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Logarithmic Vector Fields and Freeness of Divisors and Arrangements: New perspectives and applications (online meeting)

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Abstract. The central topic of the workshop was the notion of logarithmic vector fields along a divisor in a smooth complex analytic or algebraic variety, i.e., the vector fields on the ambient variety tangent to the divisor. Following their introduction by K. Saito for the purpose of studying the universal unfolding of an isolated singularity, this fundamental object has been the focus of studies in a wide range of mathematical fields such as algebra, algebraic geometry, singularity theory, root systems, (geometric) representation theory, combinatorics, (toric) topology, or symplectic geometry. In the last few years the logarithmic vector field approach has seen some unexpected and striking advances and deep applications. The aim of the workshop was to provide reports and to share these various new developments in the field.

Mathematics Subject Classification (2010): 32S22; 52C35, 14N20, 13D02.

Introduction by the Organizers

The workshop *Logarithmic Vector Fields and Freeness of Divisors and Arrangements: New perspectives and applications*, organised by Takuro Abe (Fukuoka), Alexandru Dimca (Nice), Eva-Maria Feichtner (Bremen), and Gerhard Röhrle (Bochum) was held as an online event due to the ongoing COVID-19 pandemic. The workshop was well attended with over 50 participants from around the globe.

Overall, the workshop worked surprisingly well given the circumstances. The technical support through the video assistants, Lukas Kühne and Paul Mücksch, was excellent. All online talks were prepared extremely well. What helped greatly was the very good attendance and intense focus by participants. Naturally, the greatest challenge was posed by the vast difference in time zones of all the participants locations, limiting participation on some morning and some late afternoon talks. But what seemed to work well is having a focused core session in the middle of the day allowing for most participants from outside Europe to participate.

Also, what helped making this event a success was the fact that we were able to record all talks and that most participants were willing to share their slides.

The topics of the contributions ranged from very classical topics such as simplicial arrangements and singularities of configuration spaces to entirely new developments, such as elliptic reflection arrangements and Lagrangian geometry of matroids.

We highlight some of the contributions in a number of areas separately showcasing the impact and relevance of the developments.

Simplicial arrangements
Michael Cuntz gave an overview on his recent joint work with S. Elia and J. Labbé. Simplicial hyperplane arrangements generalize finite Coxeter arrangements and the weak order through the poset of regions. For simplicial arrangements, posets of regions are always lattices. The weak order is known to be congruence normal, and congruence normality for simplicial arrangements can be determined using polyhedral cones called shards. In their work they determine which arrangements lead to congruence normal lattices of regions. Using oriented matroids this work recasts shards as covectors to determine congruence normality of large hyperplane arrangements.

Root systems and reflection groups
Root systems constitute the important origin of hyperplane arrangement theory. We can say that arrangement theory is a generalization of the "symmetry" of root systems and Weyl arrangements as their geometric realization. In this research area several mathematical strands intersect, and influence each other.

One of such developments is given in Misha Feigin's talk. He explained the relation between the quasi-invariant rings of several reflection groups, and the logarithmic derivation modules of their multiarrangements. For the former, the freeness was proved by using the action of the rational Cherednik algebra, and the latter by Terao, Yoshinaga and others. Feigin showed with his co-researchers that these freeness properties are equivalent, and gave a uniform proof of freeness of reflection arrangements with invariant multiplicities. We hope that this result will give a new bridge between geometric representation theory, integrable systems and hyperplane arrangements.

We were particularly enthusiastic about Professor Kyoji Saito's participation in the meeting, the founder of the very subject matter of this workshop more than 50 years ago! He reported on his ongoing work on elliptic reflection arrangements. Analogous to the case of a classical finite reflection arrangement, an elliptic reflection arrangement is the union of zero hyperplane loci of an elliptic root system. On the complex period domain, the elliptic Weyl group of the root system acts properly and discontinuously. Its fixed points set of the action is the elliptic reflection arrangement. In his talk Kyoji Saito focused on the homotopy type of the complement of the reflection arrangement, equivalently the regular orbit space of the Weyl group action. He proposed three very explicit conjectures which would lead to a construction of counterexamples to the long-standing assertion that the discriminant complements are $K(\pi, 1)$-spaces.

Configuration spaces
Reporting on her joint work with N. Gadish, Christin Bibby described their approach using generating functions to investigate representation stability phenomena in orbit configuration spaces. Specifically, they use the notion of twisted commutative algebras, which essentially categorify exponential generating functions. This idea allows for a factorization of the orbit configuration space "generating function" into an infinite product, whose terms turn out to be tractable. Based on this approach, Bibby gives a simple geometric technique for identifying new stabilization actions with finiteness properties.

Uli Walther reported on his joint work with G. Denham, D. Pol, and M. Schulze on configuration polynomials. These generalize the classical Kirchhoff polynomial defined by a graph and appear in the theory of Feynman integrals. Considering a linear realization of a matroid over a field, one associates with it a configuration polynomial and a symmetric bilinear form with linear homogeneous coefficients. The corresponding configuration hypersurface and its non-smooth locus support the respective first and second degeneracy scheme of the bilinear form. In their work it is shown that these schemes are reduced and describe the effect of matroid connectivity.

Geometry of matroids
In his talk, Graham Denham explained his recent exiting work with F. Ardila and J. Huh, where they study Chern–Schwartz–MacPherson (CSM) classes in connection with the geometry of matroids. They introduce the conormal fan of a matroid M, which is a Lagrangian analog of the Bergman fan of M. They use this conormal fan to give a Lagrangian interpretation of the Chern-Schwartz-MacPherson cycle of M.

Hyperplane arrangements, commutative algebra and algebraic geometry
Brian Harbourne gave a talk on H-constants, which play a key role in the Bounded Negativity Conjecture. The understanding of these H-constants in the case of line arrangements in the projective plane led to many interesting open questions, for instance the classification of all such line arrangements having only triple points.

Juan Migliore discussed his joint work with U. Nagel and H. Schenck on the various scheme structures associated to the Jacobian ideal of a hyperplane arrangement.

Piotr Pokora gave a talk on unexpected curves, a classical subject in Algebraic Geometry, and key role played in this subject by the line arrangements.

Henry Schenck discussed his joint work with L. Busé, A. Dimca and G. Sticlaru concerning the Castenuovo-Mumford regularity of the Milnor algebra of a projective hypersurface. For many classes of hypersurfaces V this regularity is bounded by a linear function in the degree d of V, but the authors have found a sequence of very singular surfaces for which this regularity has a quadratic grow in d.

Milnor fiber and monodromy
Alexander Suciu in his talk recalled first the construction of several algebraic objects associated to a finitely presented group. Then he used this objects to study the Milnor fiber of central, complex hyperplane arrangements. Important new results show that the inclusion of the Milnor fiber into the arrangement complement induces isomorphisms or epimorphisms between these objects.

Freeness and Terao's conjecture
Terao's Conjecture from the early 1980s asserts that algebraic notion of freeness only depends on the combinatorics of the underlying arrangement over a fixed field. This conjecture is still wide open and an ongoing driving force in the area. Several talks evolved around this theme.

In his talk, Lukas Kühne reported on his joint work with M. Barakat on their algorithmic approach towards this conjecture in rank 3 in arbitrary characteristic.

Paul Mücksch reported on his recent work on Yuzvinsky's lattice sheaf cohomology for hyperplane arrangements. He was able to establish the relationship between the cohomology of a certain sheaf on the intersection lattice of a hyperplane arrangement introduced by Yuzvinsky and the cohomology of the coherent sheaf on punctured affine space respectively projective space associated to the derivation module of the arrangement. His main result gives a Künneth formula connecting the cohomology theories, answering a question posed by Yoshinaga. This gives a characterization of the projective dimension of the derivation module and yields a new proof of Yuzvinsky's freeness criterion.

Masahiko Yoshinaga reported on new joint work with D. Suyama. It has been known for some time that the modules of logarithmic derivations for the (extended) Catalan and Shi arrangements associated with root systems are free - due to seminal work of Yoshinaga. Nevertheless explicit bases for such modules are not known in general. Yoshinaga and his collaborator were able to construct explicit bases for type A root systems based on a construction of discrete analogues of Bandlow-Musiker's integral formula for a basis of the space of quasiinvariants.

Stefan Tohăneanu discussed the conjecture saying that every central free hyperplane arrangement with exponents 1's, 2's and at most one 3 is supersolvable. This interesting conjecture is known to hold in many cases, in particular when all exponents are at most 2.

Workshop (online meeting): Logarithmic Vector Fields and Freeness of Divisors and Arrangements: New perspectives and applications

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Abstracts

Congruence normality for simplicial arrangements

Michael Cuntz
(joint work with Sophia Elia, Jean-Philippe Labbé)

Reflection groups and groupoids may be investigated via the simplicial arrangements of hyperplanes associated to their root systems. These arrangements produce different kinds of combinatorial objects. In the context of logarithmic vector fields, the intersection lattice (or the corresponding oriented matroid) is a central structure; here the main open problem, Terao's Conjecture, is to find the relation between the combinatorial side and the freeness of the arrangements. The root poset also gives a lot of information concerning freeness, for instance, the subarrangements defined by ideals in this poset were successfully used in [1]. The poset of regions has apparently less been in the focus in this context yet. This talk is partly an advertisement to explore the consequences of several properties of this poset[1] in the special case of simplicial arrangements.

Simplicial arrangements are real arrangements of hyperplanes which decompose the space into open simplicial cones. They were introduced in [9], Grünbaum published a first collection of examples in rank three in the 70's and revised this catalogue later [8]. Deligne [7] was the first to use simplicial arrangement to prove an important conjecture, the $K(\pi, 1)$ conjecture for finite real reflection groups.

However, since the introduction of simplicial arrangements, very few further results were achieved. Among others, a handful of examples were added to Grünbaum's catalogue, see [5] and [6], and supersolvable simplicial arrangements were classified [4]. It is still an open question whether there are more simplicial arrangements in rank three, even whether there are some further infinite series.

In order to make some progress towards a classification, it is natural to search for structures lurking behind the catalogue. There are many similarities between simplicial arrangements and Coxeter groups, for example, they correspond to normal fans of simple zonotopes. A large subclass of the known simplicial arrangements comes from finite Weyl groupoids; these are generalizations of Weyl groups and are completely classified [3]. Root systems of finite Weyl groupoids may be constructed recursively by successively adding sums of roots. This additive structure is visible within the *digraph of shards* of the arrangement. Shards are pieces of hyperplanes that cut each other with respect to a given base chamber. There is a theorem by Reading [10] telling that, in a simplicial arrangement, the shard digraph is acyclic if and only if its poset of regions is *congruence normal*.

The reason why congruence normality is important is that finite congruence normal lattices are exactly the lattices obtained from a one-element lattice by a sequence of doublings of convex sets. Since for Weyl groupoids, additivity of the

[1]Note that for many "interesting" arrangements of hyperplanes, these three posets (intersection lattice, root poset, and poset of regions) contain the same information in the sense that given one of them, it is possible to reconstruct the others.

arrangement is comparable to the acyclicity of the shard digraph, the property of being congruence normal seems to play an important role for simplicial arrangements.

In our joint work [2], for each simplicial arrangement of the extended catalogue, we decide whether its poset of regions is congruence normal or not. We achieve this goal by computing the shard digraph using the oriented matroid of the arrangement. The result is roughly speaking that those arrangements which are related to reflection arrangements are congruence normal with respect to any chamber, some others are sometimes congruence normal, and part of those which I found and which Grünbaum missed are never congruence normal. More precisely, all the finite Weyl groupoids are always congruence normal, the other two known simplicial arrangements with this property are the Coxeter arrangement of type H_3 and its dual (or equivalently a restriction of the arrangement of type H_4 to any of its hyperplanes).

Among the open questions and tasks, let me stress the following.

(1) Classify congruence normal simplicial arrangements (see also [2]).
(2) What is the relation between congruence normality and freeness of an arrangement?
(3) Find more properties of the shard digraph that are useful in the context of arrangements.
(4) What is the smallest number of hyperplanes for which there exists a tight arrangement which is not congruence normal (see also [2])?
(5) What happens with congruence normality under restrictions or other types of operations on arrangements?

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Hyperplane arrangements and k-Lefschetz properties

Elisa Palezzato
(joint work with Michele Torielli)

1. Lefschetz and k-Lefschetz properties

Let $\mathbb{K}$ be a field and $R = \bigoplus_{i \geq 0} R_i$ a graded ring over $\mathbb{K}$, where R_i are the homogeneous components of R with $\dim_{\mathbb{K}}(R_i) < \infty$. R is said to have the **weak Lefschetz property (WLP)**, if there exists an element $\ell \in R_1$ such that the multiplication map

$$\times \ell \colon R_i \to R_{i+1}$$

is full-rank for every $i \geq 0$. In this case, ℓ is called a **weak Lefschetz element**. Similarly, R is said to have the **strong Lefschetz property (SLP)**, if there exists an element $\ell \in R_1$ such that the multiplication map

$$\times \ell^s \colon R_i \to R_{i+s}$$

is full-rank for every $i \geq 0$ and $s \geq 1$. In this case, ℓ is called a **strong Lefschetz element**.

The notions of weak and strong k-Lefschetz properties were introduced in [4] as a generalization of the weak and strong Lefschetz properties. See also [3] and [5].

If we consider k a positive integer, then the graded ring R is said to have the ***k*-SLP** (respectively the ***k*-WLP**) if there exist elements $\ell_1, \dots, \ell_k \in R_1$ satisfying the following two conditions

(1) R has the SLP (respectively WLP) with Lefschetz element ℓ_1,
(2) $R/\langle \ell_1, \dots, \ell_{i-1} \rangle$ has the SLP (respectively WLP) with Lefschetz element ℓ_i, for all $i = 2, \dots, k$.

In this case we will say that $(R, \ell_1, \dots, \ell_k)$ has the k-SLP (respectively k-WLP).

As noted in Remark 6.2 of [3], if $(R, \ell_1, \dots, \ell_k)$ has the k-SLP (respectively k-WLP), then ℓ_1 is a Lefschetz element for R. However, if g_1 is another Lefschetz element for R, there do not necessarily exist $g_2, \dots, g_k \in R_1$ such that $(R, g_1, \dots, g_k)$ has the k-SLP (respectively k-WLP).

From now on, we will consider $\mathbb{K}$ a field of characteristic 0 and $S = \mathbb{K}[x_1, \dots, x_l]$ the polynomial ring with standard grading. The following result shows that we can reduce the study of the k-Lefschetz properties to the monomial case using $\operatorname{rgin}(I)$, the **generic initial ideal** with respect to the ordering DegRevLex (see [1] for more details on rgin).

Theorem 1 ([7, Theorem 3.6])**.** *Let I be a homogeneous ideal of S and $1 \leq k \leq l$. Then the following two conditions are equivalent*

(1) S/I has the k-SLP (respectively the k-WLP),
(2) $(S/\operatorname{rgin}(I), x_l, \dots, x_{l-k+1})$ has the k-SLP (respectively the k-WLP).

In the study of k-Lefschetz properties an important role is played by the so called almost revlex ideals, where a monomial ideal I of S is called an **almost revlex ideal**, if for any power-product t in the minimal generating set of I, every other power-product t' of S with $\deg(t') = \deg(t)$ and $t' >_{\text{DegRevLex}} t$ belongs to the ideal I.

Theorem 2 ([7, Theorem 4.6])**.** *Let I be an almost revlex ideal of S. Then $(S/I, x_l, \dots, x_1)$ has the l-SLP.*

Theorem 3 ([8, Theorem 5.8])**.** *Let I be a homogeneous ideal of $S = \mathbb{K}[x, y, z]$ such that S/I has the SLP. Then $\operatorname{rgin}(I)$ is an almost revlex ideal and it is uniquely determined by the Hilbert function of I.*

Given I a homogeneous ideal of S, we will denote by $\hat{I}$ the following ideal

$$\hat{I} = I + \langle x_1, \dots, x_l \rangle^{\operatorname{reg}(I)+1},$$

where $\operatorname{reg}(I)$ denotes the **Castelnuovo-Mumford regularity** of I. The introduction of $\hat{I}$ allows us to relate the non-Artinian case with the Artinian one.

Theorem 4 ([7, Theorem 5.4, Corollary 5.5])**.** *Let I be a homogeneous ideal of S and $1 \leq k \leq l$. Then the following facts are equivalent*

(1) the graded ring S/I has the k-SLP (respectively the k-WLP),

(2) the graded Artinian ring $S/\hat{I}$ has the k-SLP (respectively the k-WLP).

Theorem 5 ([7, Theorem 5.6])**.** *Let I be a monomial ideal of S. Then I is an almost revlex ideal if and only if $\hat{I}$ is an almost revlex ideal.*

2. Hyperplane arrangements and k-Lefschetz properties

A finite set of affine hyperplanes $\mathcal{A} = \{H_1, \dots, H_n\}$ in $\mathbb{K}^l$ is called a **hyperplane arrangement**. For each hyperplane H_i we fix a defining linear polynomial $\alpha_i \in S$ such that $H_i = \alpha_i^{-1}(0)$, and let $Q(\mathcal{A}) = \prod_{i=1}^{n} \alpha_i$. An arrangement $\mathcal{A}$ is called **central** if each H_i contains the origin of $\mathbb{K}^l$. In this case, each $\alpha_i \in S$ is a linear homogeneous polynomial, and hence $Q(\mathcal{A})$ is homogeneous of degree n. We denote by $\operatorname{Der}_{\mathbb{K}^l} = \{\sum_{i=1}^{l} f_i \partial_{x_i} \mid f_i \in S\}$ the S-module of **polynomial vector fields** on $\mathbb{K}^l$ (or S-derivations). We will say that a central arrangement $\mathcal{A}$ is **free** if and only if the **module of vector fields logarithmic tangent** to $\mathcal{A}$ (or logarithmic vector fields)

$$D(\mathcal{A}) = \{\delta \in \operatorname{Der}_{\mathbb{K}^l} \mid \delta(\alpha_i) \in \langle \alpha_i \rangle, \forall i\}$$

is a free S-module. If we denote by $J(\mathcal{A})$ the **Jacobian ideal** of $\mathcal{A}$, we can connect it to the study of free arrangements (see also [6]).

Theorem 6 ([2, Theorem 5.4])**.** *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement in $\mathbb{K}^l$. Then $\mathcal{A}$ is free if and only if $\operatorname{rgin}(J(\mathcal{A}))$ coincides with S or it is minimally generated by*

$$x_1^{n-1},\ x_1^{n-2}x_2^{\lambda_1},\ \dots,\ x_2^{\lambda_{n-1}}$$

with $1 \leq \lambda_1 < \lambda_2 < \dots < \lambda_{n-1}$ and $\lambda_{i+1} - \lambda_i = 1$ or 2.

Conjecture 7 ([2, Conjecture 5.7])**.** *Let $\mathcal{A}$ be a central arrangement in $\mathbb{K}^l$ and $d_0 = \min\{d \mid x_2^d \in \operatorname{rgin}(J(\mathcal{A}))\}$. If $\operatorname{rgin}(J(\mathcal{A}))$ has a minimal generator t that involves the third variable of S, then $\deg(t) \geq d_0$.*

The study of the Lefschetz properties has already been linked to the theory of arrangements, see for example [8]. The following results deepen this connection.

Proposition 8 ([7, Lemma 8.1]). *Let $\mathcal{A}$ be a central arrangement in $\mathbb{K}^2$. Then $S/J(\mathcal{A})$ has the 2-SLP.*

Putting together [7, Theorem 8.2] and [8, Theorem 8.5, Proposition 8.10], we obtain the following result.

Theorem 9. *Let $\mathcal{A}$ be a free arrangement in $\mathbb{K}^l$. Then*

(1) $S/J(\mathcal{A})$ has the l-SLP,
(2) $S/J(\mathcal{A})$ has an increasing Hilbert function,
(3) $\mathrm{rgin}(J(\mathcal{A}))$ is an almost revlex ideal.

Improving on [7, Proposition 8.5], we have the following result.

Theorem 10. *Let $\mathcal{A}$ be a central arrangement in $\mathbb{K}^3$. Then Conjecture 7 holds for $\mathcal{A}$ if and only if $S/J(\mathcal{A})$ has the 3-WLP.*

It is then natural to rewrite Conjecture 7 in $\mathbb{K}^3$ using the language of the k-WLP.

Conjecture 11. *Let $\mathcal{A}$ be a central arrangement in $\mathbb{K}^3$. Then $S/J(\mathcal{A})$ has the 3-WLP.*

Remark 12. *We are currently working on a proof of this conjecture. Moreover, notice that we have examples of central arrangements in $\mathbb{K}^4$ for which $S/J(\mathcal{A})$ does not have the WLP, and of central arrangements in $\mathbb{K}^3$ for which $S/J(\mathcal{A})$ does not have the SLP.*

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Singularities of configuration polynomials

Uli Walther

(joint work with Graham Denham, Delphine Pol, Mathias Schulze)

Notation and introduction. Fix a field $\mathbb{K}$ and a finite set E. Then let $W \subseteq \mathbb{K}^E$ be a subspace of the affine space $\mathbb{K}^E = \bigoplus_{e \in E} \mathbb{K} \cdot e$ that is not contained in a coordinate hyperplane of $\mathbb{K}^E$.

Let $\check{e}$ be the functional on $\mathbb{K}^E$ dual to $e \in E$, relative to the distinguished basis E for $\mathbb{K}^E$. The intersection of W with the union of the coordinate planes $(\check{e} = 0)$ produces a hyperplane arrangement in W, and defines a matroid M_W whose set of bases $\mathcal{B}$ consists of the maximal subsets B of E such that the restrictions to W of the functionals $\{\check{e} | e \in B\}$ are linearly independent. The rank of this matroid is $\dim_{\mathbb{K}}(W)$, and W is called a (linear) realization of M_W.

Let $x_E = \{x_e\}_{e \in E}$ be indeterminates. The generic diagonal quadratic form $Q_E = \sum_{e \in E} x_e \check{e} \otimes \check{e}$ on $\mathbb{K}^E$ restricts to a bilinear symmetric *configuration form*

$$Q_W := Q_E|_{W \times W} \colon W \times W \to \mathbb{K}[x_E]$$

on W. Via a basis for W one can express Q_W in terms of a symmetric matrix whose coefficients are linear functions in x_E. This matrix varies with the basis of W by conjugation with the appropriate element of $GL(W)$. Despite this variability, we denote it by Q_W as well. The *configuration polynomial* $\psi_W \in \mathbb{K}[x_E]$ of degree $\mathrm{rk}(\mathsf{M}_W)$ is the determinant of Q_W, defined by W up to unit factors from $\mathbb{K}$.

It is true in general that

$$\psi_W = \sum_{B \in \mathcal{B}_{\mathsf{M}}} c_{W,B} \cdot \prod_{e \in B} x_e,$$

where the coefficients $c_{W,B}$ vary covariantly under a change of basis for W. The hypersurface X_W defined by $\psi_W = 0$ is the *first degeneracy locus* of the form Q_W.

