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Geometry, Dynamics and Spectrum of Operators on Discrete Spaces (online meeting)

Organized by
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Abstract. Spectral theory is a gateway to fundamental insights in geometry and mathematical physics. In recent years the study of spectral problems in discrete spaces has gained enormous momentum. While there are some relations to continuum spaces, fascinating new phenomena have been discovered in the discrete setting throughout the last decade. The goal of the workshop was to bring together experts reporting about the recent developments in a broad variety of dynamical or geometric models and to reveal new connections and research directions.

Mathematics Subject Classification (2010): Primary: 47A10, 47B80, 47A35. Secondary: 05C50, 82B44, 52C23, 46L10.

Introduction by the Organizers

The workshop *Geometry, Dynamics and Spectrum of Operators on Discrete Spaces*, organized by David Damanik (Houston), Matthias Keller (Potsdam), Tatiana Nagnibeda (Geneva) and Felix Pogorzelski (Leipzig), brought together experts and young researchers from spectral theory in various geometric settings. A particular emphasis was laid on discrete models such as graphs with the goal of illustrating bridges to continuous models such as manifolds, but at the same time highlighting fascinating new phenomena that can be observed in one of the realms but not in the other, respectively. Moreover, the gathering aimed at identifying new research perspectives given by the interplay between geometric, dynamical and spectral theoretic aspects in either model. In this framework the participants presented results in various mathematical areas. This concerns aspects of Laplace and Schrödinger operators (Fischer, Frank, Stollmann, Veselić, Wojciechowski), aperiodic order (Baake, Beckus, Grigorchuk, Kellendonk, Lenz, Smilansky), the heat flow (Rose, Schmidt, Wirth) and curvature (Münch, Peyerimhoff).

The conference featured several talks about recent results on Laplace and Schrödinger operators in various geometric settings. For Laplace operators on graphs, Peter Stollmann explained a new uncertainty principle at low energies in terms of subsets of positive density with respect to a decomposition of the graph with a spectral gap. The approach bears witness of a general method that is applicable in various models. In analogy to Euclidean potential theory and going beyond various stochastic models, Florian Fischer presented a Fatou–Naïm–Doob-type theorem for positive sub-critical Schrödinger operators on graphs. Radoslaw Wojciechowski's talk was devoted to new insights into essential self-adjointness of Laplace operators on graphs. He described the tight connections to the ℓ^2-Liouville property and a stability criterion for subgraphs. Rupert Frank presented new insights into the spectrum of (possibly random) Schrödinger operators on the real line with constant electric field motivated by a prominent model in solid state physics. He detected the sensitive connections between the choice of the key parameters and the (type of the) occurring spectrum. In the realm of Schrödinger operators in Euclidean space, Ivan Veselić presented a quantitative scale free unique continuation principle with applications in mathematical physics, such as Wegner and observability estimates.

The theory of aperiodic order was reflected in several talks covering a wide range of research topics. Rostislav Grigorchuk reported on several results on the spectrum of Laplace and Markov operators on Schreier graphs arising from finitely generated groups defined via dynamical data. One central focus was laid on criteria leading to Cantor spectrum of Lebesgue measure zero. Taking the Heisenberg group as an example, Siegfried Beckus explained a path via dilations towards symbolic substitutions, thus revealing a source of linearly repetitive and aperiodic subshift systems in a class of non-abelian groups. In the framework of translation bounded measures on locally compact abelian groups, Daniel Lenz characterized key aspects of mathematical diffraction theory via notions of almost periodicity for the autocorrelation measure. A classification of cut-and-project sets in $\mathbb{R}^d$ via affine group invariant ergodic measures on the space of lattices (so-called "Ratner–Marklof–Strömbergsson measures") with applications to counting statistics was presented by Yotam Smilansky. Michael Baake explained that weak model sets of maximal density are pure point diffractive and illustrated some specifics via models of number-theoretic origin such as the visible lattice points and the square-free integers in the Gaussian integers. Johannes Kellendonk discussed dynamical aspects of one-dimensional symbolic shift systems via the underlying Ellis semigroup. Among other things, this included a characterization of tameness for Toeplitz systems of finite Toeplitz rank.

Another emphasis was laid on the impact of geometric-analytic conditions on the behavior of the underlying heat kernel. Christian Rose described geometric conditions for subsets of Riemannian manifolds in order to admit a bounded Sobolev extension operator. This leads to uniform Neumann heat kernel estimates for such subsets under integral Ricci curvature bounds. Marcel Schmidt explained a sharp criterion for stochastic completeness via volume growth for graphs endowed with an intrinsic metric with finite distance balls. Using globally local (GL) graphs as a tool, one obtains a discrete analogue of a celebrated theorem of Grigor'yan for manifolds. Melchior Wirth presented a logarithmic Sobolev inequality for Markov semigroups in a non-commutative setting via Wasserstein-type distances on density operators in von Neumann algebras.

Structural insight on the geometry of graphs is also obtained by discrete notions of curvature. Norbert Peyerimhoff presented in his talk a new formula relating the Bakry–Émery curvature of a vertex in a weighted graph with the smallest eigenvalue of an associated matrix arising from objects in a semidefinite programming problem. Introducing a new concept for straight lines in graphs, Florentin Münch explained the way towards a discrete version of the Cheeger–Gromoll splitting theorem for Ollivier–Ricci curvature in connected, locally finite graphs.

We dedicate this workshop to Daniel Lenz, whose broad research interests were reflected in many topics discussed at the workshop, on the occasion of his 50th birthday.

On the mode of the workshop: Due to the Covid-19 pandemic, the workshop had to be changed on short notice from a hybrid format to a pure online format. Although there is clearly no way to experience the genuine Oberwolfach spirit via Zoom, everyone was looking at the bright side. There were 16 inspiring online talks and insightful discussions in parallel Zoom meeting rooms. The slots that opened up after cancellations could be filled also on short notice and we are grateful to the MFO administration for generously allowing us to extend more invitations than originally planned. In particular, this created a unique opportunity for young researchers in the current situation. The organizers are grateful to the reporter Siegfried Beckus and to the two video conference assistants Florian Fischer and Elias Zimmermann who guaranteed a smooth technical process during the workshop.

Acknowledgement: Florian Fischer participated in the workshop as a recipient of a prize granted by the DMV for his excellent master thesis. Siegfried Beckus, Florian Fischer and Elias Zimmermann are grateful for financial support through the MFO. The MFO and the workshop organizers would like to thank the Simons Foundation for supporting Radoslaw Wojciechowski in the "Simons Visiting Professors" program at the MFO.

Workshop (online meeting): Geometry, Dynamics and Spectrum of Operators on Discrete Spaces

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Abstracts

A new uncertainty principle at low energies

Peter Stollmann
(joint work with Daniel Lenz, Gunter Stolz and Martin Tautenhahn)

1. Introduction

This talk is about a new *uncertainty principle at low energies* and its very simple proof. Based on joint works in preparation with Daniel Lenz, Gunter Stolz and Martin Tautenhahn.

- Classical **unique continuation** doesn't hold for graphs.
- There are estimates

$$\|\phi\|^2 \leq \kappa^{-1}\|\phi 1_D\|^2 \text{ for all } \phi \in \mathrm{Ran}(P_I)$$

  where,
  - D is spread out in X in the sense that for some $R > 0$

$$X \subset \bigcup_{p \in D} B_R(p),$$

  - $I = [0, E]$ where E is **small enough**,
  - and the geometry of the graph, R and E are suitably related and determine κ.

Estimates of this type are mostly based on a spectral theoretic uncertainty principle from [1]. They have been established, e.g. for the lattice in [2, 8] and for quite general graphs in [5]; see also [4].

2. A new method

Let X be a weighted graph with **energy form** $\mathcal{E}$. This means that we are given the following data:

- X is an arbitrary set, whose elements are referred to as *vertices*;
- $b : X \times X \to [0, \infty)$ is a symmetric function with $b(x, x) = 0$ for all $x \in X$.
- $m : X \to (0, \infty)$ is a function on the vertices.

An element $(x, y) \in X \times X$ with $b(x, y) > 0$ is then called an *edge* and b is denoted as *edge weight*; The positive function $m : X \to (0, \infty)$ gives a measure on X of which we think as a volume. In particular, we define

$$m(\Omega) := \sum_{x \in \Omega} m(x)$$

for $\Omega \subset X$. A natural distance to consider is

$$d(x, y) := \inf\{L(\gamma) \mid \gamma \text{ a path from } x \text{ to } y\},$$

where the length $L(\gamma)$ of a path γ is given by

$$L(\gamma) := \sum_{j=0,\dots,k-1} \frac{1}{b(x_j, x_{j+1})}.$$

Setting $d(x,x) = 0$ we obtain a pseudo-metric, i.e., d is symmetric and satisfies the triangle inequality. Clearly, in this generality, d need not separate the points of X. The energy form is given by

$$\mathcal{E}(f) := \frac{1}{2} \sum_{x,y\in X} b(x,y)(f(x) - f(y))^2 \text{ for } f \in \mathbb{R}^X,$$

(with the obvious domain) on the underlying Hilbert space

$$\ell^2(X,m) := \{f \in \mathbb{R}^X \mid \sum_{x\in X} |f(x)|^2 m(x) < \infty\}.$$

For finite [**connected**] $C \subset X$:

$$\lambda_1^N(C) = \inf\{\mathcal{E}(f) \mid f \in \ell^2(C,m) \text{ with } \langle f, 1\rangle = 0 \text{ and } \|f\| = 1\},$$

denotes the spectral gap or first nonzero eigenvalue of the Laplacian H_C^N with Neumann boundary conditions.

A decomposition $\mathcal{C}$ of X with spectral gap λ is a sequence $(C_n)_{n\in\mathbb{N}}$ of pairwise disjoint, finite and connected subsets of X such that

$$X = \bigcup_{n\in\mathbb{N}} C_n \text{ and } \lambda_1^N(C_n) \geq \lambda \quad (n \in \mathbb{N}).$$

We have the following general result:

Theorem 1. *Let $\mathcal{C}$ be as above and assume that $D \subset X$ has relative density ρ w.r.t. $\mathcal{C}$, i.e.,*

$$m(D \cap C_n) \geq \rho \cdot m(C_n) \text{ for all } n \in \mathbb{N}.$$

Assume that

$$\mathcal{E}(f) \leq \frac{1}{42}\lambda\rho\|f\|^2.$$

Then

$$\frac{1}{42}\rho\|f\|^2 \leq \|f1_D\|^2.$$

The proof is based upon the following local estimate:

Proposition 1. *Let $D \subset C \subset X$, C connected and finite. Assume that*

$$m(D) \geq \rho m(C)$$

and $f \in \ell^2(C)$,

$$\mathcal{E}(f) \leq \frac{1}{4}\lambda_1^N(C)\rho\|f\|^2.$$

Then

$$\frac{1}{5}\rho\|f\|^2 \leq \|f1_D\|^2.$$

To transfer the local result that can be used for the members of a decomposition in the above sense to the global case, we employ the following easy observation. We couldn't find a reference in the form stated below although the estimate should be well-known; however, a similar statement was communicated to us, see [6].

Lemma 1 (Neißichkait's Lemma). *Let* $0 \leq a_k, b_k$ *for* $k \in \mathbb{N}$ *and* $0 < \alpha < \beta$. *Assume that*

$$\infty > \sum_{k \in \mathbb{N}} a_k \geq \beta \sum_{k \in \mathbb{N}} b_k.$$

Then

$$\sum_{k \in \mathbb{N}: a_k \geq \alpha b_k} a_k \geq \left(1 - \frac{\alpha}{\beta}\right) \sum_{k \in \mathbb{N}} a_k.$$

The result is one possible special form of a whole class of uncertainty principles that can be established by the above method. We comment on possible variants:

- For the lattice, and more generally, Cayley graphs of polynomial volume growth, we obtain estimates that are considerably stronger than what was known before.
- For very general weighted graphs: use [5] and [4].
- Continuum results: Easy, lead to uncertainty principles complementary and partially more general than the ones in [9]. However, the lower bounds for infinite coupling obtained in the latter article are beyond the scope of the new method: this will be discussed elsewhere.
- The use of a Poincaré inequality under similar circumstances is not new at all; for a much older result in a different but somewhat related direction, see Gesztesy, Graf, Simon [3]; see also [8].
- In the same direction there are results by Pesenson [7] for graphs and metric measure spaces for spectral projectors. The key word is 'Paley Wiener spaces'; we still have to check out the precise relation to our results since we found this reference only recently!
- Application to lower bounds for Schrödinger operators on graphs: in preparation.

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A Fatou–Naïm–Doob-type Theorem for Schrödinger Operators on Graphs

Florian Fischer
(joint work with Matthias Keller)

Fatou–Naïm–Doob-type theorems state that the limit to the Martin boundary of a fraction of two positive superharmonic functions is given by a certain Radon–Nikodým derivative. This result is classical in Euclidean potential theory (see e.g. [1, 4, 10] or more recently [5]), in axiomatic Brelot-type potential theory (see [8]) as well as in probabilistic potential theory (see e.g. [3], and for random walks on trees see [2, 9]). Our version of this Fatou–Naïm–Doob theorem generalizes the probabilistic results to arbitrary graphs.

Let X be a countable discrete set. By a graph we mean a symmetric function $b\colon X \times X \to [0, \infty)$ with zero diagonal such that b is locally summable, i.e., $\sum_{y\in X} b(x, y) < \infty$ for all $x \in X$. We also assume that the graph is connected, i.e., for every vertices $x, y \in X$ there is a path $x_0, \ldots, x_n \in X$, such that $x = x_0$, $y = x_n$ and $b(x_{i-1}, x_i) > 0$ for all $i \in \{1, \ldots, n-1\}$.

On the formal space

$$\mathcal{F} = \{f \in \mathcal{C}(X)\colon \sum_{y\in X} b(x, y)\,|f(y)| < \infty \text{ for all } x \in X\}$$

we define the *Schrödinger operator* $H = H_{b,c,m}$ on $\mathcal{F}$ with respect to $0 < m \in C(X)$ and $c \in C(X)$ via

$$Hf(x) = \frac{1}{m(x)} \sum_{y\in X} b(x, y)\big(f(x) - f(y)\big) + \frac{c(x)}{m(x)} f(x), \qquad x \in X.$$

A function $s \in \mathcal{F}$ is called *(super)harmonic* on X if $Hs = 0$ ($Hs \geq 0$) on X.

Assume further that H is non-negative and subcritical, i.e., there exists a positive Green function $G\colon X \times X \to (0, \infty)$ to H. Let $\mathcal{G}$ be the set of G-integrable functions. Then we define the *Green operator* G on $\mathcal{G}$ via

$$Gf(x) = \sum_{y\in X} G(x, y) f(y),$$

and call Gf a *potential* with *charge* f. By the Riesz decomposition (cf. [7]) we have that for any non-negative superharmonic function s there is a unique decomposition

$$s = GHs + \mathrm{ghm}_s \qquad \text{on } X,$$

where $\mathrm{ghm}_s \geq 0$ denotes the greatest harmonic minorant of s.

