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New Directions in Rough Path Theory (online meeting)

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6 December – 12 December 2020

Abstract. Rough path theory emerged as novel approach for dealing with interactions in complex random systems. It settled significant questions and provided an effective deterministic alternative to Itô calculus, itself a major contribution to 20$^{\text{th}}$ century mathematics. Its impact has grown substantially in recent years: most prominently, rough paths ideas are at the core of Martin Hairer's Fields Medal-winning work on regularity structures, but there are also original and successful applications in other areas. The workshop focused on three areas that have been strongly influenced by the core ideas in rough path theory and which have witnessed considerable activity over the past few years: applications to data science, algebraic aspects and connections with stochastic analysis.

Mathematics Subject Classification (2010): 60H07, 60H10, 60H15, 60H20, 62J02, 68T05, 68T45, 76M50, 81Q30, 14A10.

Introduction by the Organizers

The workshop MFO-2050a *New Directions in Rough Path Theory*, organised by Thomas Cass, Dan Crisan (Imperial College London), Peter Friz (Technische Universität Berlin) and Massimiliano Gubinelli (University of Bonn) was attended with over 30 participants with broad geographic representation from all continents. The program consisted of 23 talks, each being followed by a discussant's presentation, leaving sufficient time for additional questions from the audience. Because of the ongoing Covid-19 pandemic, this event took place online via the Zoom platform hosted by TU Berlin. In accordance with Oberwolfach's tradition,

TABLE 1. Timetable

	Monday	Tuesday	Wednesday	Thursday	Friday
09h45	Opening				
10h00	Teichmann	Chevyrev	Gassiat	Oberhauser	Geng
11h00	Tapia	Zorin-Kranich	Leahy	Boedihardjo	De Vecchi
12h00	Preiss	Lê	Hairer	Bayer	Li
14h30	Seeger	Ebrahimi-Fard	Bruned		Hao
15h30	Bailleul	Diehl	Zambotti		Lyons
16h30					Closing

the schedule was not known in advance by the participants. Messages containing the links of the talks and the day's schedule were sent each morning to the group, which was also intended to prevent unwanted intrusions.

Further informal discussions took place in between and after the talks, based on the `gather.town` platform, a video-calling service that lets multiple people hold separate conversations in parallel easily. These sessions support LaTeX, allowing effective exchanges on mathematical topics.

Organizing an online meeting is a challenge in itself. Some of the experimental aspects were managed with the precious help of Tom Klose and Antoine Hocquet (also reporter) who both hosted the Zoom sessions, and had daily meetings with the organizers to decide on improvements.

MOTIVATIONS

The meeting brought together world-leading researchers to develop synergy on interconnected areas of rough path theory. The general topic of the workshop was rough path theory in the broad sense, including recent developments based on the following three main themes:

- The mathematics of the signature and its applications to data science;
- Connections with algebraic geometry and algebraic aspects of renormalization;
- Applications to stochastic analysis: SLE, Finance, fluid dynamical models and homogenisation.

At the heart of Lyons' rough path theory is the challenge of describing a continuous but potentially highly oscillatory vector valued path $(X_t)_{t\in[0,T]}$, so as to accurately predict the response of a nonlinear system such as the controlled ODE $dY_t = f(Y_t)dX_t$, $Y_0 = x$, for sufficiently smooth vector fields f. The fundamental concept of *signature* allows to represent such responses in terms of a *continuous* solution map, defined on a set of potentially very irregular controls. The signature of a rough input X consists in the collection of its iterated integrals seen as a group-like element of a closed tensor algebra. This is a mathematically rich object which captures many of the important analytic and geometric properties of paths, the interest of which is certainly not limited to the understanding of ODEs driven

Table 2. Speakers and discussants

Speaker	Title	Discussant
Teichmann	Semi-martingale (randomized) Signatures	Tapia
Tapia	Unified signature cumulants and generalized Magnus expansions	Ebrahimi-Fard
Preiss	Areas of areas generate the shuffle algebra	Lyons
Seeger	A Besov-type sewing lemma and applications	Gassiat
Bailleul	Regularity structures and paracontrolled calculus	Perkowski
Chevyrev	The rough path BDG inequality and Yang-Mills measure on the lattice	Gubinelli
Zorin-Kranich	Itô integrals of branched rough paths	Lê
Lê	On stochastic controlled rough paths	Zorin-Kranich
Ebrahimi-Fard	Generalized iterated-sums signatures	Riedel
Diehl	Tropical iterated-sums	Hao
Gassiat	Non-uniqueness for reflected rough differential equations	Gubinelli
Leahy	On solution properties of rough Euler's equation	Crisan
Hairer	The support of singular stochastic PDEs	Friz
Bruned	Algebraic deformation for (S)PDEs	Hairer
Zambotti	Hairer's Reconstruction Theorem without Regularity Structures	Bailleul
Oberhauser	Signatures and Filtrations	Riedel
Boedihardjo	The length of a path and the norm of its signature	Salvi
Bayer	Optimal stopping with signatures	Oberhauser
Geng	Precise local estimates for hypoelliptic differential equations driven by fractional Brownian motion	Cass
De Vecchi	An introduction to Grassmannian stochastic analysis	Li
Li	Rough homogenisation	Chevyrev
Hao	Finite radius of convergence of expected signature of stopped Brownian motion on 2D domains	Boedihardjo
Lyons	Structure theorems for information in streamed data	Bayer

by irregular signals. In a seminal paper of Hambly and Lyons [10], signatures are shown to provide faithful representations of a broad class of paths. Even though the correspondence between the two objects is not yet fully understood, this observation has been used in the past decade to offer a new way to parsimoniously capture complex streamed data, and this has led to striking benefits in the analysis of real-world applied data streams (see, e.g., [17]). In these applications the signature is exploited with other tools from data science (deep learning, kernel methods, etc.) as an observable way of describing ordered data over an interval. It is now clear that the range of possible applications is much broader than those illustrated above and that the signature offers a generic methodology that could be exploited much more widely. Realising these benefits will demand that serious mathematical questions be addressed, especially on the algebraic aspects of the theory (see below), and one of the goals of this meeting was to bring people together to make progress on these critical questions.

Understanding algebraic aspects of the set of iterated integrals is important both to statistical applications (as outlined above) and to establish relations with algebraic geometry. Renormalization theory, especially in connection to regularity structures (which can be seen as a wide generalisation of rough paths) also has very deep connections to algebraic aspects of rough path theory. In the recent work of Bruned, Hairer and Zambotti [6] two Hopf algebras in co-interaction have been shown to rule the double positive/negative renormalization taking place in regularity structures, when applied to solution theory of singular SPDEs. Similar ideas stressing the importance of pre-Lie structure were used later for rough path purposes, as ways to operate certain transforms while preserving key algebraic structures [3]. This observation was imported back to the full setting of regularity structures applied to singular SPDEs [2], illustrating well the type of reciprocal actions expected by the organizers of the workshop. Interaction of renormalization with symmetries is a deep subject in theoretical physics (covariance preservation, gauge symmetries, anomalies, Ward identities), and much of these phenomena are seen making surface in the rough path/regularity structure world, see for example the work of Hairer on the construction of a natural evolution on the space of loops with values in a Riemannian manifold [12]. Recently rough-paths/regularity structures start to be applied to gauge theories [16, 14] and it is to be expected that the interplay between geometry, analysis and algebra will have an even major role in the analysis of non-abelian gauge theories. Algebraic aspects of rough paths have also relation to other combinatorial structures like quasi-shuffle algebras and planar rooted trees and there is a vivid interest in understanding better these issues in relation to analysis [4, 11]. More than ever, a game of questions and answers is taking place between the aforementioned topics, and this workshop was a good opportunity to gather various experts of these different sub-fields.

Interest in stochastic analysis and the study of random systems, which is at the basis of the development of rough paths and regularity structures, has not waned over the years, despite the fact that the latter theories have a deterministic flavour. The workshop was an occasion to highlight areas of probability theory which have clearly benefited from these recent developments. We should indeed mention the following offshoots which have been widely discussed by the participants:

- (SLE) Only recently has there been attempts to import tools familiar to the rough path community to Schramm-Loewner evolution (SLE). This has already led to some interesting insights on the pathwise and sample path structure of SLE, see e.g. [8, 9], and naturally relates to important work by Y. Wang [13, 15].
- (Mathematical Finance) "Rough volatility" (introduced by Jim Gatheral and Mathieu Rosenbaum) presently has a major impact on Quantitative Finance. It was recently realized that a robustification of this model class is most naturally achieved using Hairer's regularity structures [5]. It is an exciting outlook to understand the full consequence, both theoretically and from a numerical perspective, of this approach.
- (Homogenisation) Ian Melbourne and coworkers have been using rough path to approach some difficult non-Markovian homogenisation problems, see [7] for a review. We believe there is much further potential in this approach to homogenization.
- (Fluid dynamic models) In recent research one can use stochastic geometric mechanics principles to incorporate observed data into variational principles, in order to derive data-driven nonlinear dynamical models. Fluid dynamics driven by rough paths are natural pathwise formulations of their classical stochastic counterparts. They offer a larger class of models otherwise unavailable through classical Brownian driven equations. There is an intrinsic connection between such models and models obtained via homogenization arguments.

Acknowledgement: The organizers acknowledge TU Berlin for providing the Zoom licence to run the workshop.

References

[1] Amendola, Carlos, Peter Friz, and Bernd Sturmfels. Varieties of signature tensors. In *Forum of Mathematics*, Sigma, vol. **7**. Cambridge University Press, 2019.

[2] Bruned, Y., A. Chandra, I. Chevyrev, and M. Hairer. Renormalising SPDEs in regularity structures. *Journal of the European Mathematical Society* (2019).

[3] Bruned, Yvain, Ilya Chevyrev, Peter K. Friz, and Rosa Preiss. A rough path perspective on renormalization. *Journal of Functional Analysis* **277**, no. 11 (2019): 108283.

[4] Bruned, Yvain, Charles Curry, and Kurusch Ebrahimi-Fard. Quasi-shuffle algebras and renormalisation of rough differential equations. *Bulletin of the London Mathematical Society* **52**, no. 1 (2020): 43-63.

[5] Bayer, Christian, Peter K. Friz, Paul Gassiat, Jorg Martin, and Benjamin Stemper. A regularity structure for rough volatility. Mathematical Finance **30**, no. 3 (2020): 782-832.

[6] Bruned, Yvain, Martin Hairer, and Lorenzo Zambotti. Algebraic renormalisation of regularity structures. *Inventiones mathematicae* **215**, no. 3 (2019): 1039-1156.

[7] Chevyrev, Ilya, Peter K. Friz, Alexey Korepanov, Ian Melbourne, and Huilin Zhang. Multiscale systems, homogenization, and rough paths. In *International Conference in Honor of the 75th Birthday of SRS Varadhan*, pp. 17-48. Springer, Cham, 2016.

[8] Friz, Peter K., and Atul Shekhar. On the existence of SLE trace: finite energy drivers and non-constant κ. *Probability Theory and Related Fields* **169**, no. 1-2 (2017): 353-376.

[9] Friz, Peter K., and Huy Tran. On the regularity of SLE trace. In *Forum of Mathematics*, Sigma, vol. 5. Cambridge University Press, 2017.

[10] Hambly, Ben, and Terry Lyons. Uniqueness for the signature of a path of bounded variation and the reduced path group. *Annals of Mathematics* (2010): 109-167.

[11] Curry, Charles, Kurusch Ebrahimi-Fard, Dominique Manchon, and Hans Z. Munthe-Kaas. Planarly branched rough paths and rough differential equations on homogeneous spaces. *Journal of Differential Equations* **269**, no. 11 (2020): 9740-9782.

[12] Hairer, Martin. The motion of a random string. *arXiv preprint arXiv:1605.02192* (2016).

[13] Rohde, Steffen, and Yilin Wang. The Loewner energy of loops and regularity of driving functions. *International Mathematics Research Notices* (2017).

[14] Shen, Hao. Stochastic quantization of an Abelian gauge theory. *arXiv preprint arXiv:1801.04596* (2018).

[15] Wang, Yilin. Equivalent descriptions of the Loewner energy. *Inventiones mathematicae* **218**, no. 2 (2019): 573-621.

[16] Yamashita, Hideyasu. Smooth approximation of Yang–Mills theory on $\mathbb{R}^2$: a rough path approach. *arXiv preprint arXiv:1705.02507* (2017).

[17] Yin, Fei, Qiu-Feng Wang, Xu-Yao Zhang, and Cheng-Lin Liu. ICDAR 2013 Chinese handwriting recognition competition. In *2013 12th International Conference on Document Analysis and Recognition*, pp. 1464-1470. IEEE, 2013.

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Abstracts

Semi-martingale (randomized) Signatures

Josef Teichmann
(joint work with Christa Cuchiero, Lukas Gonon, Lyudmila Grigoryeva, Juan-Pablo Ortega)

In many areas of theoretical or practical interest the quantitative understanding of dynamics in terms of initial values, local characteristics and noisy controls is pivotal. Classical approaches provide numerical algorithms to approximate such dynamics, where the quality of approximation depends on the regularity of the local characteristics and on the nature of the noise. Machine-learning technologies based on universal approximation theorems follow a different approach to this problem: on a sufficiently rich training data set the map from initial values, local characteristics and noisy controls to dynamic solution is approximated by universal classes of maps.

Here the paradigm of reservoir computing, which – in different forms – appears in several areas of machine learning, enters the stage: split the input-output map into a generic part (the *reservoir*), which is *not or only vaguely* trained, and a readout part, which is acurately trained and often linear. In many cases reservoirs can be realized physically, whence ultrafast evaluations are possible, and learning the readout layer is often a simple regression. In particular reservoirs are often dynamical systems themselves, but with randomly chosen characteristics.

Dynamical systems with universality properties are a well known area: stochastic Taylor expansion for Brownian noises via stochastic analysis (or for rough input signals Rough Path Theory, or for more complicated noises the Theory of Regularity Structures) enables the construction such universal dynamical systems, i.e. systems taking values in infinite dimensional algebraic systems, whose components can serve as regression bases for any other dynamical system.

This well known theory, which is perfectly in line with the paradigm of reservoir computing, is still not fully satisfying: usually reservoirs at work are of moderate dimension and have excellent regression properties. Let us take signature which takes values in the free algebra $\mathbb{A}_d$ generated by d indeterminates as an example. We do *not* apply generic numerical methods to approximate the signature reservoir, but rather try to compress the information of signature through a random projection

$$\pi : \mathbb{A}_d \to \mathbb{R}^k \,,$$

whose image can be surprisingly approximated by a dynamical system on $\mathbb{R}^k$, which has random characteristics. This is due to the fact that the characteristics of the signature process have a nilpotent structure, whence its random projections look – by the central limit theorem – almost like random matrices. The random projections are of course chosen according to the Johnson-Lindenstrauss lemma.

Therefore we can split (asymptotically in time), now fully in line with reservoir computing, the map $(B_t)_{0\le t\le T} \mapsto (Y_t)_{0\le t\le T}$, where Y solves a stochastic differential equation

$$dY_t = \sum_{i=1}^{d} V_i(Y_t) \circ dB^i(t)$$

into two parts:

- A locally linear systems on $\mathbb{R}^k$

$$dX_t = \sum_{i=0}^{d} \sigma(A_i X_t + b_i) \circ dB_t^i\,,\ X_0 \in \mathbb{R}^k\,,$$

  where σ is a componentwise applied activation function $\sigma : \mathbb{R} \to \mathbb{R}$ (notice that we explicitly allow the identity as a possible choice here) and A_i, b_i are appropriately chosen random matrices or vectors (often just by independently sampling from $\mathrm{N}(0,1)$. This is a finite dimensional system, which we call randomized signature process and which, again, does *not* depend on the specific dynamics.
- A linear map $X \mapsto WX$ which is actually trained (e.g. a linear regression) to explain $(Y_t)_{0\le t\le T}$ as good as possible.

This also yields a fresh view on describing dynamics driven by, e.g., Brownian motion: one can either describe a dynamics by providing its local characteristics, i.e. the vector fields $V_1, \dots, V_d$, or, in view of the above sketch, one can describe a dynamics by providing a reservoir, e.g. randomized signature, and the regression coefficients W. It is impressive that it is linear task to learn W from the (high-frequency) observation of one trajectory Y together with the control u. This actually leads to a fascinating econometric approach which will be investigated in subsequent work. All necessary references can be found in [1].

References

[1] Cuchiero, Gonon, Grigoryeva, Ortega and Teichmann. Discrete-time signatures and randomness in reservoir computing. arXiv preprint arXiv:2010.14615 (2020).

Unified signature cumulants and generalized Magnus expansions

Nikolas Tapia

(joint work with Peter K. Friz, Paul Hager)

In this talk, we show that *signature cumulants*, defined as logarithm of expected signatures of semimartingales, satisfy a fundamental functional relation.

Definition 1. Let X be an $\mathbb{R}^d$-valued cádlág semimartingale and denote by $\mathrm{Sig}(X)_{s,t}$ its signature over the interval $[s,t]$, defined as the unique solution to the Marcus equation

$$(1)\quad S_{s,t} = 1 + \int_{(s,t]} S_{s,u-}\,\mathrm{d}X_u + \frac{1}{2}\int_s^t S_{s,u-}\,\mathrm{d}\langle X^c\rangle_u + \sum_{s<u\le t} S_{s,u-}(\exp(\Delta X_u) - 1 - \Delta X_u).$$

We show that under suitable integrability conditions on X, the *signature moments* $\boldsymbol{\mu}_t(T) \coloneqq \mathbb{E}_t[\mathrm{Sig}(X)_{t,T}]$ are well defined, and so are the associated *signature cumulants* $\boldsymbol{\kappa}_t(T) \coloneqq \log \boldsymbol{\mu}_t(T)$. Moreover, when seen as processes in the variable $t \in [0,T]$, both $\boldsymbol{\mu}(T)$ and $\boldsymbol{\kappa}(T)$ define semimartingales with values in the completed tensor algebra $T(\!(\mathbb{R}^d)\!)$.

In what follows we deal with general semimartingales taking values in $T(\!(\mathbb{R}^d)\!)$, defined as formal word series $\mathbf{X}_t = \sum_w \mathbf{X}_t^w w$ such that each component $\mathbf{X}^w$ is a real-valued semimartingale. Given two such processes $\mathbf{X}_t$ and $\mathbf{Y}_t$, we define their *outer stochastic bracket*

$$[\![\mathbf{X},\mathbf{Y}]\!]_t \coloneqq \sum_{u,v} [\mathbf{X}^u, \mathbf{Y}^v]_t u \otimes v \in T(\!(\mathbb{R}^d)\!) \otimes T(\!(\mathbb{R}^d)\!),$$

where $[X,Y]_t = \langle X^c, Y^c\rangle_t + \sum_{u<t} \Delta X_u \Delta Y_u$ denotes the usual stochastic bracket between real-valued semimartingales. With this notation, Definition 1 extends to the case where X is allowed to be an arbitrary word series. Finally, given two linear operators L, K on $T(\!(\mathbb{R}^d)\!)$, we let $(L \odot K)(a \otimes b) \coloneqq L(a)K(b)$.

Theorem 1. *For a sufficiently integrable semimartingale $\mathbf{X}$ with values in $T(\!(\mathbb{R}^d)\!)$, the signature cumulants $\boldsymbol{\kappa}_t(T)$ are the unique solution of the functional equation*

$$(2)\quad \boldsymbol{\kappa}_t(T) = \mathbb{E}_t\Big\{ \int_{(t,T]} H(\boldsymbol{\kappa}_{u-})(\mathrm{d}\mathbf{X}_u) + \frac{1}{2}\int_t^T H(\boldsymbol{\kappa}_{u-})(\mathrm{d}\langle \mathbf{X}^c\rangle_u) + \frac{1}{2}\int_t^T H(\boldsymbol{\kappa}_{u-}) \circ Q(\boldsymbol{\kappa}_{u-})(\mathrm{d}[\![\boldsymbol{\kappa},\boldsymbol{\kappa}]\!]^c_u) + \int_t^T H(\boldsymbol{\kappa}_{u-}) \circ (\mathrm{id} \odot G(\boldsymbol{\kappa}_{u-}))(\mathrm{d}[\![\mathbf{X},\boldsymbol{\kappa}]\!]^c_u) + \sum_{t<u\le T} \Big(H(\boldsymbol{\kappa}_{u-})\Big(\exp(\Delta\mathbf{X}_u)\exp(\boldsymbol{\kappa}_u)\exp(-\boldsymbol{\kappa}_{u-}) - 1 - \Delta\mathbf{X}_u\Big) - \Delta\boldsymbol{\kappa}_u\Big)\Big\},$$

where all integrals are understood in the Itô or Riemann–Stieltjes sense. The operators H, G, Q are defined as follows:

$$G(\boldsymbol{\kappa}) := \frac{e^{\operatorname{ad}\boldsymbol{\kappa}} - 1}{\operatorname{ad}\boldsymbol{\kappa}} = \sum_{n=0}^{\infty} \frac{(\operatorname{ad}\boldsymbol{\kappa})^n}{(n+1)!}$$

$$H(\boldsymbol{\kappa}) := \frac{\operatorname{ad}\boldsymbol{\kappa}}{e^{\operatorname{ad}\boldsymbol{\kappa}} - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} (\operatorname{ad}\boldsymbol{\kappa})^n$$

$$Q(\boldsymbol{\kappa}) := \sum_{m,n=0}^{\infty} 2 \frac{(\operatorname{ad}\boldsymbol{\kappa})^n \odot (\operatorname{ad}\boldsymbol{\kappa})^m}{(n+1)!(m)!(n+m+2)}$$

This equation, in a deterministic setting, contains Hausdorff's differential equation [3], which itself underlies Magnus' expansion [6]. The (commutative) case of multivariate cumulants arise as another special case and yields a new Riccati-type relation valid for general semimartingales. Here, the accompanying expansion provide a new view on recent "diamond" and "martingale cumulants" [1, 2, 5] expansions. This is summarized in the following diagram.

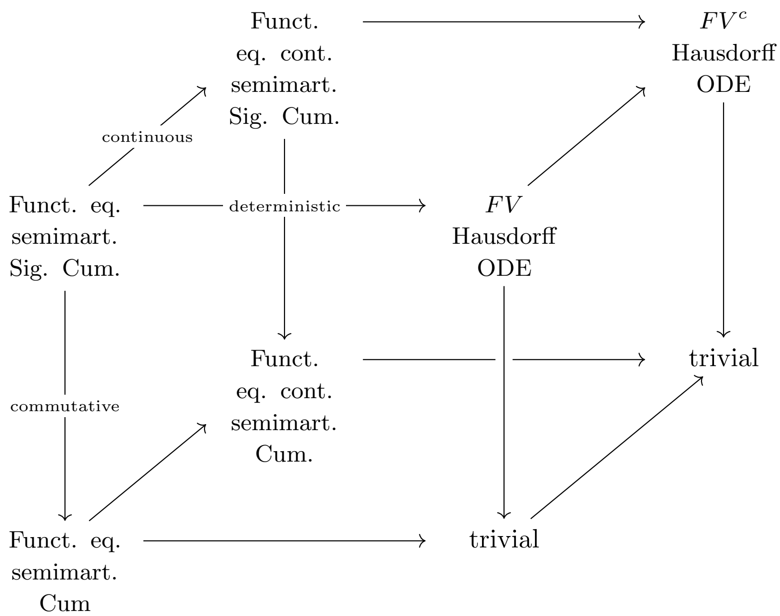


Figure 1. Theorem 1 and its implications

Projecting eq. (2) onto its graded components, we obtain a recursive formula for the computation of $\boldsymbol{\kappa}$. This recursion reduces to the diamond expansion of Friz, Gatheral and Radoičić [2] in the commutative setting. Moreover, in the deterministic setting our recursion extends the recursive formulation of Magnus expansion due to Iserles and Nørsett [4] to cádlág paths. We note however, that in our general setting, we do not have a nice representation of the terms in the series in terms of combinatorial objects such as trees, as is the case for both "diamond" expansions and the Magnus expansion. We also note that questions of convergence of the series have not yet been addressed.

