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# Spatial Networks and Percolation (hybrid meeting)

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Abstract. The classical percolation problem is to find whether there is an infinite connected component in a random set created by removing edges from a $d$-dimensional lattice, independently at random. Since its introduction into the mathematical literature by Broadbent and Hammersley (1957) the subject of percolation has developed in many ways and is now one of the most exciting and active research areas in probability and statistical mechanics. In this workshop, we focused on current trends, including percolation on point sets with correlations, on spatial random graphs and networks with scale-free degree distribution or long-range edge distribution, percolation of random sets like level set of Gaussian fields or the vacant set of interlacements, conformally invariant percolation structures in the plane, and random walks or information diffusion on percolation clusters. The workshop brought together more than sixty experts and promising young researchers from probability, statistical mechanics and computer science working on all aspects of percolation and spatial networks for an unprecedented exchange of new ideas and methods, with outstanding high quality online talks.

*Mathematics Subject Classification (2010):* 05C80 (random graphs), 60K35 (disordered systems), 82B43 (percolation), 82C44 (dynamics of disordered systems), 60K35 (interacting random processes, percolation).

## Introduction by the Organizers

The workshop *Spatial Networks and Percolation*, organised by Nina Gantert (München), Júlia Komjáthy (Eindhoven, NL), Peter Mörters (Köln) and Vincent Tassion (ETH Zürich) was well attended with over 60 participants (6 in person, 60 online) with broad geographic representation from all continents, including ten women researchers. The workshop not only brought together experts of the various modern subfields of percolation, but also gave place for talks of young and promising researchers - PhD students and postdocs: roughly half of the talks were given by this group. We had four thematic days, each centered around a subfield of modern percolation theory, and we dedicated the last day to talks by PhD students. Each thematic day started with an longer overview talk given by an expert in the topic, followed by modern developments and finished by talks by postdocs and young researchers in the afternoon.

**Percolation.** Percolation is a paradigm for global structure emerging from local rules in stochastic models. The simplest version of percolation, often called *Bernoulli percolation*, was formulated by Broadbent and Hammersley in 1957 who defined a random medium by independently erasing edges from the $d$-dimensional lattice $\mathbb{Z}^d$. The problem is to describe the probability $\theta(p)$ of the origin being in an infinite connected component in dependence on the retention probability $p$. Interest in the probability community in the model reached a new level with the seminal result of Kesten (1980) showing that in $d = 2$ the function $\theta(p)$ takes the value zero for $p \leq \frac{1}{2}$ and positive values for $p > \frac{1}{2}$. Percolation remained at the centre of attention of probability research throughout the 1980s and 1990s with many new techniques developed that had a profound influence also on other models in statistical mechanics. Examples include the van den Berg-Kesten-Reimer inequality, the Burton-Keane technique to show uniqueness of percolation clusters, the use of differential inequalities to prove sharpness results by Aizenman, Barsky and Menshikov, the continuity result of Grimmett and Marstrand, or the lace expansion technique of Hara and Slade. In 1999 Grimmett summarised the state of the art in the second edition of his book *Percolation*.

**Thematic days: Planar percolation.** The new century saw major new developments on the planar case, $d = 2$, when Smirnov showed conformal invariance of the continuum scaling limit of two-dimensional critical percolation and Schramm introduced the limiting object now known as Schramm-Loewner Evolution (SLE). This and subsequent work by Camia, Newman and Lawler, Werner and Schramm was rewarded with Fields medals for Wendelin Werner (2006) and Stanislav Smirnov (2010). The emerging theory established new, deep and mathematically rigorous links between critical lattice models in statistical mechanics and conformally invariant models in the plane, such as SLE and conformal field theories studied earlier in the physics literature. A complete rigorous picture however is yet to be established. In his invited address to ICM 2018 Hugo Duminil-Copin, a major contributor himself, summarises the current state of these ongoing developments under the title *Sixty years of percolation.*

In the thematic day related to this topic, Hugo Vanneuville started by an overview talk on sharp phase transition, noise sensitivity and their relation to the instability of critical planar percolation models, followed by the latest renormalisation methods for crossing probabilities, presented by Laurin Köhler-Schindler. Daniel Contreras presented a new proof of the celebrated Grimmett-Marstrand theorem, while Alexis Prévost described critical exponents in a model with long-range dependencies.

**Thematic day: Continuum percolation.** Beyond the case of Bernoulli percolation, percolation models incorporating strong correlations in the percolation set have become a major focus in the past decade. In an influential early paper Bollobás and Riordan (2006) investigated percolation on the Voronoi cells of a Poisson process, while the seminal paper of Sznitman (2010) investigating percolative properties of the vacant set of random interlacement was the start of a major new research direction. Dynkin-type isomorphism theorems link percolation of interlacements and the level sets of the Gaussian free field.

In the thematic day related to this topic, Nina Holden presented novel developments on the proof of conformal invariance of percolation on random planar maps, while Juhan Aru described the construction of the Gaussian free field using Brownian loops. Ioan Manulescu presented novel proofs of rotational invariance of planar FK-percolation model, while Franco Severo then described their latest results on Gaussian free field (GFF) excursion clusters, while Malin Palö-Forsström presented a novel theorem of understanding the behavior of Wilson-loops in Abelian lattice gauge theories, (that have a strong relation to Ising models via their Gibbs-measure like formulation). Avelio Sepúlveda presented a result on the Villain model, closely related to the GFF and its BKT transition point.

**Thematic day: Dynamical percolation and processes on percolation clusters.** The understanding of the geometry of percolation improved through the study of the behaviour of random walks on percolation clusters. In 2004 Sidoravicius and Sznitman proved quenched invariance principles, followed by extended and still ongoing research on the structure of the walks and their transition kernels. In a further exciting development, percolation is given a temporal structure by opening and closing edges in a time-dependent process. A rich picture is emerging from this, starting with the early work of Häggström et al (1997) and continuing with extensive current research, for example work of Jacob et al (2019) on information diffusion in evolving networks and Amir and Baldasso (2019) on percolation for majority dynamics.

In the thematic day, Jeffrey Steif started with an overview lecture. Perla Sousi presented the newest developments on uniform spanning trees and forests in high dimensions. Gábor Pete and Gideon Amir presented the latest developments with regards to noise sensitivity of Ising model and majority dynamics with iid percolation as input, while Jan Nagel gave new results on the regularity of speed of

Random Walk on branching process trees. Somewhat related to this topic, Marie Théret presented novel results on the speed of a new model of first passage percolation in the Gilbert model, where spread is immediate within the circles centered around vertices. Daniel Valesin presented their results on percolation phase transition on spread out percolation related to clusters in the stationary measure of contact process on $\mathbb{Z}^d$, while Emmanuel Jacob in his talk described the phase diagram of contact process on evolving networks. Related to this topic, Amitai Linker presented their results on the metastable density of the contact process on scale-free networks such as the spatial hyperbolic random graph, where the metastable density is characterised as a function of the underlying graph parameters, while John Fernley presented a model of "awareness", where a dynamically changing Erdös Rényi graph model is allowed to re-arrange its edges adaptively considering the surrounding infection-presence.

**Thematic day: Long-range and scale free percolation.** Concurrently, in an initially unrelated development, physicists, computer scientists and applied mathematicians studied new models for networks describing a range of objects, from the internet to relationships of scientific collaborations and personal friendships. Many of these network models embed naturally in Euclidean or hyperbolic space, the focus of these mathematical developments is therefore on continuum percolation, where bonds are between vertices given by a point process rather than a regular lattice. This creates rich connections to the traditional area of stochastic geometry, see the book Penrose (2003) on geometric random graphs. Percolation questions for spatially embedded graphs with long edges or high degree vertices, called long-range percolation or scale-free percolation, respectively, become the driving force for the developing mathematical theory, see Berger (2002) and Deijfen, Hofstad and Hooghiemstra (2013). In particular, novel multi-scale and renormalisation techniques emerge for the investigation of spatial networks, a development which is still ongoing, see for example Biskup (2004), Biskup and Lin (2019). Current research focuses on behaviour of processes such as random walk or information diffusion on these networks, see Heydenreich, Hulshof, Jorritsma (2017), Chatterjee and Dey (2016), and Komjáthy, Lodewijks (2019).

In the thematic day dedicated to this topic, Tom Hutchcroft presented cluster tail bounds using the two-host inequality, that partially proves long standing conjecture about the power-law decay of long-range percolation cluster sizes at criticality. Christoph Garban presented a novel proof of the first order phase transition of the same model with quadratically decaying connection function in dimension one, based on novel renormalisation techniques. Markus Heydenreich described high-dimensional critical behavior in the random connection models, while Dieter Mitsche presented results on the second largest component in a "scale-free" one-dimensional model called hyperbolic random graph, where they show that in the supercritical regime, the second largest cluster in a finite box of volume $n$ is poly-logarithmic with a specific exponent related to the power-law decay exponent of the degree distribution. Following up on scale free spatial random graph models, Peter Gracar presented a generalised "kernel based" setting to treat such graph in high generality and Arne Grauer showed their results on doubly-logarithmic chemical distances in these models. Related to this topic, Joost Jorritsma presented their result on the evolution of chemical distance in scale-free preferential attachment networks, as the network grows. Elisabetta Candellero presented her latest results on competing first passage percolation on non-amenable hyperbolic graphs.

## Workshop (hybrid meeting): Spatial Networks and Percolation

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# Abstracts

## An introduction and overview of dynamical percolation

Jeffrey Steif

This talk was an overview of the history of dynamical percolation from when it started in 1995 sparked by a question of Paul Malliavin. The initial paper ([8]) was by Olle Häggström, Yuval Peres and the author while the model had also been introduced by Itai Benjamini. The second and third stages of dynamical percolation were, as described in the talk, by Oded Schramm and the author ([13]), and by Christophe Garban, Gábor Pete and Oded Schramm ([3]) respectively. Very relevant to and motivation for these studies was the seminar paper ([1]) on noise sensitivity by Itai Benjamini, Gil Kalai and Oded Schramm.

In a nutshell, dynamical percolation is the model where the edges, vertices or hexagons in an ordinary percolation model evolve independently in time according to simple two state Markov Chains and one studies the question of whether there exist "exceptional" times at which the percolation configuration looks markedly different than in ordinary percolation.

Important separate developments in probability theory about 20 years ago were going on which would allow the carrying out of the second and third stages of dynamical percolation mentioned above. These include (1) the proof ([14]) by Stas Smirnov of conformal invariance of critical percolation on the hexagonal lattice, (2) Schramm's ([12]) famous invention of the Stochastic-Löwner evolution (later to be renamed the Schramm-Löwner evolution) and (3) the development by Greg Lawler, Oded Schramm and Wendelin Werner and by Stas Smirnov and Wendelin Werner of critical exponents for critical percolation on the hexagonal lattice; see [10] and [15].

Details of the results obtained prior to approximately 2010 can found in [7] or [16].

There have been newer papers on dynamical percolation in recent years and the bibliography lists some of these; however this list is certainly not exhaustive.

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## The four dimensional uniform spanning tree


Perla Sousi
(joint work with Tom Hutchcroft)


In this work we analyze the logarithmic corrections to mean-field scaling in the four dimensional *uniform spanning tree*, particularly with regard to the distribution of the *past* of the origin. Our results complement those of Lawler [8] and Schweinsberg [13], who computed the logarithmic corrections to scaling for some other features of the model. Before stating our results, let us first recall the definition of the model, referring the reader to [11, 1, 6] for further background.

A uniform spanning tree of a finite connected graph is simply a spanning tree of the graph chosen uniformly at random; the **uniform spanning forest** of the hypercubic lattice $\mathbb{Z}^d$ is defined to be the weak limit of the uniform spanning trees of the boxes $\Lambda_r = [-r, r]^d \cap \mathbb{Z}^d$, or equivalently of any other exhaustion of $\mathbb{Z}^d$ by finite connected subgraphs. This limit was proven to exist independently of the choice of exhaustion by Pemantle [12], who also proved that the uniform spanning forest of $\mathbb{Z}^d$ is almost surely connected, i.e., a single tree, if and only if $d \leq 4$: this is a consequence of the fact that two independent walks on $\mathbb{Z}^d$ intersect infinitely often a.s. if and only if $d \leq 4$ [3], and is closely related to the fact that the upper-critical dimension of the uniform spanning tree is 4. In light of these results, we refer to the uniform spanning forest of $\mathbb{Z}^d$ as the uniform spanning *tree* when $d \leq 4$. There are various interesting senses in which the four-dimensional uniform spanning tree only just manages to be connected: It can be shown that the length of the path connecting two neighbouring vertices has an extremely heavy $(\log n)^{-1/3}$ tail [8].

Besides connectivity, the other basic topological features of the uniform spanning forest are also now understood in every dimension. Indeed, following partial results of Pemantle [12], it was proven by Benjamini, Lyons, Peres, and Schramm [2] that every tree in the uniform spanning forest of $\mathbb{Z}^d$ is *one-ended* almost surely when $d \geq 2$. This means that for every vertex $x \in \mathbb{Z}^d$ there is exactly one simple path to infinity emanating from $x$ which, by Wilson's algorithm [14, 2], is distributed as an infinite loop-erased random walk. See also [9, 5, 4] for further related results. In order to quantify this one-endedness and better understand the geometry of the trees, we seek to analyze the distribution of the *finite* pieces of the tree that hang off this infinite spine.

Let us now introduce some relevant notation. Let $\mathfrak{T}$ be the uniform spanning tree of $\mathbb{Z}^4$. For each $x \in \mathbb{Z}^4$, the **past** $\mathfrak{P}(x)$ of $x$ in $\mathfrak{T}$ is defined to be the union of $x$ with the finite connected components of $\mathfrak{T} \setminus \{x\}$. We refer to the graph distance on $\mathfrak{T}$ as the **intrinsic** distance (a.k.a. chemical distance) and the graph distance on $\mathbb{Z}^4$ as the **extrinsic** distance. We write $\mathrm{rad}_{\mathrm{int}}(\mathfrak{P}(0))$ and $\mathrm{rad}_{\mathrm{ext}}(\mathfrak{P}(0))$ for the **intrinsic** and **extrinsic radii** of $\mathfrak{P}(0)$, that is, the maximum intrinsic or extrinsic distance between 0 and another point in $\mathfrak{P}(0)$ as appropriate. In high dimensions, it is proven in [6, Theorem 1.1] that the past has intrinsic diameter at least $n$ with probability of order $n^{-1}$ and volume at least $n$ with probability of order $n^{-1/2}$; the same as the probabilities that the survival time and total progeny of a critical, finite-variance branching process are at least $n$ respectively [10]. Our first main theorem computes the logarithmic corrections to this behaviour in four dimensions, giving up-to-constants estimates on the probability that the past has large intrinsic radius or volume.

**Theorem 1** (Volume and the intrinsic one-arm)**.** *Let $\mathfrak{T}$ be the uniform spanning tree of $\mathbb{Z}^4$ and let $\mathfrak{P} = \mathfrak{P}(0)$ be the past of the origin. Then*

$$\mathbb{P}(\mathrm{rad}_{\mathrm{int}}(\mathfrak{P}) \geq n) \asymp \frac{(\log n)^{1/3}}{n} \qquad \text{and} \tag{1}$$

$$\mathbb{P}(|\mathfrak{P}| \geq n) \asymp \frac{(\log n)^{1/6}}{n^{1/2}} \tag{2}$$

*for every $n \geq 2$.*

**Theorem 2** (The extrinsic one-arm)**.** *Let $\mathfrak{T}$ be the uniform spanning tree of $\mathbb{Z}^4$ and let $\mathfrak{P} = \mathfrak{P}(0)$ be the past of the origin. Then*

$$\frac{(\log n)^{2/3}}{n^2} \lesssim \mathbb{P}(\mathrm{rad}_{\mathrm{ext}}(\mathfrak{P}) \geq n) \lesssim \frac{(\log n)^{2/3+o(1)}}{n^2} \tag{3}$$

*for every $n \geq 2$.*

Our proof builds upon both the ideas developed to analyze the high-dimensional uniform spanning forest in [6] and on Lawler's results on the logarithmic corrections for *loop-erased random walk* in four dimensions [8, 7]. As in [6], our proof relies heavily on the analysis of the *interlacement Aldous–Broder algorithm* [5]. In order to perform this analysis, it is important to establish concentration estimates for the *capacity* of 4d loop-erased random walk that could be of independent interest.

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## Some results related to noise sensitivity in the Ising model

Gábor Pete

(joint work with Pál Galicza, Christophe Garban)

For a sequence of **stationary reversible Markov processes** $\omega^n(t)_{t\geq 0}$, we call a sequence of functions $f_n$ on the state spaces **noise-sensitive** if they decorrelate much faster than the full chain: $\forall\, \varepsilon > 0$, as $n \to \infty$,

$$\mathrm{Corr}\Big(f_n\big(\omega^n(0)\big),\, f_n\big(\omega^n(\varepsilon\, T^n_{\mathrm{relax}})\big)\Big) \to 0,$$

where $T_{\mathrm{relax}} := \frac{1}{1-\lambda_2}$ is the relaxation time of the chain. In the case of iid bits being updated by rate 1 exponential clocks, we have $T^n_{\mathrm{relax}} \asymp 1$, hence we get back the usual notion of noise-sensitivity, introduced by Benjamini, Kalai and Schramm [2]. A key theorem there is that a sequence of monotone functions is noise sensitive iff they are asymptotically **uncorrelated from all weighted majorities**.

The proof relies on discrete Fourier analysis and the hypercontractivity of the continuous time random walk Markov operator, or equivalently, on the logarithmic Sobolev constant being uniformly positive [6]. It is a natural wish to generalize this result from iid bits to natural local resampling processes of other spin systems, in the subcritical (i.e., low correlations, high temperature) and critical regimes:

**Question 1.** *Does the characterization of noise sensitivity via weighted majorities remain true for Ising Glauber (heat bath) dynamics on transitive graphs? How about heat bath dynamics in the FK random cluster models?*

**Question 2.** *Is the log-Sobolev constant for these spin and FK Ising dynamics are of the same order as the relaxation time?*

For sufficiently high temperature, a uniformly positive log-Sobolev constant was proved in [15, 12], while for planar Ising Glauber, for all $\beta < \beta_c$, in [11].

Famous examples of noise-sensitive functions of iid bits are the crossing events in critical planar percolation. Besides the Fourier-based approaches of [2, 14, 9], the very recent preprint [16] introduces a non-Fourier approach, suggesting that answering the following question might be possible without addressing Question 1.

**Question 3.** *Are crossing events in critical planar spin and FK Ising noise sensitive? Are there exceptional times?*

Christophe and I have the following partial results (in preparation) for Question 3. Let $f_n$ be the left-to-right crossing in a bounded domain $(\Omega, \partial_1, \partial_2) \cap \frac{1}{n}\mathbb{Z}^2$.

**Theorem 1.** *Let $\omega_t$ be the stationary heat-bath dynamics for $\mathsf{FK}(p_c(q), q)$ on $\Omega \cap \frac{1}{n}\mathbb{Z}^2$, $q \geq 1$. Let $\tau(n) = 1/\mathbb{E}|\mathsf{Piv}_n|$, where $\mathsf{Piv}_n$ is the set of pivotal edges for $f_n$. Then, for all $t > 0$ and large enough $n$, there is $c_t > 0$ s.t.*

$$c_t < \mathrm{Corr}\big(f_n(\omega_0), f_n(\omega_{t\tau(n)})\big) < 1 - c_t\,.$$

**Proof idea.** A space-time second moment method for the number of pivotal switches gives that it is 1 with positive probability, giving the result.