A special case arises from a graph G with edge set $\mathcal{E}$ and vertex set $\mathcal{V}$. If $I_G \colon \mathbb{K}^{\mathcal{E}} \to \mathbb{K}^{\mathcal{V}}$ is the incidence matrix, let ψ_{W_G} be the configuration polynomial to the realization $W_G := \mathrm{rowspan}(I_G) \subseteq \mathbb{K}^{\mathcal{E}}$ of the matroid whose bases are the spanning forests of G. It turns out that ψ_G is the Kirchhoff polynomial of G, so $c_{W,B} = 1$ for all bases. For general matroids, we found that the configuration polynomial can differ significantly in its geometry from the usual matroid polynomial (defined as the sum over the monomials indexed by the bases of the matroid).

Motivation and references. The surface $X_G := X_{W_G}$, and its projectivization $\mathbb{P}X_G$, are of great interest in the theory of particle scattering, since ψ_{W_G} appears in the denominator of the associated Feynman integral, causing anomalous behavior near the critical points of ψ_G This talk is concerned with the structure of the singular locus of X_W, and the topology of the complement Y_W of $\mathbb{P}X_W$ in projective space.

The main references for this talk are: the groundbreaking article [5] on ψ_G; the article [12] that discusses advances on X_W for the matroid setup; work of Aluffi and Marcolli on combinatorial methods; (*e.g.*, [1]); articles of Brown [6, 7] and

Marcolli [10] that discuss the relevance of Kirchhoff polynomials in physics. For more background see the bibliography trees in [5, 6, 8, 9, 10].

The singular locus of X_W. The *second degeneracy scheme* $\Delta_W \subseteq \mathbb{K}^E$ of Q_W is defined by the submaximal minors of Q_W. It is a subscheme of the *Jacobian scheme* $\Sigma_W \subseteq \mathbb{K}^E$ of X_W, defined by ψ_W and its partial derivatives. By [12], Σ_W and Δ_W have the same underlying reduced scheme:

$$\Delta_W \subseteq \Sigma_W \subseteq \mathbb{K}^E, \quad \Sigma_W^{\mathrm{red}} = \Delta_W^{\mathrm{red}}.$$

Our basic main results from are compactly presented as follows.

Theorem 1 ([8]).

- *X_W is reduced and generically smooth over $\mathbb{K}$.*
- *Δ_W is reduced and equals Σ_W^{red}, the (reduced) non-smooth locus of X_W.*
- *If 2 is a unit in $\mathbb{K}$, the Jacobian scheme Σ_W is generically reduced.*

The configuration polynomial ψ_W factors if and only if the matroid is a direct product of matroids on disjoint subsets of E. Thus, the singular locus of X_W is dominated by the intersections of its components unless the matroid is connected.

Theorem 2 ([8]). *If M_W is connected with $\mathrm{rk}(M) > 0$, then X_W is integral. If in addition $\mathrm{rk}(\mathsf{M}_W) \geq 2$ then Δ_W is Cohen–Macaulay of codimension 3 in $\mathbb{K}^E$. If, moreover, M is 3-connected, then Δ_W is integral.*

While the main objective of [8] is to establish the results above, along the way we continue the systematic study of configuration polynomials in the spirit of [5, 12]. For instance, we describe the behavior of configuration polynomials with respect to connectedness, duality, deletion/contraction and 2-separations.

Our proofs intertwine methods from matroid theory, commutative algebra and algebraic geometry. An important commutative algebra ingredient is a result of Kutz: the grade of an ideal of submaximal minors of a symmetric matrix cannot exceed 3, and equality forces the ideal to be perfect. We make use of deformation to the normal cone, which leads to disucssions of Rees rings.

On the matroid side our approach makes use of *handle decompositions* allowing proofs on connected matroids by induction. A distinguished role as induction base is played by the prism matroid derived from the biparticte complete graph $K_{2,3}$, and Tutte's wheels-and-whirls theorem.

On the complement of $\mathbb{P}X_W$. Here, $\mathsf{M} = \mathsf{M}_G$ is induced from the incidence matrix of the graph G with edge set $\mathcal{E}$. We consider the projective hypersurface $\mathbb{P}X_G$ defined by the vanishing of ψ_G, and its complement Y_G in $\mathbb{P}\mathbb{K}^{\mathcal{E}}$.

Certain aspects of Y_G have been studied for a long time. For example, let $\psi_G^\perp = \sum_{B \in \mathcal{B}_{\mathsf{M}_G}} \prod_{e \in (\mathcal{E} \setminus B)} x_e$ be the *Szymanzik polynomial.* Then [2] suggests that the Euler characteristic $\chi(Y_G^\perp)$ of the complement $Y_G^\perp = \mathbb{P}^{\mathcal{E}} \setminus \mathrm{Var}(\psi_G^\perp)$ should be within the set $\{-1, 0, 1\}$ for every graph G, and attributes this conjecture to Aluffi. For a planar graph, $\psi_G^\perp$ is the same as $\psi_{G^\perp}$ attached to the dual graph $G^\perp$.

Aluffi's conjecture is known for wheels and cycle graphs. In [11], Müller-Stach and Westrich pioneer the use of a torus action to study $\chi(Y_G^\perp)$ via the results of

[4]. They obtain a quick proof of Aluffi's conjecture for disconnected graphs or graphs with a nexus (these permit monomial torus actions), and identify a class of connected planar graphs named $\star$-graphs with a non-monomial action, verifying Aluffi's conjecture for these graphs. We found that the dual of a $\star$-graphs has a cone vertex, and prove that $\chi(Y_G) = 0$ for all non-degenerate cone graphs.

We extended this vanishing significantly further, to the case where G contains a *fat nexus* (removal of its adjacent vertices disconnects the graph). We use Hadamard products to show that the presence of such vertex induces a non-monomial torus action on Y_G whose fixed point set is a union of three nonempty projective spaces of predictable dimension. Aluffi's conjecture follows for all planar graphs whose dual has a fat nexus, as they have have $\chi(Y_G^\perp) = 0$.

As working tools we establish duality formulæ that relate Y_G and its intersections with coordinate hyperplanes with the corresponding gadgets for $Y_G^\perp$. Much of this is done on the level of the Grothendieck ring of varieties, through which χ factors. The underlying geometric construction is the Cremona transform and Möbius inversion based on inclusion/exclusion.

As an application we construct a planar connected graph G with $\chi(Y_G) = -2$, and deduce that $\chi(Y_G)$ can take any integer value. This shows that Aluffi's conjeture is not true in general, even for planar graphs. It remains an open question to develop general recursions for $\chi(Y_G^\perp)$ and to classify G by $\chi(Y_G^\perp)$.

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The cohomology of $\overline{\mathcal{M}}_{0,n+1}$ is Koszul (a proof of a conjecture of Manin)

Vladimir Dotsenko

The purpose of this talk is to report the main results of the article [2] which offers a positive solution of a conjecture of Manin on the cohomology of $\overline{\mathcal{M}}_{0,n+1}$, the Deligne–Mumford compactification of moduli spaces of complex projective lines with $n+1$ marked points. The rational cohomology algebras of those spaces are isomorphic to their Chow rings and can be explicitly presented by generators and relations and admit various nice bases, yet not all natural questions about those algebras admit easy answers. For purposes of rational homotopy theory, it would be highly advantageous to know that the rational cohomology algebras of $\overline{\mathcal{M}}_{0,n+1}$ are Koszul. This question has been open for some 15 years; Manin [8, Section 3.6.3] asked it in a more general context of genus zero components of the extended modular operad [7]; Readdy [11] mentioned that the same had been asked by Reiner, and Petersen [9] asked this a few years after that. Until recently, it has been proved only for $n \leq 6$.

The most common way to prove Koszulness of an algebra presented by generators and relations amounts to exhibiting a quadratic Gröbner basis [10] for a certain monomial ordering. There are two celebrated presentations of the cohomology algebra of $\overline{\mathcal{M}}_{0,n+1}$, one due to Keel [6] and the other due to De Concini and Procesi [1]. It turns out that those presentations are not suitable for our purposes (which perhaps explains why the problem remained open).

Proposition 1. *Neither the Keel presentation nor the De Concini–Procesi presentation for $H^\bullet(\overline{\mathcal{M}}_{0,n+1},\mathbb{Q})$ admits a (linear and) quadratic Gröbner basis (either commutative or noncommutative) for $n \geq 4$, no matter what ordering of monomials one chooses.*

The presentation which ended up crucial for our solution was found independently by Etingof, Henriques, Kamnitzer and Rains in [4] and by Singh in [12]. This presentation identifies $H^\bullet(\overline{\mathcal{M}}_{0,n+1},\mathbb{Q})$ with the quotient of the polynomial algebra in variables X_S, $S \subseteq \{1,\dots,n\}$, $|S| \geq 3$, modulo the ideal generated by the following three groups of polynomials:

- X_S^2, where $|S| = 3$,
- $X_S(X_S - X_{S\setminus\{s\}})$, where $|S| > 3$, and $s \in S$,
- $(X_{S\cup T} - X_S)(X_{S\cup T} - X_T)$, where $S \cap T \neq$, $S \not\subseteq T$, $T \not\subseteq S$.

Using the operad structure on the collection of graded vector spaces $H_\bullet(\overline{\mathcal{M}}_{0,n+1},\mathbb{Q})$ and the Gröbner basis for that operad found in [3], the author was able to find a suitable ordering of monomials to show that this presentation has a quadratic Gröbner basis of relations.

The ordering is constructed as follows. Consider the following binary relation $\prec'$ on the set of all subsets of $\{1,\dots,n\}$: we say that $I \prec' J$ if either $J = I\setminus\{a\}$ where $a \in I$, $a \neq \max(I)$, or $I = J \setminus \max(J)$. Let $\prec$ be the transitive closure of $\prec'$. One can show that $\prec$ is a partial order. Let us fix some extension of that partial order to a total order; we denote that extension by $\triangleleft$. We may use this order to define a graded lexicographic ordering of the generators X_I ($I \subseteq \{1, \ldots, n\}$, $|I| \geq 3$) of the cohomology algebra $H^\bullet(\overline{\mathcal{M}}_{0,n+1}, \mathbb{Q})$; we denote that ordering by the same symbol $\triangleleft$. It turns out that for any such ordering $\triangleleft$, the ideal of defining relations of the cohomology has a quadratic Gröbner basis. This proves the following result.

Theorem 2. *The ring* $H^\bullet(\overline{\mathcal{M}}_{0,n+1}, \mathbb{Q})$ *is Koszul.*

This theorem allows one to present the rational homotopy Lie algebra

$$\pi_*(\Omega\overline{\mathcal{M}}_{0,n+1}) \otimes \mathbb{Q}$$

by generators and relations, and obtain a formula for its Hilbert series. To the best of our knowledge, no substantial information on rational homotopy groups of spaces $\overline{\mathcal{M}}_{0,n+1}$) has been available before.

The author expects that this result can be generalised to a wide class of cohomology algebras of De Concini–Procesi wonderful models: the bravest conjecture here is that the cohomology algebra is Koszul for any building set such that the corresponding nested set complex is a flag complex. It is possible that such result is even available for the algebra associated to an arbitrary atomistic lattice in work of Feichtner and Yuzvinsky [5].

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H-constants and line arrangements

Brian Harbourne

The concept of H-constants was introduced at Oberwolfach in mini-workshop 1040b in 2010 (organized by Th. Bauer, S. Di Rocco, B. Harbourne and T. Szemberg) with the goal of studying how singular a reduced plane algebraic curve can be. The main motivation originally came from the Bounded Negativity Conjecture.

Definition: Let $C \subset \mathbb{P}^2$ be a reduced singular curve with $d = \deg(C)$, $\mathrm{Sing}(C) = \{p_1, \ldots, p_r\}$, and $m_i = \mathrm{mult}_{p_i}(C)$. Then

$$H(C) = \frac{d^2 - (m_1^2 + \cdots + m_r^2)}{r}.$$

The version of the Bounded Negativity Conjecture (BNC) most relevant to H-constants is for rational surfaces. It has two formulations. These formulations are equivalent in that one is true if and only if the other is (see [6]).

Bounded Negativity Conjecture (rational case): Let X be a smooth projective rational surface over a field $k = \overline{k}$ of arbitrary characteristic.

(1) There is a bound b_X such that $C^2 \geq b_X$ for all reduced, irreducible curves $C \subset X$.
(2) There is a bound B_X such that $C^2 \geq B_X$ for all reduced curves $C \subset X$.

It now follows that:

Theorem: If $\inf_{\substack{C \subset \mathbb{P}^2 \\ \text{reduced,} \\ \text{singular curve}}} H(C) > -\infty$, then the BNC holds.

Remarks: (1) No example is known of a reduced and irreducible singular curve $C \subset \mathbb{P}^2$ with $H(C) \leq -2$. However, if C_d is a general plane rational curve of degree d, then

$$H(C_d) = -2 + \frac{6d - 4}{(d-1)(d-2)}$$

so

$$\inf_{\substack{C \subset \mathbb{P}^2 \text{ reduced, irreducible} \\ \text{singular curve}}} H(C) \leq -2.$$

(2) No example is known of a reduced, singular curve $C \subset \mathbb{P}^2$ over $k = \mathbb{C}$ with $H(C) \leq -4$. However, examples show

$$\inf_{\substack{C \subset \mathbb{P}^2 \text{ reduced,} \\ \text{singular curve}}} H(C) \leq -4.$$

See [12]. (In any positive characteristic, however, taking collections of lines for C shows that $\inf_C H(C) = -\infty$.)

In order to understand better what happens for reduced but not irreducible plane curves, it is reasonable to study curves all of whose components are lines; i.e., line arrangements. As the parenthetical comment above suggests, this is of most interest in characteristic 0. There are a number of results, and also some open questions, the first of which is:

Open Question: What is the most negative value of $H(C)$ when C is an arrangement of lines defined over $\mathbb{Q}$?

The most negative such example C found so far has

$$H(C) = -\frac{503}{181} \approx -2.78.$$

In this case C is a simplicial arrangement of 37 lines denoted $\mathcal{A}(37, 3)$ in [9]. It came from checking the list of known simplicial line arrangements (see [9, 7, 8]).

For real line arrangements C we have the result that both $H(C) > -3$ and $\inf_C H(C) = -3$, where the infimum is obtained using simplicial, supersolvable examples (in particular, let C be the sides and lines of symmetry of a regular n-gon for odd $n \geq 3$) [5].

For line arrangements C over $\mathbb{C}$, less is known. We have $H(C) > -4$, but the most negative known example comes from a curve C of 45 lines constructed by A. Wiman in 1896 (see [13, 5, 4]). It has

$$H(C) = -\frac{225}{67} \approx -3.56.$$

It is not clear what the greatest lower bound is for H-constants of complex line arrangements, so we have the following question:

Open Question: How negative can $H(C)$ be for a complex line arrangement C?

It is also not clear what one can say about H-constants for free complex line arrangements, but as a starting point it is interesting to look at the subclass of complex supersolvable line arrangements.

Open Question: How negative can $H(C)$ be when C is a complex and supersolvable line arrangement? (Is it true, for example, that $H(C) > -3$ in this case?)

Recall: A singular point p of a line arrangement C is *modular* if every other singular point of C is on a component of C through p. We say C is *supersolvable* if it has one or more modular points.

A possible approach to the above open question is to classify the complex supersolvable C. A classification has been given for all such C with more than one modular point (see [2, 10]), and for these we do have $H(C) > -3$.

Open Problem: Classify all complex, supersolvable line arrangements with exactly one modular point.

When C is a line arrangement, we have $H(C) = (d^2 - \sum_{i=1}^{r} m_i^2)/r = (d - \sum m_i)/r = d/r - \overline{m}$. Thus having any $m_i = 2$ seems bad if you want very negative $H(C)$.

Conjecture (Anzis and Tohăneanu [3]; now a theorem of Abe [1]): Let t_2 be the number of points of multiplicity 2. For a complex supersolvable C of d lines, we have $t_2 \geq d/2$.

The Wiman arrangement of 45 lines has $t_2 = 0$. Maybe others with $t_2 = 0$ would also give very negative H-constants.

Open Problem: Classify complex line arrangements with $t_2 = 0$.

Only 4 kinds are currently known. (For $k \geq 2$, let t_k be the number of points of multiplicity k. In the list below $t_k = 0$ except as otherwise noted.)

(1) Trivial cases ($d \geq 3$ concurrent lines, so $t_d = 1$). Here $H(C) = 0$.
(2) so-called Fermat arrangements C, defined by $(x^n - y^n)(x^n - z^n)(y^n - z^n) = 0$ with $n \geq 3$. Thus C has $3n$ lines with $t_3 = 12$ if $n = 3$, and $t_3 = n^2$, $t_n = 3$ if $n > 3$. Here $H(C) = -3 + (9/(n^2 + n))$.
(3) Klein's (1879, [11]) 21 lines with $t_3 = 28$ and $t_4 = 21$. Here $H(C) = -3$.
(4) Wiman's (1896, [13]) 45 lines with $t_3 = 120$, $t_4 = 45$ and $t_5 = 36$. Here $H(C) = -225/67 \approx -3.56$.

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Quasi-invariants and free multi-arrangements

Misha Feigin

(joint work with Takuro Abe, Naoya Enomoto, Masahiko Yoshinaga)

Let $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials. Let R be a reduced root system of a Coxeter group W in $V \cong \mathbb{R}^n$. Let $m\colon R \to \mathbb{N}$ be a W-invariant multiplicity function. We denote $m_\alpha := m(\alpha)$, $\alpha \in R$. A polynomial $q \in \mathbb{C}[x]$ is called *quasi-invariant* if the following conditions hold true for any $\alpha \in R$:

$$\partial_\alpha^{2s-1} q(x) = 0 \text{ at } (\alpha, x) = 0$$

for any integer $s = 1, \ldots, m_\alpha$, where $\partial_\alpha = (\alpha, \frac{\partial}{\partial x})$ is the normal derivative to the hyperplane $\Pi_\alpha = \{x \in V : (\alpha, x) = 0\}$. It is easy to see that these polynomials form a ring which we denote by $Q_m = Q_m(R)$. For constant multiplicity function we have the following inclusion of spaces of quasi-invariants:

$$\mathbb{C}[x] = Q_0 \supseteq Q_1 \supseteq Q_2 \supseteq \ldots \supseteq \mathbb{C}[x]^W = Q_\infty.$$

Rings of quasi-invariants appeared first in the work [1] of Chalykh and Veselov on quantum Calogero–Moser systems. They showed that there exists a homomorphism χ from Q_m to the ring of differential operators such that for any homogeneous $q \in Q_m$

$$\chi(q) = q(\partial) + \text{lower order differential operator.}$$

Furthermore,

$$\chi(x_1^2 + \ldots + x_n^2) = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \sum_{\alpha \in R_+} \frac{m_\alpha(m_\alpha+1)(\alpha,\alpha)}{(\alpha,x)^2} \tag{1}$$

is the generalised Calogero–Moser operator related to the root system R.

This implies algebraic integrability of the operator (1) as it can be included in a commutative ring containing n algebraically independent differential operators corresponding to basic invariants as well as additional elements.

The study of algebraic properties of quasi-invariants was started in the work [2] of Veselov and the speaker where the term was also introduced. In particular, some conjectures were put forward which were established by Etingof and Ginzburg shortly afterwads in [3]. Thus it was shown in [3] that the space Q_m is a free module over the ring of invariants $\mathbb{C}[x]^W$ of dimension $|W|$. Furthermore, W-module $Q_m/I \cong \mathbb{C}W$, where I is the ideal in Q_m generated by homogeneous W-invariant polynomials of positive degree.

We show that the isotypic component Q_m^V of the reflection representation V of W in quasi-invariants is closely related to the logarithmic vector fields. Consider the following vector fields with polynomial coefficients:

$$D_m = \{L\colon L(\alpha, x) = (\alpha, x)^{2m_\alpha} p_\alpha \text{ for any } \alpha \in R\},$$

where $p_\alpha \in \mathbb{C}[x]$. It is known that D_m is a free module over polynomials $\mathbb{C}[x]$ and that W-invariant vector fields $D_m^W \subset D_m$ form a free module over invariant polynomials $\mathbb{C}[x]^W$ [4]. Relation with quasi-invariants is given by the following result:

$$Q_m^V \cong D_m^W \otimes V, \tag{2}$$

which is an isomorphism of modules over $\mathbb{C}[x]^W \otimes \mathbb{C}W$ [5].

Note that relation (2) is an isomorphism of free $\mathbb{C}[x]^W$-modules and that it follows that components of a free basis of the module D_m^W are elements from $Q_m^V \subset Q_m$. This has implications both for quasi-invariants and for the logarithmic vector fields. For example, for the multiplicity function $m \geq 1$ we establish the isomorphism [5]

$$\mathcal{D} : Q_m^V \to Q_{m-1}^V,$$

where $\mathcal{D}$ is the Saito primitive derivative, that is $\mathcal{D} = \frac{\partial}{\partial y_n}$ where y_n is a basic invariant of the highest degree. One also gets integral formulas for the components of basis logarithmic vector fields in some cases.

Above considerations can be generalised in two directions. Firstly, one can deal with the case of finite complex reflection groups W. The corresponding quasi-invariant polynomials were introduced and studied in [6]. Using these results we show that complex reflection multi-arrangements $\mathcal{A}$ are free provided that multiplicity function has the form

$$m(H) = c(H)n(H) + 1, \quad (H \in \mathcal{A}),$$

where $n(H)$ is the size of the stabilizer $W_H \subset W$ of the hyperplane H, and $c\colon \mathcal{A} \to \mathbb{N}$ is W-invariant [5]. The same statement holds true for the multiplicity function $m(H) = c(H)n(H)$ [5]. This strengthens the results of [7] where the freeness was established in the case $c = const$ for well-generated groups W and for the series $W = G(r, p, n)$.

The second generalisation is for the non-homogeneous version of quasi-invariants Q_m^{tr} and the extended Catalan arrangements. A polynomial $q(x)$ belongs to the space Q_m^{tr} if and only if the following conditions hold true for all $\alpha \in R$:

$$q(x + \frac{1}{2}s\alpha^\vee) = q(x - \frac{1}{2}s\alpha^\vee) \text{ at } (\alpha, x) = 0$$

where $s = 1, \ldots, m_\alpha$ and $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ (see [1], where these quasi-invariants are considered in relation with generalised Calogero–Moser systems with trigonometric potentials). One can check that the highest order term of such a polynomial q belongs to the space of quasi-invariants Q_m.

