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Classical and Quantum Mechanical Models of Many-Particle Systems (online meeting)

Organized by
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Abstract. The collective behaviour of many-particle systems is a common denominator in the challenges of a highly diverse range of applications: from classical problems in Physics (gas dynamics e.g. Boltzmann's equation, plasma dynamics e.g. various Vlasov equations, semiconductors, quantum mechanics) to current models in biology (kinetic models for collective interaction e.g. swarming, evolution of trait-structured species) to rising topics in social sciences (opinion formation, crowding phenomena) and economics (wealth distribution, mean-field games).
Key mathematical questions concern the analysis (global-in-time wellposedness, regularity), rigorous scaling resp. macroscospic limits (model reduction from many-particle models to mean-field/mesoscopic descriptions to macroscopic evolutions), efficient and asymptotic preserving numerical methods and qualitative results (e.g. large-time equilibration).

Mathematics Subject Classification (2010): Primary: 76P05 Rarefied gas flows, Boltzmann equation, 35Q40 PDEs in connection with quantum mechanics, 35Q92 PDEs in connection with biology and other natural sciences, 82C40 Kinetic theory of gases, 82C22 Interacting particle systems; Secondary: 82C70 Transport processes, 45K05 Integro-partial differential equations, 82D10 Plasmas, 81S30 Phase space methods including Wigner distributions, 81S22 Open systems, reduced dynamics, master equations, decoherence.

Introduction by the Organizers

The workshop *Classical and Quantum Mechanical Models of Many-Particle Systems*, organized by Eric Carlen (New Brunswick), Klemens Fellner (Graz), Isabelle Gallagher (Paris) and Pierre-Emmanuel Jabin (College Park), focused on partial differential equations describing the collective behavior of many-particle systems in various application fields: physics (gas dynamics, plasmas, quantum mechanics), mathematical biology (cell mobility, evolution of trait-structured species), and social sciences (wealth distribution).

The many innovative talks highlighted recent progress on analytical results (like global-in-time well-posedness, regularity of solutions), model reduction (i.e., the rigorous derivation of simpler or mesoscopic models from many-particle master equations and asymptotic limits), efficient numerical schemes (preferably preserving the physically conserved quantities), and quantitative solution properties (like convergence to the equilibrium for large time).

Due to the present circumstances, the workshop took place entirely online. The workshop was organized around a limited number of zoom talks of about 40min each, 3 per day, bridging European and American time zones. Priority was given as much as possible to younger or more junior researchers for speakers.

The workshop made used of break-up rooms for virtual coffee. This turned out to be quite effective to get the participants to interact together with an open problem session that ran for an hour the last day of the conference.

Overall we were very happy to be able to still have this workshop but obviously missed the interactions outside of the talks that form a traditional part of the Oberwolfach experience.

Workshop (online meeting): Classical and Quantum Mechanical Models of Many-Particle Systems

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Abstracts

Quasilinear Approximation of the Vlasov Equation

Claude Bardos
(joint work with Nicolas Besse)

The object of this talk was a report on joint on going program with Nicolas Besse from the Observatory of the Cote d'Azur "Nice -University". We have already two contributions one submitted [1] and one already posted on Arxiv.org [2]. It is devoted to the quasi linear approximation for solutions of the Vlasov equation. This is a very popular tool in Plasma Physic cf. [4] which proposes, for the quantity:

$$q(\int_{\mathbb{R}^d_v} f(x,v,t)dx)\,, \tag{1}$$

the solution of *a parabolic, linear or non linear evolution equation*

$$\partial_t q(t,v) - \nabla_v(D(q,t;v)\nabla_v q) = 0 \tag{2}$$

Since the Vlasov equation is an hamiltonian reversible dynamic while (2) is not reversible whenever $D(q,t,v) \neq 0$ the problem is subtle. Hence I did the following things:

(1) Give some sufficient conditions, in particular in relation with the Landau damping that would imply $D(q,t,v) \simeq 0$. a situation where the equation (2) with $D(q,t;v) = 0$ does not provides a meaning full approximation.
(2) Building on contributions of [8] and coworkers show the validity of the approximation (2) for large time and for a family of convenient randomized solutions. This is justified by the fact that the assumed randomness law is in agreement which what is observed by numerical or experimental observations (cf. [1]).
(3) In the spirit of a Chapman Enskog approximation formalize the very classical physicist approach (cf. [7] pages 514-532) one can show [3] that under analyticity assumptions this approximation is valid for short time. As in [7] one of the main ingredient of this construction is based on the spectral analysis of the linearized equation and as such it makes a link with a classical analysis of instabilities in plasma physic.

Remarks
In some sense the two approaches are complementary. The short time is purely deterministic and the stochastic is based on the intuition that over longer time the randomness will take over of course the transition remains from the first regime to the second remains a challenging open problem. The similarity with the transition

to turbulence in fluid mechanic is striking. It is underlined by the fact that the tensor

$$\lim_{\epsilon\to 0}\mathbb{D}^\epsilon(t,v)=\lim_{\epsilon\to 0}\int dx\int_0^{\frac{t}{\epsilon^2}} d\sigma\, E^\epsilon(t,x+\sigma v)\otimes E^\epsilon(t-\epsilon^2\sigma,x)$$

which involves the electric fields here plays the role of the Reynolds stress tensor.

Obtaining, for some macroscopic description, a space homogenous equation for the velocity distribution is a very natural goal. Here the Vlasov equation is used as an intermediate step in the derivation. Along such line, deeper connection with the use of the Lenard-Balescu equation as presented in the talk [5] of Mitia Duerinckx, "Propagation of chaos and corrections to mean field for classical interacting particles " in this workshop (cf. also [6]) should be considered in the future.

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Mean field limits for inertialess particles sedimenting in a Stokes flow

Richard Höfer

(joint work with Richard Schubert, Juan J.L. Velázquez)

Suspensions of many small particles are ubiquitous in nature and technology.

We consider models of identical spherical particles $B_i := B_R(X_i)$, $1\le i\le N$, where the gravitational acceleration g is the driving force for the dynamics. The fluid could be modeled by the incompressible Navier-Stokes equations

$$\rho_f(\partial_t u+u\cdot\nabla u)-\mu\Delta u+\nabla p=0,\quad \operatorname{div} u=0 \qquad \text{in } \mathbb{R}^3\setminus\bigcup_{i=1}^N\overline{B_i},$$
$$u(x)\to 0 \quad \text{as } x\to\infty,\qquad u=V_i+\Omega_i\times(x-X_i)\quad \text{in } B_i,$$

and the particle evolution by Newton's equations.

$$m_i \frac{d}{dt} V_i = (\rho_p - \rho_f)|B_i| g + \int_{\partial B_i} \sigma[u,p] n \, d\mathcal{H}^2,$$
$$\frac{d}{dt}(I_i \Omega_i) = \int_{\partial B_i} (x - X_i) \times \sigma[u,p] n \, d\mathcal{H}^2$$

Here, $I_i \in \mathbb{R}^{3\times 3}$ denotes the moment of inertia of B_i, and $\sigma[u,p] = \mu(\nabla u + (\nabla u)^T) - p\mathrm{Id}$ the fluid stress.

We are interested in the limit $N \to \infty$, $R \to 0$ in a suitable scaling such that the collective effects of the particles is of order one. The implicit, singular and long-range nature of the interaction renders the problem very difficult.

For a single particle with velocity V in a quiescent Stokes flow, it is well known that the drag force exerted on the particle by the fluid is $6\pi\mu R(u_\infty - V)$, where u_∞ is the fluid velocity at infinity.

Even for positive Reynolds numbers, the fluid is formally well approximated by the Stokes equations at the length-scale of a single particle. Thus, by a superposition principle and a action-reaction principle, one formally obtains the following Vlasov-Navier-Stokes system as the limit of the microscopic system under a suitable scaling:

$$\partial_t f + v \cdot \nabla_x f + \lambda \mathrm{div}_v \left(\hat{g} f + \frac{9}{2}\gamma (u - v) f \right) = 0,$$
$$\mathrm{Re}(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla p = 6\pi\gamma \int_{\mathbb{R}^3} (v - u) f \, dv, \ \mathrm{div} u = 0,$$

where $\mathrm{Re}, \gamma, \lambda$ are dimensionless parameters. The derivation of this coupled system is completely open. Neglecting the particle evolution the derivation of the fluid equations, the so called Brinkman equations, has been obtained (see e.g. the classical works [1, 3] and recent improvements [2, 4, 8]).

Formally setting the Reynolds number and Stokes number equal to zero ($\mathrm{Re} = \lambda = 0$ for $\gamma > 0$), one obtains the coupled transport-Stokes system for the sedimentation of inertialess particles

$$\partial_t \rho_* + (u_* + \frac{2}{9}\gamma^{-1}\hat{g}) \cdot \nabla_x \rho_* = 0,$$
$$-\Delta u_* + \nabla p = \frac{4\pi}{3}\rho_* \hat{g}, \qquad \mathrm{div} u_* = 0.$$

This has been made rigorous in [7].

This system can be derived rigorously from the microscopic inertialess dynamics

$$-\Delta u_N + \nabla p_N = 0, \quad \mathrm{div} u_N = 0 \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{i=1}^N B_i,$$
$$u_N = V_i + (x - X_i) \times \Omega_i \quad \text{in } B_i, \quad 1 \le i \le N, \qquad \frac{d}{dt} X_i(t) = V_i,$$
$$\int_{\partial B_i} \sigma[u_N, p_N] n \, d\mathcal{H}^2 = -\frac{g}{N}, \qquad \int_{\partial B_i} \sigma[u_N, p_N] n \times (x - X_i) \, d\mathcal{H}^2 = 0.$$

After the preliminary result [11] this has been achieved in [6] (see also [12]). The main assumptions are that the particles are well separated and their volume fraction tends to zero .

Recently, in [9], we extended the result to account for a first order correction in the volume fraction ϕ of the particles. It is well-known that inertialess particles increase the effective viscosity of suspensions and the first order effect in terms of the particle volume fraction has been computed by Einstein in his PhD thesis in 1905. Einstein's formula has been proved rigorously in recent years (see [13] and later works). Consequently, we obtain that the microscopic dynamics is well approximated in the p-Wasserstein distance by the macroscopic system

$$\partial_t \rho + (u_{eff} + (6\pi NR)^{-1} g) \cdot \nabla \rho = 0, \quad \rho(0, \cdot) = \rho_0,$$
$$\operatorname{div} \left(2 \left(1 + \frac{5}{2} \phi_N \rho \right) Du_{eff} \right) + \nabla p = \rho g, \quad \operatorname{div} u_{eff} = 0.$$

The proof is based on explicit approximations of the particle velocities through the fluid PDE based on the method of reflections following the work [10]. Moreover, we adapt and improve classical results by Maxime Hauray [5] to pass to the limit by controlling simultaneously the infinte Wasserstein distance and the minimal particle distance.

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The Landau equation: Particle Methods & Gradient Flow Structure

José A. Carrillo
(joint work with Matias G. Delgadino, Laurent Desvillettes, Jingwei Hu, Li Wang, Jeremy Wu)

The Landau equation is an important partial differential equation in kinetic theory. It gives a description of colliding particles in plasma physics [9], and it can be formally derived as a limit of the Boltzmann equation where grazing collisions are dominant [10]. Similar to the Boltzmann equation, the rigorous derivation of the Landau equation from particle dynamics is still a huge challenge. For a spatially homogeneous density of particles $f = f_t(v)$ for $t \in (0,\infty)$, $v \in \mathbb{R}^d$ the homogeneous Landau equation reads

(1)
$$\partial_t f(v) = \nabla_v \cdot \left(f(v) \int_{\mathbb{R}^d} |v-v_*|^{2+\gamma} \Pi[v-v_*](\nabla_v \log f(v) - \nabla_{v_*} \log f(v_*)) f(v_*) dv_* \right).$$

For notational convenience, we sometimes abbreviate $f = f_t(v)$ and $f_* = f_t(v_*)$. We also denote the differentiations by $\nabla = \nabla_v$ and $\nabla_* = \nabla_{v_*}$. The physically relevant parameters are usually $d = 2, 3$ and $\gamma \geq -d-1$ with $\Pi[z] = I - \frac{z\otimes z}{|z|^2}$ being the projection matrix onto $\{z\}^\perp$. In this paper, for simplicity we will focus in the case $d = 3$ and vary the weight parameter γ, although most of our results are valid in arbitrary dimension. The regime $0 < \gamma < 1$ corresponds to the so-called *hard potentials* while $\gamma < 0$ corresponds to the *soft potentials* with a further classification of $-2 \leq \gamma < 0$ as the moderately soft potentials and $-4 \leq \gamma < -2$ as the very soft potentials. The particular instances of $\gamma = 0$ and $\gamma = -d$ are known as the Maxwellian and Coulomb cases respectively.