Main tools: spatial decorrelation ($z$ being pivotal at time $t$ is quasi-independent of $x$ being pivotal at $s$) and the abundance of macroscopic pivotals (the alternating 4-arm exponent being less than 2), proved recently in [7] for all $1 \leq q \leq 4$. □

**Theorem 2.** *In the wired or free limit of the $\mathsf{FK}(p_c(q), q)$ heat bath dynamics on $\mathbb{Z}^2$, the Hausdorff dimension of exceptional times for having an infinite cluster is*

$$\dim_H(\mathcal{E}_q) \leq 1 - \frac{\rho_1^{FK(q)}}{2 - \rho_4^{FK(q)}}\,,$$

*if these exponents exist. This bound is conjecturally 0 for $q \geq 4\cos^2(\frac{\pi}{4}\sqrt{14}) \approx 3.83$.*

**Proof idea.** We stochastically dominate the stationary dynamics by a monotone one, always opening edges when their clocks ring. The stability argument of [10] and the above-mentioned results of [7] yield the probability that, at time $\varepsilon$ in the monotone dynamics, the origin is in an infinite cluster. □

The following answers a question of Broman and Steif [5].

**Theorem 3.** *For Ising Glauber on $\mathbb{Z}^2$, there are no exceptional times.*

**Proof idea.** No self-touches in $\mathrm{SLE}_3$, hence very few macroscopic pivotals. □

Going back to the iid case, it was a highly influential discovery of [13] and [14] that many interesting functions can be (approximately) computed by **low-revealment adaptive algorithms**, yielding sharp thresholds and noise-sensitivity:

$$\mathrm{Corr}\Big[f_n(\omega),\ \mathbb{E}\big[\,f_n(\omega)\,\big|\,\omega_{i_1},\ldots,\omega_{i_T}\,\big]\Big] \to 1\,,$$

where $i_{t+1} \in \sigma(\omega_{i_1},\ldots,\omega_{i_t},\eta)\ \forall\, t$, with $\eta$ being some randomization independent of $\omega$, and $\sup_{j\in V_n} \mathbb{P}\big[\,j\in\{i_1,\ldots,i_T\}\,\big] \to 0$. A natural question (proposed in a slightly different formulation by Itai Benjamini): are there *any* functions $f_n$ for which there exist **low-revealment non-adaptive algorithms**, i.e., random subsets $\mathcal{U}_n \subseteq V_n$, independent of the input $\omega$, with small revealment $\delta_{\mathcal{U}} := \sup_{j\in V_n} \mathbb{P}\big[\,j\in \mathcal{U}_n\,\big] \to 0$, while $\mathrm{Corr}\Big[f_n(\omega),\ \mathbb{E}\big[\,f_n(\omega)\,\big|\,\omega_{\mathcal{U}_n}\,\big]\Big] \to 1$ ? Our answer in [8] is negative:

**Theorem 4.** *There is no sparse reconstruction for iid input $\omega$ for any $f_n$.*

**Proof.** We use the Fourier spectral sample, a random subset of the bits defined by $\mathbb{P}[\,\mathscr{S}_f = S\,] := \widehat{f}(S)^2/\|f\|^2$, where $\widehat{f}(S)^2 := \mathbb{E}\big[\,f(\omega)\,\chi_S(\omega)\,\big]$ and $\chi_S(\omega)$ is the Fourier basis. Then, conditionally on $\mathcal{U}$:

$$\begin{aligned}\mathsf{clue}(f\,|\,\mathcal{U}) := \frac{\mathrm{Var}\big(\mathbb{E}[\,f\mid\omega_{\mathcal{U}}\,]\,\big|\,\mathcal{U}\big)}{\mathrm{Var}(f)} &= \frac{\sum_{\emptyset\neq S\subseteq\mathcal{U}}\widehat{f}(S)^2}{\sum_{\emptyset\neq S\subseteq V}\widehat{f}(S)^2}\\ &= \mathbb{P}\big[\,\mathscr{S}_f\subseteq\mathcal{U}\,\big|\,\mathcal{U},\ \mathscr{S}_f\neq\emptyset\,\big] \le \tilde{\mathbb{P}}[X_f\in\mathcal{U}\,|\,\mathcal{U}]\,,\end{aligned}$$

where $X_f$ is a uniform random element of $\mathscr{S}_f$ conditioned to be non-empty.

$$\tilde{\mathbb{P}}[X_f\in\mathcal{U}] = \sum_{j\in V_n}\tilde{\mathbb{P}}[X_f=j,\ j\in\mathcal{U}] \le \delta_{\mathcal{U}}\sum_{j\in V_n}\tilde{\mathbb{P}}[X_f=j] = \delta_{\mathcal{U}}\,,$$

hence $\mathbb{E}\big[\,\mathsf{clue}(f\,|\,\mathcal{U})\,\big] \le \delta_U$, and $\mathbb{E}\Big[\,\mathrm{Corr}\big[f_n(\omega),\ \mathbb{E}\big[\,f_n(\omega)\,\big|\,\omega_{\mathcal{U}_n}\,\big]\big]\,\Big] \le \sqrt{\delta_U}$. □

Since this was a tricky but simple use of the Fourier spectrum, one may hope that generalizing sparse reconstruction results from iid bits to other measures could be a first step in generalizing noise sensitivity results. This sparse reconstruction project is ongoing joint work with Pál, with occasional input from Christophe.

First we generalized Theorem 4 from $\pm1$ bits to any iid measure, using the **Efron-Stein decomposition** instead of Fourier [8]. The next step is to consider **factor of iid** (fiid) measures: spin systems $\sigma:\mathbb{Z}^d \longrightarrow \{\pm1\}$ for which there is a measurable coding map $\psi:[0,1]^{\mathbb{Z}^d} \longrightarrow \{\pm1\}$ such that $\sigma(x) = \psi\big(\omega(x+\cdot)\big)$ holds for $\omega \sim \mathrm{Unif}[0,1]^{\otimes\mathbb{Z}^d}$. This factor map is **finitary** if there is a random coding radius $R(\omega) < \infty$ such that $R(\omega)$ and $\psi(\omega)$ are determined by $\big\{\omega(x) : x\in[-R,R]^d\big\}$. Three key examples were proved by van den Berg and Steif [3]:

- For $\beta < \beta_c$, the unique **Ising** measure on $\mathbb{Z}^d$ is a finitary factor of $\mathrm{Unif}[0,1]^{\mathbb{Z}^d}$, with coding radius $\mathbb{P}[\,R > t\,] < \exp(-ct)$.
- At $\beta_c$, it is a finitary factor, but only with $\mathbb{E}[\,R^d\,] = \infty$.
- For $\beta > \beta_c$, the + measure is fiid, but not finitary.

**Theorem 5.** *If $\sigma$ is a finitary factor of iid on $\mathbb{Z}^d$ with $\mathbb{P}[\,R > t\,] < \exp(-ct)$, and $\sigma_n$ is a version on the torus $\mathbb{Z}_n^d$, then, for any function $f_n$ of the spins, and any random subset with revealment $\delta_{\mathcal{U}_n} = o(1/\log^d n)$, independent of $\sigma_n$, we have*

$$\mathbb{E}\big[\,\mathsf{clue}(f_n\,|\,\mathcal{U}_n)\,\big] := \mathbb{E}\left[\frac{\mathrm{Var}(\mathbb{E}[\,f \mid \sigma_{\mathcal{U}_n}\,])}{\mathrm{Var}(f_n)}\right] \to 0\,.$$

**Proof idea.** Take $\mathcal{W}_n := \bigcup_{u\in\mathcal{U}_n} B_{C\log n}(u)$. Then $\omega_{\mathcal{W}_n}$ determines $\sigma_{\mathcal{U}_n}$ with high probability, but the revealment $\delta_{\mathcal{W}_n}$ is still small, hence the clue is small. □

This proof seems wasteful, because $B_{C\log n}(u)$ is the worst case for each $u$. However, we have examples of fiid measures where the result is sharp.

**Theorem 6.** *On the tori $\mathbb{Z}_n^d$, $d \geq 2$, at $\beta_c$, the magnetization $M_n(\sigma) := \sum_{x\in\mathbb{Z}_n^d} \sigma_x$ and $\mathsf{Maj}_n(\sigma) := \operatorname{sign} M_n$ can be reconstructed from some low revealment subset $\mathcal{U}_n$.*

**Proof idea.** The susceptibility is $\sum_{x\in\mathbb{Z}^d} \mathrm{Cov}_\beta[\sigma_0, \sigma_x] = \infty$ at $\beta = \beta_c$, by [1]. □

A Markov random field on $\{\pm 1\}^{\mathbb{Z}^d}$ has **strong spatial mixing** (SSM) if for any finite box $V$, given boundary configurations $\sigma_{\partial V}$ and $\tilde\sigma_{\partial V}$ differing only at one vertex $v \in \partial V$, for any radius $R$, the conditional distributions inside $V$ satisfy $d_{\mathrm{TV}}\left(\sigma_{V\setminus B_R(v)},\ \tilde\sigma_{V\setminus B_R(v)}\right) \leq \exp(-cR)$. Ising for $d \leq 2$ all $\beta < \beta_c$ and any $d$ for $\beta$ low enough satisfy SSM [11, 12]. It is proved in [4] that SSM implies a uniform spectral gap for certain block dynamics (e.g., Swendsen-Wang). Inspired by this:

**Theorem 7.** *For any SSM Markov field on $\{-1,1\}^{\mathbb{Z}^d}$, any function $f_n$ and any independent random subset with revealment $\delta_{\mathcal{U}_n} \to 0$, we have $\mathbb{E}\big[\,\mathsf{clue}(f_n\,|\,\mathcal{U}_n)\,\big] \to 0$.*

**Proof idea.** A block Glauber dynamics $\sigma \mapsto \sigma^{\mathcal{U}}$: first $\mathcal{U}$ is sampled, then $\sigma_{\mathcal{U}}$ gets fixed and $\sigma_{\mathbb{Z}_n^d\setminus\mathcal{U}}$ gets resampled from the conditional distribution. If $\delta_{\mathcal{U}}$ is small, then the spectral gap of this chain is close to 1, by a path coupling argument. □

Since SSM for Ising on $\mathbb{Z}^d$ is equivalent to a uniform log-Sobolev constant for the Glauber dynamics [11], a positive answer to Question 2 would imply that Theorem 7 applies to all $\beta < \beta_c$ Ising models on all $\mathbb{Z}^d$. Then Theorem 6 would say that there is sparse reconstruction for Ising iff majority can be reconstructed; this extremal role would be analogous to Theorem 4 on iid sparse reconstruction and to Question 1 on Ising noise sensitivity.

**Question 4.** *For $\beta < \beta_c$ Ising on $\mathbb{Z}_n^d$, $d \geq 3$, shave off the $\log^d n$ from Theorem 5.*

**Question 5.** *What is the exact reconstruction threshold for critical Ising $\mathbb{Z}_n^2$?*

We know that a revealment $\delta_{\mathcal{U}_n} \gg n^{-7/4}$ is enough for magnetization, while $\delta_{\mathcal{U}_n} \ll n^{-15/8}$ is not enough for any function.

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## Contact process on evolving scale-free networks

Emmanuel Jacob
(joint work with Amitai Linker, Peter Mörters)

In a well-known paper Chatterjee and Durrett [1] show that the contact process on scale-free networks exhibits metastable behaviour. Later, the rate of decay of the metastable density of the contact process has been identified when the infection parameter $\lambda$ decays to 0, in dependance of the power-law exponent $\tau > 2$, in the context of the configuration model in Mountford et al [5], as well as in the context of the preferential attachment model in Can [2]. These results are in contrast to the corresponding mean-field model where for small infection rates, metastable behaviour is only observed for $2 < \tau < 3$.

In the present work we investigate the contact process on a network evolving according to an independent stationary dynamics, thereby interpolating between these scenarios. The speed of this stationary dynamics can be slowed down or sped up by virtue of decreasing or increasing a real parameter $\eta$, with $\eta \downarrow -\infty$ approaching the static and $\eta \uparrow +\infty$ the mean-field case.

## 1. The model

We consider the vertex set $V = \{\frac{1}{N}, \frac{2}{N}, \ldots, 1\}$. Initially, each pair of vertices $x$ and $y$ are connected independently with probability $\frac{1}{N} p(x,y) \wedge 1$, where $p$ is the factor kernel $p(x,y) = x^{-\gamma} y^{-\gamma}$, for some parameter $\gamma \in (0,1)$. Later, each vertex updates at rate $\kappa(x) = x^{-\gamma\eta}$, for some parameter $\eta \in (-\infty, +\infty)$, and then resamples all its connections simultaneously.

The network dynamics is stationary, and at a fixed time it is a rank-one approximation of a configuration model as well as a scale-free network, with power-law exponent $\tau = 1 + 1/\gamma \in (2, +\infty)$. Moreover, the expected degree of a vertex $x \in V$ is of order $x^{-\gamma}$, so taking $\eta \downarrow -\infty$ or $\eta \uparrow +\infty$ can make the updating rate of the strong vertices go to 0 or $+\infty$ respectively.

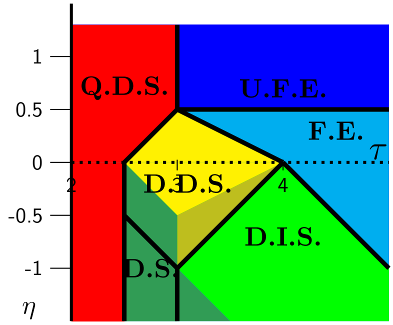
In [3, 4], several variations of this model are also studied. In particular, the factor kernel can be replaced by the preferential attachment kernel, and the vertex updating scheme can be replaced by an edge updating scheme, where the edges are updated independently.

## 2. Our results

We obtain a complete phase diagram, depending on the two parameters $\tau \in (2, +\infty)$ and $\eta \in (-\infty, +\infty)$, yielding for each particular value one of the two following scenarios:

- Whatever the infection rate $\lambda > 0$, the (expected) density of infected vertices stays close to a metastable positive value $\rho(\lambda) > 0$ up to time $\exp(\varepsilon N)$ for some $\varepsilon > 0$. We then say there is slow extinction and metastability. Moreover, we provide asymptotics for the metastable density as $\rho(\lambda) = \lambda^{\xi(1+o(1))}$ when $\lambda \to 0$, with $\xi$ some explicit exponent.
- For small infection rates, the (expected) extinction time of the infection is at most linear in $N$. We then say there is fast extinction, or even ultra fast extinction when the extinction time is at most logarithmic in $N$.

The full result with the expression of the metastable density exponent $\xi$ is provided in [3] for $\eta \geq 0$ and in the forthcoming paper [4] for $\eta < 0$. We provide here only a summary in the following picture:




The two blue phases on the right correspond to fast and ultra-fast extinction. All the other phases correspond to slow extinction. The different colors correspond to different expressions for $\xi$. The different acronyms correspond to different mechanisms that suffice to explain the survival of the infection for an exponentially long time, as well as to provide a lower bound for $\xi$.

More precisely, each mechanism is performed among the strongest vertices, namely those below some threshold $a \in (0, 1)$. We consider either Direct Spreading of the infection between those, or Indirect Spreading via a common neighbour in $(a, 1)$. Moreover, we consider either Quick mechanisms when the spreading is performed in a unit time interval, or Delayed mechanisms, when the spreading is performed in a longer time interval, corresponding to the time length where we can maintain the infection around a vertex *locally*, namely using only infections from the vertex to its neighbours, and back.

## 3. Techniques

We discuss in slightly more details the techniques used to upper bound the extinction time of the infection or the metastable density exponent $\xi$. We still define a threshold $a \in [0, 1)$, but we search for a score to associate to the configuration, that would be a supermartingale up to the hitting time of some vertex $x \in (0, a]$. The definition of an appropriate score so as to obtain this supermartingale property is actually the most delicate part of the proof. Then, standard techniques allow to upper bound the probability that a vertex $x \in (a, 1)$ is infected in the metastable state when $a > 0$, or to prove fast extinction if we can take $a = 0$.

We succeed in defining a supermartingale under some less and less restrictive requirements by an addition of several ideas, which make the definition of the score more and more involved, but can still be briefly summarized below.

- Associate a score to recovered vertices, that reflects the *probability that it will be reinfected by one of its neighbours.*
- As long as an edge $\{x, y\}$ didn't transmit an infection, do not reveal the presence of the edge. Instead, *reveal and use the edge* to transmit the infection *at the mean rate* $\lambda p(x, y)/N$.

▷ Using these two ideas, we can define a supermartingale under the requirement

$$\forall x > a, \quad \mathfrak{c} T_x^{loc} \int \lambda p(x,y) s(y) \mathrm{d}y \leq s(x),$$

where $\mathfrak{c}$ is an explicit constant and $T_x^{loc}$ corresponds to the time of local survival around $x$ of the infection, using only $x$ and its direct neighbours.

- Treat separately *slowly updating vertices* $\{x, y\}$: reveal edge $\{x, y\}$ immediately, and if absent, then reveal and use it at rate $(\kappa(x)+\kappa(y))p(x,y)/N$.

  ▷ This allows to weaken the requirement to

$$\mathfrak{c} T_x^{loc} \int_0^1 (\lambda \wedge (\kappa(x) + \kappa(y))) p(x,y) s(y) \mathrm{d}y \leq s(x).$$

- Treat separately *weak vertices* with average degree less than $1/8\lambda^2$.

  ▷ This allows to further weaken the requirement, specially for weak vertices, to

$$\mathfrak{c} \int_0^1 \lambda p(x,y) s(y) \mathrm{d}y \leq s(x).$$

Finally, these less restrictive requirements allow us to prove fast extinction in the appropriate phases, or to obtain an upper bound for $\xi$ matching the lower bound in the other phases.

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## The speed of random walk on Galton-Watson trees with vanishing conductances

Jan Nagel

(joint work with Tabea Glatzel)

It is well known since the work of [1], that a simple random walk $(X_n)_{n\geq 0}$ on a supercritical Galton-Watson tree starting at the root has a positive effective speed,

$$\lim_{n\to\infty} \frac{|X_n|}{n} = v \quad \text{a.s.}, \tag{1}$$

where $|X_n|$ denotes the distance of $X_n$ to the root and the limit holds almost surely with respect to the annealed law, conditioned on the tree being infinite. The effective speed $v$ can be given explicitly in terms of the offspring law. Such a law of large numbers also holds for a random walk on a Galton-Watson tree with random conductances. For this process, given a Galton-Watson tree $T$ without leaves, the edges of the tree are assigned random positive conductances, independently and identically distributed for different edges and the random walk traverses an edge with a probability proportional to the conductance of that edge. We assume that the marginal law $\mu$ of the conductances is uniformly elliptic, that is,

$$\mu([\delta, \delta^{-1}]) = 1. \tag{2}$$

for a $\delta > 0$. Is was proven by [2], that a convergence as in (1) still holds, when we now also average over the conductances, with a limit $v = v(\nu, \mu)$ depending on the offspring law $\nu$ of the tree and the law $\mu$ of the conductances. Unfortunately, the speed cannot be explicitly computed even for very simple distributions of conductances.