Recall also the extended Catalan arrangement

$$Cat = \{\Pi_{\alpha,j} : \alpha \in R_+, -m_\alpha \leq j \leq m_\alpha\},$$

where $\Pi_{\alpha,j} = \{x \in V : (\alpha, x) = j\}$. And let $D(Cat)$ be the corresponding module of logarithmic vector fields:

$$D(Cat) = \{L\colon (\alpha - j) | L(\alpha), \text{ for any } \alpha \in R, -m_\alpha \leq j \leq m_\alpha\}.$$

Similarly to the statement (2) we have the following isomorphism of modules over $\mathbb{C}[x]^W \otimes \mathbb{C}W$ [5]:

$$Q_m^{tr,V} \cong D(Cat)^W \otimes V. \tag{3}$$

Freeness results for quasi-invariants Q_m^{tr} (see [8]) then lead to freeness of the extended Catalan arrangements known from [9], [10]. In the case of the non-reduced root system $R = BC_n$ we establish freeness of the following arrangement. Let $k, r, l \in \mathbb{N}$. The arrangement is defined by the equation $P = 0$, where

$$P = \prod_{i=1}^{n}\left(x_i \prod_{j=1}^{k}(x_i^2 - j^2)\prod_{j=1}^{r}(x_i^2 - (k+2j)^2)\right)\prod_{i<j}^{n}\prod_{\epsilon\in\{\pm 1\}}\left((x_i - \epsilon x_j)\prod_{s=1}^{l}((x_i - \epsilon x_j)^2 - 4s^2)\right)$$

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A generating function approach to new representation stability phenomena in orbit configuration spaces

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(joint work with Nir Gadish)

Let X be a connected, locally compact, Hausdorff topological space. Let G be a finite group acting on X *almost freely*, so that the set $S = \{x \in X \colon \exists g \in G \setminus \{e\}, gx = x\}$ is finite. Following [7], define the *orbit configuration space* as

$$\begin{aligned}\mathrm{Conf}_n(G,X) :=& \{(x_1,\dots,x_n) \in X^n \colon Gx_i \cap Gx_j = \emptyset \text{ if } i \neq j; x_i \notin S\}\\ =& X^n \setminus \bigcup_{H\in\mathcal{A}_n(G,X)} H,\end{aligned}$$

where $\mathcal{A}_n(G,X) = \{H_{ij}(g), H_i^s\}_{i,j,g,s}$ with

$$\begin{aligned}H_{ij}(g) &= \{(x_1,\dots,x_n) \in X^n \colon gx_i = x_j\} && 1 \le i < j \le n, g \in G\\ H_i^s &= \{(x_1,\dots,x_n) \in X^n \colon x_i = s\} && 1 \le i \le n, s \in S\end{aligned}$$

A *layer* of $\mathcal{A}_n(G,X)$ is a nonempty connected component of some intersection $\cap_{H\in T} H$, where $T \subseteq \mathcal{A}_n(G,X)$. The *poset of layers* $P_n(G,X)$ is the set of layers of $\mathcal{A}_n(G,X)$, ordered by reverse inclusion.

Note that S_n acts on X^n by permuting coordinates and G acts on each coordinate of X^n. Together, one obtains an action of the wreath product group $S_n[G] = G^n \rtimes S_n$ on X^n, $\mathcal{A}_n(G,X)$, $P_n(G,X)$, and $\mathrm{Conf}_n(G,X)$.

Examples of interest include the ordered configuration space (G trivial; S empty), the complement of certain reflection arrangements ($G = \mathbb{Z}_d$ acting on $X = \mathbb{C}$ via multiplication by roots of unity; $S = \{0\}$), and the type C toric and elliptic arrangements ($G = \mathbb{Z}_2$ acting on $X = \mathbb{C}^\times$ or a complex elliptic curve via the group inversion; S is the set of two-torsion points).

The combinatorial structure of these arrangements is particularly fascinating. The posets $P_n(G,X)$ have a description using partitions akin to the Dowling lattice [1, Thm. C]. In particular, in the examples of hyperplane arrangements above, the lattice π_n of partitions of $\{1,2,\dots,n\}$ arises as $P_n(0,\mathbb{C})$, and the Dowling lattice $D_n(\mathbb{Z}_d)$ arises as $P_n(\mathbb{Z}_d,\mathbb{C})$. Moreover, the local structure of the poset of layers may be described in terms of these lattices: an interval $(P_n(G,X))_{\le\beta}$ is isomorphic to a product of partition and Dowling lattices [1, Thm. A].

Using this description via partitions, one obtains a factorization of the characteristic polynomial [1, Thm. B]:

$$\sum_{\beta\in P_n(G,X)} \mu(X^n,\beta)t^{\dim\beta} = \prod_{k=0}^{n-1}(t-|S|-|G|k),$$

generalizing that of partition and Dowling lattices [6, 4].

Turning to topology, I will now assume for simplicity that the action of G on X is free, so that $S=\emptyset$. The above factorization of the characteristic polynomial yields a factorization of the compactly supported Euler characteristic [1, Thm. E]:

$$\chi_c(\mathrm{Conf}_n(G,X)) = \prod_{k=0}^{n-1}(\chi_c(X)-|G|k) = n!G^n\binom{\chi_c(X)/|G|}{n},$$

which can easily be seen to have exponential generating function

$$\begin{aligned}\sum_{n\ge 0}\chi_c(\mathrm{Conf}_n(G,X))\frac{t^n}{n!} &= (1+|G|t)^{\chi_c(X)/|G|}\\ &= \prod_{n=1}^{\infty}\exp\left(|G|^{n-1}\cdot\chi_c(X)\cdot(-1)^{n-1}(n-1)!\cdot\frac{t^n}{n!}\right).\end{aligned}$$

Using the language of twisted commutative algebras, we can categorify this statement and encapsulate more information about the homology of orbit configuration spaces. A *GTCA* or *G-twisted commutative algebra* is a lax monoidal functor $A_\bullet : FB_G \to \mathcal{C}$, where $\mathcal{C}$ is a symmetric monoidal category and FB_G is the category of finite sets with G-bijections. This means that we assign to each finite set I an object $A_I \in \mathrm{ob}(\mathcal{C})$ with an action of the wreath product group $S_I[G]$, along with $S_I[G]\times S_J[G]$-equivariant maps $A_I\otimes A_J \to A_{I\sqcup J}$, satisfying some axioms.

To describe a GTCA, it suffices to consider only the finite sets $I = \{1, 2, \dots, n\}$, in which case we abbreviate $A_n := A_I$.

The example we are most interested in is the Borel–Moore homology of the orbit configuration space $\mathrm{Conf}_n(G, X)$, that is, the homology of its one-point compactification. For simplicity, we assume coefficients are in $\mathbb{Q}$ to obtain a GTCA $H_*^{BM}(\mathrm{Conf}_\bullet(G, X))$ of $\mathbb{Q}$-modules. The obvious open inclusion $\mathrm{Conf}_{n+m}(G, X) \hookrightarrow \mathrm{Conf}_n(G, X) \times \mathrm{Conf}_m(G, X)$ induces the multiplication $H_*^{BM}(\mathrm{Conf}_n(G, X); \mathbb{Q}) \otimes H_*^{BM}(\mathrm{Conf}_m(G, X); \mathbb{Q}) \to H_*^{BM}(\mathrm{Conf}_{n+m}(G, X); \mathbb{Q})$. We can study this GTCA through a spectral sequence:

Theorem 1. [2, Thm. A] *There is a spectral sequence of GTCA's converging to* $H_*^{BM}(\mathrm{Conf}_n(G, X))$ *with*

$$E^1 \cong \bigotimes_{n=1}^{\infty} \mathrm{Ind}_{G \times S_n}^{FB_G}(H_*^{BM}(X) \otimes \widetilde{H}_{n-3}(\overline{\pi}_n)),$$

where $\mathrm{Ind}_{G \times S_n}^{FB_G} V$ *indicates a GTCA freely generated by the* $G \times S_n$*-representation* V*, and where* $\widetilde{H}_{n-3}(\overline{\pi}_n)$ *denotes the reduced homology of the order complex of the partition lattice* π_n *with its top and bottom elements removed.*

Let us further assume that $\dim H_*^{BM}(X) < \infty$ and the top nonvanishing Borel–Moore homology group $H_d^{BM}(X)$ is in degree $d \geq 2$. By exploiting the explicit product structure in Theorem 1, one obtains stability structures in the homology of orbit configuration spaces. For instance, for a fixed i, the sequence $H_{d\bullet - i}^{BM}(\mathrm{Conf}_n(G, X))$ exhibits representation stability, in the sense of [3], via repeated multiplication by $H_d^{BM}(X)$ [2, Thm. B]. Using multiplication by other generators in the tensor decomposition of Theorem 1 yields higher-order notions of stability, for instance [2, Thm. C]. In particular, this approach offers a new perspective on secondary stability contrasting that of [5].

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Elliptic Reflection Arrangements

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(joint work with Yoshihisa Saito and Tadashi Ishibe)

An elliptic reflection arrangement is, similar to a classical finite reflection arrangement, the union of zero hyperplane loci of an elliptic root system [8]. On the complex elliptic period domain, the elliptic Weyl group of the root system acts properly and discontinuously. The fixed points set of the action is the elliptic reflection arrangement. Our main interest is the homotopy of its compliment, or, equivalent to say, of the regular orbit space of the elliptic Weyl group action. We propose three conjectures on them, which lead to a construction of counter examples to a long-standing question: whether discriminant compliments are $K(\pi, 1)$-spaces?

To understand the backgrounds and motivations of the present study, let us recall some results of the classical finite reflection arrangement case [1, 2, 3, 9].

Let W be a finite reflection group acting on a real vector space V of rank l irreducibly, and $\cup_\alpha H_\alpha$ be its reflection hyperplanes (= the union of fixed points of the action of W). According to the celebrated Chevalley's polynomial invariant theory, the invariants $S(V^*)^W$ (here, $S(V^*)$ is the symmetric tensor algebra of the dual space of V, i.e. the polynomial function ring on V) is generated by l-number of algebraically independent homogeneous polynomials, say $P_1, \cdots, P_l$. Denoting by $V_\mathbb{C}$ the complexification $V \otimes \mathbb{C}$, we have an identification $V_\mathbb{C}/W = \mathrm{Spec}(S(V_\mathbb{C}^*)^W) \underset{P_1,\cdots,P_l}{\simeq} \mathbb{C}^l =: S$. The square of the Jacobian $\Delta(P_1, \cdots, P_L)$ of the basic invariants defines the discriminant of the quotient morphism $V_\mathbb{C} \to S$, whose zero loci $D \subset S$, i.e. the discriminant loci, coincides with the set of irregular orbits $\cup_\alpha H_{\alpha\mathbb{C}}/W$ of the W-action on $V_\mathbb{C}$. Thus we obtain an identification:

$$S \setminus D \quad \simeq \quad (V_\mathbb{C}/W)_{reg} = (V_\mathbb{C} \setminus \cup H_\alpha)/W \ .$$

of the complement of the discriminant loci with the regular orbit space of W-action on $V_\mathbb{C}$. Let us recall three basic results on its homotopy groups.

1. Two descriptions of the fundamental group:
(a) The view point from the regular orbit space of W-action ([1]):
Using the 1-skeleton and 2 skeleton of the polyhedron dual to the Weyl chamber decomposition of V, Brieskorn got a *presentation of the fundamental group on the Coxeter-Dynkin diagram* [1] by Artin braid relations, where the Artin braid

[1]By a *presentation on the Coxeter Dynkin diagram*, we mean a presentation of a group whose generator system is in one to one correspondence with vertices of the diagram ($\simeq$ walls of a chamber), and whose relation system is described by edges of the diagram.

relations are the *homogenizations* [2] of the Coxeter relations. Such groups are nowadays called Artin groups of finite type.[3]

(b) Zariski-van Kampen presentation ([9]):
Using a real structure $S_{\mathbb{R}}$ and $D_{\mathbb{R}}$ of the quotient space S and its discriminant, we showed that the Zariski pencils in S (defined as integral orbits of the primitive vector field ∂_l) intersects with the discriminant only along the real axis. Then, for this "canonical chosen" Zariski-van Kampen generator system of the fundamental group of $S \setminus D$, the obtained relations are exactly the same Artin braid relations given in the above (a).

2. The vanishing of higher homotopy groups.
Deligne ([3]) described the nerve system for the (universal) covering system of $V_{\mathbb{C}} \setminus \cup H_{\alpha\mathbb{C}}$ using the gallery presentation of the *Artin monoid* (see Footnote 3), and, then, using the contractibility of the polyhedron dual to the chamber system, he showed the covering space of $V_{\mathbb{C}} \setminus \cup H_{\alpha\mathbb{C}}$ is contractible. That is, the space $S \setminus D = (V_{\mathbb{C}}/W)_{reg}$ is a $K(\pi,1)$-space for $\pi = A_M$ (an Artin group of finite type) and, hence, its higher homotopy groups vanish.

We switch to the elliptic case.

A subset R in a real vector space F equipped with a positive semidefinite metric I of signature $(l+2,2,0)$, i.e. $\mathrm{rank}_{\mathbb{R}}(\mathrm{rad}(I)) = 2$, satisfying some axioms, is called an *elliptic root system* ([8](I)). A rank 1 subspace $G = (a)$ of $\mathrm{rad}(I)$ for $\mathrm{rad}(I)_{\mathbb{Z}} = \mathbb{Z}a + \mathbb{Z}b$, is called a marking. All marked elliptic root system (R, G) are classified by elliptic diagrams $\Gamma(R, G)$ associated with them.[4]

The extended elliptic period domain $\widetilde{\mathbb{E}}$ is a family of complex affine subspaces of rank $l+1$ in $(F/G \oplus (\mathrm{rad}(I)/G)^*)^* \otimes \mathbb{C}$ parameterized by the upper half plane $\tau \in \mathbb{H} := \{x \in \mathrm{rad}(I)^*_{\mathbb{C}} \mid Im(b(x)/a(x)) > 0, a(x) = 1\}$, on which the extended elliptic

[2]By the "homogenization" of a Coxeter relation: $(ab)^m = 1$ for letters a, b and $m \in \mathbb{Z}_{>0}$, we mean the homogeneous relation: $ab\ldots = ba\ldots$ where the left (resp. right) hand-side is an alternating sequence of the letters a and b starting from a (resp. b) of length m. The homogeneous relation is called the Artin braid relation [2].

[3]An Artin group A_M and an Artin monoid A_M^+ for a Coxeter matrix $M \in M(n, \mathbb{Z}_{\geq 1} \cup \{\infty\})$ (a symmetric matrix whose diagonal elements are equal to 1), are defined as follows ([2]).

$$(1)\qquad \begin{array}{rcl} A_M & := & \langle a_1, \cdots, a_n \mid \underbrace{a_i a_j \cdots}_{m_{ij}-times} = \underbrace{a_j a_i \cdots}_{m_{ji}-times} \quad for\ 1 \leq i, j \leq n \rangle \\ A_M^+ & := & \langle a_1, \cdots, a_n \mid \underbrace{a_i a_j \cdots}_{m_{ij}-times} \sim \underbrace{a_j a_i \cdots}_{m_{ji}-times} \quad for\ 1 \leq i, j \leq n \rangle^+ \end{array}$$

where $\langle L \mid \mathcal{R} \rangle$ is the standard notation for a group generated by the set L of generators and defined by the fundamental relations $\mathcal{R}$, and $\langle L \mid \mathcal{R} \rangle^+$ is the notation for the quotient of the free monoid generated by the set L of letters and divided by the equivalence relation $\sim$ generated by the fundamental relations $\mathcal{R}$ (c.f. §3 (3.1) Definition). The natural localization morphism $A_M^+ \to A_M$ is known to be injective for some particular cases: (1) M is of finite types studied in above 1 ([2] and Deligne [3]), and (2) M is of affine types (L'Paris). This injectivity is one crucial step in Deligne's proof in 2. to show the contractibility of the covering system.

[4]Likewise a classical finite root system, vertices of an elliptic diagram represent certain "basis" of the root system and edges represents intersection numbers among them. Elliptic diagrams include some edge $\circ\!\!=\!\!=\!\!=\!\!\circ$, implying that the intersection number of corresponding roots is positive. This fact distinguishes elliptic diagram completely from classical Coxeter diagrams.

Weyl group $\widetilde{W}(R,G)$ acts as a reflection group properly discontinuously. The fixed point locus of the action is equal to the union of reflection hyperplanes $\cup_\alpha H_\alpha$, which is called the *elliptic reflection arrangement.* Using θ-invariant theory, we see that the ring of holomorphic $\widetilde{W}(R,G)$-invariant functions on $\widetilde{\mathbb{E}}$ is "isomorphic" to the ring of $l+1$-algebraically independent theta-functions $\theta_0,\cdots,\theta_l$ over $\mathcal{O}_{\mathbb{H}}$. This means that the orbit space $S_{ell} := \widetilde{\mathbb{E}}//_{ell}\widetilde{W}(R,G)$ is a family of $l+1$-dimensional complex affine spaces of coordinates $\theta_0,\cdots,\theta_l$, parameterized by $\mathbb{H}$.

The elliptic discriminant is defined by the square Θ_A^2 of the Jacobian $\Theta_A = Jac(\theta_0,\cdots,\theta_l)$, and its zero loci $D_{ell} \subset S_{ell}$ is identified with the union of the irregular orbits and the boundary component:$(\widetilde{\mathbb{E}}/\widetilde{W}(R,G))_{irr}\cup\mathbb{H}$.Thus, we obtain an identification ([8](II)) of two spaces:

$$S_{ell}\setminus D_{ell} \quad\simeq\quad (\widetilde{\mathbb{E}}/\widetilde{W}(R,G))_{reg}=(\widetilde{\mathbb{E}}\setminus\cup_{\alpha\in R}H_\alpha)/\widetilde{W}(R,G),$$

whose topology is the main subject of the present note. Namely, we ask the analogues of the previous results 1. (a), (b) and 2. to this new elliptic setting. Partial answers are given in the following 1.* (a)*, (b)* and 2.*

1.* (a)* The elliptic Weyl grpup $W(R)$ or its central extension $\widetilde{W}(R,G)$ act nowhere properly discontinuously on the real dual spaces F^* or $\widetilde{F}^*$ (that is, there is no analogue of Weyl chambers, alcoves and their galleries). Nevertheless, following three steps obtained recently, we get an affirmative analog of 1. (a).

Step 1. The groups $W(R)$ and $\widetilde{W}(R,G)$ are no-longer Coxeter groups, but are presented by generalizing Coxeter relations on the elliptic diagram $\Gamma(R,G)$ ([12]):

(i) Generators are in one to one correspondence with the reflections associated with the vertices of the elliptic diagram $\Gamma(R,G)$,
(ii) Relations are given according to subdiagrams of $\Gamma(R,G)$. The relations for subdiagrams consisting of two vertices are the classical Coxeter relations, but there are more relations involving 3 or 4 vertices of the elliptic diagram, which, altogether, we shall call elliptic Coxeter relations,
(iii) $W(R)$ satisfies one more relation: $\tilde{c}(R,G)^{m(R,G)}=1$, where $\tilde{c}(R,G)$ is a hyperbolic Coxeter element and $m(R,G)$ is the elliptic Coxeter number.

Step 2. Those inhomogeneous form of elliptic Coxeter relations are homogenized in order to get a description of Elliptic Hecke algebras ([10]).
Step 3. The group defined by these homogenized relations are shown ([11]) to be isomorphic to the fundamental groups of regular orbit spaces of extended Coxeter groups studied by Looijenga and Lek ([6, 5]). Thus, we shall call these homogenized elliptic Coxeter relations *elliptic braid relations* or *elliptic Arin relations.*

We define now algebraically the *elliptic Artin group* $A(R,G)$ and *Artin monoid* $A(R,G)^+$ attached to (R,G) as the quotient of the free group or the monoid generated by the elements corresponding to the vertices of the diagram $\Gamma(R,G)$ and divided by the relations generated by the elliptic braid relations, respectively. By definition, the elliptic Artin group is naturally isomorphic to the fundamental group of the elliptic discriminant compliment.

However, one unexpected surprising consequences is that the monoid $A(R,G)^+$ is no-longer cancellative ([4]).[5] In particular the natural localization morphism

[5]We say non-cancellative, if there exist $a,b,c,d\in A(R,G)^+$ such that $abd\sim acd$ but $b\not\sim c$.

$A(R,G)^+ \to A(R,G)$ is not injective. This causes a question: what is the geoemtric significance of the elliptic Artin monoid (see next paragraph for one answer).

1.* (b)* For the same reason as in 1.* (a)*, the elliptic period domains $\mathbb{E}$ and its extension $\widetilde{\mathbb{E}}$ are no-longer domains in a complexification of the real vector spaces. There seems not yet a reasonable "real structure theory" for them (due to shiftings of affine subspaces to complex directions) (cf. the classical case [9]).

Nevertheless, likewise the classical case, one can define Zariski pencils l_{t_*} by the integral orbit lines of the primitive vector field ∂_l on S_{ell} [6] , which intersect with the discriminant loci transversally exactly at $\mathrm{mult}_0(D_{ell}) = \#(|\Gamma(R,G)|)$ number of points (see [8] (I) (8.6) Similarity v)). However, the lack of the "real structure", the pencil do not intersect with discriminant along "real axis". Here, we ask:

Conjecture 1 ([4]) There exists a special choice of Zariski-van Kampen generator system of $\pi_1(l_{t_*} \setminus D_{ell}, *)$ ($*$: base point), which corresponds naturally in one to one with the vertices of the diagram $\Gamma(R,G)$, such that obtained Zariski-van Kampen relations coincide with elliptic braid relations given in 1.* (a)*.

The chosen generator system in Conjecture 1 define also the positive monoid $\mathcal{F}^+$ in $\pi_1(l_{t_*} \setminus D_{ell}, *)$. However, we have not cared in Conjecture 1, how to move the parameter t_* of the pencil l_{t_*} along paths ν in T_{ell} := the quotient space of S_{ell} by orbits of $\partial_l \underset{\tau,\theta_0,\cdots,\theta_{l-1}}{=} \mathbb{H} \times \mathbb{C}^l$. In the next, we ask this question precisely.

Conjecture 2 ([4]) There exists a special choice of generators $\nu_j \in \pi_1(T_{ell} \setminus B_{ell}, t_*)$ (B_{ell} is the bifurcation set of the projection $D_{ell} \to T_{ell}$) such that the associated Zariski-van Kampen relations $x = y$ for $x, y \in \mathcal{F}^+$ choose a unique 2-homotopy class $\mathcal{R}(x,y)$ in the set $[x,y]$ of all homotopy equivalences from x to y in $S_{ell} \setminus D_{ell}$[7] so that the generated relation choose at most one class $\mathcal{R}(x,y) \in [x,y]$ for $x, y \in \mathcal{F}^+$.

Conjecture 2 gives much more precise relations on the set of positive words in $\mathcal{F}^+$ than the usual 1-homotopy equivalence. So, the quotient monoid defined by this "sharp" homotopy equivalence is a monoid isomorphic the elliptic Artin monoid defined in 1.* (a)*, which we may call the *geometric elliptic Artin monoid.*

2.* Since elliptic Artin monoids are no-longer cancellative, a naive analogue of Deligne's proof of $K(\pi,1)$-ness for elliptic discriminant compliment spaces may no-longer work. Instead, we expect that some π_2-classes begin to appear as follows.

Let a, b, c, d be a non-cancellative quadruple in the sense of Footnote 5. According to Conjecture 2, there is a particular homotopy equivalence class $\mathcal{R}(abd, acd) \in [abd, acd]$. Then composing with the standard homotopy equivalences $a^{-1}a \sim *$

[6]A primitive derivation is the derivation of the invariant ring (either Chevalley's polynomial invariant ring in classical cases or the theta invariant ring in the elliptic cases) with respect to the highest weight basic invariant, which is unique up to a constant factor. In the primitive form theory [7], it corresponds to the shift of energy level, defining the Hodge filtration.

