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Nonstandard Finite Element Methods (hybrid meeting)

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Abstract. Finite element methodologies dominate the computational approaches for the solution to partial differential equations and nonstandard finite element schemes most urgently require mathematical insight in their design. The hybrid workshop vividly enlightened and discussed innovative nonconforming and polyhedral methods, discrete complex-based finite element methods for tensor-problems, fast solvers and adaptivity, as well as applications to challenging ill-posed and nonlinear problems.

Mathematics Subject Classification (2010): 65Nxx.

Introduction by the Organizers

The workshop *Nonstandard Finite Element Methods*, organised by Daniele Boffi (KAUST and Pavia), Carsten Carstensen (Berlin), Alexandre Ern (Paris) and Jun Hu (Peking) was well attended with 56 virtual and 7 participants at the MFO with 30 presentations and an overall broad geographic representation from all continents. The meeting was in a hybrid format and showed an enthusiastic participation despite some difficulties related to the different time zones.

Unlike meetings with an engineering-oriented focus on applications, this workshop focussed primarily on the mathematical foundation of nonstandard discretizations. Relevant discussions involved, in particular, a priori and a posteriori error analysis, stability estimates and estimates for inf-sup constants related to various operators, convergence and superconvergence properties, definition and construction of new spaces and schemes, convergence and optimality of adaptive algorithms.

The meeting brought at least three communities in the modern numerical analysis of partial differential equations together: nonconforming finite element methods in various flavors, novel finite elements based on discrete complexes, fast solvers and adaptive methods. Several applications to challenging nonlinear problems were also highlighted.

Nonconforming finite element methods have been extensively developed in the last few years, leading to new families of methods including virtual element methods, hybrid higher-order methods, discontinuous Petrov Galerkin methods, and hybridizable discontinuous Galerkin methods. As compared to the classical finite element method, the striking advantages of these novel techniques include the use of general cell domains and arbitrarily high orders of approximation, without the need for special stabilization parameters. Some links have already been uncovered between these approaches, and some more should presumably appear in the near future. Moreover, some model applications were discussed on fourth-order or shell problems, when the conforming finite element practice appears cumbersome in relation to nonconforming ones.

Complexes on the continuous and discrete level for tensor problems provoked new finite elements and there was even an interface opened to neural networks and deep learning. The Hellan-Herrmann-Johnson finite element method caught particular interest. These topics were, so far, less consolidated in the reference community and got an interesting momentum from the interactions undergone during this workshop.

Besides the theoretical nature of the workshop, an increasing interest towards fast solvers and adaptivity has been observed. A salient example is the convergence proof of an adaptive method for the Hamilton–Jacobi–Bellman equation with Cordes coefficients. Quite interestingly, several contributions focused on challenging nonlinear problems, including eigenvalue problems, contact and friction problems, and hyperbolic problems. Other innovative applications dealt with ill-posed problems as in the unique continuation problem and on indefinite second-order PDEs.

A new topic of this workshop is the finite element complex construction of tensor-based problems like the linearized Einstein-Bianchi equation and the related biharmonic equation within the mixed divdiv formulation. Given the importance of the nonlinear Einstein field equations and the challenge of the corresponding numerical methods, such a topic deserves more attention and efforts from the finite element method community.

The workshop consisted of many one hour talks but also of a few shorter presentations by younger scientists, who thankfully enjoyed the opportunity to present their ideas to the community.

Workshop (hybrid meeting): Nonstandard Finite Element Methods

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Robust Strain Gradient Finite Elements: Analysis and Numerics

Pingbing Ming
(joint work with Hongliang Li, Yulei Liao, Zhong-ci Shi, Huiyu Wang)

We consider robust finite elements approximation of a linear strain gradient elastic model proposed by Aifantis et al [2], which could be regarded as a simplification of the more general strain gradient elastic models [3] because it contains only one extra material parameter besides the Lamé constants. This simplified strain gradient model successfully eliminated the strain singularity of the brittle crack tip field.

From the mathematical point of view, this model is a singular perturbed elliptic system of fourth order. A natural candidate for approximating such problem is C^1 conforming elements such as Argyris triangle. However, such choice is not preferred in practical computations because the number of the local degrees of freedom is too large, which is more remarkable in three dimension. To overcome this difficulty, we first prove a new H^2 Korn inequality and its discrete analog. The latter is dubbed as H^2 Korn inequality. These inequalities improved the older ones in the literature, e.g., [4] and [1]. It follows from the broken H^2 Korn inequality that a tensor product of the H^1 conforming but H^2 nonconforming element could be a good element for this strain gradient elastic model. We employ the Specht triangle [6] and its three-dimensional extension [8] as representative. Based on a new established enriching operator, we prove sharp error estimates for these elements for both the smooth solution and the solutions with sharp layers. All estimates are uniform with respect to small strain gradient materials parameter. Numerical results are reported to confirm the theoretical results. It is worth mentioning that a family of rectangular and cuboidal elements for this model may be found in [5], and more three dimensional tests may be found in [7]. Possible further work is to design robust elements for the nonlinear strain gradient elastic models, thin beam and thin plate with strain gradient effect.

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Conforming Finite Element Gradgrad and Divdiv complexes

Jun Hu

This talk consists of two parts. The first part considers the finite element method of the so called linearized Einstein-Bianchi system from [1]

$$\dot{\mathbf{E}} + \operatorname{curl}\mathbf{B} = 0, \quad \operatorname{div}\mathbf{E} = 0,$$
$$\dot{\mathbf{B}} - \operatorname{curl}\mathbf{E} = 0, \quad \operatorname{div}\mathbf{B} = 0,$$

with symmetric and traceless tensor fields $\mathbf{E}$ and $\mathbf{B}$. Introducing a new variable $\sigma(t) = \int_0^t \operatorname{div}\operatorname{div}\mathbf{E}\mathrm{d}s$, the linearized Einstein-Bianchi system can be realized as a Hodge wave equation [1]

$$\begin{aligned} \dot{\sigma} &= \operatorname{div}\operatorname{div}\mathbf{E}, \\ \dot{\mathbf{E}} &= -\operatorname{grad}\operatorname{grad}\sigma - \operatorname{sym}\operatorname{curl}\mathbf{B}, \\ \dot{\mathbf{B}} &= \operatorname{curl}\mathbf{E}. \end{aligned} \tag{1}$$

Given initial conditions $\sigma(0), \mathbf{E}(0)$ and $\mathbf{B}(0)$, and with appropriate boundary conditions, the equation (1) is well-posed [1].

The mixed finite elements for the linearized Einstein-Bianchi system is closely related to the discretization of an associated differential complex. Such a Gradgrad-complex, introduced in [2] to derive a Helmholtz-like decomposition for biharmonic problems in $\mathbb{R}^3$, is given by

$$\begin{aligned} P_1(\Omega) \xrightarrow{\subset} H^2(\Omega;\mathbb{R}) \xrightarrow{\operatorname{grad}\operatorname{grad}} H(\operatorname{curl},\Omega;\mathbb{S}) \xrightarrow{\operatorname{curl}} H(\operatorname{div},\Omega;\mathbb{T}) \\ \xrightarrow{\operatorname{div}} L^2(\Omega;\mathbb{R}^3) \longrightarrow 0, \end{aligned} \tag{2}$$

where the space $H(\operatorname{div},\Omega;\mathbb{T})$ consists of square-integrable tensors with square-integrable divergence, taking value in the space $\mathbb{T}$ of traceless matrices. The complex is exact provided that the domain Ω is contractible and Lipschitz [2], that is, the range space of each map is the kernel space of the succeeding map. The purpose of this part is to construct conforming finite element spaces $U_h \subset H^2(\Omega;\mathbb{R}), \Sigma_h \subset H(\operatorname{curl},\Omega;\mathbb{S}), V_h \subset H(\operatorname{div},\Omega;\mathbb{T})$ and $Q_h \subset L^2(\Omega;\mathbb{R}^3)$ such that

$$P_1(\Omega) \xrightarrow{\subset} U_h \xrightarrow{\operatorname{grad}\operatorname{grad}} \Sigma_h \xrightarrow{\operatorname{curl}} V_h \xrightarrow{\operatorname{div}} Q_h \longrightarrow 0 \tag{3}$$

is an exact sub-complex of (2).

The first main ingredient of the construction is a crucial structure of $H(\operatorname{curl},\Omega;\mathbb{S})$ finite element spaces with additional regularity at vertices:

$$\begin{aligned} \Sigma_{k,h} = \{\sigma \in H(\operatorname{curl},\Omega;\mathbb{S}) : \sigma = \sigma_c + \sigma_b, \sigma_c \in C^0(\Omega;\mathbb{S}), \\ \sigma_c \text{ is } C^2 \text{ at all vertices}, \sigma_c\big|_K \in P_k(K;\mathbb{S}), \sigma_b\big|_K \in \Sigma^*_{K,k,b}, \forall K \in \mathcal{T}_h\}, \end{aligned}$$

where the bubble function space with additional regularity conditions on K reads

$$\Sigma^*_{K,k,b} := \{\sum_{i=0}^{3} \lambda_j\lambda_l\lambda_m P^{(i,0)}_{k-3}(K;\mathbb{R})\boldsymbol{n}_i\boldsymbol{n}_i^T\} + \{\lambda_0\lambda_1\lambda_2\lambda_3 P_{k-4}(K;\mathbb{S})\}$$

with the auxiliary space $P^{(i,0)}_k(K;\mathbb{R}) := \{p \in P^{(i)}_k(K;\mathbb{R}) : p = 0 \text{ at the vertices of } \boldsymbol{f}_i\}$.

The second main ingredient of the construction is an essential structure of $H(\mathrm{div},\Omega;\mathbb{T})$ finite element spaces with additional regularity at vertices as follows

$$V_{k,h} := \widetilde{V}_{k,h} + B_{k,h} \text{ with } B_{k,h} := \sum_K V^*_{K,k,b},$$

where

$$\widetilde{V}_{k,h} := \{\mathbf{v} \in H^1(\Omega;\mathbb{T}), \mathbf{v} \text{ is } C^1 \text{ at all vertices}, \mathbf{v}\big|_K \in P_k(K;\mathbb{T}), \forall K \in \mathcal{T}_h\}$$

and

$$V^*_{K,k,b} := \sum_{i=0}^{3} \sum_{\substack{0\le j<l\le 3\\ j,l\neq i}} \lambda_j\lambda_l P^{(j,l,0)}_{k-2}(K;\mathbb{R})\boldsymbol{n}_i\boldsymbol{t}^T_{j,l}$$

with the auxiliary space $P^{(i,j,0)}_k(K;\mathbb{R}) := \{u \in P_k(K;\mathbb{R}) : u \text{ vanishes at } \boldsymbol{x}_i, \boldsymbol{x}_j\}$.

Given $K \in \mathcal{T}_h$, define

$$Q^\perp_k(K) := \{\mathbf{q} = (q_1,q_2,q_3)^T \in P_k(K;\mathbb{R}^3) : \int_K q_i = 0, 1 \le i \le 3,$$
$$\int_K (xq_1 + yq_2 + zq_3) = 0\}.$$

and

$$R^\perp_k(K) := \{\mathbf{q} \in Q^\perp_k(K) : \mathbf{q}(\boldsymbol{x}_i) = 0, 0 \le i \le 3\}.$$

The third main ingredient of the construction is the following algebraic result

$$\mathrm{div}\, V^*_{K,k,b} = R^\perp_{k-1}(K).$$

for any $K \in \mathcal{T}_h$ when $k \ge 3$.

Define the space of piecewise polynomials of degree $\le k$ that are C^0 at vertices

$$Q_{k,h} := \{\mathbf{q} \in L^2(\Omega;\mathbb{R}^3) : \mathbf{q} \text{ is } C^0 \text{ at all vertices}, \mathbf{q}|_K \in P_k(K;\mathbb{R}^3), \forall K \in \mathcal{T}_h\}$$

Finally, the main result of this part is: Take Σ_h to be $\Sigma_{k,h}$, V_h to be $V_{k-1,h}$, Q_h to be $Q_{k-2,h}$, with $k \ge 7$, the discrete complex

$$P_1(\Omega) \xrightarrow{\subset} U_h \xrightarrow{\text{gradgrad}} \Sigma_h \xrightarrow{\text{curl}} V_h \xrightarrow{\text{div}} Q_h \longrightarrow 0$$

is exact. More details can be found in [3].

The second part introduces a new family of mixed finite elements for solving a mixed formulation of the biharmonic equations in two and three dimensions. The symmetric stress $\sigma = -\nabla^2 u$ is sought in the Sobolev space $H(\mathrm{div}\mathbf{div},\Omega;\mathbb{S})$ simultaneously with the displacement u in $L^2(\Omega)$. Stemming from the structure of $H(\mathbf{div},\Omega;\mathbb{S})$ conforming elements for the linear elasticity problems proposed by J. Hu and S. Zhang, the $H(\mathrm{div}\mathbf{div},\Omega;\mathbb{S})$ conforming finite element spaces are

constructed by imposing the normal continuity of $\mathbf{div}\sigma$ on the $H(\mathbf{div}, \Omega; \mathbb{S})$ conforming spaces of P_k symmetric tensors.

For two dimensions, the degrees of freedom are defined as follows

$$\sigma(a) \text{ for all } a \in \mathcal{V}(K); \tag{4}$$

$$(\sigma\mathbf{n}, \phi)_e \text{ for all } \phi \in P_{k-2}(e; \mathbb{R}^2),\ e \in \mathcal{E}(K); \tag{5}$$

$$(\mathbf{div}\sigma \cdot \mathbf{n},\ q)_e \text{ for all } q \in P_{k-1}(e),\ e \in \mathcal{E}(K); \tag{6}$$

$$(\sigma, \nabla^2 q)_K \text{ for all } q \in P_{k-2}(K); \tag{7}$$

$$(\sigma, \nabla \mathbf{curl}\, q)_K \text{ for all } q \in \lambda_1\lambda_2\lambda_3 P_{k-3}(K)/P_0(K); \tag{8}$$

$$(\sigma, \mathcal{J} q)_K \text{ for all } q \in (\lambda_1\lambda_2\lambda_3)^2 P_{k-4}(K). \tag{9}$$

The global finite element space is defined by

$$\begin{aligned}\Sigma_{k,\triangle_2} :=& \{\tau \in H(\mathrm{div}\mathbf{div}, \Omega; \mathbb{S}) :\ \tau|_K \in P_k(K; \mathbb{S}) \text{ for all } K \in \mathcal{T}_h,\\ & \text{all the degrees of freedom (4)–(9) are single-valued}\}.\end{aligned} \tag{10}$$

For three dimensions, the degrees of freedom are defined as follows

$$\sigma(a) \text{ for all } a \in \mathcal{V}(K); \tag{11}$$

$$(\mathbf{t}_e^{\intercal}\sigma\mathbf{n}_j, q)_e,\ (\mathbf{n}_i^{\intercal}\sigma\mathbf{n}_j, q)_e\ 1 \le i, j \le 2, \text{for all } q \in P_{k-2}(e),\ e \in \mathcal{E}(K); \tag{12}$$

$$(\sigma\mathbf{n}, \phi)_F \text{ for all } \phi \in P_{k-3}(F; \mathbb{R}^3),\ F \in \mathcal{F}(K); \tag{13}$$

$$(\mathbf{div}\sigma \cdot \mathbf{n}, q)_F \text{ for all } q \in P_{k-1}(F),\ F \in \mathcal{F}(K); \tag{14}$$

$$(\sigma, \nabla^2 q)_K \text{ for all } q \in P_{k-2}(K); \tag{15}$$

$$(\sigma, \nabla\phi)_K \text{ for all } \phi \in \mathcal{W}_{k-1}(K; \mathbb{R}^3); \tag{16}$$

$$(\sigma, \tau)_K \text{ for all } \tau \in \mathcal{M}_k(K; \mathbb{S}). \tag{17}$$

The global conforming finite element space is defined by

$$\begin{aligned}\Sigma_{k,\triangle_3} :=& \{\tau \in H(\mathrm{div}\mathbf{div}, \Omega; \mathbb{S}) :\ \tau|_K \in P_k(K; \mathbb{S}) \text{ for all } K \in \mathcal{T}_h,\\ & \text{all the degrees of freedom (11)–(17) are single-valued}\}.\end{aligned}$$

The inheritance makes the basis functions easy to compute. The discrete spaces for u are composed of the piecewise P_{k-2} polynomials without requiring any continuity. Then the mixed finite element discrete problem of the biharmonic problem finds $\sigma_h \in \Sigma_{k,\triangle_d}$, and $u_h \in P_{h,\triangle_d}$ such that

$$\begin{aligned}(\sigma_h, \tau_h) + (\mathrm{div}\mathbf{div}\,\tau_h, u_h) =& 0 && \text{for all } \tau_h \in \Sigma_{k,\triangle_d},\\ (\mathrm{div}\,\mathbf{div}\sigma_h, v_h) =& -(f, v_h) && \text{for all } v_h \in P_{h,\triangle_d}.\end{aligned} \tag{18}$$

Such mixed finite elements are inf-sup stable on both triangular and tetrahedral grids for $k \ge 3$, and the optimal order of convergence is achieved. Besides, the superconvergence and the postprocessing results are displayed. More details can be found in [4].