References

[1] E. Alos, J. Gatheral, and R. Radoičić, *Exponentiation of conditional expectations under stochastic volatility*, Quantitative Finance; SSRN **20** (2020), no. 1, 13–27.

[2] P. K. Friz, J. Gatheral, and R. Radoičić, *Forests, cumulants, martingales*, 2020, `arXiv:2002.01448 [math.PR]`.

[3] F. Hausdorff, *Die symbolische Exponentialformel in der Gruppentheorie*, Ber. Verh. Kgl. Sächs. Ges. Wiss. Leipzig., Math.-phys. Kl. **58** (1906), 19–48.

[4] A. Iserles and S. P. Nørsett, *On the solution of linear differential equations in lie groups*, Philos. Trans. Roy. Soc. A **357** (1999), no. 1754, 983–1019.

[5] H. Lacoin, R. Rhodes, and V. Vargas, *A probabilistic approach of ultraviolet renormalisation in the boundary sine-gordon model*, 2019, `arXiv:1903.01394 [math.PR]`.

[6] W. Magnus, *On the exponential solution of differential equations for a linear operator*, Commun. Pure Appl. Math. **7** (1954), no. 4, 649–673.

Areas of areas generate the shuffle algebra

Rosa Preiss

(joint work with Joscha Diehl, Terry Lyons and Jeremy Reizenstein)

We are concerned with the signed area, or Levy area, between two components of a path, which comes with very interesting properties: It is a rotation invariant in two dimensions, it is iterable, meaning all kinds of bracketings like

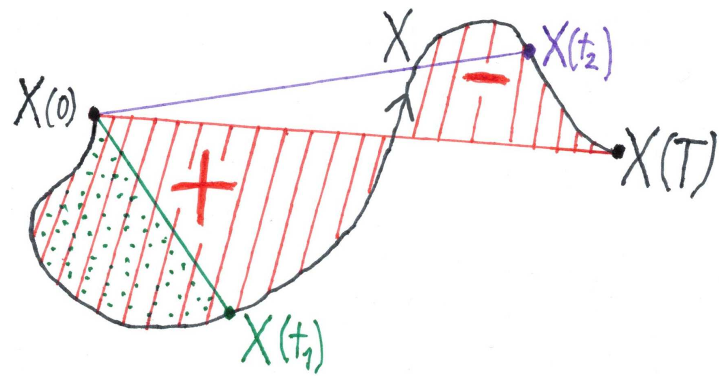
$$\mathsf{Area}(\mathsf{Area}(\mathsf{X}^1,\mathsf{X}^2),\mathsf{X}^1),\mathsf{Area}(\mathsf{Area}(\mathsf{Area}(\mathsf{X}^1,\mathsf{X}^2),\mathsf{X}^1),\mathsf{X}^1)),$$
$$\mathsf{Area}(\mathsf{Area}(\mathsf{X}^1,\mathsf{X}^2),\mathsf{Area}(\mathsf{X}^2,\mathsf{Area}(\mathsf{X}^1,\mathsf{X}^2))),\dots$$

can be computed. Area behaves well with respect to discretization and in the stochastic setting, it preserves the martingale property. It's analytic computation is also very simple:

Let $\mathsf{Area}(\mathsf{X}^1,\mathsf{X}^2)_t$ denote *two times* the signed area enclosed by the two-dimensional path $(\mathsf{X}^1,\mathsf{X}^2)$ up to time t and the straight line connecting $(\mathsf{X}^1(t),\mathsf{X}^2(t))$ with $(\mathsf{X}^1(0),\mathsf{X}^2(0))$ (see figure below). We have

$$\mathsf{Area}(\mathsf{X}^1,\mathsf{X}^2)_t=\int_0^t(\mathsf{X}^1_s-\mathsf{X}^1_0)\,\mathrm{d}\mathsf{X}^2_s-\int_0^t(\mathsf{X}^2_s-\mathsf{X}^2_0)\,\mathrm{d}\mathsf{X}^1_s.$$

The claim presented in the talk, that for Area bracketings of semimartingales it would not matter whether one uses Itô or Stratonovich integrals, turned out to be wrong however, as already the counterexample $\mathsf{Area}(\mathsf{Area}(\mathsf{B}^1,\mathsf{B}^2),\mathsf{B}^1)$ for independent Brownian motions B^1,B^2 shows.



Let us consider the signature $\sigma(\mathsf{X})$ of a path X as an element of $T((\mathbb{R}^d))$, the dual space of the space of words $T(\mathbb{R}^d)$, i.e.

$$\langle \sigma(\mathsf{X}), \mathtt{i}_1 \cdots \mathtt{i}_n \rangle = \int_0^T \int_0^{r_n} \cdots \int_0^{t_2} \mathrm{d}\mathsf{X}_{t_1}^{i_1} \ldots \mathrm{d}\mathsf{X}_{t_n}^{i_n}.$$

There exists a bilinear operation $\succ : \mathrm{T}^{\geq 1}(\mathbb{R}^d) \times \mathrm{T}^{\geq 1}(\mathbb{R}^d) \to \mathrm{T}^{\geq 1}(\mathbb{R}^d)$ such that for any regular enough path $\mathsf{X} : [0, L] \longrightarrow \mathbb{R}^d$ and any $a, b \in \mathrm{T}^{\geq 1}(\mathbb{R}^d)$, we have

$$\int_0^s \mathbf{X}_t^a \mathrm{d}\mathbf{X}_t^b = \mathbf{X}_s^{a \succ b},$$

where $\mathbf{X}_t^a := \langle \sigma(\mathsf{X}|_{[0,t]}), a \rangle$ (e.g. [5]). Thus, we get identity, Ree 1958],

$$\mathsf{Area}(\mathbf{X}^a, \mathbf{X}^b)_t = \mathbf{X}_t^{\mathsf{area}(a,b)},$$

where we define the purely algebraic area operation as $\mathsf{area}(a, b) := a \succ b - b \succ a$, which will be the object of our study.

area is obviously anticommutative, but neither associative,

$$\begin{aligned} &\mathsf{area}(\mathsf{area}(1, 2), 3) = 123 - 132 + 213 - 231 - 312 + 321 \\ &\neq 123 - 132 - 213 + 231 - 312 + 321 = \mathsf{area}(1, \mathsf{area}(2, 3)) \end{aligned}$$

nor does it satisfy the Jacobi identity,

$$\begin{aligned} &\mathsf{area}(1, \mathsf{area}(2, 3)) + \mathsf{area}(2, \mathsf{area}(3, 1)) + \mathsf{area}(3, \mathsf{area}(1, 2)) \\ &= -123 + 132 + 213 - 231 - 312 + 321 \neq 0. \end{aligned}$$

It does however satisfy the so-called *Tortkara identity* introduced in [3]:

$$\begin{aligned} &\mathsf{area}(\mathsf{area}(a, b), \mathsf{area}(c, d)) + \mathsf{area}(\mathsf{area}(a, d), \mathsf{area}(c, b)) \\ &= \mathsf{area}(\mathsf{vol}(a, b, c), d) + \mathsf{area}(\mathsf{vol}(a, d, c), b), \end{aligned}$$

where

$$\mathsf{vol}(a, b, c) := \mathsf{area}(\mathsf{area}(a, b), c) + \mathsf{area}(\mathsf{area}(b, c), a) + \mathsf{area}(\mathsf{area}(c, a), b).$$

vol corresponds to the signed volume:

$$\text{Volume}(\mathbf{X}^a, \mathbf{X}^b, \mathbf{X}^c) = \langle \sigma(X), \mathsf{vol}(a, b, c) \rangle$$

is *six times* the signed volume enclosed by the three-dimensional path $(\mathbf{X}^a, \mathbf{X}^b, \mathbf{X}^c)$, where $\mathbf{X}_t^z = \langle \sigma(\mathsf{X}|_{[0,t]}), z \rangle$ [2].

As our main result, we show that the area algebra $A(\mathbb{R}^d)$ spanned by area bracketings ("areas of areas") starting from the letters $\{\mathtt{1}, \ldots, \mathtt{d}\}$ is itself a generator of the shuffle algebra $(T(\mathbb{R}^d), ⧢)$, where $a ⧢ b = a \succ b + b \succ a$. This is a corollary of the following more general fact:

Theorem 1. *Let $X_n \subseteq \mathrm{T}_n(\mathbb{R}^d)$ and $X = \bigcup_n X_n$. Then,*

For all $n \geq 1$, for all nonzero $L \in \mathfrak{g}_n$ there is an $x \in X_n$ such that $\langle x, L \rangle \neq 0$

if and only if

X shuffle generates the shuffle algebra $T(\mathbb{R}^d)$.

Referring back to signatures, our result means that knowledge of the values of the full increments and of all areas of areas computed from a given path is equivalent to the full signature.

We furthermore look at a left bracketing of area, i.e.

$$\overleftarrow{\mathsf{area}}(\mathtt{i}_1 \ldots \mathtt{i}_n) := \mathsf{area}(\ldots \mathsf{area}(\mathsf{area}(\mathsf{area}(\mathtt{i}_1, \mathtt{i}_2), \mathtt{i}_3), \mathtt{i}_4), \ldots, \mathtt{i}_n)$$

Conjecture. *A linear basis is given by the union of the letters and $(\overleftarrow{\mathsf{area}}\mathtt{ij}w)_{(\mathtt{i},\mathtt{j},w)}$, where w runs over all words in d letters and $(\mathtt{i}, \mathtt{j})$ over all letters such that $\mathtt{i} < \mathtt{j}$.*

This was proven for $d = 2$ in [4]. It remains an open problem for $d \geq 3$.

While we didn't manage so far to make progress on the conjecture so far, we came along properties of the area left-bracketing which highlight how few is understood about the algebraic operation so far, e.g. the following result which seems quite unexpected at first.

Theorem 2. *We have*

$$\overleftarrow{\mathsf{area}}(vx) = \overleftarrow{\mathsf{area}}(v)\overleftarrow{\mathsf{area}}(x)$$

for any $\mathrm{T}^{\geq 1}(\mathbb{R}^d)$ and any Lie polynomial x without a first order term.

Further open problems include understanding the Tortkara identity in geometric terms as well as finding *free*, i.e. polynomially independent, shuffle generating set consisting of areas of areas, which would allow for a non-redundant storing of the information of the signature in area terms.

See [1] for a preprint of our project.

References

[1] Joscha Diehl, Terry Lyons, Rosa Preiß, and Jeremy Reizenstein. Areas of areas generate the shuffle algebra, 2020. arXiv:2002.02338.

[2] J. Diehl and J. Reizenstein. Invariants of Multidimensional Time Series Based on Their Iterated-Integral Signature. *Acta Applicandae Mathematicae*, 164(1):83–122, 2019.

[3] AS Dzhumadil'daev. Zinbiel algebras under q-commutators. *Journal of Mathematical Sciences*, 144(2):3909–3925, 2007.
[4] A.S. Dzhumadil'daev, N.A. Ismailov, and F.A. Mashurov. On the speciality of Tortkara algebras. *Journal of Algebra*, 540:1–19, 2019.
[5] E. Gehrig and M. Kawski. A hopf-algebraic formula for compositions of noncommuting flows. In *Proceedings of the IEEE Conference on Decision and Control*, pages 1569–1574, 2008.

A Besov-type sewing lemma and applications

Benjamin Seeger
(joint work with Peter Friz)

The sewing lemma [3, 12, 4, 10] is a result from real analysis that provides a framework for constructing a generalized integral from a given, approximate increment. It has become a versatile tool in the field of rough analysis and rough (partial) differential equations. The purpose of the present work [7] is to prove the sewing lemma in a Besov scale, that is, to measure the increments and integrals with respect to Besov-type norms.

An immediate application of our work is to give an analytical meaning to rough differential equations (RDEs) of the form

$$dY_t = f_0(Y_t)dt + f(Y_t)d\mathbf{X}, \tag{1}$$

where f_0 and $f = (f_1, f_2, \dots, f_n)$ are smooth and $\mathbf{X}$ is a Besov rough path. More precisely, $\mathbf{X}$ is an n-dimensional path $(X^1, X^2, \dots, X^n)$ belonging to B^α_{pq} for $0 < \alpha < 1$, $1/\alpha < p \le \infty$, and $1 \le q \le \infty$, augmented with sufficient extra information, typically interpreted as iterated integrals, in order to regain analytic well-posedness.

The theory for (1) for α-Hölder paths, that is, $X \in B^\alpha_{\infty,\infty}$, can be found, for instance, in [5] when $\alpha > 1/3$, and for arbitrary $\alpha > 0$ in the general "geometric" setting, in both the Hölder and variation sense, in [9] (see also [13, 15] for a discussion about branched rough paths). The case of p-variation paths, and the treatment of jumps (see [10]), is not part of the Besov scale considered here, but appears as an endpoint thereof, in the sense that $W^{\alpha,p} \subset V^{1/\alpha}$ for $p \in (1/\alpha, \infty]$ [8], which is valid also in a non-linear setting that covers rough path spaces. The extension to general Besov spaces is given in [18].

RDEs in the Besov scale, with $\alpha > 1/3$, were studied via paracontrolled distributions in [21]. Results are obtained in [6] for RDEs in a Besov–Nikolskii type scale by interpolation of non-linear rough path spaces of Hölder and variation type. Further progress on RDEs in the Sobolev scale $B^\alpha_{pp} = W^{\alpha,p}$, notably existence and uniqueness, is made in [19]. In their Remark 5.3, these authors further conjecture locally Lipschitz continuity for the solution map with respect to an inhomogeneous Sobolev rough path distance. Such a result is an immediate consequence of our sewing lemma, and is also valid for the more general Besov setting.

Adaptations of sewing and rough integration have been used, for instance, in the analysis of level sets in the Heisenberg group [20], in a "mild" semigroup setting in [14, 11], in a mean-field context in [1], and in a "stochastic rough" setting in

[17]. Our Besov sewing result is thus likely to prove useful beyond (Besov) rough integration and differential equations.

Finally, the Besov scale has also gained interest in the theory of regularity structures, starting with [16]. A central tool in the theory is the reconstruction theorem, which can be seen as a generalization of rough integration. Note also the recent works [22] and [2] on the reconstruction theorem; we suspect the latter provides an ideal framework to formulate Besov sewing in a setting of regularity structures.

References

[1] Bailleul, Ismaël; Catellier, Rémi; Delarue, François. Solving mean field rough differential equations. Electron. J. Probab. 25 (2020), Paper No. 21, 51 pp.

[2] Caravenna, Francesco; Zambotti, Lorenzo. Hairer's Reconstruction Theorem without Regularity Structures. arXiv:2005.09287 [math.AP]

[3] Feyel, Denis; de La Pradelle, Arnaud. Curvilinear integrals along enriched paths. Electron. J. Probab. 11 (2006), no. 34, 860–892.

[4] Feyel, Denis; de La Pradelle, Arnaud; Mokobodzki, Gabriel. A non-commutative sewing lemma. Electron. Commun. Probab. 13 (2008), 24–34.

[5] Friz, Peter K.; Hairer, Martin. A course on rough paths. With an introduction to regularity structures. Universitext. Springer, Cham, 2014.

[6] Friz, Peter K.; Prömel, David J. Rough path metrics on a Besov-Nikolskii-type scale. Trans. Amer. Math. Soc. 370 (2018), no. 12, 8521–8550.

[7] Friz, Peter K.; Seeger, Benjamin. A Besov-type sewing lemma and applications. To appear.

[8] Friz, Peter; Victoir, Nicolas. A variation embedding theorem and applications. J. Funct. Anal. 239 (2006), no. 2, 631–637.

[9] Friz, Peter K.; Victoir, Nicolas B. Multidimensional stochastic processes as rough paths. Theory and applications. Cambridge Studies in Advanced Mathematics, 120. Cambridge University Press, Cambridge, 2010.

[10] Friz, Peter K.; Zhang, Huilin. Differential equations driven by rough paths with jumps. J. Differential Equations 264 (2018), no. 10, 6226–6301.

[11] Gerasimovičs, Andris; Hairer, Martin. Hörmander's theorem for semilinear SPDEs. Electron. J. Probab. 24 (2019), Paper No. 132, 56 pp.

[12] Gubinelli, M. Controlling rough paths. J. Funct. Anal. 216 (2004), no. 1, 86–140.

[13] Gubinelli, Massimiliano. Ramification of rough paths. J. Differential Equations 248 (2010), no. 4, 693–721.

[14] Gubinelli, Massimiliano; Tindel, Samy. Rough evolution equations. Ann. Probab. 38 (2010), no. 1, 1–75.

[15] Hairer, Martin; Kelly, David. Geometric versus non-geometric rough paths. Ann. Inst. Henri Poincaré Probab. Stat. 51 (2015), no. 1, 207–251.

[16] Hairer, Martin; Labbé, Cyril. The reconstruction theorem in Besov spaces. J. Funct. Anal. 273 (2017), no. 8, 2578–2618.

[17] Lê, Khoa. A stochastic sewing lemma and applications. Electron. J. Probab. 25 (2020), Paper No. 38, 55 pp.

[18] Liu, Chong; Prömel, David J.; Teichmann, Josef. Characterization of nonlinear Besov spaces. Trans. Amer. Math. Soc. 373 (2020), no. 1, 529–550.

[19] Liu, Chong; Prömel, David J.; Teichmann, Josef; On Sobolev rough paths. J. Math. Anal. Appl. 497 (2021), no. 1, 124876.

[20] Magnani, Valentino; Stepanov, Eugene; Trevisan, Dario. A rough calculus approach to level sets in the Heisenberg group. J. Lond. Math. Soc. (2) 97 (2018), no. 3, 495–522.

[21] Prömel, David J.; Trabs, Mathias. Rough differential equations driven by signals in Besov spaces. J. Differential Equations 260 (2016), no. 6, 5202–5249.
[22] Singh, Harprit; Teichmann, Josef. An elementary proof of the reconstruction theorem. arXiv:1812.03082 [math.PR]

Paracontrolled calculus and regularity structures

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(joint work with Masato Hoshino)

Two different sets of tools for the study of singular stochastic PDEs have emerged recently, under the form of Hairer's theory of regularity structures [9, 6, 7, 5] and paracontrolled calculus [8, 1, 2], after Gubinelli, Imkeller and Perkowski' seminal work. Both of them implement the same mantra: Make sense of the equation in a restricted space of functions/distributions whose elements look like linear combinations of reference random quantities, for which the ill-defined terms that come form the analysis of the product problems can be defined using probabilistic tools. Within the setting of regularity structures, Taylor-like pointwise expansions and jet-like objects are used to make sense of what it means to look like a linear combination of reference quantities

$$f(\cdot) \sim \sum_\tau f_\tau(z)\big(\mathsf{\Pi}^{\mathsf{g}}_z\tau\big)(\cdot), \quad \text{near } z, \text{ for all spacetime points } z.$$

In the paracontrolled approach, one uses paraproducts to implement this mantra

$$f \sim \sum_\tau \mathsf{P}_{f_\tau}[\tau].$$

Each term $\mathsf{P}_a b$ is a function or a distribution. This approach is justified at an intuitive level by the fact that $\mathsf{P}_{f_\tau}[\tau]$ can be thought as a modulation of the reference function/distribution $[\tau]$. The two options seem technically very different from one another.

While Hairer's theory has now reached the state of a ready-to-use black box for the study of singular stochastic PDEs, like Cauchy-Lipschitz well-posedness theorem for ordinary differential equations, the task of giving a self-contained treatment of renormalisation matters within paracontrolled calculus remains to be done. It happens nonetheless to be possible to compare the two languages, independently of their applications to the study of singular stochastic PDEs. One can indeed give a one-to-one correspondence between models and modelled distributions on a given regularity structure and finite collections of distributions and functions that serve as building blocks in a paracontrolled description of the corresponding objects.

Given a (concrete) regularity structure $\mathscr{T} = (T^+, T)$, we denote by β_0 the minimum of the homogeneity of elements of T, by $\mathcal{B}$ a linear basis of T and by $\mathcal{B}^+$ a linear basis of T^+. Write $\mathcal{B}^+_X$, resp. $\mathcal{B}_{\underline{X}}$, for the canonical polynomial structure inside T^+, resp. T. Write $\mathcal{B}_\circ^+$ for the non-polynomial elements of $\mathcal{B}_X^+$. We use the notation

$$\Delta\sigma = \sum_{\mu\leq\sigma} \mu\otimes\sigma/\mu \in T\otimes T^+, \qquad \Delta^+\tau = \sum_{\nu\leq^+\tau} \nu\otimes\tau/^+\nu \in T^+\otimes T^+.$$

1. Paracontrolled systems for models. *Let a (concrete) regularity structure $\mathscr{T} = (T^+, T)$ be given, together with a model $\mathsf{M} = (\mathsf{g}, \mathsf{\Pi})$ on it. One can construct functions*

$$[\,\cdot\,]^{\mathsf{M}} : T \mapsto C^{\beta_0}(\mathbb{T}^d), \text{ and } [\,\cdot\,]^{\mathsf{g}} : T^+ \mapsto C^0(\mathbb{T}^d),$$

such that $[\sigma]^{\mathsf{M}} \in C^{|\sigma|}(\mathbb{T}^d)$, and $[\tau]^{\mathsf{g}} \in C^{|\tau|}(\mathbb{T}^d)$, for every homogeneous $\sigma \in T$ and $\tau \in T^+$, all $[\sigma]^{\mathsf{M}}$, and $[\tau]^{\mathsf{g}}$ are continuous function of the model $(\mathsf{g}, \mathsf{\Pi})$, such that for any $\tau \in \mathcal{B}^+ \setminus \mathcal{B}_X^+$ and $\sigma \in \mathcal{B} \setminus \mathcal{B}_{\underline{X}}$, one has paracontrolled representations of the $\mathsf{\Pi}$ and g maps by paracontrolled systems

$$\begin{aligned} \mathsf{g}(\tau) &= \sum_{\substack{\mathbf{1}<^+\nu<^+\tau \\ \nu\in\mathcal{B}^+\setminus\mathcal{B}_X^+}} \mathsf{P}_{\mathsf{g}(\tau/^+\nu)}[\nu]^{\mathsf{g}} + [\tau]^{\mathsf{g}}, \\ \mathsf{\Pi}\sigma &= \sum_{\substack{\mu<\sigma \\ \mu\in\mathcal{B}\setminus\mathcal{B}_{\underline{X}}}} \mathsf{P}_{\mathsf{g}(\sigma/\mu)}[\mu]^{\mathsf{M}} + [\sigma]^{\mathsf{M}}. \end{aligned} \tag{1}$$

If one assumes that there exists a finite subset $\mathcal{G}_\circ^+$ of $\mathcal{B}_\circ^+$ such that $\mathcal{B}_\circ^+$ is of the form

$$\mathcal{B}_\circ^+ = \bigsqcup_{\tau\in\mathcal{G}_\circ^+} \Big\{ D^k\tau \,;\, k\in\mathbb{N}^d,\ |\tau| - |k| > 0 \Big\}.$$

and the splitting map Δ^+ enjoys a mild recursive structure satisfied by all regularity structures that appear in the study of singular stochastic PDEs then the restriction to $\mathcal{G}_\circ^+ \times \{\tau \in \mathcal{B}\,;\, |\tau| \leq 0\} \subset T^+ \times T$ of the paracontrolled representation (1) *provides a parametrisation of the set of models on $\mathscr{T}$.*

The above assumption is Assumption (C) in [4]. This statement holds for general regularity structures. In the particular case of regularity structures built with integration operators, as in the study of stochastic singular PDEs, the g map of a model is determined by its $\mathsf{\Pi}$ map, and the paracontrolled parametrisation involves only $\{[\tau]\,;\, \tau \in \mathcal{B}, |\tau| \leq 0\}$.

