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Mini-Workshop: Nonpositively Curved Complexes (online meeting)

Organized by
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Abstract. The leading theme of the meeting was to understand nonpositively curved complexes and groups acting on them. Motivations, questions, results, and techniques being presented and discussed come from various areas of mathematics, including algebraic geometry, Lie groups, metric graph theory, geometric topology, algebraic topology, coarse geometry, K-theory, and, in general, geometric and analytic group theory. The subject discussed focused around participant's latest achievements, and important open questions in the area.

Mathematics Subject Classification (2020): 20F65 Geometric group theory, 20F67 Hyperbolic groups and nonpositively curved groups, 20F69 Asymptotic properties of groups, 57M07 Topological methods in group theory, 20J05 Homological methods in group theory, 51E24 Buildings and the geometry of diagrams, 20F55 Reflection and Coxeter groups (group-theoretic aspects), 05C12 Distance in graphs, 14E07 Birational automorphisms, Cremona group and generalizations, 53C23 Global geometric and topological methods (á la Gromov); differential geometric analysis on metric spaces.

Introduction by the Organizers

The online workshop *Nonpositively Curved Complexes*, organised by Damian Osajda (Wrocław), Piotr Przytycki (Montreal), and Petra Schwer (Magdeburg) was attended with 17 participants from Australia, the United States, Canada, and 5 countries in Europe: France, Germany, Poland, Switzerland, United Kingdom. The main area of the subjects of the talks and topics discussed were a broadly understood Geometric Group Theory, and the main fields of expertise of the participants were mostly closely related to this. However, the workshop was a nice blend of researchers with various backgrounds, including: Algebraic Geometry, Metric and Algorithmic Graph Theory, K-theory, Convex and Discrete Geometry, as well as Differential Geometry.

The spine of the workshop were 11 online talks, each lasting 30 minutes, usually not disturbed by questions, and followed only by very few end remarks. Most of the activity was concentrated in the follow-up discussion sessions. There were 7 of them, with topics related to preceding talks, but often extending to other talks as well, or beyond. The lengths of the sessions varied from one hour to above two hours, and were determined by the needs of participants and according to their activity and suggestions. Similarly, forms of various sessions were different, and shaped on the fly by participants. During such sessions, the speakers of the preceding talks were asked to explain some details of proofs, or to broaden the context. Particular emphasis was set on formulating open questions in the area by both, the speaker and other experts. The list of such questions is a part of the current report. There were vivid discussions on some questions, and few of hastily formulated ones have been already answered, and do not appear on the list. The discussions were held also outside the official schedule, via emails and other forms of private communication. Additionally, there was a welcome session on Monday evening, and a farewell session on Friday evening, both less formal, and more for socializing purposes. Still, both of them included vivid mathematical discussions. Because of the online format, and the attendance of persons from 3 continents in 3 different time zones the talks and other activities were organized, roughly, in three time slots: one for Europe together with North America, one Europe-Australia slot, and one slot available for everyone. There were four sessions, consisting of talks and the following discussions within the Europe-North America slot; three in the Europe-Australia one. The welcome session and the farewell session were held in the slots available for everyone (Europe's late evening).

The unifying theme of research of all the participants, and the main subject of the workshop were nonpositively curved complexes and groups of their automorphisms. Below, we describe briefly the content of the 11 scheduled 30 minutes talks.

Urs Lang (Zurich) presented motivations and tools coming from metric and convex geometry, focusing on the notion of spaces with geodesic bicombings as examples of a weak notion of nonpositive curvature. He and collaborators have recently developed a theory of such spaces and initiated studies of groups acting on them. Jérémie Chalopin (Marseille) turned towards a combinatorial counterpart of injective metric spaces, being examples of spaces with geodesic bicombings – towards Helly graphs. This is a classical family of graphs but only recently it has been realised that many groups appearing in the scope of geometric group theory act nicely on such graphs, and that such actions provide a lot of useful information on the groups. In particular, Thomas Haettel (Montpellier) presented new examples of groups acting geometrically on injective metric spaces and on Helly graphs, within automorphism groups of buildings and symmetric spaces. On the other side, Nima Hoda (Paris) presented his recent results showing that the crystallographic groups are Helly if and only if they are cocompact cubical providing, in particular, examples of CAT(0) biautomatic (in fact, Coxeter) groups that are not Helly. Alexandre Martin (Edinburgh) talked about applying tools of nonpositive curvature in the realm of Artin groups. It is believed that all Artin groups are CAT(0), but there are currently other nonpositive curvature techniques being used successfully to explore this notoriously resistant class of groups. Elia Fioravanti (Bonn) worked with one such recent notion – coarse-median structure – studying subgroups fixed by coarse-median preserving automorphisms. Motiejus Valiunas (Wrocław) showed that groups hyperbolic relatively to Helly groups are Helly. Anne Lonjou (Orsay) explained a recent construction of a CAT(0) cubical complex on which the Cremona group of rank 2 acts. Anne Thomas (Sydney) presented a proof of the fact that locally elliptic actions of finitely generated groups on some 2-dimensional buildings have global fixed points. Alexander Engel (Münster) described a construction of boundaries of groups, generalising Gromov and visual CAT(0) boundaries. Olga Varghese's (Münster) talk concerned the question of continuity of abstract homomorphisms from locally compact groups into groups of isometries of CAT(0) spaces.

