted at the workshop and to provide excellent training for early career participants. Workshop activities were divided between 20 hour or half-hour talks and research in group collaborations. Activities commenced on the first day with an in-depth discussion of problems to be studied. There were also two formal progress report sessions, apart from informal discussions held throughout the workshop.

The research groups focused their efforts on six main problems, labeled A–F and described below.

A. Jordan type and symmetric decomposition of an Artinian Gorenstein algebra with respect to an ideal. The group consists of the following members: Anthony Iarrobino, Leila Khatami, Pedro Macias Marques, Liena Colarte Gomez, and Johanna Steinmeyer. This work grew in part from discussions at several previous Lefschetz Properties conferences.

A local Artinian Gorenstein (AG) algebra $(A,\mathfrak{m})$ satisfies, A^*, the associated graded algebra, has a decreasing filtration by ideals $A^* = C(0) \supset C(1) \supset \cdots$ whose successive quotients $Q(a) = C(a)/C(a+1)$ are reflexive A^* modules [9, Theorem 1.5]. The Hilbert function decomposition has been mainly studied for non-homogenous AG algebras, with respect to the maximum ideal $\mathfrak{m}$. The ideals $C(a)$ are defined using the intersection of the $\mathfrak{m}$-adic and Loewy $(0 : \mathfrak{m}^i)$ filtrations of A^*. By generalizing the definitions of [9], or by using work of T. Harima and J. Watanabe on central simple modules [8] we can consider such decompositions with respect to any ideal J. When the ideal $J = (\ell)$ is principal and A is homogeneous we hypothesized that this concept leads to Jordan degree type, discussed by B. Costa and R. Gondim and others [5]. However, we found that it only leads to the Jordan type (Lemma 2). We omit the definition of $C^J(a)$ (see [9, 1]).

Lemma 1. [1, Theorem 2.5] Let A be a Gorenstein Artinian algebra of socle degree j, and $J \subset \mathfrak{m}$ an ideal of A, and suppose $A = R/\mathrm{Ann}\ F$, $F \in S$.

(i.) The sequence $\mathrm{Gr}_J(A) = C^J(0) \supset C^J(1) \supset \cdots \supset C^J(j) = 0$ is a descending sequence of ideals of $\mathrm{Gr}_J(A)$. If $J = (\ell)$, $\ell \in A_1$ the quotient $Q^J(0) = R/\mathrm{Ann}\ F_0$ where F_0 is the term of F of highest degree in L when F is written as $\sum F_i L^{j-i}$ with $\ell \circ L = 1$ and $F_i \in \ell^\perp$.

(ii.) Let $\langle \cdot,\cdot \rangle \to \mathsf{k}$ be an exact pairing on A and assume further that $J^{j_J} = 0 : \mathfrak{m}_A \cong \mathsf{k}$. Then $\langle \cdot,\cdot \rangle$ induces an exact pairing $\langle \cdot,\cdot \rangle : Q^J(a)_v \times Q^J(a)_{j_J-a-v} \to \mathsf{k}$.

Example. Take $A = \mathsf{k}\{x,y\}/I$, $I = (xy, y^2 - x^3) = \mathrm{Ann}\ F$, $F = Y^2 + X^3$, Hilbert function $H(A) = (1,2,1,1)$ and Jordan partition $P_y = (3,1,1)$: as a vector space

$A = \langle 1, x, y, x^2, x^3\rangle$ and we have that the y socle degree can be read from F and is $j_y = 2$, the degree of the highest Y term in F, as $X \in y^{\perp}$. So $Q^y(0)^{\vee} = \langle 1, Y, F\rangle$. The next term in F has degree 0 in Y, so two less, giving that $Q^y(2)_0^{\vee} = \langle X, X^2\rangle$, the derivations of the term X^2 that are linearly independent from $Q^y(0)^{\vee}$. In $\mathrm{Gr}_y(A)$ we have $Q^y(0) = \langle 1, y, y^2 = x^3\rangle$ and $H_y(Q(0) = (1_0, 1_1, 1_2)$, $Q^y(1) = 0$, but $Q^y(2)_0 = \langle x, x^2\rangle$, with $H_y(Q^y(2)) = (2_0)$, giving $H_y(\mathrm{Gr}_y(A)) = (1, 1, 1) + (2, 0, 0) = (3, 1, 1)$ which is the conjugate of $P_{A^*,y} = (3, 1, 1)$ arising from a y-symmetric decomposition of $\mathrm{Gr}_y(A)$.

Lemma 2. Each Jordan ℓ-string of any AG algebra A begins in degree zero. The Hilbert functions in the $Q^{\ell}(a)$ decomposition are entirely determined by $P_{\ell,A}$ the Jordan type. Each part p_i determines a string $(p_i)_0$ in $Q^{\ell}(j+1-p_i)$, where $j = j_{\ell}$ is the ℓ-socle degree. So each $H^{\ell}(Q(a))$ is constant, like $Q(0)$ above, of the form

$$H^{\ell}(Q(a)) = ((n_a)_0, \dots, (n_a)_{j-a}),$$

where n_a is the number of parts equal to $j_{\ell} + 1 - a$ in the Jordan type P_{ℓ}.

Proof. Let t be the number of strings (parts of P_{ℓ}). Consider $m_{\ell} : A \to A$. It has kernel $(0 : (\ell))$ and image (ℓ), but $\dim_k(0 : \ell)$ just counts the number of strings, so $\dim_k A/\ell A = \dim(0 : \ell) = \dim A - t$, so each string begins in ℓ−degree zero (there are no unexpected relations). $\square$

Double stratification, $Q^{J,K}(a, b)$ decomposition. We considered symmetric decomposition for two ideals together [1, Definition 2.13, Lemma 2.14]. We concluded there should be two different socle degrees, j_J and j_K. But we found that this was problematic when $j_J \neq j_K$. It seems that the Definition needs a modification.

Example. $F = X^2Y^3 + XY^4$ (two monomials of the same $\mathfrak{m}$-degree). Here, letting $A = R/\mathrm{Ann}\, F$, we have $|A| = 12$, $j_{(x)} = 2$, $j_{(y)} = 4$,

$$I = \mathrm{Ann}\, F = (x^3, y^4 - xy^3 + x^2y^2), \quad H_{\mathfrak{m}}(F) = (1, 2, 3, 3, 2, 1).$$

We take $J = (x)$, $K = (y)$. We have $P_x = (3, 3, 3, 3)$, $P_x^{\vee} = (4, 4, 4)$; $P_y = (5, 5, 2)$, $P_y^{\vee} = (3, 3, 2, 2, 2)$. We found that $Q(0, 0)$ and $Q(0, 1)$ strata had some expected behavior, but not a predicted duality. But there were extra terms that seemed spurious in $Q(0, 2)$ and $Q(0, 3)$ – they were linear combinations of terms in $Q(0, 0)$ and $Q(0, 1)$. On the other hand, when we considered $F = X^3Y Z + X^2Y^3$ we found no spurious terms and the projections to the x, y directions gave the Q^x and Q^y single decompositions.

Question 1A. Can we redefine the $Q^{J,K}$ stratification so as to give the expected dimension, and also maintain the connection to the single ideal $Q^J(a)$, $Q^K(a)$ stratifications? Take J, K principal and compare with Jordan types?

Question 1B. By considering the $Q^{\ell,\mathfrak{m}}$ decomposition, can we define a Jordan order type for non-graded AG algebras?

Question 2. Consider $F = \mu_1 + \mu_2$ binomial, say in X, Y. Can we determine the $Q^{x,y}$ decomposition as combinatorial invariants depending on a poset formed from μ_1, μ_2? What changes if we have binomials of different degrees or in 3 variables?

Question 3. Consider a minimal resolution of an AG $A = R/I$; can we find some stratification on the syzygy modules, related to $Q(a)$ decomposition?
Question 4. What can we say about the Jordan type of A and $Q(0)$: can A be strong Lefschetz but $Q(0)$ not be strong Lefschetz? (The opposite can occur).

B. Companion varieties. Let Z be a finite set of points in the projective space $\mathbb{P}^N$ admitting an unexpected hypersurface, see [4] for an introduction to this circle of ideas. It was realized in [7] that if for a given point $P = (p_0 : \ldots : p_N)$ the set Z admits a unique unexpected hypersurface of degree d, there is an associated bi-homogeneous polynomial $F(p, x)$. This polynomial, written in a basis $g_0, \ldots, g_M$ of the graded part $[I(Z)]_d$ has the form

$$F(p, x) = \sum_{i=0}^{M} h_i(p) g_i(x).$$

Polynomials $g_0, \ldots, g_M$ define, under favorable circumstances, a rational map $\mathbb{P}^N \dashrightarrow \mathbb{P}^M$, whose image $X(Z)$ has interesting geometric properties.

Szpond realized in [12] that also the image $Y(Z)$ of the rational map defined by $h_0, \ldots, h_M$ has interesting properties. She called $X(Z)$ and $Y(Z)$ *companion varieties* associated to Z. In [12] they were studied in detail for the case when Z is the root system B_3. The working group, consisting of Roberta Di Genarro, Giovanna Illardi, Rosa Maria Miró Roig, Tomasz Szemberg and Justyna Szpond extended this study to configurations of points defined by other root systems and by the duals of Fermat-type arrangements of lines. The works are continued and a joint publication is expected as an outcome.

C. Proper intersections. This group (Brian Harbourne, Giuseppe Favacchio, Emilia Mezzetti, Juan Migliore, Tomasz Szemberg, Justyna Szpond and Martin Vodička) worked on problems motivated by [2]. Chiantini and Migliore attempted to classify sets of points in $\mathbb{P}^3$, spanning the whole $\mathbb{P}^3$, which projections to a general plane in $\mathbb{P}^3$ are complete intersections. They showed that sets of intersection points of lines (grids) on a smooth quadric surface in $\mathbb{P}^3$ have this property. The appendix to [2] contains, however, an example of points which don't come this way, yet their projection to a general plane is a complete intersection. Shortly before the workshop, Pokora, Szpond and Szemberg [11] found another series of examples.

The works of the group was motivated by the following two problems:

(1) Is there a series of examples of non-grids in $\mathbb{P}^3$ with an unbounded number of points?
(2) Are there similar examples in higher dimensional projective spaces?

A possible path of investigations in higher dimensional projective spaces is related to proper intersections. We say that two subvarieties X and Y in the projective space $\mathbb{P}^N$ form a proper intersection, if their dimensions are complementary, i.e., $\dim(X) + \dim(Y) = N$ and they intersect in $\dim(X) \cdot \dim(Y)$ points. Then, the original question, can be stated in a slightly milder version: are there subvarieties in $\mathbb{P}^{N+1}$, which project to proper intersections in $\mathbb{P}^3$?

During the workshop partial answers to above problems have been found and the group stays in contact pushing the research further.

D. Lefschetz properties and monomial algebras. The group consists of the following members: Akihito Wachi, Junzo Watanabe, Samuel Lundqvist, Chris McDaniel, Nasrin Altafi, Martí Salat Moltó.

A result by Wiebe [13] says that if $R/\operatorname{ini}(I)$ has the WLP (SLP), then so does R/I. Thus it makes sense to find techniques to decide whether a monomial algebra has the WLP (SLP) or not.

The group has been exploiting an idea[1] which can be used to draw conclusions about the presence of the WLP (SLP) of a monomial algebra $A := R/I$ by looking at smaller pieces of A. More precisely, let m be a monomial and let $I_0 := I + (m)$ and $I_1 := I : (m)$. If $A_0 := R/I_0$ and $A_1 := R/I_1$ have the WLP (SLP), and also fulfills a "symmetricish" condition on the Hilbert series, then so does A. When this condition is fulfilled and A_0 and A_1 have the desired Lefschetz property, we say that A is glued from A_0 and A_1. The process might of course be continued, constructing A_{00} and A_{01} and so on. The hope is that we will eventually end of with something nice which we know has the WLP (SLP), say a CI or something in codim 2.

Our main focus have been on algebras with a symmetric Hilbert function. In this case our symmetricish condition can be split into two classes; "center-to-center" and "slightly shifted".

Results:

- The Gorenstein algebra with dual generator $x^2y^2 + z^4$ has the SLP and a symmetric Hilbert series, but does not decompose center-to-center, so Question 1 in the paper in the second footnote has a negative answer. But it does decompose slightly shifted, and its Gin does decompose center-to-center.
- There is a connection to the connected sum construction of Gorenstein algebras. There are indications by means of examples that the method we study is stronger than the connected sum construction.
- We have a condition in terms of the socle for when a symmetric algebra with the SLP can be decomposed center-to-center, but it is unclear how to use the condition.

Questions:

- Does a Gin coming from an SLP algebra with a symmetric series always decompose "center-to-center?
- Can all algebras in three variables with symmetric Hilbert series and the SLP be decomposed symmetricish?
- Does a Gorenstein algebra with dual generator $\sum_i \ell_i^d$ with the SLP decompose center-to-center by always choosing the variables as m in the operations $A_0 := A + m$ and $A_1 := A : m$?

[1] See https://arxiv.org/pdf/2006.14453.pdf

- Under which conditions are $R/A_1, R/A_{01}, R/A_{001}\dots$ isomorphic to $k[x]/x^d$? (Different d's.)
- What can be said about the intersection of $CI's$? When does the intersection give a symmetric HS?
- Can our method be used to provide a new proof of the fact that for $n = 2$, all artinian quotients have the SLP?
- When does a lex segment have the WLP (SLP)?

E. Lefschetz properties and toric varieties. This group consists of Karim Adiprasito, Eran Nevo and Larry Smith. They discussed two problems:

- The adaption of biased pairing theory, and thereby construction of Lefschetz elements, for toric varieties with a free involution. It seems that even in this case, significant new ideas are needed.
- Based on an idea of Smith, we investigated the relation between the Eisenbud-Levine-Khimshiashvili to maximal totally anisotropic subspaces. After understanding that the formula instead describes maximal totally isotropic subspaces, we discovered a relation to a question of Charney and Davis concerning the signature of certain toric varieties.

  Unfortunately, since Eisenbud-Levine-Khimshiashvilionly describes the absolute value of the signature, this seems to be a dead end.

F. Sym(∞)**-Lefschetz properties**. The group consists of the following members: Mats Boij, Martina Juhnke-Kubitzke, Uwe Nagel, Piotr Pokora, Tim Römer, and Larry Smith.

We studied Sym(∞)-Lefschetz properties in the following setting. Let $I_n \subset S_n := \mathbb{K}[x_1, ..., x_n]$ be a Sym(n)-invariant monomial ideal. We also assume that

$$\langle \mathrm{Sym}(m) I_n R_m \rangle \subseteq I_m \text{ with } m \geq n.$$

Because of this, there will be a limiting object $I_\infty = \bigcup_{n\geq 0} I_n \subseteq S_\infty = \bigcup_{n\geq 0} S_n$. (See e.g. [6].)

In general, we would be interested to see how the Lefschetz properties with respect to a linear form for S_n/I_n are reflected in the limiting object S_∞/I_∞. Here we have concentrated on the case of ideals generated by monomials. Since we have Sym(n)-symmetry we get that the ideals we need to consider are built up from blocks indexed by partitions of d when we consider ideals generated by monomials of degree d.

First of all, we wanted to use representation theory in order to find obstructions to Lefschetz properties.

Example 1. *(Togliatti) For $d = 3$ and $n = 3$, $I_3 = \langle x_1^3, x_2^3, x_3^3, x_1x_2x_3\rangle$ corresponds to the sum of two ideal $I_{(3)} + I_{(1,1,1)}$ – the failure of the Weak Lefschetz Property in degree 2 to 3 that is known for $n = 3$ generalizes to all $n \geq 3$ and the representation theoretic obstruction is the same for all $n \geq 3$. In particular, we have two copies of the trivial representation in degree two and only one in degree three. Observe that since we have the torus action, we only need to consider the symmetric linear* *form which makes the multiplication map equivariant and Schur's lemma shows that it cannot be injective.*

The representation stability that we see from the example has been proven to hold in general for FI-modules by Church, Ellenberg and Farb [3] and this will allow us to provide a large family of examples generalizing the Togliatti example above.

In order to get positive results we would need some kind of maximal rank property that is valid within each representation type corresponding to a specific Specht-module S_λ. This turns out to be false in general, as we can see from examples where bijectity fails in the symmetric part even though we have the same multiplicity for the trivial representation in two consecutive degrees. These examples all have in common that the failure occurs in degrees less than the degrees of the generators of I_n. For multiplication maps in higher degrees there is still hope for a maximal rank property to hold.

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Workshop (hybrid meeting): Lefschetz Properties in Algebra, Geometry and Combinatorics

Table of Contents

Abstracts

Some problems in Lefschetz theory

Karim Adiprasito

I presented some problems in Lefschetz theory. First, I discussed the problem of describing the space of Artinian reductions that are Lefschetz, for instance among those toric varieties with fixed equivariant cohomology. As observed in earlier work of the author, one of the natural sufficient conditions for the Lefschetz property are based on totally isotropy of monomial subspaces under the Hodge-Riemann pairing, and this leaves several problems, concerning in particular varieties with group actions that prevent Artinian reductions in sufficient symmetry.

On the other hand it is equally an open problem for general position linear systems, and in particular it is an open question whether for neighborly spheres, every linear system of parameters is Lefschetz. We presented some evidence towards this conjecture, by showing it true when restricted to low-degree strong Lefschetz properties.

The second problem is equally related to the construction of Lefschetz elements. Indeed, assume a space of multilinear forms. We discussed how much the Schmidt norm of a generic element, and the Schmidt norm of the subspace can differ, though in recent work with Kazhdan and Ziegler, we were able to demonstrate a linear bound in fixed degree.

Using Jordan type to determine irreducible components of families of local Artinian Gorenstein algebras of given Hilbert function

Pedro Macias Marques
(joint work with Anthony Iarrobino)

We study local Artinian Gorenstein (AG) algebras and consider the set of Jordan types of elements of the maximal ideal, i.e. the partition giving the Jordan blocks of the respective multiplication map.