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Space-time discontinuous Galerkin methods for the wave equation

Ilaria Perugia

(joint work with Pratyuksh Bansal, Andrea Moiola, Joachim Schöberl, Christoph Schwab, Paul Stocker, Christoph Wintersteiger)

Space-time finite element methods for the approximation of time-dependent PDEs present some advantages, as compared to more standard space discretization plus time-stepping: high-order approximation both in space and time is simple to obtain, spectral convergence of the space-time error can be obtained by p-refinement, stability is achieved under a *local* CFL condition, and the approximate solutions are available at all times in the interval of interest. Their main drawback is their high complexity. In fact, in order to approximate a time-dependent problem in d space dimensions, one needs to implement and solve a $(d+1)$-dimensional problem.

Here, we focus on discontinuous Galerkin (DG) space-time approximations of the acoustic wave equation, and on two ideas to reduce their complexity, asymptotically, to that of one d-dimensional elliptic solve. The first idea consists in using *Trefftz polynomial spaces*, instead of complete polynomial spaces, in combination with a *tent-pitching* mesh design [4, 6]. The second idea consists in applying the so-called *combination formula* to a sequence of anisotropic, tensor product (in time) space-time discretizations, exploiting the unconditional well-posedness of suitably designed space-time discontinuous Galerkin methods [1]. We refer to the references in [4, 1] for a literature overview.

Model problem. We consider as a model problem the wave equation formulated as a first order system. Let $Q := \Omega \times (0,T)$ be a space-time domain, where $\Omega \subset \mathbb{R}^d$ is a Lipschitz, bounded polytopic domain, and let $c = c(\boldsymbol{x}) > 0$ be a bounded function in Ω, which is assumed to be piecewise constant on a finite polytopic partition $\{\Omega_i\}$ of Ω, and which represents the wave velocity. Given $f \in L^2(Q)$,

$v_0 \in L^2(\Omega)$, and $\boldsymbol{\sigma}_0 \in L^2(\Omega)^d$, the problem reads as follows:

$$\begin{cases} \text{find } (v, \boldsymbol{\sigma}) \text{ such that} & \\ \nabla v + \dfrac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0}, \qquad \nabla \cdot \boldsymbol{\sigma} + c^{-2} \dfrac{\partial v}{\partial t} = f & \text{in } Q \\ v(\cdot, 0) = v_0, \quad \boldsymbol{\sigma}(\cdot, 0) = \boldsymbol{\sigma}_0 & \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega \times [0, T]. \end{cases}$$

We denote by $\mathcal{L}_{\mathsf{wave}}$ the wave operator $\mathcal{L}_{\mathsf{wave}}(w, \boldsymbol{\tau}) := \Big(\nabla \cdot \boldsymbol{\tau} + c^{-2} \dfrac{\partial w}{\partial t} \,,\ \nabla w + \dfrac{\partial \boldsymbol{\tau}}{\partial t} \Big)$.

Space-time DG methods. Consider polytopic meshes $\mathcal{T}_h$ of the space-time domain Q, which are aligned with the partition $\{\Omega_i\}$. Space-time DG methods are defined by taking the space-time variational formulation in each element $K \in \mathcal{T}_h$, by discretizing test and trial functions in discontinuous, piecewise polynomial spaces $\boldsymbol{V}(\mathcal{T}_h)$, and by replacing the interelement traces by numerical fluxes, which we denote by the hat symbol in the following formulation:

$$- \int_K \left[v_h \Big(\nabla \cdot \boldsymbol{\tau}_h + c^{-2} \frac{\partial w_h}{\partial t} \Big) + \boldsymbol{\sigma}_h \cdot \Big(\nabla w_h + \frac{\partial \boldsymbol{\tau}_h}{\partial t} \Big) \right] dV$$
$$+ \int_{\partial K} \left[(\widehat{v}_h \, \boldsymbol{\tau}_h + \widehat{\boldsymbol{\sigma}}_h \, w_h) \cdot \boldsymbol{n}_K^{\boldsymbol{x}} + \Big(\widehat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\tau}_h + c^{-2} \, \widehat{v}_h \, w_h \Big) n_K^t \right] dS = \int_K f \, w_h \, dV$$

for all $K \in \mathcal{T}_h$, where $(\boldsymbol{n}_K^{\boldsymbol{x}}, n_K^t) \in \mathbb{R}^{d+1}$ denotes the unit normal vector to ∂K pointing outside K. Adding over all $K \in \mathcal{T}_h$, we obtain the complete space-time DG method denoted as follows:

$$\mathcal{A}_{\mathsf{DG}}(v_h, \boldsymbol{\sigma}_h; w_h, \boldsymbol{\tau}_h) = \ell_{\mathsf{DG}}(w_h, \boldsymbol{\tau}_h).$$

Under the following assumptions:

- *i) meshes*: all interior faces are either space-like, namely $c\,|\boldsymbol{n}_F^{\boldsymbol{x}}| < n_F^t$, or time-like, namely $n_F^t = 0$,
- *ii) numerical fluxes*: we choose upwind fluxes on space-like faces and standard DG-elliptic fluxes on time-like faces,
- *iii) approximation spaces*: if $(w_h, \boldsymbol{\tau}_h) \in \boldsymbol{V}_p(\mathcal{T}_h)$, then $\mathcal{L}_{\mathsf{wave}}(w_h, \boldsymbol{\tau}_h) \in \boldsymbol{V}_p(\mathcal{T}_h)$,

existence and uniqueness of DG solutions can be proven by modifying the argument from [5]. Moreover, on any space-like surface, the L^2 norm of the Galerkin error is bounded by a DG-type norm of a projection error [4, 1]. We remark that these results hold true unconditionally. Conditions on the ratio between the size in space and the size in time of the space-time elements are required in order to achieve the highest possible convergence rates.

Notice that, if $\boldsymbol{V}(\mathcal{T}_h)$ is locally made by standard space-time polynomials, by tensor product (in time) polynomials, or by Trefftz polynomials (namely polynomials in the kernel of $\mathcal{L}_{\mathsf{wave}}$), then assumption *iii)* is satisfied. We underline that, unlike in the case of the Helmholtz problem, the wave equation in the time domain

admits *polynomial* Trefftz spaces with the same h-version approximation properties, for functions in the kernel of $\mathcal{L}_{\mathsf{wave}}$, as the complete spaces of polynomials of the same degree.

Trefftz methods. A complete analysis of space-time Trefftz DG methods for the case $f = 0$ is presented in [4], where error estimates in mesh-independent norms in Q are also derived by a modified duality argument. While local complete spaces of space-time polynomials of degree p have number of degrees of freedom that scale like p^{d+1}, the local spaces of Trefftz polynomials have number of degrees of freedom that scale like p^d. The desired reduction of the number of degrees of freedom is therefore achieved. Being based on shape functions in the kernel of $\mathcal{L}_{\mathsf{wave}}$, the Trefftz DG formulation contains integrals on the mesh skeleton only.

In order to improve computational efficiency, the Trefftz DG method can be combined with tent-pitching. Tent-pitching (see e.g. [7]) is a PDE-driven, front-advancing mesh construction technique for hyperbolic problems, which consists in stacking tent-pitched objects on top of each other. Each tent is union of $(d+1)$-dimensional simplices, whose high (local advancement in time) is chosen so that the casuality constraint of the PDE is respected (*local* CFL condition). In this way, the PDE is explicitly solvable within each tent. The solution within tents on the same level can be done in parallel. The Trefftz approach combines very well with tent-pitching. In fact, as the Trefftz DG method contains no volume terms, the evolution of the solution within each tent, from bottom to top, is performed by solving a small algebraic linear system. A discussion on the implementation and numerical results are presented in [6].

Tensor-product methods and combination formula. The unconditional well-posedness of the considered space-time DG methods can be exploited, in case of tensor-product (in time) elements, in order to reduce the complexity of the method by applying the so-called combination formula [2]. In [1], this strategy is discussed, with focus on possible solution singularities. Restricting to the case $d = 2$, the regularity theory of the wave equation (see e.g. [3]) states that acoustic waves exhibit conical singularities in space at the vertices of the partition $\{\Omega_i\}$. These singularities can be numerically resolved by using, in space, the same graded meshes as for elliptic problems. In [1], unconditional stability is proven and an error analysis is performed. With a suitable mesh grading in space, and relating the time mesh size to the space mesh size, it is shown that the same convergence rates as for smooth solutions are obtained.

The optimal h-convergence rates are achieved with a number of degrees of freedom that scales like h^{-3}. The question whether this can be achieved with a number of degrees of freedom that scales like h^{-2} is affirmatively answered by taking a weighted sum of suitably selected solutions obtained with anisotropic (in time) space-time meshes, with different levels of space and time refinements (combination formula). The unconditional well-posedness of the space-time DG method guarantees that, although not accurate, solutions obtained with strongly

anysotropic meshes still contain meaningful information. The sum of the dimensions of the problems to be solved in order to construct the combined solution scales like h^{-2}. Numerical experiments are presented in [1].

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Approximation by Piecewise Polynomials: Finite Elements versus Neural Networks

Jinchao Xu

This talk is devoted to piecewise polynomial discretization of $2m$-th-order partial differential equations in $\mathbb{R}^d$ for any $m, d \geq 1$. We first recall a family of nonconforming element of piecewise polynomial degree m by Wang and Xu [2] when $m \leq d$. We then report a recent solution in Wu and Xu [3] of the open problem for the case $m > d$ by still using piecewise polynomials of degree m by adding some additional (and minimal number of) stabilization terms. But how to construct conforming finite element methods is still an unsolved open problem for general m and d. Instead, we report a family of H^m-conforming piecewise polynomials study in Xu [4] based on the artificial neural network using ReLU^k as activation function, referred to as the finite neuron method (FNM), for numerical solution of $2m$-th-order partial differential equations in $\mathbb{R}^d$ for any $m, d \geq 1$ and then provide a convergence analysis for this method. By combing the results in Xu [4] and Siegel and Xu [1], we can prove the following error estimates

$$\|u - u_n\|_{H^m} = \mathcal{O}(n^{m-(k+1)} \log(n)) \tag{1}$$

where u_n is the finite neuron approximation with $\mathcal{O}(n)$ parameters. We further point out that the error estimate (1) for finite neuron method is significantly better than the corresponding finite element solution approximation, say u_n^{FE}, from a finite element space of $\mathcal{O}(n)$ dimensions. Roughly speaking, the following estimate holds:

$$\|u - u_n\|_{H^m} \approx \|u - u_n^{FE}\|_{H^m}^d. \tag{2}$$

A discussion is also provided on the difference and relationship between the finite neuron method and finite element methods (FEMs). For example, for the finite neuron method, the underlying finite element grids are not given a priori and the discrete solution can be obtained by only solving a non-linear and non-convex optimization problem which can be very challenging.

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Invariant domain preserving approximation of nonlinear hyperbolic systems using finite elements

Jean-Luc Guermond
(joint work with Matthias Maier, Murtazo Nazarov, Ignacio Tomas, Bojan Popov)

The objective of this talk is to report on recent advances made on the approximation of nonlinear hyperbolic systems in conservative form using nonstandard finite element techniques; say, $\partial_t \mathbf{u} + \nabla\cdot\mathbb{f}(\mathbf{u}(\mathbf{x},t)) = \mathbf{0}$, $t>0$, $\mathbf{x}\in D\subset\mathbb{R}^d$. This work is part of a long term research program involving the following collaborators: M. Maier (Texas A&M Univ., USA), M. Nazarov (Uppsala Univ., Sweden), I. Tomas (Sandia Natl. Lab., NM, USA), and B. Popov (Texas A&M Univ., USA).

The key idea behind this project is to construct approximation techniques that are explicit in time, use finite elements in space, and preserve important properties of the continuous system, like the positivity of the density and internal energy for the compressible Euler equations, or the positivity of the water height for the shallow water equations. For more general systems, these properties can be formulated in terms of invariant sets. Assuming that the dependent variable $\mathbf{u}$ takes values in $\mathbb{R}^m$, a subset $\mathcal{B}$ of $\mathbb{R}^m$ is said to be an invariant domain of the hyperbolic system if for any initial data that takes values in $\mathcal{B}$, the solution also takes values in $\mathcal{B}$. The notion of invariant domains generalizes the maximum principle, which we recall holds true only for scalar conservation equations. The existence of nontrivial invariant domains can be established for most hyperbolic systems. Moreover, under mild assumptions, one can also show that invariant domains are necessarily convex, [1, 6, 7]. Another important idea is that the approximate solution should also satisfy discrete entropy inequalities, since for most hyperbolic systems uniqueness is only guaranteed if additional entropy inequalities are satisfied. Being invariant domain preserving and satisfying discrete entropy inequalities are essential features that guarantee robustness. These ideas have been

implemented in a generic algorithm proposed in [3] and which proceeds as follows. Let $\mathbf{u}_h^n = \sum_{i\in\mathcal{V}} \mathbf{U}_i^n \varphi_i$ be the approximation of $\mathbf{u}(\cdot, t^n)$, where $\{\varphi_i\}_{i\in\mathcal{V}}$ are the scalar shape functions and $\mathbf{U}_i^n \in \mathbb{R}^m$ for all $i \in \mathcal{V}$. Here $\mathcal{V}$ is the set enumerating all the scalar shape functions. The update at time $t^{n+1} = t^n + \tau$ is obtained by setting $m_i \mathbf{U}_i^{n+1} = m_i \mathbf{U}_i^n - \tau \sum_{j\in\mathcal{I}(i)} \mathbb{f}(\mathbf{U}_i^n)\mathbf{c}_{ij} + \tau \sum_{j\in\mathcal{I}(i)\setminus\{i\}} d_{ij}^n (\mathbf{U}_j^n - \mathbf{U}_i^n)$, where m_i is the lumped mass matrix coefficient associated with the shape function φ_i, $\mathcal{I}(i)$ is the set collecting the indices of the degrees of freedom in the stencil of i, $\mathbf{c}_{ij} = \int_D \varphi_i \nabla\varphi_j \mathrm{d}\mathbf{x} \in \mathbb{R}^d$, and d_{ij}^n is an artificial viscosity coefficient. (Here the shape functions are assumed to be continuous, but $\mathbf{c}_{ij}$ can be defined for discontinuous finite elements as well.) It is established in [3] that, under an appropriate CFL restriction on the time step τ and provided $\frac{d_{ij}^n}{\|\mathbf{c}_{ij}\|_{\ell^2}}$ is larger than the maximum wave speed in the Riemann problem with left-right Riemann data $(\mathbf{U}_i^n, \mathbf{U}_j^n)$ and flux $\mathbb{f}(\mathbf{v})\frac{\mathbf{c}_{ij}}{\|\mathbf{c}_{ij}\|_{\ell^2}}$, this algorithm is invariant domain preserving and satisfies a discrete entropy inequality for all the entropies of the system. This result hods true in any space dimension d and for every flux $\mathbb{f}$ provided there exists an invariant domain $\mathcal{B}$ such that the flux $\mathbb{f}(\mathbf{v})\mathbf{n}$ is hyperbolic for all $\mathbf{v}$ in $\mathcal{B}$ and all unit vector $\mathbf{n}$ in $\mathbb{R}^d$. The only restriction on the finite element mesh is that it is a member of a shape-regular sequence of meshes.

The above algorithm is robust at the cost of being only first-order accurate in time and space (in compliance with Godunov's theorem). Higher-order accuracy in time can be achieved up to fourth order by using an explicit Runge–Kutta time stepping technique that is strong stability preserving (SSP RK). Achieving higher-order accuracy in space while maintaining the invariant domain preserving properties and satisfying discrete entropy inequalities is nontrivial. It is doable though by introducing a nonlinear post-processing called convex limiting. One must abandon the idea of enforcing every possible invariant domain preserving properties and every entropy inequalities, and instead restricts oneself to enforcing only a finite set of convex constraints. A convex limiting technique has been developed in [4] for the Euler equations and generalized in [5] for any hyperbolic flux. These algorithms are easy to implement. They also give results that are reproducible since they are unambiguously defined and do not depend on any user-dependent stabilization parameters.

There are some limitations of the techniques described above. The first one is that the approximation is based on scalar-valued shape functions; that is, the approximation in $\mathbb{R}^m$ is done by using m copies of the scalar-valued finite element space. This may be an obstacle when one solves the magnetohydrodynamic equations where Gauss' laws of magnetism and electricity must be enforced. Using edge elements in that case would naturally enforce these constraints. Another limitation is that the convex limiting method is not yet robust with respect to the polynomial degree of the finite element approximation. Finally, as mentioned in [2], it is not yet clear how the convex limiting method can incorporate traditional linear stabilization techniques (upwind dG, Galerkin Least Squares, Continuous Interior Penalty, Subgrid Stabilization, Local Projection Stabilization, Orthogonal Subgrid Scales, etc).

