Mathematisches Forschungsinstitut Oberwolfach

Report No. 7/2021

DOI: 10.4171/OWR/2021/7

Mini-Workshop: Non-semisimple Tensor Categories and Their Semisimplification (online meeting)

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7 February – 13 February 2021

Abstract. Finite tensor categories are, despite their many applications and great interest, notoriously hard to classify. Among them, the semisimple ones (called fusion categories) have been intensively studied. Those with non-integral dimensions form a remarkable class. Already more than 20 years ago, tilting modules have been proposed as a source of such fusion categories. In this way, the Verlinde categories associated to the pair of a simple complex Lie algebra $\mathfrak{g}$ and an integer level k have been recovered in a purely algebraic framework–called semisimplification of tensor categories. Recently efforts to understand how to go beyond these examples emerged. This mini-workshop aims at bringing together experts from various branches of representation theory and topological field theory to deepen our understanding of finite tensor categories and to compare new ways to understand semisimplification.

Mathematics Subject Classification (2010): 16T05, 20G42, 81T45.

Introduction by the Organizers

Due to the travel restrictions by the covid pandemic, the mini-workshop had to be held online. The number of participants was raised by the MFO to 22 plus 2 graduate students that acted as video assistants. Thus we had 24 participants from 7 countries (Argentina, France, Germany, Israel, United Kingdom, United States). Given the time difference between Israel and the Pacific coast of the US, where several of the participants live, we established the least inconvenient schedule (in our opinion) that nevertheless meant the challenge to attend the Zoom sessions after a long day for the colleagues from Israel and Europe. The program consisted of three talks of 60 minutes per day, with a total of 14 talks as one was cancelled for personal reasons.

As indicated in the abstract, the objective of the meeting was to reflect on the finite tensor categories, the main ways of describing and classifying them, and their applications. Several of the talks focused on the method known as semisimplification, which considers the quotient of a tensor category by negligible morphisms. Concretely, these were the talks by Brundan (quantum GL_N), Entova, Heidersdorf, Serganova (Lie supergroups), Ostrik (tensor envelopes), Snyder (quantum G_2 at roots of unity), Vay (small quantum groups at dihedral groups). Other methods of describing new examples of tensor categories were discussed by Nikshych (minimal non-degenerate extensions) and Plavnik (ribbon zesting). Relations of tensor categories with mathematical physics and representation theory were the subjects of the talks by Costantino (stated skein modules and algebras), Harman (Tannakian approach to representations in defining characteristic), Negron (quantum SL(2) and logarithmic CFT), Woike (the Hochschild complex of a modular tensor category). Pevtsova's talk dealt with the tensor product property of rank varieties of finite tensor categories.

The well-known atmosphere of Oberwolfach meetings, which facilitates long and deep discussions between participants, is irretrievably lost in the online version. Even so, the possibility of discussing, albeit in a reduced format, the topics of the mini-workshop was well received by the participants. The talks were animated by questions and comments from many of those present in a cordial ambiance. We thank the MFO for the opportunity to hold this virtual meeting without further delay and, as was expressed by all the participants, we hope that a new face-to-face meeting would be possible.

Understanding tensor categories, particularly finite and fusion ones, is an exciting topic with interactions with various areas such as representation theory, homological algebra, topological field theory, topology. We are confident that the talks in this mini-workshop and the discussions that they raised will lead to further progress in these directions.

Mini-Workshop (online meeting): Non-semisimple Tensor Categories and Their Semisimplification

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Abstracts

Semisimplifications and gligible quotients of categories associated with two dimensional cobordisms

Victor Ostrik
(joint work with M. Khovanov, Y. Kononov)

Given a sequence $\alpha = \{\alpha_0, \alpha_1, \alpha_2, ...\}$ of elements of a field k one defines a category where the objects are closed oriented 1-manifolds and the morphisms are $k-$linear combinations of oriented cobordisms modulo relations saying that a closed surface of genus g evaluates at the element $\alpha_g \in k$. The Karoubian envelope of this category is a symmetric tensor category $VCob_\alpha$ where the tensor product is defined as a disjoint union. The goal of this talk is to describe some results on quotients $\overline{VCob_\alpha}$ of the category $VCob_\alpha$ by the negligible morphisms.

Theorem 1. Assume that the category $VCob_\alpha$ has a nonzero negligible morphism. Then the sequence α must be linearly recursive, i.e. the generating function $Z_\alpha(T) = \sum_i \alpha_i T^i$ is a rational function of T.

Theorem 2. (see [1]) Assume the sequence α is linearly recursive. Then the category $\overline{VCob_\alpha}$ has finite dimensional Hom spaces.