Furthermore, for $\emptyset \neq A \subseteq X$, $0 \leq s$ superharmonic we define the *reduced function* r_s^A of s relative to A via

$$r_s^A(x) = \inf\{u(x)\colon 0 \leq u \text{ superharmonic on } X, u \geq s \text{ on } A\}.$$

This non-negative superharmonic function has many nice properties (cf. [6]) and some of them allow us to define the so-called *minimal fine topology* as follows:

Let $\hat{X}$ be the Martin compactification of X with corresponding Martin boundary $\mathcal{M}$, Martin kernel $K\colon X \times \hat{X}$ and unique Martin measure μ_s for $s \geq 0$ superharmonic (cf. e.g. [11] for details). Then, by the Riesz decomposition we have

$$r^A_{K(\cdot,\xi)} = GHr^A_{K(\cdot,\xi)} + \mathrm{ghm}_{r^A_{K(\cdot,\xi)}}.$$

We now call $A \subseteq X$ *minimally thin* at $\xi \in \mathcal{M}$ if

$$\mu_{\mathrm{ghm}_{r^A_{K(\cdot,\xi)}}}(\xi) = 0.$$

In this talk, we will also see a long list of characterizations of minimally thin sets (cf. [6]). Via the minimally thin sets one can define the so-called *minimal fine filter*

$$F_\xi = \{X \setminus A\colon A \text{ is minimally thin at } \xi\}, \qquad \xi \in \mathcal{M}_{\min},$$

where $\mathcal{M}_{\min}$ denotes the minimal Martin boundary. Along this filter one defines the minimal fine limits mf lim and induces the minimal fine topology. One can show that the minimal fine topology is indeed finer than the Martin topology.

Now, we are in a position to state the main result of this talk.

Theorem 1 (Fatou–Naïm–Doob's Theorem, [6])**.** *Let H be non-negative and subcritical, and $s, t > 0$ be superharmonic. Let f be the Radon–Nikodým derivative of the absolutely continuous component of μ_{ghm_s} with respect to μ_{ghm_t}. Then,*

$$\operatorname*{mf\,lim}_{x\to\xi} \frac{s(x)}{t(x)} = f(\xi) \qquad \text{at } \mu_{\mathrm{ghm}_t}\text{-a.e. } \xi \in \mathcal{M}_{\min}.$$

In the special case of p being a potential with non-negative charge and h is harmonic, we even have

$$\operatorname*{mf\,lim}_{x\to\xi} \frac{p(x)}{h(x)} = 0 \qquad \text{at } \mu_h\text{-a.e. } \xi \in \mathcal{M}.$$

Similarly to the results in the continuum, one also deduces on graphs that

$$\liminf_{x\in X, x\to\xi} \frac{s(x)}{G(o,x)} = \operatorname*{mf\,lim}_{x\to\xi} \frac{s(x)}{G(o,x)}, \quad \operatorname*{mf\,lim}_{x\to\xi} \frac{s(x)}{K(x,\xi)} = \mu_{\mathrm{ghm}_s}(\xi) = \inf_{x\in X} \frac{s(x)}{K(x,\xi)}$$

for all $\xi \in \mathcal{M}_{\min}$ and $s > 0$ superharmonic.

That this discrete theory fits perfectly beside the continuous theory seems not to be a coincidence. Using the here presented approach one can show that these graphs are also examples of balayage spaces (cf. [6]).

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Geometric aspects of Sobolev extension operators

Christian Rose

(joint work with Olaf Post, Xavier Ramos Olivé)

This talk is based on the recent article [3] where we constructed Sobolev extension operators for subsets of manifolds with norms depending quantitatively only on the curvature of the boundary and its tubular neighborhood. We explain how those operators are used to derive uniform Neumann heat kernel estimates for whole classes of subsets of manifolds satisfying integral Ricci curvature conditions.

If $M = (M^n, g)$ is a smooth Riemannian manifold of dimension $n \geq 2$ and $\Omega \subset M$ non-empty and open we call a linear and bounded $E_\Omega \colon H^1(\Omega) \to H^1(M)$ with

$$E_\Omega u \restriction_\Omega = u \quad u \in H^1(\Omega)$$

a *Sobolev extension operator*. Studying such extension operators has a long history and starts with famous work of Whitney [5]. The existence of E_Ω for given Ω depends on the properties of its boundary $\partial\Omega$. A famous construction for Lipschitz boundaries is due to Stein [4]. Nowadays Sobolev extension operators pop up in several contexts and are built based on the needs of a specific application. Usually, the norms of extension operators for fixed Ω depend crucially on the properties of a given atlas of $\partial\Omega$. Geometric applications to classes or sequences of subsets $\Omega \subset M$ require uniform bounds on the norm of E_Ω in terms of geometric quantities like curvature rather than properties of atlases, though. It is clear that curvature restrictions for $\partial\Omega$ cannot yield any control on its atlases. For the following classes of subsets we constructed Sobolev extension operators with uniform upper bounds on the norm in terms of geometric parameters.

Definition 1. *Let M be a Riemannian manifold of dimension $n \geq 2$, $R > 0$, and $H, K \geq 0$. A non-empty, open $\Omega \subset M$ is called* (R, H, K)-regular *if:*

- *$\overline{\Omega} \neq M$ is a connected manifold with (smooth) boundary $\partial\Omega$;*
- *the* exterior and interior rolling R-ball condition *holds: for any $x \in \partial\Omega$ there exist $p \in M \setminus \Omega$ and $q \in \Omega$ such that $B(p, R) \subset M \setminus \Omega$, $B(q, R) \subset \Omega$, and $\overline{B(p, R)} \cap \partial\Omega = \{x\} = \overline{B(q, R)} \cap \partial\Omega$;*
- *the second fundamental form satisfies $-H \leq \mathrm{II}_\Omega \leq H$;*
- *in the tubular neighborhood $\partial\Omega_R$ the sectional curvature satisfies $-K \leq \mathrm{Sec} \leq K$.*

Our first result is the existence of Sobolev extension operators for the classes of (R, H, K)-regular domains given R, H, and K.

Theorem 1 (Post, Ramos, R. '20, [3])**.** *Fix $H, K \geq 0$ and a complete Riemannian manifold M of dimension $n \geq 2$. There exists $R_0 = R_0(H, K) > 0$ such that for any $R \in (0, R_0]$ there exists a constant $C(R, H, K) > 0$ such that for any (R, H, K)-regular $\Omega \subset M$ there exists an extension operator*

$$E_\Omega \colon H^1(\Omega) \to H^1(M)$$

satisfying

$$\|E_\Omega\| \leq C(R, H, K).$$

For the proof of Theorem 1, we extend functions $u \in \mathcal{C}^1(\overline{\Omega})$ along geodesics perpendicular to $\partial\Omega$ to a function $E_\Omega u$ outside Ω depending on u's values inside Ω and compute its H^1-norm w.r.t. distance hypersurfaces. The main technical difficulty consists of controlling $|\nabla E_\Omega u|$ along the geodesic in terms of the geometric parameters and $|\nabla u|$ inside Ω. To this end, we construct variations of $E_\Omega u$ through geodesics for all directions in the tangent space of $\partial\Omega$, compute the partial dervatives, and show that $|\nabla E_\Omega u|$ can be controlled in terms of Jacobi fields of the variation. Moreover, one needs to bound $|\nabla E_\Omega u|$ in terms of a non-orthogonal frame along a geodesic, which might be a result of independent interest.

Our motivation to study Sobolev extension operators is based on our interest in quantitative estimates for Neumann heat kernels. In their famous paper [1], Li and Yau showed Neumann heat kernel estimates of the form

$$h_t(x, y) \leq \frac{C_1}{\sqrt{\mathrm{Vol}(B(x, \sqrt{t}))\mathrm{Vol}(B(y, \sqrt{t}))}} \exp\left(C_2 K t - C_3 \frac{d(x, y)^2}{t}\right) \tag{1}$$

for M being a compact manifold with Ricci curvature bounded below by $-K$, $K \geq 0$, and (smooth) convex boundary ∂M, or M being a geodesic ball B in a manifold with Ricci curvature bounded below by $-K$, $K \geq 0$. Wang relaxed the conditions of the first mentioned result to compact manifolds with Ricci curvature bounded below by $-K$, $K \geq 0$ and boundary not necessarily convex but satisfying the interior rolling R-ball condition (cf. Definition 1) and second fundamental form w.r.t. the outward normal bounded below.

In the last decades there was an increasing interest in relaxing uniform lower Ricci curvature bounds to integral curvature bounds, since the latter are more stable under perturbations of the metric. For $p > n/2$ and $D > 0$, we let

$$\kappa(p, D) := \sup_{x\in M} D^2 \left(\frac{1}{\mathrm{Vol}(B(x, D))} \int_{B(x,D)} |\mathrm{Ric}_-|^p \mathrm{dvol} \right)^{1/p}.$$

Petersen and Wei proved Laplacian comparison estimates for the distance function [2] and that $\kappa(p, D)$ is a scaling invariant quantity. Note that if $\mathrm{Ric} \geq -K$, $K \geq 0$, then $\kappa(p, D)$ is small for $D > 0$ small enough. Among all the generalizations of results depending on uniform lower bounds on the Ricci curvature to integral Ricci curvature assumptions that appeared during the last decades, a Neumann heat kernel estimate for classes of subsets with boundaries satisfying certain regularity assumptions was missing. Our second main result is the following variant of (1) if only integral Ricci curvature conditions hold.

Theorem 2 (Post, Ramos, R. '20, [3])**.** *Let $2p > n \geq 2$, $D > 0$, and $K, H \geq 0$. There exists an explicitly computable $R_0 = R_0(H, K) > 0$ such that for any $R \in (0, R_0]$, there are explicitly computable constants $C = C(n, p, R, D, H, K) > 0$ and $\epsilon = \epsilon(n, p, R, H, K) > 0$ such that if M is a complete Riemannian manifold of dimension n satisfying*

$$\kappa(p, D) \leq \epsilon,$$

then for any (R, H, K)-regular domain $\Omega \subset M$ with $\mathrm{diam}\Omega \leq D/2$*, the Neumann heat kernel h^Ω of Ω satisfies*

$$h_t^\Omega(x, x) \leq \frac{C}{\mathrm{Vol}(\Omega \cap B(x, \sqrt{t}))}, \quad x \in \Omega, \ t > 0. \tag{2}$$

To prove the statement, we first show that our assumptions yield a family of local Gagliardo-Nirenberg inequalities on $(0, D]$. Theorem 1 then implies quantitative global Gagliardo–Nirenberg inequalities on $(0, \infty)$ for (R, H, K)-regular $\Omega \subset M$ with $\mathrm{diam}\Omega \leq D/2$. We infer from more general statements that this is already equivalent to (2).

It is worth noting that we derived Neumann eigenvalue estimates as well, which basically follow from Theorem 2 and generalize many other earlier results on eigenvalue estimates to our (R, H, K)-regular domains even if $M = \mathbb{R}^n$. Moreover, we derived Harnack inequalities for the Neumann heat kernel, which generalize Li and Yau's result, yielding lower bounds for the Neumann heat kernel, too.

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Essential self-adjointness of the Laplacian on graphs: stability and Liouville properties

Radosław K. Wojciechowski
(joint work with Bobo Hua, Jun Masamune and Atsushi Inoue, Jun Masamune)

We discuss the essential self-adjointness of the Laplacian on weighted graphs following the setting established by Matthias Keller and Daniel Lenz in [1]. More specifically, if X is a countably infinite set of vertices, b is a symmetric edge weight with zero diagonal which is summable at each vertex and m is a strictly positive vertex measure such that $\sum_{y \in X} b^2(x,y)/m(y) < \infty$ at each vertex $x \in X$, then restricting the formal Laplacian

$$\mathcal{L}f(x) = \frac{1}{m(x)} \sum_{y \in X} b(x,y)(f(x) - f(y))$$

to the finitely supported functions results in a symmetric operator on the Hilbert space $\ell^2(X,m)$. The question then arises under which conditions this operator has a unique self-adjoint extension. In this case, the Laplacian is said to be essentially self-adjoint.

In [1] one finds a condition for essential self-adjointness in terms of the measure of infinite paths. This extended a result in [6] for the case of standard edge weights and counting measure. Furthermore, for locally finite graphs, [3] gives a condition for essential self-adjointness in terms of metric completeness with respect to intrinsic path metrics. All of these results rely on a characterization of essential self-adjointness of the Laplacian in terms of the triviality of λ-harmonic square summable functions for $\lambda < 0$. More specifically, essential self-adjointness is equivalent to the fact that functions $f \in \ell^2(X,m)$ which are in the domain of the formal Laplacian and satisfy

$$\mathcal{L}f = \lambda f$$

must be trivial for $\lambda < 0$. We first highlight the connection between this criterion for essential self-adjointness and the constancy of harmonic square summable functions, i.e., the case when $\lambda = 0$ in the equation above. More specifically, we say that a graph satisfies the ℓ^2-Liouville property if every function which satisfies $\mathcal{L}f = 0$ and is in $\ell^2(X,m)$ is constant.

Our first result can be summarized as follows: essential self-adjointness always implies the ℓ^2-Liouville property. On the other hand, the ℓ^2-Liouville property along with strict positivity and the fact that the entire space has infinite measure, implies essential self-adjointness. This comes from recent joint work with Bobo Hua and Jun Masamune [2]. Let us note that the formulation given in [2] is general enough to also cover an analogous result for Laplacians on Riemannian manifolds.

Furthermore, [2] contains examples of graphs and manifolds which show that the additional assumptions of strict positivity and infinite measure are necessary. The graph examples can be achieved using birth-death chains (i.e., graphs over the natural numbers where the only edges are those that connect subsequent numbers) which always satisfy the ℓ^2-Liouville property but for which the Laplacian is not necessarily essentially self-adjoint or by gluing birth-death chains together in a way that preserves the ℓ^2-Liouville property but so that essential self-adjointness fails.

We also mention that [2] gives a technique for gluing birth-death chains together in order to break the ℓ^2-Liouville property on a graph over the integers. The main idea for this approach is that strict positivity implies the existence of a Green's function which is in $\ell^2(X,m)$ and is harmonic except for at one vertex. By removing this vertex and attaching appropriate birth-death chains, the Green's function can be extended to the new graph so that it remains in $\ell^2(X,m)$. In particular, taking a birth-death chain whose Laplacian is strictly positive, removing the origin and gluing a second birth-death chain whose edge weights and vertex measure satisfy

$$\sum_{r=0}^{\infty}\left(\sum_{k=0}^{r}\frac{1}{b(k,k+1)}\right)^2 m(r+1)<\infty$$

allows the extension of the Green's function to the entire graph in such a way that the resulting function is harmonic everywhere and is in $\ell^2(X,m)$. In particular, the resulting graph does not satisfy the ℓ^2-Liouville property and the Laplacian is not essentially self-adjoint. This shows that both the ℓ^2-Liouville property and essential self-adjointess are not particularly stable when gluing graphs together.