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An arbitrary-order discrete de Rham complex on polyhedral meshes

Daniele Antonio Di Pietro

(joint work with Jérôme Droniou)

The design of stable and convergent methods for the numerical approximation of certain classes of partial differential equations requires to reproduce, at the discrete level, the underlying geometric, topological, and algebraic structures. This leads to the notion of *compatibility*, which can be achieved either in a conforming or non-conforming numerical setting. Relevant examples include PDEs that relate to the de Rham complex. For an open connected polyhedral domain $\Omega \subset \mathbb{R}^3$, this complex reads

$$\mathbb{R} \xrightarrow{i_\Omega} H^1(\Omega) \xrightarrow{\mathbf{grad}} \boldsymbol{H}(\mathbf{curl};\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{0} \{0\},$$

where i_Ω denotes the operator that maps a real value to a constant function over Ω, $H^1(\Omega)$ the space of scalar-valued functions over Ω that are square integrable along with their gradient, $\boldsymbol{H}(\mathbf{curl};\Omega)$ (resp. $\boldsymbol{H}(\mathrm{div};\Omega)$) the space of vector-valued functions over Ω that are square integrable along with their curl (resp. divergence).

In this talk we discuss an arbitrary-order discrete de Rham (DDR) complex suitable for the numerical approximation of PDEs on general polyhedral meshes originally introduced in [1, 2]; see also the precursor works [4, 5]. This complex is obtained by replacing the spaces and vector calculus operators by discrete counterparts. Specifically, given a polyhedral mesh $\mathcal{T}_h$ regular in the sense of [5, Chapter 1], the spaces in the DDR complex are spanned by vectors of polynomials whose components are attached to mesh entities (possibly including vertices, edges, faces, and elements according to the space) and are selected so as to:

(1) emulate, at the discrete level, the continuity properties of the corresponding continuous space;

(2) enable the reconstruction of discrete counterparts of vector calculus operators and of the corresponding potentials.

The discrete vector calculus operators, on the other hand, are constructed by emulating local discrete integration by parts formulas on edges, faces, and elements, and by taking component-wise L^2-orthogonal projections on the DDR spaces. Based on the above construction, one can define discrete counterparts of the L^2-products in the DDR spaces composed of two terms: the first, involving the L^2-product of the potentials, is in charge of consistency, whereas the second, involving a least-square penalisation of high-order residuals inside each element, ensures stability.

The DDR sequence enjoys the following properties, which play a key role for its application to the discretization of PDE problems:

(1) *Complex and exactness properties.* The sequence forms a complex, i.e., the image of each discrete vector calculus operator is contained in the kernel of the next one. Moreover, the following exactness properties are reproduced at the discrete level: $\operatorname{Im} i_\Omega = \operatorname{Ker} \mathbf{grad}$ (since Ω is connected); $\operatorname{Im} \mathbf{grad} = \operatorname{Ker} \mathbf{curl}$ if the first Betti number of Ω is zero; $\operatorname{Im} \mathbf{curl} = \operatorname{Ker} \operatorname{div}$ if the second Betti number of Ω is zero; $\operatorname{Im} \operatorname{div} = L^2(\Omega)$ (since we are in dimension three).
(2) *Uniform Poincaré inequalities.* Whenever a function from a space in the sequence lies in some orthogonal complement of the kernel of the vector calculus operator defined on this space, its (discrete) L^2-norm is controlled by the (discrete) L^2-norm of the operator up to a multiplicative constant independent of the mesh size.
(3) *Primal and adjoint consistency.* The discrete vector calculus operators satisfy appropriate commutation properties with the interpolators and their continuous counterparts. Additionally, these operators along with the corresponding (scalar or vector) potentials approximate smooth fields with sufficient accuracy. Finally, the vector calculus operators enjoy suitable adjoint consistency properties. The notion of adjoint consistency accounts for the failure, in non-conforming settings, to exactly verify global integration by parts formulas, and is relevant whenever a formal integration by parts is used to derive the weak formulation of the problem at hand.

The theoretical and numerical properties of the DDR sequence are showcased on a model problem inspired by magnetostatics. The discretization is obtained in a natural way: starting from the weak formulation of the problem set in $\boldsymbol{H}(\mathbf{curl};\Omega)\times\boldsymbol{H}(\operatorname{div};\Omega)$, the numerical scheme is obtained by replacing the continuous spaces, vector calculus operators, and L^2-products with their discrete counterparts discussed above. We show that, when the DDR complex of degree $k \geq 0$ is used as a starting point, the approximations of the magnetic field and of the corresponding vector potential converge to their exact counterparts as h^{k+1} (with h denoting the mesh size) in the natural discrete graph norms. These results are confirmed numerically on a variety of classical and polyhedral meshes.

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Some recent interpolation and stability estimates for Virtual Elements

Lourenço Beirão da Veiga
(joint work with Giuseppe Vacca)

The Virtual Element Method (VEM) was introduced in [1, 2] as a generalization of the finite element method (FEM) that is able to cope with general polytopal meshes; since its introduction, the VEM enjoyed a large success in the numerical analysis and engineering communities.

The present talk will not deal with advanced applications of the method but rather with its foundations, and may be of interest in general for schemes making use of polygonal meshes. Standard h-interpolation (and convergence) estimates for shape regular meshes in FEM and VEM involve the diameter h_E of elements as the main grid parameter; in the presence of triangular (or quadrilateral) shape regular meshes this gives a complete picture. Instead, in the presence of more general meshes one may wonder if polygons with many small edges (and an associated richer discrete space such as those used in VEM or Polygonal FEM) can yield, in some sense, better interpolation properties and if this will reflect also on the final error among the discrete and exact solutions. Basically, the answer is no, but the investigation allows to shed more light on the matter and develop an interesting variant.

Looking into the interpolation capabilities of the VEM space, by a refined analysis we show that the H^1 interpolation error on each element (polygon) E can be split into a boundary contribution and a bulk contribution. Although for basic VEM spaces the bulk contribution will dominate the error, this investigation leads to the following idea: if one increases the degree of the VEM only inside the element then the bulk approximation order improves. For such "enriched" VEM, elements with small edges indeed lead to more accurate interpolation in a sense that we will make precise.

In the VEM setting, in order for such refined interpolation property to reflect also on an improved convergence property, one needs also to ameliorate the stability estimates of the scheme. Indeed, standard VEM stabilization estimates assume a bounded number of edges, an hypothesis that we are able to eliminate leading to final convergence estimates that show an improvement in the presence of many small edges (with respect to standard estimates looking only at the element diameter). Finally, we will show some numerical test both for quadrilateral/Voronoi meshes with edge subdivision and on meshes generated by an agglomeration procedure. The numerical experiments are in accordance with the theoretical results and help to appreciate from the practical standpoint the two separate bulk and boundary contributions to the error.

The results of this talk can be found in the recent preprint [3]. Possible future investigations include the possibility of extending this approach also to H_{div} and H_{curl} exact complex spaces (see for instance [4]) recently analyzed in [5] under more stringent mesh requirements.

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H^1-conforming finite element cochain complexes on Cartesian meshes

Francesca Bonizzoni
(joint work with Guido Kanschat)

In recent years considerable effort was put into the development of H^1-conforming methods with exact divergence constrain in two and three dimensions. This talk, instead, is concerned with the full finite element cochain complex on Cartesian meshes of arbitrary dimension.

The starting point is the one-dimensional H^1-conforming finite element cochain complex based on cubic polynomials with modified Hermitian interpolation, and quadratic polynomials with Lagrangian interpolation. Applying a general result on the cochain property of interpolation operators, we provide commuting interpolation operators for differentiable functions. Quasi-interpolation operators with weighted node functionals are then derived, and proved to be commuting with the exterior derivative and L^2-stable.

Based on the tensor product construction (see [2]), we obtain H^1-conforming finite element spaces on Cartesian meshes of arbitrary dimension. Moreover, commuting tensor product interpolation operators as well as commuting L^2-stable tensor product quasi-interpolation operators are derived.

The construction of the H^1-conforming finite element cochain complex and the corresponding commuting quasi-interpolation operators is then extended to higher order polynomial spaces.

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A hybridized high-order method for unique continuation subject to the Helmholtz equation

Guillaume Delay

(joint work with Erik Burman, Alexandre Ern)

We study the following unique continuation problem for the Helmholtz equation:

$$-\Delta u - \omega^2 u = f \text{ in } \Omega, \qquad u = g \text{ in } \varpi,$$

where $\Omega \subset \mathbb{R}^d$ and $\varpi \subset \Omega$. Moreover f and g are given data and $\omega > 0$ is the wave number. This problem is ill-posed since no boundary conditions are given. In the sequel, we have only noised data $g_\delta := g + \delta$ instead of g where δ is unknown.

This problem is classically regularized at the continuous level, using for instance the Tikhonov regularization [5] or the quasi-reversibility method [3]. The regularized problem is then discretized. In the present work, we first discretize the ill-posed problem and then regularize it at the discrete level [1, 2].

A hybrid discontinuous Galerkin method is considered. For a polynomial degree $k \geq 1$, we define

$$U_{\mathcal{T}} := \{v_h \in L^2(\Omega) \quad | \ v_h|_T \in \mathbb{P}^k(T) \ \ \forall T \in \mathcal{T}_h\},$$
$$U_{\mathcal{F}} := \{v_h \in L^2(\mathcal{F}_h) \ | \ v_h|_F \in \mathbb{P}^k(F) \ \ \forall F \in \mathcal{F}_h\},$$

where $\mathcal{F}_h$ is the set of all faces of the mesh $\mathcal{T}_h$. Moreover, we define the space $U_h := U_{\mathcal{T}} \times U_{\mathcal{F}}$ and U_{h0} is the subset of U_h with null polynomials attached to the faces composing $\partial\Omega$. In the sequel, for all $v_h \in U_h$ and all $T \in \mathcal{T}_h$, we denote by u_T the polynomial of u_h attached to the cell T and by $u_{\partial T}$ the polynomials of u_h attached to the faces composing ∂T. Moreover we define $u_{\mathcal{T}} := (u_T)_{T\in\mathcal{T}_h}$.

The numerical method used is: Find $(u_h, \xi_h) \in U_h \times U_{h0}$ such that

$$(u_h, v_h)_\varpi + s_h(u_h, v_h) + a_h(\xi_h, v_h) = (g_\delta, v_h)_\varpi \qquad \forall v_h \in U_h,$$
$$a_h(u_h, \eta_h) - \sigma_h(\xi_h, \eta_h) = (f, \eta_h)_\Omega \qquad \forall \eta_h \in U_{h0},$$

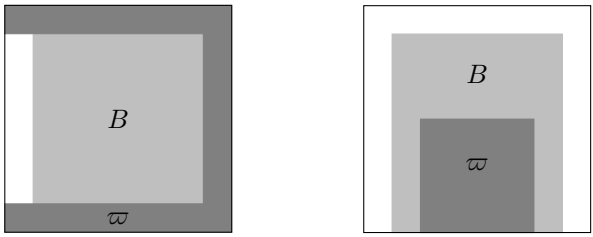

FIGURE 1. Considered configurations for ϖ and B.

where $s_h(u_h, v_h) := \sum_{T\in\mathcal{T}} h^{2k}(u_T, v_T)_T + h_T^{-1}(u_T - u_{\partial T}, v_T - v_{\partial T})_{\partial T}$, $a_h(u_h, \eta_h) := \sum_{T\in\mathcal{T}_h}(\nabla u_T, \nabla\eta_T)_T - (u_T - u_{\partial T}, \nabla\eta_T\cdot n_T)_{\partial T} - (\eta_T - \eta_{\partial T}, \nabla u_T\cdot n_T)_{\partial T} - \omega^2(u_T, \eta_T)_T$ and $\sigma_h(\xi_h, \eta_h) := \sum_{T\in\mathcal{T}_h}(\nabla\xi_T, \nabla\eta_T)_T + h_T^{-1}(\xi_T - \xi_{\partial T}, \eta_T - \eta_{\partial T})_{\partial T}$.

This discretization has the advantage of using arbitrary high-order polynomials, which is motivated by the fact that high-order is well-suited to solve the Helmholtz problem [4]. Moreover, we can use a static condensation technique so that the global system involves the face degrees of freedom only. Also note that the primal-dual formulation of the problem makes the scheme stable unconditionally, which is not always the case for the well-posed Helmholtz problem.

Using an inf-sup condition and the consistency of the scheme, we can prove the convergence of the error in residual norm:

$$|||I_h^k(u) - u_h, \xi_h||| \leq C(h^k\|u\|_{H^{k+1}(\Omega)} + \|\delta\|_\varpi),$$

where $I_h^k(u) \in U_h$ is an approximation of u and $|||v_h, \eta_h|||^2 := s_h(v_h, v_h) + \|v_h\|_\varpi^2 + \|v_h\|_R^2 + \sigma_h(\eta_h, \eta_h)$ with $\|v_h\|_R^2 := \sum_{T\in\mathcal{T}_h} h_T^2\|\Delta v_T + \omega^2 v_T\|_T^2 + \sum_{F\in\mathcal{F}_h} h_F\|[\![\nabla v_\mathcal{T}]\!]_F\|_F^2$ where $[\![\nabla v_\mathcal{T}]\!]_F$ denotes the jump of $\nabla v_\mathcal{T}$ across F.

This convergence result is then combined with the following conditional stability. Let $B \subset \Omega$. There exist $C(\omega) > 0$ and $\alpha \in (0, 1]$ (depending on ϖ and B) such that for all $v \in H^1(\Omega)$, we have

$$\|\nabla v\|_B + \omega\|v\|_B \leq C(\omega)\omega(\|v\|_\Omega + \rho(v))^{1-\alpha}(\|v\|_\varpi + \rho(v))^\alpha,$$

with $\rho(v) := \|\Delta v + \omega^2 v\|_{H^{-1}(\Omega)}$. Possible configurations for ϖ and B are given in Figure 1. For more information, the reader can report to [2].

Using the convergence of the error in residual norm and the conditional stability, we can prove that

$$\|\nabla(u - u_h)\|_B + \omega\|u - u_h\|_B \leq C(\omega)\omega h^{\alpha k}(\|u\|_{H^{k+1}(\Omega)} + h^{-k}\|\delta\|_\varpi).$$

Note that the convergence rate in energy norm is αk. Moreover, when $h^{-k}\|\delta\|_\varpi$ becomes larger than $\|u\|_{H^{k+1}(\Omega)}$, one has to stop the refinement, otherwise the error will increase.

Numerical tests have been run in order to corroborate these results on the two configurations of Figure 1. We observed convergence rate of k for the configuration on the left panel and of about $0.3 \times k$ for the configuration on the right panel.

Note that this corroborates the fact that we expect the configuration on the left to be more stable than the configuration on the right.

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Finite Elements For Divdiv-Conforming Symmetric Tensors

Long Chen

(joint work with Xuehai Huang)

Let $\Omega \subset \mathbb{R}^3$ be a three dimensional bounded domain. Let $\mathbb{S}$ be the space of all symmetric matrices of dimension $\mathbb{R}^{3\times 3}$. We study the space

$$\boldsymbol{H}(\mathrm{div}\mathbf{div}, \Omega; \mathbb{S}) := \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega; \mathbb{S}) : \mathrm{div}\mathbf{div}\boldsymbol{\tau} \in L^2(\Omega)\}$$

and construct corresponding finite element spaces, which can be applied to discretize the linearized Einstein-Bianchi system [4, Section 4.11] and the mixed formulation of the biharmonic equation [3].

Let K be a polyhedron. The set of edges of K is denoted by $\mathcal{E}(K)$, the faces by $\mathcal{F}(K)$, and the vertices by $\mathcal{V}(K)$. Based on a polynomial complex and a Koszul complex, we first obtain a decomposition of the polynomial tensor space

$$\mathbb{P}_k(K; \mathbb{S}) = \mathrm{sym}\mathbf{curl}\, \mathbb{P}_{k+1}(K; \mathbb{T}) \oplus \boldsymbol{x}\boldsymbol{x}^\top \mathbb{P}_{k-2}(K), \tag{1}$$

where $\mathbb{T}$ is the space of all traceless matrices. We can show that $\mathrm{div}\mathbf{div}: \boldsymbol{x}\boldsymbol{x}^\top \mathbb{P}_{k-2}(K) \to \mathbb{P}_{k-2}(K; \mathbb{R}^3)$ is a bijection.