Theorem 3. Assume the sequence α is linearly recursive and let $Z_\alpha = Z_{\beta_1} + Z_{\beta_2} + \ldots$ be the decomposition of the rational function $Z_\alpha(T)$ into fractions with single pole counting infinity (so this is essentially partial fractions decomposition). Then we have an equivalence of symmetric tensor categories

$$\overline{VCob_\alpha} = \overline{VCob_{\beta_1}} \otimes \overline{VCob_{\beta_2}} \otimes \cdots$$

Theorem 4. The quotient category $\overline{VCob_\alpha}$ is semisimple if and only if the function $Z_\alpha(T)$ has only simple poles (including $T = \infty$).

Also we state some conjectures and theorems on the dimensions of Hom spaces in categories $\overline{VCob_\beta}$ where the function $Z_\beta(T)$ is a fraction with single pole (note that Theorem 3 above reduces the study of general categories $\overline{VCob_\alpha}$ to this special case, at least in the case of algebraically closed field k).

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The braided Picard group and the group of minimal non-degenerate extensions of a symmetric tensor category

Dmitri Nikshych
(joint work with Alexei Davydov)

A minimal extension of a symmetric tensor category $\mathcal{E}$ is a non-degenerate braided tensor category $\mathcal{C}$ such that $\dim(\mathcal{E})^2 = \dim(\mathcal{C})$ together with an embedding of $\mathcal{E}$ into $\mathcal{C}$. In [3, 4], Lan, Kong, and Wen observed that minimal extensions of $\mathcal{E}$ form a group. For example, the group of minimal extensions of $\mathsf{Rep}(G)$ is the third cohomology group of G. In this talk, I will explain how to compute this group for a pointed symmetric tensor category using the theory of graded braided extensions [2]. I will also explain how the group of minimal extensions can be viewed as a "higher Picard group" and formulate some conjectures that extend the description of the Picard group of braided tensor category [1] to the 2-categorical setting.

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Simple modules of small quantum groups at dihedral groups

Cristian Vay
(joint work with Gastón Andrés García)

The Drinfeld double $\mathcal{D} = \mathcal{D}(\mathfrak{B}(V)\#H)$ of the bosonization of a finite-dimensional Nichols algebra $\mathfrak{B}(V)$ and a finite-dimensional Hopf algebra H is a sort of generalized small quantum group. Indeed, if we consider the canonical triangular decomposition $u_q(\mathfrak{g}) = u^+ \otimes u^0 \otimes u^-$ of a small quantum group, its positive part is a Nichols algebra and $u_q(\mathfrak{g})$ is a quotient of the Drinfeld double of $u^+\#u^0$. The Drinfeld double $\mathcal{D}$ also admits a triangular decomposition. Explicitly, $\mathcal{D} = \mathfrak{B}(V)\otimes \mathcal{D}(H) \otimes \mathfrak{B}(\overline{V})$ where $\mathfrak{B}(\overline{V})$ is a Nichols algebra over $\mathcal{D}(H)$, the Drinfeld double of H. As for $u_q(\mathfrak{g})$, the simple modules over $\mathcal{D}$ are in bijective correspondence with those over the middle term in the triangular decomposition, and can be constructed as quotients of generalized Verma modules. This is a common phenomenon for algebras with triangular decomposition, see for instance [1].

Once we have the classification of the simple $\mathcal{D}$-modules, the question regarding their weight decomposition, *i. e.* their structure as $\mathcal{D}(H)$-modules, naturally arises. In case H is a group algebra of an abelian group Γ, the weights are one-dimensional and there are several results known. However, in the non abelian case, the weights are not necessarily one-dimensional and the computations turn out to be more involved. In this case, the weight decomposition of the simple modules was only known for $\mathfrak{B}(V)$ being the Fomin-Kirillov algebra over the symmetric group $\mathbb{S}_3$ [4].

In the present work [3], we address this question for $\Gamma = \mathbb{D}_{4t}$, a dihedral group with $t \geq 3$. We consider these groups because their finite-dimensional Nichols algebras were classified in [2]. Unlike $\mathbb{S}_3$, for each dihedral group there are infinitely many Nichols algebras. On the way, we develop new techniques that can be applied to Nichols algebras over any Hopf algebra. Namely, we explain how to construct recursively the simple modules when the Nichols algebra is generated by a decomposable module, and show that the highest-weight of minimum degree in a Verma module determines its socle. We also prove that tensoring a simple module by a rigid simple module gives a semisimple module.