Let $A = R/I$ be a local AG algebra, quotient of $R = \mathsf{k}\{x_1, \dots, x_r\}$, the regular local ring in r variables over an infinite field k. We denote by $\mathfrak{m}$ the maximal ideal of A, and by j its socle degree, i.e. the unique integer satisfying $\mathfrak{m}^j \neq 0 = \mathfrak{m}^{j+1}$. The Hilbert function of A is $H(A) = (h_0, \dots, h_j)$, where $h_i = \dim_{\mathsf{k}} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. We term *Gorenstein sequence* a sequence of integers that is the Hilbert function of an AG algebra. If T is a Gorenstein sequence, we let $\mathrm{Gor}(T)$ be the the reduced variety whose closed points are the Gorenstein quotients A of R having Hilbert function T. We adress the following question:

Conjecture 1. *Given a Gorenstein sequence T, is the family* $\mathrm{Gor}(T)$ *an irreducible algebraic set? If not, how many irreducible components does it have?*

A central tool in the study of AG algebras is their dual generator, given by Macaulay's inverse systems. Let $S = \mathsf{k}_{DP}[X_1, \dots, X_r]$ be the divided power ring in r variables and let R act on S by contraction, i.e.

$$x^\alpha \circ X^\beta = \begin{cases} X^{\beta-\alpha}, & \text{if } \beta \geq \alpha \\ 0, & \text{otherwise.} \end{cases}$$

Then for any AG algebra $A = R/I$ there is an element $f \in S$ such that

$$I = \mathrm{Ann} f = \{h \in R : h \circ f = 0\}.$$

The associated graded algebra of an AG algebra A is $A^* = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$. Iarrobino showed [3, Theorem 1] that A^* admits a filtration by ideals $C(a)$, $a \geq 0$, whose successive quotients $Q(a)$ are reflexive modules over A^* (see also [4] for a further discussion). As a consequence, the Hilbert functions of the modules $Q(a)$ give a symmetric decomposition $\mathcal{D}$ of $H(A)$. We denote by $\mathrm{Gor}_{\mathcal{D}}(T)$ the subfamily of $\mathrm{Gor}(T)$ of those algebras whose symmetric decomposition is $\mathcal{D}$. Iarrobino also showed [4, Lemma 4.1A] that the dimensions $N_{i,b} = \dim_{\mathsf{k}} \mathfrak{m}^i/\big(\mathfrak{m}^i \cap (0 : \mathfrak{m}^b)\big)$, that can be read from the symmetric decomposition of $H(A)$, satisfy a semicontinuity property along flat families of AG algebras having fixed Hilbert function.

Given an element $\ell \in \mathfrak{m}$, we define the *Jordan type* P_ℓ as the partition of $n = \dim_{\mathsf{k}} A$ giving the Jordan block decomposition of the nilpotent multiplication map $m_\ell : A \to A$, $m_\ell(a) = \ell \cdot a$. The *generic Jordan type* of A is the Jordan type of ℓ in an open dense subset of $\mathfrak{m}$.

Given two partitions P and P' of n, with $P = (p_1, p_2, \dots, p_s)$, $p_1 \geq p_2 \geq \cdots \geq p_s$ and $P' = (p'_1, p'_2, \dots, p'_t)$, $p'_1 \geq p'_2 \geq \cdots \geq p'_t$, we say that $P \leq P'$ in the *dominance* order if for each $i \in \{1, \dots, \min(s,t)\}$ we have

$$\sum_{k=1}^{i} p_k \leq \sum_{k=1}^{i} p'_k.$$

Harima and Watanabe [2] have shown that a linear element ℓ in a graded Artinian algebra A is a weak Lefschetz element if and only if the number of parts in its Jordan type is the Sperner number of the Hilbert function of A, and that ℓ is a strong Lefschetz element if and only if its Jordan type is the conjugate partition of the Hilbert function of A. Together with Maeno, Morita, Numata, and Wachi, these authors have shown [1] that when the Hilbert function of A is unimodal, its conjugate partition is an upper bound for the Jordan type of ℓ. In a joint work with Chris McDaniel and Tony Iarrobino [5], we have extended this result to the case of non-standard graded Artinian algebras and modules over local Artinian algebras. Given this bound in general, we may extend the notions of weak and strong Lefschetz properties to local Artinain algebras, by saying that an Artinain algebra A satisfies the weak Lefschetz property if the number of parts in its generic Jordan type is the Sperner number of the Hilbert function, and that it satisfies the strong Lefschetz property if its generic Jordan type is the conjugate partition of the Hilbert function.

Let $(A_t)_{t\in T}$ be a flat family of AG algebras over an irreducible parameter space T. Let P_T be the generic Jordan type of A_t, for t in an open dense subset of T. Then for any $t_0 \in T$, the generic Jordan type $P_{A_{t_0}}$ of the algebra A_{t_0} satisfies $P_T \geq P_{A_{t_0}}$, in the dominance order.

We use the semicontinuty properties of the dimensions $N_{i,b}$ and Jordan types to determine irreducible components in $\mathrm{Gor}(T)$. In particular, we show:

Theorem 2. *The Gorenstein sequence $H = (1,3,4,4,3,2,1)$ has three symmetric decompositions.*

(1) (a) A generic algebra in $\mathrm{Gor}_{\mathcal{D}_1}(H)$,

$$\mathcal{D}_1 = \big(H(0) = (1,2,3,4,3,2,1),\, H(1) = H(2) = 0,\, H(3) = (0,1,1,0)\big)$$

will have generic Jordan type $P_1 = (7,5,3,2,1)$.

(b) A generic algebra in $\mathrm{Gor}_{\mathcal{D}_2}(H)$

$$\mathcal{D}_2 = \big(H(0) = (1,2,3,3,3,2,1),\, H(1) = 0,\, H(2) = (0,1,1,1,0)\big)$$

will have generic Jordan type $P_2 = (7,5,3,3)$.

(c) Finally, a generic algebra in $\mathrm{Gor}_{\mathcal{D}_3}(H)$,

$$\mathcal{D}_3 = \big(H(0) = (1,2,2,2,2,2,1),\, H(1) = (0,1,2,2,1,0)\big),$$

will have strong Lefschetz Jordan type $P_3 = (7,5,4,2)$.

There are no further symmetric decompositions of H.

(2) These three decompositions satisfy $N_{2,3}(\mathcal{D}_1) \geq N_{2,3}(\mathcal{D}_2) \geq N_{2,3}(\mathcal{D}_3)$; as a consequence

$$\overline{\mathrm{Gor}_{\mathcal{D}_3}(H)} \cap \big(\mathrm{Gor}_{\mathcal{D}_2}(H) \cup \mathrm{Gor}_{\mathcal{D}_1}(H)\big) = \emptyset$$

and

$$\overline{\mathrm{Gor}_{\mathcal{D}_2}(H)} \cap \mathrm{Gor}_{\mathcal{D}_1}(H) = \emptyset.$$

(3) In the dominance partial order, $P_3 \geq P_2 \geq P_1$. We have that no subfamily $\mathcal{A}(W) \subset \mathrm{Gor}_{\mathcal{D}_i}(H)$ can specialize to an element $\mathcal{A}(w_0)$ of $\mathrm{Gor}_{\mathcal{D}_j}(H)$ having generic Jordan type P_j when $i < j$.

(4) Each $\mathrm{Gor}_{\mathcal{D}_i}(H)$, $i \in \{1,2,3\}$, is an irreducible component of $\mathrm{Gor}(H)$.

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Expect unexpected

Tomasz Szemberg
(joint work with Piotr Pokora, Justyna Szpond)

This presentation is based on a joint work with Piotr Pokora and Justyna Szpond carried out in the period September 13-26, 2020 at the MFO, while we were Research Fellows there. Our MFO preprint [11].

1. Introduction

Finite sets of points in the complex projective plane $\mathbb{P}^2$ determined as intersection points of line arrangements or as the duals of lines forming the arrangement, or both, exhibited a number of properties interesting from the point of view of various packages of problems studied currently in algebraic geometry and commutative algebra. Arrangements of lines defined by finite reflection groups seem of particular interest. Most prominently, they appear in recent works on the containment problem between symbolic and ordinary powers of homogeneous ideals (see e.g. [8], [2]) and in works revolving around the Bounded Negativity Conjecture (see e.g. [3], [1]).

In our work we are interested in properties of Z_{60}, the set points in $\mathbb{P}^3$ determined by the reflection group labeled as G_{31} in the Shephard-Todd classification [12]. It turns out that this configuration of points was known already to Felix Klein, who studied it in connection with his early works on Icosahedron [10]. Recently, the 60 points forming the configuration have been rediscovered by Ivan Cheltsov and Konstanti Sharamov [4]. They showed that they are a union of three orbits of a cyclic extension of a finite Heisenberg group $H_{2,2}$ acting on the projective space $\mathbb{P}^3$. As this group is much smaller than G_{31}, it makes explicit calculations much more feasible.

2. Unexpected hypersurfaces

Building upon an example due to Di Gennaro, Illardi and Vallès [7], Cook II, Harbourne, Migliore and Nadel introduced in [6] the notion of unexpected curves, which in the subsequent article by the last three authors joined by Teitler [9] has been extended to hypersurfaces of arbitrary dimension. Roughly speaking, a set Z in a projective space admits an unexpected hypersurface of degree d if the naive count of conditions imposed by a fat point of multiplicity $m \geq 2$ on the linear system of hypersurfaces of degree d fails, i.e., if there are more such hypersurfaces than expected. It is worth to point out that it never happens if Z is an empty set or if Z consists of general points. It was in fact quite surprising to realize that there are indeed sets Z admitting unexpected hypersurfaces.

Our main result in this direction are the following two statements to the effect that Z_{60} admits unexpected surfaces in two unrelated ways. Such a phenomena has been not observed before.

Theorem 1. *The set Z_{60} admits, for a general point P in $\mathbb{P}^3$ a unique unexpected surface of degree 6 with a points of multiplicity 6 at P.*

Theorem 2. *Let P, Q_1, Q_2 be general points in $\mathbb{P}^3$. Then there exists a unique unexpected surface of degree* 6 *vanishing in all points of Z_{60} with a point of multiplicity* 4 *at P and multiplicity* 2 *at Q_1 and Q_2.*

3. Sets of points with the geproci property

Two years ago, in another workshop from the series of workshops on Lefschetz properties, Luca Chiantini and Juan Migliore noticed that there are sets Z of points in $\mathbb{P}^3$ spanning the whole space, such that their generic projection to a plane is a complete intersection. If this happens for a set Z, we say simply that the set Z has the *geproci* property. In [5] the authors identified a natural class of sets, which they call grids, with the geproci property. The appendix to their article contains an example of a set Z with the geproci property which is not a grid. Such sets seem very hard to come by and it is natural to wonder if they can be classified in some way. We show that Z_{60} and some of its subsets have the geproci property.

Theorem 3. *The set Z_{60} has the geproci property. More precisely, its general projection to $\mathbb{P}^2$ is a complete intersection of curves of degree* 6 *and* 10.

We show that whereas the curve of degree 6 is irreducible, the curve of degree 10 can be constructed taking projections of 10 lines, naturally associated to the Z_{60} configuration.

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Gorenstein algebras generated by determinants

Akihito Wachi

(joint work with Takahiro Nagaoka)

In this talk, it is shown that Artinian Gorenstein algebras generated by powers of the basic relative invariants of regular prehomogeneous vector spaces of commutative parabolic type have the strong Lefschetz property (SLP). See Nagaoka-Yazawa [3] for a relation to Hodge-Riemann relation, and see also Maeno-Numata [2] for a related work about Gorenstein algebras associated to matroids.

Set $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$, and let F be a homogeneous polynomial in R. Define an ideal $\mathrm{Ann}_R(F)$ of R as

$$\mathrm{Ann}_R(F) = \{p \in R \mid p \circ F = 0\},$$

where the circle ($\circ$) means the differentiation, that is, $x_i \circ F = \frac{\partial}{\partial x_i}(F)$, for example. Then it is known that the quotient algebra $R/\mathrm{Ann}_R(F)$ is a standard graded Artinian Gorenstein algebra, which we call *the Artinian Gorenstein algebra generated by a polynomial F*.

For example, if $F = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, then $\mathrm{Ann}_R(F) = (x_1^{a_1+1}, x_2^{a_2+1}, \ldots, x_n^{a_n+1})$ and the algebra $R/\mathrm{Ann}_R(F)$ generated by F is a monomial complete intersection, and it has the SLP. But, in general, the SLP of the algebra generated by a polynomial is difficult to determine, and the SLP is determined only for limited cases.

As another example, which is part of the main result, let $R = \mathbb{C}[x_{ij} \mid 1 \leq i, j \leq n]$, and $F = \det(x_{ij})$. Then the algebra $R/\mathrm{Ann}_R(F)$ generated by F turns out to have the SLP. Furthermore, we have the following theorem, which is the main result of this talk.

Theorem 1. *If R and F are in the following table, and t is a non-negative integer, then the Artinian Gorenstein algebra $R/\mathrm{Ann}_R(F^t)$ generated by F^t has the SLP.*

In the first three cases, note that R is the coordinate ring of $\mathrm{Sym}(n,\mathbb{C})$ *(the space of symmetric matrices of size n),* $\mathrm{Mat}(n,\mathbb{C})$ *(the space of square matrices of size n), or* $\mathrm{Alt}(n,\mathbb{C})$ *(the space of alternating matrices of size n).*

type	R	F
(C_n, n)	$\mathbb{C}[x_{ij} \mid 1 \le i,j \le n]/(x_{ij} - x_{ji})$	$\det(x_{ij})$
(A_{2n-1}, n)	$\mathbb{C}[x_{ij} \mid 1 \le i,j \le n]$	$\det(x_{ij})$
(D_n, n)	$\mathbb{C}[x_{ij} \mid 1 \le i,j \le n]/(x_{ij} + x_{ji})$	$\mathrm{Pf}(x_{ij})$ *(n: even)*
$(B_m, 1)$, $(D_m, 1)$	$\mathbb{C}[x_1, x_2, \ldots, x_n]$	$x_1^2 + x_2^2 + \cdots + x_n^2$
$(E_7, 7)$	$\mathbb{C}[$*27 variables*$]$	*a polynomial of degree 3*

'Type' is of regular prehomogeneous vector spaces of commutative parabolic type, which is explained in the next remark.

The set of Lefschetz elements coincides with the open orbit of the prehomogeneous vector space. In particular, it is independent of t.

Remark 2. *If a Lie group K acts on a vector space V, and there is a Zariski open K-orbit on V, then the pair (K, V) is called a* prehomogeneous vector space. *See Kimura* [1] *for the details.*

Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$, $\mathfrak{n}^+$ the nilpotent radical of $\mathfrak{p}$, and $\mathfrak{k}$ a Levi subalgebra of $\mathfrak{p}$. Denote by K a complex Lie group whose Lie algebra is $\mathfrak{k}$, then K acts on $\mathfrak{n}^+$ by the adjoint action. If $\mathfrak{n}^+$ is a commutative Lie algebra, then $\mathfrak{p}$ is automatically a maximal parabolic subalgebra of $\mathfrak{g}$, and it is known that $(K, \mathfrak{n}^+)$ becomes a prehomogeneous vector space. This prehomogeneous vector space is called a prehomogeneous vector space of commutative parabolic type.

For a prehomogeneous vector space (K, V) (assume V is $\mathbb{C}$-vector space), a polynomial $F \in \mathbb{C}[V]$ is called a relative invariant, *if there exists a group character $\chi : K \to \mathbb{C}^\times$ such that $F(kv) = \chi(k)F(v)$ for any $k \in K$ and $v \in V$.*

Prehomogeneous vector spaces of commutative parabolic type are classified, and those having relative invariants are also classified into six cases (A_{2n-1}, n), (C_n, n), (D_n, n) (n: even), $(B_m, 1)$, $(D_m, 1)$, and $(E_7, 7)$. See Rubenthaler [4] *for the details. In this notation, the first entry is the type of the simple Lie algebra $\mathfrak{g}$, and the second entry is the index of the simple root which determines the maximal parabolic subalgebra $\mathfrak{p}$. In each case, there exists a unique irreducible relative invariant up to scaling, which is called* the basic relative invariant. *The polynomials F in the table of the theorem are the basic relative invariants. These six cases are called* regular *prehomogeneous vector spaces of commutative parabolic type. The definition of 'regular' is not explained here, but a prehomogeneous vector space is regular if and only if there is a relative invariant and its Hessian is not identically zero.*

We give an outline of the proof of the theorem. The proof is independent of six cases.

Lemma 3 (condition of the SLP in terms of $\mathfrak{sl}_2$)**.** *Let $R = \mathbb{C}[x_1, x_2, \dots, x_n]$, and I be a homogeneous ideal of R. Suppose that R/I is an Artinian algebra with a symmetric Hilbert function. Then (1) and (2) are equivalent.*

(1) R/I has the SLP.

(2) There is an action of $\mathfrak{sl}_2 = \langle x, y, h\rangle$ such that

(a) the eigenspaces of h coincide with the homogeneous components of R/I, and

(b) the action of y gives the multiplication map of a linear form in R.

In view of the lemma, it suffices to find a 'nice' $\mathfrak{sl}_2$-triple. But it can not be found in $\mathfrak{k}$, since K-action on $\mathbb{C}[\mathfrak{n}^+]$ does not change degrees of monomials, while the action of y in the lemma changes degrees. So the key idea is to find a 'nice' $\mathfrak{sl}_2$-triple in a Lie algebra larger than $\mathfrak{k}$. In fact, it is found in $\mathfrak{g}$.

For a Lie algebra homomorphism $\lambda : \mathfrak{p} \to \mathbb{C}$, define the *generalized Verma module* as

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda \quad (\mathbb{C}_\lambda \text{ is the representation space of } \lambda),$$

where $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$. Then we have $M(\lambda) \simeq U(\mathfrak{n}^-) \simeq S(\mathfrak{n}^-) \simeq \mathbb{C}[\mathfrak{n}^+]$ as vector spaces, where $U(\mathfrak{n}^-)$ is the universal enveloping algebra of $\mathfrak{n}^-$, which is the opposite Lie algebra of $\mathfrak{n}^+$, and $S(\mathfrak{n}^-)$ denotes the symmetric algebra of $\mathfrak{n}^-$. The first isomorphism is by the definition of $M(\lambda)$, the second one is by the commutativity of $\mathfrak{n}^-$, and the third one is by the duality of $\mathfrak{n}^+$ and $\mathfrak{n}^-$ under the Killing form. $M(\lambda)$ is, of course, a left $\mathfrak{g}$-module, and therefore $\mathfrak{g}$ acts on $\mathbb{C}[\mathfrak{n}^+]$. So if $\mathfrak{sl}_2$-triple is given in $\mathfrak{g}$, then it acts on $\mathbb{C}[\mathfrak{n}^+]$.

Finally the 'nice' $\mathfrak{sl}_2$-triple can be found in $\mathfrak{g}$, which satisfies the condition of the lemma. In the following, we give an example of $\mathfrak{sl}_2$-triple etc.

Example 4. *For the type (C_n, n),*

$$\mathfrak{g} = \mathfrak{sp}_n = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \;\middle|\; A \in \mathfrak{gl}_n,\ B, C \in \mathrm{Sym}_n(\mathbb{C}) \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathfrak{g} \right\}, \qquad \mathfrak{k} = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \in \mathfrak{g} \right\} \simeq \mathfrak{gl}_n,$$

$$\mathfrak{n}^+ = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \right\} \simeq \mathrm{Sym}_n(\mathbb{C}), \qquad \mathfrak{n}^- = \left\{ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \in \mathfrak{g} \right\} \simeq \mathrm{Sym}_n^*(\mathbb{C}),$$

and $(K, \mathfrak{n}^+)$ is a regular prehomogeneous vector space of commutative parabolic type. The basic relative invariant is

$$F = \det(X) \qquad (X \in \mathfrak{n}^+ \simeq \mathrm{Sym}_n(\mathbb{C})).$$

The 'nice' $\mathfrak{sl}_2$-triple can be taken as

$$x = \begin{pmatrix} 0 & 1_n \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1_n & 0 \end{pmatrix}, h = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix} \in \mathfrak{sp}_n,$$

where 1_n denotes the unit matrix of size n. Then y is in the open K-orbit on $\mathfrak{n}^-$, and it corresponds to a Lefschetz element $x_{11} + x_{22} + \cdots + x_{nn}$. The open orbit is *the set of non-singular symmetric matrices of size n, and thus this set is equal to the set of Lefschetz elements.*

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The Jordan degree type of codimension three Artinian Gorenstein algebras, and punctual schemes in $\mathbb{P}^2$.