These parametrisation results give a direct access to density results for smooth models and extension results for models defined on sub-regularity structures, giving an analogue of Lyons-Victoir extension theorem.

2. Paracontrolled systems for modelled distributions. The reference functions $[\tau]^{\mathsf{M}}$ and $[\sigma]^{\mathsf{g}}$ involved in the paracontrolled representation of a model happen to be the building blocks used in the paracontrolled representation of modelled distributions. *One can associate to any modelled distribution*

$$\mathbf{f} = \sum_{\tau\in\mathcal{B};|\tau|<\gamma} f^\tau\tau \in D^\gamma(\mathscr{T}, \mathsf{g}),$$

a distribution $[\mathsf{f}]^{\mathsf{M}} \in C^\gamma(\mathbb{T}^d)$ *such that one defines a reconstruction* Rf *of* f *setting*

$$\mathsf{Rf} := \sum_{\tau\in\mathcal{B};|\tau|<\gamma} \mathsf{P}_{f^\tau}[\tau]^{\mathsf{M}} + [\mathsf{f}]^{\mathsf{M}}. \tag{2}$$

Each coefficient f^τ*, also has a representation*

$$f^\tau = \sum_{\tau<\mu;|\mu|<\gamma} \mathsf{P}_{f^\mu}[\mu/\tau]^{\mathsf{g}} + [f^\tau]^{\mathsf{g}}, \tag{3}$$

for some $[f^\tau]^{\mathsf{g}} \in C^{\gamma-|\tau|}(\mathbb{R}^d)$*. Under a mild structure assumption on the regularity structure, the paracontrolled representation of elements in* $D^\gamma(T,\mathsf{g})$ *defines a locally bi-Lipschitz homeomorphism*

$$D^\gamma(T,\mathsf{g}) \to \prod_{\tau\in\mathcal{B},\,|\tau|<\gamma} C^{\gamma-|\tau|}(\mathbb{T}^d).$$

Interestingly, the above mentioned assumption involves not the regularity structure itself but rather a linear basis $\mathcal{B}$ of T. It may then happen that one basis of T satisfies it whereas another does not. It happens that the class of regularity structures introduced by Bruned, Hairer and Zambotti in [6] for the study of singular stochastic PDEs all satisfy this assumption, despite the fact that their canonical bases do not satisfy it. The previous statement provides a much refined version of the paraproduct-based construction of the reconstruction operator from Gubinelli, Imkeller and Perkowski' seminal work [8], that was first refined in Martin and Perkowski's work [10]. Its proof happens to be equivalent to an extension problem for the g map from the set T^+ to a Hopf algebra extension of the latter that is a quotient a free Hopf algebra by some relations. Compatibility with this quotient structure of the initial data explains why the latter need to satisfy some structure conditions reminiscent of similar constraints put forward by Martin and Perkowski in [10].

References

[1] I. Bailleul and F. Bernicot and D. Frey, *Space-time paraproducts for paracontrolled calculus, 3d-PAM and multiplicative Burgers equations*, Ann. Sci. Ec. Norm. Sup. Ser.4,51(6), (2018), 1399–1456.

[2] I. Bailleul and F. Bernicot, *High order paracontrolled calculus*, Forum Math. Sigma, **7**(e44), (2019), 1–94.

[3] I. Bailleul and M. Hoshino, *Paracontrolled calculus and regularity structures (I)*, arXiv:1812.07919. To appear in J. Math. Soc. Japan, (2021), 1-43.

[4] I. Bailleul and M. Hoshino, *Paracontrolled calculus and regularity structures (II)*, arXiv:1912.08438, 1-37.

[5] Y. Bruned and A. Chandra and I. Chevyrev and M. Hairer, *Renormalising SPDEs in regularity structures*, arXiv:1711.10239.v2, to appear in J. Europ. Math. Soc., (2020^+).

[6] Y. Bruned and M. Hairer, and L. Zambotti, *Algebraic renormalization of regularity structures*, Invent. Math. **215**(3):1039-1156, (2019).

[7] A. Chandra and M. Hairer, *An analytic BPHZ theorem for regularity structures*, arXiv:1612.08138v5, (2016).

[8] M. Gubinelli and P. Imkeller and N. Perkowski, *Paracontrolled distributions and singular PDEs*, Forum Math. Pi, **3**(e6), (2015), 1–75.

[9] M. Hairer, *A theory of regularity structures*, Invent. Math., **198**(2), (2014), 269-504.
[10] J. Martin and N. Perkowski, *A Littlewood-Paley description of modelled distributions*, J. Funct. Anal., **6**, (2020), 108634.

A gauge-fixed representation of the 2D Yang–Mills measure

Ilya Chevyrev

We report on a recent gauge-fixed representation of the 2D Yang–Mills (YM) measure obtained in [2]. We will also comment on the connection with another work carried out by the author together with A. Chandra, M. Hairer, and H. Shen [1] where a canonical Markov process associated to the Langevin dynamic of the 2D YM measure is constructed.

Let G be a connected, compact Lie group with Lie algebra $\mathfrak{g}$. The YM measure on the torus $\mathbb{T}^d$ associated to the trivial principal G-bundle is formally a probability measure on the space $\mathfrak{g}$-valued 1-forms on $\mathbb{T}^d$ the density of which with respect to the (infinite-dimensional and ill-defined) Lebesgue measure is proportional to

$$\exp\Big(-\int_{\mathbb{T}^d}|F_A(x)|^2\,\mathrm{d}x\Big)\ .$$

Here A is a $\mathfrak{g}$-valued 1-form with curvature F_A. The integral $\int_{\mathbb{T}^d}|F_A(x)|^2\,\mathrm{d}x$ is known as the *YM energy.* One of the main difficulties associated to this energy is its invariance under an infinite-dimensional group of *gauge transformations* of the form $A\mapsto A^g = gAg^{-1}-(\mathrm{d}g)g^{-1}$, where $g\colon\mathbb{T}^d\to G$.

Rigorous constructions of the YM measure in dimension $d=2$ have included two approaches. The first aims to characterise gauge-invariant observables, often focusing on Wilson loops; in particular, a complete specification of such observables which are sufficient to separate any two 1-forms was derived by Lévy [5]. The second approach, taken by Driver [4] (though only for the base space $\mathbb{R}^2$), involves writing the measure in an axial-gauge in which it can be seen as a linear transformation of a $\mathfrak{g}$-valued white noise; the main drawback of this approach is that the resulting measure is supported on the Hölder–Besov space $\mathcal{C}^\alpha$ for $\alpha<-\frac{1}{2}$, while the the interpretation of the YM measure as a perturbation of the Gaussian free field suggests it should have a representation supported on $\mathcal{C}^\alpha$ for any $\alpha<0$.

The main result of [2], stated as Theorem 1 below, provides in certain cases such a representation which furthermore allows for holonomies along a family of curves to be defined in a pathwise (rather than just probabilistic) sense.

Theorem 1. *Suppose G is simply connected. For every $\alpha\in(\frac{1}{2},1)$, there exists a separable Banach space Ω^1_α of distributional $\mathfrak{g}$-valued 1-forms on $\mathbb{T}^2$ and an Ω^1_α-valued random variable A such that the Wilson loop observables induced by A for axis-parallel paths are equal in law to those of the YM measure.*

The precise statement of Theorem 1 is given in [2, Theorem 1.1] and the relevant definitions can be found in [2, Sections 3]. We mention here that Ω^1_α is a space of distributions for which line integrals along axis-parallel paths give rise to α-Hölder paths in $\mathfrak{g}$, and thus holonomies along axis-parallel paths for every $A\in\Omega^1_\alpha$ make sense using Young ODE theory. These holonomies furthermore possess reasonable stability properties. The space Ω^1_α moreover embeds into the Hölder–Besov space $\mathcal{C}^{\alpha-1}$, and thus the random variable A has the same small-scale regularity as the Gaussian free field on $\mathbb{T}^2$, which is the optimal regularity in the Hölder–Besov scale expected for any gauge-fixed version of the YM measure.

We mention that the assumption in Theorem 1 that G simply connected is used only to ensure that every principal G-connections on $\mathbb{T}^2$ can be represented as a global 1-form. If $\mathbb{T}^2$ was replaced by the $[0,1]^2$, treated as a manifold with boundary so that all principal G-bundles are now trivial, then the analogue of Theorem 1 (with even a slightly simplified proof) will remain true without the assumption that G is simply connected.

Before discussing the proof of Theorem 1, let us comment on the connection with the recent work [1]. The results in [2] can be split into two parts – the first concerns the construction of the space Ω^1_α and the second concerns the construction of the random variable A. In this regard, the first part of [2] has been significantly generalised in [1]; the spaces in the latter corresponding to Ω^1_α allow for holonomies along *all* sufficiently regular curves. They furthermore come with a natural gauge group $\mathfrak{G}^\alpha$ which allows one to build a canonical quotient space $\mathfrak{O} = \Omega^1_\alpha/\mathfrak{G}^\alpha$ with good topological properties. This quotient space $\mathfrak{O}$ should therefore be the natural state space of the YM measure and its "gauge-fixed representations" should arise, for example, from measurable selections $\mathfrak{O} \to \Omega^1_\alpha$. The main result of [1] is the construction of a Markov process on $\mathfrak{O}$ which corresponds to the YM stochastic quantisation equation. The main tool used in renormalising and solving the stochastic quantisation equation is the theory of regularity structures. The work [1] however does not generalise the second part of [2], that is, the construction of the Ω^1_α-valued random variable A. In other words, it is not proven (but is expected to be true) that the YM measure is supported on $\mathfrak{O}$ and is the unique invariant measure of the Markov process constructed in [1].

We now describe the proof of Theorem 1 which is based on lattice approximations. Consider the uniform lattice Λ on the unit square on the torus $\mathbb{T}^2$ with spacing $\varepsilon = 2^{-N}$, $N \geq 1$. A *discrete connection* is a function $U \colon \mathbf{B} \to G$, where $\mathbf{B} = \{(x,y) \in \Lambda^2 \,;\, |x-y| = \varepsilon\}$ is the set of oriented bonds (i.e. links) of Λ, such that $U(y,x) = U(x,y)^{-1}$ for all $(x,y) \in \mathbf{B}$. Denote by $\mathfrak{A}$ the group of discrete connections, which is isomorphic to $G^{|\mathbf{B}|/2}$.

A discrete approximation of the YM measure is the probability measure on $\mathfrak{A}$

$$\mu(\mathrm{d}U) = Z^{-1} \prod_{p \subset \Lambda} Q(U(\partial p))\, \mathrm{d}U \;,$$

where $\mathrm{d}U$ is the Haar measure on $\mathfrak{A}$, the product is over all plaquettes p of Λ (i.e. squares of dimension $\varepsilon \times \varepsilon$), $U(\partial p) \in G$ is the holonomy U around p, and $Q = \exp(\varepsilon^2 \Delta) \colon G \to (0,\infty)$ is the heat kernel on G at time ε^2. (This choice for Q is known as the Villain action; another common choice known as the Wilson action is $Q(x) = \exp(\varepsilon^{-2} \Re \mathrm{Tr}(x))$, where G is assumed to be a matrix group.) It is known from [5] that, as $\varepsilon \to 0$, the Wilson loop observables of μ for axis-parallel paths converge to those of the continuum YM measure. To prove Theorem 1, it therefore suffices to show tightness in $\varepsilon > 0$ of the lattice versions of the norms $|\log(U^g)|_{\Omega^1_\alpha}$ for $U \sim \mu$, where $g\colon \Lambda \to G$ a suitable gauge transformation (which necessarily depends on the realisation of U).

There are now two main steps. The first step, which is completely deterministic, is an Uhlenbeck-type theorem applicable to discrete and certain distributional connections. Roughly speaking, it states that for any $U \in \mathfrak{A}$, there exists a gauge transformation $g\colon \Lambda \to G$ such that $|\log(U^g)|_{\Omega^1_\alpha}$ is bounded by a function of

$$\sup_{r \subset \Lambda} \frac{|\log \mathrm{hol}(U, \partial r)| + |X|_{q\text{-var}}}{\mathrm{Area}(r)^{\alpha/2}} . \tag{1}$$

The supremum in (1) is taken over all rectangles r in Λ, $\mathrm{hol}(U, \partial r)$ is the holonomy of U along the boundary of r, and $|X|_{q\text{-var}}$ is the q-variation of a $\mathfrak{g}$-valued sequence X corresponding to the "anti-development" of U along r; the precise definitions can be found in [2, Section 4]. We note that (1) is gauge-invariant. This result should should be compared to the classical result of Uhlenbeck that $|A^g|_{W^{1,p}}$ can be bounded by a function of $|F_A|_{L^p}$. Both of the quantities $|\log(U^g)|_{\Omega^1_\alpha}$ and (1) are weaker than $|A^g|_{W^{1,p}}$ and $|F_A|_{L^p}$ respectively and are in particular applicable, in the $\varepsilon \to 0$ limit, to distributional 1-forms with the regularity of the Gaussian free field.

The second step involves probabilistic bounds on the quantity (1). More precisely, [2, Theorem 5.3] establishes moment estimates of (1) uniform in the lattice spacings. The proof of these estimates uses a random walk representation of the YM measure through an axial gauge. Control on (1) thus reduces to control on the q-variation of the rough path lift of this random walk, which in turn is handled by the rough path BDG inequality for càdlàg (local) martingales proven in [3].

We conclude by mentioning two open questions.

- Is some analogue of Theorem 1 true in the case that the 2D YM measure is coupled to a Higgs field? One still expects the quantity (1) to be a good measure of "non-flatness" of 1-forms which is stable in the $\varepsilon \to 0$ limit. However, the main difficulty is that the random walk representation is no longer applicable and it is not clear how to derive the necessary probabilistic bounds.
- Can similar bounds be obtained in the 3D case? One indeed expects the YM measure to again have the same regularity as the Gaussian free field (now in $\mathcal{C}^\alpha$, $\alpha < -\frac{1}{2}$), and the gauges used in [2] can be generalised to 3D. The main difficulty, however, is that (1) is no longer a good measure of "non-flatness" in 3D and therefore some new quantity needs to be found.

References

[1] A. Chandra, I. Chevyrev, M. Hairer and H. Shen. Langevin dynamic for the 2D Yang-Mills measure. *Preprint* arXiv:2006.04987, 2020.

[2] I. Chevyrev. Yang-Mills measure on the two-dimensional torus as a random distribution. *Comm. Math. Phys.* **372** (2019), no. 3, 1027–1058.

[3] I. Chevyrev and P. K. Friz. Canonical RDEs and general semimartingales as rough paths. *Ann. Probab.* **47** (2019), no. 1, 420–463.

[4] B. Driver. YM_2: continuum expectations, lattice convergence, and lassos. *Comm. Math. Phys.* **123** (1989), no. 4, 575–616.
[5] T. Lévy. Yang-Mills measure on compact surfaces. *Mem. Amer. Math. Soc.* **166** (2003), no. 790, xiv+122 pp.

On stochastic controlled rough paths

Khoa Lê
(joint work with Peter Friz, Antoine Hocquet)

Consider hybrid stochastic rough differential equations of the type

$$dY_t(\omega) = b(\omega, t, Y_t(\omega))dt + \sigma(\omega, t, Y_t(\omega))dB_t(\omega) + f(Y_t(\omega))d\boldsymbol{X}_t, \quad t \in [0, T]. \tag{1}$$

Here, B is a standard Brownian motion with respect to a filtration $\{\mathcal{F}_t\}$, $\boldsymbol{X} = (X, \mathbb{X})$ is a deterministic Hölder-rough path of regularity $\alpha \in (1/3, 1/2]$. Such equations appear in non-linear filtering theory in which $B(\omega)$ models the signal noise, whereas X plays the role of an observed (hence given) path.

When the coefficients are deterministic and time-independent, hybrid rough stochastic differential equations such as (1) have been investigated earlier. In [1], a flow transformation based on the rough flow associated to f and $\boldsymbol{X}$ is applied to transform (1) to an Itô stochastic differential equation. In [3, 2], by lifting $(B(\omega), X)$ to a joint rough path, equation (1) is considered as a *random* rough differential equation. These methods, however, require additional regularity assumptions on σ and f respectively. Although stochastic differential equations and rough differential equations are well-studied separately using Itô's stochastic analysis and Lyons' theory of rough path ([7]), the appearance of a Brownian motion and a rough path at the same time in (1) clearly has created non-trivial mathematical challenges. This is due to the lack of a unified theory which can incorporate simultaneously the martingale structure of Brownian motion and the algebraic structure of the rough path.

The talk discusses on an alternative approach to (1) which is introduced in [4] and is based upon a new class of stochastic processes called stochastic controlled rough paths. The method provides a direct analysis to (1) without any transformations nor the existence of a joint rough path lift $(B(\omega), X)$. The class of stochastic controlled rough paths contains semimartingales and (random) controlled rough paths. It is stable under compositions with regular functions and integrations against themselves. Compare with the notion controlled rough paths introduced by Gubinelli in [5], stochastic controlled rough paths take values in spaces of random variables of finite moments and exhibit Taylor-like expansions only through conditional expectations. The novelty lies in the later property, which is crucial as it allows one to average out martingale parts in the remainders. On the well-posedness of (1), we have the following result.

Theorem 1. *Assume that*

- *$y \mapsto b(\omega, t, y), \sigma(\omega, t, y)$ are adapted and bounded Lipschitz uniformly in (ω, t),*
- *f is of regularity $\mathcal{C}^\gamma$ with $\alpha\gamma \geq 1$.*

Then equation (1) *has a unique strong solution in the class of stochastic controlled rough paths.*

To study the probability laws of the solutions to (1), we also introduced in [4] rough martingale problems associated to (1). Along this direction, we have the following result.

Theorem 2. *Under the hypotheses of Theorem 1, assume additionally that b, σ are deterministic. The rough martingale problem associated to* (1) *is well-posed and the solution to* (1) *is strong Markov and has Feller property.*

These results are made available due to the recently developed stochastic sewing lemma from [6]. Nevertheless, applying the stochastic sewing lemma to study equation (1) is not straightforward and indeed presents many technical difficult issues. This is mainly because estimates for the solutions of (1) require controls of polynomial type, yet the usual moment norms are not invariant under polynomial maps. This issue is not present in the analysis for (random) rough differential equations. To overcome such problem, we replace the usual moment norms by a mixed conditional norm. For instance, we show that the solution to (1) satisfies

$$\sup_{\omega} \left(\mathbb{E}(|Y_t - Y_s|^m | \mathcal{F}_s)\right)^{\frac{1}{m}} \leq C|t-s|^\alpha \quad \forall s \leq t$$

for some constants $C > 0$ and $m \geq 2$.

References

[1] Crisan, Dan, Joscha Diehl, Peter K. Friz, and Harald Oberhauser. Robust filtering: correlated noise and multidimensional observation. The Annals of Applied Probability 23, no. 5 (2013): 2139-2160.

[2] Diehl, Joscha, Peter K. Friz, and Wilhelm Stannat. Stochastic partial differential equations: a rough paths view on weak solutions via Feynman-Kac. Annales de la Faculté des sciences de Toulouse: Mathématiques, vol. 26, no. 4, pp. 911-947. 2017.

[3] Diehl, Joscha, Harald Oberhauser, and Sebastian Riedel. A Lévy area between Brownian motion and rough paths with applications to robust nonlinear filtering and rough partial differential equations. Stochastic processes and their applications 125, no. 1 (2015): 161-181.

[4] Peter Friz, Antoine Hocquet, and Khoa Lê. Rough Markov diffusions and stochastic differential equations. *Preprint*, 2021.

[5] Gubinelli, Massimiliano. Controlling rough paths. Journal of Functional Analysis 216, no. 1 (2004): 86-140.

[6] Lê, Khoa. A stochastic sewing lemma and applications. Electronic Journal of Probability 25 (2020).

[7] Lyons, Terry J. Differential equations driven by rough signals. Revista Matemática Iberoamericana 14, no. 2 (1998): 215-310.

Generalized iterated-sums signatures

Kurusch Ebrahimi-Fard
(joint work with Joscha Diehl, Nikolas Tapia)

Iterated-integrals and -sums signatures capture rather well features of sequentially ordered data [2, 3, 9, 11]. A certain universality property combined with an inherent recursive structure make them ideal for approximating arbitrary (bounded) nonlinear mappings on sequence space by linear functionals on feature space. Both these properties are succinctly described in the context of graded connected word Hopf-algebras. Indeed, the iterated-integrals and -sums signature maps are shuffle respectively quasi-shuffle algebra morphisms satisfying Chen's relation with respect to the deconcatenation coproduct. Here, shuffle and quasi-shuffle products algebraically encode integration respectively summation by parts.

Motivated by [9], we study in [4] a family of generalized iterated-sums signatures, obtained in terms of particular nonlinear transformations, which are algebra morphisms over word Hopf algebras defined in terms of modified quasi-shuffle products and the deconcatenation coproduct. Our approach permits to consider other transformations of the iterated-sums signature and to express them in terms of the un-transformed iterated-sums signature. The first is obtained by applying a tensorized nonlinear transformation to each time slice, the second one is constructed by applying a polynomial map to increments, and the third follows by first transforming the data and then considering its increments (generalizing [1]).