The second result we highlight provides a further discussion of the stability of essential self-adjointness. To put this result into context, let us mention that [1] shows that if the Dirichlet Laplacian on a subgraph gives a stochastically incomplete process and if the weighted vertex degree from the subgraph to the entire graph is bounded, then the process on the entire graph is stochastically incomplete. Furthermore, [1] gives examples to show that the boundedness condition on the vertex degree is necessary as every stochastically incomplete graph can be made into a subgraph of a stochastically complete graph. Our result concerning the stability of essential self-adjointness can be stated as follows: if the weighted vertex degree of the boundary between a subgraph and its complement is bounded, then the Laplacian on the entire graph is essentially self-adjoint if and only if the Laplacians with Neumann conditions at the boundary between the subgraph and its complement on each of the subgraphs are essentially self-adjoint. Here, by weighted vertex degree of the boundary we mean the following: if $X_1 \subseteq X$ and $X_2 = X \setminus X_1$, then we require that the functions defined on X via

$$\mathrm{Deg}_{\partial X_k}(x) = \frac{1}{m(x)}\sum_{y\notin X_k} b(x,y)$$

are bounded on X_k for $k = 1,2$. This result can be found in work in progress with Atsushi Inoue and Jun Masamune [5]. In particular, gluing a birth-death chain whose Laplacian is essentially self-adjoint to another birth-death chain whose Laplacian is not essentially self-adjoint at one vertex results in a graph whose Laplacian is not essentially self-adjoint similar to the discussion concerning the ℓ^2-Liouville property above.

Furthermore, [5] gives an example of a birth-death chain with a Laplacian which is not essentially self-adjoint but by gluing infinitely many additional vertices, we make the Laplacian on the entire graph essentially self-adjoint. The construction is in the spirit of one found in [1] and involves attaching a terminal vertex to every vertex in the birth-death chain. Finally, let us mention that using recently established results of Atsushi Inoue concerning limit point-limit circle theorems for graphs over the integers [4] along with the stability and Liouville results discussed above, in [5] we characterize the essential self-adjointness of the Laplacian on birth-death chains as well as on the integers. The proof of these characterizations involves a reduction to the case of harmonic functions and the criterion is in terms of the double sum introduced in the context of the Green's function above. For details, see the forthcoming paper [5].

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Spectra, self-similar groups, aperiodic order and random Schrödinger operators

Rostislav Grigorchuk

(joint work with Artem Dudko, Daniel Lenz, Tatiana Nagnibeda and Daniel Sell)

The results presented in the talk are based on joint results of the speaker with Artem Dudko, Daniel Lenz, Tatiana Nagnibeda and Daniel Sell presented in articles [4, 7, 8, 9, 10].

Given an action of a finitely generated group G with a system of generators $A = \{a_1, \ldots, a_m\}$ on a set X, one can associate with each point $x \in X$ an orbital graph Γ_x with the set of vertices the orbit Gx and the set of edges of the form $\{(y, ay) : a \in A, y \in Gx\}$. Additionally, we supply the edges by the corresponding labels $a \in A$. Given an element $m = \sum_{g \in A \cup A^{-1}} c_g \cdot g \in \mathbb{C}[G]$ of the group algebra using the coefficients $c_g \in \mathbb{C}$ one can convert Γ_x into a weighted graph $\Gamma_{x,m}$ and consider the convolution operator $L_{x,m}$ in the Hilbert space $l^2(Gx)$.

If m is a self-adjoint element with real coefficients, the operator $L_{x,m}$ could be interpreted as (discrete) Laplace (or Markov) operator on $\Gamma_{x,m}$. Study of spectra of such operators (i.e. spectra of weighted graphs) is related to many problems of mathematics. The following major problems are still open. What could be the shape of the spectrum? Can Cayley graphs have Cantor spectrum? Can a torsion free group have a gap in the spectrum of a Cayley graph?

In fact, operators of the type $L_{x,m}$ can be obtained from the regular representation λ_G (the case of Cayley graphs) or quasi-regular representation λ_{G/G_x} (the case of Schreier graph) in the Hilbert spaces $l^2(G)$ and $l^2(G/G_x)$ respectively (where G_x is the stabilizer subgroup of x).

More generally, any unitary representation ρ of G in a Hilbert space $\mathcal{H}$ extends to a $\star$-representation of the group algebra $\mathbb{C}[G]$ by bounded operators in $\mathcal{H}$, and for any $m \in \mathbb{C}[G]$ one gets the operator $\rho(m)$. Study of spectral properties of such operators is a huge area of mathematics that includes the spectral problem for graphs. As an example, let us mention the "almost Mathieu operator" arising from the well known unitary representation of the Heisenberg group.

Given an action of G on a measure space (X, μ) with quasi-invariant measure μ one can consider the Koopman type representation κ in $L^2(X, \mu)$: $\kappa(g)f(x) = \sqrt{\frac{dg_\star\mu}{d\mu}(x)}f(g^{-1}x)$ or the groupoid representation π in $L^2(\mathcal{R}, \nu)$ where $\mathcal{R} \subset X \times X$ is the equivalence relation induced by the partition into orbits and where $\nu = \mu\times\theta$, θ being the counting measure on the fibers (for precise definition see [4]). With each point $x \in X$ one can associate an orbit graph Γ_x and a permutational representation ρ_x in $l^2(Gx)$ (which is unitary equivalent to the quasi-regular representation λ_{G/G_x}).

Theorem 1. *[Artem Dudko, Rostislav Grigorchuk [4]] Let (G, X, μ) be an action of a countable group G with quasi-invariant probability measure μ. Then for any $m \in \mathbb{C}[G]$*

$$sp(\kappa(m)) \supseteq sp(\rho_x(m) = sp(\pi(m))$$

μ-almost surely.

If in addition the equivalence relation given by the partition into orbits is hyperfinite (i.e. is amenable in Zimmer's sense) then in the above statement the relation $\supseteq$ can be replaced by the equality relation.

In the case when support of m is $A \cup A^{-1}$ and $\Gamma_x, x \in X$ are converted into weighted graphs with weight determined by the coefficients of m, the spectrum of the operator $\rho_x(m)$ coincides with the spectrum of the weighted Laplace operator on Γ_x. The above theorem is a modification of the result from [1] concerning the case when the group acts on the rooted tree T by automorphisms and hence acts on its boundary ∂T by homeomorphisms (or to be more precise by solenoidal maps). In that case the coincidence of spectra holds for graphs associated with all points of the boundary by [1].

The result from [1] and Theorem 1 allow to compute in many cases spectra of graphs associated with self-similar groups, as shown in [1, 12] and many other articles cited in [3, 11] for instance. One particular case concerns the group $\mathcal{G}$ of intermediate growth between polynomial and exponential, introduced in [5]. The group $\mathcal{G}$ has a presentation by generators and relations

$$\mathcal{G} = \langle a, b, c, d : 1 = a^2 = b^2 = c^2 = d^2 = bcd = \sigma^k((ad)^4)$$

$$= \sigma^k((adacac)^4), k = 0, 1, 2, \dots \rangle$$

where σ is the substitution $\sigma : a \to aca, b \to d, c \to b, d \to c$. $\mathcal{G}$ acts on the binary rooted tree T by automorphisms and Schreier graphs $\Gamma_x, x \in \partial T$ associated with the points of the boundary ∂T have a "linear" structure discovered in [1]. Let $m = xa + yb + zc + ud \in \mathbb{C}[\mathcal{G}]$ and $\Gamma_{x,m}$ be a weighted Schreier graph given by coefficients of m. If $y = z = u$ then the spectrum of $\Gamma_{x,m}$ is a union of two intervals (or one interval). If at least two numbers from $\{y, z, u\}$ are distinct then the spectrum of $\Gamma_{x,m}$ is a Cantor set of Lebesgue measure zero. This is proved by Daniel Lenz, Tatiana Nagnibeda and Rostislav Grigorchuk in [7, 9]. The proof of this fact is based on the relation of the Laplacian on $\Gamma_{x,m}$ with the dynamically defined random Jacobi-Schrödinger operator associated with the minimal subshift (Ω_σ, T) determined by the substitution σ. In [7, 8, 9], various dynamical and combinatorial properties of the subshift (Ω_σ, T) are investigated in details. Also the result of Siegfried Beckus and Felix Pogorzelski from [2] about Cantor spectrum for such operators is used.

The group $\mathcal{G}$ is in an particular case of uncountable family $\mathcal{G}_\omega, \omega \in \{0, 1, 2\}^{\mathbb{N}}$ of 4-generated groups introduced in [6]. All these groups act by automorphisms of T and hence act by homeomorphisms on ∂T. As before, the corresponding Schreier graphs $\Gamma_{\omega,x}, \omega \in \{0, 1, 2\}^{\mathbb{N}}, x \in \partial T$ can be supplied with a weight coming from coefficients of the element m of the group algebra. In [10] the notion of a *leading sequence condition* (LCS) for subshifts has been introduced, a combinatorial criterion for it has been found, and it has been proved that (LCS) implies the uniformity of locally constant cocycles, which in turn implies the Cantor spectrum of zero Lebesgue measure for the random Jacobi-Schrödinger operator with dynamically defined Jacobi matrix satisfying natural conditions.

It is shown in [10] that Sturmian subshifts and simple Toeplitz subshifts satisfy (LCS). This is used to show that in the case when the relations $y = z = u$ do not hold the spectrum of the weighted Laplace operator on graphs $\Gamma_{\omega,x}$ is a Cantor set of Lebesgue measure zero which does not depends on x .

The latter result has consequences for the dynamics of multidimensional rational maps discovered for the first time in [1] and carefully inspected in [3] in view of the alternative "integrable vs not integrable". It gives important information about the shape of a joint spectrum of the pencil of Laplace operators determined by the element m of the group algebra with the support on the generating set of $\mathcal{G}_\omega$. This spectrum is invariant set for corresponding map and has the shape of a closed set whose crossings by vertical lines are Cantor sets. The corresponding pictures can be found in [11].

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Bakry–Émery curvature on graphs as an eigenvalue problem

Norbert Peyerimhoff
(joint work with David Cushing, Supanat Kamtue, Shiping Liu)

Following an idea by David Cushing, we reformulate the semidefinite programming problem of computing Bakry–Émery curvature of a vertex $x \in V$ of a weighted graph $G = (V, b, m)$ as a problem of finding the smallest eigenvalue. This reformulation is not only of practical importance for more speedy explicit curvature computations but has also various interesting theoretical implications.

A weighted graph $G = (V, b, m)$ is given by a vertex set V, a symmetric function $b : V \times V \to [0, \infty)$ with $b(x, x) = 0$ for all $x \in V$ representing the edge weights, and a vertex measure $m : V \to (0, \infty)$. Two vertices $x, y \in V$ are adjacent (notation $x \sim y$) iff $b(x, y) > 0$. We assume that each vertex has only finitely many neighbors. The combinatorial sphere and the ball of radius $k \in \mathbb{N}$ around x is denoted by $S_k(x)$ and $B_k(x)$, respectively. Moreover, let $p_{xy} = \frac{b(x,y)}{m(x)}$ and $d_x = \sum_y b(x, y)$ be the "vertex degree". In the special case $d_x = m(x)$ for all $x \in V$, the p_{xy} can be understood as transition probabilities of a reversible Markov chain.

The Laplacian $\Delta : C(V) \to C(V)$ (with $C(V)$ the vector space of all functions $f : V \to \mathbb{R}$) is given by

$$\Delta f(x) = \sum_y p_{xy}(f(y) - f(x)).$$

Bakry–Émery curvature is a concept for spaces with a Laplace operator (or a Markov generator) and is motivated by Bochner's formula, a fundamental identity on Riemannian manifolds (M, g). Using the quadratic forms

$$\begin{aligned}\Gamma(f,g) &= \frac{1}{2}\left(\Delta(fg) - f\Delta g - g\Delta f\right), \quad \Gamma(f) = \Gamma(f,f),\\ \Gamma_2(f,g) &= \frac{1}{2}\left(\Delta\Gamma(f,g) - \Gamma(f,\Delta g) - \Gamma(g,\Delta f)\right), \quad \Gamma_2(f) = \Gamma_2(f,f),\end{aligned}$$

Bochner's formula implies for N-dimensional manifolds (M, g) with Ricci curvature bounded below by $K \in \mathbb{R}$ at $x \in M$ the curvature-dimension inequality

$$\Gamma_2(f)(x) \geq \frac{1}{N}|\Delta f(x)|^2 + K\Gamma(f)(x)$$

for all functions $f \in C^\infty(M)$. This motivates the following definition:

Definition 1. *Let $N \in (0, \infty]$. The Bakry–Émery curvature $K_{G,x}(N)$ for dimension N is the largest value $K \in \mathbb{R}$ such that*

$$\Gamma_2 f(x) \geq \frac{1}{N}(\Delta f(x))^2 + K\Gamma f(x) \quad \forall f \in C(V).$$

We also say that the vertex x satisfies $CD(K, N)$.

Variants of this curvature notion are CDE, CDE' and $CD\psi$ introduced in [1] and [3], respectively. $K_{G,x}$ is fully determined by the information of the 2-ball $B_2(x)$. Using an observation by Schmuckenschläger [4] to drop columns and rows corresponding to the center vertex x, the computation of $K_{G,x}(N)$ boils down to solution of the semidefinite programming problem

$$\begin{aligned}&\text{maximize } K\\ &\text{subject to } \Gamma_2(x)_{\hat{1}} - \frac{1}{N}v_0^2(v_0^2)^\top - K\Gamma(x)_{S_1,S_1} \geq 0,\end{aligned}$$

with matrices $\Gamma_2(x), \Gamma(x)$ given in [2, Section 10] and $v_0 = (\sqrt{p_{xy_1}}, \ldots, \sqrt{p_{xy_m}})^\top$ with $S_1(x) = \{y_1, \ldots, y_m\}$. Indexing by $\hat{1}$ means removal of first column and row, squaring a vector means squaring each entry, and indexing by "S_1, S_1" means choosing the block corresponding to the vertices in $S_1(x)$. Moreover, the smaller matrices are extended by zero-blocks to match the size of $\Gamma_2(x)_{\hat{1}}$.