The purpose of this talk is to propose a new perspective inspired from gradient flows for weak solutions to (1), which is in analogy with the relationship of the heat equation and the 2-Wasserstein metric, see [8, 2]. Moreover, we aim at showing how to use this intepretation to propose a deterministic particle method to solve efficiently the Landau equation (1). One of the fundamental steps is to symmetrize the right hand of (1). More specifically, if we consider a test function $\phi \in C_c^\infty(\mathbb{R}^d)$ we can formally characterize the equation by

(2)
$$\frac{d}{dt}\int_{\mathbb{R}^d} \phi f dv = -\frac{1}{2}\iint_{\mathbb{R}^{2d}} f f_* |v-v_*|^{2+\gamma} (\nabla\phi - \nabla_*\phi_*) \cdot \Pi[v-v_*](\nabla \log f - \nabla_* \log f_*) dv_* dv,$$

where the change of variables $v \leftrightarrow v_*$ has been exploited. Building our analogy with the heat equation and the 2-Wasserstein distance, we define an appropriate gradient

$$\tilde{\nabla}\phi := |v-v_*|^{1+\frac{\gamma}{2}} \Pi[v-v_*](\nabla\phi - \nabla_*\phi_*),$$

so that equation (2) now looks like

$$\frac{d}{dt}\int_{\mathbb{R}^d}\phi f dv = -\frac{1}{2}\iint_{\mathbb{R}^{2d}} f f_* \tilde{\nabla}\phi\cdot\tilde{\nabla}\log f dv_* dv,$$

noting that $\Pi^2 = \Pi$. To highlight the use of this interpretation, we notice that $\tilde{\nabla}\phi = 0$, when we choose as test functions $\phi = 1,\ v_i,\ |v|^2$ for $i = 1,\dots,d$ which immediately shows that formally the equation conserves mass, momentum and energy. The action functional defining the Landau metric mimics the Benamou-Brenier formula [3] for the 2-Wasserstein distance. In fact, the Landau metric is built by considering a minimizing action principle over curves that are solutions to the appropriate continuity equation, that is

$$d_L(f,g) := \min_{\substack{\partial_t\mu + \frac{1}{2}\tilde{\nabla}\cdot(V\mu\mu_*)=0\\ \mu_0=f,\ \mu_1=g}} \left\{\frac{1}{2}\int_0^1\iint_{\mathbb{R}^{2d}} |V|^2\, d\mu(v)d\mu(v_*)dt\right\}, \tag{3}$$

where the $\tilde{\nabla}\cdot$ is the appropriate divergence; the formal adjoint to the appropriate gradient. Also, we notice that analogously to the heat equation, written as the continuity equation $\partial_t f = \nabla\cdot(f\nabla\log f)$, the Landau equation can be formally re-written as

$$\partial_t f = \frac{1}{2}\tilde{\nabla}\cdot(f f_*\tilde{\nabla}\log f),$$

equivalent to the continuity equation with non-local velocity field given by

$$\begin{cases} \partial_t f + \nabla\cdot(U(f)f) = 0 \\ U(f) := -\displaystyle\int_{\mathbb{R}^d} |v-v_*|^{2+\gamma}\Pi[v-v_*]\left(\nabla\log f - \nabla_*\log f_*\right) f_* dv_*\,. \end{cases} \tag{4}$$

Considering the evolution of Boltzmann entropy we formally obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} f\log f dv =: -D(f_t) = -\frac{1}{2}\iint_{\mathbb{R}^{2d}} |\tilde{\nabla}\log f|^2 f f_* dv_* dv \le 0. \tag{5}$$

In physical terms this is referred to as the *entropy dissipation* or *entropy production* for it formally shows that the entropy functional

$$\mathcal{H}[f] := \int_{\mathbb{R}^d} f\log f dv$$

is non-increasing along the dynamics of the Landau equation. Moreover, by integrating equation (5) in time one formally obtains

$$\mathcal{H}[f_t] + \int_0^t D(f_s)ds = \mathcal{H}[f_0]. \tag{6}$$

Similar to H-solutions our approach will also be based on the entropy dissipation (6). Following De Giorgi's minimizing movement ideas [1, 2], we characterize the Landau equation by its associated Energy Dissipation Inequality. More specifically, we show that weak solutions to (1) with initial data f_0 are completely determined by the following functional inequality:

$$\mathcal{H}[f_t] + \frac{1}{2}\int_0^t |\dot{f}|^2_{d_L}(s)\, ds + \frac{1}{2}\int_0^t D(f_s)\, ds \le \mathcal{H}[f_0] \qquad \text{for a.e. every } t > 0,$$

where $|\dot{f}|^2_{d_L}(s)$ stands for the metric derivative associated to the Landau metric defined above. Our analysis is also largely inspired by Erbar's approach in viewing the Boltzmann equation as a gradient flow [7] and recent numerical simulations of the homogeneous Landau equation in [5] based on a regularized version of (4). In contrast with the classical 2-Wasserstein metric, one of the main features of the Landau equation (1) and metric (3) is that they are non-local. Hence, the convergence analysis usually relying on convexity and lower-semi continuity needs to be adapted to deal with the non-locality of this equation.

From the numerical viewpoint we will propose a deterministic particle scheme that preserves all the conserved quantities at the semidiscrete level for the regularized Landau equation and that is entropy decreasing. We will illustrate the performance of these schemes with efficient computations using treecode approaches borrowed from multipole expansion methods for the 3D relevant Coulomb case.

From the theoretical viewpoint, we use the theory of metric measure spaces for the Landau equation, we carefully study the Landau distance d_L. Moreover, we show for a regularized version of the Landau equation that we can construct gradient flow solutions, curves of maximal slope, via the corresponding variational scheme. The main result obtained for the Landau equation shows that the chain rule can be rigorously proved for the grazing continuity equation, this implies that H-solutions with certain apriori estimates on moments and entropy dissipation are equivalent to gradient flow solutions of the Landau equation. We crucially make use of estimates on Fisher-like quantities in terms of the Landau entropy dissipation developed in [6].

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The 3-particles collision integral +condensate: condensate growth?

Miguel Escobedo

The three particles collision integral for an isotropic dilute gas of bosons may be written (cf.[2]):

$$C_{1,2}(F) = \iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} (R(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) - R(\mathbf{k}_1,\mathbf{k},\mathbf{k}_2) - R(\mathbf{k}_2,\mathbf{k}_1,\mathbf{k})\, d^3\mathbf{k_1}\mathbf{d^3}\mathbf{k_2}$$

$$\begin{aligned} R(\mathbf{k},\mathbf{k_1},\mathbf{k_2}) = &\left|\mathcal{M}(\mathbf{k},\mathbf{k_1},\mathbf{k_2})\right|^2 \delta\left(\omega(\mathbf{k}) - \omega(\mathbf{k_1}) - \omega(\mathbf{k_2})\right) \times \\ &\times \delta\left(\mathbf{k} - \mathbf{k_1} - \mathbf{k_2}\right)\left(F_1F_2(1+F) - (1+F_1)(1+F_2)F\right) \end{aligned}$$

where $F(t,\mathbf{k})$ is the density of particles at time t and momentum $\mathbf{k}$ and we denote $F_j(t) \equiv F(t,\mathbf{k}_j)$ for $j = 1,2,3,4$, $\omega(\mathbf{k})$ is the energy of particles of momentum $\mathbf{k}$, $\left|\mathcal{M}(\mathbf{k},\mathbf{k_1},\mathbf{k_2})\right|^2$ is the scattering amplitude.

Different functions $\omega(\mathbf{k}), \left|\mathcal{M}(\mathbf{k},\mathbf{k_1},\mathbf{k_2})\right|^{\mathbf{2}}$ may arise for different types of particles or waves For a gas of bosons in presence of the condensate, the dispersion law is usually taken as being (cf.[2]):

$$\begin{aligned} &\omega(t,\mathbf{k}) = \sqrt{2gn(t)|\mathbf{k}|^2 + |\mathbf{k}|^4},\ (m = 1/2)\\ &g = 4\pi a,\ a:\ \text{s-wave scattering length}\\ &n(t):\text{condensate's density} \end{aligned}$$

As described in [2, 3]; the collision integral $C_{1,2}$ describes number-changing processes between superfluid component (condensate) and the normal fluid. In the moderately low temperature

$$\omega(k) = k^2, \qquad \left|\mathcal{M}(\mathbf{k},\mathbf{k_1},\mathbf{k_2})\right|^2 = 32a^2n(t).$$

For a spatially homogeneous isotropic gas:

$$F(t,\mathbf{k}) = F(t,|\mathbf{k}|);\ \ g(t,x) = |\mathbf{k}|\,F(t,|\mathbf{k}|),\ \ x = |\mathbf{k}|^2;$$

and the system may be written:

$$\frac{\partial F}{\partial t}(t,\mathbf{k}) = C_{1,2}(F)(t,\mathbf{k}),\ \ (F(t,\mathbf{k}) \equiv F(t,k)) \tag{1}$$

$$\frac{dn(t)}{dt} = -\int_{\mathbb{R}^3} C_{1,2}(F)(t,\mathbf{k})\,d^3\mathbf{k} \tag{2}$$

where $n = n(t)$ represents the condensate density. Notice that this system satisfies the formal conservation of number of particles and energy. The particles density $F(t)$ may be a measure, but the description assumes: $F(t,\{0\}) = 0$ for all $t > 0$.

If $g(t,x) = k\,F(t,k)$ where $x = k^2$ is a function defined for $x > 0$ and bounded at the origin, simple integration by parts would give:

$$n'(t) = -n(t)M_{1/2}(g)(t)$$

and n would decrease for all t.

However, it was shown in [1] that for all initial data $\forall g_0 \in \mathscr{M}_1^+([0,\infty))$, $n_0 > 0$ there exists a weak solution $g \in C([0,\infty); \mathscr{M}_r^+([0,\infty))$, $n \in L^\infty(0,\infty)$ such that, for $\varphi_\varepsilon(x) = \left(1 - \frac{x}{\varepsilon}\right)_+^2$

$$n'(t) = -n(t)\left(M_{1/2}(g)(t) - T(g)(t)\right)$$
$$T(g) = \lim_{\varepsilon\to 0} \iint\limits_{(0,\infty)^2} \frac{\varphi_\varepsilon(x+y) + \varphi_\varepsilon(|x-y|) - 2\varphi_\varepsilon(\max\{x,y\}).}{\sqrt{xy}} g(x)g(y)dxdy > 0.$$

As a Corollary, it was deduced in [1] that, if g has no atoms and $g(t,x) \underset{x\to 0}{\sim} a(t)x^\theta$ for some θ then $\theta = -1/2$. That is exactly the behavior of the equilibria $\frac{\sqrt{x}}{e^x - 1}$ and it is known (cf.[4]) that if $x^{-1/2}g(x)$ is bounded for large x, such that $x^{-1/2}g(x) \underset{x\to 0}{\to} a$ and g has some Hölder regularity for $0 < x < 1$ then

$$\lim_{\delta\to 0} \int_\delta^\infty \sqrt{x}\, I(g)(x)dx = -\frac{a^2\,\pi^2}{3} + 2M_{1/2}(g).$$

None of these conditions is proved to be satisfied by the weak solutions obtained in [1]. Our purpose is then to prove existence of regular global radially symmetric solutions to (1) that behave like $x^{-1/2}$ as $x \to 0$ for all $t > 0$. A possible strategy is to start linearizing the equation around an equilibria as follows

$$F(t, \mathbf{k}) = F_0 + F_0(1 + F_0)\Omega(t).$$

In order to simplify as much as possible the presentation we change to $x = |\mathbf{k}|$ variable. Then, $\omega(\mathbf{k}) = |\mathbf{k}|^2 = x^2$ and $F_0(\mathbf{k}) \equiv F_0(x) = \frac{1}{e^{x^2}-1}$. If one consider the new function $f(t,x) = \frac{\Omega(t,x)}{x^2}$ and keep only linear terms in the equation,

$$\frac{\partial f}{\partial t}(t,x) = n(t)\mathscr{L}(f(t))(x), t > 0, x > 0,$$
$$\mathscr{L}(f(t))(x) = \int_0^\infty (f(t,y) - f(t,x))M(x,y)dy$$
$$M(x,y) = \left(\frac{1}{\sinh|x^2 - y^2|} - \frac{1}{\sinh(x^2 + y^2)}\right)\frac{y^3 \sinh x^2}{x^3 \sinh y^2}$$

It is then possible to prove the following:

Theorem. Suppose that $u_0 \in L^1(0,\infty)$ satisfies

(3) $$\sup_{0<x<1} x^\theta |u_0(x)| + \sup_{x>1} |u_0(x)| < \infty$$

for some $\theta \in (0,1)$. Then, there exists

(4) $$u \in C([0,\infty); L^1(0,\infty)) \cap L^\infty_{loc}((0,\infty); L^\infty(0,\infty));\ \ p_c \in C([0,\infty))$$

such fhat for every $t > 0$, $u(t) \in C(0, \infty)$,

$$\frac{\partial u}{\partial t}, \mathscr{L}(u) \in L^{\infty}_{loc}((0,\infty); L^{\infty}(\delta,\infty) \cap L^1(0,\infty)),\ \forall \delta > 0, \tag{5}$$

$$\left|\frac{\partial u(t,x)}{\partial t}\right| + |\mathscr{L}(u)(t,x)| \leq \Xi\left(||u_0||_{L^\infty(1,\infty)} + ||u_0||_1; t, x\right) \tag{6}$$

for some explicit function Ξ

$$\frac{\partial u(t,x)}{\partial t} = \mathscr{L}(u(t))(x) + F(u(t))(x). \tag{7}$$

It is tempting to use the arguments in [4] using the approximated solution $F_0 + F_0(1+F_0)x^2u(t,x)$ in (2) and obtain

$$n(t) \approx \int_0^\infty I(F_0 + F_0(1+F_0)x^2u(t))(x)x^2dx = \frac{(1+a(t))^2\pi^2}{3} - \\ - \int_0^\infty (F_0 + F_0(1+F_0)x^2u(t,x))x^3dx$$

The proof of this Theorem is based on the approximation of $\mathscr{L}$, where the kernel M is replaced by its asymptotic behavior for $x << 1, y << 1$:

$$L(f)(x) = \int_0^\infty K(x,y)(f(t,y) - f(t,x))dy$$
$$K(x,y) = \left(\frac{1}{|x^2-y^2|} - \frac{1}{x^2+y^2}\right)\frac{y}{x},\ \forall x > 0,\ \forall y > 0,\ x \neq y.$$

When f is a regular function, $L(f)$ may be written,

$$L(f)(x) = \int_0^\infty H\left(\frac{x}{y}\right)\frac{\partial f}{\partial y}(y)\frac{dy}{y}$$
$$H(z) = \frac{\mathbb{H}(1-z)}{z}\log\left(\frac{1+z^2}{1-z^2}\right) + \frac{\mathbb{H}(z-1)}{z}\log\left(1 - \frac{1}{z^4}\right)$$

where $\mathbb{H}$ is the Heaviside's function and the problem

$$\frac{\partial f}{\partial t} = L(f),\ f(0) = f_0$$

is solved and studied in detail, using Mellin transform.

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The Vlasov-Poisson-Landau System with Specular Reflection Boundary Condition

Yan Guo
(joint work with Hongjie Dong, Zhimeng Ouyang)

A collisional plasma confined in a non-convex bounded domain (e.g. a tokamak) is described by the Vlasov-Poisson-Landau system, in which charged particles interact with a self-consistent electrostatic potential and their Coulombic collisions, and reflect specularly at the boundary. Global well-posedness of this model is established near Maxwellians via combining nonlinear energy method with the S_p estimate for Fokker-Planck type of equations. This is a joint work with Hongjie Dong and Zhimeng Ouyang [1].

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On the correction to Einstein's formula for the effective viscosity

Amina Mecherbet
(joint work with David Gérard-Varet)

Rigorous justification of effective models for suspensions of viscous particles has attracted a lot of attention recently. Applications range from geophysics (sedimentation), biophysics (modelling of respiratory aerosols) to biology (bacterial suspensions a low Reynold number, polymeric fluids).

This talk concerns the problem related to Einstein's formula for the effective viscosity. A first investigation by the first author in collaboration with M. Hillairet [2] leads to the conclusion that the mean-value of the second order approximation of the effective viscosity is explicitly given by a mean-field limit as soon as the microscopic velocity is close to the solution of an effective Stokes model. In this presentation based on the preprint [1], I will explain how can we prove the reciprocal property, that is, the convergence to an effective Stokes model is ensured as soon as the second order correction corresponds to a mean-field limit quantity encoding the pairwise interactions between particles.

Given n identical spherical particles defined by $B_i = B(x_i, r)$ where x_i the center of mass of the i^{th} particle and r their radius, we set $\mathcal{F}^n = \mathbb{R}^3 \setminus \bigcup_{i=1}^{n} \bar{B}_i$ the domain occupied by the fluid and consider the following Stokes equation

$$
(1) \qquad \begin{cases} -\mu\Delta u + \nabla p = 0, \ \mathrm{div}(u) = g_n, & \text{on } \mathcal{F}^n \\ u = u_i + \omega_i \times (x - x_i), & \text{on } B_i, 1 \leq i \leq n, \\ \lim\limits_{|x|\to\infty} |u(x)| = 0. & \end{cases}
$$

where μ the fluid viscosity, $g_n \in L^2(\mathbb{R}^3) \cap L^{6/5}(\mathbb{R}^3)$ models some forcing. u_i, ω_i correspond to the linear and angular velocity of the i^{th} particle. They are unknowns, associated to the newtonian dynamics of the particles: in the absence of inertia, relations are of the form

$$(2) \quad \int_{\partial B_i} \sigma(u_n, p_n)\nu = -\int_{B_i} g_n dx, \qquad \int_{\partial B_i} [\sigma(u_n, p_n)\nu] \times (x - x_i) = -\int_{B_i} g_n \times (x - x_i) dx,$$

where ν the unit outer normal on the sphere ∂B_i, (ρ_p, ρ_f) the particle and fluid density, $\Sigma(u,p)$ corresponds to the stress tensor associated to the Stokes equation

$$\Sigma(u,p) = 2\mu D(u) - p\mathbb{I}.$$

with $\mathbb{I}$ the 3×3 identity matrix and $D(u)$ the symmetric gradient of u.

We assume that the particles occupy a volume of size one in the sense that

$$\text{(A0)} \qquad \rho_n = \frac{1}{n}\sum_i \delta_{x_i} \underset{n\to\infty}{\longrightarrow} \rho$$

in the sense of measures with ρ a bounded density with support $\overline{\mathcal{O}}$ where $\mathcal{O}$ a smooth bounded domain such that $|\mathcal{O}| = 1$. we assume that we are in a regime such that the volume fraction of the particles $\lambda = \frac{4}{3}\pi r^3 n$ is small but of order one (independent of n). Note that this means that the radius of the particles r scales like $n^{-1/3}$. We also assume the following separation assumption between the particles

$$\text{(A1)} \qquad \min_{i \neq j} |x_i - x_j| \geq c n^{-1/3},$$

for a given constant $c > 0$. The aim is to show that the fluid-particle system converges to a Stokes equation with a viscosity coefficient $\mu_{\text{eff}} = \mu_{\text{eff}}(x)$ different from the fluid viscosity μ locally in $\mathcal{O}$ due to the presence of the particles. We show that the limit equation can be approximated, up to an error of order $o(\lambda^2)$, by the following effective model

$$(3) \qquad \begin{cases} -\text{div}(2[\mu + \mu_1\lambda + \mu_2\lambda^2]D(\bar{u}) - \mathbb{I}\bar{p}) &= g, \quad \text{on } \mathbb{R}^3, \\ \text{div}(\bar{u}) &= 0, \quad \text{on } \mathbb{R}^3, \end{cases}$$

with an appropriate first and second order corrections μ_1, μ_2. It is important to emphasize that these corrections are not scalar neither constants but can be seen as measures on the space $\mathbb{R}^3 \times \mathbb{R}^3$ with values in the space

$$\text{Sym}\left(\text{Sym}_{3,\sigma}(\mathbb{R})\right) := \{M : \text{Sym}_{3,\sigma}(\mathbb{R}) \to \text{Sym}_{3,\sigma}(\mathbb{R}), \quad M^t = M\}$$

of symmetric isomorphisms of the space of trace-free symmetric 3×3 matrices denoted by $\text{Sym}_{3,\sigma}(\mathbb{R})$. Using the method of reflections we are able to identify μ_2 as a mean-field limit of a measure $\mu_{2,n}$ defined as a compactly supported distribution on $\mathbb{R}^3_x \times \mathbb{R}^3_y$, with values in the space $\mathrm{Sym}\left(\mathrm{Sym}_{3,\sigma}(\mathbb{R})\right)$: for $F = F(x,y) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ (even for $F \in C^1(\mathbb{R}^3 \times \mathbb{R}^3)$),

$$(4) \quad \langle \mu_{2,n}, F \rangle = \frac{75\mu}{16\pi}\left(\frac{1}{n^2}\sum_{i\neq j} \mathcal{M}(x_i - x_j)F(x_i,x_j) \;-\; \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathcal{M}(x-y)F(x,y)\rho(x)\rho(y)dy\right),$$

where $\mathcal{M} = \mathcal{M}(x) \in \mathrm{Sym}\left(\mathrm{Sym}_{3,\sigma}(\mathbb{R})\right)$ is given by

$$\begin{aligned}\mathcal{M}(x)S : S' &= -D\left(\frac{x\otimes x : S}{|x|^5}x\right) : S',\\ &= -2\frac{Sx\cdot S'x}{|x|^5} + 5\frac{(S:x\otimes x)(S':x\otimes x)}{|x|^7}, \quad \forall S,S' \in \mathrm{Sym}_{3,\sigma}(\mathbb{R}).\end{aligned}$$

The main result is the following

Theorem 1. *Let $\lambda > 0$, $g \in L^{3+\epsilon}$, $\epsilon > 0$, $\mu_2 \in L^\infty\left(\mathbb{R}^3, Sym\left(Sym_{3,\sigma}(\mathbb{R})\right)\right)$. For all n, let r_n such that $\lambda = \frac{4\pi}{3}nr_n^3$, $g_n \in L^{\frac{6}{5}}(\mathbb{R}^3)$. Let $u_{n,\lambda}$ the solution of* (1)-(2) *in $\dot{H}^1(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$. Assume* (A0)-(A1)*, that $g_n \to g$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$, and that*

$$(\mathrm{A2}) \qquad \mu_{2,n} \to \mu_2(x)\delta_{x=y} \quad in\ \mathcal{D}'\left(\mathbb{R}^3\times\mathbb{R}^3, Sym\left(Sym_{3,\sigma}(\mathbb{R})\right)\right)$$

with $\mu_{2,n}$ defined in (4)*. Then any accumulation point u_λ of $u_{n,\lambda}$ solves*

$$(5) \qquad \left\{\begin{array}{rcll} -div(2[\mu + \frac{5}{2}\mu\rho\lambda + \mu_2\lambda^2]D(u_\lambda) - \mathbb{I}p_\lambda) &=& g + R_\lambda, & in\ \mathbb{R}^3,\\ div(u_\lambda) &=& 0, & in\ \mathbb{R}^3,\end{array}\right.$$

where R_λ satisfies for all $q \geq 3$

$$(6) \qquad |\langle R_\lambda, \phi\rangle| \leq C\lambda^{\frac{7}{3}}\|D\phi\|_q, \quad \forall \phi \in \dot{H}^1(\mathbb{R}^3)\cap \dot{W}^{1,q}(\mathbb{R}^3).$$

Estimate (6) shows in particular that the model (5) is close to the effective model (3) up to a $o(\lambda^2)$ error.

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From the many-body quantum dynamics to the Vlasov equation

Chiara Saffirio
(joint work with Laurent Laflèche)

We consider the Vlasov-Poisson system

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - (\nabla |\cdot|^{-1} * \varrho) \cdot \nabla_v f = 0\,, \\ \varrho_f(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,dv\,, \end{cases} \tag{1}$$

where the unknown $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ represents the density of charged particles in a plasma and $\varrho_f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ is the spacial density associated with f. In what follows we will restrict our analysis to the spacial dimensions $d = 2$, 3, which are the physical interesting cases.

Well-posedness of the Cauchy problem associated with (1) in dimension $d = 2$ is well known from the work of Okabe and Ukai [7]. The three dimensional case has been addressed by Pfeffelmoser [8] and Lions and Perthame [6] in the 90s. They show that a solution of Eq. (1) exists and is unique under integrability and regularity assumptions on the initial datum.