We then present a Green's identity. Let $\boldsymbol{\tau} \in \mathcal{C}^2(K; \mathbb{S})$ and $v \in H^2(K)$. Then

$$\begin{aligned}(\mathrm{div}\mathbf{div}\boldsymbol{\tau}, v)_K = (\boldsymbol{\tau}, \nabla^2 v)_K &- \sum_{F\in\mathcal{F}(K)} \sum_{e\in\mathcal{E}(F)} (\boldsymbol{n}_{F,e}^\top \boldsymbol{\tau}\boldsymbol{n}, v)_e \\ &- \sum_{F\in\mathcal{F}(K)} \left[(\boldsymbol{n}^\top\boldsymbol{\tau}\boldsymbol{n}, \partial_n v)_F - (2\mathrm{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \partial_n(\boldsymbol{n}^\top\boldsymbol{\tau}\boldsymbol{n}), v)_F\right], \end{aligned} \tag{2}$$

which motivates two trace operators

$$\mathrm{tr}_1(\boldsymbol{\tau}) = \boldsymbol{n}^\top\boldsymbol{\tau}\boldsymbol{n}, \quad \mathrm{tr}_2(\boldsymbol{\tau}) = 2\mathrm{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \partial_n(\boldsymbol{n}^\top\boldsymbol{\tau}\boldsymbol{n}).$$

Let K be a tetrahedron. Take the space of shape functions

$$\boldsymbol{\Sigma}_{\ell,k}(K) := \mathrm{sym}\mathbf{curl}\, P_\ell(\Omega; \mathbb{T}) \oplus \boldsymbol{x}\boldsymbol{x}^\top \mathbb{P}_{k-2}(\Omega)$$

with $k \geq 3$ and $\ell \geq \max\{k-1, 3\}$. By the decomposition (1), we have

$$\mathbb{P}_{\min\{\ell,k\}}(K;\mathbb{S}) \subseteq \boldsymbol{\Sigma}_{\ell,k}(K) \subseteq \mathbb{P}_{\max\{\ell,k\}}(K;\mathbb{S}) \quad \text{and} \quad \boldsymbol{\Sigma}_{k,k}(K) = \mathbb{P}_k(K;\mathbb{S}).$$

The most interesting cases are $\ell = k-1$ and $\ell = k$ which correspond to RT and BDM $H(\mathrm{div})$-conforming elements for the vector functions, respectively.

For each edge, we chose two normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$. The degrees of freedom are given by

$$\boldsymbol{\tau}(\delta) \quad \forall\, \delta \in \mathcal{V}(K), \tag{3}$$

$$(\boldsymbol{n}_i^\top \boldsymbol{\tau} \boldsymbol{n}_j, q)_e \quad \forall\, q \in \mathbb{P}_{\ell-2}(e), e \in \mathcal{E}(K),\ i,j = 1,2, \tag{4}$$

$$(\boldsymbol{n}^\top \boldsymbol{\tau} \boldsymbol{n}, q)_F \quad \forall\, q \in \mathbb{P}_{\ell-3}(F), F \in \mathcal{F}(K), \tag{5}$$

$$(2\mathrm{div}_F(\boldsymbol{\tau}\boldsymbol{n}) + \partial_n(\boldsymbol{n}^\top \boldsymbol{\tau} \boldsymbol{n}), q)_F \quad \forall\, q \in \mathbb{P}_{\ell-1}(F), F \in \mathcal{F}(K), \tag{6}$$

$$(\boldsymbol{\tau}, \boldsymbol{\varsigma})_K \quad \forall\, \boldsymbol{\varsigma} \in \nabla^2 \mathbb{P}_{k-2}(K) \oplus \mathrm{sym}(\boldsymbol{x} \times \mathbb{P}_{\ell-2}(K;\mathbb{T})), \tag{7}$$

$$(\boldsymbol{\tau}\boldsymbol{n}, \boldsymbol{n} \times \boldsymbol{x} q)_{F_1} \quad \forall\, q \in \mathbb{P}_{\ell-2}(F_1), \tag{8}$$

where $F_1 \in \mathcal{F}(K)$ is an arbitrarily but fixed face. The degrees of freedom (8) will be regarded as interior degrees of freedom to the tetrahedron K, that is the degrees of freedom (8) are double-valued on each face $F \in \mathcal{F}_h^i$ when defining the global finite element space. Unisolvence can be found in our work [2].

A two dimensional version of $\boldsymbol{H}(\mathrm{div}\mathbf{div}, \Omega; \mathbb{S})$ element has been constructed in [1]. In addition, a discrete div-div complex has been established

$$\boldsymbol{RT} \xrightarrow{\subset} V_{\ell+1} \xrightarrow{\mathrm{symcurl}} \boldsymbol{\Sigma}_{\ell,k} \xrightarrow{\mathrm{div}\mathbf{div}} \mathbb{P}_{k-2} \to 0. \tag{9}$$

We plan to construct finite element for $\boldsymbol{H}(\mathrm{symcurl}, \Omega; \mathbb{T})$ and obtain similar discrete complex in three dimensions.

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Polygonal staggered discontinuous Galerkin methods

Eun-Jae Park

(joint work with Lina Zhao, Dohyun Kim)

In this report, we propose and analyse polygonal staggered discontinuous Galerkin (DG) methods with applications to the coupled Stokes and Darcy-Forchheimer problem and Darcy flows in fractured porous media (cf. [1, 2]). The key idea of the staggered DG methods is to divide the initial partition (general polygonal meshes) into the union of triangles by connecting the interior point to all the vertices of

the initial partition (see Figure 1 for an illustration). Then the corresponding basis functions for the associated variables with staggered continuities are defined on the resulting triangulations. Consequently, the method can be defined without resorting to numerical flux. In summary, staggered DG methods offer the following salient features: It is locally conservative over each dual element; superconvergence can be obtained; it works on fairly general meshes; it handles hanging nodes naturally; it is stable without numerical flux or penalty term.

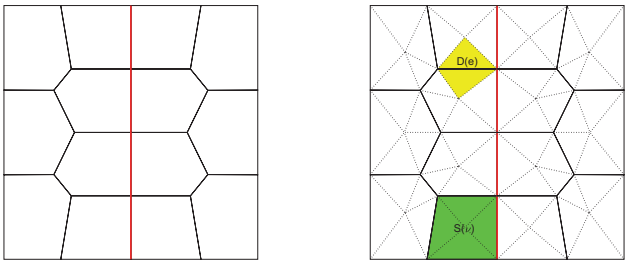

FIGURE 1. Schematic of the primal mesh, the dual mesh and the primal simplicial sub-meshes.

The fluid flow between porous media and free-flow zones can be viewed as a coupled problem with two physical systems interacting across an interface. The coupled model has received great attention and our goal is to develop a numerical scheme for the coupled Stokes and Darcy-Forchheimer problem:

$$\nabla \cdot \boldsymbol{u}_D = f_D \quad \text{in } \Omega_D,$$

$$\frac{\mu}{\rho} K^{-1} \boldsymbol{u}_D + \frac{\beta}{\rho} |\boldsymbol{u}_D| \boldsymbol{u}_D + \nabla p_D = \boldsymbol{g}_D \quad \text{in } \Omega_D$$

and

$$\begin{aligned} \nabla \cdot \boldsymbol{\sigma}_S + \nabla p_S &= \boldsymbol{f}_S && \text{in } \Omega_S, \\ \boldsymbol{\sigma}_S &= -\nu \nabla \boldsymbol{u}_S && \text{in } \Omega_S, \\ \nabla \cdot \boldsymbol{u}_S &= 0 && \text{in } \Omega_S, \end{aligned}$$

where Ω_S and Ω_D represent the Stokes region and Darcy-Forchheimer region, respectively. Here the Beavers–Joseph–Saffman interface conditions are exploited.

In the proposed scheme the interface conditions are enforced by switching the roles of the variables met on the interface, which eliminate the hassle of introducing additional variables. The main difficulty lies in the proof of the optimal convergence for the flux variable. To this end, a new discrete trace inequality and a generalized Poincaré–Friedrichs inequality are established. Our analysis can be carried out by exploiting these two inequalities and the monotone property of the nonlinear operator under certain regularity assumptions without any restrictions on the source terms. The theoretical results show that optimal convergence rates $\mathcal{O}(h^{k+1})$ can be achieved for all the variables measured in L^2 norm if k-th order polynomials are used for all the variables.

The fluid flow in fracture porous media can be described by Darcy's law with appropriate jump conditions along the fracture. Darcy's law in the porous media Ω_B and along the fracture Γ is given by

$$\begin{aligned} \boldsymbol{u} + K\nabla p &= 0 && \text{in } \Omega_B, \\ \nabla \cdot \boldsymbol{u} &= f && \text{in } \Omega_B, \\ -\nabla_t \cdot (K_\Gamma \nabla_t p_\Gamma) &= \ell_\Gamma f_\Gamma + [\boldsymbol{u} \cdot \boldsymbol{n}_\Gamma] && \text{on } \Gamma, \\ p &= p_0 && \text{on } \partial\Omega_B, \\ p_\Gamma &= g_\Gamma && \text{at } \partial\Gamma. \end{aligned}$$

Here $\boldsymbol{u}, p$ are the flow velocity and the pressure in Ω_B and p_Γ is the pressure along Γ, respectively. Then the jump conditions on Γ are given by

$$\eta_\Gamma\{\boldsymbol{u} \cdot \boldsymbol{n}_\Gamma\} = [p], \quad \alpha_\Gamma[\boldsymbol{u} \cdot \boldsymbol{n}_\Gamma] = \{p\} - p_\Gamma.$$

We discretize Ω_B by the staggered DG method and Γ by standard continuous Galerkin method. The jump conditions along the fracture are enforced similarly to the previous case.

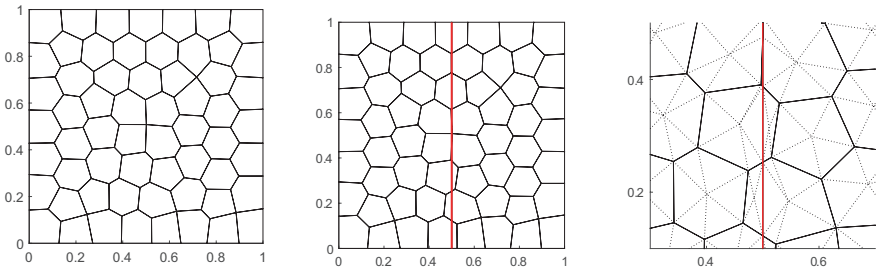

FIGURE 2. Underlying polygonal mesh ($\mathcal{T}_u$, left), modified mesh ($\tilde{\mathcal{T}}_u$, center) and its magnified view with dual edges (right). The modified mesh contains both sliver elements and small edges.

Staggered DG methods are numerically verified that they are robust to mesh distortion and work reliably even when the underlying mesh contains small edges. These properties make staggered DG method favorable since generating a mesh satisfying strong regularity assumption can be difficult and time consuming when the fracture has complex geometry. Our analysis is carried out without the assumption on the ratio between mesh size and edge size. *A priori* error estimates are carried out with optimal convergence $\mathcal{O}(h^{k+1})$ for all variables measured in L^2-norm. Several numerical examples are presented to verify the theoretical findings and to demonstrate performance of the proposed method on meshes with extreme cases. Finally, the proposed method is implemented on meshes obtained from background mesh without *a priori* knowledge on the fracture so that the resulting mesh contains small edges and sliver elements, see Figure 2. The numerical

experiments suggest that our method can handle fairly general meshes so that it can be flexibly applied to fractured porous media with complex geometries.

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Recent Developments in the TDNNS method

Joachim Schöberl

(joint work with Astrid Pechstein)

We consider mixed variational formulations for elasticity, i.e. we search for a symmetric stress tensor σ and a displacement vector u such that

$$\begin{aligned} \int A\sigma : \tau \quad + \quad \langle \operatorname{div} \tau, u \rangle \quad &= \quad 0, \\ \langle \operatorname{div} \sigma, v \rangle \quad &= \quad f(v) \end{aligned}$$

holds for all test functions τ and v. Different mixed methods differ in the interpretation of the duality pairing $\langle \operatorname{div} \sigma, v \rangle$. It may be interpreted as $(\operatorname{div} \sigma, v)_{L_2} = \int \operatorname{div} \sigma \, v$, or as $\langle \operatorname{div} \sigma, v \rangle_{H^{-1} \times H^1} = -\int \sigma : \nabla v$. Different formulations lead to different smoothness of spaces, and thus different types of finite elements. In the TDNNS method we define the duality pair as $\langle \operatorname{div} \sigma, v \rangle_{H(\operatorname{curl})^* \times H(\operatorname{curl}}$. The required smoothness is continuity for the tangential component of the displacement vector, and continuity of the normal-normal component of the stress matrix (thus the name of the method). It is easy to construct finite element spaces with this requirements, in 2D as well as in 3D. The discrete variational formulation is given by

$$\begin{aligned} \int A\sigma : \tau \qquad + \quad \textstyle\sum_T \left\{ \int_T \operatorname{div} \tau \cdot u - \int_{\partial T} \tau_{n\tau} u_\tau \right\} \quad &= \quad 0, \\ \textstyle\sum_T \left\{ \int_T \operatorname{div} \sigma \cdot v - \int_{\partial T} \sigma_{n\tau} v_\tau \right\} \quad &= \quad -\int f \cdot v. \end{aligned}$$

The continuity of the normal component of u, as well as the tangential component of the normal stress vector σ_n follows from the formulation in weak sense. In early works [1, 2] the method has been proposed, and analyzed in discrete norms.

In more recent work [3] we proved error estimates in the norm

$$\|\sigma - \sigma_h\|_{H(\operatorname{div}\operatorname{div}),h} + \|u - u_h\|_{H(\operatorname{curl})},$$

where the discrete norm

$$\|\sigma\|_{H(\operatorname{div}\operatorname{div}),h} := \|\sigma\|_{L_2} + \sup_{w_h \in W_h \subset H^1} \frac{\langle \operatorname{div}\operatorname{div} \sigma, w_h \rangle}{\|w_h\|_{H^1}}$$

mimics the norm $H(\operatorname{div}\operatorname{div}) = \|\sigma\|_{L_2} + \|\operatorname{div}\operatorname{div}\sigma\|_{H^{-1}}$. With this norm, we could extend the method as well as the analysis to the Reissner Mindlin plate model [4]. The moments are taken in $H(\operatorname{div}\operatorname{div})$, the rotations in $H(\operatorname{curl})$, and

the vertical displacement in H^1. Thus, discrete rotations in the Nédélec finite space can match exactly with gradients from Lagrangian finite elements, and shear locking is avoided.

The Hellan-Herrmann-Johnson finite element method (see [5] and references therein) for the Kirchhoff plate equation uses symmetric, nn-continuous finite elements for the moments, and continuous finite elements for the vertical displacement: Find $\sigma_h \in \Sigma_h \subset H(\operatorname{div}\operatorname{div})$ and $w_h \in W_h \subset H^1$ such that

$$\begin{aligned} \int \sigma_h \tau \quad + \quad \textstyle\sum_T \int_T \operatorname{div} \tau \, \nabla w_h + \int_{\partial T} \tau_{nt} \nabla_t w_h &= 0 \\ \textstyle\sum_T \int_T \operatorname{div} \sigma_h \nabla v + \int_{\partial_T} \sigma_{nt} \nabla_t v &= \int f v \end{aligned}$$

for all discrete τ and v. This mixed method satisfies the magic discrete kernel inclusion

$$V_{h,0} \subset V_0$$

which leads to best-approximation property of the bending moments σ without the need of approximation of the displacement:

$$\|\sigma - \sigma_h\|_{L_2} \leq \inf_{\tau_h \in \Sigma_h} \|\sigma - \tau_h\|_{L_2} + \|f - I_h f\|.$$

The distributional evaluation of the div div-operator has led to equilibrated residual error estimates for the Kirchhoff plate [6].

The discrete bilinear forms for $\langle \operatorname{div} \sigma, v \rangle$ of the TDNNS-method and the HHJ method are tightly related, i.e.

$$b^{TDNNS}(\sigma, \nabla w) = b^{HHJ}(\sigma, w).$$

This relation has lead to the design of special bubble functions, and improved error estimates for the TDNNS method:

$$\|\sigma - \sigma_h\|_{L_2} + \|\operatorname{curl}(u - u_h)\|_{L_2} \preceq \inf_{\tau_h, v_h} \|\sigma - \tau_h\|_{L_2} + \|\operatorname{curl}(u - v_h)\|_{L_2} + \|f - I_h f\|.$$

A bad approximation of gradient components in the displacement field does not hurt the stress approximation.

The following version of an elasticity complex is useful for the design of finite elements:

$$\begin{array}{ccccccc} [H^1]^2 & \xrightarrow{\text{sym-}\operatorname{curl}^T} & H(\operatorname{div}\operatorname{div}) & \xrightarrow{\operatorname{div}} & H(\operatorname{curl})^* & \xrightarrow{\operatorname{div}} & [H^1]^* \\ \bigcup & & \bigcup^{[1]} & & \bigcup^{[1]} & & \bigcup^{[1]} \\ \mathcal{W}_h & \xrightarrow{\text{sym-}\operatorname{curl}^T} & \Sigma_h & \xrightarrow{\operatorname{div}} & \mathcal{N}_h^* & \xrightarrow{\operatorname{div}} & \mathcal{L}_h^* \end{array}$$

[1] slightly non-conforming

- $\operatorname{range}(\text{sym-}\operatorname{curl}^T) = \operatorname{kern}(\operatorname{div}\operatorname{div})$
- the operator $\operatorname{div}\operatorname{div} : \Sigma_h \to \mathcal{L}_h^*$ from Hellan-Herrmann-Johnson is onto
- the operator $\operatorname{div} : \Sigma_h \to \mathcal{N}_h^*$ from TDNNS is not onto
- Adding $\text{sym-}\operatorname{curl}^T$ of bubbles from $\mathcal{W}_h$ to Σ_h leads to LBB-stability of TDNNS method and does not spoil the exactness of HHJ

This complex guided the design of the cheapest, second order accurate method: choose $\Sigma_h = P^{1,sym} + \text{sym-curl}^T(b,0) + \text{sym-curl}^T(0,b)$, where b is the cubic bubble, and $V_h = \mathcal{N}_2^I$, the second order type-I Nédélec space. The element has 11 + 8 degrees of freedom.