Anthony Iarrobino

(joint work with Nancy Abdallah, Nasrin Altafi, Leila Khatami, Alexandra Seceleanu, and Joachim Yaméogo)

Let ℓ be a linear, non-unit element of a graded Artinian algebra A. The Jordan degree type $S(A,\ell)$ of the multiplication map $m_\ell : A \to A$ is a sequence of ordered pairs giving the lengths of the cyclic $k[\ell]$ modules decomposing A, and the degrees of their generators [10, §2.6]. It has equivalent information to the central simple modules for (A,ℓ) of T. Harima and J. Watanabe [7, 8]. When A is Gorenstein $S(A,\ell)$ enjoys a symmetry, illustrated by the string diagrams of B. Costa and R. Gondim [5].

The morphism π. If the Hilbert function T of a codimension three Artinian Gorenstein (AG) algebra contains (s,s,s), where s is the Sperner number – the maximum value of T – then V. Kanev and the author showed that there is a morphism

$$\pi : \mathrm{Gor}(T) \to \mathrm{Hilb}^s(\mathbb{P}^2)$$

mapping an AG quotient $A = R/I$ of $R = k[x,y,z]$ having Hilbert function T, to the punctual Hilbert scheme $\mathrm{Hilb}^s(\mathbb{P}^2)$ parametrizing length-s subschemes of the projective space $\mathbb{P}^2$. Here $\pi(A)$ is a scheme whose one-dimensional coordinate algebra B satisfies $B = R/J$ where J is the ideal of the early generators of I. Using this, we can see how the pairs in $S(A,\ell)$ fall into three groups, having to do with a newly-defined Jordan degree type $S(B,\ell)$ for a one-dimensional ring. This viewpoint leads to restrictions on the Jordan degree types S that may occur for $A \in \mathrm{Gor}(T)$, and any $\ell \in R_1$.

The morphism π has been studied by J.O. Kleppe [11], and in special cases where the image $\pi(A)$ is smooth [1, 3]. The Hilbert scheme of length s punctual schemes in $\mathbb{P}^2$ has an extensive literature, and structure theorems for s small are known. Also, Macaulay duality has been studied for dimension one CM rings B [6]. These are ingredients that can enter into the study of JDT for codimension three Artinian Gorenstein algebras.

We are interested in all linear forms ℓ, as studied analogously in [2] in height two. We study the case $T = (1, 3, 3, 1)$ where there are seven JDT, and $T = (1, 3, 3, 3, 1)$ where there are eight: the 8 possible JDT for $T = (1, 3^k, 1)$ for any $k \geq 3$ will correspond to those for $k = 3$. We can verify the results in this case using the rank matrices as introduced by N. Altafi in [1].

A challenge to extending this approach in general is that the information in the image $\pi(A)$ is not enough to determine the Jordan degree type of A uniquely: rather the socle-degree j Artinian Gorenstein algebras in the fibre of the mapping to $B = R/J$ correspond to elements F of degree j in the perpendicular set of forms annihilated by the ideal J, and these can correspond to a finite set of JDT, even fixing Jordan type information for B.

It is an open problem in general whether a height three AG algebra is strong Lefschetz (see [1, 3, 4, 12] for special cases. We discussed the strong Lefschetz property of AG algebras $A = R/I$ having Hilbert function $(1, 3, 4, 4, 4, 3, 1)$. First, there is a non-zero-divisor $\ell \in R_1$ on $B = R/J, J = (I_{\leq} 3)$: that is enough to show that the map $\ell^2 : A_2 \to A_4$ has full rank. The non-vanishing of a Hessian [13] shows that $\ell^4 : A_1 \to A_5$ is also an isomorphism; we considered some alternate ways to show this based on the map π.

There is much that is mysterious about the map π and Jordan type.

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Rank matrices and Jordan types of Artinian Gorenstein algebras

Nasrin Altafi

We introduce rank matrices of linear forms for graded Artinian algebras that represent the ranks of multiplication maps in various degrees.

Let $S = \mathsf{k}[x_1, \dots, x_n]$, where k is a filed of characteristic zero. Let $R = \mathsf{k}[X_1, \dots, X_n]$ be Macaulay dual ring to S and F be a polynomial of degree $d \geq 2$ in R. We consider Artinian Gorenstein algebra $A = S/\mathrm{Ann}(F)$ with socle degree d.

Definition 1. *For a linear form $\ell \in A_1$ define the rank matrix, $M_{\ell,A}$, of A and ℓ to be the upper triangular square matrix of size $d+1$ with the following entries*

$$(M_{\ell,A})_{i,j} = \mathrm{rk}\left(\times \ell^{j-i} : A_i \longrightarrow A_j\right),$$

for every $0 \leq i \leq j \leq d$.

Entries of the rank matrix are equal to the ranks of higher Hessian matrices introduced by Maeno-Watanabe [3] and Gondim-Zappalà [2].

$$\mathrm{rk}\,\mathrm{Hess}_\ell^{(i,d-j)}(F) = (M_{\ell,A})_{i,j}, \quad 0 \leq i < j. \tag{1}$$

For an upper triangular square matrix M of size $d+1$ with non-negative entries we provide necessary conditions to ensure that M occurs as the rank matrix for some Artinian Gorenstein algebra A and linear form $\ell \in A_1$.
Define the diagonal vector of M by $\mathrm{diag}(i, M) := ((M)_{0,i}, (M)_{1,i+1}, \dots, (M)_{d-i,d})$, and denote $\mathrm{diag}(i, M)_+ := (0, (M)_{0,i}, (M)_{1,i+1}, \dots, (M)_{d-i,d})$.

Proposition 2. *Let M be the rank matrix of some Artinian Gorenstein algebra A and linear form $\ell \in A_1$. Then*

(*i*) $\mathrm{diag}(i, M)$ *is an O-sequence, for every* $0 \leq i \leq d$,
(*ii*) $\mathrm{diag}(i, M) - (\mathrm{diag}(i+1, M))_+$ *is an O-sequence, for every* $0 \leq i \leq d-1$ $(\mathbf{v}_+ = (0, \mathbf{v}))$,
(*iii*) $M_{i,j} + M_{i-1,j+1} \geq M_{i-1,j} + M_{i,j+1}$, *for every* $0 \leq i \leq j$, *set* $M_{i,j} = 0$ *for* $i < 0$ *or* $j < 0$.

We show that there is a one-to-one correspondence between the rank matrices and Jordan degree type partitions. We introduce Jordan degree type matrix, $J_{\ell,A}$, associated to A and ℓ to be upper triangular matrices of size $d+1$ and with entries

$$(J_{\ell,A})_{i,j} := (M_{\ell,A})_{i,j} + (M_{\ell,A})_{i-1,j+1} - (M_{\ell,A})_{i-1,j} - (M_{\ell,A})_{i,j+1},$$

where we set $(M_{\ell,A})_{i,j} = 0$ if either $i < 0$ or $j < 0$.

Proposition 3. *There is a one-to-one correspondence between the rank matrices and Jordan degree types.*

We provide a formula to obtain the Jordan type partition of ℓ and A from the rank matrix of ℓ and A. For every $0 \leq i \leq d$, where d is the socle degree of A, define Artinian Gorenstein algebra $A^{(i)} := S/\mathrm{Ann}(\ell^i \circ F)$. Denote $\mathbf{d} := (\dim_{\mathsf{k}} A^{(0)}, \dim_{\mathsf{k}} A^{(1)}, \ldots, \dim_{\mathsf{k}} A^{(d)})$.

Proposition 4. *The Jordan type partition of ℓ for A is given by*

$$P_{\ell,A} = \big(\underbrace{d+1, \ldots, d+1}_{n_d}, \underbrace{d, \ldots, d}_{n_{d-1}}, \ldots, \underbrace{2, \ldots, 2}_{n_1}, \underbrace{1, \ldots, 1}_{n_0}\big),$$

such that $\mathbf{n} = (n_0, n_1, \ldots, n_d) = \Delta^2 \mathbf{d}$, the second difference sequence of $\mathbf{d}$.

Using this approach, we provide a complete list of possible rank matrices for Artinian Gorenstein quotients of $S = \mathsf{k}[x,y,z]$ with at most three non-zero diagonals, or equivalently, rank matrices for linear forms ℓ such that $\ell^3 = 0$. We state the result for even and odd socle degree d separately.
Denote by $t = h_A(\frac{d}{2} - 1)$, $s = h_{A^{(1)}}(\frac{d}{2} - 1)$, and $r = h_{A^{(2)}}(\frac{d}{2})$.

Theorem 5. *There is an Artinian Gorenstein algebra $A = S/\mathrm{Ann}(F)$ of even socle degree $d \geq 2$ and $\ell \in A_1$ such that $\ell^2 \neq 0, \ell^3 = 0$ if and only if*

(1) *$r \in [1, \frac{d}{2} - 1]$, $s \in [2r, \frac{d}{2} + r]$ and $t \in [2s - r, \frac{d}{2} + s + 1]$, for $d \geq 4$; or*
(2) *$r = \frac{d}{2}$, $s = d - 1$ and $t \in [\frac{3d}{2} - 2, \frac{3d}{2}]$, for $d \geq 2$.*

Moreover, $M_{\ell,A}$ is completely determined by (r,s,t).

Denote by $t = h_A(\frac{d-1}{2})$, $s = h_{A^{(1)}}(\frac{d-1}{2})$, and $r = h_{A^{(2)}}(\frac{d-1}{2})$.

Theorem 6. *There is an Artinian Gorenstein algebra $A = S/\mathrm{Ann}(F)$ of odd socle degree $d \geq 2$ and $\ell \in A_1$ such that $\ell^2 \neq 0, \ell^3 = 0$ if and only if*

(1) *$r \in [1, \frac{d-1}{2} - 1], s \in [2r, \frac{d-1}{2} + r], t \in [2s - r, \frac{d-1}{2} + s + 1]$, for $d \geq 5$; or*
(2) *$r \in [1, \frac{d-1}{2} - 1], s = \frac{d-1}{2} + r + 1, t = d + r$, for $d \geq 5$; or*
(3) *$r = \frac{d-1}{2}, s \in [d-1, d]$ and $t \in [\frac{d-1}{2} + s - 1, 3\frac{d-1}{2}]$, for $d \geq 3$.*

Moreover, $M_{\ell,A}$ is completely determined by (r,s,t).

Using the above theorems and the correspondence between rank matrices and higher Hessians in (1) we are able to prove that Jordan type partitions (or more precisely, Jordan degree type partitions) of A and ℓ where $\ell^4 = 0$ area completely determined by the ranks of at most three of the Hessian matrices. Moreover, the explicit formulas for the rank matrices completely determine the Jordan (degree) types in these cases.

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The Weak Lefschetz Property for Cohomology Modules of Vector Bundles on $\mathbb{P}^2$

Zachary Flores

(joint work with Gioia Failla, Chris Peterson)

Let $\mathbb{K}$ be an algebraically closed field, and $R = \mathbb{K}[x_0, x_1, \ldots, x_r]$ the polynomial ring over $\mathbb{K}$. Any R-modules considered will be finitely generated and graded. We are interested in *finite length* R-modules with the following property.

Definition 1. *Given a finite length graded R-module M, write $M = \bigoplus_{j\in\mathbb{Z}} M_j$. We say that M has the Weak Lefschetz Property (WLP) if given a general linear form ℓ, the map $\times\ell : M_{j-1} \to M_j$ has maximal rank for all $j \in \mathbb{Z}$.*

The following result served as motivation for our main result. Moreover, each proof provided guidance on how to further utilize geometric techniques in this situation. For an ideal $I \subseteq R$, let $\alpha(I)$ denote the minimal degree of an element of R.

Theorem 2 (Theorem 2.3, [3] and Corollary 2.4, [1])**.** *If $r = 2$, $\mathbb{K}$ has characteristic zero, and I is a complete intersection with $\alpha(I) \geq 2$, then R/I has the WLP.*

We quickly note Theorem 2 fails in positive characterstic: if $r \geq 2$ and $\mathbb{K}$ has characteristic $p > 0$, then $R/(x_0^p, \ldots, x_r^p)$ does not have the WLP. Henceforth, we will always assume that $\mathbb{K}$ has characteristic zero, and also that $r = 2$, so R is a polynomial ring in three variables over an algebraically closed field in characteristic zero.

We start with some remarks on the proof Theorem 2 in [3]. Suppose $I = (f_1, f_2, f_3)$ with $\deg(f_i) = d_i$, and $\leq d_1 \leq d_2 \leq d_3$. In (Corollary 2, [5]) it was shown that if $d_3 \geq d_1 + d_2 - 3$, then R/I has the WLP. Utilizing this result in [3], it is shown that R/I has the WLP when $d_3 < d_1 + d_2 - 3$ to complete the proof of Theorem 2.

With this, we ask the following:

(1) Can we extend Theorem 2 to non-cyclic finite length R-modules? If so, what class of finite finite R-modules is an appropriate replacement for complete intersections?
(2) Provided (1) is answered, can we give a self-contained proof?

We affirmatively answer (1) and (2) in the following theorem.

Theorem 3 (Theorem 3.7, [2])**.** *Let $\mathcal{E}$ be a rank two, normalized (that is, $c_1(\mathcal{E}) \in \{-1, 0\}$) vector bundle on $\mathbb{P}^2$. Then $H^1_*(\mathbb{P}^2, \mathcal{E})$ has the WLP.*

To see how this is an extension of Theorem note that $M := H^1_*(\mathbb{P}^2, \mathcal{E})$ is a finite length R-module with presentation

$$(\star) \qquad \bigoplus_{j=1}^{n+2} R(-b_j) \xrightarrow{\varphi} \bigoplus_{j=1}^{n} R(-a_i) \to M \to 0$$

Conversely, given any such presentation $(\star)$ with M a finite length R-module, if $E = \ker(\varphi)$, then its sheafification, $\widetilde{E}$, is a rank two vector bundle on $\mathbb{P}^2$, which up to a twist, is normalized. Moreover, there is a $d \in \mathbb{Z}$ such that $H^1_*(\mathbb{P}^2, \widetilde{E})(-d) \cong M$. Since a complete intersection has presentation $(\star)$ with $n = 1$, we recover Theorem 2.

We next discuss the techniques involved in the proof of Theorem 3. Let ℓ be a general linear form and L the general line defined by ℓ in $\mathbb{P}^2$. Consider the short exact sequence for $t \in \mathbb{Z}$

$$(1) \qquad 0 \to \mathcal{E}(t-1) \to \mathcal{E}(t) \to \mathcal{E}(t)|_L \to 0$$

Now $H^1_*(\mathbb{P}^2, \mathcal{E}) = \oplus_{t\in\mathbb{Z}} H^1(\mathbb{P}^2, \mathcal{E}(t))$, and the exact sequence (1) induces the map that is multiplication by ℓ:

$$\phi_\ell(t) : H^1(\mathbb{P}^2, \mathcal{E}(t-1)) \to H^1(\mathbb{P}^2, \mathcal{E}(t))$$

Thus to show that $H^1_*(\mathbb{P}^2, \mathcal{E})$ has the WLP, we need to show that $\phi_\ell(t)$ is either injective or surjective for all $t \in \mathbb{Z}$. To obtain conditions about the rank of $\phi_\ell(t)$, we apply the global section functor to the short exact sequence (1) to obtain a long exact sequence of cohomology groups. With some basic diagram chasing, we obtain several conditions that tell us about the rank of $\phi_\ell(t)$, but all of which involve the need to understand the vector bundle $\mathcal{E}|_L$.

Now $\mathcal{E}|_L$ is a vector bundle of rank two on a copy of $\mathbb{P}^1$, so it splits as sum of line bundles

$$\mathcal{E}|_L \cong \mathcal{O}_L(a) \oplus \mathcal{O}_L(b)$$

for $(a, b) \in \mathbb{Z}^2$. The vector (a, b) is called the *generic splitting type* of $\mathcal{E}$, as the restriction of $\mathcal{E}$ to any *general line* L' will split as a sum of the line bundles $\mathcal{O}_{L'}(a) \oplus \mathcal{O}_{L'}(b)$. The key ingredient to the proof of Theorem 3 is understanding the generic splitting type of rank two bundles on $\mathbb{P}^2$.

We calculate the generic splitting type of a rank two vector bundle on $\mathbb{P}^2$ by making use of *slope-stability conditions*. The first result in this direction is the classical Grauert-Mülich theorem that computes the generic splitting type for a *semistable* bundle on $\mathbb{P}^2$.

Proposition 4 (Corollary 2, pg. 206, [4])**.** *Let $\mathcal{E}$ be a be a normalized, semistable rank two vector bundle on $\mathbb{P}^2$. Then*

- *if $c_1(\mathcal{E}) = 0$, then the generic splitting type of $\mathcal{E}$ is $(0, 0)$;*
- *if $c_1(\mathcal{E}) = -1$, then the generic splitting type of $\mathcal{E}$ is $(0, -1)$.*

A vector bundle that is not semistable is called *unstable*, and in order to prove Theorem 3, we need calculate the generic splitting type for an unstable rank two vector bundle on $\mathbb{P}^2$. In this direction, if $\mathcal{E}$ is unstable vector bundle on $\mathbb{P}^2$ of rank two, there is a $k \geq 0$ such that $H^0(\mathbb{P}^2, \mathcal{E}(-k)) \neq 0$ by (Lemma 1.2.5, [4]). The largest such k is called the *index of instability* of $\mathcal{E}$. We compute the generic splitting type of an unstable vector bundle $\mathcal{E}$ in the following proposition.

Proposition 5 (Proposition 3.5, [2]). *Let $\mathcal{E}$ be a normalized, unstable bundle of rank two on $\mathbb{P}^2$ with index of instability given by k. Then*

(1) *if $c_1(\mathcal{E}) = 0$, then $k > 0$, and the generic splitting type of $\mathcal{E}$ is $(-k, k)$;*
(2) *if $c_1(\mathcal{E}) = -1$, then $k \geq 0$, and the generic splitting type of $\mathcal{E}$ is $(-k-1, k)$.*

With Proposition 4 and Proposition 5, we can given an even more precise statement of our main result.

Theorem 6 (Theorem 3.7, [2]). *If $\mathcal{E}$ is a normalized, rank two vector bundle $\mathbb{P}^2$, with first cohomology module $H^1_*(\mathbb{P}^2, \mathcal{E}) = \oplus_{t\in\mathbb{Z}} H^1(\mathbb{P}^2, \mathcal{E}(t))$, and $\phi_\ell(t) : H^1(\mathbb{P}^2, \mathcal{E}(t-1)) \to H^1(\mathbb{P}^2, \mathcal{E}(t))$ is multiplication by ℓ, then*

(1) *$H^1_*(\mathbb{P}^2, \mathcal{E})$ has the WLP;*
(2) *if $\mathcal{E}$ is semistable, then*
  (a) *if $c_1(\mathcal{E}) = 0$, $\phi_\ell(t)$ is injective for $t \leq -1$ and surjective for $t \geq -1$;*
  (b) *if $c_1(\mathcal{E}) = -1$, then $\phi_\ell(t)$ is injective for $t \leq -1$ and surjective for $t \geq 0$;*
(3) *if $\mathcal{E}$ is unstable with index of instability k, then*
  (a) *If $c_1(\mathcal{E}) = 0$, then $\phi_\ell(t)$ is injective for $t \leq k-1$ and surjective for $t \geq -k-1$;*
  (b) *If $c_1(\mathcal{E}) = -1$, then $\phi_\ell(t)$ is injective for $t \leq k$ and surjective for $t \geq -k-1$.*

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On the canonical module of GT-varieties and the normal bundle of RL-varieties.