We now consider a system of N interacting fermions in $\mathbb{R}^d$ in the mean-field limit, whose dynamics is expected to be approximated, for N large enough, by the Hartree-Fock equation (Cf. for instance [10], [2], [1])

$$i\,\hbar\,\partial_t\,\omega_{N,t} = [\,\mathsf{h}(t)\,,\,\omega_{N,t}\,] \tag{2}$$

where $\omega_{N,t}$ is a one-particle operator on $L^2(\mathbb{R}^d)$ with tr $\omega_{N,t} = N$, $\hbar$ is the Planck constant, the parentheses $[A, B]$ denote the commutator $AB - BA$ of the operators A and B, and $\mathsf{h}(t)$ is the time-dependent Hartree-Fock Hamiltonian

$$\mathsf{h}(t) = -\hbar\Delta + |\cdot|^{-1} * \varrho - \mathsf{X}\,,$$

where ϱ is given in terms of the diagonal kernel of the operator $\omega_{N,t}$ by the identity

$$\varrho(t,x) = N^{-1}\omega_{N,t}(x,x)\,,$$

and X is the exchange operator defined through its kernel

$$\mathsf{X}(x,y) = N^{-1}|x-y|^{-1}\omega_{N,t}(x,y)\,.$$

As already pointed out in [2], the fermionic mean-field scaling for fermions initially confined in a volume of order one forces the Planck constant to scale proportionally to $N^{-\frac{1}{d}}$. In other words, the mean-field limit for fermions is coupled with the semiclassical limit. For this reason, we will fix from now on $\hbar = N^{-\frac{1}{d}}$.

Furthermore, we observe that the Hartree-Fock equation is still N dependent. It is therefore natural to investigate its limit as N goes to infinity, i.e. to study its semiclassical approximation as $\hbar$ goes to zero. Our main result (based on [4]) shows that in fact the solution of the Hartree-Fock equation (2) can be approximated by a solution of the Vlasov-Poisson equation (1) in strong topology under very general regularity assumptions on the initial datum of the Vlasov-Poisson system. Moreover, the rate of convergence is given explicitly. This improves on previous results [5] (where the convergence is in weak topology without rate of convergence),

[3] (where a rate of convergence is provided in weak-topology) and [9] (where a strong convergence with explicit rate is proven only for a very special class of initial data).

Before writing the precise theorem, we recall the notion of Wigner transform and Weyl quantization. Let ω be a one particle operator on $L^2(\mathbb{R}^d)$ with $\mathrm{tr}\,\omega = N$. We define the Wigner transform of ω as

$$f(x,v) = \left(\frac{\hbar}{2\pi}\right)^d \int \omega\left(x + \frac{\hbar y}{2}, x - \frac{\hbar y}{2}\right) e^{-iv\cdot y}\, dy\,.$$

Conversely, given a function f on the phase space $\mathbb{R}^d \times \mathbb{R}^d$, we define its Weyl quantization by

$$\omega_f(x,y) = N \int f\left(\frac{x+y}{2}, v\right) e^{iv\cdot\frac{(x-y)}{\hbar}}\, dv.$$

Define furthermore the Sobolev spaces

$$\mathcal{W}^{k,p}_m(\mathbb{R}^d \times \mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d \times \mathbb{R}^d) \mid \sum_{j=0}^{k} \|(1 + x^2 + v^2)^{\frac{m}{2}} \nabla^j f\|_{L^p} < \infty\}\,,$$

$$H^k_m(\mathbb{R}^d \times \mathbb{R}^d) = \mathcal{W}^{k;2}_m(\mathbb{R}^d \times \mathbb{R}^d)\,,$$

then our main result reads

Theorem 1. *Let f be a solution to the Vlasov-Poisson system* (1) *and $\omega_{N,t}$ a solution to the Hartree-Fock Eq.* (2) *with initial conditions*

$$f_0 \in \mathcal{W}^{k+1,\infty}_m(\mathbb{R}^d \times \mathbb{R}^d) \cap H^{k+1}_k(\mathbb{R}^d \times \mathbb{R}^d)\,,$$
$$\mathrm{tr}\,\omega_{N,0} = N\,, \quad \mathrm{tr}\,(-\hbar^2 \Delta\omega_{N,0}) \le CN\,,$$

for $m > d$ and $k > m + 6$.
Then, there exists $\lambda(t) \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R}_+)$ and $C(t) \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R}_+)$ depending only on the initial conditions on the solution of the Vlasov-Poisson Eq. (1) *such that*

$$\mathrm{tr}\,|\omega_{N,t} - \omega_f| \le (\mathrm{tr}\,|\omega_{N,0} - \omega_{f,0}| + C(t)\, N\, \hbar^1)\, e^{\lambda(t)}\,,$$

where ω_f and $\omega_{f,0}$ denote respectively the Weyl quantization of f and f_0.

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A new Lyapunov functional for the (spatially homogeneous) Landau equation with Coulomb potential

Laurent Desvillettes
(joint work with Ling-Bing He and Jin-Cheng Jiang)

Landau collision operator (Landau, 1938) for charged particles writes

$$Q(f,f)(v) = \nabla \cdot \left\{ \int_{\mathbb{R}^3} |v-w|^{-1}\, \Pi(v-w) \Big(f(w)\, \nabla f(v) - f(v)\, \nabla f(w) \Big)\, dw \right\},$$

where

$$\Pi_{ij}(z) := \delta_{ij} - \frac{z_i z_j}{|z|^2}$$

is the i,j-component of the orthogonal projection Π on $z^\perp := \{y \,/\, y \cdot z = 0\}$.

The corresponding equation (spatially homogeneous Landau equation with Coulomb potential) writes (for $f := f(t,v) \geq 0$):

$$\partial_t f(t,v) = Q(f,f)(t,v), \qquad f(0,v) = f_{in}(v).$$

For the theory of existence of solutions to this equation, we refer to [4], [6], [8], [2], and the references therein.

The Landau operator can also be viewed as a parabolic operator:

$$Q(f,f) = (a * f) \,:\, \nabla\nabla f + 8\pi\, f^2,$$

where a is a matrix-valued function whose components are given by

$$a_{ij}(z) = \Pi_{ij}(z)\, |z|^{-1},$$

and the associated equation can be compared to other parabolic equations, such as the nonlinear heat equations, the quadratic reversible reaction-diffusion system

$$\partial_t u_k - d_k\, \Delta u_k = (-1)^k\, (u_1\, u_3 - u_2\, u_4), \qquad k = 1,..,4,$$

and the 3D-incompressible Navier-Stokes equation

$$\partial_t u + (u \cdot \nabla) u + \nabla p = \Delta u, \qquad \nabla \cdot u = 0.$$

A recent result analogous to the estimate by Caffarelli, Kohn, Nirenberg (cf. [1]) of the Hausdorff dimension of the set of singular times for the solutions of the 3D-incompressible Navier-Stokes equation was obtained in [5]:

Theorem (Golse, Gualdani, Imbert, Vasseur): For initial data with finite entropy and third moment, the set of singular (positive) times for (suitable) solutions to the Landau equation (with Coulomb potential) is of Hausdorff dimension $\leq 1/2$.

Other classical results due to Leray (cf. [7]) on the 3D-incompressible Navier-Stokes equation include the following statements:

- If $||u||_{L^2}\,||u||_{\dot H^1}$ is small enough, then there is a unique strong solution to the 3D-incompressible Navier-Stokes equation,
- For all initial data with finite energy, the solutions to the 3D-incompressible Navier-Stokes equation become strong after some time $T > 0$.

We now present our main result (cf. [3]), whose parts ii) and iii) can be seen as analogues of the results by Leray described above:

Theorem: Let $f_0 \in L\log L(\mathbb{R}^3)\cap L^1_{55}(\mathbb{R}^3)\cap \dot H^1(\mathbb{R}^3)$ be a nonnegative normalized initial datum. Then there exist (explicitly computable) constants $B, C > 0$, $k_0 > 7/2$, $k > 0$ (depending only on K satisfying $\|f_0\|_{L^1_{55}(\mathbb{R}^3)} + \|f_0\,|\ln f_0|\|_{L^1(\mathbb{R}^3)} \leq K$) such that the three following statements hold:

(i)(Monotonicity of a functional). We denote by $f := f(t, v)$ a smooth and quickly decaying nonnegative solution to Landau equation (with Coulomb potential) with initial datum f_0.

We define $h := f - \mu$, where $\mu := (2\pi)^{-3/2}\exp(-|v|^2/2)$ is the centered reduced Maxwellian and $H := \int \left[f\,\ln\left(\frac{f}{\mu}\right) - f + \mu \right]$ the relative entropy.

Then the following *a priori* estimate holds:

$$\frac{d}{dt}\left[H(t) - \frac{5}{2}\left(\|h(t)\|^2_{\dot H^1} + B\,(1+t)^{-k_0+1} \right)^{-\frac{2}{5}} \right] + C\,(1+t)^k \leq 0.$$

(ii)(Global regularity for initial data below threshold). If moreover

$$H(0)\,\left(\|h(0)\|^2_{\dot H^1} + B\right)^{\frac{2}{5}} \leq \frac{5}{2},$$

then Landau equation (with Coulomb potential) admits a (unique) global and strong (that is, lying in $L^\infty(\mathbb{R}_+; H^1(\mathbb{R}^3))$) nonnegative solution satisfying that

$$\forall t > 0, \qquad \|h(t)\|_{\dot H^1}\left(H(t) + \frac{C}{k+1}\left[(1+t)^{1+k} - 1 \right] \right)^{\frac{5}{4}} \leq \left(\frac{2}{5}\right)^{-\frac{5}{4}}.$$

(iii)(No blowup after a finite time). If finally

$$H(0)\,\left(\|h(0)\|^2_{\dot H^1} + B\right)^{\frac{2}{5}} > \frac{5}{2},$$

we denote

$$T^* := \left(\frac{1+k}{C}\left[H(0) - \frac{5}{2}\,\left[\|h(0)\|^2_{\dot H^1} + B\right]^{-2/5} \right] + 1 \right)^{\frac{1}{k+1}} - 1.$$

Then one can construct a global weak (or H-) nonnegative solution of Landau equation with Coulomb potential such that for $t > T^*$, it becomes global and strong (that is, it lies in $L^\infty_{loc}(]T^*, \infty[; H^1(\mathbb{R}^3))$).

We also present an extra result, relative to a the behavior of a solution at a possible blowup time:

Proposition: Let $f := f(t, v)$ be a nonnegative solution of the Landau equation with Coulomb potential, corresponding to initial data f_0 satisfying the assumptions of the previous theorem.

We suppose that $f \in L^\infty_{loc}([0, \bar{T}[; H^1(\mathbb{R}^3))$ and that $\|f(t)\|_{\dot{H}^1(\mathbb{R}^3)}$ blows up at time $\bar{T}$.

Then for $\bar{T} - t \ll 1$ and some explicitly computable constants c, C_0 (depending only on K such that $\|f_0 \, |\ln f_0| \, \|_{L^1(\mathbb{R}^3)} + \|f_0\|_{L^1_{55}(\mathbb{R}^3)} \le K$ and $\bar{T}$),

$$\|h(t)\|_{\dot{H}^1} \ge C_0 \, (H(t) - \bar{H})^{-\frac{5}{4}}, \quad \inf_{s \in [t, \bar{T}]} \|h(s)\|_{\dot{H}^1} \le c \, (\bar{T} - t)^{-\frac{30}{7}} \exp\left\{ c \, [\bar{T} - t]^{-\frac{225}{7}} \right\},$$

where $\bar{H} := \lim_{t \to \bar{T}^-} H(t)$.

We end up with a few open questions related to the results described above:

- Is it possible to find a space which would be the equivalent of $H^{1/2}(\mathbb{R}^3)$ for the 3D incompressible Navier-Stokes equation?
- Lorentz spaces are used in the proof: is the use of those spaces unavoidable (this may have to do with the issue of possible criticality of the Coulomb case)?
- Is it possible to obtain an algebraic estimate for the solution at a possible blowup time?