Following [6], a commutative quasi-shuffle algebra $(\mathcal{A}, \succ, \bullet)$ over $\mathbb{R}$ consists of a nonunital commutative $\mathbb{R}$-algebra $(\mathcal{A}, \bullet)$ equipped with a *half-shuffle product* $\succ\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ satisfying the relations

$$x \succ (y \succ z) = (x \succ y + y \succ x + x \bullet y) \succ z \quad \text{and} \quad (x \succ y) \bullet z = x \succ (y \bullet z).$$

They imply commutativity and associativity of the quasi-shuffle product, $x * y := x \succ y + y \succ x + x \bullet y$. A quasi-shuffle morphism between $(\mathcal{A}, \succ, \bullet)$ and $(\tilde{\mathcal{A}}, \tilde{\succ}, \tilde{\bullet})$ is a linear map $\Lambda\colon \mathcal{A} \to \tilde{\mathcal{A}}$ such that for all $x, y \in \mathcal{A}$, $\Lambda(x \succ y) = \Lambda(x) \tilde{\succ} \Lambda(y)$ and $\Lambda(x \bullet y) = \Lambda(x) \tilde{\bullet} \Lambda(y)$. Any such map is an algebra morphism, $\Lambda(x * y) = \Lambda(x) \tilde{*} \Lambda(y)$. Extending $(\mathcal{A}, \succ, \bullet)$ to $\bar{\mathcal{A}} := \mathbb{R}1 \oplus \mathcal{A}$ by defining for $x \in \mathcal{A}$: $1 \bullet x = x \bullet 1 := 0$, $1 \succ x := 0$ and $x \succ 1 := x$ turns $\bar{\mathcal{A}}$ into a unital algebra [1]. If $(\mathcal{A}, \bullet)$ has trivial product, we obtain what is called a commutative shuffle algebra with shuffle product $x \sqcup y := x \succ y + y \succ x$. We refer to [4] for details and more references.

The reduced symmetric algebra, $S(A) = \bigoplus_{n=1}^{\infty} S^n A$, over a finite alphabet, $A = \{\mathtt{1}, \ldots, \mathtt{d}\}$ is the space spanned by square brackets $[\mathtt{i}_1 \cdots \mathtt{i}_n] \in S^n A$ in commuting letters $\mathtt{i}_k \in A$. The standard basis of the tensor algebra $H := \bigoplus_{n=0}^{\infty} S(A)^{\otimes n}$, where $S(A)^{\otimes 0} = \mathbb{R}\mathrm{e}$, consists of words $\mathfrak{A}^*$ over the set $\mathfrak{A}$ of all square brackets, the latter constituting a basis for $S(A)$. The length of a word $w = s_1 \cdots s_k \in H$ is defined to be $\ell(w) = k$. We endow H with a commutative quasi-shuffle product,

[1] Note that the product $1 \succ 1$ as well as $1 \bullet 1$ are excluded and that $1 * 1 := 1$.

obtained from the bracket product of $S(A)$, by defining: $u \succ va := (u \star v)a$ and $ua \bullet vb := (u \star v)[ab]$ together with $\mathrm{e} \star u := u =: u \star \mathrm{e}$, such that

$$ua \star vb = (u \star vb)a + (ua \star v)b + (u \star v)[ab], \quad u, v \in \mathfrak{A}^*,\ a, b \in S(A).$$

This product is compatible with the deconcatenation coproduct Δ and the counit ε determined by the grading, so that $H_{qsh} := (H, \star, \Delta, \mathrm{e}, \varepsilon, \alpha)$ is a Hopf algebra [7]. The free commutative unital quasi-shuffle algebra over $\mathbb{R}^d$ is isomorphic to $(H, \succ, \bullet)$ [10]. The dual space H^* can be identified with word series, $\mathrm{S} = \sum_{w \in \mathfrak{A}^*} \langle \mathrm{S}, w \rangle w$. Equipped with the non-commutative convolution product of two maps $\mathrm{R}, \mathrm{S} \in H^*$, defined by $\mathrm{RS} := \sum_{w \in \mathfrak{A}^*} \langle \mathrm{R} \otimes \mathrm{S}, \Delta w \rangle w$, it becomes an unital associative algebra.

A sequence $I = (i_1, \dots, i_k) \in \mathcal{C}(n)$ of positive integers such that $\sum_{1 \le m \le k} i_m = n$ is a composition of $n \ge 1$. Let $w = s_1 \cdots s_n \in \mathfrak{A}^*$ and $I = (i_1, \dots, i_k) \in \mathcal{C}(n)$, we define $I[w] := [s_1 \cdots s_{i_1}][s_{i_1+1} \cdots\ s_{i_1+i_2}] \cdots [s_{i_1+\cdots+i_{k-1}+1} \cdots s_n] \in \mathfrak{A}^*$. Following [5, 7, 8], given $f(t) = \sum_{n=1}^{\infty} c_n t^n \in t\mathbb{R}[[t]]$ we define $\Psi_f \in \mathrm{End}_{\mathbb{R}}(H)$ by $\Psi_f(w) := \sum_{I \in \mathcal{C}(\ell(w))} c_{i_1} \cdots c_{i_k} I[w]$. It can be shown that Ψ_f is always a graded coalgebra morphism. Moreover, if $c_1 \neq 0$, then $\Psi_f^{-1} = \Psi_{f^{-1}}$. These maps can be used to define a *deformed quasi-shuffle algebra* with deformed half-shuffle. Indeed, let $f \in t\mathbb{R}[[t]]$ be invertible. The space $(H, \succ_f, \bullet_f)$, where $u \succ_f v := \Psi_f^{-1}(\Psi_f(u) \succ \Psi_f(v))$ and $u \bullet_f v := \Psi_f^{-1}(\Psi_f(u) \bullet \Psi_f(v))$, is a commutative quasi-shuffle algebra with deformed quasi-shuffle product $u \star_f v := \Psi_f^{-1}(\Psi_f(u) \star \Psi_f(v))$, such that $H_f = (H, \star_f, \Delta, \mathrm{e}, \varepsilon)$ is a connected graded Hopf algebra and the map $\Psi_f \colon H_f \to H_{qsh}$ is a Hopf algebra isomorphism. Note that for $f(t) = \exp(t) - 1$, the corresponding deformed quasi-shuffle product coincides with the classical shuffle product on H.

For fixed positive integers d and N, we define a *d-dimensional time series of length N* as a sequence of vectors $x = (x_0, \dots, x_{N-1}) \in (\mathbb{R}^d)^N$, where $x_j = (x_j^{(1)}, \dots, x_j^{(d)})$. Its increment series, $\delta x \in (\mathbb{R}^d)^{N-1}$, has entries $\delta x_k := x_{k+1} - x_k$. Let x be a d-dimensional time series and $f \in t\mathbb{R}[[t]]$. The *generalized iterated-sums signature* is the family of linear maps $\mathrm{ISS}(x)^{(f)} := (\mathrm{ISS}(x)^f_{n,m} : 0 \le n \le m \le N)$ defined by

$$\mathrm{ISS}(x)^f_{n,m} := \overrightarrow{\prod_{n \le j < m}} \left(\varepsilon + f\left(\sum_{a \in \mathfrak{A}} \delta x_j^a a \right) \right).$$

Note that for $f = t$ we recover the canonical iterated-sums signature, $\mathrm{ISS}(x)$, introduced in [2]. In [4] it is shown that the generalized iterated-sums signature satisfies Chen's property, i.e., $\mathrm{ISS}(x)^f_{n,n'} \mathrm{ISS}(x)^f_{n',n''} = \mathrm{ISS}(x)^f_{n,n''}$, for any $0 \le n \le n' \le n'' \le N$, and for $w \in H$, we have that $\langle \mathrm{ISS}(x)^f_{n,m}, w \rangle = \langle \mathrm{ISS}(x)_{n,m}, \Psi_f(w) \rangle$. The latter implies that $\mathrm{ISS}(x)^f$ is a character over the Hopf algebra H_f.

Let $f \in t\mathbb{R}[t]$ be a polynomial. The iterated-sums signature of the transformed series $(f(\delta x_0), \dots, f(\delta x_{N-1}))$ is given by

$$\mathrm{ISS}(x)^{(f)}_{n,m} = \overrightarrow{\prod_{n \le j < m}} \left(\varepsilon + \sum_{a \in \mathfrak{A}} f(\delta x_j)^a a \right),$$

where f acts in a vectorized fashion. From this we see immediately that $\mathrm{ISS}(x)^{(f)}$ satisfies Chen's identity and that for $a_1 \cdots a_p \in \mathfrak{A}^{*\,2}$.

$$\langle \mathrm{ISS}(x)^{(f)}_{n,m}, a_1 \cdots a_p \rangle = \sum_{n \le i_1 < \cdots < i_p < m} f(\delta x_{i_1})^{a_1} \cdots f(\delta x_{i_p})^{a_p}.$$

For any $w \in H$, we have that $\langle \mathrm{ISS}(x)^{(f)}_{n,m}, w\rangle = \langle \mathrm{ISS}(x)_{n,m}, f_\diamond(w)\rangle$ holds. Here, the map $f_\diamond$ is an algebra morphism of the Hopf algebra H_{qsh} canonically associated with the polynomial f. This implies that $\mathrm{ISS}(x)^{(f)}$ is a quasi-shuffle character.

We now consider polynomial transformations of time series, generalising results from [1]. Using the fact that H_{qsh} is the free commutative quasi-shuffle algebra over $\mathbb{R}^d$, we can show the following [4]. Given a d-dimensional time series x with $x_0 = 0$ and a polynomial map $P\colon \mathbb{R}^d \to \mathbb{R}^e$ without constant term, we define the e-dimensional time series $X := P(x)$. Then, for all $0 \le k \le N$, $\langle \mathrm{ISS}(X)_{0,k}, w\rangle = \langle \mathrm{ISS}(x)_{0,k}, \Lambda_P(w)\rangle$. Here, $\Lambda_P\colon H_{qsh}(\mathbb{R}^e) \to H_{qsh}(\mathbb{R}^d)^3$ is the unique quasi-shuffle morphism, determined by its action on $[\mathtt{1}], \ldots, [\mathtt{e}]$ as $\Lambda_P\,([\mathtt{i}]) := \iota(p_i) \in H_{qsh}(\mathbb{R}^d)$, where $\iota\colon \mathbb{R}[x^{(1)}, \ldots, x^{(d)}] \to H_{qsh}(\mathbb{R}^d)$ is the unique morphism of commutative algebras satisfying $\iota(x^{(i)}) = [\mathtt{i}]$. As a corollary we obtain, that for all $0 \le k \le N$, $\mathrm{ISS}(X)_{0,k} = \mathrm{ISS}(x) \circ \Lambda_{\tilde{P}_{x_0}}$, where $\tilde{P}_{x_0} := P(\cdot + x_0) - P(x_0)$.

In [3] we consider iterated-sums signatures over any commutative semiring. In the case of the tropical semiring, this corresponds to features of (real-valued) time series that are not easily available using existing signature-type objects.

References

[1] L. Colmenarejo and R. Preiß, *Signatures of paths transformed by polynomial maps*, Beitr. Algebra Geom. **61** (2020), 695–717.

[2] J. Diehl, K. Ebrahimi-Fard, and N. Tapia, *Time-warping invariants of multidimensional time series*, Acta Appl. Math. **170** (2020), 265–290.

[3] J. Diehl, K. Ebrahimi-Fard, and N. Tapia, *Tropical time series, iterated-sums signature and quasi-symmetric functions*, 2020, arXiv:2009.08443.

[4] J. Diehl, K. Ebrahimi-Fard, and N. Tapia, *Generalized iterated-sums signatures*, 2020, arXiv:2012.04597.

[5] L. Foissy, F. Patras, and J.-Y. Thibon, *Deformations of shuffles and quasi-shuffles*, Ann. Inst. Fourier **66** (2016), 209–237.

[6] L. Foissy and F. Patras, *Lie theory for quasi-shuffle bialgebras*, Periods in Quantum Field Theory and Arithmetic (J. I. Burgos Gil, K. Ebrahimi-Fard, and D. Manchon, eds.), Springer Proceedings in Mathematics & Statistics, **314**, Springer, 2020, pp. 537?615.

[7] M. E. Hoffman, *Quasi-shuffle products*, J. Alg. Combin. **11** (2000), 49–68.

[8] M. E. Hoffman and K. Ihara, *Quasi-shuffle products revisited*, J. Algebra **481** (2017), 293–326.

[9] F. J. Király and H. Oberhauser, *Kernels for sequentially ordered data*, J. Mach. Learn. Res. **20** (2019), 1–45.

[10] J.-L. Loday, *On the algebra of quasi-shuffles*, Manuscripta Math. **123** (2007), 79–93.

[2] Here we extend notation by defining $x_j^{[\mathtt{i}_1 \cdots \mathtt{i}_n]} := x_j^{(i_1)} \cdots x_j^{(i_n)}$.

[3] The notation, $H_{qsh}(\mathbb{R}^d)$, indicates the size of the underlying alphabet.

[11] T. Lyons and H. Oberhauser, *Sketching the order of events*, 2017, arXiv:1708.09708.
[12] C. Toth, P. Bonnier, and H. Oberhauser, *Seq2tens: An efficient representation of sequences by low-rank tensor projections*, 2020, arXiv:2006.07027.

Tropical time series, iterated-sums signatures and quasisymmetric functions

Joscha Diehl
(joint work with Kurusch Ebrahimi-Fard, Nikolas Tapia)

The theory of rough paths is closely related to the iterated-integrals signature of curves. The latter has recently found remarkable success in the realm of machine learning as the *signature method*.

We propose in [1] to use the *iterated-sums signature* as the input to machine learning pipelines, since it offers several benefits: it operates immediately on the discrete time series (no, even formal, "lifting" to a continuous curve is necessary), and it gives a richer set of features (for example, it contains non-trivial information about one-dimensional time-series, whereas the iterated-integrals signature needs ad-hoc preprocessing to achieve this). signature contains the following features

For a time series[1] $z_1, z_2, \ldots, z_N \in \mathbb{R}$, the entries of the iterated-sums signature are of the form

$$(*) \qquad \sum_{1\le i_1<\cdots<i_k\le N} z_{i_1}^{\alpha_1}\cdot \ldots \cdot z_{i_k}^{\alpha_k},$$

for integers $k \geq 1$ and $\alpha_1, ..., \alpha_k \geq 1$. At first sight, the calculation of $(*)$ has a runtime of order N^k. But it is in fact linear, owing to a dynamic programming principle (known, in the realm of iterated-integrals, as *Chen's identity*). Indeed,

$$(**) \qquad \begin{aligned}\sum_{1\le i_1<\cdots<i_k\le N} z_{i_1}^{\alpha_1}\cdot \ldots \cdot z_{i_k}^{\alpha_k} &= \sum_{1\le i_1<\cdots<i_k\le N-1} z_{i_1}^{\alpha_1}\cdot \ldots \cdot z_{i_k}^{\alpha_k}\\ &+ \sum_{1\le i_1<\cdots<i_{k-1}\le N-1} z_{i_1}^{\alpha_1}\cdot \ldots \cdot z_{i_{k-1}}^{\alpha_{k-1}}\cdot z_N^{\alpha_k}.\end{aligned}$$

and by iterating, one reduces the calculation to a runtime of order $k \cdot N$.

Iterated sums such as $(*)$ have the added benefit of allowing us to vary the underlying algebraic structure. In particular, these sums are meaningful over any (commutative) semiring, [2]. Fortunately, the dynamic programming principle applies there as well, since in $(**)$ we only used associativity and distributivity.

For concreteness, we consider the tropical semiring (or min-plus semiring) $(\mathbb{R}\cup\{+\infty\}, \oplus, \odot, +\infty, 0\}$, with "addition"

$$a \oplus b := \max\{a, b\},$$

and "multiplication"

$$a \odot b := a + b.$$

[1] Often these are increments of a time series x, i.e. $z_i = x_i - x_{i-1}$, but this is not necessary.

An iterated-sum for a time-series $z_1, ..., z_N \in \mathbb{R} \cup \{+\infty\}$ is then of the form

$$\min_{1 \le i_1 < \cdots < i_k \le N} \{\alpha_1 \cdot z_{i_1} + ... + \alpha_k \cdot z_{i_k}\},$$

for integers $k \ge 1$ and $\alpha_1, ..., \alpha_k \ge 1$.

It is well-known (but, to our knowledge, not rigorously proven) that such a minimum is not well-approximated by expressions in the iterated-integrals (or, classical, iterated-sums) signature. The aim of embedding such features in a signature-type object are the impetus for [2]. There it is shown that they can be stored as the character over an appropriate (semi)-bialgebra (just as iterated-integrals yield a character over the shuffle Hopf algebra), that they satisfy Chen's identity and that they are in close correspondence to quasisymmetric expressions (just as iterated-sums are in correspondence with quasisymmetric functions). Similar to iterated-integrals' invariance to reparametrization, the resulting features of a time series are invariant to "nothing happening".

Open questions

- A Hopf algebraic setting seems out of reach, since additive inverses do not exist and consequently the usual procedure of building an antipode fails. Relatedly, it is an open question how to reduce redundancies in these iterated-sums. The usual procedure of "taking the logarithm" is not possible over a general semiring.
- Is it possible to "mix" various semiring operations in a consistent fashion? The following mixed iterated-sum, for example, *is* amenable to dynamic programming:

$$\max_{i_3} \left(\sum_{i_1, i_2 : i_1 < i_2 < i_3} z_{i_1} z_{i_2} + z_{i_3} \right)$$

- Is there a corresponding theory of iterated "integrals"? At least in the case of idempotent semirings, e.g. the tropical semiring, the integrals of idempotent analysis [3] or pseudo-analysis [5, 4] seem promising.

References

[1] Diehl, J., Ebrahimi-Fard, K. and Tapia, N., 2019. *Time warping invariants of multidimensional time series.* Acta Applicandae Mathematicae, 170, 265-290.
[2] Diehl, J., Ebrahimi-Fard, K. and Tapia, N., 2020. *Tropical time series, iterated-sums signatures and quasisymmetric functions.* arXiv preprint arXiv:2009.08443.
[3] Kolokoltsov, V.N. and Maslov, V.P., 2013. *Idempotent analysis and its applications* (Vol. 401). Springer Science & Business Media.
[4] Sugeno, M. and Murofushi, T., 1987. *Pseudo-additive measures and integrals.* Journal of Mathematical Analysis and Applications, 122(1), pp.197-222.
[5] Pap, E., 2008. Generalized real analysis and its applications. International Journal of Approximate Reasoning, 47(3), pp.368-386.

Non-uniqueness for reflected rough differential equations

Paul Gassiat

We consider differential equations with normal reflection taking values in a closed domain $D \subset \mathbb{R}^d$ and driven by a signal X, which in general take the form

$$dZ_t = f(Z_t)dX_t + dK_t, \;\; Y_0 = y_0, \tag{1}$$

where the unknown is the pair (Z, K) which must satisfy the additional constraint

$$\forall t \geq 0, \;\; Z_t \in D, \quad dK_t = 1_{\{Z_t \in \partial D\}} n(Z_t)|dK_t|$$

where $n(z)$ is an inner normal of D at $z \in \partial D$.

The classical case where the driving signal is a semi-martingale is classical and well understood since works of Tanaka, Lions-Sznitman, Saicho going back to the 70s and 80s.

An important question is to understand to which extent a (rough) pathwise theory is possible for (1). Existence results have been proven a few years ago by Aida [1, 2] when X is a rough path with finite p-variation ($p < 3$), under essentially the same assumptions on D as in the semimartingale case. Uniqueness for (1) has been proven in the following special cases :

- the Young case (X of finite p-variation, $p < 2$), by Falkowski and Słomiński [4] (in the case $D = \mathbb{R}_+^{d_1} \times \mathbb{R}^{d_2}$),
- the one-dimensional case $D = \mathbb{R}_+$, by Deya et al. [3].

However, the question of uniqueness in the case of rough signals ($p > 2$) and multidimensional domains ($d \geq 2$) had so far remained open, and the main goal of this talk was to present a simple counter-example showing that an equation of the form (1) driven by a rough signal may have infinitely many solutions, even for smooth domains (in our case the domain is just $\mathbb{R}_+ \times \mathbb{R}$).

The equation under consideration is written as :

$$\begin{aligned} dZ_t &= AZ_t d\lambda_t - e_1 d\gamma_t + e_1 dK_t \quad \text{ for } t \in [0,1],, \\ & Z \cdot e_1 \geq 0, \quad dK = 1_{\{Z \cdot e_1 = 0\}}|dK| \end{aligned} \tag{2}$$

where the unknown (Z, K) takes values in $\mathbb{R}^2 \times \mathbb{R}$, (e_1, e_2) denotes the canonical basis,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

λ is a given scalar continuous path, and γ is a nondecreasing scalar function.

Theorem 1. *There exists $\lambda \in \cap_{p>2} C_{p-var}([0,1], \mathbb{R})$, and γ continuous and increasing s.t.* (2) *admits uncountably many distinct solutions on* $[0,1]$ *with* $Z_0 = 0$, *which are all non-null at positive times.*

Let us describe how these solutions are obtained. The trajectories corresponding to the linear part of the equation (driven by λ) are given by hyperboles asymptotic to the $\{x = y\}$ line, and which cross the y axis (i.e. the reflecting boundary) in the normal direction. On the other hand, in the part of the plane where the equations are constrained to live, the drift $-e_1 d\gamma$ pushes the solution Z towards these hyperboles that are further away from the origin. The solutions from the Theorem above are then obtained by alternating intervals where λ acts by moving Z away from the y axis along a small hyperbole arc and then γ pushes Z back to the y axis, see Figure 1 below. One can then see that taking λ of infinite 2-variation, one may accumulate infinitely many such small intervals in such a way that the solution may escape from 0 in finite time.

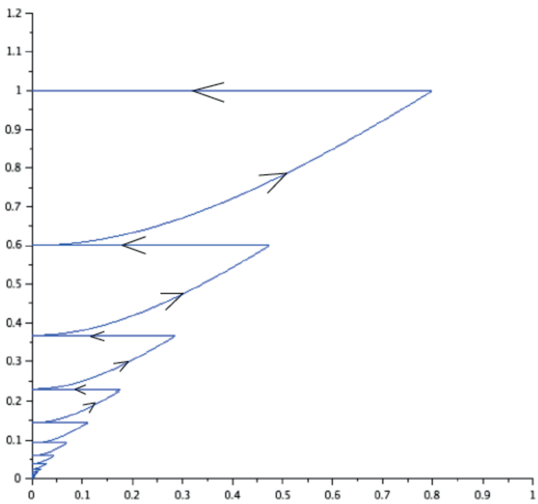

Figure 1. Trajectory of the solution obtained in the proof of Theorem 1

In the case where we fix $\gamma_t = t$ in the equation above, we obtain a sharp criterion on the modulus of continuity of λ so that the above admits a unique solution. Rather than stating the exact result here, let us just give its application to Hölder and log-Hölder moduli.

Theorem 2. *Let* $\omega(r) = r^\alpha$ *(resp.* $\omega(r) = r^{1/2}|\log(r)|^\beta$*). Then there exists* $\lambda : [0,1] \to \mathbb{R}$ *admitting* ω *as modulus of continuity, and such that* (2) *admits infinitely many solutions, if and only if* $\alpha < \frac{1}{2}$ *(resp.* $\beta > \frac{1}{2}$*).*

Note that Lévy's modulus of continuity corresponds exactly to the case of $\beta = \frac{1}{2}$ above, so that this theorem implies path-by-path well-posedness in the case where λ is a realization of a Brownian motion, and furthermore, that one can find paths of regularity slightly worse than Brownian motion (measured by the logarithmic term in the modulus) for which non-uniqueness holds.

In addition, we present some results in the case where λ is taken as the path of a fractional Brownian motion.