For the eigenvalue reformulation we employ the Schur complement of a block matrix $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$, namely $Q(M, M_{22}) = M_{11} - M_{12}M_{22}^{-1}M_{21}$. We choose $Q = Q(\Gamma_2(x)_{\hat{1}}, \Gamma_2(x)_{S_2,S_2})$ and define

$$A_\infty(x) = 2D^{-1}QD, \quad A_N(x) = A_\infty(x) - \frac{2}{N}v_0v_0^\top$$

with the diagonal matrix $D = \mathrm{diag}(v_0)$. Note that $A_N(x)$ is a rank one perturbation of $A_\infty(x)$. Our main result is

Theorem 1. *Let $G = (V, b, m)$ be a weighted graph. Then we have for $x \in V$,*

$$K_{G,x}(N) = \lambda_{\min}(A_N(x)).$$

Using the Matrix Determinant Lemma, we have

Corollary 1. *Assume that $A_\infty = A_\infty(x)$ is positive definite. Then we have $K_{G,x}(N_0) = 0$ at precisely the dimension*

$$N_0 = 2v_0^\top A_\infty^{-1} v_0 = 2\sum_{ij} \sqrt{p_{xy_i} p_{xy_j}} (A_\infty^{-1})_{ij}.$$

The variational Rayleigh quotient description

$$\lambda_{\min}(A_N(x)) = \inf_{v \neq 0} \frac{v^\top A_N(x) v}{v^\top v}$$

leads to the following upper curvature bound:

Theorem 2. *Let $G = (V, b, m)$ be a weighted graph. Then we have for $x \in V$ and $N \in (0, \infty]$,*

$$K_{G,x}(N) \leq 2\frac{v_0^\top A_N(x) v_0}{v_0^\top v_0} = \mathcal{K}_\infty^0(x) - \frac{2}{N}\frac{d_x}{m(x)} \tag{1}$$

with

$$\mathcal{K}_\infty^0(x) = \frac{1}{2}\left[\frac{d_x}{m(x)} + 3\frac{m(x)}{d_x} p_{xx}^{(2)} - \frac{m(x)}{d_x} \sum_{z \in S_2(x)} p_{xz}^{(2)}\right],$$

*where $p_{uv}^{(2)} = \sum_w p_{uw} p_{wv}$. If (1) holds with equality for some $N \in (0, \infty]$, we say that x is N-*curvature sharp.

Our main theorem is also useful to derive various further properties of the continuous curvature function $K_{G,x} : (0, \infty] \to \mathbb{R}$. Let $E_{\min}$ be the eigenspace of the smallest eigenvalue of the symmetric matrix $A_\infty(x)$. Then we have:

(i) The curvature function is either analytic, strictly concave and strictly monotone increasing on $(0, \infty]$ or there is a threshold N_1 such that $K_{G,x}$ is analytic, strictly concave and strictly monotone increasing on $(0, N_1]$ and constant on $[N_1, \infty]$.
(ii) If v_0 is N_1-curvature sharp with maximally chosen $N_1 \in (0, \infty]$, then (1) holds with equality for all $N \leq N_1$ and $K_{G,x}$ is constant on $[N_1, \infty]$.
(iii) v_0 is an eigenvalue of $A_\infty(x)$ if and only if $K_{G,x}$ is N_1-curvature sharp for some $N_1 \in (0, \infty]$.
(iv) $K_{G,x}$ is constant on a nontrivial segment $[N_1, \infty]$ if and only if v_0 is perpendicular to $E_{\min}$.

We also show that a sufficient criterion for a vertex $x \in V$ to be N-curvature sharp fo some $N \in (0, \infty]$ are the following two homogeneity properties: $p^-(y) = p_{yx}$

is independent of $y \in S_1(x)$ (x is S_1-*in regular*) and $p^+(y) = \sum_{z \in S_2(x)} p_{yz}$ is independent of $y \in S_1(x)$ (x is S_1-*out regular*).

Finally, let us mention that our main theorem can also be used to prove a Conjecture in [2]:

Theorem 3 ([2, Conjecture 6.13])**.** *Let $G = (V, E)$ be a combinatorial graph with the non-normalized Laplacian (that is $m \equiv 1$ and $b(x, y) = 1$ if $x \sim y$) and $x \in V$. Let $G' = (V', E')$ be the graph obtained from G by one of the following operations:*

- *Delete a leaf in $S_2(x)$ and its incident edge.*
- *Delete $z \in S_2(x)$ and its incident edges $\{\{y, z\} \in E : y \in S_1(x)\}$; add a new edge between every two of $\{y \in S_1(x) : \{y, z\} \in E\}$.*

Then we have for any $N \in (0, \infty]$: $K_{G',x}(N) \geq K_{G,x}(N)$.

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Primitive substitutions beyond abelian groups: The Heisenberg group

Siegfried Beckus

(joint work with Tobias Hartnick and Felix Pogorzelski)

This talk is devoted to the 50th birthday of Daniel Lenz.

Substitutions in the geometric or the symbolic setting are local rules replacing a tile or a colored vertex by a finite pattern. They are used to define tilings in abelian groups (e.g. Penrose tiling) or symbolic dynamical systems (e.g. Fibonacci sequence) producing aperiodic dynamical systems with low complexity. For abelian groups these dynamical systems are minimal and uniquely ergodic if the substitution is primitive [8, 9, 10]. Moreover, the existence of proximal pairs implies aperiodicity [3, 2]. Another approach proving aperiodicity in the geometric setting is by requiring that the substitutions are injective [1, 11].

In this talk, we present two works [6, 7] in progress with Tobias Hartnick and Felix Pogorzelski where these concepts have been extended to the non-abelian world. This has its origin in finding uniquely ergodic and minimal dynamical systems with low complexity as well as in the recent developments [4, 5].

In [6], the concept of linearly repetitive configurations is extended to (colored) Delone sets in amenable groups. Such sets are called *linearly repetitive* if there is a $C > 0$, such that each pattern supported on a ball B_r can be found in any pattern supported on ball B_{Cr}. Replacing balls by suitable Følner sequence, the concept of tempered repetitivity is introduced leading to uniquely ergodic hulls of the corresponding (colored) Delone set.

With this at hand, one can ask for the existence of such aperiodic sets. A large class of examples are constructed in [7] extending the concept of a primitive substitutions to certain nilpotent Lie groups. In order to get aperiodic elements, the concept of non-periodic substitution is introduced. Besides that we provide an algorithm to explicitly construct non-periodic primitive substitutions, which works for a large class of groups. For instance, one can find already more than 60 different groups up to dimension 7 (excluding products of lower dimensional nilpotent Lie groups) falling into our setting while only a handful of them are abelian. In order to show their beauty and the difficulties that arise in the non-abelian world, we will focus in this talk on the Heisenberg group. We need two ingredients for our framework

- *geometric data*: a locally compact second countable group G (ambient space), a family of dilations $(D_\lambda)_{\lambda\geq 0}$ on G and a lattice Γ that is invariant under some of the dilations together with a fundamental domain V
- *combinatorial data*: a finite alphabet $\mathcal{A}$, a scaling factor λ and a substitution rule S_0

Let $G := \mathbb{H}_3(\mathbb{R})$ be the 3-dimensional Heisenberg group, which we can think of as $\mathbb{R}^3$ with multiplication and inverse given by

$$(x,y,z)(a,b,c) := (x+a, y+b, z+c+xb-ya), \qquad (x,y,z)^{-1} := (-x,-y,-z).$$

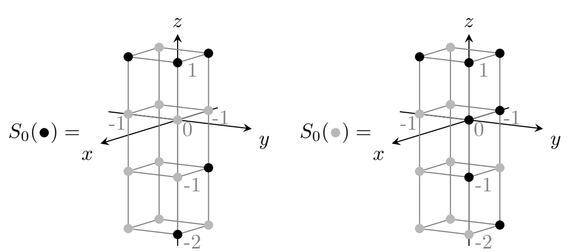
The Cygan-Korányi norm $\|(x,y,z)\| := ((x^2+y^2)^2+z^2)^{1/4}$ defines a left-invariant metric via $d(g,h) := \|g^{-1}h\|$ for $g,h \in \mathbb{H}_3(\mathbb{R})$. For $\lambda > 0$, the automorphism D_λ on $\mathbb{H}_3(\mathbb{R})$ defined by $D_\lambda(x,y,z) := (\lambda x, \lambda y, \lambda^2 z)$ satisfies $d(D_\lambda(g), D_\lambda(h)) = \lambda d(g,h)$. A lattice in G with fundamental domain $V := \big[-\frac{1}{2}, \frac{1}{2}\big)^3 \subseteq \mathbb{H}_3(\mathbb{R})$ is given by

$$\Gamma := \mathbb{H}_3(\mathbb{Z}) := \{(x,y,z) \in \mathbb{H}_3(\mathbb{R}) : x,y,z \in \mathbb{Z}\}$$

Then $D_\lambda(\Gamma) \subseteq \Gamma$ holds if and only if $\lambda \in \mathbb{N}$. In this model, we conclude

$$\Gamma = \bigsqcup_{g\in\Gamma} D_\lambda(g)\big(D_\lambda(V)\cap\Gamma\big), \qquad \text{for } \lambda\in\mathbb{N}. \tag{1}$$

Having all geometric data at hand, let us consider the combinatorial data:



$\mathcal{A} = \{\bullet, \circ\}$

$\lambda = 2$

$D_\lambda(V)\cap\Gamma = \{-1,0\}^2 \times \{-2,-1,0,1\}$

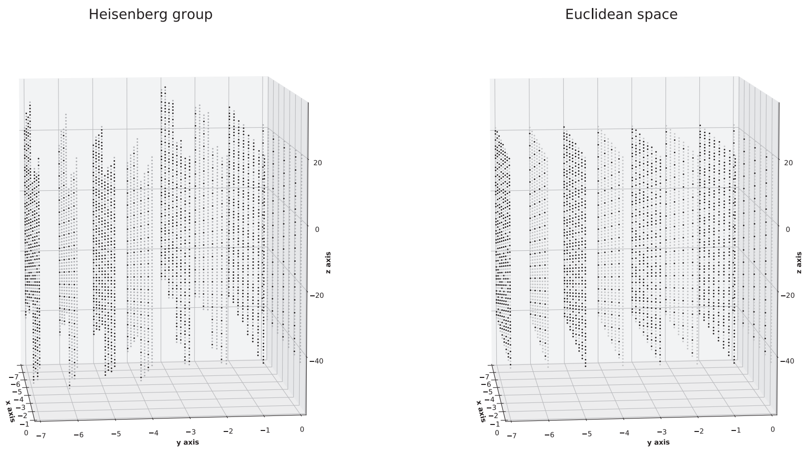


FIGURE 1. The pattern $S^3(\bullet)$ is plotted on the left for the Heisenberg group $\mathbb{H}_3(\mathbb{Z})$ and on the right for $\mathbb{Z}^3$ for the same S_0.

Invoking (1), for every $\gamma \in \Gamma$, there is a unique $g \in \Gamma$ and $h \in D_\lambda(V) \cap \Gamma$ such that $\gamma = D_\lambda(g)h$. With this at hand, the substitution rule S_0 can be extended to

$$S : \mathcal{A}^{**} \to \mathcal{A}^{**} := \bigsqcup_{K \subseteq \Gamma} \mathcal{A}^K, \qquad S(\omega)(\gamma) := S_0(\omega(g))(h), \qquad \gamma = D_\lambda(g)h.$$

The countable group Γ acts on the compact product space $\mathcal{A}^\Gamma$ via $(\gamma\omega)(g) := \omega(\gamma^{-1}g)$ and we conclude for $\lambda \in \mathbb{N}$:

Proposition 1. *The map $S : \mathcal{A}^\Gamma \to \mathcal{A}^\Gamma$ is continuous, $S(\gamma\omega) = D_\lambda(\gamma)S(\omega)$ and*

$$\Omega(S) := \{\omega \in \mathcal{A}^\Gamma \, : \, \omega \text{ admits only legal patterns w.r.t. } S\}$$

is closed, S-invariant ($S(\omega) \in \Omega(S)$ for $\omega \in \Omega(S)$) and Γ-invariant ($\gamma\omega \in \Omega(S)$ for $\gamma \in \Gamma$ and $\omega \in \Omega(S)$). If, additionally, λ is sufficiently large relative to V, then $\Omega(S)$ is non-empty.

It is worth pointing out that the main technical difficulty is to show $\Omega(S) \neq \emptyset$ for which one needs to study the growth of the support $S^n(a)$ in $n \in \mathbb{N}$ for some $a \in \mathcal{A}$. As we can observe in Figure 1, the non-abelian behavior differs from the abelian. Here, we call λ *sufficiently large relative to V* if $\lambda \geq 1 + \frac{r_+}{r_-}$ where $r_-, r_+ > 0$ satisfy $B_{r_-}(e) \subseteq V \subseteq B_{r_+}(e)$.

For our chosen metric on $\mathbb{H}_3(\mathbb{Z})$, one computes $r_- = \frac{1}{4}$ and $r_+ = \sqrt[4]{\frac{1}{2}}$ implying $\lambda \geq 5$ is sufficiently large relative to V. However, one can show that for $\lambda = 2$, the set $\Omega(S)$ is nonempty (and hence for $\lambda \geq 2$). In order to do so one first shows that $B_r\big((-1,-1,0)\big) \subseteq D_\lambda\big((D_\lambda(V) \cap \Gamma)V\big)$ where the set on the right hand side is the support $S^2(a)$ for any $a \in \mathcal{A}$ and $r := 1.682$. The choice of r leads to $\lambda = 2 > \frac{r}{r-r_+}$. Then there are arbitrarily large legal patterns for any substitution rule with $\lambda = 2$ by a straightforward adjustment of the proof of the theorem of the support growth in [7].

The substitution rule S_0 is called *primitive* if there is an $L \in \mathbb{N}$ such that the letter a occurs in $S^L(b)$ for all $a, b \in \mathcal{A}$. Furthermore, S_0 is called *non-periodic* if S_0 is injective and for all $\gamma \in \big(D_\lambda(V) \cap \Gamma\big) \setminus \{e\}$ and $a, b \in \mathcal{A}$:

$$\big(\gamma^{-1} S_0(a)\big)|_{\gamma^{-1}D_\lambda(V)\cap D_\lambda(V)} \neq S_0(b)|_{\gamma^{-1}D_\lambda(V)\cap D_\lambda(V)}.$$

It is straightforward to check that our example S_0 is primitive and non-periodic. An element $\omega \in \mathcal{A}^\Gamma$ is called *non-periodic* if $\gamma\omega = \omega$ implies $\gamma = e$ where $e \in \Gamma$ is the neutral element. Then $\Omega(S)$ is *weakly aperiodic* if there exists a non-periodic element in $\Omega(S)$ and it is *strongly aperiodic* if all elements of $\Omega(S)$ are non-periodic.

Theorem 1 ([6, 7])**.** *Let $\Gamma := \mathbb{H}_3(\mathbb{Z})$, $\lambda \geq 2$ and $S_0 : \mathcal{A} \to \mathcal{A}^{D_\lambda(V)\cap\Gamma}$.*

(a) If S_0 is primitive, then every element in $\Omega(S)$ is linearly repetitive and $\Omega(S)$ is minimal and uniquely ergodic.
(b) If S_0 is non-periodic, then $\Omega(S)$ is weakly aperiodic.
(c) If S_0 is non-periodic and primitive, then $\Omega(S)$ is strongly aperiodic.

Assertion (c) does not hold in full generality while the other statements extend to a large class of nilpotent Lie groups. It is an open question if $\Omega(S)$ is strongly aperiodic whenever S_0 is non-periodic and primitive.

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Pure point diffraction

Daniel Lenz
(joint work with Timo Spindeler and Nicolae Strungaru)

A pivotal article on the topic of diffraction for aperiodic structures was written by Bombieri–Taylor in 1986 [5]. It has as its title:

Which distributions of matter diffract? An initial investigation.