[7]Recall that Zariski-van Kampen construction of relations, moving pencils along the paths ν_j, shows not only existence of homotopy relations but also choose elements in the 2-homotopy classes in $[\tilde{\gamma}_i, \sigma(\nu_j)\tilde{\gamma}_i]$ for $1 \le i \le l+2$ and for a generator system $\{\nu_j\}_j$ of $\pi_1(T_{ell} \setminus B_{ell})$.

and $dd^{-1} \sim *$ before and after $id_{a^{-1}} \cdot \mathcal{R}(abd, acd) \cdot id_{d^{-1}}$, we obtain a class, denoted by $a\backslash\mathcal{R}(abd, acd)/d$, in $[* \cdot b \cdot *, * \cdot c \cdot *] = [b, c]$. Thus, if we have another non-cancellative quadruple a', b, c, d', then we obtain another $a'\backslash\mathcal{R}(a'bd', a'cd')/d'$ in the same $[b, c]$ so that their "difference" defines a class $\pi_2(S_{ell} \setminus D_{ell}, *)$. Actually, we know ([4]) that there are "enormous" set of such pairs of non-cancellative quadruples on the same pair $(b, c) \in (\mathcal{F}^+)^2$, whose nature is still to be studied. Then the final conjecture in the present note is the following:

Conjecture 3. ([4]) The set of all such difference classes:

$$a\backslash\mathcal{R}(abd, acd)/d - a'\backslash\mathcal{R}(a'bd', a'cd')/d,$$

attached to non-cancellative quadruples of $A(R, G)^+$ generate $\pi_2(S_{ell} \setminus D_{ell}, *)$.

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Formal line arrangements and rigid planar frameworks

Michael DiPasquale

In this abstract we give a preliminary report making a connection between the notion of **formal line arrangements** in the sense of Falk and Randell [2] and **rigid planar frameworks**; a recent survey of the latter topic can be found in [3].

We start by defining formal line arrangements. Let $\mathbb{K}$ be a field (for the sake of this abstract we may assume $\mathbb{K}$ is the real numbers). Suppose $\mathcal{A} \subset \mathbb{P}^2(\mathbb{K})$ is an arrangement of lines $L_1, \ldots, L_d$ which are the zero loci of the linear forms $\alpha_1 = a_{11}x + a_{21}y + a_{31}z, \ldots, \alpha_d = a_{1d}x + a_{2d}y + a_{3d}z$. The **coefficient matrix** of $\mathcal{A}$, which we denote by $C(\mathcal{A})$, is the $3 \times d$ matrix whose columns are the coefficients of $L_1, \ldots, L_d$. We call a vector $\mathbf{v} \in \ker(C(\mathcal{A}))$ a *linear dependence* on $\mathcal{A}$, and we

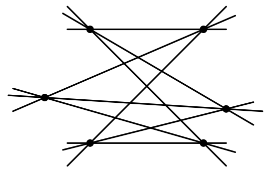

Figure 1. The example of Ziegler and Yuzvinsky

refer to the number of non-zero entries in $\mathbf{v}$ as its **length**. The following definition was first made in [2]. (The same definition extends to any dimension.)

Definition 1. A line arrangement is **formal** if all linear dependencies are generated by dependencies of length three.

Many classes of interesting arrangements are formal; Falk and Randell showed in [2] that $K(\pi, 1)$ arrangements are formal and Yuzvinsky proved in [6] that free arrangements are also formal. Even if an arrangement is not free, formality of the arrangement can have interesting consequences for the algebraic structure of the module of logarithmic derivations; see the following example which also illustrates that formality cannot be detected from the intersection lattice.

Example 2. Consider the arrangement of nine lines formed from a hexagon (not regular) by extending the six edges of the hexagon and taking also the lines passing through opposite vertices of the hexagon (see Figure 1). Yuzvinsky observes [6] that if the six triple points lie on a conic then this line arrangement is *not* formal; otherwise it is formal.

Additionally, Ziegler observes that the generators and relations of the module of logarithmic derivations for this line arrangement also changes depending on whether the six triple points lie on a conic [7].

We now turn to planar frameworks and their rigidity.

Definition 3. A **planar framework** $G(\mathbf{p})$ in $\mathbb{R}^2$ is a graph $G = (V, E)$ along with a **realization** $\mathbf{p} : V \to \mathbb{R}^2$ assigning coordinates to every vertex of G. An **infinitesimal motion** of $G(\mathbf{p})$ is a vector field $F : V \to \mathbb{R}^2$ satisfying that $F(u)-F(v)$ is orthogonal to $\mathbf{p}(u)-\mathbf{p}(v)$ for every pair of adjacent vertices $u, v \in V$.

Any framework has a three-dimensional space of **trivial** infinitesimal motions which arise from the rigid motions of $\mathbb{R}^2$. A framework is **infinitesimally rigid** if its only infinitesimal motions are the trivial infinitesimal motions. The rigidity of a framework depends heavily on the realization.

Example 4. The $K_{3,3}$ framework (see Figure 2) is infinitesimally rigid unless its vertices lie on a conic [4]. Notice that the example of Ziegler and Yuzvinsky in Example 2 is obtained by extending the edges of this framework.

Our primary observation is that (under appropriate hypotheses) a planar framework $G(\mathbf{p})$ is infinitesimally rigid if and only if the line arrangement $\mathcal{A}$ obtained

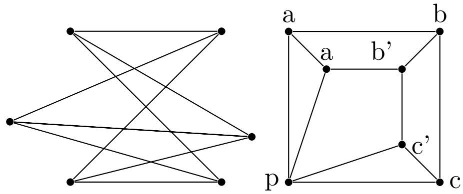

FIGURE 2. The $K_{3,3}$ framework (left). Another generically minimally rigid graph (right).

by extending its edges is formal. If $e = \{u, v\}$ is an edge of G with $\mathbf{p}(u) \neq \mathbf{p}(v)$, we write L_e for the line between $\mathbf{p}(u)$ and $\mathbf{p}(v)$.

Theorem 5. *Suppose $G(\mathbf{p})$ is a planar framework and the line arrangement $\mathcal{A}$ is obtained by extending the edges of $G(\mathbf{p})$. Suppose further that*

- *Any intersection point of $\mathcal{A}$ which is not a double point is a vertex of $G(\mathbf{p})$, i.e. every non-double point is of the form $\mathbf{p}(v)$ for some vertex v of G.*
- *A non-double point $\mathbf{p}(v)$ of $\mathcal{A}$ is contained in a line L_e of $\mathcal{A}$ if and only if v is a vertex of the edge e.*

Then $\mathcal{A}$ is formal if and only if $G(\mathbf{p})$ is infinitesimally rigid.

Proof. The proof of this theorem goes through the following steps, whose details we plan to give in a forthcoming paper. It is known that a planar framework admits a non-trivial infinitesimal motion if and only if it admits a non-trivial *parallel drawing* [5]. Furthermore, a line arrangement admits a non-trivial parallel drawing if and only if it is the truncation of an essential arrangement. A result of Yuzvinsky implies this is possible if and only if the arrangement is not formal [6]. □

We now show that Example 2 fits naturally into a larger family of arrangements. The complete bipartite graph $K_{3,3}$ is an example of a *generically minimally rigid graph.* These are graphs whose generic realization is infinitesimally rigid, and which lose this property upon the removal of any edge. The following combinatorial characterization of these graphs is well-known in rigidity theory - see [3].

Theorem 6. *A graph $G = (V, E)$ is generically minimally rigid if and only if:*

- $|E| = 2|V| - 3$ *and*
- $|E'| \leq 2|V'| - 3$ *for every subgraph $G' = (V', E')$ of G.*

White and Whiteley derive a polynomial in the coordinates of the vertices of a generically minimally rigid graph, called the **pure condition**, whose zero locus consists of the infinitesimally flexible realizations [4]. If an infinitesimally flexible realization can be extended to a line arrangement which satisfies the conditions of Theorem 5, then the generic realization of the graph extends to a formal arrangement while the flexible realization extends to a non-formal arrangement *with the same intersection lattice.* We conclude with such an example.

Example 7. The graph G depicted on the right in Figure 2 is generically minimally rigid. Let $ab, a'b', bc$, and $b'c'$ denote the lines passing through a and b, a' and b', b and c, and b' and c', respectively. A realization of this graph is infinitesimally rigid unless the three vertices of a triangle are collinear or

(1) the three points $ab \cap a'b'$, $bc \cap b'c'$, and p are collinear [4, Table 1].

Let $G(\mathbf{p}_1)$ be a generic realization of G and extend its edges to get an arrangement $\mathcal{A}_1$. Let $G(\mathbf{p}_2)$ be a realization having the property (1); extend its edges to get an arrangement $\mathcal{A}_2$. We can choose both realizations so that the conditions of Theorem 5 are met. Hence the intersection lattices of $\mathcal{A}_1$ and $\mathcal{A}_2$ are the same, $\mathcal{A}_1$ is formal, and $\mathcal{A}_2$ is not formal. A check in Macaulay2 suggests that the module of logarithmic derivations of $\mathcal{A}_1$ has a different minimal free resolution than $\mathcal{A}_2$.

The correspondence with infinitesimal motions of frameworks breaks down in higher dimensions, however an equivalence between formality and *non-trivial parallel drawings* [5] remains. We do not know if there is a 'parallel drawings' interpretation for the k-formality of Brandt and Terao [1].

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Discrete integrals and Catalan/Shi arrangements

Masahiko Yoshinaga

(joint work with Daisuke Suyama)

Let $\Phi \subset \mathbb{R}^\ell$ be an irrecudible root system and $\Phi^+ \subset \Phi$ be a positive system. For $a, b \in \mathbb{Z}$ with $a \leq b$, define $\mathcal{A}_\Phi^{[a,b]}$ by

$$\mathcal{A}_\Phi^{[a,b]} = \{H_{\alpha,k} \mid \alpha \in \Phi^+, k \in \mathbb{Z}, a \leq k \leq b\},$$

where $H_{\alpha,k}$ is the hyperplane defined by $\{x \in \mathbb{R}^\ell \mid (\alpha, x) = k\}$. The freeness of the cone of the Catalan arrangement $c\mathcal{A}_\Phi^{[-m,m]}$ and of Shi arrangement $c\mathcal{A}_\Phi^{[1-m,m]}$ was conjecture by Edelman-Reiner [4] and proved in [11].

However, except for the simplest case $c\mathcal{A}_\Phi^{[0,0]}$ by Saito [7], the explicit bases of the module $D(c\mathcal{A}_\Phi^{[a,b]})$ (where $[a,b]$ is either $[-m,m]$ or $[1-m,m]$) had not been constructed. Indeed, the proofs by Edelman-Reiner [4] and Athanasiadis [2] for $\Phi = A_\ell$ employed Terao's addition-deletion theorem of freeness, and that in [11] used cohomological arguments to guarantee the existence of global sections of certain coherent sheaves associated with the graded module $D(c\mathcal{A}_\Phi^{[a,b]})$. Since then, a number of efforts have been made to construct explicit bases for $D(c\mathcal{A}_\Phi^{[a,b]})$. First, in [8], a basis for $D(c\mathcal{A}_{A_\ell}^{[0,1]})$ was constructed using the Bernoulli polynomial. Subsequently, in [6] and [9], similar bases of $D(c\mathcal{A}_\Phi^{[0,1]})$ were constructed for $\Phi = B_\ell, C_\ell, D_\ell$. Note that these works are for Shi arrangements with $m=1$. Catalan arrangements and Shi arrangements with $m>1$ have not been covered. For larger m, the type $\Phi = A_2$ was the only known case. Namely, explicit bases were constructed for $c\mathcal{A}_\Phi^{[-m,m]}$ and $c\mathcal{A}_\Phi^{[1-m,m]}$ in [1]. The purpose of the present paper is to introduce a new method to describe the bases ([10]) for type A root system.

From now, let $\mathrm{Cat}_\ell(m)$ be the affine arrangement in $\mathbb{R}^\ell$ defined by

$$\prod_{\substack{1\le i<j\le \ell \\ -m\le k\le m}} (x_i - x_j - k) = 0.$$

(Note that this is equivalent to $\mathcal{A}_{A_{\ell-1}}^{[-m,m]}$.)

To describe the main result, we need the notion of discrete integrals. For a function $f(t)$, we define the difference operator Δ as $\Delta f(t) = f(t+1) - f(t)$. When $\Delta F(t) = f(t)$, $F(t)$ is called an indefinite summation (or antidifference) of $f(t)$, and denoted by

$$F(t) = \sum f(t)\Delta t.$$

Let $F(t)$ be an indefinite summation of $f(t)$. Then we define the definite summation as

$$\sum\nolimits_a^b f(t)\Delta t = F(b) - F(a).$$

Note that the Bernoulli polynomial $B_n(t)$ is a monic of rational coefficients defined by

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n = \frac{te^{xt}}{e^t - 1},$$

(e.g., $B_0(x) = 1, B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6}, B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \cdots$) The Bernoulli polynomial $B_n(t)$ satisfies $\Delta B_n(t) = nt^{n-1}$. Therefore, the monomial t^n has an indefinite summation $\frac{B_{n+1}(t)}{n+1}$. Furthermore, arbitrary polynomial $f(t)$ has an indefinite summation.

The following is a discrete analogue of the power of a function. Let $n > 0$ be a positive integer. We define the falling power $f(t)^{\underline{n}}$ as

$$f(t)^{\underline{n}} = f(t)f(t-1)\cdots f(t-n+1).$$

Using the notion of discrete integrals, we define ζ_k^m as follows.

$$\zeta_k^m = \sum_{i,j=1}^{\ell} \left(\sum_{x_i}^{x_j} t^k g(t)^{\underline{m}} \Delta t \right) \partial_i.$$

The main result is that the homogenezations of $\zeta_0^m, \zeta_1^m, \ldots, \zeta_{\ell-2}^m$ together with $\theta_0 = \sum \partial_i$ and the Euler vector field $\theta_E := \sum x_i \partial_i$ form a basis of $D(c\,\mathrm{Cat}_\ell(m))$. See [10] for details.

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Verifying Terao's freeness conjecture for small arrangements in arbitrary characteristic

Lukas Kühne
(joint work with Mohamed Barakat)

1. Introduction

A long-standing conjecture of Terao asks whether the freeness of an arrangement over a fixed field is determined by its underlying combinatorics. Recently, Dimca, Ibadula, and Măcinic confirmed Terao's conjecture for arrangements in $\mathbb{C}^3$ with up to 13 hyperplanes [4]. In joint work with Behrends, Jefferson, and Leuner, we confirmed Terao's conjecture for rank 3 arrangements with exactly 14 hyperplanes in arbitrary characteristic [2]. The main result discussed in this talk is a common generalization of the two aforementioned results:

Theorem 1. *Terao's freeness conjecture is true for rank* 3 *arrangements with up to* 14 *hyperplanes in any characteristic.*

Our proof rests on two pillars. Firstly, we are using the database [3] that contains all relevant matroids of that size stemming from our previous work [2]. This reduces the proof to checking that the arrangements having one of 9 exceptional matroids as intersection lattice satisfy Terao's conjecture.

Secondly, we use Yoshinaga's freeness criterion and Fitting ideals to compute the non-free locus of all arrangements within the moduli space of all realizations of a fixed matroid.

2. The 9 exceptional matroids

In [2], we generated all 815107 simple rank 3 matroids with up to 14 elements with integrally splitting characteristic polynomial and stored them in public database [3]. To investigate Terao's freeness conjecture it suffices to consider the matroids that are representable over some field, not uniquely representable, not divisionally free, and not unbalanced. See [2] for the detailed definitions of these properties.

Somewhat surprisingly, it turns out that there are only 9 rank 3 integrally splitting matroids of size up to 14 satisfying all of these conditions. There is one matroid of size 9 (the arrangement corresponding to the complex reflection group $G(3,3,3)$), one of size 11 (a pentagonal arrangement), two of size 12 (one of them is the arrangement corresponding to the group $G(4,4,3)$) and five of size 13 (one of them is the matroid underlying the smallest free but not recursively free arrangement described in [1]). This already verifies Terao's freeness conjecture for rank 3 arrangements with precisely 14 hyperplanes.

To deduce the more general Theorem 1, we will subsequently investigate the freeness of these 9 exceptional matroids.

3. Freeness of multiarrangements

Given a multiarrangement $(\mathcal{A}, m)$ with hyperplanes $H_1 = \ker \alpha_{H_1}, \dots, H_n = \ker \alpha_{H_n}$ in a vector space V with $\ell = \dim V$ its module of logarithmic derivations is

$$D(\mathcal{A}, m) := \{\theta \in \mathrm{Der}(S) \mid \theta(\alpha_H) \in \alpha_H^{m(H)} S \text{ for all } H \in \mathcal{A}\},$$

where $\mathrm{Der}(S) \cong S^\ell$ is the module of all derivations on $S = k[x_1, \dots, x_\ell]$. The algorithm we use to compute $D(\mathcal{A}, m)$ is a direct translation into the language of Gröbner bases of the following proposition.

Proposition 1. *$D(\mathcal{A}, m)$ is the projection of the kernel of the morphism*

$$\psi^{(\mathcal{A},m)} : \mathrm{Der}(S) \oplus \bigoplus_{H\in\mathcal{A}} S(-m(H)) \xrightarrow{\begin{pmatrix} \frac{\partial \alpha_{H_1}}{\partial x_1} & \cdots & \cdots & \frac{\partial \alpha_{H_n}}{\partial x_1} \\ \vdots & & & \vdots \\ \frac{\partial \alpha_{H_1}}{\partial x_\ell} & \cdots & \cdots & \frac{\partial \alpha_{H_n}}{\partial x_\ell} \\ \hline \alpha_{H_1}^{m(H_1)} & 0 & \cdots & 0 \\ 0 & \alpha_{H_2}^{m(H_2)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_{H_n}^{m(H_n)} \end{pmatrix}} \bigoplus_{H\in\mathcal{A}} S$$

between free graded modules onto the direct summand $\mathrm{Der}(S)$.

Thus, one can compute the kernel of the morphism $\psi^{(\mathcal{A},m)}$ in the category of free graded modules by computing syzygies.

4. Yoshinaga's criterion

Our first technical tool to compute the non-free locus within the moduli space of a matroid is the following remarkable theorem by Yoshinaga.

Theorem 2. [6, Corollary 3.3] *Let $\mathcal{A}$ be an arrangement of rank 3 and assume the characteristic polynomial of $\mathcal{A}$ factors as $\chi_{\mathcal{A}}(t) = (t-1)(t-d_2)(t-d_3)$ for some integers d_2, d_3. Let H be any hyperplane in $\mathcal{A}$. Then, $\mathcal{A}$ is free if and only if the Ziegler restriction $(\mathcal{A}^H, m^H)$ is free with exponents (d_2, d_3).*

5. The non-free locus of an arrangement via Fitting ideals

Definition 2. Let B be a commutative ring and $\phi : U \to W$ a morphism of free B-modules of finite rank. After choosing sets of free generators for U and W one can identify ϕ with a matrix in $B^{\mathrm{rk_B} U \times \mathrm{rk_B} W}$. Define the i-th Fitting ideal

$$\mathrm{Fitt}_i(\phi) \trianglelefteq B$$

to be the ideal generated by all $m \times m$ minors of ϕ where $m := \mathrm{rk}_B W - i$.

Theorem 3 (Fitting's Lemma, [5, Cor.-Def. 20.4])**.** *The i-th Fitting ideal of ϕ only depends on the isomorphism type of* $\operatorname{coker}\phi$*.*

In particular, one might pass from a free presentation ϕ of $\operatorname{coker}\phi$ to another, preferably smaller presentation. This is the major trick which allows us to compute the Fitting ideals.

Definition 3. Let $\{\mathcal{A}_b \mid b \in \operatorname{Spec} B\} \equiv \operatorname{Spec} B$ be the moduli space of arrangements representing a rank 3 matroid with integrally splitting characteristic polynomial $\chi(t) = (t-1)(t-d_2)(t-d_3)$ for some exponents $d_2, d_3 \in \mathbb{Z}_{>0}$. We view the family $\{\mathcal{A}_b \mid b \in \operatorname{Spec} B\}$ as an arrangement $\mathcal{A}$ over the ring B. For a fixed $H \in \mathcal{A}$ define the degree $d_2 - 1$ part of $\psi^{(\mathcal{A}^H, m^H)}$ as the morphism

$$\phi := \psi_{d_2-1}^{(\mathcal{A}^H, m^H)} : U \to W$$

of free B-modules

$$U := \left(\operatorname{Der}(S) \oplus \bigoplus_{H' \in \mathcal{A}^H} S(-m^H(H')) \right)_{d_2-1}, \quad W := \left(\bigoplus_{H' \in \mathcal{A}^H} S \right)_{d_2-1}.$$

Theorem 4. *In the notation of Definition 3 the following are equivalent for $b \in$* $\operatorname{Spec} B$*:*

(1) $\mathcal{A}_b$ is not free.
(2) $D(\mathcal{A}_b^{H_b}, m^{H_b})_{d_2-1}$ does not vanish.
(3) b is in the non-free locus $V(\operatorname{Fitt}_i(\phi))$ for $i = \operatorname{rank}_B W - \operatorname{rank}_B U$.

6. Proof of Theorem 1

Proof of Theorem 1. As explained in Section 2, verifying Terao's freeness conjecture for the 9 exceptional matroids completes the proof.

We computed the non-free locus of each of these matroids using Theorem 4. Over a fixed field, the realizations of a given matroid turn out to be either all free or all non-free. We found however that the freeness depends on the field for realizations being combinatorially equivalent to the reflection arrangement corresponding to $G(3, 3, 3)$ and the pentagonal arrangement with 11 hyperplanes; all realizations are free except in characteristic 3 and 2, respectively. □

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Properties of the singular locus of a hyperplane arrangement

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(joint work with Uwe Nagel, Hal Schenck)

We describe the results of the paper *Schemes supported on the singular locus of a hyperplane arrangement in* $\mathbb{P}^n$, which has been accepted for publication in International Mathematics Research Notices. It is available on the arXiv (see [9]). The first main result will concern hyperplane arrangements in $\mathbb{P}^n$, for which we use the polynomial ring $R = k[x_0, \ldots, x_n]$ (where k is algebraically closed of characteristic zero), and for the second main result we consider hyperplane arrangements in $\mathbb{P}^3$, so we specialize to $R = k[w, x, y, z]$.

If $\mathcal{A}$ is a hyperplane arrangement in $\mathbb{P}^n$, it is defined by the homogeneous polynomial

$$F = \prod_{i=1}^{m} L_i \subseteq R,$$

where the L_i define distinct hyperplanes. The Jacobian ideal associated to $\mathcal{A}$ is the ideal generated by the partial derivatives of F:

$$J = \langle F_{x_0}, F_{x_1}, \ldots, F_{x_n} \rangle.$$

$\mathcal{A}$ is *free* if R/J is Cohen-Macaulay. In general, obstacles to $\mathcal{A}$ being free include J not being saturated or, more generally, J not being unmixed. These two conditions can be remedied algebraically, by removing primary components of codimension ≥ 3, resulting in the ideal J^{top} that is the intersection of the codimension two primary components, hence is unmixed. We also have the radical ideal $\sqrt{J}$, which is also unmixed. These two unmixed ideals define equidimensional schemes X^{top} and X^{red}, respectively. Our focus is on when the latter two unmixed ideals are Cohen-Macaulay, i.e. on when X^{top} and X^{red} are *arithmetically Cohen-Macaulay (ACM)*. These two schemes can both claim to be the "unmixed singular locus of $\mathcal{A}$," depending on whether one views this locus as a scheme or as a set.