Theorem 3. *Let* $0 < H < \frac{1}{2}$, $\mathbb{P}^H$ *be the fBm measure of Hurst index* H *on* $C([0,1],\mathbb{R})$*, and let* $\gamma_t = t$*. Then for* $\mathbb{P}^H$*-almost every* λ*,* (2) *admits infinitely many solutions.*

To conclude, we note that while the main result presented here implies that well-posedness of rough differential equations with reflection cannot hold in general, one may still hope that uniqueness holds under further restrictions, such as :

(1) **Regularity of the signal**. In equation (2) with $\gamma_t = t$, uniqueness holds when the driving path has Brownian regularity. One may conjecture that this would still be true for a general equation of the form (1), assuming that the rough path associated to X is sufficiently regular, for instance if it has finite ψ-variation for suitable ψ. Note that the important class of *Markovian* rough paths, have sample paths with similar variation regularity as semimartingales, so that such a result would allow to consider reflected equations driven by such rough paths. This would also imply that rough path-wise methods may be applied to classical reflected SDE, which might prove useful in certain contexts.
(2) **Non-degeneracy of the equation**. In (2), if the initial condition Z_0 is any point of the boundary different from the origin, then the solution is unique. Since the origin is the only point where the coefficient in front of the noise vanishes, one may hope that some non-degeneracy of the driving vector fields suffices to recover well-posedness (say in the case of fractional Brownian motion, or more generally if the noise is rough enough in some sense).

References

[1] Shigeki Aida. Reflected rough differential equations. *Stochastic Process. Appl.*, 125(9):3570–3595, 2015.

[2] Shigeki Aida. Rough differential equations containing path-dependent bounded variation terms. *arXiv e-prints*, page arXiv:1608.03083, Aug 2016.

[3] Aurélien Deya, Massimiliano Gubinelli, Martina Hofmanová, and Samy Tindel. One-dimensional reflected rough differential equations. *Stochastic Process. Appl.*, 129(9):3261–3281, 2019.

[4] Adrian Falkowski and Leszek Słomiński. Stochastic differential equations with constraints driven by processes with bounded p-variation. *Probab. Math. Statist.*, 35(2):343–365, 2015.

On solution properties of rough Euler's equation

James-Michael Leahy
(joint work with Dan Crisan, Darryl D. Holm, Torstein Nilssen)

1. Motivation (as discussed by Dan Crisan)

The mathematical theory underlying the dynamics of fluids remains one of the most active research areas, emboldened not just by theoretical considerations such as the open problem of the global well-posedness of the Navier-Stokes equation. This activity is also inspired by urgent practical applications such as the need to increase the understanding of the dynamics of Earth's oceans and atmosphere in the context of global climate change. Incorporating perturbations into the fluid motion equation has become one of the mainstream features of fluid models, particularly in the last two decades. These include deterministic perturbations [37, 15, 24, 28, 39] and stochastic perturbations [29, 25, 34, 30, 31, 27, 35, 3, 23, 21, 26, 19, 2, 10, 36, 38]. Such perturbations can be exogenously introduced into the fluid model to account for (possibly unknown) external forces. They can also appear endogenously, for example, to model the effects of unresolved fast subgrid scale physics or of other uncertain processes. In geophysical fluid dynamics, this trend has led to many numerical developments, including the introduction of parameterization schemes used to represent model uncertainties in the interaction of disparate space and time scales, in hopes of improving the probabilistic skill of the ensemble weather forecasts [35, 2, 36, 6, 5, 33, 32, 7].

A principled choice of the perturbations can be achieved by requiring that the fluid model represents a critical path of a variational principle that incorporates both resolved and unresolved fluid motions. Many stochastic fluid dynamics models are arise in this manner [19, 20]. In [8], we introduced a new class of variational principles for fluid parameterization schemes by using a temporally rough vector field in the framework of geometric rough paths [22, 13, 12]. The critical points of these rough-path constrained variational principles yield rough partial differential equations whose dynamics incorporates both the resolved-scale fluid velocity and the modeled effects of the unresolved fluctuations. The paper [8] provides a bridge between geophysical fluid dynamics and rough path theory. It draws upon knowledge from both areas, and we hope that it will impact both areas.

2. Main results

Let us fix parameters $d \in \{2,3,\ldots\}$, $K \in \mathbb{N}$, and $\alpha \in [2,3)$. Let $\mathbf{Z} = (Z, \mathbb{Z}) \in \mathcal{C}_g^\alpha \mathbb{R}^K$ be an arbitrarily given $\mathbb{R}^K$-valued α-Hölder geometric rough path on $\mathbb{R}_+$. Let $\xi = (\xi_k)_{k=1}^K$be a collection of smooth divergence-free vector fields on the torus $\mathbb{T}^d$. Consider the Hamilton-Pontryagin action functional on $[0,T]$ given by

$$S(u,\lambda,\phi) = \int_0^T |u_t|_{L^2}^2 dt + (\lambda_t, d\phi_t \circ \phi_t^{-1} - u_t dt - \xi d\mathbf{Z}_t)_{L^2}.$$

The functional S is a function of mean and divergence-free vector fields u and λ that are smooth in space and controlled by Z. It is also a function of volume-preserving diffeomorphisms generated by $\mathbf{Z}$ of the form

$$d\phi_t = v_t \circ \phi_t \, dt + \sigma_t \circ \phi_t \, d\mathbf{Z}, \quad \phi|_{t=t_0} = \mathrm{id},$$

where v is a divergence and mean-free vector field that is smooth in space and controlled by $\mathbf{Z}$ and $\sigma = (\sigma_k)_{k=1}^K$ is a collection of divergence-free vector fields that are smooth in space and time. In the definition of the functional, we understand

$$\int_0^T (\lambda_t, d\phi_t \circ \phi_t^{-1})_{L^2} := \int_0^T (\lambda_t, v_t)_{L^2} \, dt + \int_0^T (\lambda_t, \sigma_t)_{L^2} \, d\mathbf{Z}_t.$$

If $\mathbf{Z}$ is truly rough, then the Lagrangian multiplier λ, akin to the adjoint variable in the Pontryagin maximum principle, enforces the advection constraint

$$d\phi_t = u_t \circ \phi_t dt + \xi \circ \phi_t d\mathbf{Z}, \quad \phi|_{t=t_0} = \mathrm{id}\,.$$

Moreover, by a variation of volume-preserving diffeomorphism ϕ generated by $\mathbf{Z}$, we mean a family of diffeomorphisms of the form $\phi_t^\epsilon = \psi_t^\epsilon \circ \phi_t$, $\epsilon \in [0,1]$, where $\dot{\psi}_t^\epsilon = \epsilon \dot{\bar{v}}_t \circ \psi_t^\epsilon$, $\psi_0^\epsilon = \mathrm{id}$, for an arbitrarily given divergence and mean-free vector field $\bar{v}$ that is smooth in space and time and vanishes at the endpoints.

A critical point of the action functional S is characterized by the incompressible Euler system for $\lambda \equiv u$ on $[0,T] \times \mathbb{T}^d$ given by

$$\begin{cases} du_t + u_t \cdot \nabla u_t \, dt + (\xi \cdot \nabla u_t + (\nabla \xi) u_t) \, d\mathbf{Z}_t = -\nabla dq_t - dh_t, \\ \operatorname{div} u_t = 0, \;\; \int_{\mathbb{T}^d} u_t(x) \, dV = 0, \;\; \int_{\mathbb{T}^d} q_t(x) \, dV = 0, \\ u|_{t=0} = u_0, \;\; q|_{t=0} = 0, \quad h|_{t=0} = 0. \end{cases} \tag{1}$$

The pressure q and harmonic constant h are Lagrange multipliers associated with the divergence and mean-free contraints. Equation (1) can be written in terms of the one-form $u \cdot dx = u^i dx^i$, Lie derivative $\pounds$, and exterior derivative $\mathbf{d}$:

$$d(u_t \cdot dx) + \pounds_{u_t}(u_t \cdot dx) \, dt + \pounds_\xi (u_t \cdot dx) d\mathbf{Z}_t = -\mathbf{d}(dq_t - 2^{-1}|u_t|^2) - d(h_t \cdot dx).$$

As a consequence, a Kelvin circulation theorem holds: for any smooth loop $C \subset \mathbb{T}^d$,

$$\oint_{\phi_t(C)} u_t \cdot dx = \oint_C u_0 \cdot dx.$$

The vorticity two-form $\omega = \mathbf{d}(u \cdot dx) = \text{upper.tri}(\nabla u - Du) \cdot dS$ satisfies

$$d\omega_t + \pounds_{u_t} \omega_t \, dt + \pounds_\xi \omega_t \, d\mathbf{Z}_t = 0, \tag{2}$$

where $(\pounds_v \omega)_{ij} = v^q \partial_{x^q} \omega_{ij} + (\partial_{x^i} v^q) \omega_{qj} + (\partial_{x^j} \omega^q) \omega_{iq}$. In 2d, the vorticity can be identified with a scalar-valued function, denoted also by ω, which satisfies

$$d\omega_t + (u_t \cdot \nabla \omega_t) \, dt + (\xi \cdot \nabla \omega_t) d\mathbf{Z}_t = 0. \tag{3}$$

Therefore, formally, one expects $|\omega_t|_{L^p} = |\omega_0|_{L^p}$ for all $t \in [0,T]$ and $p \in [2, \infty]$.

The proof of our main results [9] relies on the method of unbounded rough drivers [1, 11, 16, 17, 18]. Specifically, we derive a proiri estimates of the vorticity equation in L^2-Sobolev spaces $W^{m-1,2}$ for $m \geq m_* = \lfloor \frac{d}{2} \rfloor + 2$ by studying the first-order symmetric system for $\nabla^{m-1}\omega \otimes \nabla^{m-1}\omega$. We then use properties of the Biot-Savart operator (i.e., $|u|_{W^{m,2}} \sim |\omega|_{W^{m-1,2}}$) to deduce bounds for the velocity.

Theorem 1(Local existence) *Let $m \geq m_*$, $u_0 \in \dot{W}^{m,2}_{\text{div}}$, and $\xi \in W^{m+4,\infty}_{\text{div}}$. Then there exists a time $T_* = T_*(d, m, |u_0|_{W^{m,2}}, \mathbf{Z}, |\xi|_{W^{m+2,\infty}})$ and solution (u, q, h) of* (1) *on the interval $[0, T_*]$ such that*

$$(u, q, h) \in V^m = C_{T_*}\dot{W}^{m,2}_{\text{div}} \cap C^\alpha_{T_*}\dot{W}^{m-1,2}_{\text{div}} \times C^\alpha_{T_*}\dot{W}^{m-2,2} \times C^\alpha_{T_*}\mathbb{R}^d.$$

Theorem 2 (Unique maximal solution) *If $(u^1, q^1, h^1), (u^2, q^2, h^2) \in V^{m_*}$ are solutions of* (1) *on an interval $[0, T]$ such that $u^1_0 \equiv u^2_0$, then the solutions must coincide. Moreover, under the assumptions of Theorem 1, there exist a unique $T_{\max} = T_{\max}(d, m_*, |u_0|_{W^{m_*,2}}, \mathbf{Z}, |\xi|_{W^{m_*+2,\infty}})$ and solution $(u, q, h) \in V^m$ of* (1) *on the interval $[0, T_{\max})$ such that if $T_{\max} < \infty$, then $\lim_{t\uparrow T_{\max}} |u_t|_{W^{m,2}} = \infty$.*

Theorem 3 (BKM blowup criterion) *If $T_{\max} < \infty$, then $\int_0^{T_{\max}} |\omega_t|_{L^\infty}\, dt = \infty$.*

Corollary 1 (Global well-posedness in 2d) *Let $d = 2$. If $(u, q, h) \in V^{m_*}$ is a solution of* (1) *on an interval $[0, T)$, then for all $p \in [2, \infty]$ and $t \in [0, T)$, $|\omega_t|_{L^p} = |\omega_0|_{L^p}$. Thus, under the assumptions of Theorem 1, there exists a unique solution of* (1) *$(u, q, h) \in V^m$ on $[0, \infty)$.*

Corollary 2 (Stability) *The following solution mapping is continuous:*

$$\dot{W}^{m,2}_{\text{div}} \times W^{m+4,\infty}_{\text{div}} \times \mathcal{C}^\alpha_g\mathbb{R}^K \to C_{T_{\max}}\dot{W}^{m-1,2}_{\text{div}} \times C_{T_{\max}}\dot{W}^{m-2,2} \times C_{T_{\max}}\mathbb{R}^d$$
$$(u_0, \xi, \mathbf{Z}) \mapsto (u, q, h).$$

3. Open problems

Due to the divergence and mean-free constraint, we cannot derive estimates of the velocity in $W^{m,2}$ using the equation's velocity formulation. Our proofs exploit the specific structure of the vorticity equation, which is free of constraints or projections. Thus, it not clear whether one can derive similar solution properties for (1) with $(\nabla\xi)u$ replaced by a generic zero-order term cu. Similarly modifying the vorticity formulation and defining u via the Biot-Savart operator is also problematic since the vorticity needs to remain an exact two-form. In recent work, P. Gassiat proved that solutions of reflected rough differential equations are not unique, which possess a constraint [14]. This is in direct contrast to their stochastic counterparts, which use Itô calculus. Thus, it could be that our solution theory cannot be extended to a rough Euler system of general type.

A related problem is to prove uniqueness in 2d under the relaxed condition $\omega_0 \in L^\infty$. One does not have enough regularity of the vorticity to take the L^2-norm of the difference of two vorticities ω^1 and ω^2 in this setting. In the unperturbed case, one proves uniqueness by considering differences in the velocities u^1 and u^2 [40, 23]. As discussed above, we do not know how to study such differences in the rough setting. A possible route to deriving such a result could be to use a rough flow approach, which was followed in the Brownian setting [4]. This approach would require deriving properties of rough flows for log-Lipschitz drifts.

References

[1] Ismael Bailleul and Massimiliano Gubinelli. Unbounded rough drivers. In *Annales de la Faculté des sciences de Toulouse: Mathématiques*, volume 26, pages 795–830, 2017.

[2] Judith Berner, Ulrich Achatz, Lauriane Batte, Lisa Bengtsson, Alvaro de la Cámara, Hannah M Christensen, Matteo Colangeli, Danielle RB Coleman, Daan Crommelin, Stamen I Dolaptchiev, et al. Stochastic parameterization: Toward a new view of weather and climate models. *Bulletin of the American Meteorological Society*, 98(3):565–588, 2017.

[3] Luigi C. Berselli and Franco Flandoli. On a stochastic approach to eddy viscosity models for turbulent flows. In *Advances in mathematical fluid mechanics*, pages 55–81. Springer, 2009.

[4] Zdzisław Brzeźniak, Franco Flandoli, and Mario Maurelli. Existence and uniqueness for stochastic 2D Euler flows with bounded vorticity. *Archive for Rational Mechanics and Analysis*, 221(1):107–142, 2016.

[5] Colin Cotter, Dan Crisan, Darryl D. Holm, Wei Pan, and Igor Shevchenko. Data assimilation for a quasi-geostrophic model with circulation-preserving stochastic transport noise. *Journal of Statistical Physics*, pages 1–36, 2020.

[6] Colin Cotter, Dan Crisan, Darryl D. Holm, Wei Pan, and Igor Shevchenko. Numerically modeling stochastic Lie transport in fluid dynamics. *Multiscale Modeling & Simulation*, 17(1):192–232, 2019.

[7] Colin Cotter, Dan Crisan, Darryl D. Holm, Wei Pan, and Igor Shevchenko. A particle filter for stochastic advection by Lie transport: A case study for the damped and forced incompressible two-dimensional Euler equation. *SIAM/ASA Journal on Uncertainty Quantification*, 8(4):1446–1492, 2020.

[8] Dan Crisan, Darryl D. Holm, James-Michael Leahy, and Torstein Nilssen. Variational principles for fluid dynamics on rough paths. *arXiv preprint arXiv:2004.07829*, 2020.

[9] Dan Crisan, Darryl D. Holm, James-Michael Leahy, and Torstein Nilssen. On solution properties of the rough incompressible Euler system. *to appear.*, 2021.

[10] Jonathan Demaeyer and Stéphane Vannitsem. Stochastic parameterization of subgrid-scale processes: A review of recent physically based approaches. *Advances in Nonlinear Geosciences*, pages 55–85, 2018.

[11] Aurélien Deya, Massimiliano Gubinelli, Martina Hofmanova, and Samy Tindel. A priori estimates for rough pdes with application to rough conservation laws. *Journal of Functional Analysis*, 276(12):3577–3645, 2019.

[12] Peter K. Friz and Martin Hairer. *A Course on Rough Paths.* Springer International Publishing, 2014.

[13] Peter K. Friz and Nicola B. Victoir. *Multidimensional stochastic processes as rough paths: theory and applications*, volume 120. Cambridge University Press, 2010.

[14] Paul Gassiat. Non-uniqueness for reflected rough differential equations. *arXiv preprint arXiv:2001.11914*, 2020.

[15] Massimo Germano, Ugo Piomelli, Parviz Moin, and William H. Cabot. A dynamic subgrid-scale eddy viscosity model. *Physics of Fluids A: Fluid Dynamics*, 3(7):1760–1765, 1991.

[16] Antoine Hocquet and Martina Hofmanová. An energy method for rough partial differential equations. *Journal of Differential Equations*, 265(4):1407–1466, 2018.

[17] Martina Hofmanová, James-Michael Leahy, and Torstein Nilssen. On the Navier–Stokes equation perturbed by rough transport noise. *Journal of Evolution Equations*, 19(1):203–247, 2019.

[18] Martina Hofmanova, James-Michael Leahy, and Torstein Nilssen. On a rough perturbation of the Navier-Stokes system and its vorticity formulation. *arXiv preprint arXiv:1902.09348. to appear in Annal. App. Prob.*, 2021.

[19] Darryl D. Holm. Variational principles for stochastic fluid dynamics. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 471(2176):20140963, 2015.

[20] Darryl D. Holm and Erwin Luesink. Stochastic wave-current interaction in thermal shallow water dynamics. *arXiv preprint arXiv:1910.10627*, 2019.
[21] Peter Imkeller and Jin-Song Von-Storch. *Stochastic climate models*, volume 49. Birkhäuser, 2012.
[22] Terry J. Lyons, Michel Caruana, and Thierry Lévy. *Differential equations driven by rough paths*, volume 1908 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007. Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004, With an introduction concerning the Summer School by Jean Picard.
[23] Andrew J. Majda, Ilya Timofeyev, and Eric Vanden Eijnden. A mathematical framework for stochastic climate models. *Communications on Pure and Applied Mathematics*, 54(8):891–974, 2001.
[24] Paul J. Mason. Large-eddy simulation: A critical review of the technique. *Quarterly Journal of the Royal Meteorological Society*, 120(515):1–26, 1994.
[25] Paul J. Mason and David J. Thomson. Stochastic backscatter in large-eddy simulations of boundary layers. *Journal of Fluid Mechanics*, 242:51–78, 1992.
[26] Etienne Mémin. Fluid flow dynamics under location uncertainty. *Geophysical & Astrophysical Fluid Dynamics*, 108(2):119–146, 2014.
[27] Remigijus Mikulevicius and BRozovskii. On equations of stochastic fluid mechanics. In *Stochastics in Finite and Infinite Dimensions*, pages 285–302. Springer, 2001.
[28] Ugo Piomelli. Large-eddy simulation: achievements and challenges. *Progress in Aerospace Sciences*, 35(4):335–362, 1999.
[29] Ugo Piomelli, William H. Cabot, Parviz Moin, and Sangsan Lee. Subgrid-scale backscatter in turbulent and transitional flows. *Physics of Fluids A: Fluid Dynamics*, 3(7):1766–1771, 1991.
[30] Ugo Piomelli and Jeffrey Robert Chasnov. Large-eddy simulations: theory and applications. In *Turbulence and transition modelling*, pages 269–336. Springer, 1996.
[31] Stephen B. Pope. Turbulent flows, 2001.
[32] Valentin Resseguier, Long Li, Gabriel Jouan, Pierre Dérian, Etienne Mémin, and Bernard Chapron. New trends in ensemble forecast strategy: Uncertainty quantification for coarse-grid computational fluid dynamics. *Archives of Computational Methods in Engineering*, pages 1–47, 2020.
[33] Valentin Resseguier, Wei Pan, and Baylor Fox-Kemper. Data-driven versus self-similar parameterizations for stochastic advection by Lie transport and location uncertainty. *Nonlinear Processes in Geophysics*, 27(2):209–234, 2020.
[34] Ulrich Schumann. Stochastic backscatter of turbulence energy and scalar variance by random subgrid-scale fluxes. *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences*, 451(1941):293–318, 1995.
[35] Glenn Shutts. A kinetic energy backscatter algorithm for use in ensemble prediction systems. *Quarterly Journal of the Royal Meteorological Society*, 131(612):3079–3102, 2005.
[36] William C. Skamarock, Joseph B. Klemp, Jimy Dudhia, David O. Gill, Zhiquan Liu, Judith Berner, Wei Wang, Jordan G. Powers, Michael G. Duda, Dale M. Barker, et al. A description of the advanced research WRF model version 4. *National Center for Atmospheric Research: Boulder, CO, USA*, page 145, 2019.
[37] Joseph Smagorinsky. General circulation experiments with the primitive equations: I. the basic experiment. *Monthly weather review*, 91(3):99–164, 1963.
[38] Oliver D. Street and Dan Crisan. Semi-martingale driven variational principles. *arXiv preprint arXiv:2001.10105*, 2020.

[39] Chenyue Xie, Jianchun Wang, and E. Weinan. Modeling subgrid-scale forces by spatial artificial neural networks in large eddy simulation of turbulence. *Physical Review Fluids*, 5(5):054606, 2020.
[40] Victor I. Yudovich. Non-stationary flow of an ideal incompressible liquid. *USSR Computational Mathematics and Mathematical Physics*, 3(6):1407–1456, 1963.

The support theorem for singular SPDEs

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(joint work with Phillip Schönbauer)

Let us start by recalling Stroock and Varadhan's famous support theorem [5] for smooth diffusions. Given vector fields $\{V_i\}_{i=0}^m$ on $\mathbb{R}^n$ with bounded derivatives up to order 2, consider the diffusion process

$$\text{(SDE)} \qquad dX_t = V_0(X_t)\,dt + \sum_{i=1}^m V_i(X_t)\circ dW_i(t)\;, \qquad X_0 \in \mathbb{R}^n\;,$$

where the W_i are i.i.d. Wiener processes and $\circ$ denotes Stratonovich integration. Fix a final time $T > 0$ and denote by $\mathcal{L}_T(X)$ the law of X on $\mathcal{C}([0,T],\mathbb{R}^n)$. A natural question is then to characterise the topological support of $\mathcal{L}_T(X)$, namely the smallest closed set of full measure.

As a first step, [5] show that solutions to (SDE) are stable under replacing the W_i's by a piecewise linear interpolation, a result usually referred to as the "Wong–Zakai theorem". (Wong and Zakai [6] actually only showed it in the simpler one-dimensional case.) This implies that $\operatorname{supp}\mathcal{L}_T(X) \subset \overline{\mathrm{Cont}_T}$, where Cont_T denotes the set of all solutions to the associated control problem

$$\dot{X}_t = V_0(X_t) + \sum_{i=1}^m V_i(X_t)\,h_i(t)\;, \qquad X_0 \in \mathbb{R}^n\;,$$

with the h_i running over all smooth functions. In a second step, [5] show that one also has the converse inclusion, so that $\operatorname{supp}\mathcal{L}_T(X) = \overline{\mathrm{Cont}_T}$.