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A Karoubian tensor subcategory of representations of algebraic supergroups

Vera Serganova

Let G be a connected affine supergroup with Lie superalgebra $\mathfrak{g}$. We assume that $\mathfrak{g}$ is a Kac-Moody superalgebra, in particular, it has an invariant symmetric form and the root decomposition with 1-dimensional root spaces. Recall that some of roots of $\mathfrak{g}$ are isotropic with respect to the invariant form. The defect of G is the maximal number of mutually orthogonal linearly independent isotropic roots. For instance, the defect of $GL(m|n)$ is $\min(m, n)$.

Let $\operatorname{Rep} G$ denote the category of finite-dimensional representations of G and let $\operatorname{Rep}^+ G$ be the Karoubian subcategory of $\operatorname{Rep} G$ generated by all irreducible representations of G. By S_G we denote the semisimplfication functor from $\operatorname{Rep} G$ to $\overline{\operatorname{Rep} G}$.

Every isotropic root α defines an embedding $i_\alpha : GL(1|1) \to G$. Let $R_\alpha : \operatorname{Rep} G \to \operatorname{Rep} GL(1|1)$ be the corresponding restriction functor.

Theorem 1. *If G is of defect 1 or $GL(m|n)$ then the functor R_α restricts to $\operatorname{Rep}^+ G \to \operatorname{Rep}^+ GL(1|1)$.*

The proof is based on results of [HW14] for $GL(m|n)$ and [G] for defect 1 supergroups. In the case of $GL(m|n)$ the crucial argument involves all non-conjugate Borel subgroups of $GL(m|n)$ and odd reflections introduced by the author.

We believe that Theorem 1 is true for all groups G satisfying our assumptions. The remaining case $G = OSp(m|2n)$ is still open.

Using Theorem 1 we can prove that if G has defect 1 then every indecomposable object of $\mathrm{Rep}^+ G$ is a simple or projective G-module. Furthermore, we can describe the semisimplification of $\mathrm{Rep}^+ G$. Denote by N the normalizer of $GL(1|1) \subset G$, let $H := N/SL(1|1)$. Then H is a reductive supergroup, i.e., $\mathrm{Rep}\, H$ is a semisimple category. If $M \in$ RepG then $S_{GL(1|1)} R_\alpha M$ has the natural structure of H-module and thus, $S := S_{GL(1|1)} R_\alpha$ is a symmetric monoidal functor $\mathrm{Rep}\, G \to \mathrm{Rep}\, H$.

Theorem 2. *The functor $S : \mathrm{Rep}^+ G \to \mathrm{Rep}\, H$ annihilates all negligible morphisms. For a certain quotient $\bar{G} = H/\Gamma$ by a finite subgroup Γ the functor $S : \mathrm{Rep}^+ G \to \mathrm{Rep}\, \bar{G}$ is well defined and essentially surjective. Hence the category $\mathrm{Rep}\, \bar{G}$ is equivalent to the semisimplification of $\mathrm{Rep}^+ G$.*

Here is the list of $\bar{G}$ for defect 1 groups.

G	$\bar{G}$
$OSp(2\|2n)$	$\mathbb{G}_m \times Sp(2n-2)$
$GL(n\|1)$	$\mathbb{G}_m \times GL(n-1)$
$OSp(2m+1\|2)$	$O(2) \times SO(2m-1)$
$OSp(2m\|2)$	$O(2) \rtimes SO(2m-2)$
$OSp(3\|2m)$	$O(2) \times OSp(1\|2m-2)$
$D(2,1;t)$	$O(2)$
AG_2	$O(2) \times PSL(2)$
AB_3	$O(2) \rtimes PSL(3)$

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Quantum SL(2) and logarithmic vertex operator algebras at $(p, 1)$-central charge

Cris Negron

(joint work with Terry Gannon)

We consider a collection of conjectures which relate representations of quantum groups at parameter $q = e^{\pi i/p}$ to representations of vertex operator algebras at central charge $c_p = 1 - 6(1-p)^2/p$. We consider specifically the triplet W_p, singlet M_p, and Virasoro $Vir_p = Vir(c_p)$ vertex operator algebras.

By work of various authors over the past decades, we understand that each of the vertex operator algebras listed above has an associated ribbon tensor category of representations. For the triplet algebra specifically, the representation category $\mathrm{rep}(W_p)$ is a non-semisimple modular category. For the singlet and the Virasoro, one has ribbon tensor categories $\mathrm{rep}(M_p)_{\mathrm{aff}}$ and $\mathrm{rep}(Vir_p)_{\mathrm{aff}}$ of, what one might call, affine representations. This affineness is reflected in the fact that the tensor categories in question admit a distinguished tensor generator.

The representations of the vertex operator algebras W_p, M_p and Vir_p provide the most fundamental examples of *logarithmic* vertex tensor categories. One can think of the logarithmic property as simply indicating a non-semisimplicity of the associated representation categories, but it more precisely references an alternate type of analysis which one employs in deriving braided monoidal structures from VOA representations, outside of the rational setting.