Liena Colarte-Gómez

(joint work with R. M. Miró-Roig)

Through this abstract, $\mathbb{K}$ denotes an algebraically closed field of characteristic zero, $R = \mathbb{K}[x_0, \ldots, x_n]$ denotes the polynomial ring and $GL(n+1, \mathbb{K})$ represents the group of invertible matrices of size $(n+1) \times (n+1)$ with coefficients in $\mathbb{K}$.

Introduction. A homogeneous artinian ideal $I \subset R$ generated by homogeneous forms $F_1, \ldots, F_r$ of degree d is said be a *Togliatti system* if R/I fails the weak Lefschetz property, abridged WLP, in degree $d-1$. Togliatti systems were introduced by Mezzetti, Miró-Roig and Ottaviani (2013), they related the failure of the WLP to the existence of rational projective varieties of $\mathbb{P}^{\binom{n+d}{d}-r-1}$ satisfying a Laplace equation. In [2], a new family of Togliatti systems is defined, the so called GT-systems. By $G \subset GL(n+1, \mathbb{K})$, we denote a linear finite group of order d acting on R. A *GT-system with group G* is a Togliatti system $I_d = (F_1, \ldots, F_{\mu_d}) \subset R$ such that the morphism $\varphi_{I_d} : \mathbb{P}^n \to \mathbb{P}^{\mu_d - 1}$ given by $(F_1, \ldots, F_{\mu_d})$ is a Galois covering with group G. We call $X_d := \varphi_{I_d}(\mathbb{P}^n)$ a *GT-variety with group G*. In [3], we study the homogeneous ideal of certain GT-threefolds. In [1], [2] and [4], the authors use invariant theory methods to tackle GT-system and the geometry of GT-varieties.

Hereby, we present the most recent results, collected in [4], on GT-varieties with finite linear cyclic diagonal groups, the notion of RL-varieties and the outcomes on their normal bundles.

The homogeneous ideal of GT-varieties. We fix integers $2 \le n < d$, $0 \le \alpha_0 \le \cdots \le \alpha_n < d$ with $GCD(d, \alpha_0, \ldots, \alpha_n) = 1$ and we fix e a dth primitive root of $1 \in \mathbb{K}$. By $M_{d;\alpha_0,\ldots,\alpha_n}$, we denote the diagonal matrix $diag(e^{\alpha_0}, \ldots, e^{\alpha_n})$. We set $\Gamma := \langle M_{d;\alpha_0,\ldots,\alpha_n} \rangle \subset GL(n+1, \mathbb{K})$ the linear cyclic group of order d and we denote $\overline{\Gamma} \subset GL(n+1, \mathbb{K})$ the linear abelian group of order d^2 generated by $M_{d;\alpha_0,\ldots,\alpha_n}$ and $M_{d;1,\ldots,1} = eId$. By R^Γ (respectively $R^{\overline{\Gamma}}$), we represent the ring of invariants of Γ (respectively $\overline{\Gamma}$) acting on R. Let $I_d \subset R$ be the ideal generated by the set of all monomials $\mathcal{M}_d := \{m_1, \ldots, m_{\mu_d}\} \subset R^\Gamma$ of degree d. We denote $\varphi_{I_d} : \mathbb{P}^n \to \mathbb{P}^{\mu_d - 1}$ the associated morphism and we set $X_d := \varphi_{I_d}(\mathbb{P}^n)$.

Partially motivated by the long-standing problem, posed by Gröber (1965), of determining whether a projection of the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{d}-1}$ is arithmetically Cohen-Macaulay (abridged aCM), in [1] the authors prove the following.

Theorem 1. *(i) $\mathcal{M}_d$ is a basis of $R^{\overline{\Gamma}}$.*

(ii) $R^{\overline{\Gamma}}$ is the coordinate ring of X_d. Hence, X_d is an aCM monomial projection of the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{d}-1}$ from the inverse system I_d^{-1}.

(iii) If $\mu_d \le \binom{d+n-1}{n-1}$, then I_d is a GT-system with group Γ.

Let $w_1, \ldots, w_{\mu_d}$ be a new set of indeterminates and set $S := \mathbb{K}[w_1, \ldots, w_{\mu_d}]$. We denote $I(X_d) \subset S$ the homogeneous ideal of X_d. From Theorem 1, we have that $I(X_d)$ is the kernel of the morphism $\rho : S \to R$ given by $\rho(w_i) = m_i$. It holds that $I(X_d)$ is the homogeneous binomial prime ideal generated by

$$\mathcal{W}_d = \{w_{i_1} \cdots w_{i_k} - w_{j_1} \cdots w_{j_k} \in S \text{ such that } m_{i_1} \cdots m_{i_k} = m_{j_1} \cdots m_{j_k},\ k \ge 2\}.$$

We denote by $I(X_d)_k$, $k \ge 2$, the set of all binomials of $\mathcal{W}_d$ of degree exactly k. In [4], it is shown that $I(X_d)$ is generated by quadrics and cubics; and examples of GT-varieties whose homogeneous ideals are minimally generated by both types of generators are exhibited. Precisely,

Theorem 2. [4] $I(X_d) = (I(X_d)_2, I(X_d)_3)$.

The canonical module of GT-varieties. The algebraic structure of the canonical module w_X of the coordinate ring of an aCM projective variety X plays a central role on its geometry. Among others, it carries on information of the Hilbert function and series, as well as the Castelnuovo-Mumford regularity, of X.

Let X_d be a GT-variety with group Γ. We denote $H_d \subset \mathbb{Z}^{n+1}_{\geq 0}$ the semigroup associated to $\mathcal{M}_d$ and by $k[H_d]$, we represent the associated semigroup ring. From Theorem 1, it follows that $R^{\overline{\Gamma}} = k[H_d]$; and from a classical result of Danilov and Stanley, it follows that $w_{X_d} = relint(I_d)$, where $relint(I_d)$ is the ideal of $R^{\overline{\Gamma}}$ induced by $H_d^+ = \{(a_0, \ldots, a_n) \in H_d \mid a_i > 0,\ i = 0, \ldots, n\}$. For $k \geq 1$, let $relint(I_d)_k \subset relint(I_d)$ be the set of all monomials of degree kd. In [4], it is proved that:

Theorem 3. $relin(I_d) = (relint(I_d)_1, relint(I_d)_2)$.

It is then natural wondering under which conditions $R^{\overline{\Gamma}}$ is level, i.e., $relint(I_d)$ is generated in only one degree, and in particular when it is Gorenstein, i.e., $relin(I_d)$ is a principal ideal. We give the following families of examples.

Proposition 4. [4] *(i) Fix integers $k \geq 1, n \geq 2$ with n even and fix $\Gamma = \langle M_{k(n+1);0,1,2,\ldots,n}\rangle \subset GL(n+1, \mathbb{K})$. The associated GT-variety X_d is level, in particular X_{n+1} is Gorenstein.*

(ii) Fix integers $k \geq 1, n \geq 3$ with n odd and fix $\Gamma = \langle M_{k(n+1);0,1,\ldots,1,2}\rangle$. The associated GT-variety $X_{k(n+1)}$ is level, in particular X_{n+1} is Gorenstein.

By means of the description of the canonical module of $R^{\overline{\Gamma}}$, we determine the Castelnuovo-Mumford regularity of X_d. Precisely,

Theorem 5. [4]

$$n \leq reg(R^{\overline{\Gamma}}) \leq n+1.$$

The equality $reg(R^{\overline{\Gamma}}) = n+1$ holds if and only if $relint(I_d)_1 \neq \emptyset$.

Cohomology of the normal bundle of RL-varieties. In [4], we call *level* a GT-variety X_d with group Γ such that $relint(I_d) = (relint(I_d)_1)$. Moreover, Proposition 4 gives examples of level GT-varieties for any dimension. Motivated by the recent work of Alzati and Re (2020) and these examples, we introduce a new family of smooth projective varieties, the so called RL-varieties, which are naturally related to level GT-varieties.

Definition 6. [4] Let X_d be a level GT-variety with group Γ. The RL-variety $\mathcal{X}_d$ associated to X_d is defined as the projection of the Veronese variety $\nu_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{n}-1}$ from the linear system $\langle relint(I_d)_1\rangle$.

We can see $\mathcal{X}_d$ as the variety associated to the parametrization f_d given by the inverse system $relint(I_d)_1^{-1}$. We prove that $\mathcal{X}_d$ is smooth and that f_d is an embedding. Afterwards, we compute the cohomology table of the normal bundle of any RL-variety. Precisely,

Theorem 7. [4] *Fix a level GT-variety X_d with group $\Gamma = \langle M_{d;\alpha_0,\dots,\alpha_n}\rangle$. Set $\eta_d := |relint(I_d)|$ and $N_d := \binom{n+d}{d} - \eta_d - 1$. Let $\mathcal{X}_d \subset \mathbb{P}^{N_d}$ be the RL-variety associated to X_d. It holds:*
(i) for $0 < i < n-1$ and for all $k \in \mathbb{Z}$, $h^i(\mathcal{X}, \mathcal{N}_{\mathcal{X}_d})(-k) = 0$.
(ii)

$$h^0(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d})(-k) = \begin{cases} (N_d+1)\binom{n+d-k}{d-k} - (n+1)\binom{n+1-k}{1-k} & k \le 1 \\ (N_d+1)\binom{n+d-k}{d-k} & k \le d \\ 0 & \textit{otherwise.} \end{cases}$$

(iii)

$$h^{n-1}(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d})(-k) = \begin{cases} (n+1)\binom{k-2}{k-2-n} & n+2 \le k < d+n+1 \\ \nu_d + \frac{n(d-1)}{d}\binom{n+d-1}{n} & k = d+n+1 \\ (n+1)\nu_d & k = d+n+2 \\ 0 & k \le n+1 \ \textit{or} \ k \ge d+n+3. \end{cases}$$

(iv)

$$h^n(\mathcal{X}_d, \mathcal{N}_{\mathcal{X}_d})(-k) = \begin{cases} (N_d+1)\binom{k-d-1}{k-d-n-1} - (n+1)\binom{k-2}{k-2-n} + & \\ \nu_d + \frac{n(d-1)}{d}\binom{n+d-1}{n} & k = d+n+1 \\ (N_d+1)\binom{k-d-1}{k-d-n-1} - (n+1)\binom{k-2}{k-2-n} + & \\ (n+1)\nu_d & k = d+n+2 \\ (N_d+1)\binom{k-d-1}{k-d-n-1} - (n+1)\binom{k-2}{k-2-n} & k \ge d+n+3 \\ 0 & \textit{otherwise.} \end{cases}$$

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The WLP for almost complete intersections generated by uniform powers of general linear forms

Mats Boij
(joint work with Samuel Lundqvist)

For positive integers n, m and d, consider the polynomial ring $= \mathbb{k}[x_1, x_2, \dots, x_n]$ and m general linear forms $\ell_1, \ell_2, \dots, \ell_m$. We are interested in the problem of when the ring

$$R_{n,m,d} = \mathbb{k}[x_1, x_2, \dots, x_n]//\langle \ell_1^d, \ell_2^d, \dots, \ell_m^d\rangle$$

satisfies the Weak Lefschetz Property (WLP), that is when the multiplication maps with a general linear form all have maximal rank.

In the case $n = 3$, there is a powerful result by Schenck and Seceleanu [8] that shows that WLP holds for $R_{3,m,d}$ for any m and d.

For $n \geq 4$ and $m = n + 1$, which means for almost complete intersections, Migliore, Miró-Roig and Nagel [4] showed that the WLP fails for even $n \geq 4$ and $d \geq 2$ except for $(n, d) = (4, 2)$. For odd n they made the following intriguing conjecture.

Conjecture 1. [4, Conjecture 6.6] *Let $n \geq 9$ be an odd integer. Then $R_{n,n+1,d} = \Bbbk[x_1, x_2, \ldots, x_n]/\langle \ell_1^d, \ell_2^d, \ldots, \ell_{n+1}^d \rangle$ fails the WLP if and only if $d > 1$. Furthermore, if $n = 7$ then $R_{n,n+1,d}$ fails the WLP when $d = 3$.*

Some important contributions to this have been made by Sturmfels and Xu [9], Miró-Roig [5], Miró-Roig and Tran [6], Nagel and Trok [7] and Ilardi and Vallès [1].

We are able to settle this conjecture completely and we can in fact extend it to a full classification of when WLP fails for $R_{n,n+1,d}$.

Theorem 2. *Let $d, n \geq 1$. Then $R_{n,n+1,d} = \Bbbk[x_1, x_2, \ldots, x_n]/\langle \ell_1^d, \ell_2^d, \ldots, \ell_{n+1}^d \rangle$ fails the WLP except when $n \leq 3$, $d = 1$ or $(n, d) \in \{(4, 2), (5, 2), (5, 3), (7, 2)\}$, and in these cases, the WLP holds.*

Our first step is to relate this to Fröberg's conjecture [2] by taking the quotient by a general linear form. This leads to $R_{n-1,n+1,d}$ and to the question whether the Hilbert series of this ring is the one expected from Fröberg's conjecture for the ideal if $n + 1$ general forms of degree d in $n - 1$ variables.

When studying $R_{n,n+2,d}$, our work builds on the methods of inverse systems. This means that we look for polynomials F in a dual polynomial ring $R = \Bbbk[X_1, X_2, \ldots, X_n]$ where S acts by differentiation such that $\ell_i^d \circ F = 0$ for all $i = 1, 2, \ldots, n + 2$. The existence of such a form shows that the Hilbert function of $R_{n,n+2,d}$ is non-zero in that degree which in most cases will let us show that $R_{n+1,n+2,d}$ fails to have the WLP.

The degree where we construct a non-zero element in the inverse system of $R_{n,n+1,d}$ is given by

$$s(n,d) = \begin{cases} \frac{(n+1)(d-1)}{2} & \text{if } n \text{ is odd,} \\ \left\lfloor \frac{n(n+2)(d-1)}{2(n+1)} \right\rfloor & \text{if } n \text{ is even.} \end{cases}$$

This number was already shown by Nagel and Trok [7] to be an upper bound for the regularity of $R_{n,n+2,d}$. By an inductive argument building on the case $d = 2$, which was handled by Sturmfels and Xu [9], we prove the following proposition.

Proposition 3. *Let $n \geq 1$ be odd and let $d \geq 1$. Then the value of the Hilbert function of $R_{n,n+2,d}$ is non-zero in degree $s(n, d)$.*

Example 4. *As an example we look at the case $(n, d) = (4, 4)$ where we want a form of degree $s(4, 4) = \lfloor 4 \cdot 6 \cdot 3/(2 \cdot 5) \rfloor = 7$ that is annihilated by the fourth powers of the linear forms $\ell_1, \ell_2 m \ldots, \ell_6$. For each $i = 1, 2, 3$ we can find a unique quadric Q_i that is annihilated by ℓ_i and by ℓ_j^2, for $j \neq i$ since there is a unique quadric in three variables passing through five general points. We can also find a* *linear form L that is annhilated by ℓ_4, ℓ_5 and ℓ_6. Now $\ell_i^4 \circ (Q_1Q_1Q_3L_{4,5,6}) = 0$ for $i = 1, 2, \ldots, 6$ by the pigeonhole principle since ℓ_i annihilates one of the factors and ℓ_i^2 annihilates the other factors.*

In order to prove that this leads to a proof of the failure of WLP we also need to show that the Hilbert function of $R_{n,n+2,d}$ would have to be zero in this degree if $R_{n+1,n+2,d}$ satisfies the WLP. This Hilbert series that $R_{n,n+2,d}$ would have if $R_{n+1,n+2,d}$ satisfied the WLP is given by the Fröberg conjecture and can be expressed as

$$\left[\frac{(1-t^d)^{n+2}}{(1-t)^n}\right] \tag{1}$$

where the brackets indicate that we truncate the series before the first non-positive term.

We are able to show that the degree of this series is less than $s(n, d)$ in all but a few cases. These sporadic cases are handled using an explicit formula for the unique form in the top degree of the inverse system of $R_{n,n+2,2}$ in the case n is odd.

Theorem 5. *Let $n = 2k - 1$ for a positive integer k. The the form*

$$F = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma) V(a_{\sigma_1}, a_{\sigma_2}, \ldots, a_{\sigma_k}) V(a_{\sigma_{k+1}}, a_{\sigma_{k+2}}, \ldots, a_{\sigma_n}) \prod_{j=k+1}^{n} a_{\sigma_j} \prod_{j=1}^{k} X_{\sigma_j}$$

is the unique form of degree k in $\mathbb{k}[X_1, X_2, \ldots, X_n]$ that is annihilated by the squares of the linear forms $x_1, x_2, \ldots, x_n, x_1+x_2+\cdots+x_n, a_1x_1+a_2x_2+\cdots+a_nx_n$.

Theorem 6. *The WLP fails for $R_{n,n+1,d} = \mathbb{k}[x_1, x_2, \ldots, x_n]/\langle \ell_1^d, \ell_2^d, \ldots, \ell_{n+1}^d\rangle$ in the cases $(n, d) \in \{(5, 5), (7, 3), (9, 2), (9, 3), (11, 2), (11, 3)\}$. In particular we have that the Hilbert series of $R_{4,6,5}$, $R_{6,8,3}$, $R_{8,10,2}$, $R_{8,10,3}$, $R_{10,12,2}$, and $R_{10,12,3}$ are*

$$1 + 4t + 10t^2 + 20t^3 + 35t^4 + 50t^5 + 60t^6 + 60t^7 + 45t^8 + 14t^9,$$
$$1 + 6t + 21t^2 + 48t^3 + 78t^4 + 84t^5 + 43t^6,$$
$$1 + 8t + 26t^2 + 40t^3 + 16t^4,$$
$$1 + 8t + 36t^2 + 110t^3 + 250t^4 + 432t^5 + 561t^6 + 492t^7 + 171t^8,$$
$$1 + 10t + 43t^2 + 100t^3 + 121t^4 + 32t^5,$$

and

$$1 + 10t + 55t^2 + 208t^3 + 595t^4 + 1342t^5 + 2431t^6 + 3520t^7 + 3916t^8 + 2860t^9 + 682t^{10}$$

which differ in the leading term from $10t^9$, $42t^6$, $15t^4$, $135t^8$, $22t^5$ and $88t^{10}$ that are expected by the Fröberg conjecture.