Since we are interested in questions about the ACM-ness of an equidimensional scheme X, the natural object to look at is the collection of Hartshorne-Rao modules

$$\bigoplus_{t \in \mathbb{Z}} H^i(\mathcal{I}_X(t))$$

for $1 \leq i \leq \dim X$. When $\dim X = 0$, X is automatically ACM. When $\dim X \geq 1$, the scheme X is ACM if and only if these modules are all equal to zero. When X is a curve, there is only one module to worry about and we denote it by $M(X)$. If $\dim X \geq 2$ then X is ACM if and only if a general hyperplane section is ACM [6]. This fact often allows us to reduce to the case of curves in $\mathbb{P}^3$ and deduce ACM-ness in $\mathbb{P}^n$.

So suppose that $\mathcal{A}$ is a hyperplane arrangement in $\mathbb{P}^3$. We make the observation that if the support of the singular locus of $\mathcal{A}$ has a component (a line) through which $\ell \geq 3$ planes of $\mathcal{A}$ pass then X^{top} has a non-reduced component supported on that line, which is a complete intersection of type $(\ell - 1, \ell - 1)$. In particular, that component (by itself) is ACM. Our first main theorem is the following.

Theorem 1. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{P}^n$ defined by a product, F, of linear forms. Let J, $\sqrt{J}$ and J^{top} be the ideals defined above. $(*)$ Assume that no linear factor of F is in the associated prime of any two* non-reduced *components of J^{top}. Then both R/J^{top} and $R/\sqrt{J}$ are Cohen-Macaulay (i.e. both X^{top} and X^{red} are ACM.) If $(*)$ fails then both X^{top} and X^{red}* may *fail to be ACM.*

We have the following application of this result. Recall that if G is a graph with vertices $v_1, \ldots, v_n$ and certain edges $\overline{v_i v_j}$, and taking $R = k[x_1, \ldots, x_n]$, then the *graphic arrangement* $\mathcal{A}_G$ associated to G is the one defined by the hyperplanes $\mathbb{V}(x_i - x_j)$ for every edge $\overline{v_i v_j}$.

Corollary 2. *Assume that no two 3-cycles of G share an edge. Then $R/\sqrt{J_G}$ and R/J_G^{top} are Cohen-Macaulay. If two 3-cycles do share an edge, these rings may or may not be Cohen-Macaualy.*

To prepare for our second main result, we recall an example due to Mustaţă and Schenck [10], which is an arrangement in $\mathbb{P}^3$. Let $R = k[w, x, y, z]$ and consider the arrangement $\mathcal{A}$ defined by the linear forms

$$x,\ y,\ z,\ w,\ x+y,\ x+z,\ x+w,\ y+z,\ y+w,\ z+w,$$
$$x+y+z,\ x+y+w,\ x+z+w,\ y+z+w,\ x+y+z+w.$$

One can check that condition $(*)$ from the first main theorem is not satisfied. It turns out that the Jacobian ideal J coming from $\mathcal{A}$ is already saturated and unmixed, but R/J is not Cohen-Macaulay. This shows that the associated schemes are not necessarily ACM in general. The radical ideal $\sqrt{J}$ is Cohen-Macaulay in this example, but one can tweak this example to find all combinations of X^{top} and X^{red} being ACM or not.

The second main theorem says that such schemes coming from arrangements can fail to be ACM by as much as desired (as measured by the total dimension of the Hartshorne-Rao module).

Theorem 3. *Let $r \geq 1$ be a positive integer. Then:*

(i) $\exists$ a positive integer N and an arrangement $\mathcal{A}_1$ such that

$$\dim_k M(X_1^{top})_i = \begin{cases} r & \text{if } i = N; \\ 0 & \text{if } i \neq N \end{cases}$$

(ii) $\exists$ a positive integer N' and an arrangement $\mathcal{A}_2$ such that

$$\dim_k M(X_2^{red})_i = \begin{cases} r & \text{if } i = N'; \\ 0 & \text{if } i \neq N' \end{cases}$$

(iii) For each $h \geq 1$ we can replace N by $N+h$ and get the same result for both (i) and (ii), by adding h general planes to $\mathcal{A}_1$ and/or $\mathcal{A}_2$.

The curve thus obtained is evenly linked to (respectively) the curve in (i) or (ii).

The main tools used to prove the two main results are Liaison Addition, due to Phil Schwartau [11], and Basic Double Linkage, created by Lazarsfeld and Rao

[8]. While Schwartau never published his thesis, his result has been generalized for different purposes, so the reader can still see a proof [1], [2], [4], [5]. Similarly, Basic Double Linkage has been generalized in different ways [1], [7]. For lack of space we do not quote these theorems here. The first application given in the talk was to show how Basic Double Linkage gives the Cohen-Macaulayness of codimension two star configurations [3]. The second application was to show how the first main theorem follows from a combination of these two tools. The third application was to show how the two tools give the second main theorem.

There was no time in this talk to describe the connections to liaison, but some short comments were given to indicate some of the open problems created by this work, all of which related to liaison and so were necessarily vague.

I am grateful to MFO for creating the opportunity for this group of researchers to get together in this workshop, even if it was only remotely. Under the circumstances, it was a very rewarding conference. I am also grateful to the organizers for putting it together and for the kind invitation to speak.

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Matrix factorizations of disciminants of complex reflection groups

Eleonore Faber
(joint work with Ragnar-Olaf Buchweitz, Colin Ingalls)

In this talk we report about a McKay correspondence for reflection groups [3] and how this allows to identify certain matrix factorizations of the discriminants of these reflection groups.

Let G be a finite subgroup of $\mathrm{GL}(n, K)$ for a field K, whose characteristic does not divide the order of G. For this talk we assume that $K = \mathbb{C}$. The group G acts linearly on a vector space $V \cong K^n$, and thus on the ring $S = \mathrm{Sym}_K(V) \cong K[x_1, \dots, x_n]$. When G is generated by reflections, then the discriminant $V(\Delta)$ of the group action of G on S is a hypersurface in the smooth quotient V/G. In particular, by results of K. Saito and H. Terao, $V(\Delta)$ is a free divisor. Geometric properties of the reflection arrangements $\mathcal{A}(G)$ and the discriminant $V(\Delta)$ have been well studied, see e.g. [7] for an overview.

From a more algebraic point of view, any free divisor D in a complex manifold M is naturally equipped with several maximal Cohen–Macaulay (MCM)-modules: Let $\mathcal{O}_D = \mathcal{O}_M/(f)$ be the local ring of D, then the Jacobian ideal J_D in $\mathcal{O}_D$ is MCM over $\mathcal{O}_D$ (see [1, 27]), as well as the module $\mathrm{Der}_M(-\log D)$ of logarithmic derivations, and the logarithmic residues $\mathcal{R}_D^{q-1} \cong \mathrm{coker}(\Omega^q_M \to \Omega^q_M(\log D))$ (see [8]).
Note that for any hypersurface $D = \{f = 0\} \subseteq M$, a MCM-module can be described by a pair of $m \times m$ matrices (A, B) with entries in $\mathcal{O}_M$ satisfying $AB = BA = f \cdot \mathbb{I}_m$, a so-called *matrix factorization*, see [5].

We seek a representation theoretic interpretation of these natural MCM-modules for the discriminants of complex reflection groups. The idea for this is coming from the algebraic McKay correspondence à la Auslander (see e.g. [4, 2] for an overview). Using the notation above, let G be a finite complex reflection group acting on S, and denote by $R = S^G \cong K[f_1, \dots, f_n]$ the invariant ring, by $J = \det(\frac{\partial f_i}{\partial x_j}) \in S$ the Jacobian defining the reflection arrangement $\mathcal{A}(G)$ (with reduced equation z), and by $\Delta = zJ$ the equation of the discriminant in R. Then we have the following:

Theorem 1 ([3], Theorem 4.17)**.** *Let $G \ncong \mu_2$ be a true (i.e., generated by reflections of order 2) reflection group, R, S, Δ as above, and $A = S * G$ the skew group ring, $e = \frac{1}{|G|}\sum_{g \in G} g$, and $\overline{A} := A/AeA$. Then*

$$\overline{A} \cong \mathrm{End}_{R/(\Delta)}(S/(J)) \ ,$$

$\mathrm{gl.dim}(\overline{A}) = n$*, and $\overline{A}$ is itself a MCM-module over $R/(\Delta)$.*

As a corollary we obtain that the irreducible representations of G (except the determinantal representation of G) correspond to the $R/(\Delta)$-direct summands of $S/(J)$.

If $n = 2$, then one gets a complete description of $\mathrm{MCM}(R/\Delta)$, the category of MCM-modules over R/Δ: $S/(J)$ is then a representation generator for $\mathrm{MCM}(R/(\Delta))$, that is, we have found all matrix factorizations for ADE-curves.

For any finite complex reflection group G the direct summands of $S/(J)$ and $S/(z)$ yield isotypical components: From the multiplication $S \xrightarrow{J} S \to S/(J) \to 0$ one obtains G-equivariant matrix factorizations for Δ:

$$S/(J) \cong_{R/(\Delta)-\text{modules}} \bigoplus_{i=1}^{r} M_i \otimes_k V_i \ ,$$

where $V_1, \ldots, V_r$ are the irreducible representations of G, and $M_i \cong \mathrm{Hom}_{KG}(V_i, S/(J))$.

This leads to the identification of the aforementioned MCM-modules /matrix factorizations of the free divisor $V(\Delta)$ as isotypical components:

Theorem 2 ([3], Theorem 5.9)**.** *There are the following isomorphisms of $R/(\Delta)$-modules:*

- *For $V_i = \mathrm{triv}$ the corresponding module is $M_{\mathrm{triv}} \cong R/(\Delta)$,*
- *For $V_i = \det^{-1}$, the corresponding module is $M_{\det^{-1}} = 0$,*
- *For $V_i = V$, the corresponding module is $M_V \cong \mathrm{Der}(-\log D)$,*
- *For $V_i = \bigwedge^m V$, the corresponding module is $M_{\bigwedge^m V} \cong \bigwedge^m \mathrm{Der}(-\log D)$, and $\mathrm{syz}(M_{\bigwedge^m V}) \cong \mathcal{R}^{m-1}$.*

Further problems are to describe the remaining isotypical components of $S/(J)$ and $S/(z)$, and in particular, to find a geometric interpretation of them. Moreover, if G a complex reflection group that is not a true reflection group, then Theorem 1 does not hold, that is, $\overline{A} \ncong \mathrm{End}_{R/(\Delta)}(S/(J))$.

Conjecture 1. *For any complex reflection group $S/(J)$ and $S/(z)$ contain the same direct summands, i.e., $\mathrm{add}(S/(J) = \mathrm{add}(S/(z))$, and*

$$\mathrm{End}_{R/(\Delta)}(S/(J)) \cong \mathrm{End}_{R/(\Delta)}(S/(z))$$

is itself maximal Cohen–Macaulay and has finite global dimension.

This conjecture is supported by

Theorem 3 ([6])**.** *For the imprimitive complex reflection groups $G = G(m, p, 2)$, $S/(z)$ is a representation generator for $\mathrm{MCM}(R/(\Delta))$ and thus $\mathrm{End}_{R/(\Delta)}(S/(z))$ is of global dimension 2 and maximal Cohen–Macaulay over $R/(\Delta)$.*
One can furthermore explicitly determine matrix factorizations for all direct summands of $S/(z)$.

Some more promising calculations for $G = G(m, p, n)$ are work in progress.

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Bernstein–Sato polynomials and a measurement of the non-freeness of an arrangement

Daniel Bath

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a central, reduced, indecomposable hyperplane arrangement of degree $d = \deg(f)$. The Bernstein–Sato polynomial $b_f(s) \in \mathbb{C}[s]$, with s a new variable, is a classical invariant of our arrangement that encodes a wealth of information about the singular locus of the arrangement as well as its Milnor fiber. In particular, its roots speak to the Arnold exponent, multiplier ideals, Hodge ideals, the eigenvalues of the algebraic monodromy of the Milnor fiber, etc. See the survey [10]. (We stick to arrangements, though all this is true for Bernstein–Sato polynomials attached to more general f.) Broadly, the Bernstein–Sato polynomial arises from a very general sort of differential equation called the functional equation.

The purpose of this talk is to demonstrate an entirely new interpretation of certain roots of the Bernstein–Sato polynomial. It is well known that its roots are negative rational numbers, but because we are dealing with hyperplane arrangements the roots are bounded further and live in $(-2, 0) \cap \mathbb{Q}$, cf. Theorem 1 of [8]. While strictly speaking the following doesn't always capture all the roots of the Bernstein–Sato polynomial, it is morally helpful to separate the roots into three sets: those lying in $[-1, -n/d]$, those in $[-2 + n/d, -1)$, those in $(-2, -2 + n/d)$. The first set contains the jumping numbers of f (see [4]), though the containment can be strict (see Remark 3.4 [9]), and have been computed explicitly by the author in the case of tame arrangements (see Theorem 1.3 [2]); the second set relates to the first through the symmetry, or lack thereof, of the roots about -1 (see [7]);

and the third set is mysterious. Nevertheless, Walther has found two arrangements in $\mathbb{C}^3$ with the same intersection lattice but different Bernstein–Sato polynomials: one has a root from this third set and the other does not (see Example 5.10 [11]).

The main result is that the presence of roots in the third set $(-2, -1 - n/d)$ measure the distance f is from being a free arrangement. Again morally: if f has roots sufficiently close to -2 it is not free, and the closer the roots are to -2 the further f is from being a free arrangement. The measurement we are using for distance to a free arrangement is the following. Consider the finite number

$$\mu_f := \min\{\deg(g) \mid g \text{ is a central arrangement and } fg \text{ is a free}\}.$$

So the larger μ_f is, the more hyperplanes we must add to f to obtain a free arrangement. By unpublished work of Yoshinaga [12], as f is an arrangement, μ_f is finite. (While "freeing" a divisor f has been studied before in some cases, e.g. [6], it is in general unknown whether or not for an arbitrary divisor a "freeing" divisor g exists.) The talk's main result is Theorem 1.5 of [2]:

Theorem *Suppose that f and g are central arrangements such that f is tame and indecomposable and fg is free. Let $2 \leq v \leq n-1$ be an integer with v and $d = deg(f)$ co-prime. If $-2 + v/d$ is a root of the Bernstein–Sato polynomial of f, then $deg(g) \geq n - v$. That is, then $\mu_f \geq n - v$.*

Please note that: the form of the root $-2 + v/d$ is not really a restriction by Theorem 1 of [8]; the assumption v and d are co-prime is for the sake of a cleaner argument and can often be dropped given knowledge of the intersection lattice of f (Example 5.5 [2]); the assumption of tameness is not necessary but is kept for simplicity. Recall that a tame arrangement is one whose logarithmic k-forms have projective dimension at most k.

Before describing the proof of the theorem, we gesture at the construction of the Bernstein–Sato polynomial. Let D_X be the sheaf of $\mathbb{C}$-linear differential operators on X an analytic space. The reader can pretend D_X is the Weyl algebra since $X = \mathbb{C}^n$ and the stories are similar on the analytic and algebraic side since f is a polynomial. Introduce a new variable s and consider the symbol f^s. One can apply $\partial \in D_X$ to f^s by formally using the chain rule. The *Bernstein–Sato polynomial* $b_f(s)$ is the monic polynomial of minimal degree satisfying the *functional equation*

$$b_f(s)f^s = Qf^{s+1}$$

for Q a differential operator in $D_X[s]$.

A similar construction occurs if we replace the symbol f^s with a symbol $F^S := f_1^{s_1} \cdots f_d^{s_d}$ for a factorization $f = f_1 \cdots f_d$. Then the *multivariate Bernstein–Sato ideal* B_F is the $\mathbb{C}[s_1, \ldots, s_d]$-ideal generated by all $b(S)$ satisfying the functional equation $b(S)F^S = QF^{S+1}$. And for f' dividing f we also have a "weird" Bernstein–Sato ideal $B_{f'F}$ satisfying the "weird" functional equation $b(S)f'F^S = QF^{S+1}$.

Sketch of proof of **Theorem**: Let f and g be as in the theorem with factorizations F and G into irreducibles and FG denoting the induced factorization of fg.

Step 1: By [11, 1, 2] respectively, the $D_X[s]$ and $D_x[S,T]$ (here S denotes many s-terms and t-terms) annihilators of f^s, F^S, and gF^SG^T are determined by the logarithmic derivations of f, f, and fg, respectively. (Tameness and freeness are used here.) Each logarithmic derivation determines an annihilating element and in this case, these elements generate the whole annihilator. By the product rule and the definition of the Bernstein–Sato polynomial (or ideal) we obtain:

$$a \in Z(b_f(s)) \to (a_1, \dots, a_d) \in Z(B_F) \to (a, \dots, a, -1, \dots, -1) \in Z(B_{gFG})$$

where $Z(-)$ denotes the zero locus. So our promised root $(-2 + v/d)$ of the Bernstein–Sato polynomial of f is transported into the "weird" zeroes $Z(B_{gFG})$ as $(-2 + v/d, \dots, -2 + v/d, -1, \dots, -1)$.

Step 2: Maisonobe was able to construct nice element of B_F for F a factorization into linears [5]. We we able to generalize this approach to find a nice element of B_{gFG} in [2]. Said element is itself a hyperplane arrangement and consists of products of terms like $s_1 + \cdots + s_d + t_1 + \cdots + t_{\deg(g)} + j$ where j is a positive integer. Here j takes on many positive values and the issue is finding a good upper bound for j. (This term corresponds to the flat at the origin; at other flats there are similar terms where the s and t terms that appear correspond to the hyperplanes containing the desired flat.)

Step 3: Narvaez-Macarro showed that, in this case, a free hyperplane arrangement has a Bernstein–Sato polynomial that is symmetric about -1, cf. [7]. This comes about by computing the $D_X[s]$-dual of the $D_X[s]$-module generated by f^s. In [2] we computed the $D_X[S,T]$-dual of the module generated by gF^SG^T by different methods, see our Appendix B as well as [3]. Here the fact fg is free is absolutely critical; without it such duality computations are currently intractable. As a consequence $Z(B_{gFG})$ has a symmetry.

Inspired by Maisonobe's ideas in [5], in our setting we use this new symmetry to find a quite precise upper bound for the value j can take in each factor $s_1 + \cdots + s_d + t_1 + \cdots + t_{\deg(g)} + j$ of our "nice" element of B_{gFG}. This upper bound uses only the degree of f, the degree of g, and the rank of fg. (Technically each factor uses such data at the corresponding flat.)

Step 4: Because of all this, we know $Z(B_{gFG})$ is contained in somewhat small (non-central) arrangement of hyperplanes. Using Step 1 we simply check whether or not $(-2+v/d, \dots, -2+v/d, -1, \dots, -1)$ appears in any of these hyperplanes in $\mathrm{Spec}(\mathbb{C}[S,T])$. Because v and d are co-prime, there are relatively few hyperplanes to check. And because the j-terms for each factor are bounded above by $\deg(g)$, $\deg(f)$, and rank data, we find that if our zero $(-2+v/d, \dots, -2+v/d, -1, \dots, -1)$ appears in one of these hyperplanes, $\deg(g)$ must be bounded from below by $n-v$.

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On discriminantal arrangement and its combinatorics

Simona Settepanella

(joint work with S. Yamagata)

In 1989, Manin and Schechtman ([9]) introduced a family of arrangements of hyperplanes generalizing classical braid arrangements, which they called the *discriminantal arrangements* (p.209 [9]). Such an arrangement $\mathcal{B}(n,k,\mathcal{A}^0), n,k \in \mathbf{N}$ for $k \geq 2$ depends on a choice $\mathcal{A}^0 = \{H_1^0, ..., H_n^0\}$ of a collection of hyperplanes in general position in $\mathbb{C}^k$, i.e., such that $\dim \bigcap_{i\in K, |K|=k} H_i^0 = 0$. It consists of parallel translates of $H_1^{t_1}, ..., H_n^{t_n}, (t_1, ..., t_n) \in \mathbb{C}^n$ which fail to form a general position arrangement in $\mathbb{C}^k$. $\mathcal{B}(n,k,\mathcal{A}^0)$ can be viewed as a generalization of the braid arrangement with which $\mathcal{B}(n,1) = \mathcal{B}(n,1,\mathcal{A}^0)$ coincides.

These arrangements have several beautiful relations with diverse problems such as the Zamolodchikov equation with its relation to higher category theory (see Kapranov-Voevodsky [6]), the vanishing of cohomology of bundles on toric varieties ([10]), the representations of higher braid groups (see [7]) and, naturally, with combinatorics. The latter is the connection we are mainly interested in and it goes from matroids to special configurations of points, from fiber polytopes to higher Bruhat orders.

From a different perspective, unknown in the literature of discriminantal arrangement until Athanasiadis pointed it out in 1999 (see [1]), Crapo introduced for the first time in 1985 (see [3]) what he called *geometry of circuits* and which is the matroid $M(n,k,\mathcal{C})$ of circuits of the configuration $\mathcal{C}$ of n generic points in $\mathbb{R}^k$. The circuits of the matroid $M(n,k,\mathcal{C})$ are the hyperplanes of $\mathcal{B}(n,k,\mathcal{A}^0)$, $\mathcal{A}^0$ arrangement of n hyperplanes in $\mathbb{R}^k$ orthogonal to the vectors joining the origin with the n points in $\mathcal{C}$ (for further development see [4]).

Both Manin-Schechtman ([9]) and Crapo ([3]) were mainly interested in arrangements $\mathcal{B}(n,k,\mathcal{A}^0)$ for which the intersection lattice is constant when $\mathcal{A}^0$ varies within a Zariski open set $\mathcal{Z}$ in the space of general position arrangements. More recently in [1], Athanasiadis proved a conjecture by Bayer and Brandt (see [2]) providing a full description of combinatorics of $\mathcal{B}(n,k,\mathcal{A}^0)$ when $\mathcal{A}^0$ belongs to $\mathcal{Z}$. Following [1] (more precisely Bayer and Brandt), we call arrangements $\mathcal{A}^0$ in $\mathcal{Z}$ *very generic*, non very generic otherwise.

However nor Manin and Schechtman neither Crapo provided a description of $\mathcal{Z}$ even if Crapo presented the first known example of a non very generic arrangement: 6 lines in generic position in $\mathbb{R}^2$ which admit translated that are respectively sides and diagonals of a quadrilateral as in Figure 1 (Crapo calls it a quadrilateral set). Few years later in 1994, Falk provided an higher dimensional example of non very generic arrangement of 6 planes in $\mathbb{R}^3$ (see [5]).

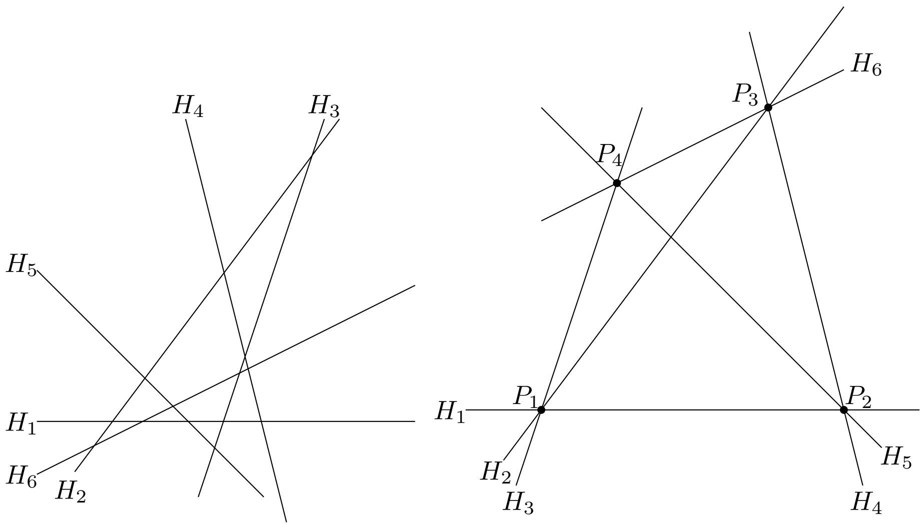

FIGURE 1. Generic arrangement of 6 lines in $\mathbb{R}^2$ and its non (very) generic translation on the right.