In collective works of Bushlanov, Creutzig, Feigin, Gainutdinov, Milas, Runkel, Semikhatov, and Tipunin, it's been conjectured that there are ribbon tensor equivalences

$$\begin{cases} \mathrm{K} : \mathrm{rep}\,\mathrm{SL}(2)_q \xrightarrow{\sim} \mathrm{rep}(Vir_p)_{\mathrm{aff}} \\ \Psi : \mathrm{rep}(\dot{\mathbf{u}}_q(\mathfrak{sl}_2)) \xrightarrow{\sim} \mathrm{rep}(M_p)_{\mathrm{aff}} \\ \Theta : \mathrm{rep}(u_q(\mathfrak{sl}_2)) \xrightarrow{\sim} \mathrm{rep}(W_p), \end{cases} \tag{1}$$

which should furthermore be tied together by a certain action of PSL(2). The quantum groups $u_q(\mathfrak{sl}_2)$, $\dot{\mathbf{u}}_q(\mathfrak{sl}_2)$, and $\mathrm{SL}(2)_q$ appearing above are, respectively, the small quantum group, torus extended small quantum group, and Lusztig's modified divided power quantum group for SL(2) at $q = e^{\pi i/p}$.

At Oberwolfach I've discussed joint work with Terry Gannon in which we prove this conjecture, and established the desired equivalences (1). Our proof relies on analyses of $\mathrm{rep}(W_p)$ (specifically) via differential equations, and understandings of quantum groups via processes of semisimplification and de-equivariantization.

Jacobson-Morozov Lemma for Lie superalgebras using semisimplification

Inna Entova-Aizenbud
(joint work with Vera Serganova)

In my talk I presented a generalization of the Jacobson-Morozov Lemma for quasi-reductive algebraic supergroups (respectively, Lie superalgebras), based on the idea of semisimplification of tensor categories. This is based on [ES20].

Given a quasi-reductive algebraic supergroup G, we used the theory of semisimplifications of symmetric monoidal categories to define a symmetric monoidal functor $\Phi_x : \mathrm{Rep}(G) \to \mathrm{Rep}(OSp(1|2))$ associated to any given element $x \in \mathrm{Lie}(G)_{\bar{1}}$.

Our construction of Φ_x is inspired by the classical Jacobson-Morozov Lemma as presented in [EtO18]. The construction is as follows (for simplicity, I present it only for nilpotent elements $x \in \mathrm{Lie}(G)_{\bar{1}}$):

Consider the additive algebraic supergroup $\mathbb{G}_a^{(1|1)}$.

The semisimplification of the category of rational representations of $\mathbb{G}_a^{(1|1)}$ is the category $\mathrm{Rep}(OSp(1|2))$ with a semisimplification functor $\mathrm{Rep}(\mathbb{G}_a^{(1|1)}) \to S : \mathrm{Rep}(OSp(1|2))$.

On the other hand, any nilpotent $x \in \mathrm{Lie}(G)_{\bar{1}}$ gives a homormorphism $\mathbb{G}_a^{(1|1)} \to \mathbb{G}$ such that the image of its differential contains x. This defines a restriction functor $R_x := Res^G_{\mathbb{G}_a^{(1|1)}}$.

Composing these functors we obtain a symmetric monoidal functor Φ_x:

$$\begin{array}{ccc} \mathrm{Rep}(G) & \xrightarrow{R_x} & \mathrm{Rep}(\mathbb{G}_a^{(1|1)}) \\ & \searrow^{S\circ R_x} & \downarrow S \\ & & \mathrm{Rep}(OSp(1|2)) \end{array}$$

We use this approach to prove an analogue of the Jacobson-Morozov Lemma for algebraic supergroups. Namely, we gave a necessary and sufficient condition on nilpotent elements $x \in \mathrm{Lie}(G)_{\bar{1}}$ for which Φ_x is exact faithful, and thus defines an embedding of supergroups $OSp(1|2) \to G$ so that x lies in the image of the corresponding Lie algebra homomorphism. Such elements are called "neat elements", and the set of neat elements in a quasi-reductive Lie superalgebra possesses some very nice properties (such as having finitely many orbits with respect to the adjoint action).

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Ribbon zesting

Julia Plavnik

(joint work with Colleen Delaney, César Galindo, Eric Rowell, Qing Zhang)

Tensor categories are rich algebraic objects that appear in many areas of mathematics, such as vertex operator algebras, representation theory, low dimensional topology, and have applications in condensed matter physics.The theory of braided fusion categories is still in its early stages and the general landscape remains largely unexplored. To deepen our knowledge of this important class of fusion categories, we look for different constructions that produce new examples. Some of the well-known constructions of braided fusion categories are the de-equivariantization, the relative tensor product, the Drinfeld center, and the gauging construction.