The article by Bombieri–Taylor is motivated by the discovery of materials with a new form of order. These materials were discovered by Shechtman in 1982 in diffraction experiments [12]. They were later called quasicrystals and Shechtman war awarded a Nobel prize in chemistry for the discovery. The diffraction experiments showed

- sharp Bragg peaks (point diffraction), indicating long range order;
- 10-fold symmetry, indicating absence of lattice structure.

In this way the diffraction experiments clearly indicated a new form of order. In mathematics the term aperiodic order was coined to describe it. Since the discovery of Shechtman there has been a lot of activity in the investigation of aperiodic order and (pure) point diffraction in physics, material sciences, chemistry and mathematics. We refer to the monographs [1, 2] for further discussion, background and references.

Here we are concerned with the harmonic analysis behind pure point diffraction. Our results can be understood to answer the question in the title of the Bombieri / Taylor article if 'diffract' is understood to mean having pure point diffraction and 'distribution of matter' is modeled by translation bounded measure. The results answer various questions posed in an influential survey article on the topic by Lagarias [8].

To be more specific, we shortly summarize the framework of mathematical diffraction theory as systematically developed by Hof [7] and later extended by various people (see e.g. the survey article [3] for details and references): Let G be a locally compact abelian group. Let $\mathcal{A} = (A_n)$ be a van Hove sequence on G. To a translation bounded measure μ on G we then associate its *autocorrelation* along $\mathcal{A}$, which is defined as the Eberlein convolution

$$\gamma := \lim_{n\to\infty} \frac{1}{|A_n|}(\mu|_{A_n} * \widetilde{\mu}|_{-A_n})$$

(if the limit exists). Here, the limit is taken in the vague sense and the translation bounded measure $\widetilde{\mu}$ is defined by $\widetilde{\mu}(\varphi) = \overline{\mu(\overline{\varphi(-\cdot)})}$ for continuous complex valued φ on G with compact support. Then γ is a positive definite measure and, hence, possesses a Fourier transform $\widehat{\gamma}$, which is a positive measure on the dual group of G. This Fourier transform is known as *diffraction measure*. Let us emphasize that existence of the autocorrelation is an assumption.

The connection of this setup to diffraction experiments is given as follows: The distribution of matter is modeled by μ and the outcome of the diffraction experiment is governed by $\widehat{\gamma}$. In particular, the case where $\widehat{\gamma}$ is a pure point measure is of utmost interest. Thus, the first question in this context is the following:

Question 1: Which conditions on μ ensure that $\widehat{\gamma}$ is a pure point measure?

Moreover, one is interested in further pieces of information concerning the atoms of $\widehat{\gamma}$. In fact, it is desirable to have a Fourier type expansion

$$\mu \sim \sum a_\xi \xi \text{ with } \widehat{\gamma} = \sum |a_\xi|^2 \delta_\xi. \quad (*)$$

Here, the sums run over all ξ in the dual group of G and only countably many of the a_ξ are non-zero. A measure μ satisfying $(*)$ is said to *solve the phase problem* (see [8] for further discussion and note that giving a precise sense to the symbol $\sim$ is part of the problem). So, the second question in this context can be formulated as follows:

Question 2: When does μ solve the phase problem?

So far, our investigation was based on a fixed van Hove sequence. Of course, it is natural to strive for independence of the van Hove sequence. This leads to the final question:

Question 3: When does μ solve the phase problem independent of the van Hove sequence?

All three questions have been answered by Lenz–Spindeler–Strungaru in [9]:

Theorem 1. *Let μ be a translation bounded measure on the locally compact abelian group G.*

(1) *Assume that μ has autocorrelation γ along the van Hove sequence $\mathcal{A}$. Then, $\widehat{\gamma}$ is a pure point measure if and only if μ is mean almost periodic.*
(2) *The measure μ solves the phase problem along the van Hove sequence $\mathcal{A}$ if and only if μ is Besicovitch almost periodic.*
(3) *The measure μ solves the phase problem along any van Hove sequence if and only if μ is Weyl almost periodic.*

Remarks.

(a) Each part of the theorem is characterization of desired property by means of almost periodicity. This is in the spirit of the mentioned article of Lagarias and, as mentioned already, answers specific questions posed in this article.
(b) Part (1) of the theorem generalizes results of Solomyak [13], Baake–Moody [4] and Gouéré [6].
(c) Part (2) of the theorem covers all classes of examples with pure point diffraction discussed so far, including weak model sets of maximal density (see [9] for detailed discussion).

(d) Part (3) of the theorem covers regular model sets as well as the smooth/ dense Dirac combs discussed by various authors as well as the 'generalized almost periodic measures' introduced by Meyer in [11] (see [9] for detailed discussion).
(e) Mean almost periodicity seems not to have been introduced explicitely before. It has been implicitly present in investigations of Solomyak [13] and Gouéré [6]. Besicovitch almost periodicity is well established for functions and so is Weyl almost periodicity. Mean, Besicovitch and Weyl almost periodicity for measures come about by dualizing the definition for functions: μ is $\Diamond$-almost periodic if $\mu*\varphi$ is $\Diamond$-almost periodic for all continuous complex valued φ on G with compact support.
(f) Related results on dynamical systems can be found in [10].

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Classification and statistics of cut-and-project sets

Yotam Smilansky
(joint work with René Rühr and Barak Weiss)

Cut-and-project point sets in $\mathbb{R}^d$ are constructed by identifying a strip of a fixed n-dimensional lattice or grid (the "cut"), and projecting the points in that strip to a d-dimensional subspace (the "project"), and are a well-studied model of aperiodic order. Dynamical results concerning the translation action on the hull of a cut-and-project set are known to shed light on certain properties of the point set itself, but what happens when instead of restricting to translations we consider all volume preserving affine actions? More specifically, we are interested in the following questions:

- What are the probability measures on cut-and-project sets in $\mathbb{R}^d$, for $d \geq 2$, that are affine group invariant and ergodic?
- What kind of counting statistics can we establish for typical cut-and-project sets, with respect to such measures?

Cut-and-project sets. Fix $n = d + m$ and a direct sum decomposition of $\mathbb{R}^n$ into a d-dimensional *physical space* V_{phys} and an m-dimensional *internal space* V_{int}, together with the associated projections π_{phys} and π_{int}. Also fix a *window* $W \subset V_{\text{int}}$ and an n-dimensional grid $\mathcal{L}$. The cut-and-project set corresponding to this information is $\Lambda = \Lambda(\mathcal{L}, W) = \pi_{\text{phys}}\left(\mathcal{L} \cap \pi_{\text{int}}^{-1}(W)\right)$. It is *irreducible* if $\overline{\pi_{\text{int}}(\mathcal{L})} = V_{\text{int}}$ and π_{phys} is injective on $\mathcal{L}$, and if W is bounded, has non-empty interior and boundary of measure zero, in which case Λ is a Delone set with a well-defined *asymptotic density* $D(\Lambda)$. A famous example is the Ammann-Beenker vertex set in $\mathbb{R}^2$, with $\mathcal{L}$ a lattice in $\mathbb{R}^4$ associated with the ring of integers of $\mathbb{Q}(\sqrt{2})$, and W a regular octagon. This set also arises via a substitution rule with substitution constant $1 + \sqrt{2}$, see [1] for more details.

RMS measures. The following construction is due to Marklof and Strömbergsson and first appeared in [2]. Let $\text{ASL}(d, \mathbb{R})$ be the group of all volume and orientation preserving affine maps on $\mathbb{R}^d$, which is the semidirect product of $\text{SL}(d, \mathbb{R})$ and $\mathbb{R}^d$. Consider the following *top-left* embedding of $\text{ASL}(d, \mathbb{R})$ in $\text{ASL}(n, \mathbb{R})$

$$(g, v) \mapsto \widetilde{(g, v)} = \left(\begin{pmatrix} g & \mathbf{0}_{d,m} \\ \mathbf{0}_{m,d} & \text{Id}_m \end{pmatrix}, \begin{pmatrix} v \\ \mathbf{0}_m \end{pmatrix} \right).$$

Denote by $Y_n = \text{ASL}(n, \mathbb{R})/\text{ASL}(n, \mathbb{Z})$ the *space of grids*. Due to the special embedding described above, for any $(g, v) \in \text{ASL}(d, \mathbb{R})$ and a grid $\mathcal{L} \in Y_n$ we have

$$(g, v).\Lambda(\mathcal{L}, W) = \Lambda\left(\widetilde{(g, v)}.\mathcal{L}, W\right),$$

and so the orbit of a cut-and-project set under $\text{ASL}(d, \mathbb{R})$ can be described via an orbit in the space of grids. By a celebrated theorem of Ratner, any orbit closure $\overline{\text{ASL}(d, \mathbb{R})\mathcal{L}} \subset Y_n$ supports an $\text{ASL}(d, \mathbb{R})$-invariant probability measure, which can be described using a Haar measure on an algebraic group H satisfying

$$\text{ASL}(d, \mathbb{R}) < H < \text{ASL}(n, \mathbb{R}), \quad \overline{\text{ASL}(d, \mathbb{R})\mathcal{L}} = H\mathcal{L}.$$

Define $\Psi(\mathcal{L}) = \Lambda(\mathcal{L}, W)$. Given an $\mathrm{ASL}(d,\mathbb{R})$-invariant ergodic measure $\overline{\mu}$ on Y_n, the measure $\mu = \Psi_*\overline{\mu}$ is a *Ratner–Marklof–Strömbergsson (RMS) measure* on a space $\{\Lambda(h\mathcal{L}, W) \,|\, h \in H\}$ of cut-and-project sets.

Classification theorem. The following new results can be found in [3]. Every $\mathrm{ASL}(d,\mathbb{R})$-invariant ergodic measure assigning full measure to the set irreducible cut-and-project sets is an RMS measure. For any such measure there exists $d \leq k \leq n$ and a real number field K so that H is a semidirect product of H' and $\mathbb{R}^n$, where $H' \subset \mathrm{SL}(n,\mathbb{R})$ arises via restriction of scalars from the field K and either the group SL_k and then $n = k \cdot \deg(K/\mathbb{Q})$, or Sp_{2k} which can only arise if $d = 2$, and then $n = 2k \cdot \deg(K/\mathbb{Q})$.

For example, if $\dim V_{\mathrm{phys}} = \dim V_{\mathrm{int}} = 2$, then the possibilities for H' are either $H' = \mathrm{SL}(4,\mathbb{R})$, $H' = \mathrm{Sp}(4,\mathbb{R})$, or a group arising via a restriction of scalars of a quadratic number field K and SL_2. This is the case for the Ammann-Beenker vertex set with $K = \mathbb{Q}(\sqrt{2})$. It would be interesting to further examine the connection between the emergence of such arithmetic examples and their possible description using substitution rules.

Statistics of typical cut-and-project sets. Following the work of Schmidt on spaces of lattices [4], we apply a *Siegel-type summation formula* and a *Rogers-type second moment bound*, for which the above classification is instrumental, to establish effective point counting results for typical cut-and-project sets. An *unbounded ordered family* is a collection of Borel subsets $\{\Omega_T \,|\, T \in \mathbb{R}_+\}$ of $\mathbb{R}^d$ so that if $0 \leq T_1 \leq T_2$ then $\Omega_{T_1} \subset \Omega_{T_2}$, for all T $\mathrm{vol}(\Omega_T) < \infty$ and $\mathrm{vol}(\Omega_T) \to \infty$. Our first result is that given an RMS measure μ, for every $\varepsilon > 0$, every unbounded ordered family and for μ-a.e. cut-and-project set Λ

$$\#(\Lambda \cap \Omega_T) = D(\Lambda)\mathrm{vol}(\Omega_T) + O(\mathrm{vol}(\Omega_T)^{1/2+\varepsilon}).$$

This matches even the best known result for lattices and centered balls.

Unlike the case of lattices, it is interesting to study the statistics of local configurations in cut-and-project sets. For a point $x \in \Lambda$ and $R > 0$, the *R-patch of Λ at x* is $P_{\Lambda,R}(x) = (\Lambda - x) \cap B(0,R)$. A well-known observation is that given an R-patch in $\Lambda = \Lambda(\mathcal{L}, W)$, the set of points $x \in \Lambda$ for which $P_{\Lambda,R}(x) = P$ is itself a cut-and-project set corresponding to the same $\mathcal{L}$ and to a window generated by intersecting finitely many translations of the original window W. This allows us to establish the following result for patches: given an RMS measure μ and a window $W \subset V_{\mathrm{int}}$ with $\dim_B \partial W < m = \dim V_{\mathrm{int}}$, There is $\theta > 0$ so that every unbounded ordered family, for μ-a.e. cut-and-project set Λ and for any patch P in Λ

$$\#\{x \in \Lambda \cap \Omega_T) \,|\, P_{\Lambda,R}(x) = P\} = D(\Lambda, P)\mathrm{vol}(\Omega_T) + O(\mathrm{vol}(\Omega_T)^{1-\theta}).$$

In the Ammann-Beenker case, $\dim_B \partial W = 1$ and any $\theta < 1/4$ would suffice.

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The spectrum of the Kronig-Penney model in a constant electric field

Rupert L. Frank
(joint work with Simon Larson)

We are interested in the nature of the spectrum of the one-dimensional Schrödinger operator

$$-\frac{d^2}{dx^2} - Fx + \sum_{n\in\mathbb{Z}} g_n \delta(x-n) \qquad \text{in } L^2(\mathbb{R})$$

with a constant $F > 0$ and for two different choices of the coupling constants g_n. In the *first model* we have $g_n \equiv \lambda \neq 0$ and in the *second model* the $g_n = \omega_n$ are i.i.d. random variables with mean zero, variance λ^2 and a compactly supported, absolutely continuous distribution. We write $H_{F,\lambda}$ and H_F^ω in the first and second model, respectively.

Our main results are

Theorem 1. *In the first model, under the assumption $F \in \pi^2\mathbb{Q} \cap (0,\infty)$,*

$$\sigma_{\rm ac}(H_{F,\lambda}) = \mathbb{R}\,, \qquad \sigma_{\rm sc}(H_{F,\lambda}) = \emptyset\,, \qquad \sigma_{\rm pp}(H_{F,\lambda}) \subseteq \left\{ \frac{\pi^2}{3p} m + \lambda :\ m \in \mathbb{Z} \right\},$$

where we wrote $F = (\pi^2/3)(q/p)$ with $p, q \in \mathbb{N}$.

Theorem 2. *In the second model, one has, almost surely,*

$$\sigma(H_F^\omega) = \mathbb{R}$$

and the spectrum is, almost surely,

- *purely singular continuous if $F > \lambda^2/2$*
- *pure point if $F < \lambda^2/2$.*

Both models were popular in solid state physics in the early eighties and there are also some rigorous mathematical results on these and related models. Delyon, Simon and Souillard [3] showed that in the second model for small F the spectrum is a.s. pure point and for large F it is a.s. purely continuous. Our contribution is to show that this transition occurs precisely at the point $F = \lambda^2/2$ and that above this point, the spectrum is a.s. purely singular continuous. A similar result was shown in a related, but different model by Minami [5]. In our proof we adapt the techniques of Kiselev, Last and Simon [4]. As in their work, (generalized) eigenfunctions have a power-like decay with exponent depending on F/λ^2.