It is natural to ask about the extension of such a result to stochastic PDEs. While many such results exist, starting with [1], they all cover the "classical" case where renormalisation is absent. The first result covering renormalisation was obtained in [3] and deals with the "generalised parabolic Anderson model" formally written as

$$\text{(gPAM)} \qquad \partial_t u = \Delta u + g(u)\,\xi\;,$$

where $u\colon \mathbb{R}_+ \times \mathbb{T}^2 \to \mathbb{R}$ and ξ denotes white noise in space (constant in time). If we write similarly to before Cont_T for the associated control problem (with ξ replaced by an arbitrary smooth control h depending on space only) and $\mathcal{L}_T(u)$ for the law of the solution to (gPAM), then it turns out that in general $\operatorname{supp}\mathcal{L}_T(u) \neq \overline{\mathrm{Cont}_T}$! Instead, given $c \in \mathbb{R}$, write $\mathrm{Cont}_T(c)$ for the solutions to the modified control problem

$$(1) \qquad \partial_t u = \Delta u + cg'(u)g(u) + g(u)\,h\;.$$

It was then shown in [3] that $\operatorname{supp}\mathcal{L}_T(u) = \overline{\bigcup_{c\in\mathbb{R}} \mathrm{Cont}_T(c)}$. To understand why this is the case, note first that the analogue of the Wong–Zakai theorem in this context states that solutions to (gPAM) can be obtained as limits as $\varepsilon \to 0$ of solutions to

$$\partial_t u = \Delta u - c_\varepsilon g'(u)g(u) + g(u)\,\xi_\varepsilon\;, \tag{2}$$

where ξ_ε denotes a mollified spatial white noise and $c_\varepsilon \approx |\log\varepsilon|$ is a suitable diverging sequence. This would suggest that one has in some sense $\operatorname{supp}\mathcal{L}_T(u) = \overline{\mathrm{Cont}_T(-\infty)}$. The second crucial remark is that one has $\overline{\mathrm{Cont}_T(c)} \subset \overline{\mathrm{Cont}_T(\bar c)}$ whenever $c > \bar c$. The reason why this is the case is that, if we set for example

$$\bar h_\varepsilon(x) = h(x) + \frac{\alpha}{\varepsilon}\cos(x_1/\varepsilon)\;,$$

then one shows that the solution to

$$\partial_t u = \Delta u + \bar c g'(u)g(u) + g(u)\,\bar h_\varepsilon \tag{3}$$

converges, as $\varepsilon \to 0$, to the solution to (1) with $c = \bar c + \alpha^2/2$. In other words, it is possible to emulate *larger* values of c by adding a highly oscillatory component to h, but it is not possible in general to emulate *arbitrary* values of c in this way.

The main question we address in [4] is how such a result generalises to more or less arbitrary singular SPDEs. To this end, we consider a class of equations (with the spatial variable taking values in a torus) of the form

$$\partial_t u = \Delta u + F(u,\nabla u) + G(u)\,\xi\;, \tag{4}$$

where u and ξ take values in some finite-dimensional vector space. For the sake of simplicity, we also assume that ξ is white, either in space-time or in space only, although this condition can be weakened. (The precise definition of a "class" of equations encompasses all equations that can be described with one given complete rule as in [2], together with a corresponding kernel assignment.)

Recall that there then exists a finite-dimensional nilpotent Lie group $\mathcal{G}_-$ (the "renormalisation group") acting on the pairs (F,G) determining SPDEs in our given class such that the following holds. Write $U(g,\xi_\varepsilon)$ for the solution to (4) with (F,G) replaced by $M^g(F,G)$ and ξ replaced by ξ_ε. Then, for every sequence of stationary mollifications ξ_ε of ξ one can choose a sequence $g_\varepsilon \in \mathcal{G}_-$ in such a way that

$$\lim_{\varepsilon\to 0} U(g_\varepsilon,\xi_\varepsilon) = U(\xi)\;, \tag{5}$$

in probability, withe the limit $U(\xi)$, which we call "the solution" to (4), independent of the choice of mollified noise ξ_ε.[1] As above, write $\mathcal{L}_T(u)$ for the law of this solution up to some fixed time horizon T and, for any given $g \in \mathcal{G}_-$, write $\mathrm{Cont}_T(g)$ for the set of all solutions to (4) with (F,G) replaced by $M^g(F,G)$ and ξ replaced by an arbitrary smooth control (and depending only on space if ξ is

[1] Different choices of g_ε can lead to different notions of solution, for example Itô solutions vs Stratonovich solutions for SDEs. Here, we assume that one such notion has been fixed once and for all.

spatial white noise). With all of these notations in place, our main result is as follows.

Theorem 1. *There exists a Lie subgroup* $\mathcal{H} \subset \mathcal{G}_-$ *as well as a fixed element* $g \in \mathcal{G}_-$ *such that*

$$\operatorname{supp} \mathcal{L}_T(u) = \overline{\bigcup_{h \in \mathcal{H}} \operatorname{Cont}_T(gh)}\ .$$

Let us conclude with a few remarks:

- The case of SDEs and the case of gPAM discussed earlier both correspond to $\mathcal{G}_- = (\mathbb{R}, +)$. In the case of SDEs however, $\mathcal{H} = \{0\}$ and g is the element selecting the Stratonovich interpretation of solutions, while in the case of gPAM one has $\mathcal{H} = \mathcal{G}_-$.
- Different choices of notion of "solution" may in general correspond to different values for the element g appearing in the statement.
- The subgroup $\mathcal{H}$ is not described explicitly, but it is guaranteed to be sufficiently large so that it is possible to choose $g_\varepsilon \in \mathcal{H}$ in (5), although the resulting "solution theory" U may be different.
- The subgroup $\mathcal{H}$ is determined by purely analytic data. In particular, different kernel assignments for a given "rule" may lead to different subgroups $\mathcal{H}$. One of the main difficulties in [4] is to establish this correspondence between "soft" analytic data and "hard" algebraic data.

References

[1] V. Bally, A. Millet, and M. Sanz-Solé, *Approximation and support theorem in Hölder norm for parabolic stochastic partial differential equations*, Ann. Probab. **23**, no. 1, (1995), 178–222.

[2] Y. Bruned, M. Hairer, and L. Zambotti, *Algebraic renormalisation of regularity structures*, Invent. Math. **215**, no. 3, (2019), 1039–1156.

[3] K. Chouk and P. K. Friz, *Support theorem for a singular SPDE: the case of gPAM*, Ann. Inst. Henri Poincaré Probab. Stat. **54**, no. 1, (2018), 202–219.

[4] M. Hairer and P. Schönbauer, *A support theorem for singular SPDEs*, arXiv:1909.05526

[5] D. W. Stroock and S. R. S. Varadhan, *On the support of diffusion processes with applications to the strong maximum principle*, Proc. Sixth Berkeley Symp. on Math. Statist. and Prob., **Vol. 3** (1972), 333–359.

[6] E. Wong and M. Zakai, *On the relation between ordinary and stochastic differential equations*, Internat. J. Engrg. Sci., **3** (1965), 213–229.

Algebraic deformation for (S)PDEs

Yvain Bruned

(joint work with Dominique Manchon)

As someone doing Analysis, Probability, Physics, Finance or Machine Learning, one has to deal with perturbative expansions and treat many terms. The main task is to resume and recombine these terms in order to approximate exact identities. This is where the need for algebraic structures emerges. Often, one designs ad hoc structures that will work for his specific problem but they are still messy. The main dream would be a toolbox for constructing complicated structures from simple ones. The work [4] presented in this workshop is the first step in this direction for (stochastic) partial differential equations ((S)PDEs). Indeed, decorated trees are used for singular SPDEs in [3] and for a numerical scheme at low regularity for dispersive equations in [5]. They are useful for describing terms in Picard iteration and allow a scale separation.

In the previous applications, one applies Taylor expansions which are encoded at the algebraic level in [3] for recentering analytical objects around a base point and for renormalising the ones that diverge at some scales. We show that the first procedure is a deformation of the Butcher-Connes-Kreimer Hopf algebra [6, 9]. The second one corresponds to a deformed extraction-contraction Hopf algebra [8, 7].

The deformation is identified at the level of pre-Lie algebras. Indeed, Guin and Oudom proposed in [11] a procedure for constructing an associative product from a pre-Lie product. In the case where this pre-Lie product is the grafting of a tree onto another one obtains a product whose dual is the Butcher-Connes-Kreimer coproduct. A pre-Lie perspective on renormalisation has already been proposed for the Rough Paths (see [1]) which was the inspiration for the work [2]. For edge-decorated trees, one needs to use Multi-pre-Lie algebras introduced in [2]. It consists of a family of grafting products indexed by the edge decorations. In fact, one can see the whole family as a pre-Lie product by considering only planted trees. In [10], the Guin-Oudom procedure has been applied to this pre-Lie product. For the deformed structures, the main steps are:

- Taylor deformation of the pre-Lie product which corresponds to a finite sum of trees of lower grading. Check that it preserves the structure.
- Construction of an associative product via the Guin-Oudom procedure.
- Identification of the coproduct, a bigrading is needed for dealing with infinite sums.

By applying this procedure to various pre-Lie products grafting, plugging and insertion, one constructs all the objects in [3] and [5]. Also as mentioned in the different steps, the deformation produces finite sums but in the dual, they will be infinite which makes the identification more involved.

We believe that this new approach provides the start of the toolbox evoked at the beginning. Various extension of [3] have been considered and this deformation could clarify them and brings also new structures.

References

[1] Y. Bruned, I. Chevyrev, P. K. Friz, R. Preiss. *A rough path perspective on renormalization.* J. Funct. Anal. **277**, no. 11, (2019), 108283. doi:10.1016/j.jfa.2019.108283

[2] Y. Bruned, A. Chandra, I. Chevyrev, M. Hairer. *Renormalising SPDEs in regularity structures.* J. Eur. Math. Soc. (2021), doi:10.4171/JEMS/1025.

[3] Y. Bruned, M. Hairer, L. Zambotti. *Algebraic renormalisation of regularity structures.* Invent. Math. **215**, no. 3, (2019), 1039–1156. doi:10.1007/s00222-018-0841-x

[4] Y. Bruned, D. Manchon, *Algebraic deformation for (S)PDEs.* arXiv:2011.05907.

[5] Y. Bruned, K. Schratz. *Resonance based schemes for dispersive equations via decorated trees.* arXiv:2005.01649.
[6] J. C. Butcher. *An algebraic theory of integration methods.* Math. Comp. **26**, (1972), 79–106. doi:10.2307/2004720.
[7] D. Calaque, K. Ebrahimi-Fard, D. Manchon. *Two interacting Hopf algebras of trees: a Hopf-algebraic approach to composition and substitution of B-series.* Adv. in Appl. Math. **47**, no. 2, (2011), 282–308. doi:10.1016/j.aam.2009.08.003.
[8] P. Chartier, E. Hairer, G. Vilmart. *Algebraic structures of B-series.* Found. Comput. Math. **10**, no. 4, (2010), 407–427. doi:10.1007/s10208-010-9065-1.
[9] A. Connes, D. Kreimer. *Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem.* Comm. Math. Phys. **210**, (2000), 249–73. doi:10.1007/s002200050779.
[10] L. Foissy, *Algebraic structures on typed decorated rooted trees*, arXiv:1811.07572
[11] D. Guin, J. M. Oudom, *On the Lie enveloping algebra of a pre-Lie algebra*, J. K-Theory **2** (2008), no. 1, 147–167.

Hairer's Reconstruction Theorem without Regularity Structures

Lorenzo Zambotti
(joint work with Francesco Caravenna)

This talk is based on the eponymous paper [1] where we have extracted a single result (the Reconstruction Theorem) from a larger theory (Regularity Structures), see [2]. We present the former in a simpler and more general version, without reference to the latter.

The main question is the following. For every $x \in \mathbb{R}^d$ we fix a distribution $F_x \in \mathcal{D}'(\mathbb{R}^d)$ and we call $(F_x)_{x\in\mathbb{R}^d}$ a *germ.* Can we find a distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ which is *locally well approximated* by $(F_x)_{x\in\mathbb{R}^d}$?

Taylor expansions. For example, let us fix $f \in C^\infty(\mathbb{R}^d)$, and let us define for a fixed $\gamma > 0$

$$F_x(y) := \sum_{|k|<\gamma} \partial^k f(x)\, \frac{(y-x)^k}{k!}, \qquad x, y \in \mathbb{R}^d\,.$$

Then the classical Taylor theorem says that there exists a function $R(x,y)$ such that

$$f(y) - F_x(y) = R(x,y), \qquad |R(x,y)| \lesssim |x-y|^\gamma$$

uniformly for every x, y on compact sets of $\mathbb{R}^d$. We say that the function f is *locally well approximated* by the germ $(F_x)_{x\in\mathbb{R}^d}$.

Let us introduce now the following fundamental tool: for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\lambda > 0$ and $y \in \mathbb{R}^d$

$$\varphi_y^\lambda(w) := \frac{1}{\lambda^d}\, \varphi\left(\frac{w-y}{\lambda}\right), \qquad w \in \mathbb{R}^d\,.$$

Then the local approximation property

$$f(y) - F_x(y) = R(x,y), \qquad |R(x,y)| \lesssim |x-y|^\gamma$$

implies

$$\left|(f - F_y)(\varphi_y^\lambda)\right| \lesssim \lambda^\gamma,$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, uniformly for y in compact sets of $\mathbb{R}^d$, $\lambda \in]0,1]$.

Another simple formula in the context of Taylor expansions is

$$\big|(F_z - F_y)(\varphi_y^\lambda)\big| \lesssim (|y-z|+\lambda)^\gamma,$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, uniformly for y, z in compact sets of $\mathbb{R}^d$, $\lambda \in]0,1]$. This comes from a simple estimate of $F_z(w) - F_y(w)$.

Coherence. We have seen that for the germ related to a Taylor expansion we have

$$\big|(F_z - F_y)(\varphi_y^\lambda)\big| \lesssim (|y-z|+\lambda)^\gamma, \qquad \big|(f - F_y)(\varphi_y^\lambda)\big| \lesssim \lambda^\gamma,$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, uniformly for y, z in compact sets of $\mathbb{R}^d$, $\lambda \in]0,1]$. We say that a general germ $(F_z)_{z\in\mathbb{R}^d} \subset \mathcal{D}'(\mathbb{R}^d)$ is *coherent* if there exist $\gamma > 0, \alpha \leq 0$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\int \varphi \neq 0$ and

$$\big|(F_z - F_y)(\varphi_y^\lambda)\big| \lesssim \lambda^\alpha (|y-z|+\lambda)^{\gamma-\alpha},$$

uniformly for z, y in compact sets of $\mathbb{R}^d$, $\lambda \in]0,1]$.

We can finally state our version of Hairer's Reconstruction Theorem, without regularity structures.

Theorem 1 (Hairer 14, Caravenna-Z. 20)**.** *Suppose that for a given $F : \mathbb{R}^d \to \mathcal{D}'(\mathbb{R}^d)$ there exist $\gamma > 0$, $\alpha \leq 0$, and a $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\int \varphi \neq 0$ and*

$$|(F_y - F_x)(\varphi_x^\lambda)| \lesssim \lambda^\alpha (|x-y|+\lambda)^{\gamma-\alpha},$$

uniformly for x, y in compact sets of $\mathbb{R}^d$, $\lambda \in]0,1]$ (coherence condition). Then there exists a unique $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ such that

$$|(\mathcal{R}F - F_x)(\psi_x^\lambda)| \lesssim \lambda^\gamma$$

uniformly for x in compact sets of $\mathbb{R}^d$, $\lambda \in]0,1]$, $\{\psi \in \mathcal{D}(B(0,1)) : \|\psi\|_{C^r} \leq 1\}$ with a fixed $r > -\alpha$.

Then $\mathcal{R}F$ is indeed locally well approximated by $(F_z)_{z\in\mathbb{R}^d}$.

References

[1] F. Caravenna, L. Zambotti, *Hairer's Reconstruction Theorem without Regularity Structures*, to apper in EMS Surveys in Mathematical Sciences (2021).

[2] M. Hairer, *A theory of regularity structures. Invent. Math.* **198**, no. 2, (2014), 269–504. doi:10.1007/s00222-014-0505-4

Filtrations and Signatures

Harald Oberhauser
(joint work with Patric Bonnier and Chong Liu)

A useful point of view of in stochastic analysis is to identify a stochastic process as a path-valued (resp. in discrete time sequence-valued) random variable. However, this ignores that a stochastic process involves a filtration. The filtration encodes how the information we have observed in the past restricts the future possibilities and thereby encodes actionable information.

For example, the solution map of the optimal stopping problem

$$\mathbf{X} := (\Omega, (\mathcal{F}_t)_t, \mathbb{P}, (X_t)_t) \mapsto \sup_{\tau \in \{0,1,\dots,T\}} \mathbb{E}[f(X_\tau)] \tag{1}$$

is not continuous in weak topology (or any other topology that ignores the filtration): consider the real-valued, discrete time processes $\mathbf{X}^n$ and $\mathbf{X}$ in Figure 1 where each consist of only two different trajectories.

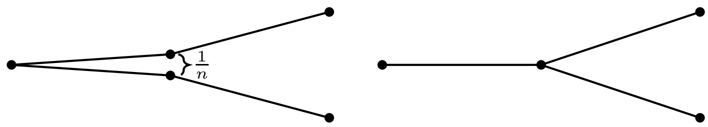

Figure 1. Both processes start at time $t = 0$ at 0 and are equal to 1 or -1 at time $t = 2$ with probability 0.5.

As $n \to \infty$, they converge in weak topology but for any finite n one would make very different choices in the model $\mathbf{X}^n$ and the model $\mathbf{X}$ since in $\mathbf{X}^n$ one already knows at time $t = 1$ what happens at time $t = 2$. More formally, what makes $\mathbf{X}^n$ so different from $\mathbf{X}$ is that their natural filtrations are very different. Such phenomena were realized in the 70's and 80's and first addressed in [2, 3] and are recently finding a revived interest [8]. Let us emphasize that the reason for discontinuities like (1) is not "highly oscillatory behaviour" (it's discrete time!), or "weird filtrations" (it happens with natural filtrations!).

What is a natural way to introduce a finer topology? One way is to use the coarsest topology that makes the solution map of interest continuous.

Definition 1. Denote with $\mathcal{S}(V)$ the space of adapted processes that evolve in a linear space V in discrete time $\{0, 1, \dots, T\}$. Denote with τ_0 the topology of weak convergence on $\mathcal{S}(V)$. Denote with τ_1 the coarsest topology on $\mathcal{S}(V)$ that makes the solution map (1) of optimal stopping continuous. We call τ_1 the adapted topology of rank 1.

It turns out that there are many different ways (not just optimal stopping) where this topology τ_1 arises which suggests that $(\mathcal{S}(V), \tau_1)$ is a natural and important object; see [8] for an overview. Moreover, seminal work of Hoover&Keisler [5] shows that τ_0 and τ_1 are just the first two topologies on $\mathcal{S}(V)$ in a sequence $(\tau_r)_{r \geq 0}$ of topologies that get finer and finer, $\tau_0 \subset \tau_1 \subset \dots$; see [5] for the precise statement. An open problem that we address with Theorem 2 is to provide a metric that induces τ_r for any r (previously this was known for $r = 0$ and $r = 1$). Either way, what follows is interesting already for $r = 1$ but what is nice is that the same approach immediately applies to any $r \geq 0$.

It turns out that the so-called *rank r prediction process* $\hat{X}^r$ of $\mathbf{X}$ defined as

$$\hat{X}^r := \Big(\mathbb{P}(\hat{X}^{r-1} \in \cdot|\mathcal{F}_t)\Big)_{t\in\{0,1,\ldots,T\}}, \quad \hat{X}^0 := X. \tag{2}$$

plays a fundamental role. Note that $\hat{X}^0 \equiv X$ evolves in V, $\hat{X}^1$ evolves in a space of measures on V, $\hat{X}^2$ evolves in the space of measures on a space of measures of sequences in V, etc. It is useful to denote these state spaces for $\hat{X}^r$ with $\mathcal{M}_r$

$$\mathcal{M}_0 := V, \quad \mathcal{M}_r := \mathrm{Meas}(\{0,\ldots,T\} \to \mathcal{M}_{r-1}) \tag{3}$$

where $\mathrm{Meas}(\{0,\ldots,T\} \to \mathcal{M}_{r-1})$ denotes the space of signed measures on sequences in $\mathcal{M}_{r-1}$ that are indexed by $\{0,\ldots,T\}$.

Theorem 1. *For any $r \geq 0$ two processes $\mathbf{X}$ and $\mathbf{Y}$ are identitical under $(\mathcal{S}(V), \tau_r)$ iff* $\mathrm{Law}(\hat{X}^r) = \mathrm{Law}(\hat{Y}^r)$.

This suggests to generalize the expected signature to characterize elements of $\mathcal{M}_r$. Therefore define the *rank r tensor algebra* T_r

$$T_0 := V, \quad T_{r+1} := T(T_r) := \prod_{m\geq 0} T_r^{\otimes m}$$

and the *rank r expected signature* $\overline{S}_r$

$$\overline{S}_r : \mathcal{M}_r \to T_r, \quad \overline{S}_r(\mu) := \int S(x^*\overline{S}_{r-1})\mu(dx)$$

where S denotes the usual signature (with an additional time coordinate to pick up time parametrization) and x^* the pullback along x.

Theorem 2. *For any $r \geq 0$ the map*

$$\mathcal{S}(V) \times \mathcal{S}(V) \to [0,\infty), \quad (\mathbf{X},\mathbf{Y}) \mapsto \|\overline{S}_{r+1}(\mathrm{Law}(\hat{X}^r)) - \overline{S}_{r+1}(\mathrm{Law}(\hat{Y}^r))\|_{T_{r+1}}$$

is a semi-metric[1] *on $\mathcal{S}(V)$ that induces the adapted topology of rank r, τ_r.*

Finally, let me emphasize that $\overline{S}_r$ yields more than a metric: it gives a graded description of how the filtration interacts with the law of the process; for $r = 0$ it completely ignores the filtration and reduces to the expected signature, for $r = \infty$ it gives the natural isomorphism class for adapted processes, see [5]. I ignored various important details in this note, such as how to put a norm on T_r (by using [7]), or how one can deal with non-compact V (by using the robust signature moments of [1]), etc.; but all are discussed in [6].

Outlook. There are many follow-up questions. One raised during the discussion is to use the higher rank signatures for optimal stopping. The recent approach [4] realizes stopping times as hitting times of linear functionals of the signature

[1]The elements of $\mathcal{S}(V)$ are filtered probability spaces carrying processes, $(\Omega, \mathcal{F}, \mathbb{P}, X) \in \mathcal{S}(V)$.

and provides a model-free approach to optimal stopping since no Markov or semimartingale assumptions are needed. In view of the above, one could develop an approach that is both: model-free (works for any model) and robust (models that are close give similar solutions).