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Quantitative Fluid Approximation in Transport Theory: A Unified Approach

Clément Mouhot
(joint work with Émeric Bouin)

The study of *transport processes*, i.e. linear collisional kinetic equations, has its theoretical roots in the mean-free path argument of Maxwell and the kinetic theory of gases of Maxwell and Boltzmann. A linear version of the Maxwell-Boltzmann equation can be written for the movement of a tagged particle within a rarefied gas, but the study of such transport processes was given a crucial new impetus in the twentieth century with (1) the *radiative transfer theory* where the kinetic distribution models the flux of photons that are transported in the plasma making up the internal layers of the sun, (2) the *nuclear reactor theory* where the kinetic distribution models the neutrons transported and scattered inside the reactor, whose flux is used to initiate and maintain the chain reaction, (3) the *semi-conductor theory* where the kinetic distribution models the flow of charge carriers in semiconductors, i.e. the evolution of the position-momentum distribution of negatively charged conduction electrons or of positively charged holes, which are responsible for the current flow in semiconductor crystals. The main mathematical object of study in *transport theory* is the linear equation

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L} f$$

on the time-dependent density of particles $f = f(t,x,v) \geq 0$ over $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, for $t \geq 0$. The left hand side accounts for free motion and the right hand side accounts for the interaction with a background, for instance scatterers, with an operator $\mathcal{L}$ that acts only on the kinetic variable v. In nuclear reactor, radiative transfer and semi-conductor theories it is common to consider *scattering operators*, sometimes also called *linear Boltzmann operators*

$$\mathcal{L} f(v) = \int_{\mathbb{R}^d} b(v,v') \Big[f(v')\, \mathcal{M}(v) - f(v) \mathcal{M}(v') \Big] \, \mathrm{d}v'$$

for some *collisional kernel* b and some *equilibrium distribution* $\mathcal{M}$. In astrophysics, one also considers *Fokker-Planck operators*

$$\mathcal{L} f := \nabla_v \cdot \left(\mathcal{M} \nabla_v \left(\frac{f}{\mathcal{M}} \right) \right).$$

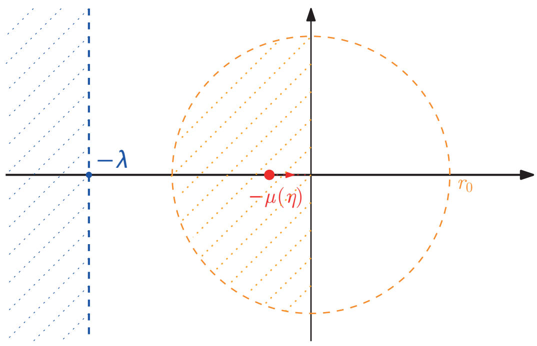
As a simplified model of long-range collisional interactions in a gas of charged particles, we also consider *Lévy-Fokker-Planck operators* for $s \in (\frac{1}{2}, 1)$ and $\alpha > s$

$$\mathcal{L}(f) = \Delta_v^s f + \nabla_v \cdot (U\, f)$$

with $U(v) = U(|v|)$ radially symmetric so that $\Delta_v^s \mathcal{M} + \nabla_v \cdot (U\, \mathcal{M}) = 0$ (the fractional Laplacian is defined by Fourier-transforming the symbol $-|\xi|^{2s}$). The transport equation is too intricate for many applications. When the relevant time and space scales of observation are much larger than the mean free time and mean free path, it is thus natural to search for a simplified *diffusive* regime, i.e. a limit of f_ε in the rescaled equation

$$\theta(\varepsilon)\partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = \mathcal{L} f_\varepsilon$$

as $\varepsilon \to 0$ and for an appropriate time-scale $\theta(\varepsilon)$. We propose a unified method for such limit in the whole space. The limit is of *fractional* diffusion type for heavy tail equilibria with slow enough decay, and of diffusive type otherwise. The proof is constructive and the diffusion matrix is obtained. A *generalised* weighted mass condition is assumed on the equilibrium $\mathcal{M}$ which allows for infinite mass. The method combines energy estimates and quantitative spectral methods to construct a 'fluid mode' for small x-frequencies (see figure), motivated by the recent papers [4, 3] and providing a new conceptual connexion with the seminal paper [1] on hydrodynamic limit. The method is applied to scattering models (without



The fluid mode $\mu(\eta) \to 0$ as $\eta = \varepsilon|\xi| \to 0$.

assuming detailed balance conditions), Fokker-Planck operators and Lévy-Fokker-Planck operators. It proves a series of new results, including the fractional diffusive limit for Fokker-Planck operators in any dimension, for which the characterization of the diffusion coefficient was not known (see [2]), and for Lévy-Fokker-Planck operators with general equilibria. It also unifies and generalises the results of ten previous papers with a quantitative method; the estimates on the fluid approximation error seem novel in these cases. The abstract method is 15 pages long, and the application to each equation is about 5 pages long.

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Propagation of chaos and corrections to mean-field for interacting particles

Mitia Duerinckx
(joint work with L. Saint-Raymond and R. Winter)

We consider a system of N classical particles in the torus $\mathbb{T}^d$, interacting via a smooth potential V, in the mean-field regime: trajectories are given by Newton's equations in the phase-space $\mathbb{D} = \mathbb{T}^d \times \mathbb{R}^d$,

$$(1) \qquad \partial_t x_j = v_j, \qquad \partial_t v_j = -\tfrac{1}{N} \textstyle\sum_{l:l\neq j}^N \nabla V(x_j - x_l), \qquad \text{for } 1 \leq j \leq N.$$

For a statistical description, we consider a random ensemble of trajectories: for simplicity, we choose initial data $\{(x_j^\circ, v_j^\circ)\}_{1\leq j\leq N}$ to be independent and identically distributed (iid) with some smooth law F° on $\mathbb{D}$. In terms of the probability density F_N for the ensemble of particles on the N-particle phase-space $\mathbb{D}^N$, Newton's equations (1) are equivalent to the Liouville equation

$$(2) \qquad \partial_t F_N + \textstyle\sum_{j=1}^N v_j \cdot \nabla_{x_j} F_N = \tfrac{1}{N} \textstyle\sum_{j\neq l} \nabla V(x_j - x_l) \cdot \nabla_{v_j} F_N,$$

with chaotic initial data $F_N|_{t=0} = (F^\circ)^{\otimes N}$. Looking for a simplified description of the system, we define the m-particle probability density as the mth marginal $F_N^m(z_1, \ldots, z_m) = \int_{\mathbb{D}^{N-m}} F_N(z_1, \ldots, z_N)\, dz_{m+1} \ldots dz_N$, with the notation $z_j = (x_j, v_j)$. For a large number $N \gg 1$ of particles, the 1-particle density F_N^1 remains close to the solution F of the Vlasov equation

$$(3) \qquad \partial_t F + v \cdot \nabla_x F = (\nabla V * F) \cdot \nabla_v F,$$

with $F|_{t=0} = F^\circ$. We refer to [1] for a review of this well-travelled mean-field result. Formally, starting from the BBGKY hierarchy, the Vlasov equation is obtained by neglecting 2-particle correlations, thus replacing F_N^2 by $(F_N^1)^{\otimes 2}$ in the equation for F_N^1. A rigorous proof was obtained in the 1970s following Klimontovich's ideas: starting from the representation $F_N^1 = \mathbb{E}[\mu_N]$ in terms of the empirical measure

$$\mu_N = \tfrac{1}{N} \textstyle\sum_{j=1}^N \delta_{(x_j, v_j)},$$

one notices that μ_N is a distributional solution of the Vlasov equation (3) and that initially $\mu_N|_{t=0}$ converges weakly to F° a.s., hence the stability of the Vlasov equation in weak topology ensures the a.s. weak convergence $\mu_N \rightharpoonup F$. This further entails $F_N^m \rightharpoonup F^{\otimes m}$ for all $m \geq 1$, which is known as propagation of chaos.

Corrections to this mean-field theory are driven by the 2-particle correlation function $G_N^2 = F_N^2 - (F_N^1)^{\otimes 2}$ and are formally obtained by only neglecting 3-particle correlations in the BBGKY hierarchy. A rigorous proof requires to show that the 2-particle correlation function is of order $G_N^2 = O(\frac{1}{N})$, while 3-particle correlations are of higher order $G_N^3 = O(\frac{1}{N^2})$. Such a refined version of propagation of chaos is provided by the following main result.

Theorem A (see [2, Theorem 1]).
For $m \geq 0$, the $(m+1)$-particle correlation function G_N^{m+1} satisfies for all $t \geq 0$,

$$\|G_N^{m+1}(t)\|_{W^{-2m,1}(\mathbb{D}^{m+1})} \leq N^{-m}\, C_m e^{C_m t},$$

where C_m only depends on d, m, V, F°.

Due to the loss of derivatives (cf. ∇_v in the right-hand side of (2)), this result cannot be deduced from the BBGKY hierarchy — in stark contrast with e.g. the quantum mean-field setting in [3]. Instead, as for the mean-field result, we develop an approach à la Klimontovich based on the empirical measure μ_N. First, we note that G_N^2 is equivalent to the variance of μ_N; the following representation formula holds more generally for all $m \geq 1$ and $\phi \in C_b(\mathbb{D})$,

$$\int_{\mathbb{D}^m} \phi^{\otimes m} G_N^m = \kappa_m[\int_{\mathbb{D}} \phi\, d\mu_N] + \text{lower-order terms},$$

where $\kappa_m[\cdot]$ stands for the mth cumulant. Next, to estimate the variance of μ_N (and its higher-order cumulants), we appeal to discrete stochastic calculus techniques with respect to iid data. For a random variable Y, we define its Glauber derivative with respect to the initial data of the jth particle as $D_j Y = Y - \mathbb{E}_j[Y]$, where $\mathbb{E}_j[\cdot]$ stands for the expectation with respect to (x_j°, v_j°) only. In these terms, a variance estimate is given by the following Efron–Stein inequality [4],

$$\mathrm{Var}[Y] \leq \sum_{j=1}^N \mathbb{E}[(D_j Y)^2].$$

Noting the similarity to Malliavin calculus and arguing as in [5], we prove corresponding cumulant estimates in form of higher-order Poincaré inequalities. We are then reduced to evaluating the multiple Glauber derivatives of μ_N with respect to iid data. Sensitivity estimates for trajectories are easily performed since the mean-field regime corresponds to weak interactions: we find for instance

$$\max_{j \neq l} |D_l(x_j^t, v_j^t)| \leq N^{-1}\, Ce^{Ct}.$$

Combining these different ingredients yields the conclusion of Theorem A.

As a consequence, the above correlation estimates can be used to rigorously truncate the BBGKY hierarchy to any order, and justify the so-called Bogolyubov corrections to the mean-field Vlasov limit. Alternatively, cumulant estimates also yield an optimal quantitative central limit theorem for μ_N, thus improving on the well-established qualitative result in [6]. We refer to [2, Sections 5–7] for details.

In a spatially homogeneous system $F^\circ(x,v) = f^\circ(v)$, the mean-field force vanishes and the Vlasov solution remains constant $F \equiv f^\circ$. The evolution of the 1-particle density is then described to leading order by the Bogolyubov correction, which takes on the following guise for the velocity density $f_N^1(v) = \int_{\mathbb{T}^d} F_N^1(x,v)\, dx$,

$$\begin{cases} \partial_t f_N^1 \sim \frac{1}{N} \int_{\mathbb{T}^d} \int_{\mathbb{D}} \nabla V(x - x_*) \cdot \nabla_v (N G_N^2)(x, v, x_*, v_*)\, dx_* dv_* dx, \\ \partial_t (N G_N^2) + i L_{f_N^1} (N G_N^2) \sim \nabla V(x_1 - x_2) \cdot (\nabla_{v_1} - \nabla_{v_2})(f_N^1 \otimes f_N^1), \end{cases}$$

where $iL_{f_N^1}$ stands for the linearized Vlasov operator at f_N^1. The effect of particle correlations takes form of a non-Markovian collision process. However, the equations display a timescale separation: f_N^1 evolves on the slow timescale $t = O(N)$

while NG_N^2 evolves on the mean-field timescale $t = O(1)$. In view of linear Landau damping in form of weak relaxation for $iL_{f_N^1}$, we may thus formally replace NG_N^2 in the first equation by its long-time limit as obtained from the second. After tedious computations, as predicted by Guernsey, Balescu, and Lenard independently in 1960 (e.g. [7, Appendix A]), this yields the so-called Lenard–Balescu equation

$$\partial_t f_N^1 \sim \tfrac{1}{N}\,\mathrm{LB}(f_N^1), \qquad \mathrm{LB}(f) = \nabla\cdot\textstyle\int_{\mathbb{R}^d} B(v, v-v_*;\nabla f)\,(f_*\nabla f - f\nabla_* f_*)\,dv_*,$$

where the collision kernel B brings a strong nonlinearity and is explicit in terms of the potential V. This equation is viewed as a correction to Landau's equation, further taking into account collective screening effects. It satisfies an H-theorem and formally describes relaxation to Maxwellian equilibrium on the slow timescale $t = O(N)$. Due to dynamical screening, even local well-posedness is a reputedly difficult open problem in the Coulomb setting. For a smooth potential V, a work in progress with R. Winter proves global well-posedness close to equilibrium.

Justifying physicists' calculations for the relaxation of NG_N^2, we obtain the following result with L. Saint-Raymond [8, 2]. Note that the exponential time growth in Theorem A requires to restrict to the intermediate timescale $1 \ll t \ll \log N$: although missing the kinetic timescale $t = O(N)$, this constitutes the first rigorous result in this direction starting from particle system (1).