The subjects of the 7 discussion sessions were chosen to be related to the preceding talks. A session on spaces with geodesic bicombings and Helly graphs was held on Monday. A session on Artin groups – but including a vivid discussion on buildings and symmetric spaces – took place on Tuesday. On Wednesday there was a morning Europe-Australia session on coarse median structures, relative hyperbolicity, and related topics, as well as an afternoon Europe-North America session on the Cremona group. Thursday's session concentrated around the question of fixed point properties of group actions on CAT(0) spaces, with a bit of boundaries topics. On Friday there was a morning session on locally compact groups and their abstract actions, and an afternoon session on Helly groups. The discussion sessions resulted in a number of open questions (and, already, some answers), that are presented within this report.

Finally we would like to thank the staff at Oberwolfach, in particular the IT group. They set up all the necessary technology and helped to make the workshop and the recording of the talks a smooth process.

Overall it is to say that the workshop, despite difficulties with time zones and general restrictions due to the online format, provided a very stimulating atmosphere and sparked many new discussions among various subgroups of the participants. The ongoing scientific exchange between the participants will certainly lead to new developments in the field.

We collect below the open problems collected and raised during this workshop.

Discussion session on spaces with bicombings and Helly graphs:

(1) Does every metric space with a bicombing also admit a convex bicombing?
(2) Does every metric space with a convex bicombing also admit a consistent bicombing?

(3) The construction of a reversible and equivariant bicombing on an injective metric space X uses the canonical embedding $X = E(X) \to \Delta(X)$ and a canonical retraction $\Delta(X) \to E(X)$. Can one get more information from this construction? Is the bicombing already convex, or consistent?
(4) A compact metric space may admit different convex bicombings. What about consistent bicombings?
(5) Is every group acting geometrically on an injective space Helly?
(6) Is there a CAT(0) group not acting on a (poly-) simplicial complex?
(7) Are groups acting geometrically on injective spaces biautomatic?
(8) Is there a connection between modular lattices and Helly graphs?
(9) Is there a notion of a modular hull of a graph?

Discussion session on Artin groups, and symmetric spaces, and buildings:

(1) Are 2-dimensional Artin groups of hyperbolic type virtually Helly?
(2) Which Artin groups act properly on CAT(0) cube complexes?
(3) Are parabolic subgroups of Artin groups stable under intersection?
(4) Is the complex of all proper parabolic (two dimensional) Artin subgroups NPC (in any sense)?
(5) Is A_Γ Helly, for Γ being the complete graph on 3 vertices with all labels equal to 3?
(6) Are all 2-dimensional Artin groups systolic?
(7) For Artin groups: is reducible equivalent to decomposition as a direct product?
(8) Are Artin groups hierarchically hyperbolic? Are dihedral Artin groups hierarchically hyperbolic?
(9) Are there non-Helly groups acting geometrically on coarse Helly graphs?
(10) Are uniform lattices in $SL_n(K)$ injective?
(11) Are uniform lattices in $GL_n(K)$ Helly?
(12) What is the injective hull of $GL_n(R)/O(n)$?
(13) What is the injective hull of the hyperbolic plane?
(14) Which Coxeter groups can be realized as reflection groups in an injective metric space?

Discussion session on "beyond hyperbolicity".

(1) Does there exist a group acting geometrically on a coarse Helly graph that is not Helly?
(2) Which Artin groups are coarse median?