We study the regularity of the speed as a function of the distribution of the conductances. The regularity of the speed on the tree as a function of the offspring law was for example studied in [3], when the offspring law is close to criticality. Using the formulas for the speed in [2], one can show that as long as we stay in the framework of uniformly elliptic conductances, the speed is continuous.

**Proposition 1.** *For any $\delta > 0$, the mapping $\mu \mapsto v(\nu, \mu)$ is continuous on the set of uniformly elliptic measures satisfying* (2)*, equipped with the weak topoplogy.*

To investigate how the speed changes when we leave the uniformly elliptic regime, we replace the measure $\mu$ by

$$\mu_\varepsilon = \alpha\delta_\varepsilon + (1-\alpha)\mu, \tag{3}$$

for $\varepsilon \geq 0$, $\alpha \in (0,1)$. That is, the conductances on a fraction $\alpha$ of edges is set to $\varepsilon$. Denote by $T_*$ the connected component of the root when edges of conductance $\varepsilon$ are removed. If $\varepsilon = 0$, the random walk is confined to $T_*$, which might be finite. In this case, and consistently with the definitions in [1] and [2], the speed $v(\nu, \mu_0)$ is defined as the a.s. limit, conditioned on $T_*$ being infinite. The following theorem gives the limit of the speed as $\varepsilon$ tends to zero.

**Theorem 2.** *It holds that*

$$\lim_{\varepsilon \searrow 0} v(\nu, \mu_\varepsilon) = \hat{\mathrm{P}}_0(|T_*| = \infty) \cdot v(\nu, \mu_0), \tag{4}$$

*where $\hat{\mathrm{P}}_0$ is the invariant measure for the tree seen from the random walk.*

Theorem 2 shows a slowdown effect for the walker as it spends time in finite parts of the tree which can only be left by traversing an edge with conductance $\varepsilon$, acting as traps in the environment.

For the proof of Theorem 2, we use that for $\varepsilon > 0$, the speed can be written as $v(\nu, \mu_\varepsilon) = \hat{\mathbb{E}}_\varepsilon[\Delta(X_0, X_1)]$, where the expectation is w.r.t. a measure $\hat{\mathbb{P}}_\varepsilon$ under which the tree seen from the random walk is a stationary process and $\Delta(X_0, X_1)$ is the change in distance to a random boundary point (the infinitely far away root) of the tree as the random walk takes its first step. To prove convergence of the speed we then show

$$\lim_{\varepsilon \searrow 0} \hat{\mathbb{E}}_\varepsilon \left[ \Delta(X_0, X_1) \mathbb{1}_{\{|T_*|<\infty\}} \right] = 0,$$

which follows from a quantitative estimate of the trapping effect, and

$$\lim_{\varepsilon \searrow 0} \hat{\mathbb{E}}_\varepsilon \left[ \Delta(X_0, X_1) \mathbb{1}_{\{|T_*|=\infty\}} \right] = \hat{\mathbb{E}}_0 \left[ \Delta(X_0, X_1) \mathbb{1}_{\{|T_*|=\infty\}} \right],$$

which requires transience estimates uniformly in $\varepsilon$.

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## Majority dynamics, median dynamics and percolation

Gideon Amir

(joint work with Rangel Baldasso and Nissan Beilin)

In this talk we discuss the well studied model of majority dynamics and its connections to the more recently introduced median dynamics process and to percolation. We prove several connections between the models and raise some new conjectures regarding the dynamics. We then discuss the dynamical percolation model defined by the opinions in majority dynamics and state some results on the behaviour of the critical probability and the model at criticality, and finally we show some new angles towards a well-known conjecture on zero-temprature Glauber dynamics on $\mathbb{Z}^d$.

In majority dynamics, each vertex $x$ of a fixed underlying graph $G$ receives an opinion that can be either zero or one. With rate one, the opinion at $x$ is updated to match the majority of the neighbors' opinions. The lecture is divided into four parts. In the first, we discuss basic properties and questions regarding majority dynamics. In the second part, we introduce a related model, median dynamics, and study its connections to majority dynamics. A central part of the talk is the statement of several new conjectures for median and majority dynamics on general graphs. In the third part of the talk we discuss majority dynamics as a model of dynamical percolation. Finally in the last part of the talk, we revisit a well-known conjecture regarding the behaviour of zero-temperature Glauber dynamics on $\mathbb{Z}^2$.

**1. Majority dynamics.** Majority dynamics is an *opinion dynamics* model in which every vertex $x$ of a fixed underlying graph $G$ holds an opinion that can be either zero or one and evolves with time. With rate one, the opinion at $x$ is updated to match the majority of the neighbors' opinions. When vertices have even degrees, one must choose a way to break ties. Unless otherwise stated, we deal with ties by letting the vertex keep its own opinion. This is equivalent to adding a self-loop in the graph. Another way to deal with tie-breaking is using a fair coin. We will refer to the latter choice as zero-temperature Glauber dynamics.

Majority dynamics has been studied since the 70's, with rigorous mathematical treatment since the 90's. The model is well-defined on any bounded degree graph, and is monotone in the initial condition. There are many aspects of majority dynamics that have been studied in the literature, on both finite and infinite graphs, both random (such as $G(n, p)$) and deterministic. A very partial list includes **Fixation** - will every vertex change its opinion infinitely often or will it stabilize on some final opinion?. If fixation occurs, **how does the final configuration behave**? **How fast** do we reach it? Will **unanimity** be reached or will both opinions survive? **Connectivity questions**- how do the clusters of opinions look like? **Marginal measures** - how does the probability that a given vertex holds the opinion "1" evolve with time? **Retention of information** - assume that the initial configuration has some bias, will we be able to detect it after time has passed? And the list goes on.

We briefly discuss some of these questions, with an emphasis on fixation.

**2. Median dynamics.** In median dynamics (First introduced in [4]), each vertex holds an opinion in the interval $[0, 1]$, with initial opinions taken to be i.i.d. $unif([0, 1])$ random variables. Each vertex updates its opinion according to an independent Poisson clock with rate 1. When the clock rings, the vertex changes its opinion to the **median** of its neighbors' opinions. In the case when the degree is even, we consider the vertex's own opinion in the pool, in order for the median to be uniquely defined (alternatively, one may use coin flips to choose between the two median values to define a median-version of the zero-temperature Glauber dynamics). This part of the talk is based on [2].

Median dynamics generalizes majority dynamics in the following sense: if one colors all vertices with opinions in $[0, p]$ as black and all vertices with opinions in $(p, 1]$ as white, one retrieves back majority dynamics between the black and white opinions. Thus it is possible to view median dynamics as a coupling of majority dynamics started with i.i.d. $Ber(p)$ initial opinions, for all $p \in [0, 1]$. This coupling offers a dictionary by which one can rewrite many results and conjectures regarding majority dynamics in terms of median dynamics. In many cases, this re-formulation seems very natural and offers new ways to approach the topics, and prompts new conjectures which we discussed next.

Fix $G$, and let $\mu_t(x)$ denote the marginal measure of the value at $x$ at time $t$. We expect the dynamics to have an averaging effect on the opinions. This brings us to the following new conjectures:

Conjecture 1: For any $G = (V, E)$, any $x \in V$ and any $\alpha > \frac{1}{2}$ the function $t \to \mu_t(x)$ is monotone non-decreasing.

Conjecture 2: For any $G = (V, E)$, any $x \in V$ and any $t$ the function $\alpha \to \frac{d}{d\alpha}\mu_t(x)([0,\alpha])$ is monotone non-decreasing in $[0, \frac{1}{2}]$. (We prove the derivative always exist)

We give equivalent formulations of the above conjectures in terms of majority dynamics. We then discuss some (very) partial results towards these conjectures on general graphs, as well as some (wrong) strengthening of these conjectures. Finally we show that the conjectures hold on $\mathbb{Z}$ and on the complete graph $K_n$.

**3. Percolation in majority dynamics.** Start majority dynamics on some infinite connected graph $G$ with i.i.d. Bernoulli-p initial opinions. One may look at the sites with opinion 1 as a dynamical percolation model in time. This model is neither stationary in time nor independent between sites.

The basic question that we address is the existence of an infinite connected cluster of vertices with opinion 1. It is easy to see that this is a $0-1$ event, monotone in $p$, and therefore, as in regular percolation, there is some critical probability $p_c(G,t)$ such that for $p > p_c$ there a.s. exists an infinite connected component at time $t$ and for $p < p_c(t)$ there a.s. doesn't. We would like to understand the behaviour at criticality and the dependance of $p_c$ on time. We restrict ourselves to the graph $G = \mathbb{Z}^2$, and write $p_c(t) = p_c(\mathbb{Z}^2, t)$.

In joint work with R. Baldasso [1] we show the following: **1** The model does not percolate at criticality: For all $t > 0$ $\mathcal{P}_{p_c(t)}[\eta_t \text{ percolates}] = 0$. **2** The function $t \to p_c(t)$ is continuous in $[0,\infty)$ and monotone non-increasing in $[0,\infty]$. **3** For any $t > 0$ $\frac{1}{2} \le p_c(t) < p_c(0) = p_c^{site}$.

We discuss the heuristics behind the proofs and the (non)-existence of exceptional times. We also mention related work by Alves and Baldasso [5] where they studied *sharp threshold* properties of this percolation model.

There are many open questions remaining, and we mention a few of them in the talk including: Is the critical probability strictly decreasing in time? Is $t \to p_c(t)$ continuous at $\infty$? Is $p_c(\infty) = \frac{1}{2}$? Can similar results be done for the hexagonal lattice, where one expects the critical probability to be $\frac{1}{2}$ for all times, and in particular are there exceptional times for the model there?

**4. Zero temperature Glauber dynamics.** In the last part of the talk we discuss a well-known conjecture regarding Zero temperature Glauber dynamics on $\mathbb{Z}^d$. Run majority dynamics on $Z^d$ starting from *i.i.d.* Bernoulli-p initial opinions, **breaking ties using coin-flips** (and not using the vertex's old opinion). That is, if a vertex has $d$ neighbours with opinion 1 This model has a very different behaviour then the regular majority dynamics and there are many less stable structures, and in particular there are no finite stable structures. Using monotonicity and symmetry one can show that there exists some $p_c = p_c(d)$ such that for every $p > p_c$ all sites in $\mathbb{Z}^d$ will eventually fixate on opinion 1 a.s., for every $p < 1 - p_c$ they will a.s. fixate on 0, and for $1 - p_c < p < p_c$ every vertex will change its opinion infinitely often.

A folklore conjecture is that $p_c(d) = \frac{1}{2}$ for every $d \ge 2$, and in particular for $\mathbb{Z}^2$.

Fontes, Schonmann and Sidoravicius [3] proved that $p_c(d) < 1$ for all $d \geq 2$. Morris [6] proved that $\lim_{d\to\infty} p_c(d) = \frac{1}{2}$.

We discuss approaching the conjecture by looking at the corresponding median dynamics, hoping that this will allow for a wider set of tools. We use the continuous opinions to rephrase the conjecture in terms of convergence of the opinions to $\frac{1}{2}$, which allows for both weaker versions (by considering weaker convergence notions) and stronger versions (regarding the rate of convergence).

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## On the threshold of spread-out contact process percolation


Daniel Valesin

(joint work with Balázs Ráth)


The spread-out contact process on $\mathbb{Z}^d$ is a continuous-time Markov process with state space $\{0,1\}^{\mathbb{Z}^d}$, usually taken as a simple model for the spread of an infection in a population. In this interpretation, each vertex of $\mathbb{Z}^d$ represents an individual, and states 0 and 1 indicate that the individual is healthy or infected, respectively. The dynamics is given by the following transition rules, from any configuration $\xi \in \{0,1\}^{\mathbb{Z}^d}$:

- for each $x$ with $\xi(x) = 1$, $x$ *recovers* (switches to state 0) with rate 1;
- for each $x$ with $\xi(x) = 1$, $x$ *transmits* the infection with rate $\lambda > 0$. When this occurs, $x$ chooses another vertex $y$ uniformly at random in the $\ell_\infty$ ball with center $x$ and radius $R$; then, $y$ becomes infected (if $y$ was already infected, nothing happens).

The parameter $\lambda$ is called the infection rate, and the parameter $R$ is the range. This model is a variant of the Harris contact process [2], and has been previously studied by Durrett, Bramson and Swindle [1]. It is well known that there exists a threshold parameter value $\lambda_c(R)$ such that, defining the probability of survival of the infection,

$$\rho(\lambda, R) := \mathbb{P}(\text{for all } t \geq 0 \text{ there exists } x \text{ such that } \xi_t(x) = 1),$$

we have $\rho(\lambda, R) > 0$ if and only if $\lambda > \lambda_c(R)$. Moreover, the process started from the configuration in which all vertices are infected is known to converge in distribution to the so-called *upper stationary distribution* $\mu_{\lambda,R}$, a probability measure on $\{0,1\}^{\mathbb{Z}^d}$ which is equal to a unit mass on the all-zero configuration if $\lambda \le \lambda_c(R)$, and is supported on configurations with density of 1's equal to $\rho(\lambda, R)$ if $\lambda > \lambda_c(R)$.

In the present work, we consider the model on dimension $d \ge 2$, we assume that $\lambda > \lambda_c(R)$, and study the upper stationary distribution from the point of view of *site percolation* on $\mathbb{Z}^d$. This means that we consider the set

$$\text{Perc} := \left\{ \begin{array}{c} \xi \in \{0,1\}^{\mathbb{Z}^d} : \text{ the subgraph of } \mathbb{Z}^d \text{ induced by the set} \\ \{x : \xi(x) = 1\} \text{ has an infinite connected component} \end{array} \right\},$$

and we study the value of $\mu_{\lambda,R}(\text{Perc})$ for different values of the parameters. Specifically, we study the threshold value

$$\lambda_p(R) := \sup\{\lambda : \mu_{\lambda,R}(\text{Perc}) = 0\}.$$

Our main result is:

**Theorem 1.** *For any $d \ge 2$, we have*

$$\lim_{R\to\infty} \lambda_p(R) = \frac{1}{1-p_c},$$

*where $p_c$ is the critical value for Bernoulli site percolation on $\mathbb{Z}^d$.*

Combining this with a result of [1], which gives the convergence $\lim_{R\to\infty} \lambda_c(R) = 1$, we also obtain:

**Corollary 1.** *For any $d \ge 2$, for $R$ large enough we have*

$$\lambda_c(R) < \lambda_p(R) < \infty.$$

In [3], the authors posed the open problem of determining whether the inequality $\lambda_c < \lambda_p$ holds for the classical (nearest-neighbor) contact process. The above corollary solves the version of this question for the sufficiently spread-out version of the process.

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## Long-range models in 1D revisited

Christophe Garban
(joint work with Hugo Duminil-Copin, Vincent Tassion)

Long-range 1-dimensional models have a rich history in statistical physics which goes back to the seminal works of Thouless and Dyson in 1969, [Tho69, Dys69]. The most interesting long-range interaction (for percolation, Ising or even Potts $q \geq 3$ models) is when sites $x, y \in \mathbb{Z}$ are subject to an interaction of order

$$J_{x,y} \equiv J_{x-y} = \frac{1}{|x-y|^2} \,.$$

Long-range percolation model with such an $1/r^2$ decay is simply defined as follows: for any $\beta > 0$ which plays the role of an *inverse temperature*, connect each pair of sites $\{x, y\}$, with $x \neq y$ in $\mathbb{Z}$ independently of the other pairs with probability

$$p_{x,y} = 1 - e^{-\frac{\beta}{|x-y|^2}} \sim_{|x-y|\to\infty} \tfrac{\beta}{|x-y|^2} \,.$$

It has been shown in [NS86] that this model undergoes a phase transition as one varies $\beta$.

Similarly, one may define a $\frac{1}{r^2}$ long-range Ising model on spin configurations $\sigma \in \{\pm 1\}^{\mathbb{Z}}$ which corresponds to the following (formal) Hamiltonian

$$H(\sigma) := -\sum_{x \neq y} \frac{\sigma_x \sigma_y}{|x-y|^2} \,.$$

A phase transition (by varying as well the inverse temperature $\beta$ in the Gibbs measure $\mathbb{P}_\beta[\sigma] \propto e^{-\beta H(\sigma)}$) had been predicted in [Tho69] and has been rigorously established in [FS82, ACCN88].

The main interesting feature of these long-range models lies in the fact that their phase transition is *discontinuous* (or first-order). This corresponds to the prediction of Thouless [Tho69] and was proved for percolation in [AN86] for percolation and then in [ACCN88] for Ising and Potts ($q \geq 1$) models.

The purpose of this talk, based on [DGT20a], was to revisit all these works (i.e. [FS82, NS86, AN86, ACCN88, IN88]) using a new and simplified renormalization scheme. The main motivation of revisiting these 1D long-range model was the application of such a renormalization scheme to the gluing of critical 2D Ising models analyzed in [DGT20b]. This second work [DGT20b] was briefly described at the end of the talk.

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## Power-law bounds for critical long-range percolation

Tom Hutchcroft

Let $d \geq 1$ and suppose that $J : \mathbb{Z}^d \to [0, \infty)$ is both **symmetric** in the sense that $J(x) = J(-x)$ for every $x \in \mathbb{Z}^d$ and **integrable** in the sense that $\sum_{x\in\mathbb{Z}^d} J(x) < \infty$. For each $\beta \geq 0$, **long-range percolation** on $\mathbb{Z}^d$ with intensity $J$ is the random graph with vertex set $\mathbb{Z}^d$ in which we choose whether or not to include each potential edge $\{x, y\}$ independently at random with inclusion probability $1 - \exp(-\beta J(y - x)) \approx \beta J(y - x)$. We are most interested in the case that $J(x)$ decays like an inverse power of $\|x\|$, so that

$$J(x) \sim A\|x\|^{-d-\alpha} \qquad \text{as } x \to \infty \tag{1}$$

for some constants $A > 0$ and $\alpha > 0$. We denote the law of the resulting random graph by $\mathbf{P}_\beta = \mathbf{P}_{J,\beta}$ and refer to the connected components of this random graph as **clusters**. Given $d \geq 1$ and a symmetric, integrable function $J : \mathbb{Z}^d \to [0, \infty)$, we define the **critical parameter**

$$\beta_c = \sup\big\{\beta \geq 0 : \mathbf{P}_\beta \text{ is supported on configurations with no infinite clusters}\big\}.$$

Elementary path-counting arguments yield that there are no infinite clusters almost surely when $\beta \sum_x J(x) < 1$, and hence that $\beta_c \geq 1/\sum_x J(x) > 0$ under the assumption that $J$ is locally finite. When $d = 1$ and $J$ is of the form (1), the model has a non-trivial phase transition in the sense that $0 < \beta_c < \infty$ if and only if $\alpha \leq 1$, while for $d \geq 2$ the phase transition is non-trivial for every $\alpha > 0$ [8, 6]. As with nearest-neighbour percolation, the model is expected to exhibit many interesting fractal-like features when $\beta = \beta_c$ (see e.g. [5, 11, 10]), but proving this rigorously seems to be a very difficult problem in general.