In 2018 the first general result on non very generic arrangements is provided. In [8] Libgober and Settepanella described a sufficient *geometric* condition on the arrangement $\mathcal{A}^0$ to be non very generic. This condition ensures that $\mathcal{B}(n,k,\mathcal{A}^0)$ admits codimension 2 strata of multiplicity 3 which do not exist in very generic case. In two subsequent papers, ([11], [12]) Sawada, Settepanella and Yamagata proved how the Pappus's (resp. Hesse's) configuration corresponds to the trace at infinity $\mathcal{A}_\infty$ in $\mathbb{P}^2$ of a non very generic arrangement $\mathcal{A}^0$ of 6 planes in $\mathbb{R}^3$ (resp. $\mathbb{C}^3$) providing a main example of what conjectured by Crapo that the intersection lattice of discriminantal arrangement represents a combinatorial way to encode special configurations of points in the space. Notice that in [11] the authors connected the non very generic arrangements $\mathcal{A}^0$ of n planes in $\mathbb{C}^3$ to well defined hypersurfaces in Grassmannian $Gr(3,n)$.

In our talk, after recollect the main results cited above, we presented a paper in which we advanced the study of non very generic arrangements and generalize the dependency condition given in [8] providing a sufficient condition for the existence in rank $r \geq 2$ of non very generic intersections, i.e. intersections which doesn't exist in $\mathcal{B}(n,k,\mathcal{A}^0), \mathcal{A}^0 \in \mathcal{Z}$. In particular we called *simple* an intersection of r hyperplanes in $\mathcal{B}(n,k,\mathcal{A}^0)$ which satisfies the property that if the arrangement $\mathcal{A}^0$ is very generic then all simple intersections of multiplicity r have rank r (that is they are r hyperplanes intersecting transversally). Then we provided both geometric and algebraic necessary and sufficient conditions for existence of simple intersections of multiplicity r in rank strictly lower than r, i.e. simple non very generic intersections. This result firstly connect configurations of non very generic points to special families of graphs (called $K_{\mathbb{T}}$-configurations) which help to understand $\mathcal{B}(n,k,\mathcal{A}^0)$ for $\mathcal{A}^0 \notin \mathcal{Z}$ (as conjectured by Crapo in [3]). Secondly it reduces the geometric problem of the existence of special (non very generic) configurations of points to a combinatorial problem on the numerical properties that r subsets of indices $L_i \subset \{1,\ldots,n\}, i = 1,\ldots,r$ of cardinality $k+1$ have to satisfy in order for the $K_{\mathbb{T}}$-configuration, $\mathbb{T} = \{L_1,\ldots,L_r\}$, to give rise to a simple non very generic intersection. The latter problem is left open together with the problem of necessary and sufficient conditions for existence of intersections in $\mathcal{B}(n,k,\mathcal{A}^0)$ which are nor simple nor very generic.

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Combinatorics and topology of abelian arrangements

Emanuele Delucchi

Let $A \in \mathbb{Z}^{d\times n}$ be an integer matrix whose columns we call $a_1, \ldots, a_n$. Write $[n] := \{1, \ldots, n\}$ and suppose $a_i \neq 0$ for all $i \in [n]$. To A we associate:

(1) A *linear arrangement* $\mathcal{A}^{\text{lin}} := \{H_i\}_{i\in[n]}$, where each $H_i := \{z \in \mathbb{C}^d \mid a_i^t z = 0\}$ is a hyperplane in $\mathbb{C}^d$. The real part $\mathcal{A}^{\text{lin}}_{\mathbb{R}} := \{H_i^{\text{lin}} \cap \mathbb{R}^d\}_{i\in[n]}$ is the associated *real arrangement* in $\mathbb{R}^d$.

(2) A *toric arrangement* $\mathcal{A}^{\text{tor}} := \{H_i\}_{i\in[n]}$, where the $H_i := \{z \in (\mathbb{C}^*)^d \mid z^{a_i} = 1\}$[1] are hypersurfaces in $(\mathbb{C}^*)^d$. Correspondingly, we set $\mathcal{A}^{\text{tor}}_{\mathbb{R}} := \{H_i^{\text{tor}} \cap (S^1)^d\}_{i\in[n]}$, an arrangement in $(S^1)^d$.

(3) An *elliptic arrangement* $\mathcal{A}^{\text{ell}} := \{H_i\}_{i\in[n]}$, where $H_i := \{z \in \mathbb{E}^d \mid \sum_j z_j^{a_{i,j}} = 0\}$ is a hypersurface in $\mathbb{E}^d$ for a given elliptic curve $\mathbb{E}$.

In the following, $\mathcal{A}$ (resp. $\mathcal{A}_{\mathbb{R}}$) will denote an arrangement in either of the aforementioned categories (resp. the associated "real" arrangements). We will refer to $\mathbb{C}^d$, $(\mathbb{C}^*)^d$ and $\mathbb{E}^d$ as the *ambient space* of $\mathcal{A}$. The ambient space of $\mathcal{A}_{\mathbb{R}}$ is, accordingly, $\mathbb{R}^d$ or $(S^1)^d$.

1. Combinatorics of intersections (see [2, 3])

The *poset of layers* $\mathcal{P}(\mathcal{A})$ is the set of connected component of intersections of the elements of $\mathcal{A}$, partially ordered by reverse inclusion (i.e., $X \leq Y$ if $X \supseteq Y$).

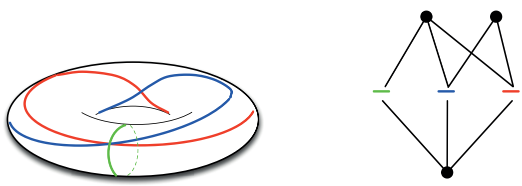

Figure 1. A depiction of $\mathcal{A}^{\text{tor}}_{\mathbb{R}}$ and $\mathcal{P}(\mathcal{A}^{\text{tor}})$ in the case where $A := \left(\begin{smallmatrix}1 & 1 & 1\\ 0 & -1 & 1\end{smallmatrix}\right)$, and an example of the toric pseudoarrangements defined in Section 3.

In the linear case, the poset of layers is a geometric lattice, and it defines the *matroid* of the arrangement. For a treatment of the general case, notice that every arrangement $\mathcal{A}$ lifts to an arrangement $\tilde{\mathcal{A}}$ of affine hyperplanes in the universal

[1] We write, as usual, $z^{a_i} = z_1^{a_{i,1}} \cdots z_d^{a_{i,d}}$

covering space of its ambient space (in all cases: the vectorspace $\mathbb{C}^d$). The poset $\mathcal{L}$ of intersections of the hyperplanes in $\tilde{\mathcal{A}}$ ordered by reverse inclusion is a *geometric semilattice*. The group Λ of deck transformations acts by translations on $\tilde{\mathcal{A}}$ and this induces an action by poset automorphisms $\alpha : \Lambda \circlearrowright \mathcal{L}$ to which is naturally associated a Tutte-type polynomial $T_\alpha(x, y)$, see [3, §10]. On the set of orbits of this action a partial order relation is defined as follows: given $x, y \in \mathcal{L}$, set $\Lambda x \leq \Lambda y$ if $x \leq \lambda y$ for some $\lambda \in \Lambda$. The resulting partial order is the quotient poset $\mathcal{L}/\Lambda$. The natural poset isomorphism $\mathcal{P}(\mathcal{A}) \simeq \mathcal{L}/\Lambda$ implies the following. *For every arrangement $\mathcal{A}$:*

- *$\mathcal{P}(\mathcal{A})$ is bounded-below and all its intervals are geometric lattices;*
- *the characteristic polynomial of $\mathcal{P}(\mathcal{A})$ is $\chi_{\mathcal{P}(\mathcal{A})}(t) = (-1^d)T(1-t, 0)$;*
- *$\mathcal{P}(\mathcal{A})$ is Cohen-Macaulay in characteristic 0 and every characteristic not dividing an explicitely computable number* [2, Theorem 5].

Notice that the above results hold more generally, i.e., for every action α of a group on a geometric semilattice that satisfies some abstract conditions [2, 3].

2. Topology

The *complement* of an arrangement $\mathcal{A}$, resp. $\mathcal{A}_\mathbb{R}$ in the ambient space X, resp. $X_\mathbb{R}$, is $M(\mathcal{A}) := X \setminus \cup\mathcal{A}$, resp. $M(\mathcal{A}_\mathbb{R}) := X_\mathbb{R} \setminus \cup\mathcal{A}_\mathbb{R}$. A main question is to understand how the combinatorics of an arrangement relates to the topology of its complement. In the linear case the integer cohomology algebra of $M(\mathcal{A}^{\text{lin}})$ is fully determined by $\mathcal{P}(\mathcal{A}^{\text{lin}})$ via the associated matroid. In the toric case, the poset $\mathcal{P}(\mathcal{A}^{\text{tor}})$ does determine the cohomology algebra over the rationals but not over $\mathbb{Z}$, see [6, 7]. Combinatorial models for the homotopy type of $M(\mathcal{A})$ are available in all cases [1, 5, 8], and all rely on the structure of the cellularization of $X_\mathbb{R}$ induced by $\mathcal{A}_\mathbb{R}$ (the model for $M(\mathcal{A}^{\text{ell}})$ relies on $\mathcal{A}_\mathbb{R}^{\text{tor}}$).

3. Cell complexes for the real case (following [4])

Every arrangement $\mathcal{A}_\mathbb{R}$ induces a cellularization of the ambient space $X_\mathbb{R}$. In the linear case, this is a polyhedral fan whose cell structure is a mainstay of the theory of Oriented Matroids. The following is an attempt at developing an abstract theory for the cellularization of the compact torus by $\mathcal{A}_\mathbb{R}^{\text{tor}}$.

Let E be a set. A *sign vector* on E is any $X \in \{0, -, +\}^E$. The *support* of a sign vector X is $\underline{X} := \{e \in E \mid X(e) \neq 0\}$ and its *zero set* is $\text{ze}(X) := \{e \in E \mid X(e) = 0\}$. The *separator* of two sign vectors X, Y is $S(X, Y) := \{e \in \underline{X} \cap \underline{Y} \mid X(e) \neq Y(e)\}$, and their *composition* is $X \circ Y$, defined as follows: for every $e \in E$ set $X \circ Y(e) = X(e)$ if $e \in \underline{X}$, and $Y(e)$ otherwise. For $e \in E$ and $X, Y \in \mathcal{L}$ define $I_e(X, Y; \mathcal{L}) := \{Z \in \mathcal{L} \mid Z(e) = 0, \forall f \notin S(X, Y) : Z(f) = X(f) \circ Y(f)\}$ and set $I(X, Y; \mathcal{L}) := \bigcup_{e \in S(X,Y)} I_e(X, Y; \mathcal{L})$. Moreover, let $X \oplus Y$ be the sign vector defined by setting, for every $e \in E$, $X \oplus Y(e) := 0$ if $e \in S(X, Y)$, and $X \oplus Y(e) = X \circ Y(e)$ otherwise. A partial order on sign vectors is defined by $X \leq Y$ if and only if $X(e) \leq Y(e)$ for all $e \in E$ where $0 < +$, $0 < -$, $+$ and $-$ incomparable.

From now on let $\mathcal{L}$ denote a *system of sign vectors* on E, that is, a subset $\mathcal{L} \subseteq \{0, -, +\}^E$. The poset $(\mathcal{L}, \leq)$ will be denoted by $\mathcal{F}(\mathcal{L})$. Moreover, set $\mathcal{P}(\mathcal{L}) := \{X \oplus (-Y) \mid X, Y \in \mathcal{L},\ I(X, -Y; \mathcal{L}) = I(-X, Y; \mathcal{L}) = \emptyset\}$.

A system $\mathcal{L}$ is called a *finitary affine oriented matroid* (FAOM) if:

(FS) $\mathcal{L} \circ (-\mathcal{L}) \subseteq \mathcal{L}$,
(SE) $X, Y \in \mathcal{L} \implies \forall e \in S(X, Y) : I_e(X, Y; \mathcal{L}) \neq \emptyset$,
(P) $\mathcal{P}(\mathcal{L}) \circ \mathcal{L} \subseteq \mathcal{L}$;
(S) $X, Y \in \mathcal{L} \implies |S(X, Y)| < \infty$ (finite separators),
(Z) $X \in \mathcal{L} \implies |\operatorname{ze}(X)| < \infty$ (finite zero sets),
(I) $|\mathcal{F}(\mathcal{L})_{\leq X}| < \infty$ (finite intervals).

Let $\{H_e\}_{e \in E}$ be a locally finite set of affine hyperplanes in $\mathbb{R}^d$ indexed by E. For every $e \in E$ the space $\mathbb{R}^d \setminus H_e$ has two connected components that we label H_e^+, H_e^-. Let $H_e^0 := H_e$. Then every $p \in \mathbb{R}^d$ has an associated sign vector X_p defined by $X_p(e) = \sigma$ if and only if $p \in H_e^\sigma$. The collection of all such X_ps is a FAOM. Notice that the definition of X_e makes sense more generally, i.e., if H_e is a tame embedding of $\mathbb{R}^{d-1}$ (a "pseudoplane"), see, e.g., the l.-h.s. of Figure 2.

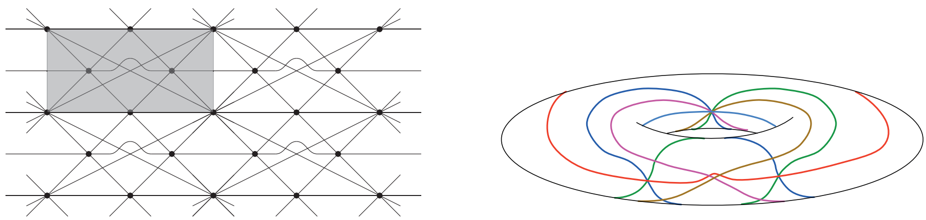

FIGURE 2. A periodic pseudoline arrangement (an instance of $\|\mathcal{F}(\mathcal{L})\|$) and the associated pseudoarrangement in $\|\mathcal{F}(\mathcal{L})/\!\!/\mathbb{Z}^2\|$.

Suppose from now on that $\mathcal{L}$ is a FAOM. Then, the poset $\mathcal{F}(\mathcal{L})$ is countable, ranked and of finite length d. Assume $d > 0$. *The order complex $\|\mathcal{F}(\mathcal{L})\|$ is a shellable, contractible PL d-manifold. When it has no boundary, $\|\mathcal{F}(\mathcal{L})\|$ is PL-homeomorphic to $\mathbb{R}^d$.*

We say that $e, f \in E$ are parallel if there is no $X \in \mathcal{L}$ with $e, f \in \operatorname{ze}(X)$. A group action $\Lambda \circlearrowright E$ that induces a free action $\Lambda \circlearrowright \mathcal{F}(\mathcal{L})$ is called *sliding* if, for every $e \in E$ and every $\lambda \in \Lambda$, $\lambda(e)$ is parallel to e. The main example for sliding actions is the action by translations on the set of sign vectors associated to a translation-periodic (pseudo-)arrangement, as in Figure 2.

In this case, *there is a homeomorphism $\|\mathcal{F}(\mathcal{L})\|/\Lambda \simeq (S^1)^d$. In this torus we have a pseudoarrangement $\mathcal{A}_{\mathbb{R}} := \{H_i/\Lambda\}$, where each H_i/Λ is a $(d-1)$-torus. The category $\mathcal{F}(\mathcal{L})/\!\!/\Lambda$ is*[2] *the face category of the polyhedral CW-complex structure defined by $\mathcal{A}_{\mathbb{R}}$ on the torus. The number of top-dimensional cells of this complex is* $T_\alpha(0, 1)$, where $\alpha : \Lambda \circlearrowright \underline{\mathcal{F}}$ denotes the induced action on the geometric semilattice $\underline{\mathcal{F}} := \{\operatorname{ze}(X) \mid X \in \mathcal{F}(\mathcal{L})\}$, ordered by inclusion.

[2] Here $/\!\!/$ denotes the quotient in the category of small categories without loops, see [5].

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Characteristic and Ehrhart quasi-polynomials for root systems

Tan Nhat Tran
(joint work with Ahmed Umer Ashraf, Masahiko Yoshinaga, and Akiyoshi Tsuchiya)

A typical problem in enumerative combinatorics is to count the size of a set depending upon a positive integer q. Often the result is a polynomial in q (e.g., the chromatic polynomial of a graph), and sometimes a quasi-polynomial. Generally speaking, a quasi-polynomial is a generalization of polynomials, of which the coefficients may not come from a ring but instead are periodic functions with integral periods. One of the most classical examples is the Ehrhart quasi-polynomial counting the number of integral points in the q-fold dilation of a rational polytope. In arrangement theory, a quasi-polynomial appears when we count the size of the complement of an integral hyperplane arrangement modulo q – the characteristic quasi-polynomial due to Kamiya-Takemura-Terao.

The presentation consists of two parts:

(1) Let Φ be an irreducible (crystallographic) root system and fix a positive system $\Phi^+ \subseteq \Phi$. We introduce two new concepts: the *$\mathcal{A}$-Eulerian polynomial* $E_\Psi(t)$ for every subset $\Psi \subseteq \Phi^+$ based on a generalization of cyclic descents of the classical Eulerian polynomial, and the *(Worpitzky-)compatible subset* of Φ^+ having to do with how the affine hyperplanes corresponding to the subset sit relative to the alcoves of the affine Weyl arrangement. We show that the characteristic quasi-polynomial $\chi^{\mathrm{quasi}}_\Psi(q)$ of Ψ can be expressed in terms of the Ehrhart quasi-polynomial of the fundamental alcove shifted by $E_\Psi(t)$. The formula specializes to two known formulas in the extreme cases: $\Psi = \Phi^+$ (e.g., [1]), and $\Psi = \emptyset$ (e.g., [4]). We found a smaller class of compatible subsets called *strongly (Worpitzky-)compatible subsets* which can be described combinatorially using the root poset. The second main result in this part is that the class of strongly compatible subsets

actually contains the *ideals* of the root system. Thanks to the discussions with Christian Stump and Takuro Abe at the MFO workshop, later on, we were able to show that the notion of strongly compatible subsets is essentially equivalent to that of *coconvex subsets* of the root system.

(2) In the second part, we continue the discussion in the first part with a focus on type A root systems. We show that the compatible graphic arrangements are characterized by *cocomparability graphs*. This can be regarded as a counterpart of the characterization by Stanley and Edelman-Reiner of free and/or supersolvable graphic arrangements in terms of chordal graphs. In addition, our main result yields new formulas for the chromatic and graphic Eulerian polynomials of cocomparability graphs.

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CSM classes and Lagrangian geometry of matroids

Graham Denham

(joint work with Federico Ardila, June Huh)

1. Some history

The logarithmic derivations on a (complex) hyperplane arrangement $\mathcal{A}$ give the defining equations of an algebraic variety which is now usually called the maximal likelihood variety [4]. The geometry of the maximal likelihood variety $\mathcal{X}(\mathcal{A})$ reflects properties of the module of logarithmic derivations: for example, an arrangement $\mathcal{A}$ is free if and only if $\mathcal{X}(\mathcal{A})$ is a complete intersection [4, Thm. 2.13]. Since $\mathcal{X}(\mathcal{A})$ is a subvariety of a product of projective spaces, it has a bidegree, a sequence of integers, which are (up to sign) the coefficients of a shift of the characteristic polynomial [5]. In the special case of a free arrangement, this amounts to a geometric reformulation of Terao's Factorization Theorem [10].

To be more precise, let $\mathcal{A}$ be an essential arrangement in an $r+1$-dimensional affine space W, with hyperplanes given by the vanishing of some nonzero linear forms $f_0, f_1, \ldots, f_n \in \mathbb{C}[W^*]$, and $f = f_0 f_1 \cdots f_n$. Under the embedding $(f_0, f_1, \ldots, f_n)$, we can consider W as a linear subspace of $\mathbb{C}^{n+1}$. Let $\hat{U} = D(f)$

denote the complement of hyperplanes in W, and U its image in $\mathbb{P}^n$. Then the closure of

$$\Big\{(p,\lambda)\in \hat{U}\times \mathbb{C}^{n+1}\setminus\{0\}\colon \sum_{i=0}^{n}\lambda_i \mathrm{d}\log f_i(p)=0\Big\}\subseteq W\times\mathbb{C}^{n+1}\subseteq \mathbb{C}^{n+1}\times\mathbb{C}^{n+1}$$

admits an action of $\mathbb{C}^\times\times\mathbb{C}^\times$, and the variety $\mathfrak{X}(\mathcal{A})$ is defined to be the quotient in $\mathbb{P}^n\times\mathbb{P}^n$. This variety[1] parameterizes sets of critical points of rational functions

$$\prod_{i=0}^{n} f_i^{\lambda_i}\colon \hat{U}\to\mathbb{C}$$

when $\lambda\in\mathbb{Z}^{n+1}$. Then

$$\chi_{\mathcal{A}}(1+t)=(-1)^r t\sum_{i=0}^{r} v_i(-t)^{r-i}$$

for integers $v_0,\dots,v_r$, where

$$[\mathfrak{X}(\mathcal{A})]=\sum_{i=0}^{r} v_i[\mathbb{P}^{r-i}\times\mathbb{P}^{n-r+i-1}]\in A_{\cdot}(\mathbb{P}^n\times\mathbb{P}^n).$$

Huh [8] showed that the results above held in a wider context of smooth (schön) very affine varieties $U\subseteq(\mathbb{C}^\times)^{n+1}$. A notable ingredient in his explanation is the Chern–Schwartz–MacPherson (CSM) class. By work of Aluffi [2], this invariant agrees with the total Chern class of the logarithmic tangent bundle of U inside a compactification Y, provided $D:=Y\setminus U$ is a normal crossings divisor:

$$c_{\mathrm{SM}}(\mathbf{1}_{U\subseteq Y})=c(\mathrm{Der}(-\log D))\cap[Y]\in A_{\cdot}(Y).$$

If $\pi\colon Y\to\mathbb{P}^n$ is a resolution of $\mathbb{P}^n\backslash U$, then $c_{\mathrm{SM}}(\mathbf{1}_{U\subseteq\mathbb{P}^n})=\pi_* c_{\mathrm{SM}}(\mathbf{1}_{U\subseteq Y})$ in $A_{\cdot}(\mathbb{P}^n)$. Huh showed that, if U is a schön very affine variety and $\mathfrak{X}(U)$ is its maximal likelihood variety, one can compute $c_{\mathrm{SM}}(\mathbf{1}_{U\subseteq Y})$ from linear slices of $\mathfrak{X}(U)$. In particular, the bidegree of $\mathfrak{X}(U)$ recovers the coefficients of $c_{\mathrm{SM}}(\mathbf{1}_{U\subseteq\mathbb{P}^n})\in A_{\cdot}(\mathbb{P}^n)$.