Another recent developement of TDNNS methods is the extension to geometric non-linear elasticity, including robustness of nearly incompressible materials [7, 8].

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Mixed Finite Element Methods for Nonlinear Elasticity and Shells

Michael Neunteufel

(joint work with Astrid Pechstein, Joachim Schöberl)

Nonlinear elasticity. The TDNNS method [6] is a mixed formulation for linear elasticity, where the stresses are normal-normal continuous and the displacements Nédélec elements. We extend it to nonlinear elasticity by lifting the distributional gradient ∇u to a regular field $\boldsymbol{F}$ with the first Piola–Kirchhoff stress tensor $\boldsymbol{P}$

$$\mathcal{L}(u, \boldsymbol{F}, \boldsymbol{P}) = \int_{\mathcal{T}} \mathcal{W}(\boldsymbol{F}) - f \cdot u \, dx - \langle \boldsymbol{F} - (\nabla u + \boldsymbol{I}), \boldsymbol{P} \rangle_h,$$

where $\mathcal{W}(\cdot)$ denotes an energy potential. Considering a lifting of the distributional Cauchy–Green strain tensor $\boldsymbol{C}(u)$ to an independent $\boldsymbol{C}$ leads to

$$\mathcal{L}(u, \boldsymbol{C}, \boldsymbol{\Sigma}, \hat{u}) = \sum_{T \in \mathcal{T}} \int_T \mathcal{W}(\boldsymbol{C}) + \frac{1}{2}(\boldsymbol{C} - \boldsymbol{C}(u)) : \boldsymbol{\Sigma} \, dx - \int_{\partial T} (\boldsymbol{F}(u)\boldsymbol{\Sigma})_{nn}(u - \hat{u})_n \, ds.$$

It is crucial to break the normal-normal continuity of $\boldsymbol{\Sigma}$ and enforcing this continuity of $\boldsymbol{P} = \boldsymbol{F}(u)\boldsymbol{\Sigma}$ with $\hat{u}$ [2]. Incompressible materials can be handled by a lifting of the distributional determinant to a new field J [5].

Nonlinear shells. The HHJ method for plates [1] uses continuous displacements and normal-normal continuous moments. We extend it to nonlinear Koiter shells given by a triangulation $\mathcal{T}$, skeleton $\mathcal{E}$, thickness t, and (co-)normal vector $\hat{\mu}$, $\hat{\nu}$ [3]

$$\mathcal{L}(u,\boldsymbol{\sigma}) = \frac{t}{2}\|\boldsymbol{E}_\tau(u)\|^2_{\boldsymbol{M}} - \frac{6}{t^3}\|\boldsymbol{\sigma}\|^2_{\boldsymbol{M}^{-1}} + \sum_{T\in\mathcal{T}}\int_T (\boldsymbol{F}_\tau^\top \nabla_\tau \nu - \nabla_\tau \hat{\nu}) : \boldsymbol{\sigma}\, dx +$$
$$+ \sum_{E\in\mathcal{E}}\int_E (\arccos(\nu_L\cdot\nu_R) - \arccos(\hat{\nu}_L\cdot\hat{\nu}_R))\boldsymbol{\sigma}_{\hat{\mu}\hat{\mu}}\, ds.$$

$\boldsymbol{M}$ denotes the material tensor, ν the deformed normal vector, and $\boldsymbol{E}_\tau(u) = 0.5(\boldsymbol{F}_\tau^\top\boldsymbol{F}_\tau - \boldsymbol{P}_\tau)$, $\boldsymbol{P}_\tau = \boldsymbol{I} - \hat{\nu}\otimes\hat{\nu}$, the Green strain tensor. By adding shearing unknowns [5] the TDNNS method for plates [6] gets extended to Naghdi shells. To alleviate membrane locking for shells discretized by triangles we propose to use the interpolation operator $\mathcal{I}_h^{\mathcal{R}}$ into the Regge elements. Inserted into the membrane term $\|\mathcal{I}_h^{\mathcal{R}}\boldsymbol{E}_\tau(u)\|^2_{\boldsymbol{M}}$ it relaxes the kernel constraints [4].

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A normal-tangential stress space - The MCS method for incompressible flows and pressure robust equilibration

Philip L. Lederer

(joint work with Jay Gopalakrishnan, Christian Merdon, Joachim Schöberl)

Let u, p be the velocity and pressure respectively, and f, ν be a given force and viscosity. On a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ the Stokes problem reads as

$$\begin{cases} \frac{1}{\nu}\sigma - \nabla u = 0 & \text{in } \Omega, \qquad \operatorname{div}(\sigma) - \nabla p = -f \quad \text{in } \Omega, \\ \operatorname{div}(u) = 0 & \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

On a given triangulation $\mathcal{T}$, the **MCS method** (see [1, 2]) approximates (u, p) in $\mathrm{RT}_k(\mathcal{T}) \times P^k(\mathcal{T})$, i.e. the Raviart-Thomas space and the space of element-wise polynomials of order k, respectively. The **normal-tangential continuous** stress space is denoted by

$$\Sigma_h(\mathcal{T}) := \{\tau_h \in P_k(\mathcal{T})^{d\times d} : [\![(\tau_h)_{nt}]\!]_F = 0 \text{ for all facets } F,\ \operatorname{tr}\sigma_h = 0\},$$

where $[\![\cdot]\!]_F$ denotes the jump operator and $(\cdot)_{nt}$ the normal-tangential trace. Then we define the method, find $(\sigma_h, u_h, p_h) \in \Sigma_h \times \mathrm{RT}_k \times P^k$ such that

$$\begin{aligned}(\nu^{-1}\sigma_h, \tau_h) + \langle \mathrm{div}(\tau_h), u_h\rangle_{V_h} &= 0 && \text{for all } \tau_h \in \Sigma_h(\mathcal{T}),\\ \langle \mathrm{div}(\sigma_h), v_h\rangle_{V_h} + (\mathrm{div}\, v_h, p_h) &= (-f, v_h)_\Omega && \text{for all } v_h \in \mathrm{RT}_k(\mathcal{T}),\\ (\mathrm{div}\, u_h, q_h) &= 0 && \text{for all } q_h \in P^k(\mathcal{T}),\end{aligned}$$

with the discrete duality pair

$$\langle \mathrm{div}(\tau_h), v_h\rangle_{V_h} := \sum_{T\in\mathcal{T}} \int_T \mathrm{div}(\tau_h)\cdot v_h - \sum_{F\in\mathcal{F}} \int_F [\![(\tau_h)_{nn}]\!](v_h)_n.$$

The MCS method provides exactly divergence-free discrete velocities leading to pressure robustness, energy stability (for the Navier-Stokes equations) and converges with optimal order. Further it can be used to derive a pressure robust equilibration with guaranteed upper bounds and local efficiency (see [3]), and a novel method for the stream function formulation (see [4]).

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A posteriori error analysis of the inf-sup constant for the divergence

Dietmar Gallistl

In this contribution, two a posteriori error estimates for the numerical approximation scheme from [1] for the inf-sup constant for the divergence (also known as the LBB constant) are shown. Given a polyhedral, connected, bounded, open Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, the inf-sup constant $\beta = \beta(\Omega)$ is defined by

$$\beta := \inf_{p\in L^2_0(\Omega)\setminus\{0\}} \sup_{v\in H^1_0(\Omega;\mathbb{R}^n)\setminus\{0\}} \frac{(p, \mathrm{div}\, v)_{L^2(\Omega)}}{\|p\|_{L^2(\Omega)}\|Dv\|_{L^2(\Omega)}},$$

where $L^2_0(\Omega)$ and $H^1_0(\Omega;\mathbb{R}^n)$ are the usual Lebesgue and Sobolev spaces for the pressure and velocity variable in a Stokes flow problem. Under the two assumptions that

(A) the squared inf-sup constant $\mu := \beta^2$ is an eigenvalue of the Cosserat operator separated from the remaining part of the spectrum by some spectral gap $\delta > 0$

and

(B) that the mesh size of the finite element triangulation is so small that the first normalized discrete Cosserat eigenfunction ξ_h and its projection $\Pi\xi_h$ to the continuous eigenspace satisfy

$$\lambda \leq (\Pi\xi_h, \xi_h)_{L^2(\Omega)} \leq 1$$

for some $\lambda > 0$ and all meshes in the class of possible refinements,

the first reliability estimate bounds the eigenvalue error from above and below by an error estimator up to multiplicative constants. More precisely, for any positive $0 < \varepsilon < \infty$, the eigenfunction error satisfies

$$(\mu_h - \mu)\left[(\Pi\xi_h, \xi_h)_{L^2(\Omega)} - \frac{\varepsilon}{2}\right] \leq \mu_h \left[\frac{n}{2\varepsilon\delta} + 1\right] \|R\|^2_{L^2(\Omega)}$$

$$\text{and} \qquad \|R\|^2_{L^2(\Omega)} \leq (\mu_h - \mu)\left(\frac{1}{\mu} + \frac{1}{\delta}\right).$$

Here, R is the divergence-free part in the Helmholtz decomposition of the discrete eigenfunction ξ_h, and is regarded as a basically computable quantity.

In the second reliability error estimate

$$(\mu_h - \mu) b(\Pi\xi_h, \xi_h) \leq \sqrt{n\mu_h}\|R\|_{L^2(\Omega)},$$

the reliability constant converges to 1 as the mesh size decreases (at the expense of a suboptimal efficiency estimate) and so allows for guaranteed enclosures of the inf-sup constant on sufficiently fine meshes. The results, their proofs, and actual computations can be found in [2].

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Multigrid injection operators for hybridized mixed and discontinuous Galerkin methods

Guido Kanschat

(joint work with Peipei Lu, Andreas Rupp)

Multigrid methods in the literature on mixed and discontinuous Galerkin methods in hybridized form (summarized as HDG methods following [5]) were confined to heterogeneous methods where the HDG discretization is first projected to a nonconforming [1, 8] or conforming [4, 6] finite element space in which standard multigrid methods can be applied. In our work, we apply the geometric multigrid method in a homogeneous way to the HDG method, such that we employ an HDG discretization on every level.

While we can use standard smoothers like the Jacobi and Gauß-Seidel methods analyzed in [2], the construction of injection operators from coarse to fine level requires additional effort for methods defined on the skeleton of the mesh. The

first successful effort in this direction was [3], and indeed we were able to prove uniform convergence using this injection operator in [7].

The injection operator from [3] uses interpolation into a conforming subspace. From the implementation point of view, this is a drawback, since it involves a wide stencil and thus breaks data locality. Therefore, we decided to look further for more local injection operators. On this quest, we were guided by two crucial properties of the injection operator I_ℓ from mesh level $\ell - 1$ to mesh level ℓ:

(1) Weak stability of the injection operator: there is a constant c independent of ℓ, such that

$$\|I_\ell \lambda\|_\ell \leq c\|\lambda\|_{\ell-1} \qquad \forall \lambda \in M_{\ell-1}.$$

Here, $M_{\ell-1}$ is the space of finite elements on the skeleton and the norm $\|\cdot\|_\ell$ is a scaled L^2-norm on the skeleton commensurate with the L^2-norm in the domain.

(2) Trace identity for conforming linear finite elements:

$$I_\ell \gamma_{\ell-1} w = \gamma_\ell w \qquad \forall w \in V^c_{\ell-1},$$

where $V^c_{\ell-1}$ is the lowest order conforming finite element space on the coarser level, and γ_ℓ denotes the trace operator from the domain to the faces on mesh level ℓ. In words, the injection operator must transform a function which is coplanar on the boundary of a coarse level cell into a function which is coplanar on all the boundary faces of the cells obtained by refinement.

The multigrid convergence analysis can be based solely on these two assumptions on the injection operator and a set of assumptions on the particular HDG method, yielding proven, uniform convergence. These are met by a wide class including hybridized LDG, Raviart-Thomas and Brezzi-Douglas-Marini methods. Furthermore, we currently require elliptic regularity. Note that the stability assumption is fairly weak, since it does not involve the energy norm. Nevertheless, paired with the second assumption, energy stability can be shown for a wide class of HDG methods.

Given now sufficient conditions for uniform convergence on the injection operator, can we improve on the previously known one? First, observe that the weak stability assumption is met very easily, just by assuring that the values of $I_\ell \lambda$ are bounded by those of λ due to a finite dimensional argument. Second, consider refining a mesh $\mathcal{T}_{\ell-1}$ from level $\ell - 1$ to level ℓ. The refinement will result in faces, which are refinements of the faces on the boundary of a cell $T \in \mathcal{T}_{\ell-1}$ plus faces in positions where there were no faces before. Observe, that an injection operator which acts as identity on the first set of edges does not violate the second condition. Therefore, we propose to employ this class of injections, which now must be completed by the definition on the second set of edges, those inside the cell T. Here, we propose two options:

(1) interpolating the values on the boundary faces in a linear preserving way,
(2) or using the HDG reconstruction by the local solver, in which case the consistency of the local solver ensures preservation of linear functions.

Both versions yield a local operator which can be computed cell by cell on the coarse mesh. The results presented will be published on arXiv in short time.

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Adjoint-based superconvergent approximations of linear functionals by Galerkin methods

Bernardo Cockburn

(joint work with Zhu Wang, Shiqiang Xia)

To approximate a functional $J(u)$, where u solves a PDE, the standard way is to obtain a numerical approximation u_h of the exact solution u and then use $J(u_h)$ as an approximation of the functional. The adjoint-based method we study here improves the accuracy of these approximations in two ways. First, by increasing the accuracy of the numerical solution u_h by means of the filtering technique devised in [1]. Then, by modifying the formula to approximate the functional by using the adjoint-correction approach proposed in [4].

The filtering is carried out by a simple convolution. It takes advantage of the well-known fact that the Galerkin solution must oscillate around the exact solution in a certain pattern (provided the mesh is translation invariant) because of the Galerkin orthogonality property. Hence, convolving the Galerkin solution with a specific B-spine kernel filters out these oscillations and provides a more accurate solution. For example, the continuous Galerkin method using polynomials of degree $k \geq 1$ converges with order $k+1$ whereas the filtered approximation converges with order $2k$, provided the exact solution is smooth enough.

The adjoint-based method for approximating functionals more accurately is the adjoint-correction method of [4]. This simple and powerful technique consists in numerically solving the adjoint problem for the functional $J(\cdot)$ so that, an extra,

computable correction term can be added to $J(u_h)$ which results in a much better approximation.

In [2], these two components were put together which resulted in the method we are reporting on here. Therein, numerical experiments were carried out using the hybridizable discontinuous Galerkin method with polynomial degree k. The results indicate that the approximation defined by this new adjoint-based method converges to the functional $J(u)$ with order h^{4k}. Compared to the standard approximation $J(u_h)$, which converges with order h^{2k+1}, this new method essentially doubles the order of convergence by only doubling the computational effort for obtaining $J(u_h)$. These results were theoretically proven in [3].

In this talk, we describe the method, display a numerical experiments confirming the theoretical results and provide examples which point to the necessity of incorporating adaptivity in order to compensate the loss of convergence due to the lack of regularity of the exact solution and the functional.

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Residual-based a posteriori error analysis for symmetric mixed Arnold–Winther FEM

Joscha Gedicke

(joint work with Carsten Carstensen, Dietmar Gallistl)

This talk introduces an explicit residual-based a posteriori error analysis for the symmetric mixed finite element method in linear elasticity after Arnold–Winther with pointwise symmetric and $H(\mathrm{div})$-conforming stress approximation. The residual-based a posteriori error estimator is reliable and efficient and truly explicit in that it solely depends on the symmetric stress and does neither need any additional information of some skew symmetric part of the gradient nor any efficient approximation thereof. Numerical experiments verify the proven reliability and efficiency of the residual a posteriori error estimator and illustrate the improved convergence rate in comparison to uniform mesh-refining. A higher convergence rate for piecewise affine data is observed in the L^2 stress error and reproduced in non-smooth situations by the adaptive mesh-refining strategy.

For mixed finite element methods like the symmetric Arnold–Winther finite element schemes, the subtle term is the nonconforming residual: Given any piecewise

polynomial $\sigma_h \in H(\mathrm{div}, \Omega; \mathbb{S})$, compute an upper bound $\eta(\mathcal{T}, \sigma_h)$ of

$$\inf_{v \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^2)} \|\mathbb{C}^{-1/2}\sigma_h - \mathbb{C}^{1/2}\varepsilon(v)\|_{L^2(\Omega)} \lesssim \eta(\mathcal{T}, \sigma_h).$$

This talk presents a reliable and efficient residual-based a posteriori error estimator of the nonconforming residual with the typical contributions to $\eta(\mathcal{T}, \sigma_h)$ computed from the (known) Green strain approximation $\boldsymbol{\varepsilon}_h := \mathbb{C}^{-1}\sigma_h$. Besides oscillations of the applied forces in the volume and along the Neumann boundary, there is a volume contribution $h_T^2 \|\operatorname{rot}\operatorname{rot}\boldsymbol{\varepsilon}_h\|_{L^2(T)}$ for each triangle $T \in \mathcal{T}$ and an edge contribution with the jump $[\boldsymbol{\varepsilon}_h]_E$ across an interior edge E with unit normal ν_E, tangential unit vector τ_E, and length h_E, namely

$$h_E^{1/2}\|\tau_E \cdot [\boldsymbol{\varepsilon}_h]_E \tau_E\|_{L^2(E)} + h_E^{3/2}\|\tau_E \cdot [\operatorname{rot}_{NC}\boldsymbol{\varepsilon}_h]_E - \partial(\nu_E \cdot [\boldsymbol{\varepsilon}_h]_E\tau_E)/\partial s\|_{L^2(E)},$$

and corresponding modification on the edges on the Dirichlet boundary with the (possibly inhomogeneous) Dirichlet data.