It is a surprising fact that our understanding of long-range percolation models is better than our understanding of their nearest-neighbour counterparts in many situations. Indeed, it is a remarkable theorem of Noam Berger [4] that long-range percolation on $\mathbb{Z}^d$ undergoes a continuous phase transition in the sense that there is no infinite cluster at $\beta_c$ whenever $d \geq 1$ and $0 < \alpha < d$. The corresponding statement for nearest-neighbour percolation with $d \geq 2$ is of course a notorious open problem needing little further introduction. While it is widely believed that the phase transition should be continuous for all $\alpha > 0$ and $d \geq 2$, it is a theorem of Aizenman and Newman [7] that the model undergoes a *discontinuous* phase transition when $d = \alpha = 1$, so that the condition $\alpha < d$ cannot be removed from Berger's result in general.

Berger's proof works by showing that the set of $\beta$ for which an infinite cluster exists a.s. is open, and gives little quantitative control of percolation at the critical parameter $\beta_c$ itself. In this paper we give a new, quantitative proof of Berger's result that yields an explicit power-law upper bound on the tail of the volume of the cluster of the origin at criticality under the same assumptions.

In this talk, based on [1], I will describe a new, quantitative proof of Berger's theorem that yields power-law upper bounds on the distribution of critical clusters under the same conditions. The theorem is most interesting when $d < 6$ and $\alpha > d/3$, in which case the model is *not* expected to have mean-field behaviour and high-dimensional techniques such as the lace expansion [9, 11, 10] should not apply. Indeed, we believe that this theorem represents the first rigorous, non-trivial power-law upper bound for a critical Bernoulli percolation model that is neither two-dimensional nor expected to be described by mean-field critical exponents. The proof is based in part on techniques first developed to analyze percolation on certain infinite-dimensional Cayley graphs [2, 3].

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## Weight-dependent random connection models - Percolation and random walks

Peter Gracar
(joint work with Lukas Lüchtrath, Markus Heydenreich, Christian Mönch and Peter Mörters)

We investigate a large class of random graphs on the points of a Poisson process in $d$-dimensional space, $d \geq 1$, which combine scale-free degree distributions and long-range effects. Every Poisson point carries an independent random weight and given weight and position of the points we form an edge between two points independently with a probability depending on the two weights and the distance of the points.

More precisely, two vertices $\mathbf{x} = (x, s)$ and $\mathbf{y} = (y, t)$, where $x$ and $y$ are the locations of the two vertices and $t$ and $s$ are their $\mathrm{Unif}(0, 1)$ distributed marks, are connected by an edge with probability $\varphi(\mathbf{x}, \mathbf{y})$. Connections between different (unordered) pairs of vertices occur independently. We work with functions $\varphi$ of the form

$$\varphi\big((x,s),(y,t)\big) = \rho\big(g(s,t)|x-y|^d\big)$$

for a non-increasing, integrable *profile function* $\rho\colon \mathbb{R}_+ \to [0, 1]$ and a suitable *kernel* $g\colon (0, 1) \times (0, 1) \to \mathbb{R}_+$, which is non-decreasing in both arguments.

We define the two functions in terms of parameters $\gamma \in (0, 1)$ and $\beta \in (0, \infty)$. and $\delta \in (1, \infty)$. The parameter $\gamma$ determines the strength of the influence of the vertex weight on the connection probabilities, large $\gamma$ correspond to strong favouring of vertices with large weight. The edge density can be controlled by the parameter $\beta$, increasing $\beta$ increases the expected number of edges connected to a vertex at the origin. Finally, $\delta$ controls the spatial embedding.

We consider the following kernels:

- The *plain kernel* $g^{\mathrm{plain}}(s,t) = \frac{1}{\beta}$.
- The *sum kernel* $g^{\mathrm{sum}}(s,t) = \frac{1}{\beta}\big(s^{-\gamma/d} + t^{-\gamma/d}\big)^{-d}$ and the *min-kernel* $g^{\min}(s,t) = \frac{1}{\beta}(s \wedge t)^{\gamma}$.
- The *max-kernel* $g^{\max}(s,t) = \frac{1}{\beta}(s \vee t)^{1+\gamma}$,
- The *product kernel* $g^{\mathrm{prod}}(s,t) = \frac{1}{\beta}s^{\gamma}t^{\gamma}$,
- The *preferential attachment kernel* $g^{\mathrm{pa}}(s,t) = \frac{1}{\beta}(s \vee t)^{1-\gamma}(s \wedge t)^{\gamma}$,

Except for the first example, our models are scale-free with power-law exponent $\tau = 1 + \frac{1}{\gamma}$.

The listed kernels in combination with various choices of the profile function lead to many well established random graph models and generalisations thereof - for details see [2, Table 1].

## 1. Percolation phase transition

We fix $g$ to be the sum, min or preferential attachment kernel, as well as $\gamma$, $\beta$ and $\delta$. Let $p \in [0,1]$ and perform Bernoulli bond percolation with retention parameter $p$ on the graph, i.e., every edge remains intact independently with probability $p$, or is removed with probability $1-p$. We denote the graph we obtain by $\mathcal{G}^p$ and ask whether there exists an infinite cluster. If so, we say that the graph *percolates*. We define the *critical percolation parameter* $p_c$ as the infimum of all parameters $p \in [0,1]$ such that the percolation probability is positive. We then have the following result.

**Theorem 1** ([4]). *Suppose $\delta > 1$. Then, for the weight-dependent random connection model with preferential attachment kernel, sum kernel or min kernel and parameters $\beta > 0$, $0 < \gamma < 1$, we have that*

(a) *if $\gamma < \frac{\delta}{\delta+1}$, then $p_c > 0$.*
(b) *If $\gamma > \frac{\delta}{\delta+1}$, then $p_c = 0$.*

Our result [4] proves the conjecture of Jacob and Mörters [5] that the phase transition between the two phases ($p_c > 0$ and $p_c = 0$) occurs at $\gamma = \frac{\delta}{\delta+1}$. This phase transition matches the transition between the ultrasmall and non-ultrasmall regime within the same models, see [3] and the *Chemical distance in weight-dependent random connection models* extended abstract in these proceedings. In order to prove this sharp phase transition, we introduce a novel path counting strategy, relying on what we call *skeletons* of paths. For details, see [4].

As we will see momentarily, $\gamma = \frac{\delta}{\delta+1}$ is not the only parameter choice where graph properties change. At $\gamma = \frac{\delta-1}{\delta}$ the graphs experience a crossover in the behaviour of random walks on the graph, which we explore next.

## 2. Transience and recurrence

We first expand the kernels we study and consider again also the product and max kernels. Our main interest is whether the infinite cluster is recurrent (i.e., whether a simple random walk in the cluster returns to the starting point almost surely), or transient (i.e., simple random walk on the cluster has positive probability of never returning to the starting point).

**Theorem 2** ([2]). *Consider the weight-dependent random connection model with $\delta > 1$.*

(a) *For preferential attachment kernel, sum kernel, or min kernel, and $\beta > \beta_c$, the infinite component is*
   - transient *if either $1 < \delta < 2$ or $\gamma > \delta/(\delta+1)$;*
   - recurrent *in $d = 2$ if $\delta > 2$ and $\gamma < (\delta-1)/\delta$.*

(b) *For the product kernel and* $\beta > \beta_c$*, the infinite component is*
- transient *if either* $1 < \delta < 2$ *or* $\gamma > 1/2$*;*
- recurrent *in* $d = 2$ *if* $\delta > 2$ *and* $\gamma < 1/2$.

(c) *For the max kernel and* $\beta > \beta_c$*, the infinite component is* transient.

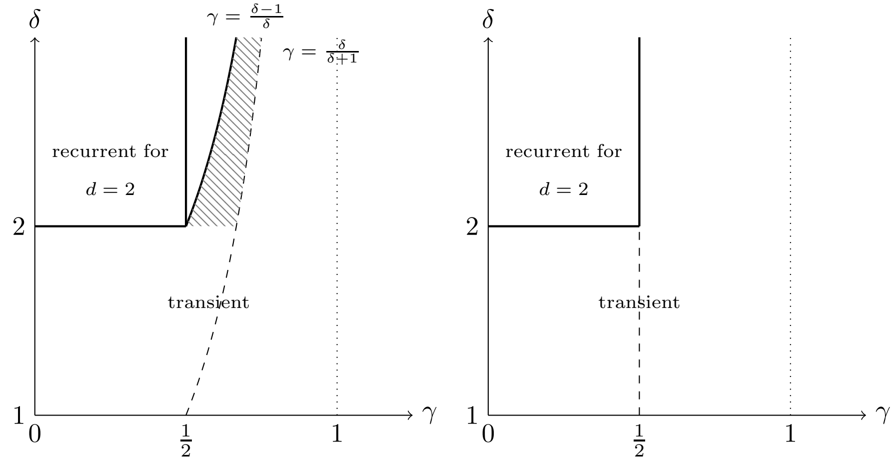


Figure 1. The left diagram represents preferential attachment, sum and min kernels. Surprisingly, in dimension two the transient regime includes an area (shaded) where $p_c > 0$, but the graphs remains transient. The right diagram represents the product kernel, where this effect does not happen.

In order to obtain the above result and expand the work of Berger [1] to inhomogeneous random graphs models, we build the graphs with the help of sprinkling. We first ensure that powerful vertices are well behaved, after which we use the remaining vertices in order to establish connections. This allows us to prove transience despite the spatial correlations not present in homogeneous models seen in related works.

## 3. Outlook

We are currently looking at expanding the results of the second theorem and filling out the gaps in the phase diagram above. Showing transience for $d \geq 3$ in the upper left corner of the diagram remains an open problem even in the simplest case of long-range percolation (i.e. the plain kernel). We believe a novel technique will have to be developed in order to precisely control the contribution of rare, but extraordinary long edges to the connectivity of the graph.

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## Competition processes on hyperbolic non-amenable graphs


Elisabetta Candellero

(joint work with Alexandre Stauffer)


We consider two first-passage percolation processes $\mathrm{FPP}_1$ and $\mathrm{FPP}_\lambda$, spreading with rates 1 and $\lambda > 0$ respectively, on a graph $G$ with bounded degree. $\mathrm{FPP}_1$ starts from a single source at the origin of $G$, while the initial configuration of $\mathrm{FPP}_\lambda$ consists of countably many seeds distributed according to a product of iid Bernoulli random variables of parameter $\mu > 0$ on $V(G) \setminus \{o\}$. Seeds start spreading $\mathrm{FPP}_\lambda$ after they are reached by either $\mathrm{FPP}_1$ or $\mathrm{FPP}_\lambda$.

This model is known as *First passage percolation in a hostile environment* (FPPHE), and was introduced in [4] as an auxiliary model for investigating a notoriously challenging model called Multiparticle Diffusion Limited Aggregation.

We consider several questions about this model, focusing on the case when $G$ is a non-amenable hyperbolic graph. We show that for any such graph $G$, and any fixed value of $\lambda > 0$ there is a value $\mu_0 = \mu_0(G, \lambda) > 0$ such that for all $0 < \mu < \mu_0$ the two processes coexist with positive probability (as proven in [1]). This shows a fundamental difference with the behavior of such process on $\mathbb{Z}^d$ (cf. [4] and the most recent results in [3]).

A natural question is whether by increasing $\mu$ one would harm the spread of $\mathrm{FPP}_1$. At first sight it seems reasonable to believe that there should be a coupling showing that this is true. However, in [2] we answer this question in the negative by constructing a non-amenable hyperbolic graph $G$ where there is no such a coupling. More precisely, we find two values $0 < \mu_1 < \mu_2 < 1$ such that the following occurs. When the initial density of seeds $\mu = \mu_1$, then $\mathrm{FPP}_1$ has a positive chance of occupying an infinite connected component of the graph, whereas when $\mu = \mu_2$, then $\mathrm{FPP}_1$ will occupy only an almost-surely finite region of $G$.


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## Random hyperbolic graphs

Dieter Mitsche
(joint work with Marcos Kiwi)

It has been empirically observed that complex networks such as social networks, scientific collaborator networks, computer networks and others (see [1]) are typically scale-free and exhibit a non-vanishing clustering coefficient. Furthermore, there is increasing evidence that the hidden underlying geometry of such networks is hyperbolic [4].

A model of complex networks that naturally exhibits clustering and scale-freeness is the Random Hyperbolic Graph model (RHG) introduced by [5]: in this model vertices of the network are points in a bounded region of the hyperbolic plane, and connections exist if their hyperbolic distance is small. More formally, the RHG model is defined as follows: fix $\alpha \geq \frac{1}{2}$ and $\nu \in \mathbb{R}^+$. For each $n \in \mathbb{N}$, consider a Poisson point process on the hyperbolic disk of radius $R := 2\log(n/\nu)$ and denote its point set by $V$ (the choice of $V$ is due to the fact that we will identify points of the Poisson process with vertices of the graph). The intensity function at polar coordinates $(r, \theta)$ for $0 \leq r < R$ and $0 \leq \theta < 2\pi$ is equal to

$$g(r, \theta) := \nu e^{\frac{R}{2}} f(r, \theta),$$

where $f(r, \theta)$ is the joint density function with $\theta$ chosen uniformly at random in the interval $[0, 2\pi)$ and independently of $r$, which is chosen according to the density function

$$f(r) := \begin{cases} \dfrac{\alpha \sinh(\alpha r)}{\cosh(\alpha R) - 1}, & \text{if } 0 \leq r < R, \\ 0, & \text{otherwise.} \end{cases}$$

Note that this choice of $f(r)$ corresponds to the uniform distribution inside a disk of radius $R$ around the origin in a hyperbolic plane of curvature $-\alpha^2$. Identify then the points of the Poisson process with vertices (that is, identify a point with polar coordinates $(r_v, \theta_v)$ with vertex $v \in V$) and make for each $n \in \mathbb{N}$ the following graph $G = G(n)$: for $u, u' \in V$, $u \neq u'$, there is an edge with endpoints $u$ and $u'$ provided the distance (in the hyperbolic plane) between $u$ and $u'$ is at most $R$, i.e., the hyperbolic distance between $u$ and $u'$, denoted by $d_H := d_H(u, u')$, is such that $d_H \leq R$, where $d_H$ is obtained by solving

$$\cosh d_H := \cosh r_u \cosh r_{u'} - \sinh r_u \sinh r_{u'} \cos(\theta_u - \theta_{u'}). \tag{1}$$

The restriction $\alpha \geq \frac{1}{2}$ and the role of $R$, informally speaking, guarantee that the resulting graph has bounded average degree (depending on $\alpha$ and $\nu$ only): if $\alpha < \frac{1}{2}$, then the degree sequence is so heavy tailed that this is impossible (the graph is with high probability connected in this case, as shown in [3]; in fact therein it is shown that for $\alpha = \frac{1}{2}$ the graph is with constant probability depending on $\nu$ connected as well). We also assume that $\alpha \leq 1$, since otherwise, if $\alpha > 1$, the largest component of a random hyperbolic graph is known to have sublinear order [2] with high probability (for $\alpha = 1$ again the existence of a giant component depends on $\nu$). For the range $\frac{1}{2} < \alpha < 1$ it is well known that with high probability there exists a giant component, but the graph is not connected, see [2].

## 1. Statements of results

We say that an event holds *asymptotically almost surely (a.a.s.)*, if it holds with probability tending to 1 as $n \to \infty$. Denoting by $L_2(G)$ the order of the second largest component of a graph $G$, we have:

**Theorem 1.** *Let $\frac{1}{2} < \alpha < 1$ and $\nu \in \mathbb{R}^+$. Let $G = G(n)$ be chosen according to the distribution $g(r, \theta)$. Then, a.a.s.,*

$$L_2(G) = \Theta(\log^{\frac{1}{1-\alpha}} n).$$

*Moreover, for some sufficiently small constant $b > 0$, there are $\Omega(n^b)$ components in $G$, each one of order $\Theta(\log^{\frac{1}{1-\alpha}} n)$.*

**Remark 1.** *For $\alpha = \frac{1}{2}$ and $\nu$ small enough, with constant probability, $L_2(G) = \Theta(\log n)$. For $\alpha = 1$ there exists $\gamma$, $0 < \gamma < 1$ such that a.a.s., $L_2(G) = \Omega(n^\gamma)$. Moreover, there exists some $0 < \delta < \gamma$ so that for some sufficiently small constant $b > 0$, a.a.s., there are $\Omega(n^b)$ components in $G$, each one of order $\Omega(n^\delta)$.*

Denote by $d(v)$ the degree of $v$ in a graph $G$. The *normalized Laplacian of $G$* is the (square) matrix $\mathcal{L}_G$ whose rows and columns are indexed by the vertex set of $G$ and whose $(u, v)$-entry takes the value

$$\mathcal{L}_G(u,v) := \begin{cases} 1, & \text{if } u = v, \\ -\dfrac{1}{\sqrt{d(u)d(v)}}, & \text{if } uv \text{ is an edge of } G, \\ 0, & \text{otherwise.} \end{cases}$$

It is well known that $\mathcal{L}_G$ is positive semidefinite, and that the multiplicity of the eigenvalue 0 is equal to the number of connected components of $G$. For a connected graph $G$, denote by $\lambda_1(G)$ the second smallest eigenvalue of $\mathcal{L}_G$. We have the following theorem for the giant component of a RHG:

**Theorem 2.** *Let $\frac{1}{2} < \alpha < 1$ and $\nu \in \mathbb{R}^+$. Let $G = G(n)$ be chosen according to the distribution $g(r, \theta)$, and let $H$ denote the giant component of $G$. Let $\omega$ be a function tending to infinity with $n$ arbitrarily slowly. Then, there exists a constant $c > 0$ so that a.a.s.,*

$$cn^{-(2\alpha-1)}/\log n \leq \lambda_1(H) \leq \omega n^{-(2\alpha-1)} \log n.$$

For a connected graph $G = (V, E)$, and a set $S \subseteq V$, define $\mathrm{Vol}(S) = \sum_{v \in S} d(v)$. The *conductance* of $S$ in $G$, $\emptyset \subsetneq S \subsetneq V$, is defined as

$$h(S) = \frac{|E(S, V \setminus S)|}{\min\{\mathrm{Vol}(S), \mathrm{Vol}(V \setminus S)\}}, \tag{2}$$

and the *conductance* of $G$ is $h(G) = \min h(S)$, where the minimum is taken over all sets $S \subseteq V$, with $S \neq \emptyset$, $S \neq V$. We can show that for small subsets of the giant component of a RHG the conductance (always only restricted to the connected graph given by the giant component) is not too small:

**Theorem 3.** *Let $\frac{1}{2} < \alpha < 1$ and $\nu \in \mathbb{R}^+$. Let $G = G(n)$ be chosen according to the distribution $g(r, \theta)$, and let $H$ denote the giant component of $G$. Let $0 < \varepsilon < 1$. A.a.s., for every set $S \subseteq V(H)$ with $Vol(S) = O(n^\varepsilon)$, we have $h(S) = \Omega(n^{-(2\alpha-1)\varepsilon+o(1)})$.*

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## Continuity of the time constant in a continuous model of first passage percolation

Marie Théret

(joint work with Jean-Baptiste Gouéré)

We consider a Boolean model $S$: informally, we throw homogeneously and independently balls with random radii in the space $\mathbb{R}^d$ and we define $S$ as their union. A particle travels at speed 1 outside $S$ and at infinite speed inside $S$, in such a way that it minimizes its travel time. We denote by $T(x, y)$ the minimal time needed by the particle to travel from a point $x$ to a point $y$. By classical subadditive arguments, it can be proved that $T(0, x)$ behaves like $\mu\|x\|$ when $\|x\|$ goes to infinity, where $\mu$ is called the time constant of the model. This constant $\mu$ depends on the parameters of the underlying Boolean model (the density of the centers of the balls, the distribution of the radii), and we study here the properties of $\mu$ as a function of those parameters.