References

[1] I. Chevyrev, H. Oberhauser, *Signature Moments to Characterize Laws of Stochastic Processes* (2019)
[2] D. Aldous, *Weak Convergence and General Theory of Stochastic Processes* (1981).
[3] A. M. Vershik. *Decreasing sequences of measurable partitions and their applications* (1970).
[4] C. Bayer, P. Hager, S. Riedel, J. Schoenmakers. *Optimal Stopping with Signatures* (2020)
[5] D. Hoover, J. Keisler. *Adapted Probability Distributions* (1984).
[6] P. Bonnier, C. Liu, H. Oberhauser. *Adapated Topologies and Higher Rank Signatures* (2020).
[7] K. Ebrahimi-Fard, F. Patras. *Cumulants, free cumulants and half-shuffles* (2015).
[8] J. Backhoff-Veraguas, D. Bartl, M. Beiglböck, M. Eder *All Adapted Topologies are Equal* (2019)

The length of a path and the norm of its signature

Horatio Boedihardjo
(joint work with Xi Geng)

The talk is based on the preprint [2] which is joint work with Xi Geng.

Let $\gamma : [0,T] \to \mathbb{R}^d$ be a bounded variation function. The signature $S(\gamma)_{0,T}$ of γ is defined by

$$S(\gamma)_{0,T} = (1, \mathbf{X}^1_{0,T}, \mathbf{X}^2_{0,T}, \ldots),$$

where

$$\mathbf{X}^n_{0,T} = \int_{0<t_1<\ldots<t_n<T} \mathrm{d}\gamma_{t_1} \otimes \ldots \otimes \mathrm{d}\gamma_{t_n}.$$

A natural norm on bounded variation path is

$$\|\gamma\|_{1-var} = \sup_{\mathcal{P}} \sum_{i=0}^{m} \|\gamma_{t_{i+1}} - \gamma_{t_i}\|.$$

Hambly-Lyons [1] conjectured an isometry result of the following form between γ and $S(\gamma)_{0,T}$: for all reduced bounded variation path x,

$$\|\gamma\|_{1-var} = \limsup_{n\to\infty} \|n!\mathbf{X}^n_{0,T}\|^{\frac{1}{n}}, \tag{1}$$

where $\|\cdot\|$ is the projective norm on $\mathbb{R}^d$

Lyons and Xu [6] show that (1) is true if $\|\gamma'_t\| = 1$ for all t and γ' is continuous, following earlier work by Hambly and Lyons that [1] holds under the additional assumption that $\gamma \in C^3$.

However, the isometry result (1) should hold not just for γ with continuous derivative, but for all reduced bounded variation path. In the preprint [2] we are trying to make progress towards removing the continuous derivative assumption. The inequality

$$\|\gamma\|_{1-var} \geq \limsup_{n\to\infty} \|n!\mathbf{X}^n_{0,T}\|^{\frac{1}{n}}$$

is known to hold for all bounded variation path.

Therefore we just need a lower bound

$$\|\gamma\|_{1-var} \leq \limsup_{n\to\infty} \|n!\mathbf{X}^n_{0,T}\|^{\frac{1}{n}}.$$

To do so, we make use of the following Lemma crucially, which was contained implicitly in [1]:

Lemma 1. *Let $A \in L(\mathbb{R}^d, L(\mathbb{R}^m, \mathbb{R}^m))$ and let*

$$\mathrm{d}Y^\lambda_t = \lambda A(\mathrm{d}\gamma_t)Y^\lambda_t, \quad Y_t = v. \tag{2}$$

Suppose that $\|v\|_{\mathbb{R}^m} = 1$,

$$\sup_{\|w\|=1} \sup_{\|v\|=1} \|A(w)v\|_{\mathbb{R}^m} = 1.$$

Then

$$\limsup_{\lambda\to\infty} \frac{\log\|Y^\lambda_t\|}{\lambda} \leq \limsup_{n\to\infty} \|n!\mathbf{X}^n_{0,T}\|^{\frac{1}{n}}.$$

Note that this lemma provides a lower bound for $\limsup_{n\to\infty} \|n!\mathbf{X}^n_{0,T}\|^{\frac{1}{n}}$ which is what we need. At the same time, the lemma means that the problem now reduced to analysing an ODE of the type (2). Generally, analysing such ODE is quite difficult. We focus on the $d = 2$ path $\gamma = (x_t, y_t)$ and choose a relatively explicit equation given by

$$A((\mathrm{d}x_t, \mathrm{d}y_t)) = \begin{pmatrix} \mathrm{d}x_t & \mathrm{d}y_t \\ \mathrm{d}y_t & -\mathrm{d}x_t \end{pmatrix}.$$

We will also reparametrise (x_t, y_t) so that

$$(x'_t)^2 + (y'_t)^2 = 1. \tag{3}$$

This means we can write

$$x'_t = \cos\alpha_t$$
$$y'_t = \sin\alpha_t.$$

The equation (2) can be written out explicitly so that if $Y_t = \begin{pmatrix} a_t \\ b_t \end{pmatrix}$,then

$$\begin{aligned} a'_t &= \lambda a_t x'_t + \lambda b_t y'_t, \quad a_0 = \cos(\frac{\alpha_0}{2}) \\ b'_t &= \lambda a_t y'_t - \lambda b_t x'_t, \quad b_0 = \sin(\frac{\alpha_0}{2}). \end{aligned} \tag{4}$$

(The choice of the initial condition will be explained later.) We will carry out the transformation

$$a_t = \rho_t^\lambda \cos\psi_t^\lambda$$
$$b_t = \rho_t^\lambda \sin\psi_t^\lambda.$$

This transformation will transform (4) to

$$\mathrm{d}\rho_t^\lambda = \lambda\rho_t^\lambda \cos(2\psi_t^\lambda - \alpha_t)\mathrm{d}t, \quad \rho_0^\lambda = 1$$
$$\mathrm{d}\psi_t^\lambda = -\lambda \sin(2\psi_t^\lambda - \alpha_t)\mathrm{d}t, \quad \psi_0^\lambda = \frac{\alpha_0}{2}.$$

In fact, $\rho_T^\lambda = \|Y_T^\lambda\|$ can be explicitly solved as

$$\rho_T^\lambda = e^{\lambda\int_0^T \cos(2\psi_t^\lambda - \alpha_t)\mathrm{d}t}$$

By (4), our original isometry conjecture (1) is reduced to showing that

$$\|(x,y)\|_{1-var} \le \limsup_{\lambda\to\infty} \frac{\log\rho_T^\lambda}{\lambda}.$$

By the unit speed parametrisation (3),

$$\|(x,y)\|_{1-var} = T.$$

By the explicit solution for ρ_T^λ, the problem is reduced to showing that

$$T \le \limsup_{\lambda\to\infty} \int_0^T \cos(2\psi_t^\lambda - \alpha_t)\mathrm{d}t. \tag{5}$$

The key difficulty here is that α may not be continuous, and one needs to identify suitable conditions on α which ensure that (x,y) is reduced.

Points arising from the discussion:
1. The discussion mentioned a related work by Cass, Foster, Lyons, Salvi and Yang [4] where they studied an inner product $\langle\cdot,\cdot\rangle$ on the signature. The norm is defined so that if $\{e_1,\dots,e_d\}$ is the standard basis of $\mathbb{R}^d$, then the set $\{e_{i_1}\otimes\dots\otimes e_{i_n}\}_{i_1,\dots,i_n}$ is orthonormal. This inner product norm can be used to define loss functions in terms of norms of signatures in machine learning probloems. One of the main results in [4] is that, for paths x and y, the function $k_{x,y} : (s,t) \to \langle S(x)_{0,s}, S(y)_{0,t}\rangle$ satisfies the hyperbolic PDE

$$\frac{\partial^2 k_{x,y}}{\partial s\partial t} = \langle \dot{x}_s, \dot{y}_t\rangle k_{x,y} \tag{6}$$

with initial conditions $k_{x,y}(u,\cdot) = k_{x,y}(\cdot,v) = 1$ and $\dot{x}_s = \frac{dx_p}{dp}\big|_{p=s}$, $\dot{y}_t = \frac{dx_q}{dq}\big|_{q=t}$.

2. A second point arising from the discussion is whether similar results to the isometry (1) holds for some rough paths. There have been work in relating the limiting asymptotics of Brownian motion to the quadratic variation of Brownian motion (up to an unknown deterministic constant), as well as to rough paths whose signature is the expeonential of a Lie polynomial, but there hasn't been any results in general. In fact, it is unclear what would be replacing the 1-variation norm on the left hand side of (1) when γ is a rough path.

3. A third point clarified during the discussion is that the "lim sup" appearing in (1) in fact exists as a limit for a general bounded variation path γ, which is a consequence of results in [5] and [3]. It is also a consequence of result in [3] that the limit in (1) is non-zero if a reduced path has a strictly positive length.

4. A fourth point raised is whether the norm of signature in (1) is in fact attained by taking modulus of certain coordinate terms in the signature. This is not known and may require understanding of the algebraic structure of signature.

5. A final point raised was whether (1) would hold for norms other than the projective norm, such as the inner product norm mentioned earlier. The techniques in the proof used properties of the projective norm, but it is not known whether the result holds for other norms even for C^1 paths.

References

[1] B. Hambly, T. Lyons, *Uniqueness for the Signature of a Path of Bounded Variation and the Reduced Path Group*, Annals of Mathematics **171** (2010), 109–167.

[2] H. Boedihardjo, X. Geng, *$SL_2(\mathbb{R})$-developments and Signature Asymptotics for Planar Paths with Bounded Variation*, arXiv:2009.13082.

[3] H. Boedihardjo, X. Geng, *A non-vanishing property for the signature of a path*, Comptes Rendus Mathematique **357** (2019), 120–129.

[4] T. Cass, J. Foster, T. Lyons, C. Salvi, W. Yang, *Computing the untruncated signature kernel as the solution of a Goursat problem*, arXiv:2006.14794.

[5] J. Chang, T. Lyons, H. Ni, *Super-multiplicativity and a lower bound for the decay of the signature of a path of finite length*, Comptes Rendus Mathematique **356** (2018), 720–724.

[6] T. Lyons, W. Xu, *Hyperbolic development and inversion of signature*, Journal of Functional Analysis **272** (2017), 2933–2955.

Optimal stopping with signatures

Christian Bayer

(joint work with Paul Hager, Sebastian Riedel, John Schoenmakers)

Given a stochastic process $(X_t)_{t\in[0,T]}$ such that $\hat{X}_t := (t, X_t)$ extends to a p-rough path $\hat{\mathbb{X}}$ and a continuous *reward process* $(Y_t)_{t\in[0,T]}$ adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by $\hat{\mathbb{X}}$ such that $\mathbb{E}\|Y\|_\infty < \infty$, we consider the *optimal stopping problem*

$$\sup_{\tau\in\mathcal{S}} \mathbb{E}Y_\tau,$$

where $\mathcal{S}$ denotes the set of $(\mathcal{F}_t)_{t\in[0,T]}$-stopping times taking values in $[0, T]$. Of course, this problem is extremely well studied theoretically, and numerous numerical algorithms were proposed when X is a Markov process. In contrast, considerably fewer numerical algorithms are known in the non-Markovian case. In fact, [5] – based on Wiener chaos expansion of Y – is one of very few examples from the recent literature. Of course, approaches based on machine learning are also put forth (see, e.g., [2]), but there is usually no accompanying theory.

In this talk, we suggest an approach based on rough path signatures, inspired by several recent papers on optimal control, such as [4]. The signature approach to optimal control problems can be roughly summarized as follows:

(1) Controls u_t are *continuous functions* of the path $\phi(\hat{X}|_{[0,t]})$ and, hence, of the signature $\theta(\hat{\mathbb{X}}^{<\infty}_{0,t})$ – and similarly for the loss function.
(2) We may approximate $\theta(\hat{\mathbb{X}}^{<\infty}_{0,T})$ by *linear functionals* $\langle l, \hat{\mathbb{X}}^{<\infty}_{0,T}\rangle$.
(3) Interchange expectation and *truncate* the signature at level N.
(4) *Optimize* $l \mapsto \left\langle l, \mathbb{E}\left[\hat{\mathbb{X}}^{\leq N}_{0,T}\right]\right\rangle$.

Note that the method only relies on minimal requirements on the underlying process X, which, in particular, does not need to have the Markov property. Moreover, [4] indicate convincing performance in several numerical examples. On the other hand, theoretical justification is largely missing.

Indeed, the known theoretical results suggest that any given fixed continuous function ϕ of the (random) path $\hat{X}|_{[0,T]}$ can be approximated arbitrarily well with arbitrarily high probability by functionals of the form $\langle l, \hat{\mathbb{X}}^{<\infty}_{0,T}\rangle$. Quantitative error controls are so far unknown.

Results of these type are difficult to apply to optimal control problems such as optimal stopping, since candidate controls often fail to be even continuous. Indeed, it is well known that optimal stopping times are often *hitting times* of certain sets, and such hitting times are not continuous as functions of the underlying path. Following [1] we get around this issue by considering *randomized stopping times*. This essentially means that we consider independent random counting processes which jump with an intensity depending on the path $\hat{X}$. Interestingly, if we now integrate out the additionally introduced randomness, we obtain smooth functions of those intensities. Using these techniques, we can prove that the supremum over all stopping times defined in terms of linear functionals of the signature gives the value of the full optimal stopping problem.

Theorem 1. *Given* $l \in T((\mathbb{R}^{1+d})^*)$*, set the* signature stopping time

$$\tau_l := \inf\left\{t \in [0,T] \,\middle|\, \langle l, \hat{\mathbb{X}}^{<\infty}_{0,t}\rangle \geq 1\right\}.$$

Assuming $\mathbb{E}[\|Y\|_\infty] < \infty$*, we have*

$$\sup_{l \in T((\mathbb{R}^{1+d})^*)} \mathbb{E}[Y_{\tau_l \wedge T}] = \sup_{\tau \in \mathcal{S}} \mathbb{E}[Y_{\tau \wedge T}].$$

In addition, we also show how to linearize the function $l \mapsto \mathbb{E}[Y_{\tau_l \wedge T}]$ in terms of the *exponential shuffle* operation on $T((\mathbb{R}^{1+d})^*)$, leading to the following result.

Theorem 2. *Let* $\mathbb{E}[\|Y\|_\infty] < \infty$. *Given* $\kappa > 0$, *define the stopping time* $\sigma = \sigma_\kappa$ *by* $\sigma := \inf\left\{t \geq 0 \mid \left\|\hat{\mathbb{X}}\right\|_{p-\mathrm{var};[0,t]} \geq \kappa\right\} \wedge T$. *Then,*

$$\sup_{\tau \in \mathcal{S}} \mathbb{E}[Y_{\tau\wedge T}] = \mathbb{E}[Y_0] + \\ + \lim_{\kappa\to\infty} \lim_{K\to\infty} \lim_{N\to\infty} \sup_{|l|+\deg(l)\leq K} \mathbb{E}\left[\int_0^{\sigma_\kappa} \left\langle \exp^{\sqcup\!\sqcup}(-(l \sqcup\!\sqcup l)\mathbf{1}), \hat{\mathbb{X}}^{\leq N}_{0,t} \right\rangle \mathrm{d}Y_t\right].$$

If Y *is a linear functional of* $\hat{\mathbb{X}}^{<\infty}$, *this formula can be further simplified. E.g., if* $d = 1$ *and* $Y = X$, *then*

$$\sup_{\tau \in \mathcal{S}} \mathbb{E}[Y_{\tau\wedge T}] = \mathbb{E}[Y_0] + \\ + \lim_{\kappa\to\infty} \lim_{K\to\infty} \lim_{N\to\infty} \sup_{|l|+\deg(l)\leq K} \left\langle \exp^{\sqcup\!\sqcup}(-(l \sqcup\!\sqcup l)\mathbf{1})\mathbf{2}, \mathbb{E}\left[\hat{\mathbb{X}}^{\leq N}_{0,\sigma_\kappa}\right] \right\rangle.$$

During the discussion, connections with the adapted topologies based on higher rank signatures were raised, which could be interesting to explore in the future.

References

[1] C. Bayer, R. Tempone, and S. Wolfers. *Pricing American options by exercise rate optimization.* Quant. Fin. 20(11), 1749–1760, 2020.
[2] S. Becker, P. Cheridito, and A. Jentzen. *Deep optimal stopping.* J. Machine Learning Research 20(74):1–25, 2019.
[3] B. Dupire. *Functional Itō calculus.* Quant. Fin. 19(5), 721–729, 2019.
[4] J. Kalsi, T. Lyons, and I. Perez Arribas. *Optimal execution with rough path signatures.* SIAM J. Financial Math. 11(2):470–493, 2020.
[5] J. Lelong. *Dual pricing of American options by Wiener chaos expansion.* SIAM J. Financial Math. 9(2), 493–519, 2018.

Precise Local Estimates for Hypoelliptic Differential Equations driven by Fractional Brownian Motion

Xi Geng

(joint work with Cheng Ouyang and Samy Tindel)

Consider the following RDE

$$\begin{cases} dX_t = V(X_t) \otimes dB_t, & 0 \leqslant t \leqslant 1, \\ X_0 = x, \end{cases}$$

where B_t is a d-dimensional fractional Brownian motion with Hurst parameter $H > 1/4$, $V = \{V_1, \cdots, V_d\}$ are C_b^∞-vector fields on $\mathbb{R}^N$. We assume that the vector fields satisfy a Hörmander-type hypoellipticity condition of order $l_0 \geqslant 1$, i.e. the vector fields together with their Lie brackets generate the tangent space at every point in space. It is well known that under such conditions, the solution X_t admits a smooth density with respect to the Lebesgue measure on $\mathbb{R}^N$. The main motivation of the talk concerns with quantitative properties of the density function $p(t,x,y)$ of X_t in small time.

Our starting point is to observe that the small time behaviour of $p(t,x,y)$ is closely related to the so-called *control distance function* associated with the equation defined by

$$d(x,y) \triangleq \inf\left\{\|h\|_{\bar{\mathcal{H}}} : h \in \Pi_{x,y}\right\}, \quad x,y \in \mathbb{R}^N.$$

Here $\bar{\mathcal{H}}$ is the intrinsic Cameron-Martin subspace of the fractional Brownian motion and $\Pi_{x,y}$ denotes the set of Cameron-Martin paths joining x to y in the sense that the time one map of the associated ODE flow driven by h sends x to y. According to the Varadhan-type asymptotics result, the density function $p(t,x,y)$ satisfies

$$\lim_{t\to 0^+} t^{2H} \log p(t,x,y) = -\frac{1}{2} d(x,y)^2.$$

This suggests that the small time behaviour of $p(t,x,y)$ is controlled by the geometry of the distance function $d(x,y)$.

Our first main result provides a sharp uniform local estimate on the control distance function.

Theorem 1. *There exist constants $C_1, C_2, \delta > 0$, such that*

$$C_1|x-y| \leqslant d(x,y) \leqslant C_2|x-y|^{1/l_0}$$

for all $x,y \in \mathbb{R}^N$ with $|x-y| < \delta$.

The essential difficulty for proving Theorem 1 is that one cannot easily construct a Cameron-Martin path joining x to y in the aforementioned sense of differential equation in large scale. As a result, one needs to rely on local constructions and delicate patching argument. The strategy relies on the use of rough Taylor expansions and fractional calculus in a crucial way. As a byproduct, the analysis for this part also yields the following interesting result which asserts that all the control distance functions associated with different Hurst parameters are locally Lipschitz-equivalent.

Theorem 2. *Suppose that the vector fields are C_b^∞-function satisfy the uniform hypoellipticity condition. For any $H \in (1/4,1)$, let $d_H(x,y)$ be the control distance function associated with the fractional Brownian motion with Hurst parameter H. Then for any $H_1, H_2 \in (1/4,1)$, there exist constants $C, \delta > 0$ such that*

$$C^{-1} d_{H_1}(x,y) \leqslant d_{H_2}(x,y) \leqslant C d_{H_1}(x,y).$$

Our second main result concerns with a sharp uniform local estimate for the density function $p(t,x,y)$ of the solution X_t.

Theorem 3. *There exist constants $C, \tau > 0$, such that*

$$p(t,x,y) \geqslant \frac{C}{\left|B_d(x,t^H)\right|}$$

for all $(t,x,y) \in (0,1] \times \mathbb{R}^N \times \mathbb{R}^N$ with $d(x,y) \leqslant t^H$ and $t < \tau$.

Our main strategy for proving Theorem 3 can be summarised by the following three steps which was initiated by Kusuoka and Stroock in the context of diffusions. The essential point is to compare the actual solution with the order-l Taylor approximation process at the density level.

(i) *Step one.* Obtain a precise local lower estimate for the density of the fBM truncated signature process $S_l(\mathbf{B})_{0,t}$ on the nilpotent group $G^{(l)}$ with respect to the Haar measure. Due to scaling properties of the signature, it turns out that the key ingredient for this step is to show that the density of $S_l(\mathbf{B})_{0,t}$ is everywhere strictly positive. For this purpose, we establish the following result which may be of independent interest.

Theorem 4. *The density $p(t, x, y)$ of X_t is everywhere strictly positive.*

(ii) *Step two.* Due to the non-degeneracy of the Taylor approximation function as a consequence of hypoellipticity, the lower bound for the density of the signature process $S_l(\mathbf{B})_{0,t}$ obtained in Step One implies a lower bound for the density of the Taylor approximation process $X_l(t, x)$ via a Riemannian disintegration formula.
(iii) *Step three.* Compare the density of $X_l(t, x)$ with the density of the actual solution X_t in small time via the method of Fourier transform.

Comments on discussant's summary. The discussant gave a wonderful summary of the talk. In his presentation several questions were raised and brief answers are recaptured in what follows.

Question 1: How are the results related to short-time asymptotics of density obtained by other authors? The existing works on short time asymptotics deal with elliptic SDEs and off-diagonal asymptotics for hypoelliptic SDEs. Short-time asymptotics is more precise in the sense that the coefficient functions in the expansion are described explicitly. On the other hand, our result provides sharp uniform estimates that are not easy to obtain from asymptotic expansions.

Question 2: Is it possible to deal with the case with drift? The case with drift is non-trivial and challenging. The main reason lies in two aspects. In the first place, the signature process loses the fractional Brownian scaling property when a drift is introduced, and as a result the first step of the strategy requires new method. Secondly, geometric constraints on the drift need to be imposed to ensure that non-horizontally accessible sets can be reached by the drift vector field. This involves delicate sub-Riemannian considerations.

Question 3: Is it possible to extend the result to other Gaussian processes? In order to obtain sharp estimates, the current work relies critically on the explicit representation of the fBM Cameron-Martin structure and fractional calculus. Although the main philosophy should be robust to treat general Gaussian processes, the precise development of the parallel analysis in the general context is not so clear at the moment.

References

[1] X. Geng, C. Ouyang and S. Tindel, Precise local estimates for differential equations driven by fractional Brownian motion: elliptic case, *arXiv preprint*, 2020.
[2] X. Geng, C. Ouyang and S. Tindel, Precise local estimates for differential equations driven by fractional Brownian motion: hypoelliptic case, *arXiv preprint*, 2020.
[3] S. Kusuoka and D. Stroock, Applications of the Malliavin calculus, Part III, *J. Fac. Sci. Univ. Tokyo* 34 (1987) 391–442.