The first model was studied in the physics literature by Ao [1], who showed that the delta potential is, in a certain sense, critical. Ao also has predictions in the case where F/π^2 is irrational, which we do not address. In the rational case, the absolute continuity of the spectrum away from a discrete set is suggested in non-rigorous work of Buslaev [2], which was made rigorous by Perelman [6] for periodic potentials slightly more regular than delta potentials. We emphasize that our rationality assumption enters only at the end of our proof and that our arguments make the predictions of Buslaev rigorous, both in the rational and in the irrational case. To do so, we proceed, however, rather differently from Buslaev and instead similarly as Perelman.

For the proof of both theorems, we fix an energy $E \in \mathbb{R}$ and consider a real solution ψ of the corresponding differential equation, that is,

$$-\psi'' - Fx\psi = 0 \quad \text{in } \mathbb{R}\setminus\mathbb{Z}\,, \qquad \psi'(n_{+0}) - \psi'(n_{-0}) = g_n\psi(n) \quad \text{for all } n \in \mathbb{Z}\,.$$

We can write

$$\psi(x) = \alpha(n)\omega(x) + \overline{\alpha(n)\omega(x)} \qquad \text{for } x \in (n-1, n)$$

for a sequence $(\alpha(n))$ and a certain fixed solution ω of $-\omega'' - Fx\omega = 0$ on $\mathbb{R}$, which can be explicitly expressed in terms of Airy functions. Of relevance for us is only its phase γ, whose derivative satisfies

$$\gamma'(x) \sim \sqrt{F}\, x^{1/2} \qquad \text{as } x \to \infty\,.$$

The equation for ψ is equivalent to an equation for α which in transfer matrix form reads

$$\begin{pmatrix} \alpha(n+1) \\ \overline{\alpha(n+1)} \end{pmatrix} = A_n \begin{pmatrix} \alpha(n) \\ \overline{\alpha(n)} \end{pmatrix}$$

with

$$A_n = 1 + \frac{U(n)}{2i}\begin{pmatrix} 1 & e^{-2i\gamma(n)} \\ -e^{2i\gamma(n)} & -1 \end{pmatrix}, \qquad U(n) := -\frac{g_n}{\gamma'(n)}\,.$$

This system is reminiscent of that arising for a discrete Schrödinger operator on $\mathbb{Z}$ with effective potential $U(n)$ (this is only correct modulo an additional phase, which, however, is irrelevant in the second model). In fact, the system is even more reminiscent of that arising for CMV matrices. This is, of course, consistent with the F-periodicity of the spectrum of our operators because of their behavior under translations.

In the situation of the second model one is therefore essentially in the situation of a Schrödinger operator with decaying randomness, and the techniques of [4] apply with some modifications.

The analysis in the first model is more complicated. Following Perelman [6] we coarse grain the system and consider $\tilde{\alpha}(\ell) := \alpha(n(\ell))$ with $n(\ell) \sim \frac{\pi^2}{F}\ell^2$ for $\ell \geq 1$. It turns out that $\tilde{\alpha}$ satisfies a similar equation as α with an effective potential $\tilde{U}(\ell)$ given essentially by $-\lambda/\sqrt{F\ell}$. In this coarse graining process, however, the oscillating factor $e^{2i\gamma(n)}$ changes to $e^{2i\Gamma(\ell)}$ with

$$\Gamma(\ell) := -\frac{\pi^3\ell^3}{3F} + \frac{\pi\ell}{F}(E-\lambda) + \frac{5\pi}{8}\,.$$

The equation for $\tilde{\alpha}$ is closely related to the tilted band picture of Ao and the effective equation of Buslaev and its rigorous derivation constitutes the technical main result of our work. We emphasize again that this does not use the rationality of F/π^2. The latter assumption is only used to deduce absolute continuity of the spectrum of $H_{F,\lambda}$, away from a discrete set, from the equation for $\tilde{\alpha}$.

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On the uniqueness class, stochastic completeness and volume growth for graphs

Marcel Schmidt

(joint work with Xueping Huang and Matthias Keller)

In 1980 Azencott [1] gave an example of a complete Riemannian manifold on which Brownian motion has finite lifetime. Such manifolds are referred to as stochastically incomplete and typically are of very large volume growth. On the other hand it was shown that stochastic completeness is guaranteed under certain volume bounds which were improved over the years, see Gaffney [4], Karp/Li [12], Davies [2] and Takeda [14]. An optimal result was obtained by Grigor'yan [5] (see also [6]) who proved stochastic completeness of a geodesically complete manifold under the condition

$$\int^{\infty} \frac{r}{\log^{\sharp} \mathrm{vol}(B_r)} dr = \infty,$$

where $\log^{\sharp} = \max\{\log, 1\}$. He also showed by examples that his criterion is sharp. Later, Grigor'yan's result was extended by Sturm [13] to strongly local Dirichlet forms where the phenomenon is referred to as conservativeness and distance balls are considered with respect to a so called intrinsic metric. Indeed, in spirit the proof in this more general situation follows Grigor'yan's. A remarkable feature of Grigor'yan's proof is that it not only yields stochastic completeness but directly implies a uniqueness class statement for the heat equation. Precisely, while stochastic completeness is equivalent to uniqueness of bounded solutions to the heat equation, the uniqueness class statement extends this uniqueness to a class of unbounded solutions which satisfy a certain growth bound.

In recent years the phenomenon of stochastic completeness was intensively studied for graphs. The interest in this topic was sparked by the PhD thesis [15] and follow up work [16] of Wojciechowski who presented examples of graphs of polynomial volume growth, which are stochastically incomplete. This showed that there is no analogous result to Grigor'yan's for graphs when one considers volume growth of balls with respect to the combinatorial graph distance. However, in view of the work of Sturm [13] for local Dirichlet forms, which uses intrinsic metrics, it seemed promising to consider distance balls with respect to a metric that is adapted to the heat flow on the graph. While such a theory of intrinsic metrics was developed at this time also for non-local (and thus for all regular) Dirichlet forms, this idea was used by Grigor'yan/Huang/Masamune [7] to prove a first result in this direction that guaranteed stochastic completeness of the graph provided

$$\mathrm{vol}(B_r) \leq \exp(Cr \log r)$$

for r large enough and some constant $0 < C < 1/2$. Shortly afterwards Grigor'yan's result for manifolds was recovered for graphs using so called intrinsic (or adapted) metrics by Folz [3] and shortly after that an alternative proof was given by Huang [9]. See also [11] for results on the closely related problem of escape rates.

In spirit, the proofs of these results used techniques that relate the non-local graph to a more local object. Specifically, Folz [3] compared the heat flow on the combinatorial graph with a corresponding metric (or quantum) graph and Huang and Shiozawa [11] decreased non-locality of the graph by inserting additional vertices in the edges (which probabilistically decreased the jump size of the process). Although this was a breakthrough, there are two aspects in which the results are not completely satisfying – one of technical the other of structural nature. The technical aspect is that the results were proven under rather restrictive conditions such as local finiteness of the graphs, finite jump size of the metric and uniform lower bounds on the measure. Moreover, the only metrics considered were special path metrics. These restrictions did not inspire much hope that the proof strategies can be carried over to more general jump processes. The second aspect, which may be seen as a shortcoming of more fundamental nature, is that the proofs do not allow to recover Grigor'yan's uniqueness class for the heat equation. Indeed, this is not a shortcoming of the proofs but the optimal uniqueness class that is known for manifolds does not hold for general graphs. In his PhD thesis [8] Huang gave an example of a nontrivial solution to the heat equation with initial value 0 on the integer line $\mathbb{Z}$, which showed that the corresponding uniqueness class statement of Grigor'yan is already wrong for this simple graph.

In this talk we present recent results from [10], where we amend these shortcomings. In order to obtain Grigor'yan's uniqueness class for the heat equation we introduce the class of globally local graphs (GL graphs for short). These are graphs whose jump size decays fast enough outside large balls. On GL graphs we establish Grigor'yan's uniqueness class and directly use it to obtain Grigoryan's optimal volume growth criterion for stochastic completeness (with respect to an intrinsic metric). This part of the results is general and also applies to jump processes associated with regular Dirichlet forms. In a second step we establish the optimal volume growth criterion for stochastic completeness for general graphs under the only assumption that they admit an intrinsic metric with finite distance balls. For this we use the ideas from [11] to refine the given graph to a GL graph of the same volume growth, establish stochastic completeness of the refined graph and then use stability of stochastic completeness under refinements.

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Uncertainty relations and applications in spectral and control theory

Ivan Veselić

Unique continuation is a prominent phenomenon encountered in the study of several classes of functions, e. g. subsets of holomorphic functions or spaces of solutions of partial differential equations. The study of this phenomenon, in the multidimensional setting, goes back at least to Carleman and Müller. It is impossible to give here an adequate overview of the gradual development of understanding of this phenomenon. Let us just mention that it has crucial consequences in a plethora of applied problems: Absence of eigenvalues embedded in the continuous spectrum of Schrödinger operators, size and dimensionality of nodal sets of Laplace-Beltrami operators on compact manifolds, inverse problems for partial differential equations, like the Calderon problem or the control problem for evolution equations.

More recently, the unique continuation principle has been successfully applied to the study of mathematical models in solid state and condensed matter physics. A particular feature of these models is the presence of a microscopic and a macroscopic length scale, where the latter is many orders of magnitude larger than the former. For periodic and ergodic Schrödinger operator, the ratio between the macroscopic and microscopic scale is determined by the spacing between atoms in crystals, which is typically a few Ångströms.

Consequently, the unique continuation principles have to take into account this geometric structure to be applicable in the mentioned physical context. This means that on the technical level we have to consider the unique continuation problem on large domains — 'large' compared to some reference scale, e. g. the atomic scale in the case of ergodic Schrödinger operator. It is this reference scale at which changes of the coefficient functions (e. g. the electric potential) are observed.

As in other situations of statistical and condensed matter physics, it is natural to approximate large bounded domains by unbounded ones, and hence include the latter in our analysis. Now we have set the stage to formulate unique continuation estimates on unbounded and (large) bounded domains and spell out thereafter applications in three different questions of mathematical physics.

Quantitative scale free unique continuation principles. Let $d \in \mathbb{N}$, $G > 0$, $\delta > 0$ and $\Gamma = \times_{i=1}^{d}(\alpha_i, \beta_i) \subset \mathbb{R}^d$ with $\alpha_i, \beta_i \in \mathbb{R} \cup \{\pm\infty\}$. Assume that $\Lambda_G := (-G/2, G/2)^d \subset \Gamma$. We say that a sequence $Z = (z_j)_{j \in (G\mathbb{Z})^d} \subset \mathbb{R}^d$ is (G,δ)-*equidistributed*, if

$$\forall j \in (G\mathbb{Z})^d: \quad B(z_j, \delta) \subset (-G/2, G/2)^d + j.$$

For a (G,δ)-equidistributed sequence Z define

$$S_{\delta,Z} = \bigcup_{j \in (G\mathbb{Z})^d} B(z_j, \delta) \cap \Gamma.$$

For a real $V \in L^\infty(\Gamma)$ define $H = -\Delta + V$ on $L^2(\Gamma)$ with Dirichlet or Neumann boundary conditions.

Theorem 1 ([1])**.** *There is an $N > 0$ depending only on d, such that for all $G > 0$, all $\Lambda_G \subset \Gamma \subset \mathbb{R}^d$ as above, all $\delta \in (0, G/2)$, all (G,δ)-equidistributed sequences Z, all $V \in L^\infty(\Gamma)$, all $E \in \mathbb{R}$, and all $\psi \in \operatorname{Ran} \mathbf{1}_{(E,\infty)}(H)$ we have*

$$\|\psi\|^2_{L^2(S_{\delta,Z})} \geq C_{\mathrm{uc}}(E) \|\psi\|^2_{L^2(\Gamma)}, \quad \textit{where} \quad t_+ := \max\{0, t\} \textit{ for } t \in \mathbb{R}$$

and $$C_{\mathrm{uc}}(E) = \sup_{\lambda \in \mathbb{R}} \left(\frac{\delta}{G}\right)^{N\left(1 + G^{4/3}\|V-\lambda\|_\infty^{2/3} + G\sqrt{(E-\lambda)_+}\right)}.$$

The estimate is called *scale free* since it is independent of the domain Γ.

Lifting estimates of edges of essential spectrum by potentials. The min max principle for hermitean matrices shows and quantifies the lifting of eigenvalues under the influence of a positive definite perturbation. In the case of Schrödinger operators, eigenvalues may have accumulation points and/or be located inside gaps of the essential spectrum. The question is whether analogous shifting estimates hold in these cases. Furthermore: Do similar lifting estimates hold for the edges of essential spectrum; and: Are positive semi-definite potentials sufficient to produce such lifting? We have a positive answer to these questions formulated in the following

Theorem 2 ([1]). *Let N, G, Γ, δ, Z, V, H, and $C_{\mathrm{uc}}(E)$ be as in Theorem 1. Let $W \in L^\infty(\Gamma)$ be real-valued with $W \geq \vartheta \mathbf{1}_{S_{\delta,Z}}$ for some $\vartheta > 0$.*

Let $a, b \in \sigma_{\mathrm{ess}}(H)$, and $a < b$ such that $(a,b) \cap \sigma_{\mathrm{ess}}(H) = \emptyset$. We set $t_0 = (b-a)/\|W\|_\infty$, and $t_+ = \max\{0,t\}$, $t_- = \max\{0,-t\}$. Then the functions $f_\pm : (-t_0, t_0) \to \mathbb{R}$

$$f_-(t) = \sup\left(\sigma_{\mathrm{ess}}(H+tW) \cap (-\infty, b - t_-\|W\|_\infty)\right),$$
$$f_+(t) = \inf\left(\sigma_{\mathrm{ess}}(H+tW) \cap (a + t_+\|W\|_\infty, \infty)\right),$$

satisfy for all $t \in (-t_0,t_0), \epsilon > 0$ such that $t+\epsilon \in (-t_0,t_0)$ the two Lipschitz bounds

$$\epsilon\, \vartheta\, C_{\mathrm{uc}}(b + \|W\|_\infty) \leq f_\pm(t+\epsilon) - f_\pm(t) \leq \epsilon\, \|W\|$$

Analogous bounds hold for discrete eigenvalues in all gaps of the essential spectrum (and below it).

Anderson localization for random Schrödinger operators. It is well known from previous work that unique continuation estimates are a powerful tool for deriving Wegner estimates, which in turn play a crucial role in proving localization. We present here a result for a model with non-linear parameter dependence.

For $0 \leq \omega_- < \omega_+ < \frac{1}{4}$ set $\Omega = \times_{j\in\mathbb{Z}^d}\mathbb{R}$, $\mathbb{P} = \bigotimes_{j\in\mathbb{Z}^d}\mu$ where μ is a probability measure with $\operatorname{supp}\mu \subset [\omega_-, \omega_+]$ and a bounded density ν. Hence, $\pi_j(\omega) \mapsto \omega_j, j \in \mathbb{Z}^d$, are continuous iid random variables. The standard random breather model is defined as

$$H_\omega = -\Delta + V_\omega^{\mathrm{br}}(x) \quad \text{with} \quad V_\omega^{\mathrm{br}}(x) = \sum_{j\in\mathbb{Z}^d} \chi_{B_{\omega_j}}(x-j), \tag{1}$$

and its restriction to the box Λ_L (with Dirichlet or Neumann boundary conditions) is denoted by $H_{\omega,L}$.