Discussion session on Cremona group:

(1) Consider a finitely generated subgroup G of the Cremona group such that each of its elements is regularizable. Does this imply that G is regularizable?
(2) When do locally elliptic actions on infinite dimensional CAT(0) cube complexes have global fixed points?

Discussion session on group actions and fixed points, and on boundaries:

(1) Does every locally elliptic action of a finitely generated group on a finite dimensional CAT(0) space have a global fixed point? In particular, does it hold in the case of Euclidean buildings (of arbitrary dimension)?
(2) Does every locally elliptic action of a finitely generated group on a finite dimensional Helly complex have a global fixed point?
(3) Do central extensions of hyperbolic groups admit coherent and expanding combings?
(4) How to construct an automatic group which does not admit an expanding combing?
(5) Are the automatic structures on: Artin groups of finite type, mapping class groups, groups acting geometrically and in an order preserving way on Euclidean buildings of type $\tilde{A}_n$, $\tilde{B}_n$ or $\tilde{C}_n$ expanding?

Discussion session on actions of locally compact groups:

(1) Are irreducible lattices in products of (two) trees finitely generated?
(2) Are non-uniform lattices in (2-dimensional) right-angled-hyperbolic buildings finitely generated?

Mini-Workshop (online meeting): Nonpositively Curved Complexes

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Spaces with convex bicombings: a survey

Urs Lang

For the purpose of this survey, a *bicombing* σ on a metric space (X, d) is a map $\sigma\colon X \times X \times [0,1] \to X$ such that $\sigma_{xy} := \sigma(x, y, \cdot)\colon [0,1] \to X$ is a (minimizing) constant speed geodesic from x to y and

$$d(\sigma_{xy}(t), \sigma_{x'y'}(t)) \le (1-t)\, d(x, x') + t\, d(y, y')$$

whenever $(x, y), (x', y') \in X \times X$ and $t \in [0,1]$. (In [5], this is called a *conical geodesic bicombing.*) A *convex* bicombing satisfies the stronger condition that the functions $t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))$ are convex on $[0,1]$, and a bicombing σ is called *consistent* if σ_{pq} is a (reparametrized) subsegment of σ_{xy} whenever p and q occur in this order on σ_{xy}. Every consistent bicombing is convex. Thus, for a geodesic metric space X, the following conditions satisfy (A) $\Rightarrow$ (B) $\Rightarrow$ (C) $\Rightarrow$ (D) $\Rightarrow$ (E):

(A) X is a CAT(0) space;
(B) X is a Busemann space;
(C) X admits a consistent bicombing;
(D) X admits a convex bicombing;
(E) X admits a bicombing.

If X is uniquely geodesic, then (E) $\Rightarrow$ (B). If X is a normed real vector space, then (A) holds if and only if the norm is induced by an inner product, (B) holds if and only if the norm is strictly convex, and (C) is always satisfied, for $\sigma_{xy}(t) := (1-t)x + ty$. Even if the underlying metric space is compact, an individual bicombing may fail to be convex [5, Example 2.2], and a convex bicombing need not be consistent [2, Theorem 1.1]. However, the implication (E) $\Rightarrow$ (D) holds if X is proper [5, Theorem 1.1], and (D) $\Rightarrow$ (C) holds if X has finite combinatorial dimension in the sense of Dress [7] (see below). In fact, in the latter case, any convex bicombing σ on X is necessarily unique, consistent, satisfies the *reversibility* condition $\sigma_{yx}(t) = \sigma_{xy}(1-t)$, and is *equivariant* with respect to any isometry γ of X, thus $\gamma \circ \sigma_{xy} = \sigma_{\gamma(x)\gamma(y)}$ for all $(x, y) \in X \times X$ [5, Theorem 1.2]. It is unkown whether (E) $\Rightarrow$ (D) or (D) $\Rightarrow$ (C) in general. The assumption of finite combinatorial dimension is rather restrictive, but the result still covers, for example, injective hulls of hyperbolic groups, as discussed below. Uniqueness of convex bicombings may fail on compact (infinite dimensional) metric spaces [5, Example 3.5]. On the other hand, if E is a dual or injective Banach space and $C \subset E$ is a closed convex subset with non-empty interior, then the only consistent bicombing on C is the linear one [2, Theorem 1.5]. Every complete metric space with a bicombing also admits a reversible bicombing [2, Proposition 1.3].