An introduction to Grassmannian stochastic analysis

Francesco De Vecchi

(joint work with Sergio Albeverio, Luigi Borasi and Massimiliano Gubinelli)

1. Motivations

Quantum field theory is a theoretical framework describing the behavior of physical fields when the energy of the system is very large or the size of the system is very small. A rigorous mathematical description of this theory is given by Osterwalder-Schrader axioms: they are a list of the properties under which a set of tempered distributions $S_n(x_1, ..., x_n)$ (where $x_i = (x_i^0, ..., x_i^{d-1}) \in \mathbb{R}^d$) are the Swinger functions of a quantum system, namely

$$\mathcal{S}_n(x_1, ..., x_n) = \langle e^{x_1^0 H}\phi(x_1^1, ..., x_1^{d-1}) \cdots e^{x_n^0 H}\phi(x_n^1, ..., x_n^{d-1})\Omega, \Omega\rangle_{\mathcal{H}},$$

where, heuristically, $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is the Hilbert space of the quantum system, $\Omega \in \mathcal{H}$ is a non-zero ground state, H is the Hamiltonian, i.e. a (generally unbounded) positive operator describing the evolution of the field, and $\phi : \mathbb{R}^{d-1} \to \mathcal{L}(\mathcal{H})$, where $\mathcal{L}(\mathcal{H})$ is the set of linear operators on $\mathcal{H}$, is the quantum field at time 0 defined on the space $\mathbb{R}^{d-1}$.

The two kind of particles existing in nature are bosons, represented by commuting quantum fields, i.e. $\phi_{\text{bos}}(x)\phi_{\text{bos}}(x') = \phi_{\text{bos}}(x')\phi_{\text{bos}}(x)$ where $x, x' \in \mathbb{R}^{d-1}$, and fermions, described by anticommuting quantum fields, i.e. $\phi_{\text{fer}}(x)\phi_{\text{fer}}(x') = -\phi_{\text{fer}}(x')\phi_{\text{fer}}(x)$ where $x, x' \in \mathbb{R}^{d-1}$. In the bosonic case Nelson discovered that the Swinger functions can be realized as the expectation of a (commutative) random field $\mathcal{S}_n(x_1, ..., x_n) = \mathbb{E}_\nu[\varphi(x_1)\cdots\varphi(x_n)]$. The measure ν is supported on the space of tempered distribution $\mathcal{S}'(\mathbb{R}^d)$ and has, heuristically, the form dν = "$e^{-S(\varphi)}\mathcal{D}\varphi$", where $S = S_{\text{free}} + S_{\text{int}}$ is the classical action of the bosonic field and $\mathcal{D}\varphi$ is an heuristic Lebesgue measure on $\mathcal{S}(\mathbb{R}^d)$. qudratic part of the action S_{free}. It is possible to give a precise mathematical meaning to the measure ν through some suitable renormalization procedures (see, e.g., [4, 11] for an introduction to the topic). One of these methods is *stochastic quantization*, introduced by Parisi and Wu [10], who proposed to build the measure ν as the invariant measure of the following singular stochastic partial differential equation (SSPDE)

$$\frac{\partial \Psi}{\partial t}(t, x) = -\frac{\delta S}{\delta \varphi}(\Psi(t, x)) + \xi(t, x)$$

where $x \in \mathbb{R}^d$, $t \in \mathbb{R}_+$ is an additional "computer time", $\frac{\delta S}{\delta \varphi}$ is the functional derivatives of the action S, and ξ is a $\mathbb{R}^{d+1}$ white noise. The relatively recent advances in the study of SSPDEs (see, e.g. [5, 6]), have generated a renewed interested in stochastic quantization.

2. Fermionic stochastic quantization

In the case of fermionic fields ϕ_{fer}, the described (standard) probabilistic representation does not hold, but an analogous theory can be developed using *quantum probability* (see, e.g., [1, 8]). Quantum probability is a non-commutative extension of (standard) probability where the space of bounded random variable is replaced by a general C^* algebra $\mathcal{A}$ and the expectation is replaced by a functional $\omega : \mathcal{A} \to \mathbb{C}$ which is continuous, linear and positive (i.e. $\omega(aa^*) \geq 0$). Osterwalder and Schrader applied these ideas to fermions: they consider an anticommutative random field $\psi : \mathbb{R}^d \to \mathcal{A}$ for which $\mathcal{S}_n(x_1, ..., x_n) = \omega(\psi(x_1) \cdots \psi(x_n))$. They also proposed a Feynman-Kac formula (see [6]) for defining the QFT with classical action $S = S_{\text{free}} + S_{\text{int}}$:

$$\mathcal{S}_n(x_1, ..., x_n) = \omega_{\text{free}}(\text{“}e^{-S_{\text{int}}(\psi_{\text{free}})}\text{”}\psi_{\text{free}}(x_1) \cdots \psi_{\text{free}}(x_n)). \tag{1}$$

Feynman-Kac formula (1) was used to prove the existence of 2 and 3 dimensional interacting fermionic QFTs (see [7] for a review). In our paper [2] we introduce a possible theory of stochastic quantization of QFTs involving fermions by exploiting formula (1) for Schwinger functions. In order to achieve this aim we have developed in [2] a stochastic calculus for Grassmann (i.e. anticommutative) algebras. The description of this theory is the main topic of the talk.

3. Grassmann stochastic analysis

If V is a (real) vector space, a *Grassmann V-random variable* X is a homomorphism $X : \Lambda V \to \mathcal{A}$ (where ΛV is the exterior algebra generated by V). In particular this means that for any $v_1, v_2 \in V$ we have $X(v_1)X(v_2) = -X(v_2)X(v_1)$. We denote by $\mathcal{G}(V)$ the space of Grassmann V-random variables.

Thanks to this definition, we can interpret the elements $F \in \Lambda V$, i.e. $F = \sum v_{a_1} \wedge \cdots \wedge v_{a_k}$, as functions from $\mathcal{G}(V)$ into $\mathcal{A}$ (the space of random variables), $F(X) := X(F) = \sum X(v_{a_1} X(v_{a_2}) \cdots X(v_{a_k})$.

If V is equipped with a pre-Hilbert scalar product and C is an antisymmetric operator on V, we say that $X \in \mathcal{G}(V)$ is a Gaussian V-random variable with covariance C if the Wick theorem holds, namely

$$\omega(X(v_1 \wedge \cdots \wedge v_{2n})) = \sum_{\mathcal{M} \in \{\text{perfect matchings of } \{1,...,2n\}\}} (-1)^{\mathcal{M}} \prod_{(i,j) \in \mathcal{M}} \langle v_i, C v_j \rangle.$$

Exploiting similar analogies we can define: stochastic processes on V as Grassmann $(V \otimes L^2(\mathbb{R}))$-random variables, C-Gaussian-noises Ξ as Gaussian random variables on $(V \otimes L^2(\mathbb{R}))$ with covariance $C \otimes I_{L^2(\mathbb{R})}$ and Brownian motions B_t as the integrals of the Gaussian noises Ξ with respect to time, i.e. $B_t(v) = \Xi(v \otimes \mathbb{I}_{[0,t]}(\cdot))$. Finally, when V is finite dimensional, we introduce a natural topology on $\mathcal{G}(V) \subset \mathcal{L}(V, \mathcal{A})$

as the one induced by the operator norm on the set of linear operators $\mathcal{L}(V, \mathcal{A})$ from V into $\mathcal{A}$.

We say that a function $\Psi_{\cdot} \in C^0(\mathbb{R}, \mathcal{G}(V))$ is a solution to the additive noise SDE with coefficient G and starting at $-T \in \mathbb{R}$ if, for any $t \geq -T$ and any $v \in V$, we have

$$\Psi_t(v) - \Psi_{-T}(v) = \int_{-T}^{t} G(\Psi_s, v)\mathrm{d}s + B_t(v) - B_{-T}(v), \tag{2}$$

where $\Psi_{-T}(v)$ is an element of $\mathcal{G}(V)$, $G : V \to \Lambda_{\mathrm{odd}}V$ is a linear map and the integral is the Bochner integral with respect to the norm of $\mathcal{A}$.
We proved existence uniqueness of solutions to equation (2). Furthermore when G is of contractive and gradient form, namely $G(v) = Av + \lambda\partial U(v)$ (where A is a linear map with only strictly negative eigenvalues and commuting with the covariance C of B_t, $\lambda \in \mathbb{R}$ is a small constant, and $U \in \Lambda_{\mathrm{even}}V$ is a function on $\mathcal{G}(V)$) we proved the convergence of the solution (when $-T \to -\infty$) to a stationary process. Finally by using an Ito formula for the solutions to additive-type equations (2), we are able to prove that the invariant stationary solution Ψ^s_t to equation (2) satisfies a Feynman-Kac formula, i.e. for any $H \in \Lambda V$

$$\omega(H(\Psi^s_t)) = \frac{\omega(H(X)e^{-2U(X)})}{\omega(e^{-2U(X)})} \tag{3}$$

where X is a Gaussian V-random variable with covariance $C_A = \int_0^\infty e^{A^T s}Ce^{As}\mathrm{d}s$.

4. Discussion and some final remarks

Thanks to the results presented above we are able to prove a stochastic quantization procedure for *regularized Yukawa model in* $\mathbb{R}^2$, and other regularized models of fermionic quantum fields (see [2] for more details). In any case, the shown theory is only the first step in order reach a stochastic quantization procedure for singular quantum fields theory in 2, 3 and 4 dimension and many interesting observations has been risen during the discussion, led by Xue-Mei Li, of the seminar.

The first problem is the removal of the regularization in the fermionic stochastic quantization equation. The main difficulty is the solutions to singular Grassmannian SPDEs cannot take values in a C^*-algebra but in a more generic $*$-algebra. The reason is that, when the regularization is removed, the fermionic fields must be realized by a unbounded operators on a Hilbert space.

A second interesting problem is the stochastic quantization of mixed bosonic-fermionic models. The main difficulty here is that we have to combine both the commutative probabilistic techniques for bosonic quantum fields and the anticommutative ones of fermionic case, without the possibility of choosing a canonical realization for the bosonic field (which, in the case of purely bosonic setting, is the one where the operators are self-adjoint).

An other possible development is the study of the perturbation expansion (in the interaction constant λ) or the semiclassical expansion (in Plank constant $\hbar$) for Grassmannian SPDEs. In this case the fermionic theory seems to be simpler than the bosonic one, since the research using renormalization group techniques (see, e.g., [7]) shows that these expansions are not only asymptotic, as in the bosonic case, but actually convergent.

References

[1] L. Accardi, A. Frigerio, and J. T. Lewis. *Quantum stochastic processes.* Publ. Res. Inst. Math. Sci., **18(1)** (1982) 97–133.

[2] S. Albeverio, L. Borasi, F. C. De Vecchi, and M. Gubinelli. *Grassmannian stochastic analysis and the stochastic quantization of Euclidean Fermions.* ArXiv preprint, (2020) arXiv:2004.09637.

[3] P. H. Damgaard and H. Hüffel. *Stochastic quantization.* Physics Reports, **152(5-6)** (1987) 227–398.

[4] J. Glimm, and A. Jaffe. *Quantum Physics:A functional integral point of view.* 2nd ed. Springer, New York (1987).

[5] M. Gubinelli, P. Imkeller, and N. Perkowski. *Paracontrolled distributions and singular PDEs.* In Forum of Mathematics, Pi, volume 3. Cambridge University Press, (2015).

[6] M. Hairer. *A theory of regularity structures.* Inventiones mathematicae, **198(2)** (2014) 269–504.

[7] V. Mastropietro. *Non-perturbative renormalization.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2008).

[8] P. A. Meyer. *Quantum probability for probabilists*, volume 1538 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, (1993).

[9] K. Osterwalder and R. Schrader. *Euclidean Fermi fields and a Feynman-Kac formula for boson-fermion models.* Helv. Phys. Acta, **46** (1973/74) 277-302.

[10] G. Parisi, Y. S. WU. *Perturbation theory without gauge fixing.* Sci. Sin. **24** (1981) 483–496.

[11] B. Simon. *The $P(\varphi)_2$ Euclidean (Quantum) Field Theory.* Princeton Univ. Press, Princeton, NJ. Princeton Series in Physics (1974).

Rough Homogenisation — an abstract

Xue-Mei Li

We are concerned with two scale slow fast stochastic equations where the noise does not have the Markov property. We briefly reviewed the stochastic averaging problem

$$dx_t^\varepsilon = f(x_t^\varepsilon, y_t^\varepsilon)dB_t + f(x_t^\varepsilon, y_t^\varepsilon)dt$$

where $y_t^\varepsilon = y_{\frac{t}{\varepsilon}}$ for y_t an ergodic Markovian process and (B_t) a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Our main focus is a Functional Central Limit Theorem for y the one dimensional fractional Ornstein-Uhlenbeck process of Hermite rank $H \neq \frac{1}{2}$ in both the Hölder and in the rough path topology. This is then applied to obtain a homogenisation theorem for a random ODE with such fast noise input. Standard Functional CLT and non-CLT's are stated for the weak topology. Although there are many interests on such problems from statisticians and practitioners, the homogenisation problem was not solved. Once the problem is stated in the stronger Hölder topology and in the rough path topology, we can use the tools from the rough paths to study random ODE's. Unlike fro Markovian systems, we have to limit to admissible functions with higher Hermite ranks. Functions with different Hermite ranks may have different scaling properties, and different types of self-similar processes as its limit. In particular the limiting process does not necessarily have the Markov property (they may). We state a result from [Gehringer-L.'19, '20] to illustrate this.

Functional CLT in the rough path topology. The functional limit theorem plays the role of Kipnis-Vardhan's theorem (for strong mixing or Markov processes). The functional CLT's take the following form (for the precise optimal conditions please consult the references listed below. Let $H^*(m) = m(H-1)+1$ and let $X^{k,\epsilon} = \alpha\left(\varepsilon, H^*(m_k)\right)) \int_0^t G_k(y_s^\varepsilon)ds$ where

$$\alpha\left(\varepsilon, H^*(m_k)\right) = \begin{cases} \frac{1}{\sqrt{\varepsilon}}, & \text{if } H^*(m_k) < \frac{1}{2}, \\ \frac{1}{\sqrt{\varepsilon|\ln(\varepsilon)|}}, & \text{if } H^*(m_k) = \frac{1}{2}, \\ \varepsilon^{H^*(m)-1}, & \text{if } H^*(m_k) > \frac{1}{2}. \end{cases}$$

- We gather those components of $X^{k,\epsilon}$ with high Hermite rank and therefore the scaling constant $\frac{1}{\sqrt{\varepsilon}}$ first and denote the sub-vector valued process by $X^{W,\epsilon}$, with the remaining part denoted by $X^{Z,\varepsilon}$. Then,

$$(X^{W,\varepsilon}, X^{Z,\varepsilon}) \xrightarrow{W} (X^W, X^Z)$$

  For $G_k \in L_p$, convergence in $\mathbf{C}^\gamma([\mathbf{0},\mathbf{T}], \mathbf{R}^\mathbf{N})$.
- Furthermore, X^W and X^Z are independent with covariance of the components in each vector as below

$$\mathbb{E}\left(X_t^i X_s^j\right) = 2(t\wedge s)A^{i,j}, \qquad A^{i,j} = \int_0^\infty \mathbb{E}\left(G_i(y_s)G_j(y_0)\right)ds.$$

- If furthermore, $G_k \in L^{p_k}(\mu)$, $p_k > 2$, $m_k \geq 1$, p_k large and G_k has fast chaos decay (this condition is a condition on the regularity), for every $\gamma \in (\frac{1}{3}, \frac{1}{2} - \frac{1}{\min_{k\leq n} p_k})$,

$$\left(X_t^\varepsilon, \mathbb{X}_{s,t}^\varepsilon\right) \overset{(W)}{\to} (X_t, \mathbb{X}_{s,t} + (t-s)A)$$

  in the rough path topology $\mathbf{C}^\gamma$. Here

$$\mathbb{X}_{u,t}^{i,j,\epsilon} := \int_u^t (X_s^{i,\epsilon} - X_u^{i,\epsilon})d\,X_s^{j,\epsilon} = \alpha_i(\varepsilon)\alpha_j(\varepsilon)\int_u^t\int_u^s G_i(y_s^\varepsilon)G_j(y_r^\varepsilon)\,drds.$$

The application of such a theorem is of the following type, again we do not specify here the precise conditions.

Theorem. Assume $G_k : \mathbf{R} \to \mathbf{R}$ are centered and in L^2.

$$\begin{cases} \dot{x}_t^\varepsilon = \sum_{k=1}^N \alpha_k(\varepsilon)\, f_k(x_t^\varepsilon)\, G_k(y_t^\varepsilon), \\ x_0^\varepsilon = x_0, \end{cases}$$

Then the solutions converge to that of

$$dx_t = \sum_{i=1}^m c_i \sigma_k(x_t) dZ_t^{H,m},$$

this is considered as a rough differential equation or a stochastic differential equation with mixed Itô-Young equation. The processes $Z_t^{H,m}$ are the Hermite processes with Hermite rank m and self-similarity exponent H (the parameters come from the Hermite rank of the functions G_k and the Hermite parameter of y_t). The above homogenisation theorem involves iterated integral. Even simpler interesting example are presented in [2].

The results discussed here are from the articles below. The first one is an unpublished article which is then split into [2] and [3], in the latter two articles we are able to include noises with a large range of Hermite parameter.

1 Homogenization with fractional random fields, arXiv: 1911.12600, Nov 2019 (preliminary version) J. Gehringer+[Li]
2 Functional limit theorems for the fractional Ornstein-Uhlenbeck process (Journal of Theoretical Probability, arXiv: 2006.11540). J. Gehringer+[Li]
3 Diffusive and rough homogenisation in fractional noise field, (arXiv: 2006.11544) submitted. J. Gehringer+ [Li]
4 Averaging dynamics driven by fractional Brownian motion. M. Hairer +[Li]

Comment on the Q&A sessions In the Q&A session, we discussed several problems. The dicussant Chevyrev gave a clear summary of our results, pointing out the differences from the classical result. For example, he pointed out that the averaging problem in [4] is between that for ODE and for the usual stochastic differential equations. We also discussed whether using variation norm instead of the Hölder norm will bring something new, this is an interesting question which was also breifly discussed in [1,3] and deserve further investigations.

Another problem we discussed is whether we could extend the results include the right hand side not in a product form. This particular problem concerning non-product form, which we have been considering for a while, presented a real challenge. This is now solved in an upcoming article entitled "Functional Limit Theorems for Volterra Processes and Applications to Homogenization" by J. Gehringer, J. Sieber and myself. This work is now completed, see arXiv: 2104.06364.

The finite radius of convergence of expected signature of stopped Brownian motion on 2D domains

Hao Ni
(joint work with Siran Li)

A fundamental question in rough path theory asks if the expected signature of geometric rough paths completely determines the law of signature. One sufficient condition for the affirmative answer is that the expected signature has an infinite radius of convergence. Recently it is proved that the expected signature of stopped Brownian motion up to the unit 2-D disk has the finite radius of convergence [1], which utilises the rotational invariance of the domain. In this presentation, I shall give a complete solution to the finiteness problem of the expected signature of stopped Brownian motions on 2-dimensional bounded domains under mild regularity assumptions:

Theorem 1. *Let Ω be bounded $C^{2,\alpha}$-domain in $\mathbb{R}^2$. The expected signature Φ of a Brownian motion stopped upon the first exit time from Ω has finite radius of convergence everywhere on Ω.*

Following [1], to show the finite radius of convergence of the expected signature suffices to show the blow up of the hyperbolic development Φ denoted by $\mathcal{H}_\lambda(z) := (\lambda H)\Phi_\Omega(z)$ when λ converges to some $\lambda^* > 0$.

A key ingredient of our proof is the introduction of a "domain-averaging hyperbolic development", which allows us to symmetrise the PDE system for the hyperbolic development of expected signature by averaging over rotated domains (see the illustration of the rotated domain in Figure 1).

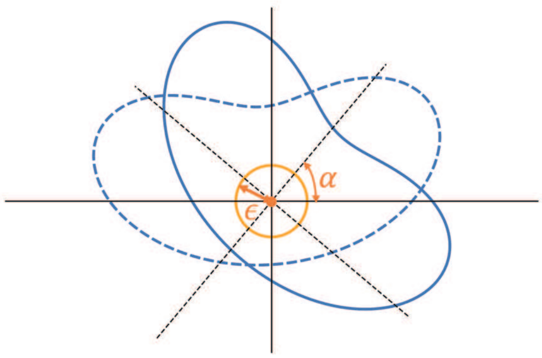

FIGURE 1. Rotation of the domain Ω (its boundary is marked in solid curve) by degree $\alpha = 45°$, with centre point marked in circle. The boundary of the rotated domain $R_\alpha(\Omega)$ is marked in dash curve.

First, by averaging the hyperbolic development of expected signature over the rotated images of the domain, we construct a quantity $\overline{\mathcal{H}_{\lambda,\epsilon}}(z)$, the "*domain-averaging development*", that has the following features:

- $\overline{\mathcal{H}_{\lambda,\epsilon}}(z)$ is rotationally invariant (with respect the domain rotations);
- $\overline{\mathcal{H}_{\lambda,\epsilon}}(z)$ inherits the finiteness of the radius of convergence from the non-averaged hyperbolic developments $\mathcal{H}_{\lambda,\Omega}$ of the expected signature;
- $\overline{\mathcal{H}_{\lambda,\epsilon}}(z)$ satisfies the same PDE as $\mathcal{H}_{\lambda,\Omega}$; thus, the domain-averaging development preserves the lower bound for the radius of convergence of the hyperbolic development.

Working locally near $z \in \Omega$, we arrive at the same PDE for $\overline{\mathcal{H}_{\lambda,\epsilon}}(z)$ as for its non-averaged analogue on a small ball around z. Loosely speaking, half of the boundary conditions are missing for the PDE for $\overline{\mathcal{H}_{\lambda,\epsilon}}(z)$; nevertheless, geometric properties of the hyperbolic development enable us to establish uniform lower bounds for the third component of $\overline{\mathcal{H}_{\lambda,\epsilon}}(z)$. Moreover, similar to the 2-D disc case, $\overline{\mathcal{H}_{\lambda,\epsilon}}(z)$ admits the explicit formula. Combined with the uniform lower bound of the third component, we are able to show the blowup of $\overline{\mathcal{H}_{\lambda,\epsilon}}(z)$.

We remark that the "domain-averaging development" may be adaptable to arbitrary dimensions. It can be viewed as a symmetrization argument for PDEs. However, it is still yet clear whether for $d \geq 3$, the radius of convergence of stopped Brownian motion up to a bounded domain is finite or not.

This presentation is based on the paper [2]. In the Q&A session, we discussed several questions, including whether the PDE approaches for the expected signature can be applied to other stopped stochastic processes.

References

[1] Boedihardjo, H., Diehl, J., Mezzarobba, M. and Ni, H.. *The expected signature of Brownian motion stopped on the boundary of a circle has finite radius of convergence.* Bulletin of the London Mathematical Society (2020).

[2] Li, S. and Ni, H. *The expected signature of stopped Brownian motions on 2D domains has finite radius of convergence everywhere.* arXiv preprint arXiv:2011.07917 (2020).

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