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Associated Prime Ideals of Equivariant Coinvariant Algebras

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Recall for a finite group G and a representation $\theta : G \hookrightarrow \mathrm{GL}(n, \mathbb{F})$ of G over a field $\mathbb{F}$ that the **equivariant coinvariant algebra** of θ is the algebra $\mathbb{F}[V]\otimes_{\mathbb{F}[V]^G}\mathbb{F}[V]$, where $V = \mathbb{F}^n$ is the representation space of θ, $\mathbb{F}[V]$ the algebra of polynolmial functions on V, and $\mathbb{F}[V]^G$ the subalgebra of G-invariant forms of $\mathbb{F}[V]$. The minimal prime ideals of $\mathbb{F}[V]\otimes_{\mathbb{F}[V]^G}\mathbb{F}[V]$ were determined in [1 − 4] (by quite distinct methods). In this note we will show that if θ is defined over a finite field $\mathbb{F}$ then the associated prime ideals of the equivariant coinvariant algebra $\mathbb{F}[V]\otimes_{\mathbb{F}[V]^G}\mathbb{F}[V]$ can only be the minimal primes or the maximal ideal. We do so by making use of Steenrod operations to obtain information concerning prime ideals in unstable algebras over the Steenrod algebra $\mathbf{P}^*$, in particular for the associated prime ideals, of an equivariant coinvariant algebra over a finite field which is such an unstable algebra. Among we obtain are the following.

- The Going Down Theorem of W. Krull holds for prime ideals with respect to the natural inclusion of either tensor factor of $\mathbb{F}[V]$ in
$$\mathbb{F}[V] \overset{\lambda}{\hookrightarrow} \mathbb{F}[V]\otimes_{\mathbb{F}[V]^G}\mathbb{F}[V] \overset{\rho}{\hookleftarrow} \mathbb{F}[V].$$
- If $\mathfrak{p} \subset \mathbb{F}[V]$ is a prime ideal and $P \subset \mathbb{F}[V]\otimes_{\mathbb{F}[V]^G}\mathbb{F}[V]$ a prime ideal lying over it, then $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(P)$, $\mathrm{coht}(\mathfrak{p}) = \mathrm{coht}(P)$, and $\mathrm{ht}(\mathfrak{p}) + \mathrm{coht}(\mathfrak{p}) = \dim(\mathbb{F}[V]) = n = \dim(\mathbb{F}[V]\otimes_{\mathbb{F}[V]^G}\mathbb{F}[V]) = \mathrm{ht}(P) + \mathrm{coht}(P)$.
- If $\mathfrak{p} \subset \mathbb{F}[V]\otimes_{\mathbb{F}[V]^G}\mathbb{F}[V]$ is a $\mathbf{P}^*$-invariant prime ideal then it is generated by the linear forms that it contains.
- The minimal prime ideals of $\mathbb{F}[V]\otimes_{\mathbb{F}[V]^G}\mathbb{F}[V]$ are $\mathbf{P}^*$-invariant so are generated by the linear forms that they contain.

Notice that the first two results are statements purely in commutative algebra: The statements have nothing to do with Steenrod operations. It is in their proofs that Steenrod algebra technology plays a role.

For the sake of completeness we also supply proofs of several basic results concerning the interaction of the Steenrod algebra with commutative algebra that seem to be new or, if they are extracted from the literature, then are given new proofs. Among which are following:

- J-P. Serre's Theorem: $\mathbf{P}^*$-invariant ideals in $\mathbb{F}[V]$ are generated by the linear forms that they contain. This is used in the proof of almost all the results that follow it.
- A standard graded $\mathbf{P}^*$-unstable integral domain A that is finitely generated as an algebra is a polynomial algebra.
- The minimal prime ideals of an unstable $\mathbf{P}^*$-algebra, are $\mathbf{P}^*$-invariant.

We hope that the method of proof used in these results will find interest among commutative algebracists in general.

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Togliatti systems associated to the dihedral group and the weak Lefschetz property.

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(joint work with Liena Colarte-Gómez, Emilia Mezzetti and Rosa M. Miró-Roig)

Introduction. *Togliatti systems* were introduced in [5], where the authors related the existence of homogeneous artinian ideals failing the weak Lefschetz property to the existence of projective varieties satisfying at least one Laplace equation. Precisely, a Togliatti system is an artinian ideal $I_d \subset k[x_0, \ldots, x_n]$ generated by $r \leq \binom{n+d-1}{d-1}$ forms $F_1, \ldots, F_r$ of degree d which fails the weak Lefschetz property in degree $d-1$. The name is in honour of E. Togliatti who gave a complete classification of rational surfaces parameterized by cubics and satisfying at least one Laplace equation of order 2. Since then, this topic and related problems has centered the attention of many works. Notwithstanding, most expositions and results deal with *monomial* Togliatti systems, while the non monomial case remains barely known.

Recently, in [4] and [2] the authors studied GT-systems, a new family of monomial Togliatti systems having a special geometric property. A GT-system is a Togliatti system I_d whose associated morphism $\varphi_{I_d} : \mathbb{P}^n \to \mathbb{P}^{\mu_{I_d}-1}$ is a Galois covering with cyclic group $\mathbb{Z}/d\mathbb{Z}$. This geometric property establishes a new link between Togliatti systems and Invariant Theory. Precisely in [1] and [3], the authors apply Invariant Theory techniques to investigate both GT-systems and their images $X_d = \varphi_{I_d}(\mathbb{P}^n)$, the so called GT-varieties. Afterwards, in [2], this notion was extended to tackle the action of any finite group, including non abelian ones. This is a starting point to the study of non monomial Togliatti systems.

Invariants of the dihedral group. From now on, we focus on the dihedral group action on $k[x_0, x_1, x_2]$. More precisely, we fix an integer $d \geq 3$ and ε a $2d$th primitive root of 1. We set $e = \varepsilon^2$ and let $\rho : D_{2d} \to GL(3, k)$ be the linear representation of $D_{2d} = \langle \tau, \varepsilon | \tau^d = \varepsilon^2 = (\varepsilon\tau)^2 = 1\rangle$, the dihedral group of order $2d$, defined by:

$$\rho(\tau) = M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^{d-1} \end{pmatrix} \qquad \text{and} \qquad \rho(\varepsilon) = \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

By $\overline{D_{2d}}$ we denote the subgroup of $GL(3, k)$ of order $(2d)^2$ generated by $\rho(D_{2d})$ and εId. Also, we denote by Γ (resp. $\overline{\Gamma}$) the cyclic subgroup generated by M (resp. M and εId). Using the results of [1] and [3] about cyclic groups, we can use the subgroup $\overline{\Gamma}$ to describe the invariants of $\overline{D_{2d}}$. For instance, using this method we find the Hilbert function:

Proposition 1. [2] *With the above notation,*

$$H(\overline{D_{2d}}, t) = \frac{2dt^2 + (d + GCD(d, 2) + 2)t + 2}{2}.$$

Then we can obtain an explicit basis for the graded pieces of the $k-$algebra $R^{\overline{D_{2d}}}$ of invariants of $\overline{D_{2d}}$, as well as a $k-$algebra basis.

Theorem 2. [2] *A k-basis $\mathcal{B}_{2td}$ of the vector space $R_t^{\overline{D_{2d}}}$ is formed by:*

(i) *the set of $td+1$ monomials $x_0^{2td}, x_0^{2td-2}x_1x_2, x_0^{2td-4}x_1^2x_2^2, \dots, x_1^{td}x_2^{td}$ of degree $2td$ which are invariants of Γ; and*

(ii) *the set of all binomials $x_0^{a_0}x_1^{a_1}x_2^{a_2} + x_0^{a_0}x_1^{a_2}x_2^{a_1}$ of degree $2td$ such that $a_1 \neq a_2$ and $x_0^{a_0}x_1^{a_1}x_2^{a_2} \in R^{\Gamma}$.*

Moreover, $R^{\overline{D_{2d}}}$ is a graded k-algebra generated in degree 1 by $\mathcal{B}_{2d}$.

This allows us to obtain information about the Cohen-Macaulayness of $R^{\overline{D_{2d}}}$:

Corollary 3. [2]

(i) *$R^{\overline{D_{2d}}}$ is a Cohen-Macaulay level algebra with Cohen-Macaulay type $d - GCD(d, 2)$ and Castelnuovo-Mumford regularity 3.*

(ii) *$R^{\overline{D_{2d}}}$ is Gorenstein if and only if $d = 3$ or 4.*

GT-systems with dihedral group. Finally, considering $I_{2d} \subset R$ the ideal generated by all invariants of degree $2d$ of D_{2d} we obtain the first large class of non monomial Togliatti systems, which are $GT-$systems with group D_{2d}. We establish a link between the ring $R^{\overline{D_{2d}}}$ and the coordinate ring of the associated varieties of these GT-systems. We call these varieties, $GT-$surface with group

D_{2d}, and we denote them $S_{D_{2d}}$. Using the previous results about the invariants of D_{2d}, we obtain geometric information about $S_{D_{2d}}$ and its homogeneous ideal $I(S_{D_{2d}})$. For instance, the last corollary translates as:

Proposition 4. [2] *$S_{D_{2d}}$ is an arithmetically Cohen-Macaulay surface of degree $deg(S_{D_{2d}}) = 2d$, regularity 3, codimension $\frac{1}{2}(3d + GCD(d,2) - 2)$ and Cohen-Macaulay type $\frac{1}{2}(d - GCD(d,2))$. In particular, $S_{D_{2d}}$ is Gorenstein if and only if $d = 3, 4$.*

Moreover, we define $\mathcal{W}_d := \{w_{(r,\gamma)} \mid 0 \leq r \leq 2(d-1)$ and $max\{0, \lceil \frac{(r-2)d}{d-2} \rceil\} \leq \gamma \leq r\}$ and a new polynomial ring $S = k[w_{(r,\gamma)}]_{w_{(r,\gamma)} \in \mathcal{W}_d}$. Then, the homogeneous ideal $I(S_{D_{2d}})$ can be seen as the kernel of $\varphi_d : S \to R$ defined by

$$\varphi_d(w_{(r,\gamma)}) = \begin{cases} x_0^{2d-2\gamma} x_1^{\gamma} x_2^{\gamma} & \text{if } r = \gamma \\ x_0^{(2-r)d+(d-2)\gamma}(x_1^{rd-(d-1)\gamma} x_2^{\gamma} + x_1^{\gamma} x_2^{rd-(d-1)\gamma}) & \text{otherwise.} \end{cases}$$

Together with the previous results, this allows us to finally give explicitly a minimal free $S-$resolution of $S/I(S_{D_{2d}})$ in terms of the codimension C and $h = 2d - C - 2$:

Theorem 5. [2] *With the above notation, $S/I(S_{D_{2d}})$ has a minimal free S-resolution*

$$\begin{aligned} 0 \to S^{b_{C,2}}(-C-2) &\to \oplus_{l=1,2} S^{b_{C-1,l}}(-C+1-l) \to \oplus_{l=1,2} S^{b_{C-2,l}}(-C+2-l) \\ &\to \cdots \to \oplus_{l=1,2} S^{b_{C-h,l}}(-C+h-l) \to S^{b_{C-h-1,1}}(-C+h) \\ &\to \cdots \to S^{b_{1,1}}(-2) \to S \to S/I(S_{2d}) \to 0 \end{aligned}$$

where

$$b_{i,j-i} := \begin{cases} i\binom{C}{i+1} + (C-i-h)\binom{C}{i-1} & \text{if } 1 \leq i \leq C-h-1, j = i+1 \\ i\binom{r}{i+1} & \text{if } C-h \leq i \leq C, j = i+1 \\ (i-C+h+1)\binom{C}{i} & \text{if } C-h \leq i \leq C, j = i+2 \\ 0 & \text{otherwise} \end{cases}$$

In particular we have a description of the structure of the homogeneous ideal $I(S_{D_{2d}})$:

Corollary 6. [2] *$I(S_{D_{2d}})$ is minimally generated by $\frac{9d^2+2d+8}{8}$ quadrics if d is even and by $\frac{9d^2-4d+3}{8}$ quadrics if d is odd.*

Using the explicit description of the morphism ϕ_d, in [2] we are able to find an explicit set of binomials and trinomials of degree two generating $I(S_{2d})$.

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Generic initial ideals and unexpected hypersurfaces

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(joint work with Elena Guardo, Brian Harbourne, Juan Migliore)

Given a subscheme X of $\mathbb{P}^n$, its defining saturated homogeneous ideal $I_X \subseteq R = K[\mathbb{P}^n] = K[x_0, \ldots, x_n]$ (where K is a field) and integers $t \geq m \geq 1$, we define two numbers associated to (X, t, m). The *actual dimension*, $\mathrm{adim}(X, t, m)$, is the dimension of the vector space of the forms in I_X of degree t vanishing at a general point P with multiplicity m. That is,

$$\mathrm{adim}(X, t, m) = \dim[I_X \cap I_P^m]_t.$$

Next, the *virtual dimension*, $\mathrm{vdim}(X, t, m)$, is the dimension of the linear system of the forms of degree t in I_X minus the expected number of conditions imposed by taking P with multiplicity m. That is,

$$\mathrm{vdim}(X, t, m) = \dim[I_X]_t - \binom{m-1+n}{n}.$$

Of course, $\mathrm{adim}(X, t, m) \geq \mathrm{vdim}(X, t, m)$ for any t, m. We say (as introduced in [1] and then in [3]) that X *admits an unexpected hypersurface of degree t vanishing at a general point P with multiplicity* m when $\mathrm{adim}(X, t, m) > 0$ and $\mathrm{adim}(X, t, m) > \mathrm{vdim}(X, t, m)$.
For a given subscheme X of $\mathbb{P}^n$ and a non-negative integer $j \geq 0$, we define the AV sequence of X with respect to j as $AV_{X,j} : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$ where

$$AV_{X,j}(m) := \mathrm{adim}(X, m+j, m) - \mathrm{vdim}(X, m+j, m).$$

For any subscheme X and integer $j \geq 0$, the sequence $AV_{X,j}$ is actually an O-sequence, up to a shift (see [2, Theorem 3.4]). Indeed, it is the Hilbert function of the standard K-algebra $R/(\mathrm{gin}(I_X) : x^{j+1})$, where $\mathrm{gin}(I_X)$ denotes the generic initial ideal with respect to the lexicographic order with $x_0 > x_1 > \cdots > x_n$.

This result is a consequence of the connections between the actual and virtual dimensions of a scheme with its the generic initial ideal since, see [2, Lemma 3.1], for any scheme X and non-negative integers t and m, we have

(i) $\mathrm{adim}(X, t, m) = \dim[\mathrm{gin}(I_X) \cap I_Q^m]_t$, where $Q = (1, 0, \ldots, 0)$;
(ii) $\mathrm{vdim}(X, t, m) = \mathrm{vdim}(\mathrm{gin}(I_X), t, m)$.

This fact can be used to obtain results which ensure the non-existence of unexpected hypersurfaces. In particular, if X lies on a hyperplane or if $\mathrm{gin}(I_X)$ is a lex-segment ideal then X does not admit *any* unexpected hypersurfaces of any type, and hence $\mathrm{adim}(X, t, m)$ *always* has the expected value.

We also notice that the failure of $\mathrm{gin}(I_X)$ to be a lex-segment ideal is a weaker condition for X than admitting some unexpected hypersurfaces. In fact, the set of points in $\mathbb{P}^n$ coming from a root system A_{n+1} has no unexpected hypersurfaces for any degree and multiplicity but its generic initial ideal fails to be a lex-segment.

As an application we study the case of a codimension two complete intersection C in $\mathbb{P}^n$. We apply a new method to study the question of unexpected hypersurfaces, namely the theory of *partial elimination ideals*. Using standard tools we show that $\dim[I_C \cap I_P^m]_t > 0$ for prescribed values of t and m.

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Unexpectedly found unexpected curves

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(joint work with Grzegorz Malara, Halszka Tutaj-Gasińska)

Let $Z = P_1 + ... + P_s$ be a reduced scheme of mutually distinct points in $\mathbb{P}^2_{\mathbb{C}}$. We say that Z admits an unexpected curve of degree d if for a general point $P \in \mathbb{P}^2_{\mathbb{C}}$ of multiplicity m we have that

$$\dim_{\mathbb{C}}[I(Z + mP)]_d > \max\left\{\dim_{\mathbb{C}}[I(Z)]_d - \binom{m+1}{2}, 0\right\}$$

with $I(Z + mP) = I(P_1) \cap ... \cap I(P_s) \cap I(P)^m$.

A general question that we can ask is to classify all those configurations of mutually distinct points Z that admit unexpected curves. In [2], Cook II, Harbourne, Migliore, and Nagel study unexpected curves from the viewpoint of line arrangements in the complex projective plane, i.e., in their setting Z denotes the set of points which are dual to lines of a given arrangement $\mathcal{A} \subset \mathbb{P}^2_{\mathbb{C}}$.

Consider a set of points $Z = \{z_1, \ldots, z_d\}$ in $\mathbb{P}^2_{\mathbb{C}}$ and the dual line arrangement $\mathcal{A}_Z = \{\ell_1, \ldots, \ell_d\}$ given by the defining polynomial f. Let us recall that the module of logarithmic derivations D is a submodule of $\mathrm{Der}(S)$ (the module of all $\mathbb{C}$-linear derivations) consisting of all elements $\delta \in \mathrm{Der}(S)$ such that $\delta(f) \in S \cdot \langle f \rangle$. Obviously D contains the Euler derivation $\partial_E = x\partial_x + y\partial_y + z\partial_z$. We know that the quotient $D_0 = D/\partial_E$ is isomorphic to the twist of $\mathrm{Syz}\, J_f$, namely we have the following exact sequence of sheaves

$$0 \to \mathcal{D}_0 \to \mathcal{O}^3_{\mathbb{P}^2_{\mathbb{C}}} \to \mathcal{J}_f(d-1) \to 0,$$

where $\mathcal{J}_f$ is the sheafification of the Jacobian ideal J_f and $\mathcal{D}_0$ is a locally free sheaf of rank 2. It is well-known that $\mathcal{D}_0$ restricted to a generic line splits, according to Grothendieck's theorem, as a sum of line bundles $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-a_Z) \bigoplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-b_Z)$. If the line is generic, then the pair (a_Z, b_Z) is called the splitting type of $\mathcal{D}_0$ and it satisfies $a_Z + b_Z = |Z| - 1$.

In [2] the authors consider the case when an unexpected curve is of degree $d = m + 1$, where m is the multiplicity that the curve has in a given general point P. They proved the following theorem, which we quote in a version changed according to Dimca's paper [3].

Theorem 1. *Let Z be the finite set of points in $\mathbb{P}^2_{\mathbb{C}}$. Let (a_Z, b_Z) be the splitting type of the derivation bundle $\mathcal{D}_0$. Let $m(\mathcal{A}_Z)$ be the of maximal multiplicity among the singular points of the arrangement $\mathcal{A}_Z$. Then Z admits and unexpected curve of degree m with a general point Q of multiplicity $m-1$ if and only if*

$$m(\mathcal{A}_Z) \leq a_Z + 1 < \frac{|Z|}{2}.$$

Nearly free arrangements

Let us recall [2, Example 6.1] which motivates our investigations in this section.