Theorem 3 (Wegner estimate for the standard random breather model [2]). *Fix $E_0 \in \mathbb{R}$ and set $\epsilon_{\max} = (1/4)\cdot 8^{-N(2+|E_0+1|^{1/2})}$, where N is the constant from Theorem 1. Then there is $C = C(d,E_0) \in (0,\infty)$ such that for all $\epsilon \in (0,\epsilon_{\max}]$ and $E \geq 0$ with $[E-\epsilon, E+\epsilon] \subset (-\infty, E_0]$, we have*

$$\mathbb{E}\left[\mathrm{Tr}\left[\chi_{[E-\epsilon,E+\epsilon]}(H_{\omega,L})\right]\right] \leq C\|\nu\|_\infty \epsilon^{[N(2+|E_0+1|^{1/2})]^{-1}} \left|\ln\epsilon\right|^d L^d.$$

Of similar importance for the proof of localization is the so-called initial scale estimate. Seelmann and Täufer succeeded in proving one (and consequently localization) for energies near spectral band edges of randomly perturbed periodic potentials. It is remarkable that they do not need to assume anything about the extrema for the associated Floquet eigenvalues.

Theorem 4 (Localization at band edges [4]). *Let* $V_{\mathrm{per}} \colon \mathbb{R}^d \to \mathbb{R}$ *be bounded and periodic,* $H_{\mathrm{per}} = -\Delta + V_{\mathrm{per}}$*,* $V_\omega^{\mathrm{an}} = \sum_{j\in\mathbb{Z}^d} \omega_j u(\cdot - j)$ *with iid bounded random variables* ω_j *with bounded density* ν *and* $L^\infty_c(\mathbb{R}^d) \ni u \geq c\,\chi_{B_\delta}$*, for some* $c, \delta > 0$*, and* $H_\omega = H_{\mathrm{per}} + V_\omega^{\mathrm{an}}$*. Assume that* $a < b$*,* $(a, b) \subset \rho(H_\omega)$*, and* $b \in \sigma(H_\omega)$*. Then there is an* $\epsilon > 0$ *such that in* $[b, b + \epsilon]$ *the operator* H_ω *exhibits Anderson localization.*

Null control and observability estimates for the heat equation. The following observability (and hence the associated control cost) estimates for the generalized heat equation (on Γ as above) have been derived from Theorem 1.

Theorem 5 ([3]). *Let* N*,* G*,* Γ*,* δ*,* Z*,* V*, and* H *be as in Theorem 1. Then for all* $\phi \in L^2(\Gamma)$*, and all* $T > 0$ *we have*

$$\|\mathrm{e}^{-HT}\|^2_{L^2(\Gamma)} \leq C_{\mathrm{obs}}(\delta, G, \|V\|_\infty, T)^2 \int_0^T \|\mathrm{e}^{-Ht}\phi\|^2_{L^2(S\cap\Gamma)} \mathrm{d}t,$$

where for some $C_1, C_2, C_3 > 0$ *depending only on the dimension we have*

$$C_{\mathrm{obs}} = \left(\frac{\delta}{G}\right)^{-C_2(1+G^{4/3}\|V\|_\infty^{2/3})} \frac{C_1}{T} \exp\left(\frac{C_3 G^2 \ln^2(\delta/G)}{T}\right) \quad \text{if } \kappa := \inf \sigma(H) \geq 0,$$

$$C_{\mathrm{obs}} = \left(\frac{\delta}{G}\right)^{-C_2(1+G^{4/3}\|V-\kappa\|_\infty^{2/3})} \inf_{t\in[0,T)} \frac{C_1}{T-t} \exp\left(\frac{C_3 G^2 \ln^2(\delta/G)}{T-t} - 2\kappa t\right),$$

if $\kappa > 0$*.*

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Weak model sets and number-theoretic dynamical systems

Michael Baake

The study of cut and project sets has become a powerful tool for many problems, ranging from number theory [8] all the way to theoretical and applied questions from the theory of aperiodic order [2, 7]. While the main focus so far has been on regular model sets [9], where the window in internal space is a topologically regular set with a boundary of measure zero, the theory is way more general. In fact, interesting examples of number-theoretic origin need the setting of *weak model sets*, where the window is still compact, but may have a boundary of positive measure, or even consist of boundary only. Here, we present two results on such systems, one of a spectral nature [3] and another of a more algebraic flavour [1].

The setting within the class of locally compact abelian groups is commonly summarized by the underlying *cut and project scheme* (CPS)

$$
\begin{array}{ccccc}
G & \xleftarrow{\;\pi\;} & G \times H & \xrightarrow{\;\pi_{\text{int}}\;} & H \\
\cup & & \cup & & \cup \ \text{dense} \\
\pi(\mathcal{L}) & \xleftarrow{\;1-1\;} & \mathcal{L} & \longrightarrow & \pi_{\text{int}}(\mathcal{L}) \\
\| & & & & \| \\
L & & \xrightarrow{\qquad\star\qquad} & & L^{\star}
\end{array}
\tag{1}
$$

with G assumed σ-compact and H compactly generated, which is a mild, but natural restriction in a spectral context. Further, $\mathcal{L}$ is a lattice in $G \times H$, and the CPS is commonly abbreviated by $(G, H, \mathcal{L})$.

Given a compact subset $W \subseteq H$, a projection set of the form

$$\varLambda = \curlywedge(W) = \{x \in L : x^{\star} \in W\}$$

is called a *weak model set.* Given an averaging sequence $\mathcal{A}$ in G of van Hove type, $\varLambda$ is said to have *maximal density* when the density of $\varLambda$ relative to $\mathcal{A}$ exists and satisfies $\mathrm{dens}(\varLambda) = \mathrm{dens}(\mathcal{L})\mathrm{vol}(W)$, where the volume of the window is measured via the (canonically normalized) Haar measure of H.

One recent result, in line with the original theory envisioned by Yves Meyer, is the following, where we refer to [2] for underlying notions and basic results from diffraction theory.

Theorem 1. *Let $(G, H, \mathcal{L})$ be the CPS from (1), and let an averaging sequence $\mathcal{A}$ in G of van Hove type be specified. Then, a weak model set of maximal density with respect to $\mathcal{A}$ is pure point diffractive, with a constructive and explicit formula for the diffraction measure, which is supported in the group $\pi_{\text{int}}(\mathcal{L}^0)$. Here, $\mathcal{L}^0$ is the annihilator of $\mathcal{L}$, which is a lattice in the dual group $\widehat{G \times H} \simeq \widehat{G} \times \widehat{H}$.*

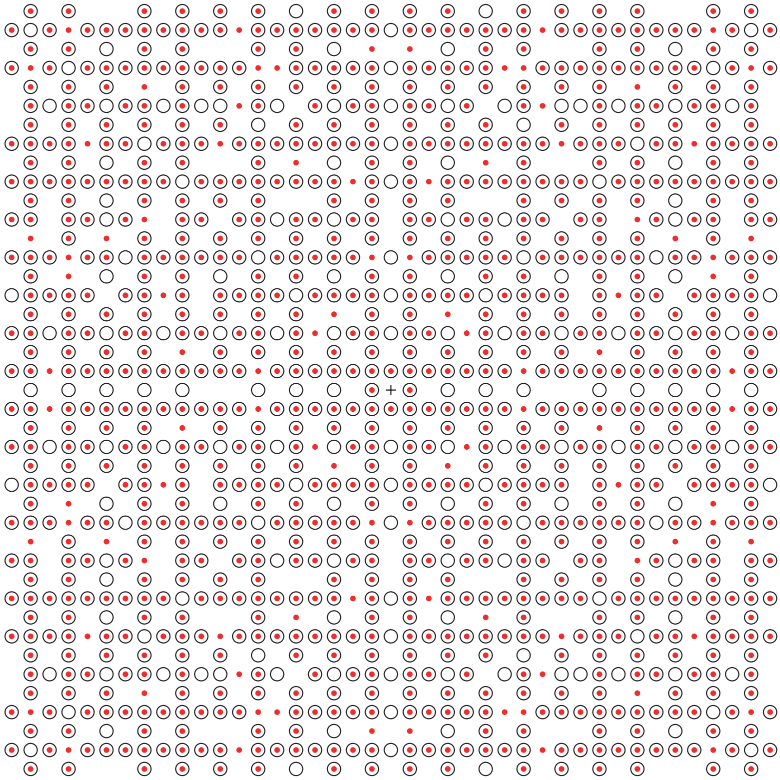

FIGURE 1. Central patch of the visible (or primitive) points of $\mathbb{Z}^2$ (dots) and of the square-free Gaussian integers (circles), with the symbol + marking the origin, which belongs to neither set.

Let us break this down for two concrete examples, namely the visible (or primitive) points within the lattice $\mathbb{Z}^2$ and the square-free integers in the ring $\mathbb{Z}[\mathrm{i}]$ of Gaussian integers, where the latter can canonically be viewed as a subset of the square lattice; see Figure 1. Both are weak model sets of maximal density, relative to an averaging sequence of centered disks around 0 of increasing radius. In fact, there are simple closed formulas for the diffraction measures in terms of Dirichlet series and Euler product expressions for the intensities; see [4, 2] for more.

Both sets define topological dynamical systems under the (continuous) shift action of $\mathbb{Z}^2$, denoted by $(\mathbb{X}_{\mathrm{V}}, \mathbb{Z}^2)$ and $(\mathbb{X}_{\mathrm{G}}, \mathbb{Z}^2)$, where the two compact spaces are obtained as the orbit closure of the sets from Figure 1 under the shift action, with the closure taken in the product topology of $\{0,1\}^{\mathbb{Z}^2}$, also known as the *local topology*; see [2] for background and details. For the spectral result, we use the patch frequency measure as invariant probability measure on the two spaces, with the (natural) patch frequencies determined via the above averaging sequence. This measure is also known as the *Mirsky measure*; see [3] and references therein.

Invoking the spectral equivalence theorem from [5], we know that these dynamical systems have pure point dynamical spectrum, which (in additive notation) is $\pi_{\text{int}}(\mathcal{L}^0)$ in the setting of Theorem 1. The dynamical spectra of the two systems turn out to be *different* subgroups of $\mathbb{Q}^2$.

Consequently, by the Halmos–von Neumann theorem, the two systems cannot be measure-theoretically conjugate, and thus certainly not topologically conjugate either. But one valid question is whether the topological distinction is possible more directly, that is, without invoking an invariant measure and the Halmos–von Neumann theorem. This is indeed possible as follows.

Let $\text{Aux}(\mathbb{X})$ denote the group of homeomorphisms of the compact space $\mathbb{X}$, where we do *not* demand any commutation properties with the shift action (this is the automorphism group in the Smale sense, in contrast to what is mostly used in symbolic dynamics). With $G = \mathbb{Z}^2$, we now have two natural algebraic objects, namely the topological *centralizer* and the topological *normalizer*,

$$\mathcal{S}(\mathbb{X}) = \text{cent}_{\text{Aut}(\mathbb{X})}(G) \quad \text{and} \quad \mathcal{R}(\mathbb{X}) = \text{norm}_{\text{Aut}(\mathbb{X})}(G). \tag{2}$$

Clearly, these groups are topological invariants, and they are often explicitly accessible. Note that the centralizer agrees with what is often called the automorphism group in symbolic dynamics, while it is only the normalizer that captures the obvious 'symmetries' of the systems; see [6] for background and examples. Here, we get the following result [1], with D_4 the dihedral group of order 8.

Theorem 2. *One has* $\mathcal{S}(\mathbb{X}_{\text{V}}) = \mathcal{S}(\mathbb{X}_{\text{G}}) = \mathbb{Z}^2$*, while the normalizers are different, namely* $\mathcal{R}(\mathbb{X}_{\text{G}}) = \mathbb{Z}^2 \rtimes D_4$ *versus* $\mathcal{R}(\mathbb{X}_{\text{V}}) = \mathbb{Z}^2 \rtimes \text{GL}(2,\mathbb{Z})$.

Here, the group $D_4 \simeq C_4 \rtimes C_2$ has a natural origin in algebraic number theory, which extends to similar systems for general quadratic or cyclotomic fields.

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Ellis semigroup of symbolic shifts

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(joint work with Marcy Barge, Gabriel Fuhrmann and Reem Yassawi)

A topological dynamical system (X, T, α) is a compact metrizable space X with an action α by homeomorphisms of a group or semi-group (with unit) T. The enveloping or Ellis semigroup $E(X, T)$ of (X, T, α) is the closure of the set of homeomorphisms $\{\alpha^t : t \in T\}$ in X^X, the semigroup (under composition) of all functions from X to X equipped with the topology of pointwise convergence.

In our talk we reviewed recent results obtained in collaboration with the above named authors about

(1) tameness of Toeplitz shifts [3],
(2) near simplicity and complete regularity of the Ellis semigroup of symbolic $\mathbb{Z}$-shifts [2],
(3) explicit computations of the Ellis semigroup for bijective substitutions [4].

1. Tameness of Toeplitz shifts

There are several equivalent formulations of what it means for a dynamical system (or its Ellis semigroup) to be tame. We mention two such formulations:

- (X, T, α) is tame if all elements of $E(X, T)$ are limits of sequences (as opposed to nets) of continuous functions.
- (X, T, α) is wild (non-tame) if it admits an independence sequence.

In the context of symbolic shifts, that is, when $(X, \mathbb{Z}, \alpha)$ is a subsystem of the full shift $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z}, \sigma)$ over a finite alphabet $\mathcal{A}$, the second condition means the following: $(X, \mathbb{Z}, \alpha)$ admits an independence sequence if there are two distinct symbols $\{a, b\}$ from $\mathcal{A}$ and an infinite subset $J \subset \mathbb{Z}$ such that, for any function $\varphi : J \to \{a, b\}$ there exists $x = (x_n)_{n\in\mathbb{Z}} \in X$ such that for all $j \in J$ we have $x_j = \varphi(j)$.

A Toeplitz shift is a symbolic shift $(X, \mathbb{Z}, \alpha) \subset (\mathcal{A}^{\mathbb{Z}}, \mathbb{Z}, \sigma)$ whose maximal equicontinuous factor $(Y, \mathbb{Z})$ is an odometer and such that the factor map $\pi : X \to Y$ admits a singleton fibre. The set $\{y \in Y : |\pi^{-1}(y)| = 1\}$ is thus not empty and we refer to its complement as the set of singular points. Toeplitz shifts can be realized by means of Bratteli-Vershik systems but in a particular way, namely for any given level n, the number of outgoing edges from each vertex of the vertex set at level n depends only on n. This property is referred to as the "equal path number property". We say that a Toeplitz shift has finite Toeplitz rank, if it can be realized by such a Bratteli-Vershik system which in addition has an upper bound on the number of vertices at level n. Primitive, aperiodic, constant length substitutions with a coincidence are examples of Toeplitz shifts which have finite Toeplitz rank. We obtain

Theorem 1 ([3]). *A Toeplitz shift with finite Toeplitz rank is tame if and only if the set of singular points is countable.*

Counterexamples show that the condition that the Toeplitz rank is finite cannot be dropped.