Theorem B (see [2, Corollary 4]).
Given F° spatially homogeneous, compactly supported, and linearly Vlasov-stable, and given V smooth and small enough, there holds for $1 \ll t_N \ll \log N$,

$$N\partial_t f_N^1|_{t=t_N\tau} \;\sim\; \mathrm{LB}(f^\circ), \qquad \textit{in } \mathcal{D}'_{\tau,v}(\mathbb{R}^+\times\mathbb{R}^d).$$

In order to reach the kinetic timescale $t = O(N)$, the correlation estimates of Theorem A are no longer applicable: propagation of chaos needs to be complemented with some decorrelation mechanism. In [8], with L. Saint-Raymond, we consider a linearized setting: time-uniform estimates on linear correlation functions then follow from an orthogonality argument as in [9]. Yet, due to resonant effects that are reminiscent of plasma echoes, these estimates only allow to extend Theorem B to $1 \ll t_N \ll N^{\frac{1}{4}}$, and all improvements remain open questions.

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Rigorous derivation of cross-diffusion equations from interacting particle systems

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(joint work with L. Chen, E. Daus, and A. Holzinger)

Cross-diffusion models describe the evolution of multicomponent systems arising in, for instance, cell biology, gas mixture theory, and population dynamics. Their derivation from microscopic models is important to determine the range of validity of the diffusive equations and to understand their possible formal gradient-flow or entropy structure. We review in this note two many-particle limits from stochastic interacting particle systems, based on the works [2, 3].

The aim is the rigorous derivation of quasilinear parabolic systems of the form

$$(1)\qquad \partial_t u_i = \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j\right) + \operatorname{div}(u_i\nabla U_i), \quad u_i(0) = u_{0,i} \quad \text{in } \mathbb{R}^d,\ t > 0,$$

where $i = 1, \ldots, n$ is the species index, $u = (u_1, \ldots, u_n)$ is the vector of (particle) densities, $A_{ij}(u)$ are the diffusion coefficients, and $U_i(x)$ are environmental potentials, from stochastic interacting particle systems of the type

$$(2)\qquad \mathrm{d}X_{k,i} = -a^{N,\eta}(X)\mathrm{d}t + b^{N,\eta}(X)\mathrm{d}W_{k,i}(t), \quad X_{k,i}(0) = \xi_{k,i}, \quad i = 1, \ldots, n,$$

where $k = 1, \ldots, N$ is the particle number, $X = (X_{k,i})$ is the vector of random positions of the particles, the parameter $\eta > 0$ models the interaction radius, $(W_{k,i})$ are d-dimensional Brownian motions, and $\xi_{1,i}, \ldots, \xi_{N,i}$ are independent and identically distributed random variables. We wish to prove the limit $N \to \infty$ and $\eta \to 0$ (in a certain sense) in (2) leading to (1). For this, we consider two examples for $a^{N,\eta}(X)$ and $b^{N,\eta}(X)$.

1. First model: interactions in the drift term

The first model is given by

$$(3)\qquad \mathrm{d}X_{k,i}^{N,\eta}(t) = -\sum_{j=1}^n \frac{1}{N}\sum_{\ell=1}^N \nabla B_{ij}^\eta\big(X_{k,i}^{N,\eta} - X_{\ell,j}^{N,\eta}\big)\mathrm{d}t + \sqrt{2\sigma_i}\mathrm{d}W_{k,i}(t),$$

where $\sigma_i > 0$ are constant diffusion coefficients and the smooth interaction potentials B_{ij} satisfy $B_{ij}^\eta(x) = \eta^{-d}B_{ij}(|x|/\eta)$ for $x \in \mathbb{R}^d$ with $\int_{\mathbb{R}^d} B_{ij}(|x|)dx =: a_{ij}$ and $B_{ij}^\eta \to a_{ij}\delta_0$ in the sense of distributions as $\eta \to 0$.

The limit $N \to \infty$, $\eta \to 0$ has to be understood in the following sense (see [6, 8]). For fixed $\eta > 0$, system (3) is approximated for $N \to \infty$ by the intermediate system

$$\mathrm{d}\bar{X}^\eta_{k,i}(t) = -\sum_{j=1}^n (\nabla B^\eta_{ij} * u_{\eta,j})(\bar{X}^\eta_{k,i}(t), t)\mathrm{d}t + \sqrt{2\sigma_i}\mathrm{d}W_{k,i}(t), \tag{4}$$

where $u_{\eta,j} = u_{\eta,j}(x,t)$ satisfies the nonlocal cross-diffusion system

$$\partial_t u_{\eta,i} = \sigma_i \Delta u_{\eta,i} + \operatorname{div}\left(\sum_{j=1}^n u_{\eta,i}\nabla B^\eta_{ij} * u_{\eta,j}\right) \quad \text{in } \mathbb{R}^d,\ t>0.$$

System (4) depends on the particle index $k = 1,\ldots,N$ only via the initial data $\bar{X}^k_{\eta,i}(0) = \xi_{k,i}$, i.e., $\bar{X}^\eta_{k,i}(t)$ are N independent copies of the solution to (4). Since $\nabla B^\eta_{ij} * u_{\eta,j} \to a_{ij}\nabla u_j$ in L^2, the limit $\eta \to 0$ in (4) leads to the limiting system

$$\mathrm{d}\widehat{X}_{k,i}(t) = -\sum_{j=1}^n a_{ij}\nabla u_j(\widehat{X}_{k,i}(t))\mathrm{d}t + \sqrt{2\sigma_i}\mathrm{d}W_{k,i}(t),$$

where the law of $\widehat{X}_{k,i}$, $u_i = \mathrm{law}(\widehat{X}_{k,i})$, is a solution to

$$\partial_t u_i = \sigma_i \Delta u_i + \operatorname{div}\left(\sum_{j=1}^n a_{ij} u_i \nabla u_j\right) \quad \text{in } \mathbb{R}^d,\ t>0, \tag{5}$$

and $u_i(0) = u_{i,0}$ is the common probability density function of $\xi_{k,i}$. This model describes segregation effects in multi-species populations [1, 5].

The main result of [2] is the proof of the estimate

$$\sup_{k=1,\ldots,N} \mathrm{E}\left(\sum_{i=1}^n \sup_{0<s<t}\left|X^{N,\eta}_{k,i}(s) - \widehat{X}_{k,i}(s)\right|\right) \le C(t)\eta,$$

under the condition that $\eta^{-(2d+4)} \le \varepsilon \log N$, where $\varepsilon > 0$ is sufficiently small. This estimate implies propagation of chaos [9]. The idea of the proof is to estimate the differences $|X^{N,\eta}_{k,i} - \bar{X}^\eta_{k,i}|$ and $|\bar{X}^\eta_{k,i} - \widehat{X}_{k,i}|$. The first difference is of order $N^{-1}\eta^{-d-2}$, coming from the properties of ∇B^η_{ij}, while the second difference is of order η, which comes from estimating $|\nabla B^\eta_{ij} * \nabla u_j - a_{ij}\nabla u_j|$ in terms of $\eta|D^2 u_j|$.

2. Second model: interactions in the diffusion term

The second model is given by

$$\mathrm{d}X^{N,\eta}_{k,i} = -\nabla U_i(X^{N,\eta}_{k,i})\mathrm{d}t + \left(2\sigma_i + 2\sum_{j=1}^n \frac{1}{N}\sum_{\ell=1}^N B^\eta_{ij}(X^{N,\eta}_{k,i} - X^{N,\eta}_{\ell,j})\right)^{1/2}\mathrm{d}W_{k,i}(t), \tag{6}$$

where we exclude $(\ell, j) \neq (k,i)$ in the sum over ℓ. The idea of the many-particle limit is as before. We first pass to the limit $N \to \infty$, leading to an intermediate nonlocal system, and then perform the limit $\eta \to 0$, giving

$$\mathrm{d}\widehat{X}_{k,i} = -\nabla U_i(\widehat{X}_{k,i})\mathrm{d}t + \left(2\sigma_i + 2\sum_{j=1}^n a_{ij}u_j(\widehat{X}_{k,i})\right)^{1/2} \mathrm{d}W_{k,i}(t),$$

and the function $u_i = \mathrm{law}(\widehat{X}_{k,i})$ satisfies

$$\partial_t u_i = \mathrm{div}(u_i \nabla U_i) + \Delta\left(\sigma_i u_i + u_i \sum_{j=1}^n a_{ij}u_j\right), \quad u_i(0) = u_{i,0}, \tag{7}$$

where $i = 1, \ldots, n$. Model (7) corresponds to the population system suggested by Shigesada, Kawasaki, and Teramoto [7]. It distinguishes from the first model (5) by the additional diffusion $\mathrm{div}(\sum_{j=1}^n a_{ij}u_j\nabla u_i)$. It is shown in [3] that if $U_i(x) = -\frac{1}{2}|x|^2$ and $\eta^{-(2d+2)} \le \varepsilon \log N$ for some sufficiently small $\varepsilon > 0$ then

$$\sup_{k=1,\ldots,N} \mathrm{E}\left(\sum_{i=1}^n \sup_{0<s<t} \left|X_{k,i}^{N,\eta}(s) - \widehat{X}_{k,i}(s)\right|\right) \le C(t)\eta.$$

This result can be extended in various directions. First, we may choose general smooth potentials U_i such that ∇U_i is globally Lipschitz continuous, D^2U_i is negative semidefinite, and D^kU_i is sufficiently small for $k \ge 3$. Second, the diffusion coefficient in (6) can be replaced by

$$\left(2\sigma_i + 2\sum_{j=1}^n f_\eta\left(\frac{1}{N}\sum_{\ell=1}^N B_{ij}^\eta(X_{k,i}^{N,\eta} - X_{\ell,j}^{N,\eta})\right)\right)^{1/2},$$

where f_η is a globally Lipschitz continuous approximation of a function f that may be only locally Lipschitz continuous (for instance, $f(z) = z^p$ for $p > 1$). Then the sum $\sum_{j=1}^n a_{ij}u_j$ in (7) has to be replaced by $\sum_{j=1}^n f(a_{ij}u_j)$. This generalization provides a derivation of the porous-medium equation from interacting particle systems. Indeed, let $n = 1$, $\sigma_1 = 0$, $U_1 = 0$, and $a_{11} = 1$. Then (7) can be written as $\partial_t u = \Delta(uf(u))$. We remark that another derivation was published in [4] assuming a double-convolution potential in the drift term.

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Short- and long-time behavior in (hypo)coercive ODE-systems and Fokker-Planck equations

Anton Arnold
(joint work with Franz Achleitner, Eric Carlen; Christian Schmeiser, Beatrice Signorello)

Abstract

We are concerned with the short- and large-time behavior of Fokker-Planck equations with linear drift, i.e. $\partial_t f = \operatorname{div}(\mathbf{D}\nabla_x f + \mathbf{C}xf)$. A coordinate transformation can normalize these equations such that the diffusion and drift matrices are linked as $\mathbf{D} = \mathbf{C}_s$, the symmetric part of $\mathbf{C}$.

The first main result of this talk is the connection between normalized Fokker-Planck equations and their drift-ODE $\dot{x} = -\mathbf{C}x$: Their L^2-propagator norms actually coincide. This implies that optimal decay estimates on the drift-ODE (w.r.t. both the maximum exponential decay rate and the minimum multiplicative constant) carry over to sharp exponential decay estimates of the Fokker-Planck solution towards the steady state.

Secondly, we define an "index of hypocoercivity", both for ODEs and Fokker-Planck equations that describes the interplay between the dissipative and conservative part of their generator. This index characterizes the polynomial decay of the propagator norm for short time.

Hypocoercive estimates

The goal of this presentation is to analyze the short and long-time behavior of linear evolution equations $\frac{d}{dt}f = -L\,f,\ t \geq 0$ with a constant-in-t operator L, such that $-L$ is dissipative and L has a unique (normalized) steady state: $L\,f_\infty = 0$.

Optimal long-time decay estimates: We aim at deriving exponential decay estimates of the form

$$\|f(t) - f_\infty\| \leq ce^{-\mu t}\|f(0) - f_\infty\|\,, \quad t \geq 0,$$

possibly with the sharp (= maximum) rate $\mu > 0$ and the minimal multiplicative constant $c \geq 1$ (uniform for all initial conditions $f(0)$). To this end we shall consider specially constructed Lyapunov functionals.

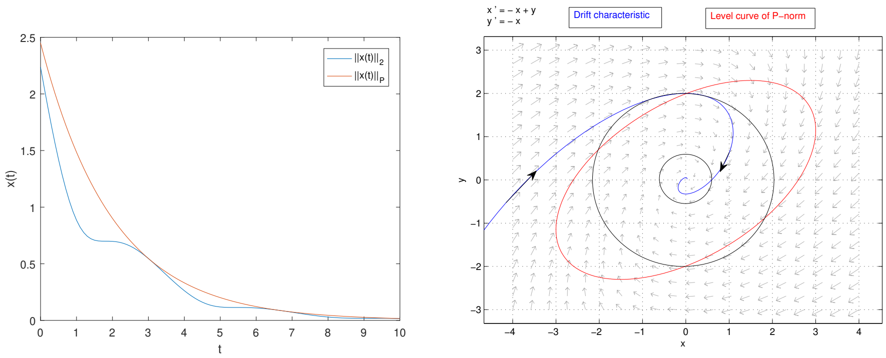

Figure 1. Time decay of a solution to a hypocoercive ODE. left: $\|x(t)\|$ in the Euclidean norm (blue) and in the **P**-norm (red); right: solution trajectory in $\mathbb{R}^2$ (blue), level curves of the Euclidean norm (black) and the **P**-norm (red).