A source of bicombings is the fact that if $\bar{X}$ is a metric space with a bicombing $\bar{\sigma}$ (for example, a Banach space), and if $\pi\colon \bar{X} \to X$ is a 1-Lipschitz retraction onto some subset X, then $\sigma := \pi \circ \bar{\sigma}|_{X \times X \times [0,1]}$ is a bicombing on X [5, Lemma 2.1]. There is also a more intrinsic view. A metric space X is an *absolute* 1-*Lipschitz* *retract* if for every isometric embedding $i\colon X \to Y$ into another metric space there is a 1-Lipschitz retraction $\pi\colon Y \to i(X)$. Equivalently, X is an *injective metric space*, i.e., for every metric space B and every 1-Lipschitz map $f\colon A \to X$ defined on a subset of B there exists a 1-Lipschitz extension $\bar{f}\colon B \to X$ of f (in other words, X is an injective object in the metric category with 1-Lipschitz maps as morphisms). Basic examples include $\mathbb{R}$, all l_∞ spaces, complete ($\mathbb{R}$-)trees, as well as l_∞ products of injective metric spaces. However, this list is by far not exhaustive. Isbell [9] showed that every metric space X has an essentially unique *injective hull* $(e, E(X))$. This means that $E(X)$ is an injective metric space, $e\colon X \to E(X)$ is an isometric embedding, and every isometric embedding of X into some injective metric space factors through e. Briefly, the construction starts from the set $\Delta(X)$ of all functions $f\colon X \to \mathbb{R}$ satisfying $f(x)+f(y) \geq d(x,y)$ for all $x, y \in X$, and $E(X)$ is the subset of all pointwise minimal ("extremal") functions, equipped with the supremum distance. The isometric embedding e takes x to the distance function d_x. Isbell's construction was rediscovered and further explored by Dress [7], who also exhibited a *canonical* retraction $\Delta(X) \to E(X)$. This can be used to equip $E(X)$, and hence every injective metric space, with a reversible and equivariant bicombing [10, Proposition 3.8]. For a finite metric space V, the injective hull $E(V)$ is a finite polyhedral complex of dimension at most $\frac{1}{2}\#V$ with l_∞ metrics on the cells. The *combinatorial dimension* of a metric space X is defined as the supremum of $\dim(E(V))$ over all finite subsets V. If X is a proper injective metric space of finite combinatorial dimension, then it follows from the aforementioned results that X possesses a unique convex bicombing which is furthermore consistent, reversible, and equivariant.

A remarkable property of the operator $X \mapsto E(X)$, observed in [8], is that it preserves δ-hyperbolicity ($\delta \geq 0$), defined via the four point condition

$$d(w,x) + d(y,z) \leq \max\{d(w,y) + d(x,z), d(x,y) + d(w,z)\} + \delta.$$

This provides a most efficient way to embed a general δ-hyperbolic space X into a complete, geodesic, and contractible δ-hyperbolic space with a reversible and equivariant bicombing. Furthermore, if X is itself geodesic or if X is the vertex set of a connected graph with the canonical metric, then $E(X)$ is within distance at most δ or $\delta + \frac{1}{2}$, respectively, of the image of e [10, Proposition 1.3]. For the vertex set X of a connected δ-hyperbolic graph with uniformly bounded valence, the injective hull $E(X)$ is proper and has a polyhedral structure with only finitely many isometry types of (l_∞) cells; in particular $E(X)$ has finite combinatorial dimension [10, proof of Theorem 1.1]. It follows that every hyperbolic group acts geometrically on a proper metric space with a consistent, reversible and equivariant bicombing [10, Theorem 1.4].

There are also analogues of the flat plane and flat torus theorems for CAT(0) spaces in the present context. Let X be a proper metric space with a consistent and reversible bicombing σ. If the isometry group of X is cocompact, then X is hyperbolic if and only if X does not contain an isometrically embedded normed plane [6, Theorem 1.1]. If Γ is a group acting geometrically on X and σ is Γ-equivariant, then every isometry in Γ has either a fixed point or a σ-axis (with all compact subsegments determined by σ), and if Γ has a free abelian subgroup A of rank $n \geq 1$, then X contains a normed n-plane on which A acts by translations [6, Proposition 5.5 and Theorem 1.2]). These results were partly extended to general bicombings in [4]: if a proper and cocompact metric space X with a bicombing admits a quasi-isometric embedding of $\mathbb{R}^n$, then X also contains a normed n-plane. Much of this is based on an averaging process, discussed next.