First, we investigate when $\mu$ is strictly positive: it happens exactly when the underlying Boolean model is in the subcritical regime of percolation, in a strong sense (i.e., when the probability of crossing an annulus inside $S$ goes to 0 when the width of the annulus goes to infinity). We then prove the continuity of $\mu$ with respect to the density of the centers of the balls and to the distribution of the radii, under some domination assumption. The difficulty and the interest of the proofs rely on the control of the impact of balls with very large radii in $S$. Indeed, those balls induce correlations at a very large scale in the model. The greedy lattice paths, the continuous counterpart of the greedy lattice animals on $\mathbb{Z}^d$, play a key role to control the influence of large balls of $S$ on the pseudo-metric $T$, and thus on $\mu$.

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## Critical behaviour of the random connection model


Markus Heydenreich

(joint work with Remco van der Hofstad, Kilian Matzke, Günter Last)


**Percolation on high-dimensional lattices.** We consider bond percolation on the hypercubic lattice $\mathbb{Z}^d$ ($d \geq 2$), and are most interested in the critical case. In a seminal paper, Hara and Slade [4] proved in 1990 that critical percolation satisfies an infrared bound when dimension is large enough, say $d > \bar{d}$ for some $\bar{d} \geq 6$. A consequence of the infrared bound is the triangle condition

$$\sum_{x,y \in \mathbb{Z}^d} \mathbb{P}_{p_c}(0 \leftrightarrow x)\,\mathbb{P}_{p_c}(0 \leftrightarrow x)\,\mathbb{P}_{p_c}(0 \leftrightarrow x) < \infty, \tag{1}$$

which in turn implies that various critical exponents exist and take on their mean-field values [1, 2, 10, 11] (see also the overview in [9]). The method of proof is the lace expansion. While it was announced that $\bar{d} = 18$ suffices (cf. [5]), Fitzner and van der Hofstad [3] verified that a modified analysis of the lace expansion even allows $\bar{d} = 10$. An infrared bound for *site* percolation in high dimensions was presented only recently [8].

Already in the original paper [4], Hara and Slade proved that the triangle condition (1) is also satisfied for a spread-out version of the model above the upper critical dimension, that is, for $d > 6$. This has been extended to *long-range percolation* in [7], where the edge between vertices $x, y \in \mathbb{Z}^d$ is occupied with probability proportional to $|x-y|^{-d-\alpha}$, and the triangle condition is confirmed to hold for dimension $d > 3\min\{\alpha \wedge 2\}$ (under suitable technical assumptions).

**The random connection model.** The emphasis of the present talk is on continuum percolation. More precisely, we are considering the random connection model, which is defined as follows. We consider a random graph embedded into $\mathbb{R}^d$ whose vertices are given by a Poisson process $\eta$ with intensity $\lambda > 0$. For any pair of vertices $x, y \in \eta$ we insert an edge $\{x, y\}$ with probability $\varphi(|x-y|)$, independently from each other, where $\varphi\colon [0,\infty) \to [0,1]$ is a non-increasing function.

If $\varphi$ is an indicator function, then we get the Gilbert disk model as a special case. Under general assumptions on $\varphi$ and for $d \geq 2$, the model undergoes a percolation phase transition: for small intensity $\lambda$ there are only finite connected component, whereas for large $\lambda$ there is one infinite component. Again, we are focussing on the critical threshold

$$\lambda_c = \inf\{\lambda > 0 \mid \text{ an infinite component exists}\}.$$

Our main result, proven in [6], is an infrared bound on the probability that two (augmented) points are in the same connected component provided that the dimension is sufficiently large. To this end, we devise a lace expansion for Poisson processes. When establishing the expansion, we use Poisson thinning to separate the different parts of a cluster. As part of our proof we also establish a BK-inequality for the random connection model. As a consequence of the infrared bound, we get the triangle condition for the random connection model, similar to (1), and derive the mean-field value of the critical exponent characterising the divergence of the expected cluster size.

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## Conformal invariance of percolation on random planar maps

Nina Holden
(joint work with M. Albenque, O. Bernardi, C. Garban, E. Gwynne,
G.F. Lawler, X. Li, A. Sepúlveda, and X. Sun)

Conformal invariance of critical percolation on the triangular lattice was proved by Smirnov [Smi01]. His proof is hard to extend to critical percolation on other lattices since his proof relies on a combinatorial identity which is only true on the triangular lattice. This talk is about conformal invariance of percolation on certain *random* lattices known as random planar maps. One reason the problem is, in some sense, easier on random planar maps is the Markov property as seen in the peeling process.

A random planar map is a connected graph drawn on the sphere such that no two edges cross, viewed modulo continuous deformations. Uniformly sampled planar maps have been proved to converge for the Gromov-Hausdorff-Prokhorov topology to the so-called Brownian map, which is a continuum random metric measure space. The result also holds for maps with disk topology and maps with various local constraints (triangulations, quadrangulations, etc.). See e.g. [AHS19, BM17, Le 13, Mie13].

Gwynne and Miller [GM17a] proved that the percolation interface on uniform quadrangulations converge to the random fractal curve known as the Schramm-Loewner evolution with parameter $\kappa = 6$ ($\mathrm{SLE}_6$). By iterating the triangulation counterpart of their result, we extend the result to convergence of all percolation interfaces on a uniform triangulation towards the conformal loop ensemble with parameter $\kappa = 6$ ($\mathrm{CLE}_6$) [GHS19a]. The convergence results mentioned so far are annealed, and in [HS19] we upgrade to *quenched* convergence. The proof of the quenched convergence result relies on a mixing result for so-called Liouville dynamical percolation [GHSS19], which builds on techniques developed for dynamical percolation on the regular triangular lattice [GPS10, GPS13, GPS18a], along with a good understanding of the percolation pivotal points [HLLS18, HLS18, BHS18].

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## Bricolage of the 2D continuum Gaussian free field

Juhan Aru
(joint work with Titus Lupu, Avelio Sepúlveda)

The aim of this talk was to explain how an enthusiastic handyman can construct the 2D continuum Gaussian free field (GFF) from a big bag of Brownian loops.

More precisely, we aimed to explain how to make mathematical sense of an excursion decomposition of the 2D GFF $\Gamma$ using the Brownian loop soup. In other words, we explained how to decompose this random Schwartz distribution $\Gamma$ into positive and negative parts, given by critical Brownian loop soup clusters.

It might be a bit surprising that such a decomposition is even possible, given that we are dealing with a non-pointwise-defined function. Throughout the talk we wanted to emphasise the interplay between discrete results, continuum results and continuum limit results.

To state the results more precisely, let us recall some basic definitions. First the main player, the 2D continuum GFF can be defined as follows (see e.g. [9, 11]:

**Definition 1** (Gaussian free field)**.** *Let $D \subseteq \mathbb{C}$ denote a simply-connected domain. Then the 2-dimensional zero boundary continuum GFF in $D$ is the centred Gaussian process $(\Gamma, f)_{f\in C_c^\infty(D))}$ whose covariance is given by*

$$\mathbb{E}((\Gamma, f)(\Gamma, g)) = \int\int_{D^2} f(z)G^D(z,w)g(w),$$

*where $G^D$ denotes the zero boundary Green's function of the Laplacian in $D$.*

Second, we recall the Brownian loop soup, introduced in [5]. In a domain $D \subseteq \mathbb{C}$ we consider the following Brownian measure on loops

$$\mu^D_{\text{loop}}(\cdot) = \int_D \int_0^{+\infty} \mathbb{P}^{z,z}_t(\cdot, T_{\partial D} > t)\frac{1}{2\pi t}\frac{dt}{t}dz,$$

where $\mathbb{P}^{z,w}_t$ denote the bridge probability measures corresponding to the Brownian motion $(B_t)_{t\geq 0}$ on $\mathbb{C}$, $dz$ denotes the Lebesgue measure on $\mathbb{C}$ and $T_{\partial_D}$ the exit time from $D$.

The Brownian loop soup of intensity $\alpha > 0$ is then the Poisson point process with intensity measure $\alpha\mu^D_{loop}$. It comes out [10] that $\alpha > 0$ plays the role of a percolation parameter: if one considers the graph, whose vertices are the loops and two loops are adjacent iff they intersect, then for $\alpha > 1/2$ this graph is fully connected and for $\alpha \leq 1/2$ the graph consists of countably many non-intersecting connected components. We call the union of loops in such a connected component a cluster and any cluster that is not surrounded by another cluster an outermost cluster. In the same paper it is shown that the outer boundaries of such clusters in the subcritical regime are given by closed loops.

We can now state our theorem, but some further comments are due straight thereafter:

**Theorem 3** (A., Lupu, Sepúlveda)**.** *In a domain $D$, consider the 2D Brownian loop soup at the critical intensity $\alpha = 1/2$. Now,*

- *form the outermost loop soup clusters and order them as $C_1, C_2, C_3 \dots$*
- *for each cluster $C_i$ we consider its Minkowski content measure $\nu_i$ (a certain positive measure on $D$ described below) and an independent $\pm 1$ valued $Ber(1/2)$ random variable $s_i$.*

*Then for any $f \in C^\infty_c(D)$, we have that*

$$\Gamma_n(f) := \sum_{i=1}^n s_i\nu_i(f)$$

*converge almost surely and have the law of $\Gamma(f)$, and in particular the so defined fields $\Gamma_n$ converge almost surely in the space of distribution to a random field $\Gamma$ with the law of a 2D GFF in $D$.*

*Moreover, the collection $(s_i, \nu_i)_{i\geq 1}$ can be recovered from $\Gamma$ in a measurable way.*

Some remarks are to be made, to answer the questions that were immediately asked during the talk:

- Without the independent signs, the positive measures $\sum_{i=1}^n \nu_i$ would diverge on every open set.
- The Minkowski content measure $\nu_i$ on a cluster $C_i$ can be seen as a sign excursion of the GFF, and is more precisely determined by setting for every $f \in C^\infty_c(D)$

$$\nu_i(f) = \frac{1}{2}\lim_{\epsilon\to 0}\sqrt{|\log \epsilon|}\int_D f(z)1_{d(z,C_i)\leq\epsilon}dz.$$

We do not know if $\nu_i$ is a (multiple of) the equilibrium measure of $C_i$, or its Hausdorff measure (in a correct gauge).
- The sign excursions are indeed deterministic w.r.t. their support! This is very different from the excursion decomposition for the Brownian motion.
- The outermost cluster boundaries have the law of $\mathrm{CLE}_4$ - this basically comes from [10].
- The sign clusters $C_i$ themselves, conditioned on their outer boundary, have the law of so called first passage sets of the 2D GFF with a certain parameter - see [1, 2].

In the talk, we started from explaining that a part of this theorem – the fact that there is some writing of the 2D continuum GFF as a sum of sign excursions – can be probably rather easily derived by taking subsequential limits of the excursion decomposition of the metric graph GFF on lattices with finer and finer mesh size. The missing parts would be:

- The uniqueness of such a limit, and relatedly the fact that this decomposition is actually measureable with respect to the underlying GFF, i.e. that the sign excursions are intrinsically defined for the 2D continuum GFF.
- The identification of the sign excursions as Minkowski content measures.

Both of these results can be drawn as rather simple consequences of previous work on first passage sets of the 2D GFF by the same authors [1, 2], at least in simply-connected domains. More precisely, to prove the first point, we can use the identification of the sign clusters as first passage sets and use the convergence results proved for those in [2]. To prove the latter, we use description of the restriction of the GFF using the Minkowski content measure of first passage sets with, proved in [1]. The case of non-simply connected domains requires more work, and borrows results also from [3].

Nicely, the results on first passage sets are themselves based on previous results from the discrete world, the continuum world and the passage from one to the other. For example,

- The relation between Brownian loop soup clusters and first passage sets of the 2D GFF comes from the discrete world [5].
- The ideas to define geometric subsets like first passage sets of the 2D Gaussian free field come from studies in the continuum [8, 4].
- The limiting arguments for first passage sets are for example using convergence results of random walk loop soups [6].

Thus, to get a nice excursion decomposition of the 2D GFF, there is an interesting interplay between discrete results, continuum results and continuum limit results. Interesting open questions are for example,

- What happens in higher dimensions?
- Can one do something similar to other continuum fields in 2D, e.g. the limit of the XOR-Ising field given by imaginary multiplicative chaos?

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# Rotational invariance of the critical planar FK-percolation model

Ioan Manolescu

(joint work with Hugo Duminil-Copin, Karol Kajetan Kozlowski, Dmitry Krachun, Mendes Oulamara)

We prove the asymptotic rotational invariance of the critical FK-percolation model $\phi_{p,q}$ on the square lattice $\mathbb{Z}^2$ with any cluster-weight $q \in [1,4]$. These models are expected to exhibit conformally invariant scaling limits that depend on $q$, thus covering a continuum of universality classes. The rotation invariance of the scaling limit is a strong indication of the wider conformal invariance, and may indeed serve as a stepping stone to the latter. Specific cases include Bernoulli percolation ($q = 1$), the FK-Ising model ($q = 2$, for which stronger results are known [5, 1]) and have implications for the $q$-state Potts model with $q = 3, 4$. Note that for $q > 4$, FK-percolation exhibits a discontinuous phase transition, which renders the scaling limit trivial.

Our result is based on the following universality theorem for FK-percolation on rectangular isoradial lattices. For $\alpha \in (0, \pi)$, we consider a particular embedding $\mathbb{L}(\alpha)$ of the square lattice with an inhomogeneous FK-percolation model $\phi_{\mathbb{L}(\alpha)}$ associated to it; for $\alpha = \pi/2$ we retrieve the regular embedding of $\mathbb{Z}^2$ and the homogeneous critical model. When rescaled, all these models are shown to be asymptotically identical in the sense of the Schramm-Smirnov and Camia-Newman topologies. Since $\mathbb{L}(\alpha)$ and $\phi_{\mathbb{L}(\alpha)}$ are invariant under the symmetry with respect to the line $e^{i\alpha/2}\mathbb{R}$, we conclude that $\phi_{\mathbb{L}(\pi/2)}$ is asymptotically invariant under all orthogonal symmetries, and thus under rotations.

The universality among rectangular isoradial lattices is proved via the star-triangle transformation. Indeed, this transformation may be used to gradually change $\mathbb{L}(\pi/2)$ into any $\mathbb{L}(\alpha)$, while preserving certain features of the model. It was proved in [4] that throughout this transformation, the large scale geometry of the model is distorted by at most a limited amount; in the present work we prove that the distortion becomes insignificant as the scale increases. This is the core of our argument and relies on a computation on the eigenvalues of the transfer matrix of the six-vertex model performed via the Bethe ansatz in [3].

This talk is based on [2].

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## Instability properties in planar percolation models

Hugo Vanneuville

(joint work with Christophe Garban, Stephen Muirhead, Alejandro Rivera, Vincent Tassion)

In this talk, we are interested in instability properties in planar percolation models constructed by using Gaussian fields. Our main goal is to explain how covariance formulas can help to study these properties. Let $\mathcal{V}$ denote the set of sites of the triangular lattice and let $(X_v)_{v\in\mathcal{V}}$ be a centred Gaussian field whose covariance depends on the distance between points: $\mathbb{E}X_vX_w = \kappa(|v-w|)$ for some function $\kappa : \mathbb{R}_+ \to \mathbb{R}$. Given a level $\ell \in \mathbb{R}$, we consider the following coloring of $\mathcal{V}$: $v$ is colored in black (resp. white) if $X_v < \ell$ (resp. $X_v > \ell$), and we study the monochromatic components.

If $(X_v)_{v\in\mathcal{V}}$ are i.i.d. $\mathcal{N}(0,1)$, the model is Bernoulli percolation (and one can note that $\ell = 0$ corresponds to $p = 1/2$). In this talk, we also consider the case $0 \leq \kappa(|v-w|) \asymp |v-w|^{-\alpha}$ for some exponent $\alpha > 0$.

Let us define the two instability properties that we are interested in. To this purpose, let us concentrate on a specific event: let $A_n$ be the event that an $n\times n$ rhombus (drawn on the triangular lattice) is crossed by a black path from left to right. We first note that, by symmetry, we have $\mathbb{P}_{\ell=0}[A_n] = 1/2$ for every $n$ and any choice of $\kappa$.

- We say that the model satisfies a **sharp threshold** if $\mathbb{P}_\ell[A_n] \to 0$ if $\ell < 0$ and $\mathbb{P}_\ell[A_n] \to 1$ if $\ell > 0$.

- Let $\tilde{X}$ be an independent copy of $X$ and let $X_t = \sqrt{1-t^2}X + t\tilde{X}$ for every $t \in [0,1]$. We say that the model is **noise sensitive** if there exists $t_n \to 0$ such that $\mathrm{Cov}_{\ell=0}(X \in A_n, X_{t_n} \in A_n) \to 0$.

We have the following theorems.

**Theorem 1.** *Sharp threshold holds for Bernoulli percolation* [3]. *This holds more generally if $\kappa \to 0$ at infinity* [4].

**Theorem 2.** *Bernoulli percolation is noise sensitive* [1]. *This holds more generally if $0 \leq \kappa(|v-w|) \ll |v-w|^{-2}$ (and under some other strong condition on $\kappa$)* [2].

The goal of this talk is to explain how covariance formulas can help to i) study sharp threshold and noise sensitivity and ii) see these phenomenons as instability properties. In particular, the covariance formulas (see below) enabled us [5] to prove noise sensitivity properties with a strategy that does not rely on any spectral tool, contrary to the previous works about this phenomenon. Spectral tools are very rich but have not been developed for other models of statistical mechanics, hence the importance to find other strategies.

Let us now introduce some objects that will enable us to state two covariance formulas. We let $T_n$ denote the threshold map:

$$T_n := \inf\{\ell : A_n \text{ holds at level } \ell\}.$$

Under some weak non-degenerescence properties on $\kappa$, there exists a unique vertex $v$ such that $X_v = T_n$. We call this vertex the saddle point and denote it by $S_n$.

When we fix a level (e.g. $\ell = 0$), we define the pivotal set $\mathrm{Piv}_n$ as follows: $v \in \mathrm{Piv}_n$ if changing the color of $v$ changes the outcome of $A_n$.

In the case of Bernoulli percolation (i.e. $(X_v)_{v\in\mathcal{V}}$ i.i.d.), we have the two following covariance formulas:

$$\mathrm{Var}(T_n) = \int_0^1 t\mathbb{P}[S_n = S_n^t]dt;$$

$$\mathrm{Cov}_{\ell=0}(X \in A_n, X_t \in A_n) = \int_t^1 \mathbb{E}_{\ell=0}[|\mathrm{Piv}_n \cap \mathrm{Piv}_n^s|]ds.$$

These formulas (and some monotonicity properties: one can show that $\mathbb{P}[S_n = S_n^t]$ and $\mathbb{E}_{\ell=0}[|\mathrm{Piv}_n \cap \mathrm{Piv}_n^t|]$ are non-increasing in $t$) imply that sharp threshold and noise sensitivity are respectively equivalent to the following instability properties:

- There exists $t_n \to 0$ such that $\mathbb{P}[S_n = S_n^{t_n}] \to 0$.
- There exists $t_n \to 0$ such that $\mathbb{E}_{\ell=0}[|\mathrm{Piv}_n \cap \mathrm{Piv}_n^{t_n}|] \to 0$.