To illustrate the situation we first consider the simple ODE

$$\dot{x} = -\mathbf{C}x, \quad \text{with } \mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since the symmetric part of **C** is not coercive, the Euclidean norm of solutions does not decay uniformly, but rather in waves, having occasionally horizontal tangents. In the phase plane this can be seen from the fact that solution trajectories are occasionally tangential to level curves of the Euclidean norm, see Figure 1. But when introducing a "distorted" vector norm $\|x\|_{\mathbf{P}} := \sqrt{x^T \mathbf{P}\, x}$, with an appropriate matrix $\mathbf{P} > 0$, $\|x(t)\|_{\mathbf{P}}$ decays exponentially.

In the talk we are mainly interested in the long-time behavior of (possibly degenerate, i.e. *hypocoercive*, see [6]) Fokker-Planck equations with linear drift, i.e.

$$\partial_t f = \operatorname{div}(\mathbf{D}\nabla_x f + \mathbf{C}xf), \quad x \in \mathbb{R}^d, \quad t \geq 0, \tag{1}$$

with $\mathbf{D} \geq 0$, **C** positive stable. In particular, we shall reduce it to the long-time behavior of its drift ODE, i.e. $\dot{x} = -\mathbf{C}x$.

Short-time decay estimate: When decomposing the matrix $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2$ into its anti-Hermitian and, respectively, its non-negative Hermitian part, we define the *hypocoercivity index* of **C** a the smallest integer

$$m_{HC} \in \mathbb{N}_0, \quad \text{such that} \quad \sum_{j=0}^{m_{HC}} \mathbf{C}_1^j \mathbf{C}_2 (\mathbf{C}_1^*)^j > 0\,.$$

This index m_{HC} describes the structural complexity of the ODE $\dot{x} = -\mathbf{C}x$ and, in particular, the interplay between its (anti-)Hermitian parts. An analogous definition can be made for (1), using the matrices **C**, **D**.

We shall show that this index characterizes the short-time behavior of both ODEs and Fokker-Planck equations – in the following sense:

$$\|f(t) - f_\infty\| \leq \left[1 - ct^a + \mathcal{O}(t^{a+1})\right] \|f(0) - f_\infty\|, \quad t \to 0+ \tag{2}$$

holds with $a = 2m_{HC} + 1$.

Notation

$f(x,t)$	solution to the Fokker-Planck equation (1)
$f_\infty(x)$	unique (normalized) steady state of the Fokker-Planck equation (1)
$\mathbf{D} \geq 0$	diffusion matrix in the Fokker-Planck equation
$\mathbf{C}$	drift matrix in the Fokker-Planck equation
m_{HC}	hypocoercivity index of the linear ODE $\dot{x} = -\mathbf{C}x$
$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2$	decomposition of the ODE matrix into its anti-Hermitian and Hermitian part

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The Cauchy problem for The Boltzmann Equation Modeling Polyatomic Gases

Irene Martínez Gamba

(joint work with Milana Pavić-Čolić)

This presentation focused on the Boltzmann equation describing a homogeneous flow of a polyatomic gas in three dimensions modeled in $\mathbb{R}^{4+} := \mathbb{R}^3 \times [0, \infty)$ after the addition of a continuous microscopic internal energy variable. In [5], we have established the existence and uniqueness theory under an extended Grad assumption on transition probability rates, that comprises hard potentials for both the relative speed and internal energy with the rate in the interval $(0, 2]$, proportional to integrable angular and partition transition functions. The Cauchy problem's solution is obtained by means of an abstract ODE flow in Banach spaces, for an initial data with finite and strictly positive gas mass and energy, finite momentum, and additionally finite number polynomial moment k_* depending on transition probability rates as well as on the structure of a polyatomic molecule or its internal degrees of freedom. Moreover, we prove that, both, the solution's polynomially and exponentially weighted Banach norms are propagated and generated uniformly in time, respectively.

More precisely, we study the scalar Boltzmann flow for interacting polyatomic gases [2], interchanging binary pairs of pre and post molecular velocities $v \in \mathbb{R}^3$ and internal energies $I \in [0,\infty)$. This collisional model describes the statistical time evolution of probability distribution density $f(t,v,I)$ in the Banach space $\mathcal{C}([0,\infty 0); L^1_k(\mathbb{R}^{4+}))$, of integrable functions in the upper half space $\mathbb{R}^{4+} := \mathbb{R}^3 \times [0,\infty)$ with the Lebesgue weight function $\langle v, I\rangle^k := (1+\frac{|v|^2}{2}+\frac{I}{m})^{k/2}$, where m is the molecular mass. Consequently, $\|f\|_{L^1_k(\mathbb{R}^{4+})}(t)$ are also referred as the $k-$Lebesgue moments associated to the solution of the Boltzmann flow, given by the evolution of colliding pair of molecules (v', I'), $(v'_*, I'_*) \in \mathbb{R}^{4+}$, under the assumption of binary elastic interactions, are linked through the conservation laws of local momentum and total (kinetic + microscopic internal) molecular energy, written in center of mass and relative velocities coordinates,

$$v + v_* = v' + v'_*, \qquad \frac{m}{4}|u|^2 + I + I_* = \frac{m}{4}|u'|^2 + I' + I'_* =: E,$$

whose local conservation equations interchange energy according to the Borgnakke-Larsen procedure [3]. For the parameter $R \in [0,1]$, the local energy

$$\frac{m}{4}|u'|^2 = RE, \quad I' + I'_* = (1-R)E \quad \text{and} \quad I' = r(1-R)E, \qquad I'_* = (1-r)(1-R)E.$$

distributes the energy proportion R of the total energy E into a pure kinetic part RE and a pure internal part $(1-R)E$; and a parameter $r \in [0,1]$ is set to distribute the proportion of total internal energy $(1-R)E$ to each interacting states corresponding to the incoming molecular internal energy pair I', I'_* . In addition, the classical scattering direction associated to the elastic theory, $\sigma \in S^2$, parametrizes pre-collisional relative molecular velocities $u' = |u'|\,\sigma = 2\sqrt{\frac{RE}{m}}\sigma$. This relation reduces to the classical monatomic single species model in the absence of internal energy modes for which $|u'| = |u|$. This Boltzmann type collision operator, written in strong bilinear form, is modeled by the non-local operator acting on a pair of probability density measures $(f,g)(v,I)$ defined by

$$Q(f,g)(v,I) = \iint_{\mathbb{R}^{4+}\times\mathcal{K}} \left({}'f' g_* \left(\frac{I I_*}{'I' I_*}\right)^{\alpha} - f g_* \right) \mathcal{B}\, \varphi_\alpha(r)\psi_\alpha(R)(1-R)R^{1/2}\, dR\, dr\, d\sigma\, dI_*\, dv_*,$$

$\alpha > -1$, with partition functions $\varphi_\alpha(r) = (r(1-r))^\alpha$, $\psi_\alpha(R) = (1-R)^{2\alpha}$. The region of integration is the upper half 4-dimensional space of definition of molecular velocity v and internal energy I, and $\mathcal{K} := [0,1]^2 \times S^2$ a compact manifold embedded in the four dimensional space. The transition probability rate (or collision kernel) $\mathcal{B} = \mathcal{B}(v, v_*, I, I_*, R, r, \sigma)$, such that $\mathcal{B}\,\varphi_\alpha(r)\psi_\alpha(R)(1-R)R^{1/2} \in L^1(\mathcal{K}, dRdrd\sigma)$, i.e. it is integrable on the compact manifold $\mathcal{K}$. In addition, the dependance of $\mathcal{B}$ with respect to the interacting pairs (v, I) and (v_*, I_*) must satisfy sufficient conditions to obtain upper and lower estimates that yield an existence and uniqueness global in time solution that propagates and generates statistical moment of any order, for a sufficiently high $k_* > 1$, depending only on the initial data and the constitutive form of the transition function $\mathcal{B}$ as described in [5], Section 3. This binary collisional form conserves total mass, momentum and energy transfer, as well as dissipates initial entropy.

Thus, the corresponding Cauchy problem consists in solving the initial value problem in a suitable subspace of $\mathcal{C}(0,\infty,L^1_k(\mathbb{R}^{4+})$ to

(1) $\partial_t f(t,v,I)=Q(f,f)(t,v,I), \quad \text{for}\, f(0,v,I)=f_0(v,I), \forall\,(t,v,I)\in\mathbb{R}^+\times\mathbb{R}^3\times\mathbb{R}^+,$

where$f_0(v,I)$ must have at least a finite k_*-moment higher than the energy. No assumption of needed on finiteness of the initial entropy, yet if initially bounded, the entropy will globally bounded as well.

Inspired in a recent review for the classical Boltzmann theory [1], an analytical keypoint consists in showing that the bilinear collisional operator, for polyatomic gas model, generates dissipation, which is manifested in the decay of its k-polynomial moment for $k \geq k_*$, depending not only on the data with initial finite mass and energy, zero momentum and a k-moment of order $k \geq k_*$, with $k_* > 1$, (where the $k = 1$-moment is the macroscopic mass plus the total (kinetic + internal) energy), but also depends on the transition probability rate $\gamma \in (0, 2]$ and the parameter α related to the degrees of freedom for translational as well as rotational and vibrational motion associated to the binary interaction. The choice of these transition probability forms, not only have physical meaning in the framework of extended thermodynamics for macroscopic polyatomic gas models calculated near equilibrium [4], but also enables a rigorous analysis on such transition rates which lead to two fundamental estimates. The first one consists in estimating from above the positive contributions from the k-Lebesgue moments of the collisional form. The second one controls from below the k-Lebesgue moments of the negative contributions from collisional form, in this case referred as the loss operator.

The former is obtained by a detailed the k-Lebesgue bracket of pre-collisional pairs $('v,'I)$ multiplied by $\mathcal{B}\,\varphi_\alpha(r)\psi_\alpha(R)(1-R)R^{1/2}$ and averaged across the compact manifold $\mathcal{K}$. Such result follows from the Compact Manifold Averaging Lemma, proved in [5], Section 4, which essentially calculates a contractive constant μ_k, for $k > 1$, with $\mu_1 = 1$ and $\mu_k \searrow 0$, for $k \to \infty$. As a consequence, the k-moments of the collision gain operator are dominated by the negative superlinear contributions corresponding to k-moments of the loss operator, when calculated for any moment order $k > k_*$. On the other hand, the later estimate from below, finds a lower bound for the negative contribution of the polyatomic collisional operator, that enables a fundamental component of the coerciveness estimate in the natural non-reflexive Banach space $L^1_k(\mathbb{R}^{4+})$. Such lower control is obtained by estimating the collision frequency associated to the potential rate $\gamma \in (0, 2]$ from below by a constant $c_{lb} := c_{lb}(m_0[f_0], m_1[f_0], m_{k_*}[f_0], \gamma)$, proportional to the γ-Lebesgue bracket, as shown in the Lower Bound Lemma, proved in [5], Section 5, which can be viewed as a functional estimate in a subset of the Banach space $L^1_k(\mathbb{R}^{4+})$ given by all positive elements, whose mass and variance are bounded by a positive and finite constant, zero mean, and a 1^+-moment, independently of being solutions of the Boltzmann flow. In addition, the negative contributions from the loss operator are proportional to the *coercive factor* $A_{k_*} = (1-\mu_{k_*})c_{lb}m_0[f_0]^{-\gamma/(2k_*)}$, such that $1 - \mu_{k_*} > 0$.

The factor A_{k_*} can be viewed as the analog to the coercive constant associated to elliptic and parabolic flows in continuum mechanics modeling, where coerciveness is crucial for the existence and uniqueness and global stability theories in Sobolev functional spaces. In this kinetic of collisional modeled flow context, coerciveness estimate enables the super-linear negative contribution for the evolution of the Ordinary Differential Inequality (ODI), with global supersolutions that control, a priori, the k-moments of the Boltzmann flow [5], Section 6. Thus the dissipative contractive constant μ_k and the coerciveness factor A_{k_*}, and its consequential properties on the k-moments of the solution of the Cauchy problem, are sufficient for the solvability of the Boltzmann flow under consideration in a suitable invariant region Ω of Banach space $\mathcal{C}([0,\infty); L^1_k(\mathbb{R}^{4+}))$ to be solvable globally in time [5], Section 7.