2. Near simplicity and complete regularity

A semigroup is called completely simple, if it does not admit a non-trivial bilateral ideal and contains an idempotent. Completely simple semigroups are well-understood, they are isomorphic to so-called matrix semigroups, which are disjoint unions of isomorphic groups.

A semigroup is called completely regular, if it is a disjoint union of groups.

We call a semigroup nearly simple if it admits a minimal bilateral ideal (called its kernel) and all its non-invertible elements belong to this ideal. A nearly simple semigroup is completely regular, provided its kernel contains an idempotent.

It is a consequence of compactness that the Ellis semigroup admits a kernel and the kernel contains an idempotent. The kernel is thus a matrix semigroup.

Aujogue has shown [1] that the Ellis semigroup associated to the dynamical system of an almost canonical cut & project set is completely regular, but not nearly simple. In the context of one-dimensional symbolic shifts we can say the following

Theorem 2 ([2])**.** *Consider a dynamical system $(X, \mathbb{Z}, \alpha)$ for which X is totally disconnected. The following are equivalent*

(1) Any forward proximal pair is forward asymptotic and any backward proximal pair is backward asymptotic,
(2) $E(X, \mathbb{Z})$ is nearly simple,
(3) $E(X, \mathbb{Z})$ is completely regular.

We can furthermore say that, if forward proximality and backward proximality are transitive relations then $E(X, \mathbb{Z})$ has either one minimal left ideal (which is the case precisely if two-sided proximality is transitive) or two minimal left ideals, namely the minimal left ideals of $E(X, \mathbb{Z}^+)$ and $E(X, \mathbb{Z}^-)$, the Ellis semigroups associated to the forward and the backward motions.

3. Ellis semigroup of bijective substitutions

A bijective substitution θ, of length ℓ, on a finite alphabet $\mathcal{A}$, is a concatenation of ℓ bijections $\theta_i : \mathcal{A} \to \mathcal{A}$, $i = 0, \cdots, \ell - 1$, such that for all $a \in \mathcal{A}$

$$\theta(a) = \theta_0(a)\, \theta_1(a)\, \cdots \theta_{\ell-1}(a).$$

The dynamical system associated to θ is the subshift $(X_\theta, \mathbb{Z}, \sigma) \subset (\mathcal{A}^{\mathbb{Z}}, \mathbb{Z}, \sigma)$ of bi-infinite sequences whose finite parts occur in $\theta^n(a)$ for some $a \in \mathcal{A}$ and $n \geq 1$. The ℓ-adic odometer $\mathbb{Z}_\ell$ is an equicontinuous factor of $(X_\theta, \mathbb{Z})$ and the factor map $\pi : X_\theta \to \mathbb{Z}_\ell$ has a unique orbit of singular points (points $z \in \mathbb{Z}_\ell$ for which $|\pi^{-1}(z)| > |\mathcal{A}|$), namely that of $0 \in \mathbb{Z}_\ell$. The Ellis semigroup of $(X_\theta, \mathbb{Z}, \sigma)$ is never tame. We denote by $E^{fib}(X_\theta, \mathbb{Z})$ the elements of $E(X_\theta, \mathbb{Z})$ which preserve the fibres of π and by $E_0^{fib}(X_\theta, \mathbb{Z})$ their restriction to the fibre $\pi^{-1}(0)$. On $(X_\theta, \mathbb{Z}, \sigma)$ forward and backward proximality coincide with forward and backward asymptoticity, resp., so that $E(X_\theta, \mathbb{Z})$ is the disjoint union of the acting group $\mathbb{Z}$ with a matrix semigroup $M[G; I, \Lambda; A]$. As a set, $M[G; I, \Lambda; A] = I \times G \times \Lambda$ where G is a group and I and Λ

are sets. The semigroup product is encoded by a G-valued matrix $A = (a_{\lambda\, i})_{\lambda\in\Lambda, i\in I}$ and given by

$$(i, g, \lambda)(j, h, \mu) = (i, g a_{\lambda\, j} h, \mu).$$

The minimal left ideals are given by $I \times G \times \{\lambda\}$, $\lambda \in \Lambda$ and here there are exactly two of them, one associated to the forward and one to the backward motion; we denote them with $\lambda = +, -$, resp..

When G is a topological group and I and Λ equipped with a topology then $M[G; I, \Lambda; A]$, equipped with the product topology, is semi-topological, that is, the semigroup product is continuous in both variables. We have

Theorem 3 ([4]). *Let $\theta = \theta_0 \cdots \theta_{\ell-1}$ be a primitive, aperiodic, bijective substitution such that $\theta_0 = \theta_{\ell-1} = \mathrm{id}$.*

(1) $E_0^{fib}(X_\theta, \mathbb{Z}) \cong \{\mathrm{id}\} \sqcup M[G_\theta; I_\theta, \{\pm\}, A_\theta]$, a finite semigroup, with

$$I_\theta = \{\theta_i \theta_{i-1}^{-1} | 1 \le i \le \ell\},$$

G_θ the group generated by I_θ, and $A_\theta = (a_{\epsilon\, i})_{\epsilon=\pm, i\in I_\theta}$, $a_{+i} = 1, a_{-,i} = i$.

(2) If $E_0^{fib}(X_\theta, \mathbb{Z})$ is generated by its idempotents then

$$E^{fib}(X_\theta, \mathbb{Z}) \cong \{\alpha^0\} \sqcup M[G_\theta^{\mathbb{Z}_\ell/\mathbb{Z}}; I_\theta, \{\pm\}, A_\theta]$$

the isomorphism preserving the topology. Here $G_\theta^{\mathbb{Z}_\ell/\mathbb{Z}}$ is the set of all functions from the space of $\mathbb{Z}$-orbits $\mathbb{Z}_\ell/\mathbb{Z}$ to G, equipped with the topology of pointwise convergence, and the matrix elements of A are seen as functions supported on the orbit of 0.

(3) If $E_0^{fib}(X_\theta, \mathbb{Z})$ is generated by its idempotents then

$$E(X_\theta, \mathbb{Z}) \cong \mathbb{Z} \sqcup M[G_\theta^{\mathbb{Z}_\ell/\mathbb{Z}} \rtimes \mathbb{Z}_\ell; I_\theta, \{\pm\}, A_\theta]$$

however the isomorphism is only algebraic.

If $E_0^{fib}(X_\theta, \mathbb{Z})$ is not generated by its idempotents then it is $\mathbb{Z}/h\mathbb{Z}$-graded, for some $h \in \mathbb{N}$ which is called the generalized height of θ, and similar expressions for $E(X_\theta, \mathbb{Z})$ incorporating this grading can be obtained [4].

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Logarithmic Sobolev inequalities for quantum Markov semigroups – an optimal transport approach

Melchior Wirth
(joint work with Haonan Zhang)

Logarithmic Sobolev inequalities are classic tools to quantify the return to equilibrium of classical and quantum Markovian evolutions. For the heat semigroup on a complete Riemannian manifold it is known that a logarithmic Sobolev inequality can be deduced from a positive lower bound on the Ricci curvature by optimal transport methods, using convexity properties of the entropy in Wasserstein space [5]. In this talk we discuss how this approach can be transferred to noncommutative geometries.

We work in the following setting. Let $\mathcal{M}$ be a von Neumann algebra and $\tau\colon \mathcal{M} \to \mathbb{C}$ a normal faithful tracial state. A τ-symmetric quantum Markov semigroup (P_t) is a weak* continuous semigroup of unital completely positive linear maps on $\mathcal{M}$ that satisfy

$$\tau((P_t x)y) = \tau(x P_t y).$$

If (P_t) admits a carré du champ, then its L^2-generator can be written as $\mathcal{L} = \partial^*\partial$, where ∂ is a derivation with values in a normal Hilbert bimodule $\mathcal{H}$ over $\mathcal{M}$ [4].

With the aid of this differential structure one can define a noncommutative transport distance on the space of density operators [6] by

$$\mathcal{W}^2(\rho_0, \rho_1) = \inf \left\{ \int_0^1 \langle \hat{\rho}_t \xi_t, \xi_t \rangle \, dt \mid \dot{\rho}_t = \partial^*(\hat{\rho}_t \xi_t) \right\},$$

where

$$\hat{\rho}\xi = \int_0^1 \rho^\alpha \cdot \xi \cdot \rho^{1-\alpha} \, d\alpha.$$

If (P_t) is the heat semigroup on a complete Riemannian manifold with lower bounded Ricci curvature, then $\mathcal{W}$ coincides with the L^2-Wasserstein distance by means of the Benamou–Brenier formula.

Our first result [6] show that this metric establishes a connection between (P_t) and the entropy $D(\rho) = \tau(\rho \log \rho)$. More precisely, if (P_t) satisfies the gradient estimate

$$(\mathrm{GE}(K,\infty)) \qquad \langle \hat{\rho}\partial(P_t x), \partial(P_t x)\rangle_{\mathcal{H}} \leq \langle \widehat{P_t\rho}\partial(x), \partial(x)\rangle_{\mathcal{H}},$$

then (P_t) satisfies the evolution variational inequality

$$(\mathrm{EVI}_K) \qquad \frac{1}{2}\frac{d}{dt}\mathcal{W}^2(P_t\rho, \sigma) + \frac{K}{2}\mathcal{W}^2(P_t\rho, \sigma) + D(P_t\rho) \leq D(\sigma).$$

If (P_t) is a classical diffusion semigroup, then $\mathrm{GE}(K,\infty)$ is equivalent to the Bakry–Émery criterion $\Gamma_2 \geq K\Gamma$, but this equivalence does not hold if the generator of (P_t) is non-local or the underlying von Neumann algebra is noncommutative.

By abstract theory of gradient flows in metric spaces, EVI_K with $K > 0$ implies that (P_t) satisfies the modified logarithmic Sobolev inequality

$$D(P_t\rho) \leq e^{-2Kt} D(\rho),$$

provided (P_t) is ergodic. For non-ergodic (P_t) the same conclusion remains true if one replaces D by a suitable relative entropy conditioned on the fixed-point algebra [7].

It is an open question whether the gradient estimate $\mathrm{GE}(K,\infty)$ is stable under taking tensor products. However, if one assumes that $(P_t \otimes I_{\mathcal{N}})$ and $(Q_t \otimes I_{\mathcal{N}})$ satisfy $\mathrm{GE}(K,\infty)$ for every tracial von Neumann algebra $(\mathcal{N}, \tau_{\mathcal{N}})$, we can show [7] that both the tensor product $(P_t \otimes Q_t)$ and the free product $(P_t * Q_t)$ satisfy $\mathrm{GE}(K,\infty)$. We also establish an infinite-dimensional version of an intertwining criterion for $\mathrm{GE}(K,\infty)$ going back to Carlen and Maas in the finite-dimensional case [2, 3].

As an application we prove the modified logarithmic Sobolev inequality with optimal constant for the QMS on free group factors given by $P_t\lambda_g = e^{-t\ell(g)}\lambda_g$, where $\ell(g)$ is the number of letters of g as a reduced word in the generators and their inverses. Our proof relies on the free product structure of (P_t), an approximation argument using ultraproducts and a new explicit Lindblad form for the generator of (P_t) using the cocycle associated with ℓ. The same result was obtained independently using different methods by Brannan, Gao and Junge [1].

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Discrete Cheeger–Gromoll splitting theorem

Florentin Münch

(joint work with Shing-Tung Yau)

An important result in Riemannian geometry is the Cheeger–Gromoll splitting theorem stating that if a manifold with non-negative Ricci curvature possesses a straight line, then it is a Cartesian product of $\mathbb{R}$ and another manifold. Since a decade ago, there is growing interest in various discrete Ricci curvature notions, see e.g. the seminal paper by Ollivier [1]. According to [2], the Ollivier curvature of an edge in a locally finite graph $G = (V, E)$ is given by

$$\kappa(x, y) = \inf_{\substack{f \in Lip(1) \\ f(y)-f(x)=1}} \Delta f(x) - \Delta f(y)$$

where $\Delta f(x) = \sum_{(x,y)\in E} f(y) - f(x)$ for $f \in \mathbb{R}^V$ and $Lip(1)$ is the set of 1-Lipschitz functions on V with respect to the combinatorial graph distance d. The curvature $Ric(G)$ of the whole graph is the infimum of the curvatures of all edges. For stating a discrete version of the Cheeger–Gromoll splitting theorem it turns out that defining straight lines in terms of the combinatorial graph distance cannot give splitting in the sense of Cartesian graph products. The intuitive reason is that balls are not round enough to give reasonable Busemann functions. In order to overcome this issue, we define a modified graph distance in the following way:

- For $x, y \in V$, let $\ell(x, y)$ be the second shortest path length from x to y.
- Let $\delta, \varepsilon : \mathbb{N} \to (0, \frac{1}{2})$ with $\varepsilon(n) \to 0$ as $n \to \infty$.

Define $w : V^2 \to [0, \infty)$,

$$w(x, y) := \begin{cases} d(x, y) - 1 + \varepsilon(d(x, y)) & : \ell(x, y) = d(x, y), \\ d(x, y) - \delta(d(x, y)) & : \ell(x, y) = d(x, y) + 1, \\ d(x, y) & : \text{ otherwise.} \end{cases}$$

The function w is not a metric, but can serve as a weight for a path metric ρ given by

$$\rho(x, y) := \inf \left\{ \sum_{i=1}^{n} w(x_{i-1}, x_i) : x = x_0, y = x_n \right\}.$$

A sequence of vertices $(x_i)_{i\in\mathbb{Z}}$ is called a straight line if

$$\rho(x_i, x_j) = |i - j| \text{ for all } i, j \in \mathbb{Z}.$$

With the definition of a straight line, the main theorem of the talk states that for a connected, locally finite graph $G = (V, E)$, the following statements are equivalent:

(1) $Ric(G) \geq 0$, and there is a straight line.
(2) $G = \mathbb{Z} \times H$ with $Ric(H) \geq 0$.

Here, $\mathbb{Z} \times H$ denotes the Cartesian product of the infinite both sided infinite path $\mathbb{Z}$ with another graph H.

The proof of (2) ⇒ (1) is easy and uses the compatibility of the curvature with Cartesian products. The proof of (1) ⇒ (2) consists of three steps: First, one shows local splitting around the straight lines, i.e., the one tube around a straight line has Cartesian product structure and allows for parallel straight lines. Second, one shows that one can do a parallel transport of a straight line through the entire graph. Third, one defines Busemann functions with respect to straight lines and shows that the other factor H can be expressed as a level set of a Busemann function.

It stays open whether one can achieve some weaker kind of splitting, e.g. in the quasi isometry sense, when only assuming straight lines in terms of the combinatorial graph distance d.

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