The main result is reliability and efficiency to control the stress error robustly in the sense that the multiplicative generic constants neither depend on the (local or global) mesh-size nor on the parameter $\lambda > 0$ but may depend on $\mu > 0$ and on the shape regularity of the underlying triangulation $\mathcal{T}$ of the domain Ω.

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Convergence of adaptive discontinuous Galerkin and C^0-interior penalty finite element methods for Hamilton–Jacobi–Bellman and Isaacs equations

Iain Smears

(joint work with Ellya L. Kawecki)

Hamilton–Jacobi–Bellman (HJB) and Isaacs equations arise in stochastic optimal control problems and two-player stochastic games, and find diverse applications in engineering, industry, economics, and finance. In many cases, these equations are fully nonlinear second-order PDE, i.e. the nonlinearity includes the second derivatives. As a model problem, we consider the Isaacs equation

$$\begin{aligned} \inf_{\alpha\in\mathscr{A}} \sup_{\beta\in\mathscr{B}} \left[L^{\alpha\beta}u - f^{\alpha\beta}\right] &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a nonempty bounded convex polytopal open set in $\mathbb{R}^d$, $d \in \{2, 3\}$, where $\mathscr{A}$ and $\mathscr{B}$ are nonempty compact metric spaces, and where the second-order nondivergence form elliptic operators $L^{\alpha\beta}$, $\alpha \in \mathscr{A}, \beta \in \mathscr{B}$, are defined by

$$L^{\alpha\beta}v := a^{\alpha\beta} : \nabla^2 v \quad \forall v \in H^2(\Omega). \tag{2}$$

It is also possible to consider problems with lower-order terms in the operators $L^{\alpha\beta}$. The infimum and supremum in (1) are understood in the pointwise sense over Ω, and it is also possible to consider (1) with the infimum and supremum in reverse order without affecting our results. If either $\mathscr{A}$ or $\mathscr{B}$ is a singleton set, then the Isaacs equation (1) reduces to a HJB equation for the value function of the associated stochastic optimal control problem. Many other nonlinear PDE can also be reformulated as HJB and Isaacs equations, a notable example being the (simple) Monge–Ampère equation, which admits a reformulation as an HJB equation. The importance of these equations therefore stems from both their applications and their connections to other PDE.

Due to the full nonlinearity of the problem, finite element approximations of (1) are necessarily nonstandard. One approach is to consider *monotone* finite element methods, based on *discrete maximum principles*, that can be shown to converge to the viscosity solution [1]. In recent years there has also been significant progress in the design of stable, convergent and high-order methods, without discrete maximum principles, for the class of problems with *Cordes coefficients* [4, 5, 6, 7], where it is assumed that there exists a $\nu \in (0, 1]$ such that

$$\frac{|a^{\alpha\beta}(x)|}{\mathrm{Tr}(a^{\alpha\beta}(x))} \leq \frac{1}{\sqrt{d-1+\nu}} \quad \forall x \in \overline{\Omega}, \quad \forall (\alpha, \beta) \in \mathscr{A} \times \mathscr{B}. \tag{3}$$

We are interested here in *adaptive* methods for Isaacs and HJB equations with Cordes coefficients, based on successive mesh refinements driven by computable error estimators. Building on our recent analysis of *a posteriori* error estimators in [3], we prove in [2] the convergence of a broad family of adaptive discontinuous Galerkin and C^0-interior penalty methods using adaptively refined conforming simplicial meshes in two and three space dimensions, with fixed but arbitrary polynomial degrees greater than or equal to two. Convergence is shown for all choices of penalty parameters that are sufficient for stability of the discrete problems.

Our analysis rests upon a detailed theory for the limit spaces, which are nonstandard function spaces that describe the limiting structure of the finite element spaces under adaptive mesh refinement. A key ingredient of our approach is a novel intrinsic characterization of the limit space in terms of characterizations of the distributional derivatives, which enables us to identify the weak limits of bounded sequences of finite element functions. We provide a detailed theory for the limit spaces, and also some original auxiliary function spaces, that resolves some foundational challenges and that is of independent interest to adaptive nonconforming methods for more general problems. These include Poincaré and trace inequalities, a proof of the density of functions with nonvanishing jumps on only finitely many faces of the limit skeleton, symmetry of the Hessians, approximation results by finite element functions and weak convergence results.

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Non-Lagrange finite element approximations of linear elliptic equations in non-divergence form and Hamilton–Jacobi–Bellman equations with Cordes coefficients

Shuonan Wu

This talk is concerned with C^0 (non-Lagrange) finite element approximations of the linear elliptic equations in non-divergence form and the Hamilton–Jacobi–Bellman (HJB) equations with Cordes coefficients. Motivated by the Miranda–Talenti estimate, a discrete analog is proved once the finite element space is C^0 on the $(n-1)$-dimensional subsimplex (face) and C^1 on $(n-2)$-dimensional subsimplex. The main novelty of the non-standard finite element methods is to introduce an interior stabilization term to argument the PDE-induced variational form of the linear elliptic equations in non-divergence form or the HJB equations. As a distinctive feature of the proposed methods, no stabilization parameter is involved in the variational forms. As a consequence, the coercivity constant (resp. monotonicity constant) for the linear elliptic equations in non-divergence form (resp. the HJB equations) at discrete level is exactly the same as that from PDE theory. The quasi-optimal order error estimates as well as the convergence of the semismooth Newton method are established. Numerical experiments are provided to validate the convergence theory and to illustrate the accuracy and computational efficiency of the proposed methods.

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New Development of Conforming Finite Elements - Beyond Nédélec

Zhimin Zhang
(joint work with Qian Zhang)

In this survey, we discuss the conforming finite element discretization of high-order equations involving operators such as $(\operatorname{curl}\operatorname{curl})^2$, $\operatorname{grad}\Delta\operatorname{div}$, and $-\operatorname{curl}\Delta\operatorname{curl}$. These operators appear in various models, such as continuum mechanics, inverse electromagnetic scattering theory, magnetohydrodynamics, and linear elasticity. Naively discretizing these operators and their corresponding eigenvalue problems using the existing H^2-conforming element might lead to wrong solutions or spurious eigenvalues. On the other hand, using non-conforming and DG methods to deal with these operators would have difficulty in implementing boundary conditions. Therefore, it is essential to design conforming finite elements for equations containing these high-order differential operators.

The curl curl-conformity or grad curl-conformity requires that the tangential component of $\operatorname{curl}\mathbf{u}_h$ is continuous. Recall that the Nédélec element requires only $\mathbf{u}_h \in H(\operatorname{curl})$ (or $H(\operatorname{div})$). Due to the continuity requirement and the naturally divergence-free property of the curl operator, it is challenging to construct curl curl-conforming elements. We start from the two dimensional (2D) case, where $\operatorname{curl}\mathbf{u}_h$ is a scalar. Our construction [6, 4] is based on the existing polynomial spaces $Q_{k-1,k}\times Q_{k,k-1}$, $\mathcal{R}_k$, and $\mathbf{P}_k$. The restriction of $k\geq 4$ for a triangular element or $k\geq 3$ for a rectangular element has to be imposed since an interior bubble should be included in the shape function space of $\operatorname{curl}\mathbf{u}_h$, and hence the simplest triangular or rectangular element has 24 degrees of freedom (DOFs). To remove the restriction, in a subsequent study, we resort to the discrete de Rham complex to construct curl curl-conforming elements. The Poincaré operator enables us to tailor the shape function space to our needs (not necessarily the existing polynomial spaces). As a result, the new triangular and rectangular finite elements [2] have only 6 and 8 DOFs, respectively.

Unlike the 2D case, $\operatorname{curl}\mathbf{u}_h$ in three dimensions (3D) should be a divergence-free vector in the space $\mathbf{H}^1$, which relates the $(\operatorname{curl}\operatorname{curl})^2$ problem to the Stokes problem. However, it is challenging to construct an inf-sup stable finite element Stokes pair that preserves the divergence-free condition at the discrete level. Neilan [5] constructed a finite element complex that includes a stable Stokes pair and an $H^1(\operatorname{curl})$-conforming element on tetrahedral meshes. Based on the same Stokes pair, we proposed a curl curl-conforming element in [8]. Compared to the $H^1(\operatorname{curl})$-conforming element [5], our curl curl-conforming element has weaker continuity ($\mathbf{u}_h$ is in $H(\operatorname{curl})$ instead of $\mathbf{H}^1$) and thus fewer DOFs. However, our element still has at least 315 DOFs. Recently, Guzmán and Neilan stabilized the lowest-order 3D Scott-Vogelius pair by enriching the velocity space with modified Bernardi-Raugel bubbles [1]. This work inspires us to construct simple curl curl-conforming elements. To obtain a family of elements, we first generalize their construction to an arbitrary order by enriching the velocity space with modified face or/and interior

bubbles. Then we construct a finite element complex which contains curl curl-conforming elements on tetrahedral meshes. The lowest-order element has only 18 DOFs [3].

The grad div-conformity requires that the normal component and divergent of the finite element function $\mathbf{u}_h$ are continuous. Since div $\mathbf{u}_h$ is a scalar, the construction of grad div-conforming elements is similar with the 2D grad curl elements. The simplest tetrahedral and cuboid elements have only 8 and 14 DOFs, respectively [7].

To summarize, we have constructed not only the grad curl-conforming or grad div-conforming elements but also the whole finite element complexes. One may further investigate robust solvers in the framework of subspace correction. We will also extend our construction to quadrilateral and hexahedral meshes and study superconvergence properties of the newly developed elements.

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A Basis for High-Order Crouzeix-Raviart Elements

Stefan Sauter

In this work, we discuss a construction of a basis for high-order Crouzeix-Raviart spaces.

1. Introduction

Crouzeix-Raviart (CR) finite element spaces have been introduced in the seminal paper [4] as a non-conforming finite element space. A main motivation for introducing these spaces is to obtain a stable pair of finite elements for discretizing the Stokes equation with relatively few degrees of freedom.

The definition in [4] is implicit via moment conditions across the mesh interfaces and it is a non-trivial problem to define a basis for these spaces. In this work, we introduce a basis for general polynomial degree p in two dimensions.

2. Crouzeix-Raviart Elements – the Implicit Definition

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz polytope with boundary $\Gamma := \partial\Omega$. We consider finite element meshes $\mathcal{G}$ consisting of (closed) simplices K, where hanging nodes are not allowed. We set $\mathcal{G}_d := \mathcal{G}$ and decompose the boundaries of the simplices in $\mathcal{G}_d$ recursively into lower-dimensional simplices.

The boundary of a simplex $K_d \in \mathcal{G}_d$ can be split into $d+1$ simplices of dimension $d-1$. The set of these facets are denoted by $\mathcal{G}_{d-1}$ with facet elements K_{d-1}; again, by convention, (relatively) closed sets. The boundary of each $K_{d-1} \in \mathcal{G}_{d-1}$ can be split into d simplices of dimension $d-2$. We iterate this process iteratively so that $\mathcal{G}_0$ denotes the set of simplex vertices.

The interior of a simplex K is denoted by $\overset{\circ}{K}$ and we write $\overset{\circ}{K}_j$ to denote the (relative) interior of a lower-dimensional simplex $K_j \in \mathcal{G}_j$, $0 \leq j \leq d-1$. For $0 \leq j \leq d$, the subset consisting of all elements $K_j \subset \mathcal{G}_j$ with $\overset{\circ}{K}_j \subset \Omega$ is denoted by $\mathcal{G}_{\Omega,j}$ while those $K_j \subset \mathcal{G}_j$ with $K_j \subset \partial\Omega$ are collected in $\mathcal{G}_{\partial\Omega,j}$.

We recall the definition of conforming *hp*-finite element spaces. For $p \in \mathbb{N}_0$, let $\mathbb{P}_p^d$ denote the space of d-variate polynomials of total degree $\leq p$. For a connected subset $\omega \subset \Omega$, we write $\mathbb{P}_d^p(\omega)$ for polynomials of degree $\leq p$ defined on ω. For a connected m-dimensional manifold $\omega \subset \mathbb{R}^d$, for which there exists a subset $\hat{\omega} \in \mathbb{R}^m$ along an affine bijection $\chi_\omega : \hat{\omega} \to \omega$, we set $\mathbb{P}_p^m(\omega) := \left\{ v \circ \chi_\omega^{-1} : v \in \mathbb{P}_p^m(\hat{\omega}) \right\}$. If the dimension m is clear from the context, we write $\mathbb{P}_p(\omega)$ short for $\mathbb{P}_p^m(\omega)$.

The conforming *hp*-finite element space (for problems with homogeneous Dirichlet boundary conditions) is given by

$$S_{\mathcal{G},\mathrm{c}}^p := \left\{ u \in C^0\left(\overline{\Omega}\right) \mid \forall K \in \mathcal{G} \quad u|_K \in \mathbb{P}_p(K) \right\} \cap H_0^1(\Omega). \tag{1}$$

Let $\mathcal{N}^p$ denote the usual set of nodal points for the p-th order Lagrange interpolation on $\mathcal{G}$ and we denote by $\mathcal{N}_\Omega^p := \mathcal{N}^p \cap \Omega$ the inner ones. The Lagrange basis for $S_{\mathcal{G},\mathrm{c}}^p$ can be indexed by the nodal points $\mathbf{N} \in \mathcal{N}_\Omega^p$ and is characterized by

$$B_{p,\mathbf{N}}^{\mathcal{G}} \in S_{\mathcal{G},\mathrm{c}}^p \quad \text{and} \quad \forall \mathbf{N}' \in \mathcal{N}_\Omega^p \quad B_{p,\mathbf{N}}^{\mathcal{G}}(\mathbf{N}') = \delta_{\mathbf{N},\mathbf{N}'}, \tag{2}$$

where $\delta_{\mathbf{N},\mathbf{N}'}$ is the Kronecker delta.

Definition 1. *For $0 \leq j \leq d$ and $K_j \in \mathcal{G}_{\Omega,j}$, the* conforming space $S_{\mathrm{c}}^p(K_j)$ *is*

$$S_{\mathrm{c}}^p(K_j) := \operatorname{span}\left\{ B_{p,\mathbf{N}}^{\mathcal{G}} \mid \mathbf{N} \in \overset{\circ}{K}_j \cap \mathcal{N}_\Omega^p \right\}.$$

Further, we set

$$\omega(K_j) := \bigcup_{\substack{K \in \mathcal{G} \\ K_j \subset K}} K. \tag{3}$$

It is well known that

$$\operatorname{supp} v \subset \omega(K_j) \quad \forall v \in S_{\mathrm{c}}^p(K_j)$$

and that these spaces give rise to a direct sum decomposition

$$S_{\mathcal{G},\mathrm{c}}^p = \bigoplus_{K\in\mathcal{G}} S_{\mathrm{c}}^p(K) \oplus \bigoplus_{K_{d-1}\in\mathcal{G}_{\Omega,d-1}} S_{\mathrm{c}}^p(K_{d-1}) \oplus \ldots \oplus \bigoplus_{K_0\in\mathcal{G}_{\Omega,d-1}} S_{\mathrm{c}}^p(K_0). \tag{4}$$

The non-conforming Crouzeix-Raviart finite element spaces will be a subspace of

$$H_{\mathcal{G}}^1(\Omega) := \left\{u \in L^\infty(\Omega) \mid \forall K \in \mathcal{G} \quad u|_K \in H^1(K)\right\}.$$

For the inner facets $K_{d-1} \in \mathcal{G}_{\Omega,d-1}$, let K', K'' be the two simplices in $\mathcal{G}$ which share K_{d-1} as a common facet. The jump $[\cdot]_{K_{d-1}} : H_{\mathcal{G}}^1(\Omega) \to H^{1/2}(K_{d-1})$ across K_{d-1} is defined by

$$[w]_{K_{d-1}} = \gamma_{K_{d-1}}(w_{K''}) - \gamma_{K_{d-1}}(w_{K'}), \tag{5}$$

where for $K \in \mathcal{G}$ we denote by w_K the restriction of $w \in H_{\mathcal{G}}^1(\Omega)$ to K and for a facet $K_{d-1} \subset \partial K$ we denote by $\gamma_{K_{d-1}} : H^1(K) \to H^{1/2}(K_{d-1})$ the standard trace operator.