After the existence and uniqueness theory for global in time solutions of problem (1), solved in [5], Section 8, we also showed that n-partial sums of ks-moments are also globally bounded, for $0 < s \leq 1$. Again, appealing to estimates, we are able to generate a set of ODIs, whose solutions propagate the order $2s$ and below the rate β_0 of the initial data, while only generate up to an order $2s \leq \gamma$, for $0 < s < 1$. The exponential rate depends on the coercive constant $A_{k_*} = (1-\mu_{k_*})c_{lb}m_0[f_0]^{-\gamma/(2k_*)}$ directly proportional to the calculated rate, uniformly in the partial sum parameter. This exponential rates exhibit the connection of the coercive estimates of k-moments. It reflects the thermodynamic properties: the higher the coerciveness parameter, the lower the exponential rate and the faster decay rate to the statistical equilibrium. These conclusions have implications in the spectral gap calculation, as it will be shown in a forthcomming work.

These three pages are fully developed in [5] including extenuating explanations and rigorous proofs. We also refer the readers for further references to such manuscript as well.

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L^2 hypocoercivity, inequalities and applications

Jean Dolbeault

Let us consider the kinetic equation

$$\partial_t f + \mathsf{T} f = \mathsf{L} f \tag{1}$$

where $\mathsf{L}f$ is either the Fokker-Planck operator $\mathsf{L}_1 f = \nabla_v \cdot \big(\mathcal{M}\,\nabla_v(\mathcal{M}^{-1} f)\big)$ or a scattering collision operator $\mathsf{L}f = \int_{\mathbb{R}^d} \sigma(\cdot,v')\,\big(f(v')\,\mathcal{M}(\cdot) - f(\cdot)\,\mathcal{M}(v')\big)\,dv'$, for instance the simplest possible one, the linear BGK operator $\mathsf{L}_2 f = \rho\,\mathcal{M}(v) - f$ where $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,dv$ is the spatial density and $\mathcal{M}(v) = (2\pi)^{-d/2}\,e^{-|v|^2/2}$. The transport operator T on the phase space (with position x and velocity v) can be rewritten as $\mathsf{T} = i\,v\cdot\xi\,\hat f$ in the Fourier variable ξ associated to x. With the operators Π and A defined respectively by $\Pi\hat f := \mathcal{M}\int_{\mathbb{R}^d} \hat f(\xi,w)\,dw$ and $\mathsf{A}\hat f(\xi,v) := -\,i\,\xi\,(1+|\xi|^2)^{-1}\cdot\int_{\mathbb{R}^d} w\,\hat f(\xi,w)\,dw\,\mathcal{M}(v)$, the L^2 entropy, or L^2 Lyapunov functional $\mathsf{H}[\hat f] := \frac12\,\|\hat f\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}\hat f,\hat f\rangle$ where $\|g\|^2 := \iint_{\mathbb{R}^d} |g(w)|^2\,d\gamma$, $d\gamma(w) := \mathcal{M}(w)^{-1}\,dw$, is such that, if f solves (1), then the *entropy – entropy production inequality*

$$\frac{d}{dt}\mathsf{H}[\hat f(t,\xi,\cdot)] \le -\lambda\,\mathsf{H}[\hat f(t,\xi,\cdot)])$$

holds if

$$Q(X,Y) := \Big(1 - \tfrac{\delta\,|\xi|^2}{1+|\xi|^2} - \tfrac{\lambda}{2}\Big)\,X^2 - \tfrac{\delta\,|\xi|}{1+|\xi|^2}\,\big(1+\sqrt3\,|\xi| + \lambda\big)\,X\,Y + \Big(\tfrac{\delta\,|\xi|^2}{1+|\xi|^2} - \tfrac{\lambda}{2}\Big)\,Y^2$$

is a nonnegative quadratic form of X and Y, where $X := \|(1-\Pi)\hat f\|$ and $Y := \|\Pi\hat f\|$. Here it is clear that $\xi\in\mathbb{R}^d$ can be considered as a parameter, that is, we can perform a *mode-by-mode* analysis. Proving the exponential decay of $\mathsf{H}[\hat f]$ for some $\delta = \delta(|\xi|)$ with a rate $\lambda = \lambda(|\xi|)$ is reduced to the discriminant condition which guarantees that $Q \ge 0$. It turns out that $\mathsf{H}[\hat f]$ is equivalent to $\|\hat f\|^2$ if $\delta < 2$ and one can prove by the method of [9, 5] that

$$\|\hat f(t,\xi,\cdot)\|^2 \le C(|\xi|)\,\|\hat f_0(\xi,\cdot)\|^2\,e^{-\lambda(|\xi|)\,t} \quad \forall\,t\ge 0\,,\quad \xi\in\mathbb{R}^d\,,$$

where $C(|\xi|) = (2+\delta(|\xi|))/(2-\delta(|\xi|))$. This has been analysed in [5] in terms of asymptotic decay rates of $\|f(t,\cdot,\cdot)\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^d,dx\,d\gamma)}$ for a choice of δ which is independent of ξ but can be refined by taking a ξ-dependent value of δ.

Theorem 1. [2] *If f solves* (1) *with $\mathsf{L} = \mathsf{L}_1$ or $\mathsf{L} = \mathsf{L}_2$ for some nonnegative initial datum $f_0 \in L^2(\mathbb{R}^d\times\mathbb{R}^d, dx\,d\gamma)\cap L^2\big(\mathbb{R}^d, d\gamma; L^1(\mathbb{R}^d,dx)\big)$, then*

$$\|f(t,\cdot,\cdot)\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^d,dx\,d\gamma)} \le (2\,\pi)^{-d}\,\Psi_{M,Q}(t) \quad \forall\,t\ge 0$$

with $M = \|f_0\|_{L^2(\mathbb{R}^d,d\gamma;L^1(\mathbb{R}^d,dx))}$, $Q = \|f_0\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d,dx\,d\gamma)}$, and

$$\Psi_{M,Q}(t) := \inf_{R>0}\Big(\int_0^R C(s)\,e^{-\,\lambda(s)\,t}\,s^{d-1}\,ds\,\omega_d\,d\,M^2 + \sup_{s\ge R} C(s)\,e^{-\,\lambda(R)\,t}\,Q^2\Big)\,.$$

The proof of this result is reminiscent of the proof in [11] of *Nash's inequality*

$$\|u\|^{2+\frac4d}_{L^2(\mathbb{R}^d)} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|^{\frac4d}_{L^1(\mathbb{R}^d)}\,\|\nabla u\|^2_{L^2(\mathbb{R}^d)} \tag{2}$$

and the definition of the operator A is inspired by the diffusion limit: see [9, 2]. It is well known that a solution to the heat equation

$$\partial_t u = \Delta u \quad \forall\,(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$$

decays according to

$$\|u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d,dx)} \le \mathcal{C}\,\|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^d,dx)}\,(1+t)^{-\frac{d}{2}}$$

after computing $\frac{d}{dt}\|u(t,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d,dx)} = -2\,\|\nabla u(t,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d,dx)}$ and taking (2) into account. See [7] for a discussion of the optimality of such an estimate. This decay rate can be recovered also at the kinetic level for the solution of (1): see [5].

The next question is of course to understand what happens in presence of an external potential. Let us start at diffusive level with the Fokker-Planck equation

$$\partial_t u = \Delta u + \nabla\cdot(u\,\nabla V) \quad \forall\,(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \tag{3}$$

where V is a given external potential. To fix ideas, we shall assume that $V(x) = |x|^\alpha$ for some $\alpha > 0$ and discuss the cases depending on the value of α.

$\triangleright$ For $\alpha \ge 1$, we have the *Poincaré inequality*

$$\int_{\mathbb{R}^d} |v-\bar{v}|^2\,d\mu_\alpha \le \mathcal{C}\int_{\mathbb{R}^d} |\nabla v|^2\,d\mu_\alpha$$

with $\bar{v} := \int_{\mathbb{R}^d} v\,d\mu_\alpha$ and $d\mu_\alpha := U_\alpha(x)\,dx$, $U_\alpha(x) := Z_\alpha^{-1}\,e^{-|x|^\alpha}$, $Z_\alpha := \int_{\mathbb{R}^d} U_\alpha\,dx$. It is then standard to prove that a solution $u(t,\cdot)$ of (3) is such that $v := u/U_\alpha$ satisfies

$$\int_{\mathbb{R}^d} |v(t,\cdot)-\bar{v}|^2\,d\mu_\alpha \le \int_{\mathbb{R}^d} |v(0,\cdot)-\bar{v}|^2\,d\mu_\alpha\,e^{-\frac{2\,t}{\mathcal{C}}} \quad \forall\,t \ge 0\,.$$

$\triangleright$ The case $\alpha \in (0,1)$ has been studied in [10] using the weak Poincaré inequality. This approach requires the existence of a uniform bound. Alternatively, we can consider the *weighted Poincaré inequality*

$$\int_{\mathbb{R}^d} |\nabla v|^2\,d\mu_\alpha \ge \mathcal{C}\int_{\mathbb{R}^d} |v-\bar{v}|^2\langle x\rangle^{-\beta}\,d\mu_\alpha \tag{4}$$

with $\beta = 2\,(1-\alpha)$ and the same notations as above for $\bar{v}$ and $d\mu_\alpha$. Here we use the notation $\langle x\rangle = \sqrt{1+|x|^2}$ and notice that β vanishes as $\alpha \to 1_-$. In order to compensate for the additional weight in the right hand side in (4), it is convenient to introduce a weighted L^2 norm with a weight $\langle x\rangle^k$.

Theorem 2. [4] *Assume that $\alpha \in (0,1)$. If u solves (3) with initial datum $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d, d\mu_\alpha) \cap \mathrm{L}^2(\mathbb{R}^d, \langle x\rangle^k\,d\mu_\alpha)$ for some $k > 0$, $v = u/U_\alpha$ and $v_0 = u_0/U_\alpha$, then*

$$\int_{\mathbb{R}^d} |v(t,\cdot)-\bar{v}|^2\;d\mu_\alpha \le \left(\left(\int_{\mathbb{R}^d} |v_0-\bar{v}|^2\;d\mu_\alpha\right)^{-\beta/k} + \mathcal{K}\,t\right)^{-k/\beta} \quad \forall\,t \ge 0\,,$$

for some constant $\mathcal{K}$ depending on k and u_0.

▷ In the limit case as $\alpha \to 0_+$, it makes sense to consider $V(x) = \gamma \log|x|$. In the range $\gamma \in (0,d)$, (3) admits no stationary solution in $\mathrm{L}^1(\mathbb{R}^d)$. In that case, we can again introduce weights and consider the *Caffarelli-Kohn-Nirenberg inequality*

$$\int_{\mathbb{R}^d} |x|^\gamma u^2\,dx \le \mathcal{C}\left(\int_{\mathbb{R}^d} |x|^{-\gamma}\,|\nabla(|x|^\gamma u)|^2\,dx\right)^a \left(\int_{\mathbb{R}^d} |x|^k\,|u|\,dx\right)^{2\,(1-a)}$$

which generalizes Nash's inequality (2). A decay result goes as follows.

Theorem 3. [6] *Let $d \ge 1$ and $\gamma \in (0,d)$, $k \ge \max\{2,\gamma/2\}$. If u solves (3) with initial datum $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d,\langle x\rangle^k\,d\mu_\alpha) \cap \mathrm{L}^2(\mathbb{R}^d,d\mu_\alpha)$, then there is a constant $c > 0$ depending on u_0 such that*

$$\|u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d,|x|^\gamma\,dx)} \le \|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^d,|x|^\gamma\,dx)}\,(1+c\,t)^{-\frac{d-\gamma}{2}} \quad \forall\,t \ge 0\,.$$

Similar results can be obtained at kinetic level when the transport operator is defined by $\mathsf{T}f = v\cdot\nabla_x f - \nabla_x V\cdot\nabla_v f$. With $\mathsf{L} = \mathsf{L}_1$ or $\mathsf{L} = \mathsf{L}_2$, and appropriate estimates involving $\langle x\rangle^k$, results are obtained which are all consistent with a diffusion limit given by (3) and rely on the same functional inequalities. So far we have considered only Maxwellian local equilibria, but a similar discussion can be done when $\mathcal{M}(v) = Z_\beta^{-1}\exp(-|v|^\beta)$ for some $\beta > 0$, depending whether $\beta \ge 1$ or not, and the case $F(v) = \langle v\rangle^{-\gamma}$ has also been studied. Notice that a fractional diffusion limit has to be considered when $\int_{\mathbb{R}^d}|v|^2\,F(v)\,dv$ is infinite. The method also adapts to equations with a Poisson coupling. See [6, 4, 8, 3, 1] for detailed statements.

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