Thus, sharp threshold and noise sensitivity are respectively equivalent to an instability property of the saddle point and of the pivotal set, for a small noise $t_n \to 0$. However, in practice, one can use the two following properties at $t = 1$, rather than for some small $t$. The main tool in the proof of these properties is hypercontractivity. These theorems hold for Bernoulli percolation and for increasing events.

**Theorem 3** ([4]). *There is sharp threshold if and only if $\mathbb{P}[S_n = S_n^1] \to 0$ (which is equivalent to $\max_v \mathbb{P}[S_n = v] \to 0$).*

**Theorem 4** ([1]). *The model is noise sensitive if and only if* $\mathbb{E}_{\ell=0}[|\mathrm{Piv}_n \cap \mathrm{Piv}_n^1|] = \sum_v \mathbb{P}_{\ell=0}[v \in \mathrm{Piv}_n]^2 \to 0$.

The above analysis implies that noise sensitivity is equivalent to an **instability property** of the pivotal set and that the study for $t = 1$ already tells us a lot about noise sensitivity, although this is a property about the behaviour of the system for a small noise $t$. However, with Vincent Tassion [5], we have developed a strategy in order to prove (quantitative) noise sensitivity by using a **stability** property of the model when $t$ is (very) close to 0. This is one of the subjects of the talk. Another subject is the use of covariance formulas for non-i.i.d. Gaussian vectors in order to prove estimates (e.g. lower bounds) on $\mathrm{Var}(T_n)$.

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## Crossing probabilities for planar percolation


Laurin Köhler-Schindler

(joint work with Vincent Tassion)


**Framework.** The study of crossing probabilities for planar percolation has been initiated by Russo [9, 10] and independently by Seymour and Welsh [12] for Bernoulli percolation in 1978. Russo-Seymour-Welsh (RSW) theory has quickly become a fundamental tool to study the critical and near-critical regime. For example, it plays an important role in Kesten's proof [7] that $p_c = 1/2$ for bond percolation on the square lattice, in Smirnov's proof [11] of conformal invariance for critical site percolation on the triangular lattice, in the study of arm exponents, as well as in establishing scaling relations.

Consequently, RSW theory was extended to other planar percolation models, leading to the resolution of important conjectures. In the spirit of the original proof for Bernoulli percolation, exploration arguments have been developed for positively associated models satisfying a spatial Markov property such as FK-percolation with cluster-weight $q \geq 1$ [1, 3, 4, 6], where an additional difficulty arises due to the effect of boundary conditions. An orthogonal and far-reaching approach has been developed by Bollobás and Riordan [2] who used a renormalization strategy to prove a weaker version of RSW for Voronoi percolation and to deduce $p_c = 1/2$.

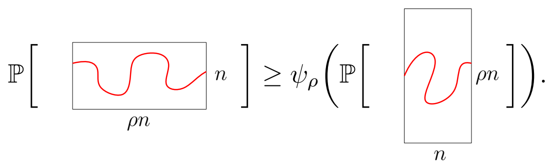
Inspired by the new approach, a RSW result for positively associated models satisfying a quasi-independence assumption has been proven in [13] Without a quasi-independence assumption, a weaker version of RSW providing bounds along a subsequence of scales has also been obtained in [13].

**Main result.** In this talk, we consider a bond percolation measure $\mathbb{P}$ on the square lattice $\mathbb{Z}^2$ satisfying the following two properties:

- $\mathbb{P}$ is invariant under the symmetries of $\mathbb{Z}^2$ (translation, rotation, reflection);
- $\mathbb{P}$ is positively associated ($\mathbb{P}[\mathcal{E} \cap \mathcal{F}] \geq \mathbb{P}[\mathcal{E}] \cdot \mathbb{P}[\mathcal{F}]$ for increasing events $\mathcal{E}, \mathcal{F}$).

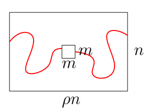
We prove the following lower bound for crossing probabilities of rectangles in the long direction in terms of the short direction:

**Theorem 1** ([8]). *For every $\rho \geq 1$, there exists a homeomorphism $\psi_\rho : [0,1] \to [0,1]$ such that for every invariant, positively associated measure $\mathbb{P}$ and for all $n \geq 1$,*




The statement of the theorem was conjectured in [5]. We refer to that paper for a more exhaustive background and motivations about RSW-type results.

**Idea of proof.** The main difficulty in proving RSW results lies in the fact that crossings might "typically" look like fractal curves. Contrary to previous approaches, we do not try to rule out this behaviour, but rather use it in our favour. To this end, we formalize the fact that crossings at scale $n$ look fractal and reach down to scale $m$. This is achieved by introducing *quasi-crossing* events:




Using geometric constructions, we establish new relations between the probabilities of quasi-crossings and crossings at different scales, which impose that the universal relation between short and long crossing probabilities must hold. Our renormalization approach is inspired by [2, 13].

**Comments.**

(1) Box-crossing-property: For self-dual models such as critical Bernoulli percolation, square crossings at any scale occur with probability 1/2. As an immediate corollary, one obtains lower and upper bounds on the crossing probabilities of rectangles with any aspect ratio that uniformly hold at all scales.

(2) Unification: Most of the previously known approaches involve positive association and symmetry together with an additional assumption that is specific to the considered model. In this sense, the theorem above unifies the previous results, and we hope that it will serve as a general tool to study critical and near-critical percolation processes in the future.
(3) Robustness: While we have stated the theorem for bond percolation on the square lattice for presentational purposes, our proof applies to more general lattices (with sufficient symmetries) and more general processes (including site percolation, continuum models, or level lines of random fields).
(4) Finite-volume extensions: For several applications, the framework of an invariant measure on the whole lattice is too restrictive. In particular, in the study of models from statistical mechanics, it is more natural to work with a measure defined in a finite box. Our methods can be extended to this finite-volume framework, where one needs a replacement for the hypothesis of translation invariance.
(5) Minimal hypotheses: The theorem is no longer valid if one removes the symmetry assumption or the positive association assumption.

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## On the radius of Gaussian free field excursion clusters

Franco Severo
(joint work with Subhajit Goswami, Pierre-François Rodriguez)

Let $\varphi$ be the Gaussian free field (GFF) on $\mathbb{Z}^d$, $d \geq 3$. For a height parameter $h$, consider the excursion sets $\{\varphi \geq h\}$. As $h$ varies this defines a natural strongly correlated percolation model, which was first investigated in the 80's by Bricmont, Lebowitz and Maes [1]. This model has received a lot of attention in the last decade. In particular, the existence of phase transition has been proved in all dimensions [11], many properties of the infinite cluster have been studied [3, 10, 12] and large deviation results for percolation events have been investigated [13, 7, 6, 2, 14]. More recently, sharpness of the phase transition was established [4].

As a direct consequence of the sharpness result mentioned above, for any level $h$ different from percolation critical parameter (denoted by $h_*$), the distribution of the radius of a finite cluster decays at least stretched exponential fast. More precisely, for every $h \neq h_*$, there exists $c, \beta > 0$ such that

$$\mathbb{P}[0 \overset{\varphi \geq h}{\longleftrightarrow} \partial B_N,\, 0 \overset{\varphi \geq h}{\not\longleftrightarrow} \infty] \leq e^{-cN^{\beta}}. \tag{1}$$

We are interested in optimal bounds for the probability in (1). In the non-percolative phase (i.e. $h > h_*$), exponential decay (i.e. (1) with $\beta = 1$) was proved for $d \geq 4$ in [9, 8] along with (sub-optimal) logarithmic corrections for $d = 3$. We prove optimal bounds (to principal exponential order) for $d = 3$ in both off-critical phases (i.e. $h \neq h_*$), whereas for $d \geq 4$ we show that exponential decay also holds in the percolative phase (i.e. $h < h_*$).

**Theorem 1** ([5])**.** *If $d = 3$, then for every $h \neq h_*$ one has*

$$\lim_{N\to\infty} \frac{\log N}{N} \log \mathbb{P}[0 \overset{\varphi \geq h}{\longleftrightarrow} \partial B_N,\, 0 \overset{\varphi \geq h}{\not\longleftrightarrow} \infty] = -\frac{\pi}{6}(h - h_*)^2. \tag{2}$$

*If $d \geq 4$, then for every $h \neq h_*$, there exist $C = C(d, h)$, $c = c(d, h) > 0$ such that*

$$e^{-CN} \leq \mathbb{P}[0 \overset{\varphi \geq h}{\longleftrightarrow} \partial B_N,\, 0 \overset{\varphi \geq h}{\not\longleftrightarrow} \infty] \leq e^{-cN}. \tag{3}$$

Our proof is based on a coarse-graining argument that expresses the probability in question as the sum two terms. One of them corresponds to a truncated version of $\varphi$ (a local field, independent at large scales), for which a corresponding one-arm event decays exponentially in $N$, regardless of the dimension $d$. The other term, which carries the long-range dependence, stems from the behavior of the harmonic field in a collection of well-separated boxes, and turns out to behave in a manner proportional to $\mathrm{cap}\big(([0, \dots, N] \cap \mathbb{Z}) \times \{0\}^{d-1}\big)$ to leading exponential order, where $\mathrm{cap}(K)$ stands for the random walk capacity of $K \subset \mathbb{Z}^d$. Since $\mathrm{cap}\big(([0, \dots, N] \cap \mathbb{Z}) \times \{0\}^{d-1}\big)$ is of order $\frac{N}{\log N}$ for $d = 3$ and of order $N$ for $d \geq 4$, the harmonic term clearly dominates in dimension 3, whereas the two terms live at the same exponential scale in dimension four and higher. This explains the discrepancy between the cases $d = 3$ and $d \geq 4$ in Theorem 1. The techniques used in the proof are inspired by the previous works on large deviation for GFF percolation mentioned above.

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## Wilson loop expectations in Abelian lattice gauge theories, with and without an Higgs field

Malin Palö Forsström
(joint work with Jonatan Lenells and Fredrik Viklund)

Lattice gauge theories have since their introduction been used to predict properties of elementary particles, as approximations of Yang-Mills gauge theory. However, many of these predictions have no rigorous mathematical backing. Moreover, it is not known how to obtain a continuum limit as the lattice spacing tends to zero even in the simplest cases.

To define the model, let $B_N = [-N,N]^4 \cap \mathbb{Z}^4$, and let $E_N$ and $P_N$ be the set of oriented edges and oriented plaquettes inside $B_N$ respectively. Let $G$ be a finite group, and let $\rho$ be a unitary and faithful representation of $G$. Finally, let $\Sigma_{E_N}$

denote the set of $G$-valued 1-forms on $E_N$. Given $\beta \geq 0$, the Gibbs measure $\mu_{N,\beta}$ on $\Sigma_{E_N}$ is then defined by

$$\mu_{N,\beta}(\sigma) := \frac{\exp\bigl(\beta \sum_{p \in P_N} \operatorname{tr} \rho((d\sigma)_p)\bigr)}{\sum_{\sigma' \in \Sigma_{E_N}} \exp\bigl(\beta \sum_{p \in P_N} \operatorname{tr} \rho((d\sigma')_p)\bigr)}, \quad \sigma \in \Sigma_{E_N}.$$

Alternatively, this measure can be seen as a measure on the set $\Sigma_{P_N}$ of $G$-valued closed 2-forms on $P_N$. When the group $G$ is Abelian, the corresponding lattice model is often referred to as *Abelian lattice gauge theory* with structure group $G$.

A natural observable in this model are so-called Wilson loops $W_\gamma$, which given a simple oriented loop $\gamma$ in $E_N$ is defined as the sum of the group elements assigned to its constituting edges. The behavior of the expected value $\mu_{N,\beta}(\operatorname{tr} \rho(W_\gamma))$ in terms of $\gamma$ and $\beta$ is believed to predict various properties of the standard model, thus giving the Wilson loops physical relevance. The main result we present during this talk is the following theorem, which describes these observables' first-order behavior.

**Theorem 1** (Theorem 1.1 in [3], see also Theorem 1.2.1 in [1] for general finite groups, and Theorem 1.1 in [2] for $G = \mathbb{Z}_2$)**.** *Let $G$ be a finite Abelian group, and let $\rho$ be a unitary and faithful $d$-dimensional representation of $G$. Consider lattice gauge theory on $B_N$ with structure group $G$ and representation $\rho$. Then there are $\beta_0 > 0$ and constants $C'$ and $C''$, such that the following holds when $\beta \geq \beta_0$. Let $\gamma$ be a simple oriented loop in $\mathbb{Z}^4$ of length $\ell$, and let $\ell_0$ be the number of corner edges in $\gamma$. Then*

$$\Bigl| \lim_{N\to\infty} \mu_{N,\beta}(W_\gamma) - \sum_{i=1}^{d} e^{-\ell(1-\theta_i(\beta))} \Bigr| \leq C' \Bigl[ \sqrt{\frac{\ell_0}{\ell}} + \lambda(\beta) \Bigr]^{C''}. \tag{1}$$

*where $\theta_i(\beta)$ denotes the $i$th eigenvalue of the matrix*

$$\Theta(\beta) = \frac{\sum_{g \in G} \rho(g) e^{12\beta \Re \operatorname{tr} \rho(g)}}{\sum_{g \in G} e^{12\beta \Re \operatorname{tr} \rho(g)}} \tag{2}$$

*and*

$$\lambda(\beta) := \max_{g \in G \setminus \{0\}} \frac{e^{-12\beta \Re \operatorname{tr} \rho(g)}}{e^{-12\beta \Re \operatorname{tr} \rho(0)}}.$$

In essence, this result says that if $\ell \min_{i \in [d]}(1-\theta_i) \gg 1$, then $\langle W_\gamma \rangle_\beta \approx 0$, if $\ell \max_{i \in [d]}(1-\theta_i) \ll 1$, then $\langle W_\gamma \rangle_\beta \approx d$, and otherwise $\langle W_\gamma \rangle_\beta$ has a non-trivial behavior.

The main idea in the proof is to analyze the behavior of so-called vortices, which are non-trivial, irreducible and closed 2-forms $\nu$ which agree with $d\sigma$ on their support. An essential tool in the proof of Theorem 1 is a Peierls argument, which shows that the probability of seeing a large vortex with support in a given plaquette is small. In constrast, the analog result in [1] uses cluster expansions, while the analog result in [2] uses duality and Dobrushin's criterion.

In the talk, we also present the corresponding theory in the presence of a Higgs field and describe how this affects our results.

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## A new point of view of the BKT-transition.

Avelio Sepúlveda
(joint work with Christophe Garban)

The aim of this talk was to explain a new point of view on the Berezinskii-Kosterlitz-Thouless transition (BKT transition). We give a new interpretation of this phenomenon as a possibility or not of the reconstruction of a GFF $\phi$ at inverse-temperature $\beta$ out of its fractional part [4].

**Some context.** BKT or topological phase transitions are a type of transition discovered by Berenziskii, Kosterlitz and Thouless. Models that undergo this phenomenon are typically 2-dimensional and do not have a classical phase transition. We call them topological because at low temperatures the topological defects (vortices) of the model are local, however at large temperature these topological defects appear everywhere. The KT phase transition is in duality with the problem of localisation of integer-valued fields. This duality was famously used by Fröhlich and Spencer to formally prove the first case of topological phase transition in [2]. To do this, they show that the GFF conditioned to have integer values has fluctuations of order 1 at low temperatures, however it fluctuates as a constant times the GFF at large temperatures.

A main example of BKT-transition is given by the Villain model in $\mathbb{Z}^2$. For a given finite graph $\lambda$ this models gives to a function $\theta : \Lambda \mapsto [-\pi, \pi)$ probability density proportional to

$$\mathbb{P}_\beta^{Vil}(d\theta) \propto \prod_{u\sim v} \sum_{m\in\mathbb{Z}} \exp\left(-\frac{\beta}{2}(\theta(u)-\theta(v)+2\pi m)^2\right) d\theta.$$

The BKT-transition can be seen from studying the model on the graph $\Lambda_n := [-n, n]^2 \cap \mathbb{Z}^2$ with boundary condition 0:

- At high temperature ($\beta \ll 1$) one has that as $n \to \infty$
$$\mathbb{E}_\beta^{Vil}[\cos(\theta)] \sim e^{-Cn}.$$
In other words, the model behaves like a supercritical model.
- At low temperature ($\beta \gg 1$) one has that as $n \to \infty$
$$\mathbb{E}_\beta^{Vil}[\cos(\theta)] \sim n^{-\alpha}.$$

In other words, the model behaves like a supercritical model. Furthermore, Frölich and Spencer conjectured in [3] that the scaling limit of $e^{i\theta}$ should behave as the imaginary exponential of the continuum Gaussian free field (GFF).

**Question of interest.** Let $\phi$ be a discrete Gaussian free field in the graph $\Lambda_n$. That is to say the GFF $\phi : \Lambda_n \mapsto \mathbb{R}$ is the random function taking value 0 in the boundary of $\Lambda_n$ with density

$$\mathbb{P}^{GFF}(d\phi) \propto \exp\left(-\frac{1}{2}\sum_{u\sim v}(\phi(u)-\phi(v))^2\right) d\phi.$$

Motivated by the behaviour of the Villain model at low temperature we ask the following question:

*Is it possible to recover the information of $\phi$ by just looking at $\exp(iT\phi)$?*

Let us note that it is of course it is not possible to recover all the information of $\phi$ be only looking at $\exp(iT\phi)$, one will always lose some microscopic information. However, we are only interested in the recovery of macroscopic information. For simplicity, we will say that we can recover the macroscopic information of $\phi$ from the information of $\exp(iT\phi)$ if for any $u \in \Lambda$ there exists a deterministic function $F_u$ such that

$$\mathbb{E}^{GFF}\left[(F_u(\exp(iT\phi)) - \phi(u))^2\right] = O(1).$$

The main result I will present is the following.

**Theorem 1.** *There exists $T_c^+ > T_c^-$ such that*

(1) *For all $T < T_c^-$, there is no way to recover the macroscopic information of $\phi$ as a function of $\exp(iT\phi)$.*
(2) *For all $T > T_c^+$, one can recover the macroscopic information of $\phi$ from $\exp(iT\phi)$.*

Furthermore, we know, using the recent result of Aru and Junnila [1], that $T_c^- > 2\sqrt{2\pi}$. We also discussed conjectures on the value of $T_c^-$ and $T_c^+$ in the talk.

**Ideas of the proof.** The main idea of the proof is to study the conditional law of $\phi$ given $\exp(iT\phi)$. We note that this law can be thought of that of an integer-valued GFF in an inhomogeneous medium.

To prove the first part of the theorem, we generalise the delocalisation results obtained by Fröhlich and Spencer [2] to the integer-valued GFF. The main difficulty of this is to overcome the lack of symmetry of the quenched model.

To show that when $T$ is small enough one can, in fact, recover the GFF we show that conditional measure concentrates around its mean, and thus the function $F_u(e^{iT\phi}) := \mathbb{E}^{GFF}\left[\phi(u) \mid e^{iT\phi}\right]$ is a good recovery function. To do this, we sample $e^{iT\phi}$ and conditionally on that we sample two functions $\phi_1$ and $\phi_2$ that are conditionally independent given $e^{iT\phi}$ and that have the law of $\phi$ conditionally on $e^{iT\phi}$. The problem is now simplified to study the difference between $\phi_1$ and $\phi_2$.