Every complete metric space X with a bicombing admits a 1-Lipschitz barycenter map $\beta\colon P_1(X) \to X$ taking each Dirac mass δ_x to the corresponding point $x \in X$, where $P_1(X)$ is the space of Borel probability measures μ on X with $\int_X d(x_0, x)\, d\mu(x) < \infty$ for some $x_0 \in X$, equipped with the L_1 Wasserstein or Kantorovich–Rubinstein metric. The construction of β goes back to Es-Sahib and Heinich for Busemann spaces and was further developed by Navas, Descombes, and Basso (see [1]). Conversely, any contracting barycenter map β on a metric space X provides a reversible bicombing, defined by $\sigma_{xy}(t) := \beta((1-t)\delta_x + t\delta_y)$. Barycenters are intimately related to fixed point properties. Unlike for CAT(0) spaces or injective metric spaces (compare [10, Proposition 1.2]), a group of isometries with bounded orbits on a Busemann space may fail to have a global fixed point [1, Section 2]. However, if X is a complete metric space with a semigroup Γ of isometries leaving a non-empty compact subset K invariant, and if X possesses a Γ-equivariant reversible bicombing σ, then Γ has a global fixed point in the closed σ-convex hull of K [1, Theorem 1.1]. More recently, Creutz [3] used the barycenter method to establish a sharp isoperimetric inequality: in a complete metric space X with a bicombing, every closed curve of length L possesses a disk filling with 2-dimensional Hausdorff measure at most $\frac{1}{2\pi} L^2$.

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Helly Graphs

Jérémie Chalopin
(joint work with Victor Chepoi, Anthony Genevois, Hiroshi Hirai, Damian Osajda)

A geodesic metric space is *injective* if any family of pairwise intersecting balls has a non-empty intersection [1]. Injective metric spaces appear independently in various fields of mathematics and computer science: in topology and metric geometry – also known as *hyperconvex spaces* or *absolute retracts* (in the category of metric spaces with 1-Lipschitz maps); in combinatorics – also known as *fully spread spaces*; in functional analysis and fixed point theory – also known as *spaces with binary intersection property*; in theory of algorithms – known as *convex hulls*, and elsewhere. They form a very natural and important class of spaces and have been studied thoroughly. The distinguishing feature of injective spaces is that any metric space admits an *injective hull*, i.e., the smallest injective space into which the input space isometrically embeds; this important result was rediscovered several times in the past [9, 7, 6].

A discrete counterpart of injective metric spaces are *Helly graphs* – graphs in which any family of pairwise intersecting (combinatorial) balls has a non-empty intersection. Again, there are many equivalent definitions of such graphs, hence they are also known as e.g. *absolute retracts* (in the category of graphs with nonexpansive maps) [2, 3, 10, 13, 11, 12]. We are interested in the nonpositive-curvature-like properties of Helly graphs and of the associated Helly complexes (given a Helly graph, its clique complex is the corresponding Helly complex).

1. Helly graphs

The following result is well known, see [10, 11, 12], and is the discrete counterpart of Isbell's Theorem about injective metric spaces [9]:

Theorem 1. *For any graph G, there exists a smallest Helly graph into which G is isometrically embedded. This graph is the* Hellyfication *of G, and is contained as an isometric subgraph in any Helly graph G' containing G as an isometric subgraph.*

From this theorem, one can see that any graph can appear as an isometric subgraph in a Helly graph. Consequently, one cannot define Helly graphs by local conditions, such as forbidden subgraphs for example.

However, assuming that the clique complex $X(G)$ of G is simply connected, we established that Helly graphs can be characterized by local conditions, establishing a local-to-global characterization of Helly graphs.

A family of subsets $\mathcal{F}$ of a set V satisfies the *Helly property* if for anysubfamily $\mathcal{F}'$ of $\mathcal{F}$, the intersection $\bigcap \mathcal{F}' = \bigcap\{F : F \in \mathcal{F}'\}$ is non-empty if and only if $F \cap F' \neq \emptyset$ for any pair $F, F' \in \mathcal{F}'$. Hence a Helly graph G is a graph in which its family of balls $\mathcal{B}(G) = \{B_k(v) : v \in V(G), k \in \mathbb{N}\}$ satisfies the Helly property. A graph G is a 1–*Helly graph* if its family of ball of radius 1 $\mathcal{B}_1(G) = \{B_1(v) : v \in$

$V(G)\}$ satisfies the Helly property. A *clique-Helly graph* is a graph G in which its family of maximal cliques satisfies the Helly property. Observe that 1-Helly and clique-Helly graphs are defined by local conditions and that a Helly graph is 1-Helly and that a 1-Helly graph is a clique-Helly graph.