Example 4. *Consider the arrangement $\mathcal{A} \subset \mathbb{P}^2_{\mathbb{C}}$ of 19 lines given by the following defining polynomial:*

$$\begin{aligned} Q(x,y,z) = {} & xyz(x+y)(x-y)(2x+y)(2x-y)(x+z)(x-z)(y+z)(y-z) \\ & (x+2z)(x-2z)(y+2z)(y-2z)(x-y+z)(x-y-z) \\ & (x-y+2z)(x-y-2z). \end{aligned}$$

We can compute the minimal free resolution of the Milnor algebra $M(Q) := S/J_Q$ obtaining

$$0 \to S(-30) \to S(-29)^2 \oplus S(-26) \to S(-18)^3 \to S.$$

This arrangement, as it was said in [2], is close *to be free in the sense of the addition-deletion procedure, namely if we remove the line $2x+y$, then the new arrangement $\mathcal{A}'$ is free with exponents $d_1 = 7$ and $d_2 = 10$. It turns out that the set of duals to lines in $\mathcal{A}$ admits an unexpected curve of degree 9 having at a general point P multiplicity 8.*

Now we are going to explain what actually means to be close to be free. Let us denote by $\mathfrak{m} = \langle x, y, z\rangle$ the irrelevant ideal. Consider the graded S-module

$$N(f) = I_f/J_f = H^0_{\mathfrak{m}}(S/J_f),$$

where I_f is the saturation of J_f with respect to $\mathfrak{m}$.

Definition 5. *We say that a reduced plane curve C is* nearly free *if $N(f) \neq 0$ and for every k one has $\dim N(f)_k \leq 1$.*

A lot of work has been done to understand geometrical and combinatorial properties of nearly free curves, both from a viewpoint of vector bundles [6] and homological properties of those curves [4].

The description of the Milnor algebra $M(f)$ for a nearly free curve $C : f = 0$ comes from [4].

Theorem 2 (Dimca-Sticlaru). *If C is a nearly free curve given by $f \in S$, then the minimal resolution of the Milnor algebra $M(f)$ has the following form:*

$$0 \to S(-b-2(d-1)) \to S(-d_1-(d-1)) \oplus S(-d_2-(d-1)) \oplus S(-d_3-(d-1)) \to S^3(-d+1) \to S$$

for some integers d_1, d_2, d_3, b such that $d_1 + d_2 = d$, $d_2 = d_3$, and $b = d_2 - d + 2$. In that case, the pair (d_1, d_2) is called the set of exponents of nearly free curve C.

Our main result of this section provides a whole family of nearly free arrangements which are non-examples with respect to [6, Proposition 3.1.].

We start with the well-known family of Fermat line arrangements $\mathcal{F}_n$ in $\mathbb{P}^2_{\mathbb{C}}$ given by the following defining polynomial

$$Q(x,y,z) = (x^n - y^n)(y^n - z^n)(z^n - x^n)$$

with $n \geq 3$. This arrangement has exactly n^2 triple points and 3 points of multiplicity n, and it is known to be free with the exponents $(n+1, 2n-2)$. We consider the arrangement $\mathcal{NF}_n$ defined by the following equation

$$\widetilde{Q}(x,y,z) = (x^n - y^n)(y^n - z^n)(z^n - x^n)/(x-y) =$$
$$(x^{n-1} + yx^{n-2} + ... + y^{n-2}x + y^{n-1})(y^n - z^n)(z^n - x^n)$$

with $n \geq 3$.

Let us present the most important combinatorial properties of $\mathcal{NF}_n$.

(1) For $n = 3$ we obtain an arrangement which is isomorphic to the famous MacLane arrangement of 8 lines – according to our best knowledge this fact is not written explicitly in the literature.
(2) For given $n \geq 3$ the arrangement $\mathcal{NF}_n$ has exactly two points of multiplicity n, one point of multiplicity $n-1$, $n^2 - n$ triple points, and n double points.

Now we show main results devoted to nearly free arrangements – [5]. The first result is devoted to deletion arrangements.

Theorem 3. *For $n \geq 3$ the arrangement $\mathcal{NF}_n$ is nearly free with the exponents $d_1 = n + 1$ and $d_2 = 2n - 2$.*

Here is our main result devoted to unexpected curves associated with nearly free arrangements which comes from

Theorem 4. *Let $Z = \{z_1, \ldots, z_d\} \subset \mathbb{P}^2_{\mathbb{C}}$ be a set of points such that the dual lines give a nearly free arrangement $\mathcal{A}$ with the exponents (d_1, d_2). Then Z admits an unexpected curve of degree $d_1 + 1$ with a general point Q of multiplicity d_1 if and only if $d_2 - d_1 \geq 3$.*

Example 6. *Let us come back to family $\mathcal{NF}_n$ with $n \geq 3$. Then the dual set of points Z_n to $\mathcal{NF}_n$ admits an unexpected curve of degree $d_1 + 1$ with a general point Q of multiplicity d_1 if and only if*

$$d_2 - d_1 = 2n - 2 - (n+1) = n - 3 \geq 3,$$

so exactly when $n \geq 6$.

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Complete quadrics and linear spaces of symmetric matrices

Martin Vodicka

(joint work with Laurent Manivel, Mateusz Michalek, Leonid Monin, Tim Sennaeve)

Let L be a general a-dimensional linear subspace of $n \times n$ symmetric matrices L. Consider the variety L^{-1}, which is obtained by inverting all matrices in L. More precisely, we invert only regular matrices and take the closure of the image of this rational map. What is the degree of L^{-1}? We denote the answer by $\phi(a, n)$.

This question can be reformulated. Firstly we pass to the projective space $\mathbb{P}(S^2(V))$. The degree L^{-1} is equal to the number of points of $\mathbb{P}(S^2(V))$ which lie in L and their inverses satisfy $a-1$ given linear condition. If we think of elements of $\mathbb{P}(S^2(V))$ as quadrics in $\mathbb{P}^{n-1}$ we get another equivalent problem. How many quadrics in $\mathbb{P}^{n-1}$ pass through $\binom{n-1}{2} - a$ given points and are tangent to $a-1$ given hyperplanes?

One of our goals is to compute these numbers (effecitvely). The answer has also meaning for statisticians, because it is maximum likelyhood degree of linear covariance model. On the other hand, it is also interesing theoretical problem. We are also interested in nice properties of these numbers. So far, only few values was known - for $n \leq 7$ or $a \leq 4$. In the latter case it was known to be a polynomial in n and it was conjectured by Sturmfels, Uhler [6] that it is always a polynomial.

Our first idea was to compute numbers $\phi(a, n)$ using intersection theory. However, we can not work in the space $\mathbb{P}(S^2(V))$ because the low rank matrices will cause problem here, since their cofactor matrices are zero and satisfy any linear condition. Instead we pass to the space of complete quadrics.

Definition 1. *The space of complete quadrics* $\mathrm{CQ}(V)$ *is the closure of* $\phi(\mathbb{P}(S^2(V))^\circ)$, *where*

$$\phi : \mathbb{P}(S^2(V))^\circ \to \mathbb{P}(S^2(V)) \times \mathbb{P}\left(S^2\left(V \wedge V\right)\right) \times \ldots \times \mathbb{P}\left(S^2\left(\bigwedge^{n-1} V\right)\right),$$

given by

$$A \mapsto (A, \bigwedge^2 A, \ldots, \bigwedge^{n-1} A).$$

There is a lot of literature about this space and there are many nice results [2, 3, 4]. However, the answer to our problem was not known.

In the space of complete quadrics there are two types of divisors $L_1, \ldots, L_{n-1}$ and $S_1, \ldots, S_{n-1}$. L_i are pullbacks of hyperplanes under

$$\pi_i : \mathrm{CQ}(V) \to \mathbb{P}\left(S^2\left(\bigwedge^i V\right)\right).$$

The divisors S_i consist of tuples $(A_1, \ldots, A_{n-1})$ where rank of A_i is 1. The classes of this divisors are not independent. In fact L_i generate $\mathrm{Pic}(\mathrm{CQ})$ and the following relations hold:

$$S_i = -L_{i-1} + 2L_i - L_{i+1}$$

This allows us to express $\phi(a, n)$ as the intersection product in the cohomology ring of the space of complete quadrics:

$$\phi(a, n) = L_1^{\binom{n+1}{2}-a} L_{n-1}^{a-1}$$

Using relations between divisors and intersection theory on Grassmanian, and a lot of known formulas from [4] we were able to find recurrent formulas for computing numbers $\phi(a, n)$. Using Hard Lefschetz Theorem and Hodge-Riemann relations it can be shown that for the fixed n they form a log-concave sequence. We were also prove our main result:

Theorem 2. *Let a be a positive integer. Then $\phi(a, n)$ is a polynomial in n.*

However, there are many other questions we can ask. For example our computations suggest that he coefficients of these polynomials also form a log-concave sequence. Another interesting thing to look at is what happen when we consider different intersection product, i.e. not only those with L_1 and L_{n-1}.

When one asks the same question for diagonal matrices instead of symmetric matrices, then one can get to study the permutohedral variety. Therefore, our approach is, in some sense, a generalization of the work of Adiprasito, Huh and Katz [1], where they proved famous Rota conjecture that the coefficients of characteristic polynomial of a matroid form a log-concave sequence. We believe that studying the variety of complete quadrics its cohomology and using Lefschetz properties it is possible to obtain more interesting results.

Despite the fact that the variety of complete quadrics have been intensively studied for a long time, there are still a lot of unanswered problems. Understanding them and solving them may have applications in various branches of mathematics.

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Radical and Perfection of Two-Rowed Specht Ideals

Chris McDaniel
(joint work with Junzo Watanabe)

Let $R = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring in n-variables over a field of characteristic zero. For a subset indices $S \subset \{1, \ldots, n\}$ define the S-Vandermonde polynomial $\Delta_S = \prod_{i<j \in S}(x_i - x_j)$. A tableau T on a partition $\lambda \vdash n$ is a labeling of the boxes of the Young diagram for λ with numbers $\{1, \ldots, n\}$. The Specht polynomial of a tableau T on λ is the product of C-Vandermonde polynomials as C ranges over the columns of T, i.e.

$$F_T = \prod_{C \in \mathrm{col}(T)} \Delta_C.$$

Their $\mathbb{F}$-span, as T ranges over $\mathrm{tab}(\lambda)$, the set of all tableau on λ, forms an irreducible representation for the symmetric group $\mathfrak{S}_n$, called the Specht module of λ, i.e.

$$V(\lambda) = \mathrm{span}_{\mathbb{F}}\left(F_T \mid T \in \mathrm{tab}(\lambda)\right).$$

The Specht ideal of λ is the ideal in R generated by the Specht module, i.e.

$$\mathfrak{a}(\lambda) = V(\lambda).R.$$

In his recent paper [3], Yanagawa asks for which λ is the ideal $\mathfrak{a}(\lambda)$ perfect, or equivalently, for which λ is the quotient ring $R/\mathfrak{a}(\lambda)$ Cohen-Macaulay? He proves the following results:

Proposition 1. *If $\lambda = (n-k, k)$, then $\mathfrak{a}(\lambda)$ is radical.*

Proposition 2. *If $\lambda = (n-k, k)$ then $\mathfrak{a}(\lambda)$ is perfect.*

Proposition 2 essentially follows from Proposition 1, and a deep theorem from the representation theory of rational Cherednik algebras obtained by Etingof, Gorsky, and Losev [1]. Our motivating goal in this project is to give an independent, self-contained proof of Proposition 2 which does not appeal to Cherednik alegbras. In this joint talk, I will discuss Proposition 1 and radicals of two rowed Specht ideals, and Junzo will discuss their perfection.

In an attempt to understand Yanagawa's proof of Proposition 1 we introduce the so-called shifted Specht polynomials, modules, and ideals. For a tableau T on $\lambda = (n-k, k)$, fix an integer $k \leq d \leq n-k$, and if

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline i_1 & \cdots & i_k & i_{k+1} & \cdots & i_d & i_{d+1} & \cdots & i_{n-k} \\ \hline j_1 & \cdots & j_k \\ \cline{1-3}\end{array}$$

then define its d-shifted Specht polynomial as

$$F_T(d) = (x_{i_1} - x_{j_1}) \cdots (x_{i_k} - x_{j_k}) x_{i_{k+1}} \cdots x_{i_d}.$$

Their $\mathbb{F}$-span is a finite dimensional $\mathfrak{S}_n$ representation called the d shifted Specht module

$$V(n,k,d) = \operatorname{span}_{\mathbb{F}} \left(F_T(d) \mid T \in \operatorname{tab}(\lambda) \right)$$

and the ideal it generates is the d-shifted Specht ideal

$$\mathfrak{a}(n,k,d) = V(n,k,d).R.$$

In case $k = 0$, $V(n,0,d)$ and $\mathfrak{a}(n,0,d)$ are generated by square free monomials of degree d; for this ideal we also use the notation $(x_1, \ldots, x_n)^{(d)}$. In the other extreme, $d = k$, $V(n,k,k)$ and $\mathfrak{a}(n,k,k)$ are the usual Specht objects, generated by Specht polynomials of degree k. In this sense we regard the shifted Specht objects as interpolating between square free monomial objects and Specht objects.

A well known fact from representation theory of the symmetric group, e.g. [2], states that a basis for the Specht module $V(n,k,k)$, and hence a minimal generating set for the Specht ideal $\mathfrak{a}(n,k,k)$ are the Specht polynomials indexed by the standard tableaux, i.e. tableaux in which the rows and columns are increasing. We denote this set of standard tableaux as $\operatorname{stab}(\lambda)$, and hence we have:

$$V(n,k,k) = \operatorname{span}_{\mathbb{F}} \left(F_T \mid T \in \operatorname{stab}(\lambda) \right).$$

In general, the standard tableaux on λ are insufficient to generate the shifted Specht modules, as the following example shows.

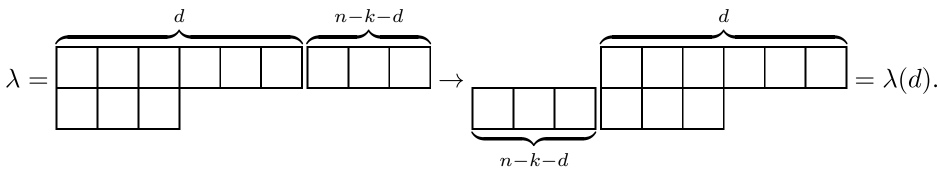
Example 3. Let $n = 4$, $k = 1$, and $d = 2$ so that $\lambda = (3,1)$. Then the standard tableaux on λ are

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$$

but to generate $V(4,1,2)$ we need the additional, non-standard tableaux

$$\begin{array}{|c|c|c|}\hline 1 & 4 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 4 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

It turns out that a minimal generating set for $V(n,k,d)$ can still be indexed by standard tableaux, not on λ but on the shifted shape $\lambda(d)$ obtained from the shape λ by moving the excess $n-k-d$ boxes from the right end of row one to the left end of row two, i.e.



We identify tableaux on λ with tableaux on $\lambda(d)$ in the obvious way.

Theorem 4 (M.-Watanabe, 2020)**.** *A basis for $V(n,k,d)$, and hence a minimal generating set for $\mathfrak{a}(n,k,d)$ are Specht polynomials indexed by standard tableaux on $\lambda(d)$, i.e.*

$$V(n,k,d) = \operatorname{span}\left(F_T(d) \mid T \in \operatorname{stab}(\lambda(d))\right).$$

Example 5. Let $n=4$, $k=1$, and $d=2$ so that $\lambda=(3,1)$. Then the standard tableaux on $\lambda(2)$ are

$$\begin{array}{ccc} & 1 & 2 \\ 3 & 4 & \end{array}, \quad \begin{array}{ccc} & 1 & 3 \\ 2 & 4 & \end{array}, \quad \begin{array}{ccc} & 1 & 4 \\ 2 & 3 & \end{array}, \quad \begin{array}{ccc} & 2 & 3 \\ 1 & 4 & \end{array}, \quad \begin{array}{ccc} & 2 & 4 \\ 1 & 3 & \end{array},$$

which form a basis for $V(4,1,2)$.

Using a familiar bijective correspondence with lattice paths, we get a dimension count, and a surprising application of Lefschetz properties (the namesake of this workshop!) yields an $\mathfrak{S}_n$-module decomposition of the shifted Specht module:

Theorem 6 (M.-Watanabe, 2020)**.** *The shifted Specht module has dimension*

$$\dim\left(V(n,k,d)\right) = \binom{n}{d} - \binom{n}{k-1}.$$

It decomposes as a direct sum of irreducible $\mathfrak{S}_n$ modules by

$$V(n,k,d) = \bigoplus_{i=k}^{d} V(n,i,i)[d-i].$$

Finally, using Theorem 4, we prove the following generalization of a result of Yanagawa.

Theorem 7 (M.-Watanabe, 2020)**.** *For any integers $0 \le k \le d \le n-k$ we have an equality of ideals*

$$\mathfrak{a}(n,k,d) = \mathfrak{a}(n,k,k) \cap (x_1,\ldots,x_n)^{(d)}.$$

Yanagawa essentially proved Theorem 7 for $d=k+1$, and using this in conjunction with some basic commutative algebra, went on to prove Proposition 1. It turns out that his arguments apply to our more general set up, and we obtain the following:

Theorem 8 (M.-Watanabe, 2020)**.** *For all $0 \le k \le d \le n-k$ the shifted Specht ideal $\mathfrak{a}(n,k,d)$ is radical.*

It also follows from Theorem 7 that the shifted Specht ideals cannot be pure, or, in particular, perfect, unless $d = k$ or $d = k + 1$. In fact we have the following result, which Junzo will discuss further in his talk:

Theorem 9 (M.-Watanabe, 2020)**.** *The shifted Specht ideal* $\mathfrak{a}(n, k, d)$ *is perfect if and only if* $d = k$ *or* $d = k + 1$.

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The perfection of Specht ideals corresponding to two-row Young diagrams and related topics

Junzo Watanabe

(joint work with Chris, McDaniel)

$\mathbb{F}$ denotes a field. The characteristic of $\mathbb{F}$ is assumed to be zero unless otherwise specified. Let $R = \mathbb{F}[x_1, \dots, x_n]$ be the polynomial ring and $E \subset R$ the graded vector subspace spanned by the square free monomials. We are interested in the algebra $A = R/(x_1^2, \dots, x_n^2)$, the quadratic monomial complete intersection and the element $l = \sum x_i$, the sum of variables. It is well known that A has the SLP and l is an SL element. This means that

$$\times l^{n-2i} : A_i \to A_{n-i}$$

is an isomorphism of vector spaces for all $i = 0, 1, \dots, [n/2]$.

The composition of maps

$$E \hookrightarrow R \twoheadrightarrow A$$

is an isomorphism of graded vector spaces. With this isomorphism we identify E and A. Let $\mathfrak{S}_n$ be the symmetric group acting on R by permutation of the variables. The action of $\mathfrak{S}_n$ can be restricted on E. Hence E is an $\mathfrak{S}_n$-module.

The multiplication map $\times l : R \to R$ induces an $\mathfrak{S}_n$-module map on A. With the identification $E = A$, it is possible to define the map $\times l : E \to E$. We denote this map by L. For $f \in E$, the image $Lf \in E$ by L is obtained by expanding lf as an element in R and ignoring the terms which include squared variables. Let $D = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}$. It is easy to see that the three operations of $L, D, [L, D]$ on E are an $\mathfrak{sl}_2$-triple.