The definition of the non-conforming finite elements involves orthogonal polynomials on facets which we introduce first. For $K_{d-1} \in \mathcal{G}_{d-1}$ we define the set of orthogonal polynomials on K_{d-1} by

$$\mathbb{P}_{p,p-1}^{\perp}(K_{d-1}) := \begin{cases} \mathbb{P}_0(K_{d-1}) & p = 0, \\ \left\{u \in \mathbb{P}_p^{d-1}(K_{d-1}) \mid \int_{K_{d-1}} uv = 0 \quad \forall v \in \mathbb{P}_{p-1}^{d-1}(K_{d-1})\right\} & p \geq 1. \end{cases} \tag{6}$$

The orthogonal polynomials on facets allows us to formulate the *weak compatibility condition (wcc)* which is employed for the definition of non-conforming finite element spaces. For $u \in H_{\mathcal{G}}^1(\Omega)$, we set

$$\begin{aligned} &[u]_{K_{d-1}} \in \mathbb{P}_{p,p-1}^{\perp}(K_{d-1}), \ \forall K_{d-1} \in \mathcal{G}_{\Omega,d-1} \\ \text{and} \quad &u|_{K_{d-1}} \in \mathbb{P}_{p,p-1}^{\perp}(K_{d-1}), \ \forall K_{d-1} \in \mathcal{G}_{\partial\Omega,d-1}. \end{aligned} \tag{7}$$

We have collected all ingredients for the (implicit) characterization of the non-conforming Crouzeix-Raviart finite element space.

Definition 2. *The non-conforming finite element space $S_{\mathcal{G},\mathrm{wcc}}^p$ with weak compatibility conditions across facets is given by*

$$S_{\mathcal{G},\mathrm{wcc}}^p := \left\{u \in H_{\mathcal{G}}^1(\Omega) \mid \forall K \in \mathcal{G} \quad u|_K \in \mathbb{P}_p(K) \ \textit{and } u \textit{ satisfies (7)}\right\}. \tag{8}$$

We are interested to derive a local decomposition of this space as a direct sum. For $K \in \mathcal{G}$ and $K_{d-1} \in \mathcal{G}_{\Omega,d-1}$, we set

$$\begin{aligned} S_{\mathrm{wcc}}^p(K) &:= \left\{v \in S_{\mathcal{G},\mathrm{wcc}}^p \mid \operatorname{supp} v \subset K\right\}, \\ S_{\mathrm{wcc}}^p(K_{d-1}) &:= \left\{v \in S_{\mathcal{G},\mathrm{wcc}}^p \mid \operatorname{supp} v \subset \omega(K_{d-1})\right\}. \end{aligned}$$

Clearly, there exist linearly independent functions $B_{p,j}^{K,\mathrm{nc}} \in S_{\mathcal{G},\mathrm{wcc}}^p$ with $B_{p,j}^{K,\mathrm{nc}}\Big|_{\Omega\setminus K} = 0$, $1 \le j \le n_d$, such that

$$S_{\mathrm{wcc}}^p(K) = S_{\mathrm{c}}^p(K) \oplus \tilde{S}_{\mathrm{nc}}^p(K) \quad \text{with} \quad \tilde{S}_{\mathrm{nc}}^p(K) := \operatorname{span}\left\{ B_{p,j}^{K,\mathrm{nc}} : 1 \le j \le n_d \right\}.$$

Let $K_{d-1} \in \mathcal{G}_{\Omega,d-1}$ with adjacent simplices $K', K'' \in \mathcal{G}$. There exist linearly independent functions $B_{p,j}^{K_{d-1},\mathrm{nc}} \in S_{\mathcal{G},\mathrm{wcc}}^p$, $1 \le j \le n_{d-1}$, with

$$\begin{aligned} &S_{\mathrm{wcc}}^p(K_{d-1}) = S_{\mathrm{c}}^p(K_{d-1}) \oplus S_{\mathrm{wcc}}^p(K') \oplus S_{\mathrm{wcc}}^p(K'') \oplus \tilde{S}_{\mathrm{nc}}^p(K_{d-1}) \\ &\text{with} \quad \tilde{S}_{\mathrm{nc}}^p(K_{d-1}) := \operatorname{span}\left\{ B_{p,j}^{K_{d-1},\mathrm{nc}} : 1 \le j \le n_{d-1} \right\} \end{aligned} \tag{9}$$

and we set

$$\tilde{S}_{\mathcal{G},\mathrm{nc}}^p := S_{\mathcal{G},\mathrm{c}}^p + \bigoplus_{K\in\mathcal{G}} \tilde{S}_{\mathrm{nc}}^p(K) + \bigoplus_{K_{d-1}\in\mathcal{G}_{\Omega,d-1}} \tilde{S}_{\mathrm{nc}}^p(K_{d-1}). \tag{10}$$

Clear, it holds

$$\tilde{S}_{\mathcal{G},\mathrm{nc}}^p \subseteq S_{\mathcal{G}}^{\mathrm{wcc},p}.$$

In most cases (including the case of two-dimensional triangulations, cf. [2]), the inclusion "$\subseteq$", holds with an "=" sign. However, there exist examples in 3D where the inclusion is strict and we refer for a detailed discussion to [3, Ex. 36].

It is not guaranteed that both sums "+" in (10) are direct – in fact this does not even hold for the $\mathbb{P}_1$ Crouzeix-Raviart elements. Hence, it is common to employ subspaces $S_{\mathcal{G},\mathrm{c}}^{p,-} \subseteq S_{\mathcal{G},\mathrm{c}}^p$, $S_{\mathrm{nc}}^p(K) \subseteq \tilde{S}_{\mathrm{nc}}^p(K)$, $S_{\mathrm{nc}}^p(K_{d-1}) \subseteq \tilde{S}_{\mathrm{nc}}^p(K_{d-1})$ such that all sums in

$$S_{\mathcal{G},\mathrm{nc}}^p := S_{\mathcal{G},\mathrm{c}}^{p,-} \oplus \bigoplus_{K\in\mathcal{G}} S_{\mathrm{nc}}^p(K) \oplus \bigoplus_{K_{d-1}\in\mathcal{G}_{\Omega,d-1}} S_{\mathrm{nc}}^p(K_{d-1})$$

are direct. We do not require that $S_{\mathcal{G},\mathrm{nc}}^p = S_{\mathcal{G},\mathrm{wcc}}^p$ but impose as a minimal requirement that

$$S_{\mathcal{G},\mathrm{c}}^p \subset S_{\mathcal{G},\mathrm{nc}}^p$$

so that standard approximation properties for conforming *hp* finite elements carry over to $S_{\mathcal{G},\mathrm{nc}}^p$.

Theorem 3. *Let Ω be a two-dimensional polygonal domain and $\mathcal{G}$ a triangulation as in §2. For $F \in \mathcal{G}_{\Omega,1}$ with endpoint A_1 and A_2 and adjacent triangles K_1, K_2 and $K \in \mathcal{G}$ with vertices A_1, A_2, A_3, we define*

$$B_{p,1}^{F,\mathrm{nc}} := \begin{cases} 1 + \sum_{j=1}^2 L_p\left(-1 + 2\lambda_{K_i,A_j}\right) & i = 1,2, \\ 0 & \text{in } \Omega\setminus\omega_F, \end{cases}$$

$$B_{p,1}^{K,\mathrm{nc}} := \frac{1}{2}\left(\sum_{j=1}^3 L_p\left(-1 + 2\lambda_{i,A_j}\right) - 1 \right)$$

with barycentric coordinates $\lambda_{K,A_j} \in \mathbb{P}_1(K)$ *such that* $\lambda_{K,A_j}(A_i) = \delta_{i,j}$, $1 \leq i,j \leq 3$. *We set*

$$S^p_{\mathrm{nc}}(F) := \begin{cases} \operatorname{span}\left\{B^{F,\mathrm{nc}}_{p,1}\right\} & p \text{ odd}, \\ \{0\} & p \text{ even}, \end{cases} \qquad S^p_{\mathrm{nc}}(K) := \begin{cases} \{0\} & p \text{ odd}, \\ \operatorname{span}\left\{B^{K,\mathrm{nc}}_{p,1}\right\} & p \text{ even}. \end{cases}$$

Then,

$$S^p_{\mathcal{G},\mathrm{wcc}} = \begin{cases} \bigoplus_{K\in\mathcal{G}} S^p_{\mathrm{c}}(K) \oplus \bigoplus_{F\in\mathcal{G}_{\Omega,1}} S^p_{\mathrm{c}}(F) \oplus \bigoplus_{F\in\mathcal{G}_{\Omega,1}} S^p_{\mathrm{nc}}(F) & p \text{ odd}, \\ \bigoplus_{K\in\mathcal{G}} S^p_{\mathrm{c}}(K) \oplus \bigoplus_{F\in\mathcal{G}_{\Omega,1}} S^p_{\mathrm{c}}(F) \oplus \bigoplus_{N\in\mathcal{G}_{\Omega,0}} S^p_{\mathrm{c}}(N) \oplus \bigoplus_{K\in\mathcal{G}} S^p_{\mathrm{nc}}(K) & p \text{ even}. \end{cases}$$

For a proof we refer to [1], [5], [2].

Remark 4. *The three-dimensional case is more involved and we refer for an explicit basis for the high-order Crouzeix-Raviart spaces to* [3, Ex. 36].

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DPG schemes for thin structures

Norbert Heuer

(joint work with Thomas Führer, Antti H. Niemi)

The DPG method (discontinuous Petrov–Galerkin method with optimal test functions) is a paradigm for numerical schemes to solve PDEs, proposed by Demkowicz and Gopalakrishnan [12, 13]. It aims at automatically ensuring discrete stability, and combines, in its standard form, an ultra-weak formulation [10] with independent trace variables [7], optimal test functions [5], and product test spaces induced by an underlying mesh. Important advantages are, besides the automatic discrete stability, that systems are Hermitian positive definite and that there is inherent a posteriori error estimation made up of local error indicators [14, 9].

Our motivation is to make the potential of this technique available for the solution of problems from the mechanics of thin structures. Prototype cases are the Reissner–Mindlin and Kirchhoff–Love models for plate bending, and the Naghdi and Koiter shell models combining, respectively, the Reissner–Mindlin and Kirchhoff–Loves models with membrane elasticity. The numerical solution of such

problems is challenging due to issues of reduced regularity, singular perturbations, locking phenomena, both individually and combined. Whereas there has been much progress during the last decades, see, e.g., [1, 2, 3, 4, 6, 11, 19, 20, 21], the numerical treatment of singularities and locking phenomena in thin structures is still a non-trivial challenge. Our conjecture is that the DPG paradigm can open efficient strategies to tackle these problems. Initial results for DPG applied to thin structures are reported in [18, 8].

In our talk we discuss DPG techniques for the Kirchhoff–Love plate bending model ("KL") and the case of a shallow Koiter shell ("sK"). The solutions to both problems suffer from low regularity at incoming corners and, depending on data, geometry, and the type of boundary conditions, numerical methods for sK suffer from various locking phenomena. We present trace operators for these cases [16, 15] that are uniformly well posed (uniformly bounded from above and from below) and allow for efficient approximations of the KL and sK models. We discuss stability properties of the sK model to show which test norms have to be chosen to achieve uniform discrete stability, and be potentially locking free. Several numerical experiments underline our theoretical results. Similarly to the KL-sK case, we expect to combine our Reissner–Mindlin techniques [17] with the Naghdi shell model.

Our conjecture is that the DPG framework can be adapted to different cases of shell models so that numerical approximations are locking free, or show at least reduced locking. It is motivated by the fact that DPG schemes in the ultraweak setting are essentially characterized by their traces. Their conforming approximations are substantially easier to construct and analyze than those of field variables with corresponding regularity.

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Error analysis of the lowest-order nonconforming VEM for second-order linear indefinite elliptic problems

Rekha Khot

(joint work with Carsten Carstensen, Amiya K. Pani)

The finite element method (FEM) is the most widely used numerical method for solving model problems involving differential equations. FEM allows to divide the domain on which problem is posed into triangles or rectangles in 2D and similarly in higher dimension. Recently this idea has been extended to 'fairly general" polygonal/polyhedral finite elements and various methods are developed to handle such elements efficiently - mimetic finite difference method, virtual element method, hybrid high order method, discontinuous Galerkin finite element method to name a few. The virtual element method (VEM) introduced in [1] is one of the well-received polygonal methods for approximating the solutions to partial differential equations (PDEs) in the continuation of the mimetic finite difference method [2]. On the account of its versatility in shape of polygonal domains, the

local finite-dimensional space (the space of shape functions) comprises polynomial as well as non-polynomial functions. The novelty of this approach lies in the fact that it does not demand for the explicit construction of non-polynomial functions and the knowledge of degrees of freedom along with projections onto polynomials is sufficient to implement the method.

The nonconforming virtual element method approximates the weak solution $u \in H_0^1(\Omega)$ to the second-order linear elliptic boundary value problem

$$\mathcal{L}u := -\mathrm{div}(\mathbf{A}\nabla u + \mathbf{b}u) + \gamma u = f \quad \text{in} \quad \Omega \tag{1}$$

for a given $f \in L^2(\Omega)$ in a bounded polygonal Lipschitz domain $\Omega \subset \mathbb{R}^2$ subject to homogeneous Dirichlet boundary conditions. Beirão da Veiga *et al.* discuss a conforming VEM for the indefinite problem in [3]. Cangiani *et al.* [4] develop a nonconforming VEM under the additional condition which makes the bilinear form coercive and significantly simplifies the analysis. The two papers [3, 4] prove *a priori* error estimates for a solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ in a convex domain Ω. The *a priori* error analysis for the nonconforming VEM in [4] can be extended to the case when the exact solution $u \in H^{1+\sigma}(\Omega) \cap H_0^1(\Omega)$ with $\sigma > 1/2$ as it is based on traces. We prove it for all $\sigma > 0$ and circumvent any trace inequality. Medius analysis [5] is known in FEM literature for problems with minimal regularity assumptions. Very recently, this analysis has been applied to the nonconforming VEM for Poisson and Biharmonic problems by Huang *et al.* in [6]. An *a posteriori* error estimate in [7] explores the conforming VEM for (1) assuming coercive problem. We investigate in [8] both the *a priori* and *a posteriori* error estimates for the nonconforming VEM assuming reduced regularity, but under the assumption that the Fredholm operator $\mathcal{L}$ is injective. We assume that the coefficients $\mathbf{A}, \mathbf{b}$ and γ are piecewise Lipschitz continuous functions, and also $\mathbf{A}$ is symmetric positive definite and satisfies uniformly elliptic condition. These assumptions imply the existence of a unique solution to the model problem.

The first step of discrete setting is the decomposition of Ω into polygonal domains. $\mathcal{T}$ is an admissible polygonal mesh if any two polygonal domains in $\mathcal{T}$ are disjoint, share either a finite number of edges or vertices. Mesh regularity assumption in [1] essentially implies that each polygonal domain in $\mathcal{T}$ can be divided into triangles such that the resultant sub-triangulation $\widehat{\mathcal{T}}$ of Ω is shape regular. Since the discrete space is not a subset of $H_0^1(\Omega)$, the substitution of a discrete function in a continuous problem leads to an error and this error is regarded as a nonconformity error. The main tool to analyze such error is the construction of mapping between nonconforming virtual element space (V_h) and the Sobolev space $V := H_0^1(\Omega)$. First we map V_h to Crouzeix-Raviart finite element space on $\widehat{\mathcal{T}}$ through an interpolation operator I_{CR} and then map it to V using the well-known enrichment or averaging operator E_h in [9]. This enrichment operator E_h is extended to a companion operator J' which has additional special properties [10]. So the composition $J' \circ I_{\mathrm{CR}}$ is another linking between V_h and V. Then we reconstruct a conforming companion operator J which is a right inverse of an interpolation

operator $I_h : V \to V_h$ and has properties like orthogonality, approximation estimates etc. This newly constructed linear operator J is an important tool in error analysis. The stability of the discrete solution allows for the proof of existence of a unique discrete solution, of a discrete inf-sup estimate and, consequently, for error estimates in the piecewise H^1 and L^2 norms. We develop an explicit residual-based *a posteriori* error estimate, which is reliable and efficient up to the stabilization and oscillation terms. The four local quantities namely the volume residual, the stabilization term, the inconsistency term, and the nonconformity term contribute to the upper bound.

The numerical experiments provide evidence for the sharpness of the mathematical a *priori* and a *posteriori* error analysis in this work and illustrate the superiority of adaptive over uniform mesh-refining. The empirical convergence rates in all examples for the H^1 and L^2 errors coincide with the theoretically predicted convergence rates. The ratio of the estimator by the error, sometimes called efficiency index, remains bounded. The volume residual clearly dominates the a *posteriori* error estimates, while the stabilisation term remains significantly smaller for the natural stabilisation. The analysis can be extended to the nonconforming VEM space of higher-order and also to 3D. The future work on the theoretical investigation of the performance of adaptive mesh-refining algorithm is clearly motivated by the successful numerical experiments.