In the case of hyperplane arrangements, this explains the earlier results. The scissors relation satisfied by CSM classes makes them compatible with deletion–contraction arguments, so additionally

$$c_{\mathrm{SM}}(\mathbf{1}_{U\subseteq\mathbb{P}^n})=\sum_{i=0}^{r}(-1)^{r-i}v_i[\mathbb{P}^{r-i}]\in A_{\cdot}(\mathbb{P}^n), \tag{1}$$

where the v_i's are the coefficients of $\chi_{\mathcal{A}}(1+t)$, a result due to Aluffi [3].

2. Beyond arrangements

In joint work with Federico Ardila and June Huh [1], we construct a tropical version of the story above. This has the advantage (from the point of view of applications) of providing a construction that works for arbitrary matroids M, rather than just matroids with complex linear realizations (complex hyperplane arrangements). In the realizable case, the Bergman fan Σ_{M} is the tropicalization of the linear space

[1] For history and motivation, see the lecture notes of Huh and Sturmfels [7].

$W \subseteq \mathbb{C}^{n+1}$. We introduce the notion of a conormal fan of a matroid, denoted $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$, which is the Lagrangian analogue of the Bergman fan. The conormal fan takes the place of the maximal likelihood variety $\mathfrak{X}(U)$.

For a rational polyhedral fan Σ, we associate a normal toric variety X_Σ, and let $A(\Sigma)$ denote the Chow ring of X_Σ. For example, if Δ_n denotes the standard coordinate simplicial fan, then $A(\Delta_n) = A(\mathbb{P}^n)$. Again in the realizable case, the closure of the complement U in $X(\Sigma_\mathrm{M})$ gives the wonderful compactification $Y(\mathcal{A})$, in which the boundary $Y(\mathcal{A}) \setminus U$ has simple normal crossings. It is known that $A(Y(\mathcal{A})) \cong A(\Sigma_\mathrm{M})$ (by [6]).

Let $N = \mathbb{R}^{n+1}/\mathbb{R}(1,1,\dots,1)$, the tropical n-torus. The fan $\Sigma_{\mathrm{M},\mathrm{M}^\perp}$ is contained in $N \oplus N$, and it is fine enough so that the addition map

$$\mu\colon N \oplus N \to N, \qquad (x,y) \mapsto x+y$$

induces a map of fans $\Sigma_{\mathrm{M},\mathrm{M}^\perp} \to \Delta_n$. In degree 1, the Chow ring can be expressed as the piecewise linear functions supported on the fan, modulo global linear functions. Let $z_0,\dots,z_n$ denote the coordinate functions on N, and consider the piecewise linear function

$$\alpha_j(z) = \max_{0\le i\le n}(z_j - z_i),$$

for each $0 \le j \le n$. All of the α_j's are equivalent modulo global linear functions, and we let α denote their common equivalence class. Under the isomorphism $A(\Delta_n) \cong A(\mathbb{P}^n)$, it is the class of a hyperplane.

López de Medrano, Rincón and Shaw [9] have recently introduced the CSM cycle of a matroid, $\mathrm{csm}(\mathrm{M}) \in A_.(\Sigma_\mathrm{M})$, which is a combinatorially-defined Minkowski weight on the cones of the Bergman fan. Their construction agrees with $c_{\mathrm{SM}}(\mathbf{1}_{U\subseteq Y(\mathcal{A})})$ when M is linearly realizable ([9, Thm. 1.2]). The Bergman fan Σ_M refines Δ_n, and the corresponding map of fans $p\colon \Sigma_\mathrm{M} \to \Delta_n$ behaves as one would hope ([9, Thm. 1.4]), in that

$$p_*\mathrm{csm}(\mathrm{M}) = \sum_{i=0}^{r}(-1)^{r-i}v_i\alpha^i \cap 1_\Delta \in A_.(\Delta_n), \tag{2}$$

just as in (1), where the v_i's are the coefficients of the shifted characteristic polynomial, $\chi_\mathrm{M}(1+t)$, and 1_Δ denotes the top-dimensional constant Minkowski weight on Δ_n.

Motivated by the situation for maximal likelihood varieties, we obtain the CSM cycle of a matroid from the conormal fan in the following way. We let $w_0,\dots,w_n$ denote coordinates for a second copy of N, and define

$$\delta_j(z,w) = \max_{0\le i\le n}(z_j + w_j - z_i - w_i).$$

Then δ is defined to be the equivalence class of the δ_j's, modulo global linear functions on $N \oplus N$. It is easy to see that $\delta = \mu^*\alpha$.

The coordinate projections

$$N \leftarrow N \oplus N \to N$$

induce maps of fans: let $\pi\colon \Sigma_{\mathrm{M},\mathrm{M}^\perp} \to \Sigma_{\mathrm{M}}$ be the first one. By working closely with the combinatorics of the conormal fan, we show that

Theorem 1 (Thm. 1.1, [1])**.**

$$\mathrm{csm}_i(\mathrm{M}) = (-1)^{r-i}\pi_*(\delta^{n-i-1} \cap 1_{\mathrm{M},\mathrm{M}^\perp}) \quad \textit{for } 0 \le i \le r,$$

were $1_{\mathrm{M},\mathrm{M}^\perp}$ *is the top-dimensional constant Minkowski weight on the conormal fan.*

Then one can push further with (2) in order to study the coefficients of $\chi_{\mathrm{M}}(1+t)$ using the Chow ring $A(\Sigma_{\mathrm{M},\mathrm{M}^\perp})$.

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Unexpected curves and line arrangements

Piotr Pokora

Here I would like to report on recent developments devoted to unexpected curves.

Let $Z = P_1 + ... + P_s$ be a reduced scheme of mutually distinct points in $\mathbb{P}^2_{\mathbb{C}}$. We say that Z admits an unexpected curve of degree d if for a general point $P \in \mathbb{P}^2_{\mathbb{C}}$ of multiplicity m we have that

$$\dim_{\mathbb{C}}[I(Z+mP)]_d > \max\left\{\dim_{\mathbb{C}}[I(Z)]_d - \binom{m+1}{2}, 0\right\}$$

with $I(Z+mP) = I(P_1) \cap ... \cap I(P_s) \cap I(P)^m$.

In [2], Cook II, Harbourne, Migliore, and Nagel study unexpected curves from the viewpoint of line arrangements in the complex projective plane, i.e., in their setting Z denotes the set of points which are dual to lines of a given arrangement $\mathcal{A} \subset \mathbb{P}^2_{\mathbb{C}}$.

Consider a set of points $Z = \{z_1, \ldots, z_d\}$ in $\mathbb{P}^2_{\mathbb{C}}$ and the dual line arrangement $\mathcal{A}_Z = \{\ell_1, \ldots, \ell_d\}$ given by the defining polynomial $f \in S := \mathbb{C}[x, y, z]$. Denote by $\mathcal{J}_f$ the Jacobian sheaf – it is the sheafification of the Jacobian ideal generated by the partials $\partial_x f, \partial_y f, \partial_z f$. We can define the derivation (or syzygy) bundle $\mathcal{D}$ be the following exact sequence

$$0 \to \mathcal{D} \to \mathcal{O}^3_{\mathbb{P}^2_{\mathbb{C}}} \to \mathcal{J}_f(d-1) \to 0.$$

It is well-known that $\mathcal{D}$ restricted to a line L splits, according to Grothendieck's theorem, as a sum of line bundles $\mathcal{O}_L(-a) \oplus \mathcal{O}_L(-b)$. If L is generic, then the pair (a, b) is called the (generic) splitting type of $\mathcal{D}$ with $a + b = d - 1$.

In [2] the authors consider the case when an unexpected curve is of degree d with a general point of multiplicity $d-1$. They prove the following theorem which we recall here in a version changed according to Dimca's paper [3].

Theorem 1 (Dimca; Cook II, Harbourne, Migliore, and Nagel)**.** Let Z be the finite set of points in $\mathbb{P}^2_{\mathbb{C}}$. Let (a, b) be the (generic) splitting type of the derivation bundle $\mathcal{D}$.

Let $m(\mathcal{A}_Z)$ denotes the maximal multiplicity of the singular points of the arrangement $\mathcal{A}_Z$. Then Z admits and unexpected curve of degree d with a general point Q of multiplicity $d-1$ if and only if

$$m(\mathcal{A}_Z) \leq a + 1 < \frac{|Z|}{2}.$$

Examples

We start with the following well-known family of Fermat line arrangements $\mathcal{F}_n \subseteq \mathbb{P}^2_{\mathbb{C}}$ with the defining equation

$$Q(x, y, z) = (x^n - y^n)(y^n - z^n)(z^n - x^n).$$

One can show that $\mathcal{F}_n$ is a free arrangement which means that the derivation bundle $\mathcal{D}$ splits, and we can find its decomposition, namely

$$\mathcal{D} = \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-(n+1)) \oplus \mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(-(2n-2)).$$

We focus on the case when $n \geq 5$ is odd.

Example 2. The set Z_n of the duals to lines in $\mathcal{F}_n$ consists of $3n$ points with the ideal $I(Z_n) = \langle x^n + y^n + z^n, xyz \rangle$. One can show that Z_n admits an unexpected curve of degree $n + 2$ with a general point of multiplicity $n + 1$.

Now we can look at a different class of arrangements which was introduced by Dimca and Sticlaru in [4].

Definition 3. A reduced curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ given by $f \in S$ is *nearly free* if the minimal resolution of the Milnor algebra $M(f) = S/\langle \partial_x f, \partial_y f, \partial_z f \rangle$ has the following form

$$0 \to S(-d_2 - d) \to S(-d_1 - (d-1)) \oplus S(-d_2 - (d-1)) \oplus S(-d_2 - (d-1)) \to S^3(-d+1) \to S,$$

where d_1, d_2 are non-negative integers such that $d_1 \leq d_2$ and $d = d_1 + d_2$. In that case, the pair (d_1, d_2) is called the set of exponents of nearly free curve C.

Consider the (deletion) arrangement $\mathcal{NF}_n$ defined by the following equation

$$\widetilde{Q}(x, y, z) = (x^n - y^n)(y^n - z^n)(z^n - x^n)/(x - y)$$

with $n \geq 3$.

One can show that for $n \geq 3$ the arrangement $\mathcal{NF}_n$ is nearly free with the exponents $(n+1, 2n-2)$. Using this family of arrangements we can construct an infinite sequence of unexpected curves.

In order to explain the reason why we can do that, let us present the following result which tells us that the sets of duals to nearly free arrangements of lines, with some combinatorial restrictions, admit unexpected curves [5, Proposition 5.4].

Theorem 4 (Malara, Pokora, and Tutaj-Gasińska)**.** Let $Z = \{z_1, \ldots, z_d\} \subset \mathbb{P}^2_{\mathbb{C}}$ be a set of points such the set of duals $\mathcal{A}_Z$ is a nearly free arrangement of lines with the exponents (d_1, d_2). Then Z admits an unexpected curve of degree $d_1 + 1$ with a general point of multiplicity d_1 if and only if $d_2 - d_1 \geq 3$.

Example 5. Consider $\mathcal{NF}_n$ with $n \geq 3$. Then the set of points Z_n, dual to lines in $\mathcal{NF}_n$, admits an unexpected curve of degree $d_1 + 1$ with a general point Q of multiplicity d_1 if and only if

$$d_2 - d_1 = 2n - 2 - (n+1) = n - 3 \geq 3,$$

so exactly when $n \geq 6$.

Finally, let us present an interesting example of a finite set of points admitting two unexpected curves. In this case the set of duals is a plus-one generated arrangement of lines – this notion was introduced very recently by Abe [1].

Example 6. Consider the following set of 18 points

$$Z := \Big\{(0,1,0), (-1,1,0), (-2,1,0), (-3,1,0), (-3,2,0), (4,0,-1)$$
$$(1,1,-1), (2,1,-1), (3,1,-1), (4,1,-1), (0,2,-1), (1,2,-1),$$
$$(2,2,-1), (0,3,-1), (1,3,-1), (-2,3,-1), (-1,3,-1), (-2,4,-1)\Big\},$$

and the dual line arrangement $\mathcal{A}_Z$ with its defining polynomial Q.

The minimal free resolution of the Milnor algebra $M(Q)$ has the following form

$$0 \to S(-30) \to S(-29) \oplus S(-28) \oplus S(-24) \to S(-17)^3 \to S,$$

from which we can read the exponents $(d_1, d_2, d_3) = (7, 11, 12)$. Since we have $d_1 + d_2 = 18$, $\mathcal{A}_Z$ is a *plus-one generated arrangement* of level 12.

We can compute the (generic) splitting type which is $(7, 10)$. It means that there exists an unexpected curves of degree j with a general point of multiplicity $j - 1$ for $j \in \{8, 9\}$.

Let us finish with the following question which motivates our current (and future) research.

Question 7. Is it possible to find combinatorial constraints on plus-one generated arrangements of lines (in a spirit of the result for nearly free arrangements) which would allow us to conclude that the dual sets of points admit unexpected curves?

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Free arrangements with low exponents

Ştefan O. Tohăneanu

The talk is based on the article [6].

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a central essential hyperplane arrangement in V a vector space of dimension k over $\mathbb{K}$ a field of characteristic zero. Let $R := Sym(V^*) = \mathbb{K}[x_1, \ldots, x_k]$ and fix $\ell_i \in R, i = 1, \ldots, n$ the linear forms defining the hyperplanes of $\mathcal{A}$ (i.e., $H_i = V(\ell_i), i = 1, \ldots, n$).

Let $D(\mathcal{A})$ denote the R-module of logarithmic derivations, and whenever this module is free (of rank k) one says that the hyperplane arrangement is *free*. For a free arrangement, the degrees of the basis elements of $D(\mathcal{A})$ are called the *exponents* of $\mathcal{A}$, denoted $\operatorname{Exp}(\mathcal{A}) := (1, d_2, \ldots, d_k)$. The exponent 1 comes from the fact that the cyclic R-module generated by the Euler derivation is a direct summand of $D(\mathcal{A})$, for <u>any</u> central hyperplane arrangement $\mathcal{A}$.

Let $L(\mathcal{A})$ denote the intersection lattice of a hyperplane arrangement $\mathcal{A}$. A subspace $X \in L(\mathcal{A})$ is said to be *modular* if $X + Y \in L(\mathcal{A})$ for all subspaces $Y \in L(\mathcal{A})$. A central essential hyperplane arrangement $\mathcal{A}$ of rank k is called *supersolvable* if $L(\mathcal{A})$ has a maximal chain of modular elements:

$$V = X_0 < X_1 < \cdots < X_k = 0.$$

Every supersolvable arrangement is free (see [3]), and in this talk we are investigating which free arrangements could be supersolvable (a first example of a free arrangement that is not supersolvable is the non-Fano arrangement; it has exponents $(1, 3, 3)$).

Conjecture. Every central free hyperplane arrangement with exponents 1's, 2's, and at most one 3, is supersolvable.

The conjecture has been verified in [6] for the following cases:

(a) When the exponents are only 1's and 2's. Note: this result adds a fifth equivalent result "$\mathcal{A}$ is free, and its exponents are each 1 or 2", to the list of equivalent results in the statement of [1, Theorem 5.11].
(b) For any inductively free hyperplane arrangement. Note: supersolvable $\subset$ inductively free $\subset$ free, and the inclusions are strict (first one because of the non-Fano arrangement, and second one because of [4, Example 4.59].
(c) For any free arrangement of rank 3, 4, or 5.

The proofs of the cases of the conjecture listed above are based on some key lemmas and results:

i. The existence of a linear logarithmic derivation different than the Euler derivation is equivalent to the hyperplane arrangement being decomposable $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. If $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, then $\mathcal{A}$ is free, respectively supersolvable, if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ are free, respectively supersolvable (see [4, Proposition 4.28] and [2, Proposition 2.6]). With this, we can assume that there is no exponent equal to 1, other than the one coming from the Euler derivation. This also helps us with result **iv.** below.

ii. If $\mathcal{A}$ is free, then $\mathcal{A}_Y$ is free for any $Y \in L(\mathcal{A})$, and since since $D(\mathcal{A}) \subset D(\mathcal{A}_Y)$, $\mathrm{Exp}(\mathcal{A}_Y)$ are also 1's, 2's, and possibly exactly one 3. $rank(\mathcal{A}_Y) < k$, so by inductive hypotheses $\mathcal{A}_Y$ is supersolvable. So, to resolve the conjecture, everything boils down to constructing a modular coatom (i.e., a flat of rank $k-1$ that is modular).

iii. Since any free hyperplane arrangement is formal (see [7]), the following result helps us with **ii.** above (for cases (a), (c), and most of (b)): If H_i and H_j belong to a unique circuit of size 3, then $X = \cap_{H \in \mathcal{A} \setminus \{H_i, H_j\}} H$ is a coatom and is modular.

iv. In dealing with the general version of the conjecture, a good enumerative understanding of rank 2 flats is required. For this purpose, if $\mathcal{A}$ is free with $\mathrm{Exp}(\mathcal{A}) = (1, d_2, \ldots, d_k)$, first we have Terao's Factorization Theorem ([4, Theorem 4.61]):

$$\pi(\mathcal{A}, t) = (1+t)(1+d_2t)\cdots(1+d_kt) = 1 + |\mathcal{A}|t + \left(\sum_{Y \in L_2(\mathcal{A})} \mu(Y)\right) t^2 + \cdots,$$

and we also have [5, Lemma 5.2]: for any $Y \in L_2(\mathcal{A})$, $\mu(Y) \leq \max\{d_2, \ldots, d_k\}$, where $\mu(Y)$ is the value of the Möbius function.

These enumerative aspects are captured in [6, Lemmas 4.2, 4.5, Remarks 4.3, 4.4, 4.6]. Below is an example that reflects the analysis that was done in proving case (c) of the conjecture.

Example. Suppose $k = 4$. $\mathrm{Exp}(\mathcal{A}) = (1, 2, 2, 3)$, so $|\mathcal{A}| = 8$. Let u be the number of rank two flats with Möbius value 2, and v be the number of rank two flats with Möbius value 3. Then $u + 3v = 5$, and $0 \leq v \leq 1$.

Suppose further that $v = 0$ (and hence $u = 5$). For $i = 1, \ldots, 8$, let u_i be the number of circuits of size 3 that contain the hyperplane H_i. Then, $1 \leq u_i \leq 3$, and $u_1 + \cdots + u_8 = 15$. If, after some reordering, $u_1 = \cdots = u_6 = 1$, since $u = 5$, at least two of the hyperplanes $H_1, \ldots, H_6$ will belong to a unique circuit of size 3, and we can use result **iii.**. In [6] we show that if the number of u_j's that are equal to 1 is less than or equal to five, we obtain a contradiction (often a contradiction with the fact that the rank of $\mathcal{A}$ is 4).

For example, suppose, after some reordering, $u_1 = u_2 = u_3 = 1$ and $2 \leq u_4, \ldots, u_8 \leq 3$. Then, $u_4 + \cdots + u_8 = 12$, so after some reordering, $u_4 = u_6 = u_8 = 2$, and $u_5 = u_7 = 3$. Then, after some renumbering, considering that any two distinct rank two flats cannot have two or more hyperplanes in common, the only way to construct the five circuits of size 3 is the following:

$$A = \{1, 4, 7\}, B = \{2, 5, 8\}, C = \{3, 5, 7\}, D = \{4, 5, 6\}, E = \{6, 7, 8\}.$$

Then we have:

- From C, ℓ_3 is a linear combination of ℓ_5 and ℓ_7.
- From D, ℓ_4 is a linear combination of ℓ_5 and ℓ_6.
- From E, ℓ_8 is a linear combination of ℓ_6 and ℓ_7.
- From B, ℓ_2 is a linear combination of ℓ_5 and ℓ_8, and so it is a linear combination of ℓ_5, ℓ_6, ℓ_7.
- From A, ℓ_1 is a linear combination of ℓ_4 and ℓ_7, and so it is a linear combination of ℓ_5, ℓ_6, ℓ_7.

All of these lead to the fact that the rank of $\mathcal{A}$ is 3, and not 4 as assumed.

We end with a note about the desire for the development and the implementation of a computer algorithm that will skip doing by hand the calculations similar to those in the above example, especially the calculations that will list all possible rank two flats of Möbius values 2 and 3, unique up to a reordering/renumbering of hyperplanes.

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On Yuzvinsky's lattice sheaf cohomology for hyperplane arrangements

Paul Mücksch

Let $\mathcal{A}$ be a hyperplane arrangement in a $\mathbb{K}$-vector space V of dimension $\ell \geq 2$ for some field $\mathbb{K}$. The intersection lattice $L(\mathcal{A})$ which encodes the combinatorics of $\mathcal{A}$ is the lattice consisting of all intersections of subsets of hyperplanes ordered by reverse inclusion. Let $S = \mathbb{K}[x_1, \dots, x_\ell]$ be the coordinate ring of the vector space V. The arrangement $\mathcal{A}$ is called *free* if the associated finitely generated graded S-module $D(\mathcal{A}) = \{\theta \in \mathrm{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A}\}$ of *logarithmic vector fields* or $\mathcal{A}$-*derivations* is a free S-module, a notion first introduced and studied by Kyoji Saito [7] and Hiroaki Terao [10]. In the study of hyperplane arrangements it is of central interest to relate properties of $D(\mathcal{A})$ to the combinatorial structure of $\mathcal{A}$ given by its intersection lattice. The ultimate solution is proposed by the following conjecture, first stated by Terao in the 1980s, cf. [6, Conj. 4.138].

Conjecture (Terao's conjecture)**.** For a fixed field $\mathbb{K}$ the freeness of $\mathcal{A}$ only depends on its intersection lattice $L(\mathcal{A})$.

Assume that $\mathcal{A}$ is central and essential, that is $\cap_{H\in\mathcal{A}} H = \{0\}$ and set $L_0 := (L(\mathcal{A}) \setminus \{\{0\}\})^{\mathrm{op}}$, i.e. the order relation in L_0 is inclusion. In a series of papers [13], [14], [15] Sergey Yuzvinsky studied the functor $\mathcal{D} : L_0 \to \mathbf{Mod}_S$, $(X \subseteq Y) \mapsto (\mathcal{D}(X) = D(\mathcal{A}_X) \hookrightarrow D(\mathcal{A}_Y) = \mathcal{D}(Y))$ regarded as a sheaf on the finite topological space associated to the poset L_0 and its cohomology. An arrangement $\mathcal{A}$ is called *locally free* if all localization subarrangements $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$ for $X \in L_0$ are free. He showed [14, Thm. 1.1] that a locally free hyperplane arrangement $\mathcal{A}$ is free if and only if the lattice sheaf cohomology groups $H^n(L_0, \mathcal{D})$ vanish for all $0 < n < \ell - 1$.

A classical theorem by Geoffrey Horrocks [3] asserts that a vector bundle $\mathcal{E}$ on projective space $\mathbb{P}^{\ell-1} = \mathrm{Proj}\, S$ splits into a direct sum of line bundles if and only if the sheaf cohomology groups $H^n(\mathbb{P}^{\ell-1}, \mathcal{E}(d))$ vanish for all $0 < n < \ell - 1$ and all $d \in \mathbb{Z}$. It turns out that the coherent sheaf $\widetilde{D}$ on $\mathbb{P}^{\ell-1}$ associated to the derivation module $D = D(\mathcal{A})$ of a locally free arrangement is a vector bundle, cf. [5, Thm. 2.3]. Applying Horrocks' criterion to $\widetilde{D}$ of a locally free hyperplane arrangement yields a freeness criterion resembling Yuzvinsky's criterion, cf. [12, Prop. 1.20]. A related similarity with local cohomology was already noticed by Yuzvinsky in [13, Rem. 2.7].