In 2014, when classifying modular categories of Frobenius-Perron dimension 36, an exotic rank 10 fusion algebra was discovered.This fusion algebra was similar but distinct from the one of the quantum group category $SU(3)_3$. To realize this new fusion algebra it was enough to "twist" a little the fusion rules of $SU(3)_3$ by tensoring them by certain invertible objects. That was the first appearence of the so-called zesting construction. Later, a similar procedure arose when studying minimal modular extensions (MMEs) of super-modular categories. Given a MME of a super-modular category, 8 of the 16 MMEs can be realized via zesting. In [1], a systematic approach to zesting was presented by giving a complete obstruction theory and parameterization of this construction.

Zesting is a procedure that consists in deforming the categorical structure (tensor product, associativity constraint, braid, and twist/ribbon structure) in a simple way (via some cohomological structures) of the input category in order to obtain a new category in which one can do explicit computations. Notice that zesting is a particular case of the extension theory developed in [2]. One of the highlights of zesting is that given that certain obstructions (in each step) vanish, there are explicit formulas to compute the modular data -the S- and T-matrices- (and other link invariants) of the resulting category in terms of the zesting data and the modular data of the original category. A similar result is true for the image of the braid group representation associated to the zested category. This construction has already proven to be effective in different contexts, it has been used to categorify both modular data and fusion rings, which in general are hard task to achieve.

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Quantum G_2 at roots of unity: diagrams vs. algebra

Noah Snyder

(joint work with Victor Ostrik)

For generic q, Kuperberg's G_2 spider [2, 3] gives a diagrammatic description of the category of representations of the quantum group G_2. We extend this correspondence to q a root of unity. Our main result is that with finitely many exceptions, the Karoubi completion of the category of G_2 spiders at a root of unity q is equivalent as a braided tensor category to the category of tilting modules for the quantum group G_2 at the same q. As an immediate consequence, the semisimplified spider category is equivalent to the semisimple quantum group fusion category (again with finitely many exceptions).

We sketch the main idea of the argument. Kuperberg's results (and generalizations thereof by Morrison–Peters–Snyder [4]) give a functor from the spider category to the category of tilting modules. A counting argument of Westbury [5] and a general homological calculation, show that the dimensions of the Hom spaces agree on the two sides, so it is enough to either show that the functor is faithful or that it is full. We show that it is faithful by writing down a diagrammatic basis on the spider side and showing that it remains linearly independent when interpreted as maps of tilting modules by evaluating it on a basis of weight vectors and checking that this evaluation matrix is upper-triangular. This is closely related to an argument in Vitale's thesis [6], and to ideas of Elias [1].

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On stated skein modules of 3-manifolds

Francesco Costantino
(joint work with T. Le, Georgia Tech)

We study the algebraic and geometric properties of stated skein algebras of surfaces with punctured boundary. We show that the skein algebra of the bigon is isomorphic to the quantum group $\mathcal{O}_{q^2}(\mathrm{SL}(2))$ that provides a topological interpretation for its structure morphisms. We also show that its stated skein algebra lifts in a suitable sense the Reshetikhin-Turaev functor and in particular we recover the dual R-matrix for $\mathcal{O}_{q^2}(\mathrm{SL}(2))$ in a topological way. We deduce that the skein algebra of a surface with n boundary components is a comodule-algebra over $\mathcal{O}_{q^2}(\mathrm{SL}(2))^{\otimes n}$ and that cutting along an ideal arc corresponds to Hochshild cohomology of bicomodules. We give a topological interpretation of braided tensor product of stated skein algebras of surfaces as "glueing on a triangle"; then we recover topologically some bialgebras in the category of $\mathcal{O}_{q^2}(\mathrm{SL}(2))$-comodules, among which the "transmutation" of $\mathcal{O}_{q^2}(\mathrm{SL}(2))$. We relate these facts with other constructions as the moduli algebras of Alekseev-Grosse-Schomerus [1] and the theory of integration of ribbon categories over surfaces of Ben-Zvi-Brochier-Jordan [2].

We then pass to study three-manifolds and state a recent result (in writing) showing that state skein modules provide a functor from a suitable category of marked surfaces and their cobordisms to a Morita category of algebras and their bimodules.

All the results are joint work with Thang Le.