The difference $\phi_1 - \phi_2$ is study using percolation techniques. In fact, we study the percolation properties of the set $\phi_1 = \phi_2$ and we show that when $T$ is low enough this set percolates. A result like this is not true for all possible values of $e^{iT\phi}$, however we circumvent this problem by using an *annealead Peierl's argument*. For this result, we use that both $\phi_1$ and $\phi_2$ have the law of a GFF.

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## Critical exponents for a percolation model on transient graphs

Alexis Prévost

(joint work with Alexander Drewitz, Pierre-François Rodriguez)

Computing the exact values of the critical exponents for a percolation model has proved to be a very challenging problem, especially in transient dimensions below the critical dimension, since it requires a deep understanding of the critical and near-critical phase. We present a specific percolation model, for which the rigidity induced by its strong long-range correlation let us derive explicitly some critical exponents.

More precisely, we consider the cable system $\widetilde{\mathcal{G}}$ associated to a transient weighted graph $\mathcal{G}$, that is the metric space where each edge of the graph is replaced by a corresponding continuous interval, on which a natural diffusion $X$ can be defined. This diffusion can be constructed as the limit of the canonical random walk on a graph corresponding to $\mathcal{G}$, but adding extra vertices on the edges, and can thus be seen as an equivalent of the Brownian motion on the cable system. The main object of interest, first introduced in [4], is the Gaussian free field $(\varphi_x)_{x\in\widetilde{\mathcal{G}}}$ on $\widetilde{\mathcal{G}}$, that is the centered Gaussian field with covariance

$$(1) \qquad \mathbb{E}[\varphi_x\varphi_y] = g(x,y),\ x,y \in \widetilde{\mathcal{G}}.$$

Here $g(x,y)$ denotes the Green function associated to the diffusion $X$, that is the average time spent in $y$ when $X$ starts in $x$. Fixing some $x_0 \in \widetilde{\mathcal{G}}$, we consider the connected component of $x_0$ in the level sets above level $h$

$$\mathcal{K}^h = \{x \in \widetilde{\mathcal{G}} : x \stackrel{\geq h}{\longleftrightarrow} x_0\},\ h \in \mathbb{R},$$

where $x \overset{\geq h}{\longleftrightarrow} x_0$ means that $x$ is connected to $x_0$ by a continuous path $\pi \subset \widetilde{\mathcal{G}}$ along which $\varphi \geq h$. One can ask the classical percolation question for this random subset of $\widetilde{\mathcal{G}}$: for which values of $h$ is $\mathcal{K}^h$ unbounded with positive probability? The answer is that, if $\mathcal{G}$ is a massless vertex-transitive graph

$$\mathbb{P}(\mathcal{K}^h \text{ is unbounded}) > 0 \text{ if and only if } h < 0.$$

This follows from combining results from [1] and [2], and highlights the interest of this model, since it shows that the associated critical parameter is always equal to 0, and that there is never percolation at criticality. We obtain a deeper understanding of this phase transition, by studying the capacity functional $\mathrm{cap}(\mathcal{K}^h)$ at any level $h \in \mathbb{R}$.

**Theorem 1.** *Assume that $\mathcal{K}^0$ is bounded a.s. For all $h \in \mathbb{R}$ the density of $cap(\mathcal{K}^h)1_{\varphi_{x_0} \geq h, \mathcal{K}^h \text{ bounded}}$ is*

$$t \mapsto \frac{1}{2\pi t\sqrt{g(x_0,x_0)(t-g(x_0,x_0)^{-1})}} e^{-h^2 t} 1_{t \geq g(x_0,x_0)^{-1}}. \tag{2}$$

Let us now state two first interesting consequences of Theorem 1 for the behavior of our phase transition near-criticality. Integrating the density (2) over $[0,\infty)$ provides us directly with an exact formula for the percolation probability

$$\mathbb{P}(\mathcal{K}^{-h} \text{ is unbounded}) = \mathbb{P}(\varphi_{x_0} \in (-h,h)) \sim ch \text{ as } h \nearrow 0. \tag{3}$$

Moreover, integrating the density (2) for $h = 0$ between $N$ and $\infty$ gives us

$$\mathbb{P}(\mathrm{cap}(\mathcal{K}^0) \geq N) \sim \frac{c}{\sqrt{N}}. \tag{4}$$

When $\mathcal{G}$ verifies

$$g(x,y) \asymp d(x,y)^{-\nu} \text{ for some } \nu \in (0,1], \tag{5}$$

then one can precisely compare the radius of a set to its capacity, and the result (4) about the tail of the capacity can be turned into a result about the radius of the cluster at criticality

$$\frac{c}{N^{\nu/2}} \leq \mathbb{P}(\mathrm{rad}(\mathcal{K}^0) \geq N) \leq \frac{c' \log(N)^{1\{\nu=1\}/2}}{N^{\nu/2}}. \tag{6}$$

One can translate the results (3) and (6) in the language of critical exponents: $\beta = 1$ and $\rho = \frac{2}{\nu}$ (see for instance Section 9 in [3] for a precise definition of these critical exponents). Assuming that the usual scaling and hyperscaling relations between the critical exponents hold, see (9.13), (9.14), (9.22) and (9.23) in [3], and assuming additionally that the weights on $\mathcal{G}$ are bounded and

$$|B(x,R)| \asymp R^{\alpha} \text{ for some } \alpha > 0, \tag{7}$$

we obtain the following expected table of exponents.

| Exponent | $\alpha_c$ | $\beta$ | $\gamma$ | $\delta$ | $\Delta$ | $\rho$ | $\nu_c$ | $\eta$ |
|---|---|---|---|---|---|---|---|---|
| Value | $2-\frac{2\alpha}{\nu}$ | 1 | $\frac{2\alpha}{\nu}-2$ | $\frac{2\alpha}{\nu}-1$ | $\frac{2\alpha}{\nu}-1$ | $\frac{2}{\nu}$ | $\frac{2}{\nu}$ | $\nu-\alpha+2$ |

TABLE 1. Expected critical exponents as a function of the parameters $\nu$ ($\leq 1$) and $\alpha$.

A canonical example of a graph verifying the conditions (5) and (7), and thus for which the above table of exponents is expected to hold, is $\mathbb{Z}^3$ with unit weights, for which $\nu = 1$ and $\alpha = 3$. It was moreover proved in [4] that $\eta = \nu - \alpha + 2$, and we are able to prove that the correlation length exponent $\nu_c = 2/\nu$, which further confirms the expected exponents in Table 1. We also obtain upper bounds on $\delta$ and $\gamma$, which are expected to be optimal if Table 1 holds.

Finally, one can conjecture that the values of the exponents in Table 1 is not only valid for the model that we consider here, but for a universality class of percolation models with long-range correlations, decaying polynomially fast with power $\nu$, see (1) and (5). In the case of the exponent $\nu_c$, this was already conjectured in the physics literature, see [5].

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## The phase transition for planar percolation models without positive associations


STEPHEN MUIRHEAD

(joint work with Alejandro Rivera, Hugo Vanneuville, Laurin Köhler-Schindler)


One of the most useful properties that a percolation model can possess is **positive associations**, i.e. that increasing events are positively correlated (for Bernoulli percolation this is the 'Harris/FKG inequality'). Yet there are many important models which lack this property, for example:

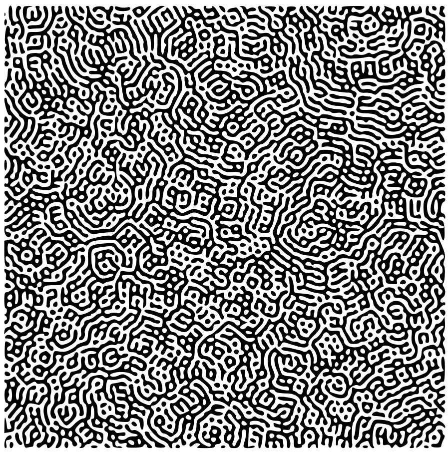
- Models with hard-core constraints or repulsion, e.g. Boolean models on repulsive point processes;
- Certain regimes of classical statistical physics models, e.g. anti-ferromagnetic Ising models, the $q < 1$ regime of the random cluster model, etc.

Existing results on these models are mostly ad hoc, with few general techniques. We present a technique to study the phase transition for **planar Gaussian** percolation models without positive associations, that we believe has the potential to be adapted to a wider class of such models.

Our motivating example is the percolation model defined by the excursion sets $(\{f \le \ell\})_{\ell\in\mathbb{R}}$ of the **random plane wave** (RPW), which is the planar centred smooth stationary Gaussian field with covariance $\kappa(x) = \mathbb{E}[f(0)f(x)] = J_0(|x|)$, where $J_0$ is the zeroth Bessel function

$$J_0(r) := \frac{1}{\pi}\int_0^\pi \cos(r\sin x)\,dx \sim \frac{c_1}{\sqrt{r}}\cos(r-c_2).$$

The RPW can be viewed as the isotropic 'Gaussian object' in the space of solutions to the planar Helmholtz equation $\Delta f = -f$; as such the RPW 'models' generic high-energy Laplace eigenfunctions on chaotic domains/manifolds.



The excursion sets $\{f \le 0\}$ (black) and $\{f \ge 0\}$ (white) for the RPW.

Since the RPW has negative correlations, its excursion sets $\{f \le \ell\}$ are not positively associated. Indeed, due to elliptic regularity these sets have hard-core constraints: there exist $0 < c < C$ such that each component of $\{f \le 0\}$ contains a ball of radius $c$ and does not contain a ball of radius $C$. The RPW also lacks other useful properties such as rapid decay of correlations or the Markov property.

It has long been conjectured [7, 14] that the excursion sets of the RPW, and planar smooth stationary Gaussian fields more generally, undergo a percolation phase transition at the critical level $\ell_c = 0$. Recently there has been much progress on this conjecture [1, 3, 13, 11, 12, 8], yet so far all known results assume either (i) positive associations (equivalent to $\kappa \ge 0$), or (ii) rapid decay of correlations, and so do not apply to the RPW.

We establish this conjecture for a very wide class of planar Gaussian fields assuming neither of these properties, in particular for the RPW:

**Theorem.** [M., Rivera & Vanneuville '20] Let $f$ be a planar centred stationary Gaussian field satisfying:

- (Smoothness) $f$ is a.s. $C^3$-smooth.
- (Non-degeneracy) The Gaussian vector $(f(0), \nabla f(0), f(x), \nabla f(x))$ is non-degenerate for $x \neq 0$.
- (Mild covariance decay) There exists $\delta > 0$ such that, as $|x| \to \infty$,

$$|\kappa(x)|(\log\log|x|)^{2+\delta} \to 0.$$

- (Symmetry) $f$ is isotropic, i.e. $\kappa(x)$ is rotationally symmetric.

Then:

- For $\ell < 0$, $\{f \leq \ell\}$ has bounded components a.s.
- For $\ell > 0$, $\{f \leq \ell\}$ has a unique unbounded component a.s.

To outline of the proof of this result let us recall the classical three-step strategy to prove that the self-dual point of a planar model is critical (see, e.g., [9, 5, 2]):

(1) Use self-duality to establish 'box-crossing estimates' at the self-dual point.
(2) Combine with a differential inequality (e.g. BKKKL, OSSS etc) to establish that crossing events undergo a sharp threshold at the self-dual point.
(3) Deduce that the self-dual point is critical.

We cannot carry this strategy out, since known methods to prove box-crossing estimates rely crucially on positive associations (or other strong properties such as the Markov property). Instead we **reverse the order of the first two steps**:

(1) Prove a sharp threshold result for crossing events on large scales **without identifying the threshold point**. For this we cannot use the differential inequalities mentioned above since they (implicitly) rely on positive associations. Instead we use a new technique based on (i) the concept of the 'threshold map', introduced recently in this context in [12], and (ii) Chatterjee's theory of superconcentration [6].
(2) Inject the sharp threshold result into the geometric constructions that underpin the proof of box-crossing estimates, in particular those in the recent proof due to L. Köhler-Schindler. The upshot is we prove a 'sprinkled' version of the box-crossings estimates.
(3) Deduce that $\ell = 0$ is critical.

Our result is only a very small step towards a full understanding of the percolation phase transition for the RPW. Here is a sample of open problems:

- Is the phase transition continuous, i.e. does $\{f \leq 0\}$ have bounded components? Since our argument uses 'sprinkling' (i.e. a small raising of the level) it implies nothing about the critical level.
- Do the zero level lines $\{f = 0\}$ have a conformal invariant scaling limit? For the RPW physicists have conjectured [4] that these lines converge to the scaling limit of critical Bernoulli percolation interfaces CLE(6).

- Is the phase transition sharp, i.e. for $\ell < 0$ is there a $c > 0$ such that
$$\mathbb{P}[0 \text{ is connected to } \partial B(R) \text{ in } \{f \leq \ell\}] \leq e^{-cR}?$$
For the RPW our argument gives the much weaker bound $e^{-c\sqrt{\log R}}$.

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## The OSSS inequality for Poisson processes and its application in continuum percolation


Günter Last

(joint work with Giovanni Peccati, D. Yogeshwaran)


Let $\eta$ be a Poisson process on a Borel space $(\mathbb{X}, \mathcal{X})$ with $\sigma$-finite and diffuse intensity measure $\lambda$; see [4]. Let $Z$ be a stopping set. This is a random subset of $\mathbb{X}$ which depends on $\eta$ in such a way, that, roughly speaking, changing the points of $\eta$ in the complement of $Z$, does not change $Z$. Assume that $f(\eta)$ is an integrable function of $\eta$ which is determined by the restriction of $\eta$ to $Z$. Let $\eta'$ be an independent copy of $\eta$. Under an additional assumption on $Z$ we prove that

$$\mathbb{E}[|f(\eta) - f(\eta')|] \leq 2 \int \mathbb{P}(x \in Z)\mathbb{E}[|D_x f(\eta)|] \, \lambda(dx), \tag{1}$$

where $D_x f(\eta) := f(\eta+\delta_x) - f(\eta)$, with $\delta_x$ the Dirac mass at $x$. If $f$ is $\{0,1\}$-valued, then this is equivalent to

$$\mathbb{V}[f(\eta)] \leq \int \mathbb{P}(x \in Z)\mathbb{E}[|D_x f(\eta)|] \, \lambda(dx), \tag{2}$$

where $\mathbb{V}$ denotes the variance operator. These inequalities remain valid for randomized stopping sets. They are the Poisson versions of the variance inequalities, proved by O'Donnell, Saks, Schramm and Servedio in their seminal paper [5]. If the so-called revealment probabilities $\mathbb{P}(x \in Z)$, $x \in \mathbb{X}$, are small, then the OSSS inequality (2) is a significant improvement of the Poincaré inequality

$$\mathbb{V}[f(\eta)] \leq \int \mathbb{E}[|D_x f(\eta)|] \, \lambda(dx).$$

We have proved (1) in [3] under the assumption that $Z$ is the union of an increasing family $(Z_t)_{t\geq 0}$ of stopping sets, satisfying certain continuity assumptions. Whether the result is valid for an arbitrary stopping set, is an open problem. Our main technical tool is the spatial Markov property of $\eta$ with respect to stopping sets and the Mecke equation; see e.g. [4].

We apply our results to $k$-percolation of the Poisson Boolean model. In this case $\eta$ is a Poisson process of deterministically bounded compact subsets (grains) of $\mathbb{R}^d$ with intensity $\gamma > 0$. We consider the random set $\mathcal{O}$ of points covered by at least $k$ of the grains. There exists a critical intensity $\gamma_c > 0$ above which $\mathcal{O}$ has an unbounded connected component. Combining the OSSS inequality (2) with tools developed in [1, 2] we show a sharp phase transition at $\gamma_c$. The probability $\theta_s(\gamma)$ that the origin is connected via $\mathcal{O}$ to a ball with radius $s$, decreases exponentially fast for $\gamma < \gamma_c$. On the other hand, in any right-hand neighborhood of $\gamma_c$ the function $\theta(\gamma) := \lim_{s\to\infty} \theta_s(\gamma)$ (the probability that the origin belongs to an infinite component) increases at least linearly.

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**Distance evolutions in growing preferential attachment graphs**

Joost Jorritsma
(joint work with Júlia Komjáthy)

We discuss the evolution of the graph distance and weighted distance between two fixed vertices in (non-spatial) preferential attachment graphs as introduced in [5]. A realization of a classical preferential attachment graph can be constructed according to an iterative procedure. One starts with an initial graph $\mathrm{PA}_1 = (V_1, E_1)$ on the vertex set $\{1\}$, after which vertices arrive sequentially at deterministic times $t \in \{2, 3, \dots\}$. We denote the graph at time $t$ by $\mathrm{PA}_t$ and label all the vertices by their arrival time. The arriving vertex $t$ connects to present vertices such that it is more likely to connect to vertices with a high degree at time $t$. Let $\mathbb{P}(\{t \to v\} \mid \mathrm{PA}_{t-1})$ denote the probability that $t$ connects to $v < t$. We consider two classical (non-spatial) variants of the model, based on [1] and [3]. Both models assume that there exists $\tau \in (2, 3)$ such that

$$\mathbb{P}(\{t \to v\} \mid \mathrm{PA}_{t-1}) = \frac{D_v(t-1)}{t(\tau-1)} + O(1/t),$$

where $D_v(t-1)$ denotes the degree of vertex $v$ directly after the arrival of vertex $t-1$. As a result, the asymptotic degree distribution has a power-law decay with exponent $\tau$ [3, 4].

In these growing graphs, we sample two vertices $u_t$ and $v_t$ uniformly from $\mathrm{PA}_t$ and study how their graph distance $d_G^{(t')}(u_t, v_t)$ evolves as $t'$ grows. The graph distance between two vertices $x$ and $y$ at time $t'$, $d_G^{(t')}(x, y)$, is defined as the minimum number of edges on a path from $x$ to $y$ that is present in $\mathrm{PA}_{t'}$. We call this discrete-time stochastic process $\big(d_L^{(t')}(u_t, v_t)\big)_{t' \geq t}$ the distance evolution. Clearly, the distance evolution is non-increasing in $t'$, since arriving edges may form a shorter path between $u_t$ and $v_t$. Our main result shows that

$$\sup_{t' \geq t} \left| d_G^{(t')}(u_t, v_t) - 4\frac{\log\log(t) - \log(1 \vee \log(t'/t))}{|\log(\tau-2)|} \vee 2 \right| \tag{1}$$

is a tight sequence of random variables, i.e., the distance evolution never leaves a tight strip with high probability as the size of the inititial graph $t$ tends to infinity.

The proof of this statement is split in showing upper and lower tightness, respectively. For both bounds, the main challenge is to achieve summable error probabilities in $t'$, which cannot be achieved via previous arguments from literature, e.g. as in [2].

For the upper bound, we show that whp there exists a sufficiently short path at all times $(t_i)_{i \geq 0}$ when the second term in (1) crosses an integer. To prove this, we establish bounds on the degree of fixed vertices (that are at bounded graph distance from $u_t$ and $v_t$ at time $t$) that hold for all times $t' \geq t$, using martingale techniques. From these vertices whose degree is controlled, we prove using existing techniques that there is a short path connecting them at the times $t_i$ with small summed error probability. Constructing this path is done via an iterative procedure where in each iteration a higher degree vertex can be reached. Applying this procedure from both sides i.e., $u_t$ and $v_t$, yields a path, as the highest degree vertices in $\mathrm{PA}_{t'}$ are within a bounded graph distance of each other. It is at the times $t_{4i}$ that one less iteration is needed for the construction of this path.