A graph is *weakly modular* if it satisfies the following two distance conditions (for every $k > 0$):

- *Triangle condition* (TC): For any vertex u and any two adjacent vertices v, w at distance k to u, there exists a common neighbor x of v, w at distance $k-1$ to u.
- *Quadrangle condition* (QC): For any vertices u, z at distance k and any two neighbors v, w of z at distance $k-1$ to u, there exists a common neighbor x of v, w at distance $k-2$ from u.

Observe that Helly graphs are weakly modular (by considering appropriate balls centered at u, v, w).

A vertex x of a graph G is *dominated* by another vertex y if the unit ball $B_1(y)$ includes $B_1(x)$. A graph G is *dismantlable* if its vertices can be well-ordered $\prec$ so that, for each v there is a neighbor w of v with $w \prec v$ which dominates v in the subgraph of G induced by the vertices $u \preceq v$.

Here is the local-to-global characterization we established for Helly graphs:

Theorem 2 ([5]). *For a graph G, the following conditions are equivalent:*

(i) *G is Helly;*
(ii) *G is a weakly modular 1–Helly graph;*
(iii) *G is a dismantlable clique-Helly graph;*
(iv) *G is clique-Helly with a simply connected clique complex.*

Moreover, if the clique complex $X(G)$ of G is finite-dimensional, then the conditions (i)-(iv) are equivalent to

(v) *G is clique-Helly with a contractible clique complex.*

For finite graphs, the equivalences $(i) \iff (ii)$ and $(i) \iff (iii)$ were establised respectively in [2] and [3]. The implication $(iii) \implies (iv)$ is trivial. We established the implication $(iv) \implies (ii)$ by constructing the universal cover of the clique complex $X(G)$ a clique-Helly graph G and establishing its properties inductively during the construction.

2. Helly Groups

A group is *Helly* if it acts geometrically on a Helly graph (necessarily, locally finite). Examples of Helly groups include (Gromov) hyperbolic groups, CAT(0) cubical groups, finitely presented C(4)-T(4) small cancellation groups. It was also shown recently in [8] that FC-type Artin groups and weak Garside groups of finite type are Helly.

We established several properties satisfied by Helly groups as shown in the following theorem.

Theorem 3 ([4]). *Let Γ be a group acting geometrically on a Helly graph G, that is, Γ is a Helly group. Then:*

(1) Γ is biautomatic.
(2) Γ has finitely many conjugacy classes of finite subgroups.
(3) Γ is (Gromov) hyperbolic if and only if G does not contain an isometrically embedded infinite ℓ_∞–grid.
(4) The clique complex $X(G)$ of G is a finite-dimensional cocompact model for the classifying space $\underline{E}\Gamma$ for proper actions. As a particular case, Γ is always of type F_∞; and of type F when it is torsion-free.
(5) Γ acts geometrically on a proper injective metric space, and hence on a metric space with a convex geodesic bicombing.
(6) Γ admits an EZ-boundary ∂G.
(7) Γ satisfies the Farrell-Jones conjecture with finite wreath products.
(8) Γ satisfies the coarse Baum-Connes conjecture.
(9) The asymptotic cones of Γ are contractible.

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Injective metrics, symmetric spaces and buildings

Thomas Haettel

We are interested in group actions on Helly graphs and injective metric spaces. Recall that a geodesic metric space is called *injective* if the family of closed balls satisfies the Helly property: any family of pairwise intersecting balls has a non-empty global intersection, see for instance [Lan13]. A graph is called *Helly* if the family of combinatorial balls satisfies the Helly property, see for instance [CCG+20].

A group is called:

- *Helly* if it acts properly and cocompactly by automorphisms on a Helly graph.
- *injective* if it acts properly and cocompactly by isometries on an injective metric space.
- *coarsely Helly* if it acts properly and coboundedly on an injective metric space.

Note that Helly implies injective, which in turn implies coarsely Helly.

Such groups enjoy various nice properties reminiscent of nonpositive curvature, such as the following.

If a finitely generated group G is coarsely Helly then

- G is semihyperbolic in the sense of Alonso-Bridson (see [BH99]).
- G has finitely many conjugacy classes of finite subgroups (see [Lan13]).