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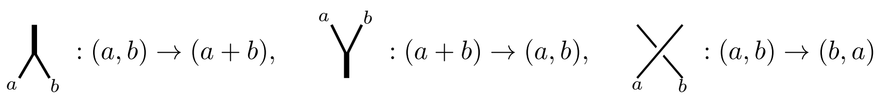
Semisimplification of the category of tilting modules for rational representations of quantum GL_n at a root of unity

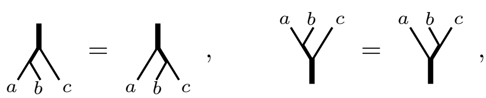
Jonathan Brundan

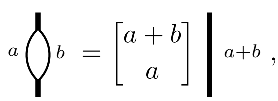
This talk explains the structure of the semisimple tensor category that is the title of the talk. The main result is the q-analog of previous joint work [1] with Entova-Aizenbud, Etingof and Ostrik.

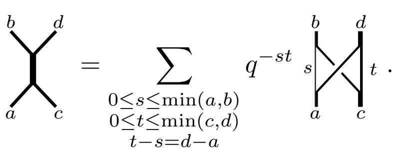
The approach is based on a fundamental strict monoidal category which is defined by some remarkably elegant generators and relations, namely, the *q-Schur category* $\mathcal{S}chur_q$. By definition $\mathcal{S}chur_q$ is a $\mathbb{Z}[q, q^{-1}]$-linear monoidal category with objects that are all strict compositions, tensor product being by concatenation. The one-part compositions (a) for $a > 0$ give a family of generating objects. In string diagrams, we represent the identity morphism of this object as a string labeled by the *thickness* a. Then there are generating morphisms



for $a, b > 0$, which we call the two-fold merge, the two-fold split and the positive crossing, respectively. The generating morphisms are subject to the following relations for $a, b, c, d > 0$ with $d - a = c - b$:







The q-Schur category arises from considerations involving the coordinate algebra of quantum GL_n in the stable limit as $n \to \infty$. In turn, the coordinate algebra of quantum GL_n can be constructed by applying the coend construction to a "fiber functor" for the HOMFLY-PT skein category, and the latter strict monoidal category also plays a role in this work. The q-Schur category is also related to the web category of Kamnitzer, Cautis and Morrison [2], but there are several features which makes the q-Schur category more accessible than the web category, for example, its morphism spaces have easy-to-construct integral bases.

The q-Schur category is related to tilting modules for quantum GL_n due to the existence of a full monoidal functor

$$\mathcal{S}chur_q \to \mathcal{T}ilt(q\text{-}GL_n)$$

taking (a) to the a-th quantized exterior power $\bigwedge^a V$ of the natural representation. The positive crossing $(a, b) \to (b, a)$ goes to $(-1)^{ab}$ times the inverse of the isomorphism $\bigwedge^a V \otimes \bigwedge^b V \to \bigwedge^b V \otimes \bigwedge^a V$ defined by the inverse of the R-matrix. The seemingly unnatural choice of signs here is a shadow of Ringel duality, which underpins the entire construction. Indeed, the final result describing the semisimplification of the category $\mathcal{T}ilt(q\text{-}GL_n)$ can be viewed as a statement that is Ringel dual to the Steinberg tensor product theorem for quantum GL_n.

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Tensor product property, rank variety and hypersurface support

Julia Pevtsova

This is a report on two independent on-going projects: one is joint with D.Benson, S. Iyengar and H. Krause and the other - with C. Negron.

Let u be a finite dimensional Hopf algebra over a field k. Since u is self-injective, we can form the stable category $\operatorname{Stab}\mathsf{u}$ which carries the structure of a tensor triangulated category. Explicitly, the objects of $\operatorname{Stab}\mathsf{u}$ are (all) u-modules whereas for morphisms we quotient out u-module homomorphisms by the subset of maps factoring through a projective module. Effectively, we kill projective u-modules, taking out injectives on the way since projectives and injectives are the same by the self-injectivity of u.

The stable category $\operatorname{Stab}\mathsf{u}$ has a "small" counterpart - the stable category $\operatorname{stab}\mathsf{u}$ of all finite-dimensional u-modules.

The main focus of this talk is the notion of support on the stable category $\operatorname{stab}\mathsf{u}$ which goes back to the work of Quillen, Alperin-Evens, Carlson and others in modular representation theory of finite groups and Friedlander-Parshall for restricted Lie algebras. The original definition of the *cohomological support* $\operatorname{supp} M$ of a u-module M goes through the action of the cohomology algebra $H^*(\mathsf{u}, k)$ on $\operatorname{Ext}^*(M, M)$ (via Yoneda product, for example). The variety of the annihilator of this action inside $\operatorname{Proj} H^*(\mathsf{u}, k)$ is the support of M. Thanks to the work of

Benson-Iyengar-Krause, this notion can be extended to the "big" stable category $\mathrm{Stab}\,\mathsf{u}$ via local cohomology functors.