Our aim is to establish the exact relationship between Yuzvinsky's lattice sheaf cohomology and the sheaf cohomology on projective space and explain the resemblance of Yuzvinsky's and Horrocks' criteria for freeness. This clarifies the resemblance with local cohomology already noted by Yuzvinsky [13, Rem. 2.7] and answers a question posed by Masahiko Yoshinaga [12, Prob. 1.49].

Set $\mathfrak{X} := \operatorname{Spec} S \setminus \{\mathfrak{m}\}$ where $\mathfrak{m} = (x_1, \ldots, x_\ell)$ is the homogeneous maximal ideal and let $\mathcal{O}_{\mathfrak{X}} = \widetilde{S}|_{\mathfrak{X}}$ be the structure sheaf (the restriction of the structure sheaf of the affine scheme $\operatorname{Spec} S$ to the open complement $\mathfrak{X}$ of the origin).

Our principal theorem establishes the exact relationship of the cohomology of the sheaf $\mathcal{D}$ on L_0 studied by Yuzvinsky with the cohomology of the coherent sheaf $\widetilde{D}|_{\mathfrak{X}}$ on the punctured spectrum $\mathfrak{X}$ associated to the derivation module.

Theorem 1 ([4, Thm. 1.1]). *For all $n \neq \ell - 1$ we have*

$$H^n(\mathfrak{X}, \widetilde{D}|_{\mathfrak{X}}) \simeq \bigoplus_{i+j=n} H^i(L_0, \mathcal{D}) \otimes_S H^j(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$$

and for $n = \ell - 1$ we have a short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=\ell-1} H^i(L_0, \mathcal{D}) \otimes_S H^j(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \longrightarrow H^{\ell-1}(\mathfrak{X}, \widetilde{D}|_{\mathfrak{X}})$$

$$\longrightarrow \operatorname{Tor}_1^S(H^1(L_0, \mathcal{D}), H^{\ell-1}(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})) \longrightarrow 0.$$

In particular, $H^n(\mathfrak{X}, \widetilde{D}|_{\mathfrak{X}}) \simeq H^n(L_0, \mathcal{D})$ for $n < \ell - 1$.

Note that sheaf cohomology on the scheme $\mathfrak{X}$ and sheaf cohomology on projective space are connected as follows, see e.g. [8, n° 69: Remarque].

Remark 1*:* Let M be a finitely generated graded S-module. Denote by $\widetilde{M}|_{\mathfrak{X}}$ the coherent sheaf associated to M on $\operatorname{Spec} S$ restricted to the open subset $\mathfrak{X} = \operatorname{Spec} S \setminus \{\mathfrak{m}\}$ and by $\widetilde{M}$ the coherent sheaf on $\mathbb{P}^{\ell-1} = \operatorname{Proj} S$ associated to M. Then $H^n(\mathfrak{X}, \widetilde{M}|_{\mathfrak{X}}) \simeq \bigoplus_{d \in \mathbb{Z}} H^n(\mathbb{P}^{\ell-1}, \widetilde{M}(d))$ for $n \geq 0$.

As a direct consequence of Remark 1 and Theorem 1 we obtain the following result which establishes the relationship between the lattice sheaf cohomology studied by Yuzvinsky and the sheaf cohomology on projective space. This answers a question posed by Yoshinaga [12, Prob. 1.49] and readily yields another proof, using Horrocks' criterion, of Yuzvinsky's freeness criterion.

Theorem 2 ([4, Thm. 1.3]). *For $n < \ell - 1$ we have*

$$H^n(L_0, \mathcal{D}) \simeq \bigoplus_{d \in \mathbb{Z}} H^n(\mathbb{P}^{\ell-1}, \widetilde{D}(d)).$$

The connection to local cohomology and projective dimension is as follows.

Remark 2*:* Recall that local cohomology is related to the cohomology on punctured affine space as follows, cf. [2, Prop. 2.2]. For $i > 0$ we have:

$$H^{i+1}_{\mathfrak{m}}(D) \simeq H^i(\mathfrak{X}, \widetilde{D}|_{\mathfrak{X}}).$$

Furthermore, by [2, Thm. 3.8] the depth, respectively the projective dimension $\operatorname{pd}(D)$ (by the Auslander-Buchsbaum formula) of the module D is tied to local cohomology by

$$\operatorname{pd}(D) \leq p \text{ if and only if } H^i_{\mathfrak{m}}(D) = 0 \text{ for } i < \ell - p.$$

Already in his original work, Yuzvinsky suspected a more direct connection to local cohomology, see [13, Rem. 2.7]. The module D is reflexive (cf. [7, p. 268]) and as such, it has projective dimension at most $\ell - 2$. Consequently, Theorem 1 together with the preceding remark directly yields the following characterization of the projective dimension of D.

Theorem 3. *The following two conditions are equivalent:*

i) $\operatorname{pd}(D) \leq p$*;*
ii) $H^n(L_0, \mathcal{D}) = 0$ *for* $0 < n < \ell - 1 - p$*.*

Thus, by Theorem 3, we obtain the following stronger form of Yuzvinsky's freeness criterion [14, Thm. 1.1] making the assumption of $\mathcal{A}$ being locally free superfluous.

Corollary ([4, Thm. 1.5])**.** The arrangement $\mathcal{A}$ is free if and only if

$$H^n(L_0, \mathcal{D}) = 0 \quad \text{for} \quad 0 < n < \ell - 1.$$

Now, we may reformulate Terao's conjecture in the following way:

Conjecture (Terao's conjecture)**.** The vanishing of the lattice sheaf cohomology groups $H^n(L_0, \mathcal{D})$ for all $0 < n < \ell - 1$ does only depend on the poset L_0.

In his recent work [1], Takuro Abe asks the problem whether even the projective dimension of the derivation module is combinatorial. This generalization of Terao's conjecture can be reformulated with Theorem 3 as follows.

Problem. Does the vanishing of the lattice sheaf cohomology groups $H^n(L_0, \mathcal{D})$ (for arbitrary n) only depend on the combinatorics of the arrangement, i.e. on the poset L_0?

Finally, in view of the long exact sequence in cohomology obtained from a short exact sequence of sheaves, we note the following problem.

Problem. Are there short exact sequences relating the sheaves $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{D}''$, $(\mathcal{D}^H, \mathbf{m}^H)$ where $\mathcal{D}'$, $\mathcal{D}''$, $(\mathcal{D}^H, \mathbf{m}^H)$ are the sheaves respectively of a deletion, restriction or Ziegler-restriction of the original arrangement?

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Castelnuovo-Mumford regularity of the Milnor ring of a hypersurface

Henry Schenck

(joint work with Laurent Busé, Alexandru Dimca, Gabriel Sticlaru)

For a reduced hypersurface $V(f) \subseteq \mathbb{P}^n$ of degree d, the Castelnuovo-Mumford regularity of the Milnor algebra $M(f)$ is well understood when $V(f)$ is smooth, as well as when $V(f)$ has isolated singularities. We study the regularity of $M(f)$ when $V(f)$ has a positive dimensional singular locus. In certain situations, we prove that the regularity is bounded by

$$T = (d-2)(n+1),$$

which is the degree of the Hessian polynomial of f. However, this is not always the case, and we prove that in $\mathbb{P}^3$ the regularity of the Milnor algebra can grow quadratically in d.

Definition 1. Let $S = \oplus_k S_k = \mathbb{C}[x_0, ..., x_n]$ be the graded polynomial ring, where S_k denotes the vector space of degree k homogeneous polynomials. For a homogeneous polynomial $f \in S_d$, the Jacobian ideal J_f is generated by the partial derivatives of f, and the Milnor algebra $M(f)$ is the graded ring S/J_f.

The ring $M(f)$ is of interest because it encodes the singular subscheme $\Sigma = \Sigma(f)$ of the projective hypersurface $V(f) \subseteq \mathbb{P}^n$. When $V(f)$ is smooth, $M(f)$ is an Artinian complete intersection and plays a central role in the Hodge theory of $V(f)$. A landmark result of Griffiths [19] shows that the Hodge numbers of $V(f)$ can be extracted from $M(f)$, and recent work of Dimca [8] shows that one can obtain related results for an even dimensional nodal hypersurface. The Milnor algebra also has applications in physics, where it is known as the Chiral ring [5], in the study of Bernstein-Sato polynomials [31], in the study of multiplier ideals [15], and in Torelli type theorems. We study the Castelnuovo-Mumford regularity of $M(f)$, which can be read from a graded minimal finite free resolution. Let

$$0 \to F_m \to \cdots \to F_0 \to M(f) \to 0,$$

be a minimal graded free resolution of the graded S-module $M(f)$, where

$$F_k = \bigoplus_j S(-a_{k,j}) \text{ for } k = 0, \ldots, m.$$

By the Hilbert Syzygy Theorem, $m \leq n+1$ and $m = \operatorname{pd} M(f)$ is the projective dimension of $M(f)$. The Castelnuovo-Mumford regularity of $M(f)$ is

$$\operatorname{reg} M(f) = \max_{i,j}\{a_{i,j} - i\}.$$

Theorem 1: For the following classes of reduced hypersurfaces, $\operatorname{reg} M(f) < T$:

(1) $V(f)$ is a generic hyperplane arrangement $\subseteq \mathbb{P}^n$.
(2) $V(f)$ is a generic determinantal hypersurface $\subseteq \mathbb{P}^n$.
(3) $V(f)$ is a hypersurface $\subseteq \mathbb{P}^n$ of degree $d \geq 3$ which is free or nearly free.
(4) $V(f)$ is a generic arrangement of surfaces with isolated singularities in $\mathbb{P}^3$.

The proof of items (1)-(3) follows from analyzing the finite free resolution of $M(f)$. The proof of (4) involves a delicate spectral sequence argument involving the Buchsbaum-Rim complex and local cohomology.

Theorem 1 leads one to hope that there could be an upper bound on regularity of $M(f)$ that is linear in d. Theorem 2 below shows that this hope is vain. We prove:

Theorem 2: There exist reduced, irreducible hypersurfaces of degree d in $\mathbb{P}^3$ for which

$$\operatorname{reg} M(f) \sim \mathcal{O}(d^2).$$

The key to proving the result is to work in the setting of bigraded hypersurfaces, and we prove that for every degree d there exists a reduced, irreducible surface in $\mathbb{P}^3$ whose Milnor ring has a minimal first syzygy of degree $\frac{d^2+d-2}{3}$.

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Milnor fibrations of arrangements with trivial algebraic monodromy

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1. **Descending series and graded Lie algebras.** Among all the descending series of subgroups associated to a group G, the most prominent are the lower central series, $\{\gamma_k(G)\}_{k\geq 1}$, and the derived series, $\{G^{(r)}\}_{r\geq 0}$. Following Stallings [9], we also consider the rational and mod-p versions of these series. All these series start at G, and obey the following recursion formulas:

$$\gamma_{k+1}(G) = [G, \gamma_k(G)] \qquad G^{(r)} = [G^{(r-1)}, G^{(r-1)}], \tag{1}$$

$$\gamma^{\mathbb{Q}}_{k+1}(G) = \sqrt{[G, \gamma^{\mathbb{Q}}_k(G)]} \qquad G^{(r)}_{\mathbb{Q}} = \sqrt{\left[G^{(r-1)}_{\mathbb{Q}}, G^{(r-1)}_{\mathbb{Q}}\right]}, \tag{2}$$

$$\gamma^{p}_{k+1}(G) = (\gamma^p_k G)^p [G, \gamma^p_k G] \qquad G^{(r)}_{p} = (G^{(r-1)}_{p})^p \left[G^{(r-1)}_{p}, G^{(r-1)}_{p}\right]. \tag{3}$$

Each one of the series on the left forms an *N-series* for G, that is, a descending filtration, $N = \{N_k\}_{k\geq 1}$, of subgroups such that $N_1 = G$ and $[N_k, N_\ell] \subseteq N_{k+\ell}$. Therefore, each subgroup N_k is normal; moreover, each quotient N_k/N_{k+1} lies in the center of G/N_{k+1}, and thus is an abelian group. The direct sum of these quotients, $\operatorname{gr}^N(G) \coloneqq \bigoplus_{k\geq 1} N_k/N_{k+1}$, acquires the structure of a graded Lie algebra. When N is one of the aforementioned N-series, the corresponding associated graded Lie algebra is denoted by $\operatorname{gr}(G)$, $\operatorname{gr}^{\mathbb{Q}}(G)$, and $\operatorname{gr}^p(G)$, respectively.

2. **Alexander invariants and characteristic varieties.** Let $G' = [G, G]$ and $G'' = [G', G']$ be the first two terms in the derived series of G. The *Alexander invariant* of G is the abelian group $B(G) \coloneqq G'/G''$, viewed as a $\mathbb{Z}[G_{\text{ab}}]$-module. The module structure is induced from conjugation in the maximal metabelian quotient, G/G''; that is, $gG' \cdot xG'' = gxg^{-1}G''$ for $g \in G$ and $x \in G'$. In like fashion, we define the rational and mod-p Alexander invariants as $B_{\mathbb{Q}}(G) \coloneqq G'_{\mathbb{Q}}/G''_{\mathbb{Q}}$ and $B_p(G) \coloneqq G'_p/G''_p$, viewed as modules over $\mathbb{Z}[G_{\text{abf}}]$ and $\mathbb{Z}[H_1(G, \mathbb{Z}_p)]$, respectively.

Assume now that G is finitely generated. Then the group of complex-valued characters, $\mathbb{T}_G \coloneqq \operatorname{Hom}(G, \mathbb{C}^*)$, is a complex algebraic group, with identity component $\mathbb{T}^0_G \cong (\mathbb{C}^*)^n$, where $n = \operatorname{rank} G_{\text{ab}}$. The (depth k) *characteristic varieties* of G are the algebraic subsets $\mathcal{V}_k(G) \subseteq \mathbb{T}_G$ consisting of those characters $\rho\colon G \to \mathbb{C}^*$ for which $\dim_{\mathbb{C}} H^1(G, \mathbb{C}_\rho) \geq k$. The set $\mathcal{V}_1(G)$ coincides, at least away from the identity $1 \in \mathbb{T}_G$, with the zero locus of the annihilator ideal of $B(G) \otimes \mathbb{C}$. Likewise, $\mathcal{W}_1(G) \coloneqq \mathcal{V}_1(G) \cap \mathbb{T}_G^0$ coincides, away from 1, with $V(\operatorname{ann}(B_{\mathbb{Q}}(G) \otimes \mathbb{C}))$.

3. **Split extensions.** Given a split extension of groups, $G = K \rtimes_\varphi Q$, we consider a certain series of normal subgroups of K. This series, $L = \{L_n\}_{n\geq 1}$, was recently introduced by Guaschi and Pereiro in [5], who showed that $\gamma_n(G) = L_n \rtimes_\varphi \gamma_n(Q)$ for all $n \geq 1$. In [13], we prove that the series L is, in fact, an N-series, and recover their result. As a corollary, we show that $\operatorname{gr}(G)$ splits as a semidirect product of graded Lie algebras, $\operatorname{gr}^L(G) \rtimes_{\bar\varphi} \operatorname{gr}(Q)$.

In the case when Q acts trivially on the abelianization K_{ab}, we show that $L_n = \gamma_n(K)$ for all $n \geq 1$. As a corollary, we recover a well-known theorem of Falk and Randell [4]. If, moreover, Q is abelian and the inclusion $\iota\colon K \to G$ induces an injection $\iota_{\mathrm{ab}}\colon K_{\mathrm{ab}} \to G_{\mathrm{ab}}$ (this always happens if $Q = \mathbb{Z}$), we prove in [14] that:

(a) The map $\operatorname{gr}(\iota)\colon \operatorname{gr}_{>1}(K) \to \operatorname{gr}_{>1}(G)$ is an isomorphism.
(b) The map $B(\iota)\colon B(K) \to B(G)$ is a $\mathbb{Z}[K_{\mathrm{ab}}]$-linear isomorphism.
(c) The map $\iota^*\colon \mathbb{T}_G \twoheadrightarrow \mathbb{T}_K$ restricts to a surjection $\iota^*\colon \mathcal{V}_1(G) \twoheadrightarrow \mathcal{V}_1(K)$.

In the case when G is a right-angled Artin group and K is the corresponding Bestvina–Brady group, this recovers the main results from [7].

For the rational lower central series we start by showing that $\gamma_n^{\mathbb{Q}}(G) = \sqrt[G]{\gamma_n(G)}$, where $\sqrt[G]{S} \coloneqq \{g \in G \mid g^k \in S \text{ for some } k > 0\}$ for $S \subseteq G$. Work of Massuyeau [6] now implies that $\{\gamma^{\mathbb{Q}}(G)\}_{n\geq 1}$ and $\{L_n\}_{n\geq 1}$ are N-series for G and K, respectively. We then show in [13] that $\gamma_n^{\mathbb{Q}}(G) = \sqrt[K]{L_n} \rtimes_\varphi \gamma_n^{\mathbb{Q}}(Q)$ for all $n \geq 1$, and thus $\operatorname{gr}^{\mathbb{Q}}(G) = \operatorname{gr}^{\sqrt[K]{L}}(G) \rtimes_{\bar\varphi} \operatorname{gr}^{\mathbb{Q}}(Q)$. In the case when Q acts trivially on the torsion-free abelianization $K_{\mathrm{abf}} = \operatorname{gr}_1^{\mathbb{Q}}(K)$, we show that $\sqrt[K]{L_n} = \gamma_n^{\mathbb{Q}}(K)$ for all $n \geq 1$, and thus $\operatorname{gr}^{\mathbb{Q}}(G) = \operatorname{gr}^{\mathbb{Q}}(K) \rtimes_{\bar\varphi} \operatorname{gr}^{\mathbb{Q}}(Q)$. If, moreover, Q is torsion-free abelian and the map $\iota\colon K \hookrightarrow G$ induces an injection $K_{\mathrm{abf}} \hookrightarrow G_{\mathrm{abf}}$, we prove in [14] that:

(a′) The map $\operatorname{gr}(\iota)\colon \operatorname{gr}_{>1}^{\mathbb{Q}}(K) \to \operatorname{gr}_{>1}^{\mathbb{Q}}(G)$ is an isomorphism.
(b′) The map $B(\iota)\colon B^{\mathbb{Q}}(K) \to B^{\mathbb{Q}}(G)$ is a $\mathbb{Z}[K_{\mathrm{abf}}]$-linear isomorphism.
(c′) The map $\iota^*\colon \mathbb{T}_G^0 \twoheadrightarrow \mathbb{T}_K^0$ restricts to a surjection $\iota^*\colon \mathcal{W}_1(G) \twoheadrightarrow \mathcal{W}_1(K)$.

Though a parallel theory in characteristic p is not yet fully developed, some of the above results do have analogues over $\mathbb{Z}_p$. For instance, if $G = K \rtimes_\varphi Q$ is a split extension with Q acting trivially on $H_1(K, \mathbb{Z}_p)$, it is shown in [2] that $\gamma_n^p(G) = \gamma_n^p(K) \rtimes_\varphi \gamma_n^p(Q)$ for all $n \geq 1$; therefore, $\operatorname{gr}^p(G) = \operatorname{gr}^p(K) \rtimes_{\bar\varphi} \operatorname{gr}^p(Q)$.

4. **Milnor fibrations of arrangements.** A construction due to Milnor associates to each homogeneous polynomial $f \in \mathbb{C}[z_0, \dots, z_d]$ a fiber bundle, with base space $\mathbb{C}^*$, total space the complement $M = \mathbb{C}^{d+1} \setminus \{f = 0\}$, and projection map $f\colon M \to \mathbb{C}^*$. The Milnor fiber $F = f^{-1}(1)$ is a Stein manifold, and thus has the homotopy type of a finite CW-complex of dimension d. The monodromy of the fibration, $h\colon F \to F$, is given by $h(z) = e^{2\pi i/n} z$, where $n = \deg f$. If the polynomial f has an isolated singularity at the origin, then F is homotopy equivalent to a bouquet of d-spheres, whose number can be determined by algebraic means. In general, though, it is a hard problem to compute the homology groups of the Milnor fiber, even in the case when f completely factors into distinct linear forms.

This situation is best described by a hyperplane arrangement, that is, a finite collection, $\mathcal{A}$, of codimension-1 linear subspaces in $\mathbb{C}^{d+1}$, for some $d > 0$. Choosing a linear form f_H with kernel H for each hyperplane $H \in \mathcal{A}$, we obtain a homogeneous polynomial, $f = \prod_{H\in\mathcal{A}} f_H$. The long exact sequence in homotopy of the Milnor fibration $F \to M \to \mathbb{C}^*$ yields a split extension at the level of fundamental groups, $G = K \rtimes_\varphi \mathbb{Z}$, where $G = \pi_1(M)$, $K = \pi_1(F)$, and $\varphi(1) = h_*\colon K \to K$. In general, the monodromy action of $\mathbb{Z}$ on K_{ab} is highly non-trivial, and the determination of $b_1(F) = \operatorname{rank} K_{\mathrm{ab}}$ is far from known, except in some cases, see for instance [8, 11, 12]. It is also known that $H_*(F,\mathbb{Z})$ may have non-trivial torsion (see [3]), and that such torsion can, in fact, occur even in $H_1(F,\mathbb{Z})$ (see [16]). Finally, it is known that the ranks of the groups $\operatorname{gr}_k(G)$ are determined by the intersection lattice, yet $\operatorname{gr}_k(G)$ may have torsion (as noted in [10]), and such torsion is not necessarily combinatorially determined (see [1]).

In forthcoming work, [15], we use the general theory described above to study Milnor fibrations of arrangements for which the monodromy $h\colon F \to F$ acts trivially on either $H_1(F,\mathbb{Z})$, or $H_1(F,\mathbb{Z})/\mathrm{Tors}$, or $H_1(F,\mathbb{Z}_p)$ for some prime p.

In the first case, we have by (a) and (c) that the inclusion $F \hookrightarrow M$ induces an isomorphism $\operatorname{gr}_{>1}(\pi_1(F)) \cong \operatorname{gr}_{>1}(\pi_1(M))$ and a surjection $\mathcal{V}_1(M) \twoheadrightarrow \mathcal{V}_1(F)$. Nevertheless, examples from [12] show that the map $\mathcal{V}_2(M) \to \mathcal{V}_2(F)$ may not be surjective. In fact, there are pairs of arrangements $\mathcal{A}$ and $\mathcal{A}'$ with trivial monodromy in first homology such that $M \simeq M'$ and yet $F \not\simeq F'$, with the difference picked up by the depth 2 characteristic varieties.

In the second case, we have by (a′) and (c′) an isomorphism $\operatorname{gr}^{\mathbb{Q}}_{>1}(\pi_1(F)) \cong \operatorname{gr}^{\mathbb{Q}}_{>1}(\pi_1(M))$ and a surjection $\mathcal{W}_1(M) \twoheadrightarrow \mathcal{W}_1(F)$. We illustrate this phenomenon with Yoshinaga's icosidodecahedral arrangement from [16]. Trying to better understand this example and those from [12] has motivated much of this work.

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