We may write $\ker\big(D : E \to E\big)$ as a sum of homogeneous spaces:

$$\ker\big(D : E \to E\big) = \sum_{i=0}^{[n/2]} V(n, i), \ V(n, i) \subset R_i.$$

Since $D : E_i \to E_{i-1}$ is surjective for $i \le [n/2]$, we see that the dimension of $V(n,i)$ is

$$|V(n,i)| = \binom{n}{i} - \binom{n}{i-1}.$$

It seems noteworthy that if $i = n/2$ or $i = (n-1)/2$, this number is the Catalan number. Also note that the kernel of the operator $D : R \to R$ is the set of solutions of the linear partial differential equation $\left(\frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}\right)F = 0$. One sees easily that the kernel of $D : R \to R$ is the subring

$$K[x_1 - x_2, \ldots, x_1 - x_n].$$

Thus we have

$$V(n,i) = K[x_1 - x_2, \ldots, x_1 - x_n] \cap E_i,$$

and we are almost forced to conclude that $V(n,i)$ is spanned by products of i linear binomials $x_p - x_q$ without overlaps of indices. We take it for granted that $V(n,i)$ is an irreducible $\mathfrak{S}_n$-modules. $V(n,i)$ is the Specht module corresponding to the partition $\lambda = (n-i, i)$ of the integer n. As an isomorphism type of $\mathfrak{S}_n$-module we write $V(n,i) \cong V^{\lambda}$. To each $V(n,i)$ we may apply L as many times as possible. Then

$$E = \bigcup_{i,j} L^j V(n,i)$$

is the decomposition of the space E into irreducible $\mathfrak{S}_n$-modules. This can be illustrated by the following diagram: $(n = 6)$

			$V(6,3)$			
		$V(6,2)$	$LV(6,2)$	$L^2V(6,2)$		
	$V(6,1)$	$LV(6,1)$	$L^2V(6,1)$	$L^3V(6,1)$	$L^4V(6,1)$	
$V(6,0)$	$LV(6,0)$	$L^2V(6,0)$	$L^3V(6,0)$	$L^4V(6,0)$	$L^5V(6,0)$	$L^6V(6,0)$

Horizontally all boxes are isomorphic to $V^{(n-i,i)}$ and vertically no two boxes are isomorphic.

Definition 1. For a triple of integers n, k, d satisfying $0 \le k \le d \le n - k$, we define $\boxed{\mathfrak{a}(n,k,d)}$ to be the ideal in R generated by elements in

$$\{L^j V(n,i) \mid i + j = d, i \ge k\}.$$

(These are the boxes at degree d and at level k or higher.)

Recently we succeeded in proving the following theorem extending the results in [2] and [8].

Theorem 2. *(1) $\mathfrak{a}(n,k,d)$ is radical over $\mathbb{F}$ in any characteristic.*
(2) $\mathfrak{a}(n,k,k+1)$ is perfect over $\mathbb{F}$ with char $\mathbb{F} = 0$.
(3) $\mathfrak{a}(n,k,d)$ is perfect if and only if it is grade unmixed, i.e., $d = k$ or $k+1$. (char $\mathbb{F} = 0$)

Remark 3. Etingof et al. [2] proved that $\sqrt{\mathfrak{a}(n,k,k)}$ is perfect over $\mathbb{F}$ with char $\mathbb{F} = 0$. Yanagawa [8] proved that $\mathfrak{a}(n,k,k)$ is radical (i.e., $\sqrt{\mathfrak{a}(n,k,k)} = \mathfrak{a}(n,k,k)$) in any characteristic.

The ideal $\mathfrak{a}(n,k,k)$ is defined by products of linear forms. So it can be viewed as a linear space arrangement. In fact

$$\mathfrak{a}(n,k,k) = \bigcap_{1\le i_1<\ldots<i_{n-k+1}\le n} (x_{i_1} - x_{i_2}, x_{i_1} - x_{i_3}, \ldots, x_{i_1} - x_{i_{n-k+1}}).$$

The RHS will be denoted by

$$\overline{(x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_{n-k+1})}.$$

("Overline" means to take the intersection of all such linear primes obtained by permutation of indices.)

A similar ideal is the intersection of coordinate primes of an equal height. Define the reduced power of the maximal ideal $(x_1,\ldots,x_n)^{(k)}$ to be the ideal generated by the square-free monomials of degree k. Then we have

$$\begin{aligned}(x_1,\ldots,x_n)^{(k)} &= (\{x_{i_1}x_{i_2}\cdots x_{i_k} \mid i_j \ne i_l\}) \\ &= \bigcap_{1\le i_1<\cdots<i_{n-k+1}\le n} (x_{i_1}, x_{i_2}, \ldots, x_{i_{n-k+1}}) \\ &= \overline{(x_1, x_2, \ldots, x_{n-k+1})}.\end{aligned}$$

Recall Hochster-Eagon [5] proved the following result.

Proposition 4 ([5]). *Suppose that I, J are perfect of grade g. Then $I\cap J$ is perfect if and only if $I + J$ is perfect of grade $g + 1$.*

Proof is easy by Meyer–Vietoris exact sequence:

$$0 \to R/I\cap J \to R/I \oplus R/J \to R/(I + J) \to 0.$$

It is easy to see that both $\mathfrak{a}(n,k,k)$ and $(x_1,\ldots,x_n)^{(k+1)}$ have the same grade $n-k$ and their sum $n-k+1$. Hence we have

Proposition 5. *The following conditions are equivalent.*

(1) $\mathfrak{a}(n,k,k) + (x_1,\ldots,x_n)^{(k+1)}$ *is perfect.*
(2) $\mathfrak{a}(n,k,k) \cap (x_1,\ldots,x_n)^{(k+1)}$ *is perfect.*

Claim (2) can be proved by the following two equalities:

$$\mathfrak{a}(n+1,k+1,k+1) + (x_{n+1}) = \mathfrak{a}(n,k,k+1) + (x_{n+1})$$

(provided $k < n/2$ if n is even) and

$$\mathfrak{a}(n,k,k+1) = \mathfrak{a}(n,k,k) \cap (x_1,\ldots,x_n)^{(k+1)} = \text{ideal } (2).$$

So Theorem 2 implies the following:

Corollary 6 (to Theorem 2). *$\mathfrak{a}(n,k,k) + (x_1,\dots,x_n)^{(k+1)}$ is perfect of grade $n-k+1$.*

Surprisingly enough we have the primary decomposition of this ideal as follows:

Theorem 7. *Under the assumption char of $\mathbb{F}$ is zero we have*

$$\begin{aligned}\mathfrak{a}(n,k,k) &+ (x_1,\dots,x_n)^{(k+1)}\\ &= \overline{(x_1-x_2, x_1-x_3, \dots, x_1-x_{n-k+1}, x_1^2)}.\end{aligned}$$

If char $\mathbb{F} > 0$, then an embedded component can appear.

In this primary decomposition if we replace x_i^2 by x_i^r, it is not easy to describe a set of generators (see Example 10) but it is possible to prove that the ideal is perfect.

Theorem 8. *The ideal*

$$\bigcap_{1\le i_1<\cdots<i_{n-k+1}\le n} (x_{i_1}-x_{i_2}, \cdots, x_{i_i}-x_{i_{n-k+1}}, x_{i_1}^r)$$

is perfect (independent of characteristic).

Note that if $r=2$, this can be proved as a consequence of results of Etingof et al. [2], but for $r>2$ we need a new proof. Tracing back the argument for Theorem 8 we can prove the perfection of the Specht ideal $\mathfrak{a}(n,k,k)$.

Example 9. Let $R = \mathbb{F}[x_1,x_2,x_3,x_4]$. Let $f_1 = (x_1-x_2)(x_3-x_4)$, $f_2 = (x_1-x_3)(x_2-x_4)$, $g_1 = x_2x_3x_4$, $g_2 = x_1x_3x_4$, $g_3 = x_1x_2x_4$, $g_4 = x_1x_2x_3$. Then by definition

$$\mathfrak{a}(4,2,2) = (f_1, f_2),$$

and

$$(x_1,x_2,x_3,x_4)^{(3)} = (g_1,g_2,g_3,g_4).$$

Theorem 7 says that, if $\mathbb{F} = \mathbb{Q}$,

$$\begin{aligned}\mathfrak{a}(4,2,2) + (x_1,\dots,x_4)^{(3)} =& (x_1^2, x_1-x_2, x_1-x_3) \cap (x_1^2, x_1-x_2, x_1-x_4)\\ &\cap (x_1^2, x_1-x_3, x_1-x_4) \cap (x_2^2, x_2-x_3, x_2-x_4)\end{aligned}$$

This is not true if char $\mathbb{F} = 2$. Indeed, over $\mathbb{F} = \mathbb{Z}/(2)$ a primary decomposition of $\mathfrak{a}(4,2,2) + (x_1,\dots,x_4)^{(3)}$ is obtained as

$$\begin{aligned}\mathfrak{a}(4,2,2) + (x_1,\dots,x_4)^{(3)} =& (x_1^2, x_1-x_2, x_1-x_3) \cap (x_1^2, x_1-x_2, x_1-x_4)\\ &\cap (x_1^2, x_1-x_3, x_1-x_4) \cap (x_2^2, x_2-x_3, x_2-x_4)\\ &\cap (x_1^2, x_2^2, x_3^2, x_4^2, f_1, f_2, x_2x_3x_4)\end{aligned}$$

Furthermore it turns out that

$$\begin{aligned}(h_1,h_2,h_3) =& (x_1^2, x_1-x_2, x_1-x_3) \cap (x_1^2, x_1-x_2, x_1-x_4)\\ &\cap (x_1^2, x_1-x_3, x_1-x_4) \cap (x_2^2, x_2-x_3, x_2-x_4)\end{aligned}$$

where $h_1 = x_2x_3 + (x_2 + x_3)x_4$, $h_2 = x_1(x_3 + x_4) + x_3x_4$, $h_3 = x_1x_2 + (x_1 + x_2)x_4$. The three dimensional vector space $\langle h_1, h_2, h_3\rangle$ is an irreducible $\mathfrak{S}_4$-module, which is not isomorphic to any Specht module.

Example 10. Let $R = \mathbb{Q}[x_1, x_2, x_3, x_4]$. Put

$$I = \bigcap_{1\le i_1<i_2<i_3\le 4} (x_{i_1}^r, x_{i_1} - x_{i_2}, x_{i_1} - x_{i_3}).$$

Then $I = (f_1, f_2, g_1, g_2, g_3, g_4)$, where

$$\begin{aligned}
f_1 &= (x_1 - x_2)(x_3 - x_4),\\
f_2 &= (x_1 - x_3)(x_2 - x_4)\\
g_1 &= (x_1 - x_2)x_4^r,\\
g_2 &= (x_1 - x_3)x_4^r,\\
g_3 &= (x_1 - x_4)x_3^r,\\
g_4 &= x_3x_4(x_2h_{r-2}(x_3, x_4) - x_3x_4h_{r-3}(x_3, x_4))
\end{aligned}$$

$h_d(u, v)$ is the complete symmetric function of degree d in two variables.

Note that $\langle f_1, f_2\rangle \cong V^{(2,2)}$. It is possible to choose g_i so that $\langle g_1, g_2, g_3\rangle \cong V^{(3,1)}$ and $\langle g_3\rangle \cong V^{(4)}$.

Related results can be found in [1], [3], [4], [6], [7].

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Progress in Tropical Hodge Theory

Graham Denham
(joint work with Federico Ardila, Graham Denham, June Huh)

One of the key ideas in Adiprasito, Huh and Katz's proof [2] of the conjectures of Rota, Heron and Welsh is a notion of Hodge theory for matroids. To a matroid M one associates a tropical linear space Σ_{M}, the Bergman fan. This fan has a Chow ring $A(\Sigma_{\mathrm{M}})$. They showed $A(\Sigma_{\mathrm{M}})$ possesses Poincaré duality, the Hard Lefschetz property, and the Hodge–Riemann relations with respect to strictly convex, piecewise-linear functions ℓ on Σ_{M}.

The Hodge–Riemann relations provide inequalities resembling those of mixed volumes in convex geometry. Such inequalities can be useful in conjunction with combinatorial interpretations of intersection indices of particular choices of the function ℓ. Indeed, we make use of this idea again in [3] to prove conjectures of Dawson and Brylawski about certain matroid h-vectors, where we introduce and study the conormal fan $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ of a matroid.

To be more precise, let Σ be a simplicial fan in a vector space $N \cong \mathbb{R}^n$. Let $A(\Sigma)$ be the ring of real-valued piecewise polynomial functions on Σ modulo the ideal generated by linear functions on Σ, and let $\mathcal{K}(\Sigma)$ be the cone of *strictly convex* piecewise-linear functions on Σ.

Definition 1. A d-dimensional simplicial fan Σ is *Lefschetz* if it satisfies the following.

(1) (Fundamental weight) The group of d-dimensional Minkowski weights on Σ is generated by a positive Minkowski weight w. We write deg for the corresponding linear isomorphism

$$\deg : A^d(\Sigma) \longrightarrow \mathbb{R}, \qquad \eta \longmapsto \eta \cap w.$$

(2) (Poincaré duality) For any $0 \le k \le d$, the bilinear map given by multiplication

$$A^k(\Sigma) \times A^{d-k}(\Sigma) \longrightarrow A^d(\Sigma) \xrightarrow{\deg} \mathbb{R}$$

is nondegenerate.

(3) (Hard Lefschetz property) For any $0 \le k \le \frac{d}{2}$ and any $\ell \in \mathcal{K}(\Sigma)$, the multiplication map

$$A^k(\Sigma) \to A^{d-k}(\Sigma), \qquad \eta \longmapsto \ell^{d-2k}\eta$$

is a linear isomorphism.

(4) (Hodge–Riemann relations) For any $0 \le k \le \frac{d}{2}$ and any $\ell \in \mathcal{K}(\Sigma)$, the bilinear form

$$A^k(\Sigma) \times A^k(\Sigma) \longmapsto \mathbb{R}, \qquad (\eta_1, \eta_2) \longmapsto (-1)^k \deg(\ell^{d-2k}\eta_1\eta_2)$$

is positive definite when restricted to the kernel of the multiplication map ℓ^{d-2k+1}.

(5) (Hereditary property) For any $0 < k \leq d$ and any k-dimensional cone σ in Σ, the star of σ in Σ is a Lefschetz fan of dimension $d - k$.

We show that the Lefschetz property is geometric, in the following sense.

Theorem 2 ([3]). *Let Σ_1 and Σ_2 be simplicial fans that have the same support. If $\mathcal{K}(\Sigma_1)$ and $\mathcal{K}(\Sigma_2)$ are nonempty, then Σ_1 is Lefschetz if and only if Σ_2 is Lefschetz.*

For example, it follows that the reduced normal fan of any simple polytope is Lefschetz, since the reduced normal fan of a simplex is Lefschetz. This was first shown by McMullen in [5] and revisited in [6, 4], and the proof we give of Theorem 2 is modelled on those arguments. Using the toric Weak Factorization Theorem [1, 7], we reduce the problem to showing that the theorem holds when the fans Σ_1 and Σ_2 differ by a stellar subdivision of a two-dimensional cone.

Our motivating application is to show that the conormal fan $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ is Lefschetz. By construction, $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ has the same support as a product of Bergman fans $\Sigma_{\mathrm{M}} \times \Sigma_{\mathrm{M}^\perp}$. The two factors are Lefschetz [2], and the Lefschetz property is preserved by products.

Question 3. *Can one find other, combinatorially interesting families of tropical varieties with the Lefschetz property?*

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Asymptotic Behaviour of Hilbert function of Gorenstein algebras of socle degree four

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(joint work with Armando Cerminara, Rodrigo Gondim, Giuseppe Zappala)

For a standard graded Artinian Gorenstein algebra, when the codimension of the algebra is less than or equal to 3, all its Hilbert vectors are unimodal and they have been characterized (see [4] and [6]).

Problem 1. *While it is known that non unimodal Gorenstein h-vectors exist in every codimension greater than or equal to 5 (see [1]), it is open whether non unimodal Gorenstein h-vectors of codimension 4 exist.*

Historically, the first such example of a non unimodal Gorenstein h-vector was given by Stanley, that showed that the h-vector $(1,13,12,13,1)$ is indeed a Gorenstein h-vector and the non unimodality occurs here in degree 2 (see [4, Example 4.3]). Stanley's example is optimal and for our purposes we call it minimal, (see [3]). We study special Gorenstein h-vectors of type $(1,r,h_2,r,1)$. For the codimension r and denoting the least possible value that h_2 may assume by $f(r)$, we study the asymptotic behavior of $f(r)$.
Stanley in [5] conjectured:

Conjecture 2. *There exists the following limit*

$$\lim_{r\to\infty}\frac{f(r)}{r^{\frac{2}{3}}}$$

and the precise value is $6^{\frac{2}{3}}$.

The precise limit was only proved in 2006 (see [2]).
The ideas are the following:

- We construct a family of Gorenstein algebras called **Full Perazzo algebras** and our main result is that, for small m, the Hilbert vectors of Full Perazzo algebras of type m, are always minimal. Moreover, we are able to give a simple proof of Stanley's conjecture and we pointed out that the h-vector of the Stanley's example is a special case of a Full Perazzo algebra.
- We introduce another family of Artinian Gorenstein algebras having non unimodal Gorenstein h-vectors: the **Turan algebras** that are Artinian Gorestein algebra presented by quadrics. We have a conjecture about the asymptotic behaviour of Artinian Gorenstein algebra presented by quadric.

Definition 3. *Let* $\mathbb{K}[x_1,\dots,x_n,u_1,\dots,u_m]$ *be the polynomial ring in the* n *variables* $x_1,\dots,x_n$ *and in the* m *variables* $u_1,\dots,u_m$. *A* ***Perazzo polynomial*** *is a reduced bihomogeneous polynomial* $f\in\mathbb{K}[x_1,\dots,x_n,u_1,\dots,u_m]_{(1,d-1)}$, *of degree* d, *of type*

$$f=\sum_{i=1}^{n}x_ig_i \tag{1}$$

with $g_i\in\mathbb{K}[u_1,\dots,u_m]_{d-1}$, *for* $i=1,\dots,n$, *linearly independent and algebraically dependent polynomials in the variables* $u_1,\dots,u_m$.

Remark 4. *A Perazzo polynomial* $f\in\mathbb{K}[x_1,\dots,x_n,u_1,\dots,u_m]_{(1,d-1)}$ *of degree* d *is a Nagata polynomial, hence the algebra* $A=Q/\mathrm{Ann}(f)$, *associated to* f, *where* $Q=\mathbb{K}[X_1,\dots,X_n,U_1,\dots,U_m]$ *is the ring of the differential operators, can be realized as a Nagata idealization of order* 1, *socle degree* d *and codimension* $n+m$.

By above Remark (4), we can give the following definition:

Definition 5. *Let* $f\in\mathbb{K}[x_1,\dots,x_n,u_1,\dots,u_m]_{(1,d-1)}$ *be a Perazzo polynomial of degree* d. *The algebra* $A=Q/\mathrm{Ann}(f)$ *associated to* f *is called* ***Perazzo algebra*** *and it is a bigraded algebra of socle degree* d *and codimension* $n+m$.