The lower bound, i.e., proving that there is never a too short path, is more involved and requires a strong control of the dynamics in the graph. In particular, we employ the observation that if there is no too short path at time $t'$, then the only way that there can be a too short path at time $t'+1$ is when the youngest vertex $t'+1$ is on the shortest path. Using inductive arguments we can show that the expected number of too short paths that pass through the youngest vertex is summable for $t' \geq t$ and tends to zero as $t$ tends to infinity.

Our work [5] initiates a research line that studies how certain graph properties defined on a fixed set of vertices evolve as the surrounding graph grows. Studying the evolution of a property on fixed vertices may sound as a natural mathematical question. Yet, only the evolution of the degree of fixed vertices has been addressed before in the preferential attachment literature [3, 6]. We encourage other people to study the evolution of (other) properties on similar (spatial) models as it yields a deeper understanding of how the dynamics form the graph.

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## The contact process on random hyperbolic graphs: Metastability and critical exponents

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(joint work with Dieter Mitsche, Bruno Schapira and Daniel Valesin)

The contact process is a continuous-time Markov process introduced by T. Harris in [1] as a model for the spread of infection, where each vertex of a given connected graph $G = (V, E)$ is at any point in time either healthy or infected. Calling $\xi_t$ the set of infected vertices at time $t$, its continuous-time dynamics is defined through the following transition rules:

- for each $x \in \xi_t$, $x$ recovers at rate 1, and
- for each pair $x, y$ with $x \in \xi_t$ and $y \notin \xi_t$, $x$ infects $y$ at rate $\lambda > 0$.

In the case of infinite networks it is well known that there is a threshold value $\lambda_0(G) \geq 0$ such that defining the survival probability

$$\gamma(\lambda, x) := \mathbb{P}\big(\forall t \geq 0,\, \xi_t \neq \emptyset \,\big|\, \xi_0 = \{x\}\big),$$

we have $\gamma(\lambda, \cdot) > 0$ as soon as $\lambda > \lambda_c$, whereas $\gamma(\lambda, \cdot) = 0$ if $\lambda < \lambda_c$. In the case of finite networks there is no survival of the infection and the typical approach is to run the process on a sequence of networks $\{G_n\}_{n\in\mathbb{N}}$ which converges locally to some infinite network $G$, and where we start with $\xi_0 = V_n$ for all $n$. In this setting there is usually some $\lambda_c(\{G_n\})$ (which coincides with $\lambda_c(G)$) such that:

- If $\lambda > \lambda_c$ then $\mathbb{E}_n(T_{ext}) = \Omega(e^{c|V_n|})$ for some $c > 0$.
- If $\lambda < \lambda_c$ then $\mathbb{E}_n(T_{ext}) = O(|V_n|^{c'})$ for some $c' > 0$.

Here $\mathbb{P}_n$ stands for the law of the process running on $G_n$ and $T_{ext}$ the hitting time of the configuration in which all vertices are healthy. In the case of scale-free networks such as the configuration model and preferential attachment networks, it was shown that $\lambda_c = 0$ (see [2, 3]). Even further, it was shown in [9] that for the contact process running on the configuration model with $n$ vertices and degree distribution $\mathbb{P}(deg(v) = k) \asymp k^{-a}$ the density $\frac{|\xi_t|}{|V_n|}$ stabilizes around some value $\rho_a(\lambda)$ for an exponentially long time, where

$$\rho_a(\lambda) \asymp \begin{cases} \lambda^{\frac{1}{3-a}} & \text{if } 2 \leq a \leq 2.5 \\ \frac{\lambda^{2a-3}}{\log^{a-2}(\lambda^{-1})} & \text{if } 2.5 < a \leq 3 \\ \frac{\lambda^{2a-4}}{\log^{a-2}(\lambda^{-1})} & \text{if } 3 < a \end{cases}$$

Similar results were proven for preferential attachment networks in [4], and some dynamic networks in [5].

In this work, we run the contact process on a sequence $\{G_n\}_{n\in\mathbb{N}}$ of finite networks, where for each $n$, $G_n$ corresponds to the Poissonized version of the model introduced in [6], described as follows;

- Defining $R = 2\log(n/\nu)$ for some $\nu > 0$, the vertices of $V_n$ are given by a Poisson point process on $B(0, R) \subseteq \mathbb{R}^2$ with intensity function given in polar coordinates by
$$g(r, \theta) = \frac{n\alpha \sinh(\alpha r)}{\cosh(\alpha R) - 1}\mathbf{1}_{[0,R]}(r)$$
for some parameter $\alpha \in (\frac{1}{2}, 1)$.
- The edge $\{x, x'\}$ belongs to $E_n$ iff $x$ and $x'$ are at hyperbolic distance $d_h(x, x')$ smaller than $R$, where $d_h$ solves
$$\cosh(d_h(x, x')) \;=\; \cosh(r)\cosh(r') - \sinh(r)\sinh(r')\cos(\theta - \theta')$$

This network has been shown to be scale-free with exponent $\tau = 2\alpha + 1$, it also exhibits a non-vanishing clustering coefficient, has a heterogeneous degree structure, the typical distance between two vertices is very small, and the maximal distance is also small (see [10, 7, 8] for additional properties). Our main results can be summarized in the following theorem:

**Theorem 1** (Mitsche, L, Schapira, Valesin). *For any* $\lambda > 0$ *and* $\alpha \in (\frac{1}{2}, 1)$, *there exist* $c > 0$ *and* $\beta \in (0,1)$, *such that* $\forall n \geq 1$,

$$\mathbb{P}_n(T_{ext} > e^{cn}) > 1 - e^{-cn^{\beta}}.$$

*Even further, for any sequence* $t_n \to \infty$ *with* $t_n = O(e^{cn})$ *and any* $\varepsilon > 0$,

$$\mathbb{P}_n\left(\left|\frac{|\xi_{t_n}|}{n} - \gamma(\lambda)\right| > \varepsilon\right) \underset{n\to\infty}{\longrightarrow} 0$$

*where for* $\lambda \approx 0$, $\gamma(\lambda)$ *satisfies*

$$\gamma(\lambda) \asymp \begin{cases} \lambda^{\frac{1}{3-\tau}}, & \text{if } 2 < \tau \leq 2.5 \\ \frac{\lambda^{2\tau-3}}{\log^{\tau-2}(\lambda^{-1})} & \text{if } 2.5 < \tau < 3 \end{cases}$$

In this talk we will discuss the similarities between our results and the ones in [9], as well as the main difficulties found in our proofs. In particular, I will introduce the concept of *ordered traces*, which we used to control the growth of the infection on networks that are not locally tree-like.

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## The Contact Process on a Graph Adapting to Infection Density

John Fernley
(joint work with Peter Mörters, Marcel Ortgiese)

The Erdős-Rényi graph model on $[N] := \{1, \ldots, N\}$ with mean degree parameter $\beta$ places each possible edge independently with probability $\beta/N$. Recently, [BNNS21] found a phase transition for the contact process on a graph with this distribution.

The graph can be made dynamic, as in [JM17], by *updating* each vertex at some constant rate $\kappa > 0$. A vertex update at some $i \in [N]$ entails simply deleting all edges incident to $i$ and generating a number of new ones distributed as $\mathrm{Bin}(N-1, \beta/N)$, with a uniformly chosen new set of neighbours from $[N] \setminus \{i\}$. Observe that this dynamic is stationary and reversible with respect to the measure of the original Erdős-Rényi graph. Then we can write $\xi_t \in \{0,1\}^{[N]}$ for the contact process infection state, with the usual infection rate parameter $\lambda > 0$, and consider on a dynamic graph the historical infection sets

$$I_t := \{v \in [N] : \exists s \in [0,t] : \xi_s(v) = 1\}.$$

We are interested in assessing the limit set $I_t \uparrow I_\infty$ from a single initial infected vertex, to understand the outbreak probability. This probability has the following two parameter cases of interest: infection supercriticality, where the probability of an outbreak to epidemic level is asymptotically positive

$$\exists \epsilon > 0 : \quad \liminf_{N\to\infty} \mathbb{P}\left(|I_\infty| > \epsilon N \,\big|\, |\xi_0| = 1\right) > 0;$$

and infection subcriticality, where this is asymptotically of zero probability

$$\forall \epsilon > 0 : \quad \lim_{N\to\infty} \mathbb{P}\left(|I_\infty| > \epsilon N \,\big|\, |\xi_0| = 1\right) = 0.$$

We call the simpler model, where every vertex updates at constant rate, the non-adaptive dynamic Erdős-Rényi. For this model we find when $\lambda$ the infection rate and $\beta$ the mean after update satisfy $\lambda\beta > 1$ then the infection is supercritical for $\kappa$ sufficiently large. Conversely, when $\lambda\beta < 1$ and $\kappa$ is sufficiently large the infection is subcritical. Thus, at least for $\kappa$ sufficiently large, this model has similar epidemic properties to the mean-field version.

More interestingly, we can make this model *adaptive* by only allowing updates at those vertices adjacent to an infected vertex. This introduces three significant technical difficulties: the graph dynamic is dependent on the infection and so the graphical construction is not easy to use; the graph dynamic is not stationary and so we can get no handle on its distribution after time 0; and because it responds to move away from infection we further lose monotonicity of infection states in this model with respect to the presence of more initial infection.

We can nevertheless find a lower bound on this infection by approximating by a local forest environment, on which we see an SIR with dropping of edges to uninfected vertices as in [BBLS19]. This leads to a region

$$\frac{\lambda\beta}{1+2\kappa+\lambda} > \sqrt{1 - \frac{\kappa\lambda\left(1-e^{-\beta}\right)}{(1+\kappa)(1+2\kappa+\lambda)}}$$

for which we prove infection supercriticality. The upper bound is more difficult. We separate it into the case of small $\kappa$, where we consider the regular contact process, and of large $\kappa$, where uninfected vertices disconnect fast enough to prevent the spread of infection, and conclude two regions of subcriticality

$$\lambda\beta < 0.21 \quad \text{or} \quad (2\beta - 1)\lambda < \kappa.$$

We have thus found, by developing new techniques to overcome the serious technical difficulties, that this targeted skipping of vertex updates can go far to prevent epidemics. Precisely, the critical line in $\lambda\beta$ increases asymptotically linearly with large $\kappa$ rather than converging as it does in the non-adaptive case.

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## A new proof of the Grimmett Marstrand Theorem

Daniel Contreras Salinas
(joint work with Vincent Tassion)

The Grimmett-Marstrand Theorem is a powerful tool in the study of supercritical percolation on $\mathbb{Z}^d$, where $d \geq 3$. It states that for any $p > p_c(\mathbb{Z}^d)$, the percolation of parameter $p$ restricted to a sufficiently thick slab $\mathbb{S}_k := \mathbb{Z}^2 \times \{0, 1, \ldots, k\}^{d-2}$ is stochastically dominated by a supercritical percolation on $\mathbb{Z}^2$, showing that the infinite cluster on $\mathbb{Z}^d$ actually lives on a thick slab of it.

The original proof in [1] uses a dynamical renormalisation argument based on an exploration process, where at each step one *seed* is targeted. Seeds are small boxes (say of size $n$) all whose edges are open, which are on the boundary of a much larger box, say of size $N$. The first novelty of our work is the use of a *seedless* renormalisation scheme, based on ideas of [2]. In fact, having explored clusters of size of order $N$ allows us to use these explored regions as seeds.
As in [1], we start with an exploration process which also carries a mixture of positive and negative information, so a sprinkling argument is needed. Using this, we can construct paths of finite length that stay on well determined regions inside a slab. We call these regions *corridors*.

The main new idea in this proof is our renormalisation argument. From a finite size criterion we are able to perform a static renormalisation. To construct this criterion, we use a very simple recurrence relation based on the paths we found in our corridors and the local uniqueness due to the uniqueness of infinite cluster in the supercritical phase.

This new renormalisation provides a very short proof in models with rotational invariance such as Voronoi and Boolean percolation models, where the corridors are not necessary.

We hope these ideas open new doors in the understanding of supercritical percolation for a variety of models. We also hope that this can serve as a basis to improve quantitative estimates such as the established in [3].

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## Chemical distance in weight-dependent random connection models

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(joint work with Peter Gracar, Peter Mörters)

We study a large class of random graphs defined on a Poisson process in $d$-dimensional space equipped with independent uniform random marks, first introduced as the *weight-dependent random connection model* in [6]. Two vertices $\mathbf{x}$ and $\mathbf{y}$ are connected by an edge independently of other edges with probability $\varphi(\mathbf{x},\mathbf{y})$, where this probability depends on the marks of the vertices and their relative locations. The possible choices of $\varphi$ offer a large class of geometric random graphs which combine scale-free degree distributions and long-range effects. The influence of the marks and of the Euclidean distance can be parametrized by $\gamma \in (0,1)$ and $\delta > 1$, such that the power-law exponent of the graphs is $1 + \frac{1}{\gamma}$ and $\delta$ describes the polynomial decay of the probability of an edge connecting two typical points in terms of their Euclidean distance. In particular, the two following motivating examples fall in this class.

In the *Boolean model* on $\mathbb{R}^d$ the vertices $\mathbf{x}$ carry random radii, which are derived from the uniform marks $t$ as $t^{-\gamma/d}$. In the classical version of the model two points are connected if their respective balls with the associated random radii intersect, whereas in the soft version of the Boolean model, a i.i.d. positive random variable $X = X(\mathbf{x},\mathbf{y})$ is associated to every unordered pair of vertices and a connection is made if and only of

$$\frac{|x-y|}{s^{-\gamma/d}+t^{-\gamma/d}} \leq X,$$

where $s,t$ are the marks of the vertices and $x,y$ their locations. Considering the soft Boolean model with a heavy-tailed random variable $X$ with decay $\mathbb{P}(X>r) \sim r^{-\delta d}$ as $r \to \infty$, for some $\delta > 1$ leads to the connection probability

$$\varphi(\mathbf{x},\mathbf{y}) = \rho(\beta^{-1}(s^{-\gamma/d}+t^{-\gamma/d})^{-d}|x-y|^d)$$

with $\beta > 0$ and function $\rho$ such that $\lim_{v\to\infty} \rho(v)v^\delta = 1$. A further variant which shows qualitatively the same behaviour aries by using the connection probability $\varphi(\mathbf{x},\mathbf{y}) = \rho(\frac{1}{\beta}(s \wedge t)^\gamma |x-y|^d)$.

The second example is the *age-based random connection model*, which was introduced in [5]. Here the mark of a vertex is considered as its birth time and two vertices are connected by an edge with probability

$$\varphi(\mathbf{x},\mathbf{y}) = \rho(\beta^{-1}(s \vee t)^{1-\gamma}(s \wedge t)^{-\gamma}|x-y|^d).$$

As $(t/s)^\gamma$ is the asymptotic order of the expected degree of a vertex with birth time $s \downarrow 0$ restricted to its neighbours younger than $t$ this infinite graph model mimics the behaviour of spatial preferential attachment networks [8].

For these three models we give a sharp criterion for the absence of ultrasmallness of the graphs and in the ultrasmall regime establish a limit theorem for the chemical distance of two points. Here the boundary of the ultrasmall regime depends not only on the power-law exponent of the graph but also on a geometric quantity, described by $\delta$. This also holds for the limit theorem for the chemical distance in the ultrasmall case, as

$$\frac{d(\mathbf{x},\mathbf{y})}{\log\log(|x-y|)} \to \frac{4}{\log\frac{\gamma}{\delta(1-\gamma)}} \text{ with high probability as } |x-y| \to \infty.$$

This displays universal behaviour markedly different from the class of spatial scale-free graphs inverstigated in [3] and [2] and the non-spatial scale-free models studied, for example, in [4].

## The optimal path structure

To establish lower bounds for the chemical distance between two vertices $\mathbf{x}$ and $\mathbf{y}$ we use a truncated first moment method to bound the expected number of paths between them. We do this by determining lower bounds for the marks of the vertices of so-called *good* paths, which represent the majority of paths in the graph. The contribution of this expectation is dominated by paths which connect as quickly as possible to vertices with small marks, i.e. powerful vertices. In particular, it is necessary to establish an upper bound for the probability of the existence of a *monotone* path from $\mathbf{x}$ to a vertex $\mathbf{z}$ where $\mathbf{z}$ has the smallest mark of all vertices in the path. Here, the possible connections between two powerful vertices are of interest. In the ultrasmall phase, i.e. if $\gamma > \frac{\delta}{\delta+1}$ we expect that two powerful vertices are typically not connected directly, but via a *connector*, a vertex with a larger mark which is connected to the two powerful vertices. If $\gamma < \frac{\delta}{\delta+1}$ this type of connection is not beneficial when compared to a direct connection as has been shown in [7]. If $\gamma > \frac{\delta}{\delta+1}$ the connection of two powerful vertices via a connector is used to ensure the existence of a path between $\mathbf{x}$ and $\mathbf{y}$ and consequently to show the upper bound for the chemical distance. We are thus able to provide an upper bound for the probability of this type of connection. This leads to a first estimate of the lower bounds for the chemical distance between $\mathbf{x}$ and $\mathbf{y}$, if we start by considering paths only containing the typical type of connection between powerful vertices. Then, in the ultrasmall regime the above described existence of a monotone path can be decomposed into two events. First, the existence of a path two steps shorter starting from $\mathbf{x}$, where the endvertex is the most powerful vertex of this path but has a larger mark than $\mathbf{z}$ and second the connection of this endvertex and $\mathbf{z}$ via some connector. This leads to a recursive representation of such monotone path's existence and enables computations which lead to the lower bound of the limit theorem.

### Connections via multiple connectors

To extend the results to the case where all path structures are considered we again rely on a recursive representation for the existence of a monotone path starting from $\mathbf{x}$ with its endvertex being the most powerful vertex of the path. Then, connections of two powerful vertices via not only one connector but via multiple connectors has to be considered. We are able to establish an upper bound for the probability of this type of connection via multiple connectors, which scales similarly to the upper bound for the connection via one connector, provided the Euclidean distance of the two powerful vertices is large. Together with controlling the combinatorical aspects of the problem this leads to the observation that the above described optimal path structure in fact captures the dominant behaviour of typical paths between two given vertices $\mathbf{x}$ and $\mathbf{y}$ and yields the same lower bound in the ultrasmall case, and furthermore the absence of ultrasmallness for $\gamma < \frac{\delta}{\delta+1}$.

### Further directions

While we are able to prove a limit theorem for the chemical distance of two vertices in the ultrasmall regime, this remains an open problem in the absence of ultrasmallness. In this regime, the proof of the lower bound seems to be a challenging task. Our method achieves a lower bound of sublogarithmic order, which we conjecture is not sharp. Instead, we expect the existence of a regime where the chemical distance of two vertices is of linear order and one where it is of polylogarithmic order. If $\delta < 2$ an upper bound for the chemical distance of polylogarithmic order is already shown in [1]. Additionally it is not clear how different the behaviour of three examples is in this regime, unlike in the ultrasmall regime, where there is no qualitative difference between the three.

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