Although the definition of the support makes little reference to the tensor structure on $\mathrm{Stab}\,\mathsf{u}$, there are well-studied classes of Hopf algebras for which the support does respect the tensor product. The relationship is expressed in what is known as the *tensor product formula*:

$$\mathrm{supp}(M \otimes N) = \mathrm{supp}\, M \cap \mathrm{supp}\, N$$

This property is known for $\mathrm{Stab}\,\mathsf{u}$ where u is a group algebra of a finite group or a finite group scheme (see [2], [5], [3]). The proof goes through a *realization* of the cohomological support variety via an alternative construction with no reference to cohomology, *rank variety*. It was introduced by Carslon for elementary abelian p-groups and developed for all finite group schemes via the theory of π-points by Friedlander-Pevtsova. In this talk we introduce and study another alternative construction of the support, that of a *hypersurface support* and apply it to establish the tensor product formula in two different situations: for finite unipotent supergroup schemes ([4]) and small quantum borels in type A ([6]).

In the case of supergroup schemes, an alternative construction of the support combines the hypersurface support for commutative complete intersections ([1]) and the theory of π-points introduced in [5] developed in the super setting. For the quantum groups, we develop the notion of the hypersurface support for integrable finite dimensional Hopf algebras which we informally refer to as non-commutative complete intersections to emphasis the analogy with the commutative case.

We say that a finite dimensional Hopf algebra u is *integrable* if there is a deformation sequence

$$Z \hookrightarrow \mathsf{U} \to \mathsf{u}$$

such that

(1) U is a Hopf algebra of a finite global dimension and $\mathsf{U} \to \mathsf{u}$ is a Hopf algebra homomorphism,
(2) Z is a smooth central subalgebra which is a (right) coideal subalgebra of U

Denoting by m_Z the augmentation ideal in Z, we set $X = \mathbb{P}(\mathsf{m}_Z/\mathsf{m}_Z^2)$ to be the *ambient* support space in which the hypersurface support $\mathrm{supp}^{hyp}(M)$ lives for any $M \in \mathrm{Stab}\,\mathsf{u}$. For any geometric point $c : \mathrm{Spec}\, K \to X$ we pick $f \in \mathsf{m}_Z$ representing c and consider the deformation sequence

$$Z/f \hookrightarrow \mathsf{U}/f \xrightarrow{\pi_f} \mathsf{u}.$$

Definition 1. $\mathrm{supp}^{hyp}(M) := \{c \in X \,:\, \mathrm{proj\ dim}_{\mathsf{U}/f} \pi_f^*(M) = \infty\}$.

This definition has a glaring problem which is taken care of by the following theorem ([7]).

Theorem 1 (Legitimacy)**.** *The finiteness of the projective dimension of $\pi_f^*(M)$ as an U_f-module does not depend on the choice of a representative $f \in \mathsf{m}_Z$ of the point $c \in X$.*

We also prove an analogue of the famous "Dade's lemma" which is a necessary ingredient on the quest to identify the hypersurface support with the cohomological support ([7]).

Theorem 2 (Dade's lemma). $\mathrm{supp}^{hyp}(M) = \emptyset$ *if and only if* $M \cong 0$ *in* $\mathrm{Stab}\,\mathsf{u}$.

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Semisimplification for algebraic (super)groups

Thorsten Heidersdorf

(joint work with Maria Gorelik, Rainer Weissauer)

The quotient $Rep(G)$ of finite dimensional representations of an algebraic supergroup by the negligible morphisms is of the form $Rep(G^{red}, \varepsilon)$ where G^{red} is an affine supergroup scheme and $Rep(G^{red}, \varepsilon)$ is the full subcategory of representations in $Rep(G)$ such that their $\mathbb{Z}/2\mathbb{Z}$-gradation is given by the operation of $\varepsilon : \mathbb{Z}/2\mathbb{Z} \to G^{red}$ [2]. It is better to semisimplify instead the full monoidal subcategory $Rep(G)^I$ of direct summands in iterated tensor products of irreducible representations of $Rep(G)$. One major problem is the computation of the Picard group of the quotient category $Rep(G)^I/\mathcal{N} =: Rep(G_I^{red}, \varepsilon)$.

In [4] the authors determined the connected derived groups $G_{n|n}$ of the group $H_{n|n} = G_I^{red}$ in case $G = GL(n|n)$. These results are based on semisimplicity statements about the Duflo-Serganova functor $DS : Rep(GL(m|n)) \to Rep(GL(m-k|n-k))$ as proven in [3]. The DS functor gives rise to a tensor functor between the semisimplifications and allows for an inductive determination of the semisimplification.