Now we fix $m \geq 2$ and we consider the m variables $u_1, \ldots, u_m$. Let us consider $M_j = u_{j_1}^{\alpha_{j_1}} \cdots u_{j_m}^{\alpha_{j_m}}$ for $j = 1, \ldots, \tau_m$ where $\tau_m := \binom{m+d-2}{d-1}$ and $\alpha_{j_1} + \cdots + \alpha_{j_m} = d - 1$.

Definition 6. *A bihomogeneous polynomial $f \in \mathbb{K}[x_1, \ldots, x_{\tau_m}, u_1, \ldots, u_m]_{(1,d-1)}$ of degree d of type:*

$$f = \sum_{j=1}^{\tau_m} x_j M_j \tag{2}$$

is called ***Full Perazzo polynomial*** *of type m.*

Remark 7. *As in Remark (4), let $f \in \mathbb{K}[x_1, \ldots, x_{\tau_m}, u_1, \ldots, u_m]_{(1,d-1)}$ be a Full Perazzo polynomial of type m and of degree d, the algebra $A = Q/\mathrm{Ann}(f)$, associated to f, where $Q = \mathbb{K}[X_1, \ldots, X_{\tau_m}, U_1, \ldots, U_m]$ is the ring of the differential operators, can be realized as a Nagata idealization of order 1, socle degree d and codimension $m + \tau_m$.*

By above Remark (7), we can give the following definition:

Definition 8. *Let $f \in \mathbb{K}[x_1, \ldots, x_\tau, u_1, \ldots, u_m]_{(1,d-1)}$ be a Full Perazzo polynomial of degree d. The algebra $A = Q/\mathrm{Ann}(f)$ associated to f is called* ***Full Perazzo algebra*** *and it is a bigraded algebra of socle degree d and codimension $m + \tau_m$.*

Proposition 9. *Let A be a Full Perazzo algebra of type $m \geq 2$ and socle degree d. Then, for $k = 0, \ldots, \lfloor \frac{d}{2} \rfloor$, $h_k = \dim A_k = \binom{m+k-1}{k} + \binom{m+d-k-1}{d-k}$.*

For small m, the Full Perazzo algebra of type $m \geq 3$ and socle degree 4 has minimal h_2. Denoting h_2 by $\mu(r)$, where r is codimension of the Full Perazzo Algebra, we have the following conjecture:

Conjecture 10. *Consider the set of Gorenstein algebras of codimension $r = m + \binom{m+2}{3}$ and socle degree 4. Then $\mu(r) = m(m+1)$.*

The following result is a generalization of the main result of [3].

Theorem 11. *Let A be a Gorenstein $\mathbb{K}$-algebra of socle degree 4, codimension r, with $r = m + \binom{m+2}{3}$ and $m \geq 3$. Let $m = (m)_{(2)} = \binom{s}{2} + \binom{t}{1}$ with $m > s > t \geq 0$ and $\varepsilon = \binom{t-1}{0}$. Let*

$$\tau = \lfloor m + \frac{3}{2} - s - \varepsilon - \sqrt{s^2 - 3s + 2t + \frac{1}{4}} \rfloor + 1.$$

Then $m(m+1) - \tau < \mu(r) \leq m(m+1)$. Moreover, for $m = 3, 4, 5$, we get $\mu(r) = m(m+1)$ and the Full Perazzo algebras have minimal second entry of the Hilbert vector.

Let r and d be two integers, we consider the family of standard graded Artinian Gorenstein $\mathbb{K}$-algebras of codimension r and socle degree d, $\mathcal{G}(r, d) :=$

$\left\{A : \ A \simeq \frac{Q}{\mathrm{Ann}(f)}\right\}$, with some $f \in \mathbb{K}[x_1, \dots, x_r]_d$ homogeneous polynomial of degree d. We define, for $0 < k < d$ an integer, the following non-decreasing function:

$$\mu_k(r,d) = \min_{A \in \mathcal{G}(r,d)} \{\dim A_k\}, \quad \delta_k(r,d) = n - \mu_k(r,d).$$

Fixed d, the above functions can be written without dependence by d, hence $\mu_k(r,d) = \mu_k(r)$ and $\delta_k(r,d) = \delta_k(r)$. Then, analyzing the case $d = 4$, we give a new short proof of the Theorem in [2] solving Stanley's conjecture.

Theorem 12. *Let $A \in \mathcal{G}(r)$ be a Gorenstein algebra of codimension r. Then* $\lim_{r\to\infty} \dfrac{[\mu(r)]}{r^{2/3}} = 6^{2/3}$.

Finally a family of Artinian Gorestein algebras, presented by quadrics, having non unimodal Hilbert function, are the Turan algebras, inspired by the famous Turan's Graph Theorem. Let $\mathcal{QG}(r)$ be the family of Artinian Gorenstein algebras presented by quadrics with socle degree 4 and codimension r. Let us call $\nu(r) = \min_{A \in \mathcal{QG}(r)} \{\dim A_2\}$.

Conjecture 13. *Let $A \in \mathcal{QG}(r)$ be a Gorenstein algebra of codimension r. Then* $\lim_{r\to\infty} \dfrac{[\nu(r)]}{r^{2/3}} = 6$.

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Waring problems and the Lefschetz properties

Rodrigo Gondim

(joint work with Thiago Dias, Rodrigo Gondim)

The Waring problem, in number theory, asks for each exponent k, what is the minimum s such that every positive integer can be decomposed as a sum of at least s perfect k-th powers. Hilbert proved that for every $k \geq 2$ the Waring problem is well posed, that is, there is a s such that every positive integer can be decomposed as a sum of at least s perfect k-th powers.

In analogy, the algebraic Waring problem asks what is the minimum s such that any homogeneous polynomial $f \in \mathbb{C}[x_0, \dots, x_n]_d$, of degree d, can be decomposed as a sum of at least s perfect d-th powers of linear forms. This problem was solved by Alexander and Hirschowitz, at least generically. They studied the higher secant defect of Veronese varieties.
We are interested in three variants of the Waring problem, our focus are special forms.

We consider these notions of rank for $f \in R_d = \mathbb{C}[x_0, \dots, x_n]_d$.

(1) The *Waring rank* of f is its algebraic rank: it is the minimum $s = wrk(f)$ such that f can be decomposed as a sum of d-th powers of s linear forms.
(2) The *Border rank* of f is its geometric rank: it is the minimum $s = \underline{rk}(f)$ such that the class $[f] \in \mathbb{P}(R_d)$ belongs to the s-th secant variety of the Veronese variety $\mathcal{V}_d(\mathbb{P}^n) \subset \mathbb{P}(R_d)$.
(3) The *Cactus rank* of f is its schematic rank: it is the minimum $s = cr(f)$ such that there is a finite scheme K of length s, $K \subset \mathcal{V}_d(\mathbb{P}^n) \subset \mathbb{P}(R_d)$ such that $[f] \in < K >$.

We know that $\underline{rk}(f) \leq wrk(f)$ and $cr(f) \leq wrk(f)$, while in general $cr(f)$ and $\underline{rk}(f)$ are incomparable. Very few examples are known satisfying $cr(f) > \underline{rk}(f)$, they are called *wild forms*. According to the best of our knowledge, the first example of wild form was constructed by W. Buczyńska, J. Buczyński.

Example 1 (*BB = Un esempio semplicissimo, from Perazzo)*. Consider the cubic $f = xu^2 + y(u+v)^2 + zv^2 \in \mathbb{C}[x, y, z, u, v]_3$. It is easy to compute its Waring rank $wrk(f) = 9$. They showed, explicitly, that $\underline{rk}(f) \leq 5$. On the other hand, $cr(f) = 6$ which agrees with the description of f as the sum of three double points in the Veronese. To conclude that $cr(f) = 6$ the authors studied the saturation of the annihilator of f.

This idea was generalized for concise forms with minimal border rank and vanishing Hessian by Huang, Michałek and Ventura.

Theorem 2 (Huang, Michałek and Ventura). *Let $f \in R_d$ be a concise form with minimal border rank. If $\mathrm{hess}_f = 0$, then f is wild.*

Theorem 3 (Double annihilator Theorem of Macaulay). *Let $R = \mathbb{K}[x_0, x_1, \dots, x_n]$ and let $Q = \mathbb{K}[X_0, X_1, \dots, X_n]$ be the ring of differential operators. Let $A = \bigoplus_{i=0}^{d} A_i = Q/I$ be a standard graded Artinian $\mathbb{K}$-algebra. Then A is a standard graded Gorenstein algebra of socle degree d if and only if there exists $f \in R_d$ such that $A \simeq Q/\operatorname{Ann}(f)$.*

We say that f is concise if $\dim A_1 = n+1$, or equivalently $I_1 = 0$. In this case $codim A = n+1$.

Let $A = Q/\operatorname{Ann}(f)$ be a standard graded Artinian Gorenstein K-algebra of socle degree d. Let $k \leq l \leq d$ be two integers and let $B_k = (\alpha_1, \dots, \alpha_{m_k})$ be an ordered $\mathbb{K}$-linear basis of A_k and let $B_l = (\beta_1, \dots, \beta_{m_l})$ be an ordered $\mathbb{K}$-linear basis of A_l. The *mixed Hessian* of f of order (k,l) with respect to the basis B_k and B_l is the matrix $\mathrm{Hess}_f^{(k,l)} := [\alpha_i\beta_j(f)]_{m_k\times m_l}$. Moreover, we define $\mathrm{Hess}_f^k = \mathrm{Hess}_f^{(k,k)}$ and $\mathrm{hess}_f^k = \det(\mathrm{Hess}_f^k)$.

Theorem 4 (-,Zappala, 2018)**.** *Let $A = Q/\operatorname{Ann}_Q(f)$ be a n AG algebra and $L \in A_1$. Let B_k and B_l be ordered basis ofr A_k and A_l. The matrix of the map $\bullet L^{l-k} : A_k \to A_l$, for $k < l \leq \frac{d}{2}$, with respect to the basis B_k and B_l coincides with $\mathrm{Hess}_f^{(d-l,k)}(L^{\perp})$, using basis B_l^* and B_k. In particular:*

$$\mathrm{rk}\left(\bullet L^{l-k}\right) = \mathrm{rk}\left(\mathrm{Hess}_f^{(d-l,k)}(L^{\perp})\right).$$

We recall that a form is wild if $cr(f) > \underline{rk}(f)$. Our strategy to construct wild forms is to find an upper bound for the border rank and a lower bound for the cactus rank and compare them.

Lemma 5. *Let $f \in \mathbb{C}[x_1,\dots,x_n,u,v]_{(k,d-k)}$ be a bi-homogeneous form of bi-degree $(k, d-k)$ with $1 \leq k \leq d-k$. The border rank of f satisfies:*

$$\underline{rk}(f) \leq k(d+2).$$

A form $f \in R_d$ is called *k-concise*, with $d \geq 2k+1$, if $I_j = 0$ for $j = 1,2,\dots,k$. It is equivalent to $a_j = \binom{n+j}{j}$ for $j = 0,\dots,k$. As usual, 1-concise forms are called concise.

Lemma 6. *Let $f \in R_d$ be a k-concise form and $A = Q/I$ be the associated algebra with $I = \mathrm{Ann}_f$. Suppose that $a_k \leq a_{d-s}$ and $k+s \leq d$. If $\mathrm{Hess}_f^{(k,s)}$ is degenerated, then exists $\alpha \in I_k^{sat} \setminus I_k$.*

Lemma 7. *Let $f \in R_d$ be a $k-$concise form with $2k < d$ and let $I = \mathrm{Ann}(f) \subset Q$. Let $J = (I_{d-k}) \subset Q$ be the ideal generated by the degree $d-k$ part of I. If $J_l^{sat} \neq \emptyset$ for some $l \leq k$, then*

$$cr(f) > a_k = \binom{n+k}{k}.$$

Theorem 8. *Let $f \in R_d$ be a k-concise homogeneous form, with $2k \leq d$. If $\mathrm{hess}_f = 0$, then*

$$cr(f) > \binom{n+k}{k}.$$

In particular, if $\underline{rk}(f) \leq \binom{n+k}{k}$, then f is wild.

The following Corollary is the main result of [1].

Corollary 9. *Let $f \in R_d$ be a concise form with minimal border rank. If $\mathrm{hess}_f = 0$, then f is wild.*

Proof. Minimal border rank means $\underline{rk}(f) = n+1$. Since f is 1-concise and $\mathrm{hess}_f = 0$, by Theorem 8, we get $cr(f) > n+1$. ∎

Example 10 (A wild form with non minimal border rank)**.** Consider the forms $f \in \mathbb{C}[x,y,z,u,v]_{288}$, given by $f = g^{16}$ with

$$f = xu^{17} + yu^{16}v + zv^{17}.$$

We know that f has vanishing Hessian. Indeed, by Gordan-Noether criteria, since the partial derivatives of g satisfy $g_x^{16} g_z = g_y^{17}$, they are algebraically dependent, therefore, $\operatorname{hess} f = 0$. Moreover, the choice of g was in such a way that its polar image has degree d. If the polar degree was lower, then the f could be not 16-concise. We checked the 16-conciseness of f which implies that its border rank is non minimal. In this case $cr(f) > a_{16} = 4845$ and $\underline{rk}(f) \leq 4640$, hence f is wild.

Theorem 11. *Let $f \in R_d$ be a k-concise homogeneous form with $2k \leq d$ and let l, s be integers such that $l \leq k \leq s$ and $s+l \leq d$. Let $I = \operatorname{Ann}(f)$ and $A = Q/I$ and suppose that* $\operatorname{Hilb}(A)$ *is unimodal. Suppose that* $\operatorname{Hess}_f^{(l,s)}$ *is degenerated. Then:*

$$cr(f) > \binom{n+k}{k}.$$

In particular, if $\underline{rk}(f) \leq a_k$, then f is wild.

The first example of a form with vanishing second Hessian whose Hessian is non vanishing was given by Ikeda.

Example 12 (A wild form without vanishing hessian)**.** Let $f = xu^3v + yuv^3 + x^2y^3 \in \mathbb{C}[x,y,u,v]_5$. Let $A = Q/\operatorname{Ann}_f$, we get

$$\operatorname{Hilb}(A) = (1,4,10,10,4,1).$$

Therefore f is 2-concise. We know that $\operatorname{hess}_f^2 = 0$. By Proposition 5, $\underline{rk}(xu^3v + yuv^3) \leq 7$. We know that, $\underline{rk}(x^2y^3) = 3$, then $\underline{rk}(f) \leq 10$. By Theorem 11 we get that $cr(f) > 10$, therefore f is wild.

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Asymptotic properties of invariant chains

Tim Römer

In algebraic geometry and commutative algebra several objects of interest occur in (symmetric) families depending on n variables. Often some "limit behavior" is visible and it is a natural question whether such a result is an accident or not. For an explicit example consider the ideal I_n which is generated by all quadratic monomials in n variables, i.e. $I_n = \langle X_1^2, X_1X_2, \ldots, X_1X_n, X_2^2, X_2X_3, \ldots, X_n^2 \rangle$. Then one is interested in associated invariants like the projective dimension or the Castelnuovo-Mumford regularity.

Following an approach of Hillar-Sullivant [5] and others we fix a suitable setup. For this let K be a field and $c \geq 1$ an integer. For $n \geq 1$ let $R_n = K[X_{c \times n}] =$

$K[X_{i,j} : 1 \leq i \leq c,\ 1 \leq j \leq n]$ be a polynomial ring in $c \times n$ variables over K. Note that these form an ascending chain $R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n \subseteq \ldots$ of polynomial rings. Let $Sym(n)$ denote the symmetric group on $\{1, \ldots, n\}$. Considering it as stabilizer of $n+1$ in $Sym(n+1)$, similarly one gets an ascending chain of symmetric groups. Define an action of $Sym(n)$ on R_n induced by

$$\sigma \cdot X_{k,j} = X_{k,\sigma(j)} \quad \text{for} \quad \sigma \in Sym(n),\ 1 \leq k \leq c, 1 \leq j \leq n.$$

A *Sym-invariant chain* $(I_n)_{n\geq 1}$ is a sequence of ideals $I_n \subseteq R_n$ satisfying: (1) Each I_n is invariant under the action of $Sym(n)$; (2) $Sym(n+1) \cdot \langle I_n R_{n+1} \rangle \subseteq I_{n+1}$. Note that such a chain is essentially the same as an FI-module over $\mathbf{X}^{FI,1}$ as studied in [9], but this point of view will not be stressed in the following.

The central problem is to study the asymptotic behavior of invariants of ideals in such Sym-invariant chains.

A first remarkable observation follows from Aschenbrenner-Hillar [1] and Hillar-Sullivant [5] and states that Sym-invariant chains stabilize, which means that there is some integer n_0 such that, for $n \geq n_0$, a generating set of I_n can be obtained from a generating set of I_{n_0} by applying permutations, i.e.

$$I_n = \langle Sym(n) I_{n_0} \rangle \text{ for all } n \geq n_0.$$

In the example $I_n = \langle X_1^2, X_1X_2, \ldots, X_1X_n, X_2^2, X_2X_3, \ldots, X_n^2 \rangle$ from above, $n_0 = 2$ and *up to symmetry* I_n is generated by X_1^2 and X_1X_2. See Draisma [4] for a survey on this and other results in the last decade. See also related approaches and developments by Church-Ellenberg-Farb-Nagpal (see, e.g., [2, 3]) and by Sam-Snowden (see, e.g., [10, 11]).

Considering again Sym-invariant chains and having the mentioned result from above in mind, several other properties and objects of interested were studied very recently. In [8] for a Sym-invariant chain $\mathcal{I} = (I_n)_{n\geq 1}$ of graded ideals its *bigraded Hilbert series* was defined as

$$H_{\mathcal{I}}(s,t) = \sum_{n\geq 0,\ j\geq 0} \dim_K [R_n/I_n]_j \cdot t^j s^n = \sum_{n\geq 0} H_{R_n/I_n}(t) \cdot s^n$$

and one of the main results of [8] states that this series is a rational function of a particular form.

As a consequence one gets, e.g., that there exists integers A, B with $0 \leq A \leq c$ such that, for all $n \gg 0$,

$$\dim R_n/I_n = An + B.$$

In [6] and [7] the following conjecture was formulated and studied:

Conjecture 1. *Let $(I_n)_{n\geq 1}$ be a Sym-invariant chain of ideals. Then pdR_n/I_n and $regI_n$ are eventually linear functions, that is,*

$$pdR_n/I_n = An + B \text{ and } regI_n = Cn + D$$

for some integers A, B, C, D whenever $n \gg 0$.

For evidences for this conjecture and especially cases where it is known to be true we refer to [6, 7]. The asymptotic behavior of syzygies in general was studied in [9]. One of the main results of that paper states informally that for every integer $p \geq 0$, the p-syzygies of the ideals I_n look alike eventually.

Having seen the (conjectured) asymptotic behavior of ideals in Sym-invariant chains, it is a natural question to ask whether Lefschetz properties can be understood from this point of view. More precisely, let $(I_n)_{n \geq 1}$ be a Sym-invariant chain of artinian ideals. Then decide whether either I_n has SLP/WLP for all $n \gg 0$ or I_n has not SLP/WLP for all $n \gg 0$. This question is studied in a working group as part of the workshop "Lefschetz Properties in Algebra, Geometry and Combinatorics".

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