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Nonstandard adaptive FEMs with smoothing

Ornela Mulita
(joint work with Stefano Giani, Luca Heltai)

In this work we propose a new nonstandard approach for Adaptive Finite Element Methods (AFEMs) based on smoothing iterations (S-AFEM), for linear, second-order, elliptic partial differential equations (PDEs). The algorithm is inspired by the ascending phase of the V-cycle multigrid method: accurate algebraic solutions in intermediate cycles of the classical AFEM are replaced with the application of a prolongation step, followed by the application of a smoother. Even though these intermediate solutions are far from the exact algebraic solutions, their a-posteriori error estimation produces a refinement pattern that is substantially equivalent to the one that would be generated by classical AFEM, at a considerable fraction of the computational cost.

We provide a qualitative analysis of how the error propagates throughout the algorithm, and we present a series of numerical experiments that highlight the efficiency and the computational speedup of S-AFEM. In particular, we consider some variants of our algorithm, where different smoothers are used in the intermediate cycles (respectively Richardson iteration, the CG method, and the GMRES method). We conclude that, in general, CG and GMRES act as robust smoothers in SAFEM also for high order approximations, and for non-symmetric problems, like, for example, drift-diffusion problems with dominant transport.

In [1] we analyze the error propagation properties of the S-AFEM algorithm, and provide a bound on the a-posteriori error estimator applied to the approximated algebraic solution. The results are not sharp, and do not provide a definitive answer on the convergence of the final S-AFEM solution to the AFEM one, but could be used as a ground state for further investigation, which is currently ongoing.

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A Hybrid High-Order discretization combined with Nitsche's method for contact and Tresca friction in small strain elasticity

Nicolas Pignet

(joint work with Franz Chouly, Alexandre Ern)

We present a Hybrid High-Order (HHO) method to discretize unilateral and bilateral contact problems with Tresca friction in small strain elasticity [1] (see [2] for the pioneer work using a conforming Lagrange discretization). The nonlinear frictional contact conditions are enforced weakly by means of a consistent Nitsche's technique with symmetric, incomplete, and skew-symmetric variants. The present HHO-Nitsche method supports polyhedral meshes and delivers optimal energy-error estimates for smooth solutions under some minimal thresholds on the penalty parameters for all the symmetry variants. An explicit tracking of the dependency of the penalty parameters on the material coefficients is carried out to identify the robustness of the method in the incompressible limit, showing the more advantageous properties of the skew-symmetric variant.

HHO methods have been introduced for linear elasticity in [3] and are formulated in terms of face unknowns that are polynomials of arbitrary order $k \geq 1$ on each mesh face and in terms of cell unknowns which are polynomials of order $l \in \{k, k \pm 1\}$, with $l \geq 1$, in each mesh cell. The devising of HHO methods hinges on two operators, both defined locally in each mesh cell: a strain reconstruction operator and a stabilization operator. The cell unknowns can be eliminated locally by static condensation leading to a global problem posed solely in terms of the face unknowns.

Since HHO methods involve both cell unknowns and face unknowns, this leads to different formulations of Nitsche's consistency and penalty terms, either using the trace of the cell unknowns (cell version) or using directly the face unknowns (face version). This talk focuses on the face version, which has better robustness properties than the cell version in the incompressible limit. The key idea to prove optimal convergence rates for this face version is to increase the polynomial order to $(k+1)$ for all the faces that are on the contact boundary (which increase slightly the size of the global problem).

This work can be pursued in several directions, such as extending the analysis to Coulomb friction which is more physically relevant but more difficult mathematically; or add a plastic behavior which implies to modify the discrete formulation since nonlinear terms appear for the test functions.

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Adaptive mixed finite element methods for non-selfadjoint indefinite second-order elliptic PDEs

Rui Ma

(joint work with Carsten Carstensen)

In this talk, we establish optimality of adaptive mixed finite element methods for non-selfadjoint indefinite second-order elliptic problems in three dimensions. The error is measured in L^2 norms and then allows for an adaptive algorithm with collective marking. The axioms of adaptivity apply to this setting and guarantee the rate optimality for sufficiently small initial mesh-sizes and bulk parameter. This talk introduces an alternative proof for the discrete reliability of the mixed Poisson problem.

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On a discrete embedding inequality for two types of fourth order elliptic equations (Biharmonic and Quad-Curl)

Weifeng Qiu

(joint work with Gang Chen, Huangxin Chen, Amiya Pani, Liwei Xu)

Biharmonic equation and Quad-Curl problem are two fourth-order elliptic partial differential equations with wide applications. First of all, we will show regularities of solutions spaces for these two PDEs. Then we will present a discrete embedding inequality to handle the stability estimates of C^0-IP type methods for these two PDEs. In fact, this discrete embedding inequality can be utilized to show stability of several numerical methods including DG.

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Distributed solution of symmetric eigenvalue problems

Antti Hannukainen
(joint work with Jarmo Malinen, Antti Ojalammi)

I discuss solution of large-scale eigenvalue problems in distributed computing environments where communication between tasks is expensive, such as a cluster in cloud computing service or on networked workstations running the HTCondor batch system. As a model problem, I consider computing eigenvalues on $(0, \Lambda)$ of the Dirichlet Laplacian discretized using conforming first order FEM. However, I believe that the applied techniques are more general and can also be used with other elliptic PDEs. The talk is based on [1, 2].

Consider the problem: find $(\lambda, 0) \in (0, \Lambda) \times V_{h0}(\Omega) \setminus \{0\}$ satisfying

$$(\nabla u, \nabla v) = \lambda (u, v) \quad \text{for all } v \in V_{h0}(\Omega). \tag{1}$$

Here, and in the following, $V_{h0}(\omega)$ and $V_h(\omega)$ denote the FE-space over the set $\omega \subset \Omega \subset \mathbb{R}^d$ for $d = 2, 3$ with and without imposed zero Dirichlet b.c.. The corresponding algebraic EVP is: find $(\lambda, x) \in (0, \Lambda) \times \mathbb{R}^n \setminus \{0\}$ satisfying $A\vec{x} = \lambda M \vec{x}$.

We develop a Ritz method using method subspace constructed from local Ritz spaces associated to an overlapping decomposition of the computational domain into subdomains $\{U^{(p)}\}_{p=1}^P$. Each local Ritz space is independent of others. In [2] the partition of unity method is used to combine the local spaces into a conforming global space. If the overlap is of the size of one element layer, one can simply work with matrices.

Instead of $A\vec{x} = \lambda M \vec{x}$, we solve $Q^T A Q \vec{y} = \mu Q^T M Q \vec{y}$. Here $Q \in \mathbb{R}^{n \times k}$, $k \ll n$. Accuracy of approximations $\lambda \approx \mu$ and $\vec{x} \approx Q\vec{y}$ depends on subspace $V = range(Q)$. Our method uses a "non-standard" choice of $V = V_{U^{(1)}} \times \ldots \times V_{U^{(P)}}$ corresponding to

$$Q = \begin{bmatrix} Q_1 & & \\ & \ddots & \\ & & Q_P \end{bmatrix} \quad \text{where} \quad V_{U^{(i)}} = range\, Q_i \quad \text{for} \quad i \in \{1, \ldots, P\}.$$

We call $\{V_{U^{(p)}}\}_{p=1}^P$ local subspaces, as each $V_{U^{(p)}}$ is associated to the subdomain $U^{(p)}$. Under sufficient assumptions on V and $\sigma(A, M)$, the relative eigenvalue error $(\lambda - \mu)\lambda^{-1}$ on $(0, \Lambda)$ is bounded by the approximability of the corresponding eigenvector $\vec{x}$ s.t. $A\vec{x} = \lambda M \vec{x}$, $\|\vec{x}\|_M = 1$ in V as

$$(\lambda - \mu)\lambda^{-1} \leq C \min_{v \in V} \|\vec{x} - \vec{v}\|_A^2 \leq C \left(\sum_{p=1}^P \|\vec{x}_{U^{(p)}} - v_p\|_{A^{(p)}} \right)^2.$$

Here $A^{(p)}$ is the FE-matrix block corresponding to interior degrees of freedom on $U^{(p)}$. The local Ritz space $V_{U^{(p)}}$ is designed accordingly to approximate $\vec{x}_{U^{(p)}}$.

Designing the subspace $V_{U^{(p)}}$ requires information on eigenvectors corresponding to eigenvalues on $(0, \Lambda)$. For this purpose, we attach an *extended subdomain* $\widehat{U}^{(p)}$ to $U^{(p)}$. For notational convenience, denote $U = U^{(p)}$ and $\widehat{U} = \widehat{U}^{(p)}$. Let

$(\lambda, u) \in (0, \Lambda) \times V_{h0}(\Omega) \setminus \{0\}$ satisfy (1) and $\vec{x}$ be the coordinate vector of u. Then $u|_{\widehat{U}}$ satisfies

$$(\nabla u|_{\widehat{U}}, \nabla v_0) - \lambda(u|_{\widehat{U}}, v_0) = 0 \quad \text{for all } v_0 \in V_{h0}(\widehat{U}).$$

We use the above equation to express $\vec{x}_U$ as a function of the unknown boundary trace $\gamma_{\partial\widehat{U}} u|_{\widehat{U}}$. This function features a finite-dimensional part and λ-dependent boundary-to-interior mapping, a linear operator mapping boundary traces from $\partial\widehat{U}$ to U. In [2], we show that in the continuous setting the boundary-to-interior mapping is compact for $\lambda \in (0, \Lambda)$.

We obtain the space V_U from the finite dimensional part, and by approximating the range of the λ-dependent boundary-to-interior mapping as follows: First, we approximate it by interpolation. Then we linearise the approximation. Finally, singular value decomposition is used to obtain V_U. For detailed description of this process and error analysis, see [2].

Numerical examples indicate that the proposed method has potential. As an example, we approximate 200 lowest eigenvalues of three dimensional Dirichlet Laplacian using cluster of 25 workstations. The largest example computation has ten million unknowns and took approximately two hours to solve.

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Least-Squares and discontinuous Petrov-Galerkin methods for the approximation of eigenvalues

Fleurianne Bertrand

(joint work with Daniele Boffi, Henrik Schneider)

Accurate flux approximations are of interest in many applications and a lot of attention has recently been devoted to the reconstruction of the flux from a primal formulation since they are usually not $H(\mathrm{div})$-conforming. The reconstruction procedures for fluxes are also of particular importance for a posteriori error estimation and have a long history with ideas dating back at least as far as [1] and [2]. They have received large attention recently: a unified framework for Stokes is presented in [3], polynomial-degree robustness is shown in [4], extensions to three space dimensions [5]. An alternative approach uses flux-based variational formulations involving the flux as an independent variable approximated in a suitable $H(\mathrm{div})$-conforming finite element spaces. Such approaches may either lead to a saddle-point problem or a symmetric positive definite system. The first one, have been intensively studied and we refer to [9] for an overview. This talk, therefore, focuses on the second type and covers the Least-Squares Method (see [10] for a

comprehensive overview) and the discontinuous Petrov-Galerkin method (introduced in a series of papers [11, 12, 13]). Even if the proposed methods may not be competitive with other solution techniques, the presented analysis should shed some light on the fundamental properties of these formulations.

1. Computation of the eigenvalues with the Least-Squares method

In this part, we aim at investigating the least-squares finite element approximation of the eigensolutions of operators associated with second-order elliptic equations. Given $f \in L^2(\Omega)$, the simplest least-squares formulation for the source problem $-\Delta u = f$ with homogeneous Dirichlet boundary conditions, is given by the minimization of the following functional $\mathcal{F}(\boldsymbol{\tau}, v) = \|\boldsymbol{\tau} - \nabla v\|^2 + \|\operatorname{div}\boldsymbol{\tau} + f\|^2$. The corresponding variational formulation can be used in a natural way to consider the following eigenvalue problem: find $\lambda \in \mathbb{C}$ and $u \in H_0^1(\Omega)$ with $u \neq 0$ such that for some $\boldsymbol{\sigma} \in H(\operatorname{div};\Omega)$ it holds

$$\text{(LS)} \qquad \begin{cases} (\boldsymbol{\sigma},\boldsymbol{\tau}) + (\operatorname{div}\boldsymbol{\sigma},\operatorname{div}\boldsymbol{\tau}) - (\nabla u,\boldsymbol{\tau}) = -\lambda(u,\operatorname{div}\boldsymbol{\tau}) & \forall \boldsymbol{\tau} \in H(\operatorname{div};\Omega) \\ -(\boldsymbol{\sigma},\nabla v) + (\nabla u,\nabla v) = 0 & \forall v \in H_0^1(\Omega) \end{cases}$$

A symmetric equivalent formulation follows from the fact that $(u,\operatorname{div}\boldsymbol{\tau}) = -(\nabla u,\boldsymbol{\tau})$ and in particular the eigenvalues are real. Let $\Sigma_h \subset H(\operatorname{div};\Omega)$ and $U_h \subset H_0^1(\Omega)$ be conforming finite element spaces. The discretization of (LS) reads: find $\lambda_h \in \mathbb{R}$ and $u_h \in U_h$ with $u_h \neq 0$ such that for some $\boldsymbol{\sigma}_h \in \Sigma_h$ it holds

$$(LS_h) \qquad \begin{cases} (\boldsymbol{\sigma}_h,\boldsymbol{\tau}) + (\operatorname{div}\boldsymbol{\sigma}_h,\operatorname{div}\boldsymbol{\tau}) - (\nabla u_h,\boldsymbol{\tau}) = -\lambda_h(u_h,\operatorname{div}\boldsymbol{\tau}) & \forall \boldsymbol{\tau} \in \Sigma_h \\ -(\boldsymbol{\sigma}_h,\nabla v) + (\nabla u_h,\nabla v) = 0 & \forall v \in U_h \end{cases}$$

Regarding the source problem, the choice of finite element spaces for the approximation of the different variables is not restricted by compatibility conditions. However, for the eigenvalue problem, our framework is restricted to finite element spaces Σ_h and U_h satisfying the following approximation properties

$$\inf_{\boldsymbol{\tau}\in\Sigma_h} \|\boldsymbol{\chi} - \boldsymbol{\tau}\|_{H(\operatorname{div};\Omega)} \leq Ch^s \left(\|\boldsymbol{\chi}\|_{\mathbf{H}^s(\Omega)} + \|\operatorname{div}\boldsymbol{\chi}\|_{H^{1+s}(\Omega)}\right),$$
$$\inf_{v\in U_h} \|p - v\|_{H^1(\Omega)} \leq Ch^s \|p\|_{H^{1+s}(\Omega)}.$$

Using duality arguments, we are then able to derive refined L^2-estimates that directly imply the uniform convergence of the discrete solution operator to the continuous one and thus, the convergence of the eigenvalues, see [14]. If standard (nodal) finite elements are used for the definition of Σ_h, then it is not clear in this case if the uniform convergence is satisfied and if the eigenmodes are well approximated. We extended successfully these results in [15] to the linear elasticity problem, although we were not able to obtain a symmetric formulation of the corresponding problem. Our numerical results show that in several cases the computed eigenvalues are real and positive. Some pairs of complex conjugate eigenvalues may be present which converge to real numbers according to the developed analysis.

2. Computation of the eigenvalues with the discontinuous Petrov-Galerkin method

Another approach consists of the discontinuous Petrov-Galerkin method, which is a stress-based mixed method that can be seen as a Least-Squares method simultaneously. The main idea is to use a suitable discontinuous trial and test functions that are tailored for stability. The *ideal* dPG formulation is turned into a *practical* dPG formulation [16] where the test function space is easily computable and arbitrary close to the optimal one. The dPG formulation comes with a natural a posteriori error indicator that can be used for driving a robust *hp* adaptivity. Usually, the trial space U consists of two components and can be presented as $U = U_0 \times U_1$, where U_0 is a functional space defined on Ω (volumetric part) and U_1 is the remaining part that can be defined on Ω or the skeleton of a given triangulation. The continuous eigenvalue problem then reads: find eigenvalues $\lambda \in \mathbb{C}$ and eigenfunctions $u = (u_0, u_1) \in U = U_0 \times U_1$ with $u_0 \neq 0$ such that

$$b(u, v) = \lambda m(u_0, v) \quad \forall v \in V. \tag{DPG}$$

The fundamental characterization of the solution u_h of the DPG system by a mixed problem defined only via the discrete spaces U_h and V_h leads to the following eigenvalue problem: find $\lambda_h \in \mathbb{C}$ such that for some $u_h = (u_{0,h}, u_{1,h}) \in U_h = U_{0,h} \times U_{1,h}$ with $u_{0,h} \neq 0$ and some $\varepsilon_h \in V_h$ it holds

$$(DPG_h) \qquad \begin{cases} (\varepsilon_h, v_h)_V + b(u_h, v_h) = \lambda_h m(u_{0,h}, v_h) & \forall v_h \in V_h \\ \overline{b(z_h, \varepsilon_h)} = 0 & \forall z_h \in U_h. \end{cases}$$

The available error estimates allow us to prove the uniform convergence of the eigenvalue, using the classical Babŭska–Osborn theory [17] for the primal and the ultra-weak formulation of the Laplacian. The natural error estimator associated with the energy residual reads $\eta = \|\varepsilon_h\|$. With natural modifications of the analysis of [18] global efficiency and reliability can be proved. Both properties rely on the following (usual) higher-order term $\lambda u_0 - \lambda_h u_{0,h}$.

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