The entire $GL(m|n)$-case, $m \geq n$, can be reduced to the $m = n$-case as shown in upcoming work of Heidersdorf and Weissauer. Indeed one gets $G_{m|n} \cong SL(m-n) \times G_{n|n}$. Crucial here are two ingredients: One can basically decompose an irreducible representation of non-vanishing superdimension into a $GL(m-n)$-part and a $GL(n|n)$-part; and the explicit computation of $GL(m|2)$-tensor products to get the induction started.

Parts of this picture are now emerging for the orthosymplectic superalgebra $\mathfrak{osp}(m|2n)$ as well. In joint work with Maria Gorelik [1] we proved the semisimplicty of the DS functor for $\mathfrak{osp}$ and OSp. More precisely DS sends any semisimple to a semisimple representation and satisfies some purity property. This result implies that the DS functor gives rise to a tensor functor between the semisimplifications, so that the inductive determination of the groups $H_{m|2n}$ should work for $\mathfrak{osp}(m|2n)$ and $OSp(m|2n)$ similarly to the $GL(m|n)$-case.

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The Hochschild complex of a modular tensor category and differential graded modular functors

Lukas Woike

(joint work with Christoph Schweigert)

It is well-known that for a semisimple modular tensor category, the Reshetikhin-Turaev construction [3] yields an extended three-dimensional topological field theory and hence by restriction to surfaces a modular functor, i.e. a consistent system of projective mapping class group representations. By work of Lyubashenko [2] the construction of a modular functor from a modular tensor category remains possible in the non-semisimple case. This construction, however, produces vector space valued quantities and hence is insensitive to the rich homological algebra of a modular tensor category.

In the articles [4, 5] extending [1], we establish that for any (not necessarily semisimple) modular tensor category $\mathcal{C}$ over an algebraically closed field k, Lyubashenko's construction is the zeroth homology shadow of a *differential graded modular functor*, i.e. a symmetric monoidal functor

$$\mathfrak{F}_{\mathcal{C}} : \mathcal{C}\text{-}\mathsf{Surf}^{\mathsf{c}} \to \mathsf{Ch}_k \tag{1}$$

from a certain $\mathcal{C}$-labeled surface category (or rather a central extension thereof) to chain complexes. This functor satisfies a version of excision, i.e. its values (the so-called *differential graded conformal blocks*) can be computed via homotopy coends. The differential conformal block $\mathfrak{F}_{\mathcal{C}}(\mathbb{T}^2)$ for the torus is quasi-isomorphic to the

Hochschild complex of $\mathcal{C}$. As a consequence, the Hochschild complex of a modular tensor category comes with a homotopy coherent projective action of $\mathrm{SL}(2,\mathbb{Z})$, the mapping class group of the torus.

The chain complexes assigned by (1) are explicitly computable by choosing a marking on the surface, i.e. a cut system and a certain embedded graph. For our construction, we replace the connected and simply connected groupoid of cut systems that appears in the Lego Teichmüller game by a contractible Kan complex.

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A Tannakian Approach to Representations in Defining Characteristic

Nate Harman

Suppose G is a simply connected semisimple algebraic group defined over a finite field $\mathbb{F}_q$. Motivated by a question posed by Geordie Williamson I explained two results, both of which attempt to make sense of the sentence: "The representation theory of $G(\mathbb{F}_q)$ is obtained by taking the representation theory of G and forcing the Frobenius twist to act by the identity".

The first result takes place at the level of Grothendieck rings and says that

$$K_0(Rep(G)) \underset{[Fr]}{\overset{Id}{\rightrightarrows}} K_0(Rep(G)) \xrightarrow[{[Res]}]{} K_0(Rep(G(\mathbb{F}_q)))$$

is a coequalizer diagram in the category of rings. The proof was given by a direct calculation using the Steinberg tensor product theorem and the restriction theorem describing the irreducible representations if $G(\mathbb{F}_q)$ (see [2]).

The second result is a categorical version of the first, saying that

$$Rep(G) \underset{Fr}{\overset{Id}{\rightrightarrows}} Rep(G) \xrightarrow[Res]{} Rep(G(\mathbb{F}_q))$$

is a coequalizer diagram in a certain (2-)category of "based" Tannakian categories – that is Tannakian categories equipped with a fiber functor. Here the heavy lifting is done by the Deligne-Milne framework for Tannakian formalism [1], and the proof was essentially a tautology once we properly translated the problem into that framework.

Also explained was how to extend these methods to understand representations of Frobenius kernels, as well as certain infinitesimal thickenings of Chevalley groups. Moreover it was explained that despite the similarities between the statements of these two results, that the first theorem does not actually follow from the second.

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