If G is injective then

- G satisfies the Farrell-Jones conjecture (see [KR17]).

If G is Helly then

- G is biautomatic (see [CCG+20]).

Most of the recent examples of such groups are actually Helly groups, such as the following:

- Cocompactly cubulated groups are Helly.
- Hyperbolic groups are Helly (see [Lan13]).
- Artin groups of type FC are Helly (see [HO19]).
- Lattices in Euclidean buildings of type $\tilde{C}_n$ are Helly (see [CCG+20]).

One recent example of coarsely Helly groups come from the following.

Theorem 4 (H - Hoda - Petyt 2020 [HHP20])**.** *Every hierarchically hyperbolic group, and in particular every mapping class group, is coarsely Helly.*

In this report, we will describe a recent work giving numerous other examples of Helly groups and injective groups among higher rank lattices, see [Ha21] for details. The core of the argument is based on the following.

Proposition 5. *Let $\mathbb{K}$ denote a valued field, V a finite-dimensional vector space over $\mathbb{K}$, and let H denote either $\mathbb{R}$ or a cyclic subgroup of $\mathbb{R}$. Let X denote a non-empty set of norms on V such that:*

- $\forall a \in H, \forall \eta \in X, e^a \eta \in X.$
- $\forall \eta, \eta' \in X, \exists a \in H, e^{-a}\eta \leq \eta' \leq e^a \eta.$
- *for every non-empty $F \subset X$ bounded above, the set $\{\eta \in X | F \leq \eta\}$ has a minimum.*

For any $\eta, \eta' \in X$, let us define

$$d(\eta, \eta') = \sup_{v \in V \setminus \{0\}} \left| \log \frac{\eta(v)}{\eta'(v)} \right|.$$

Then closed balls in (X, d) satisfy the Helly property.

We can then apply this proposition to two different settings.

Application 1
Let p denote a prime number, and $n \geq 2$. Let X denote the set of all norms on $\mathbb{Q}_p^n$, it is the extended Bruhat-Tits building of $GL(n, \mathbb{Q}_p)$, also called the Goldman-Iwahori space (see [GI63]). Its vertex set identifies with the subset $X^{(0)}$ of all norms with values in $p^{\mathbb{Z}} \cup \{0\}$. The metric d from Proposition 5 coincides with the natural ℓ^∞ metric on apartments in X, which naturally identify with $\mathbb{R}^n$. We prove the following.

Theorem 6 (H 2021)**.** *The extended Bruhat-Tits buildings (X, d) is injective. Its vertex set $(X^{(0)}, d)$ is a Helly graph. As a consequence, uniform lattices in $GL(n, \mathbb{Q}_p)$ are Helly, and also biautomatic.*

Note that we obtain in [Ha21] a similar result for all classical semisimple groups over non-Archimedean local fields. Note that in the particular case of $GL(n)$, the Helly property is a consequence of work of Hirai (see [Hir20]), and biautomaticity for lattices in Euclidean buildings is a result of Swiatkowski (see [Ś06, Theorem 6.1]).

Application 2
Fix $n \geq 2$, and let X denote the set of all Euclidean norms on $\mathbb{R}^n$, it is the symmetric space of $GL(n, \mathbb{R})$. The metric d from Proposition 5 coincides with the natural ℓ^∞ metric on apartments in X, which naturally identify with $\mathbb{R}^n$. Also denote by $\hat{X}$ the set of all norms on $\mathbb{R}^n$. We prove the following.

Theorem 7 (H 2021)**.** *The space $(\hat{X}, d)$ is injective, and $GL(n, \mathbb{R})$ acts properly and cocompactly on it by isometries. As a consequence, uniform lattices in $GL(n, \mathbb{R})$ are injective.*

Similar results hold for other classical real semisimple groups, see [Ha21]. However, inspired by work of Hoda (see [Hod20]), the situation for $SL(n)$ is quite different.

Theorem 8. *Let $\mathbb{K}$ denote a local field and $n \geq 3$. Then $SL(n, \mathbb{K})$ is not coarsely Helly.*

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Non-positively curved features of Artin groups

Alexandre Martin

(joint work with Mark Hagen, Alessandro Sisto)

Artin groups form a large class of groups encompassing braid groups and right-angled Artin groups. Given a finite simplicial graph Γ such that every edge of Γ between vertices a, b is labelled by an integer $m_{ab} \geq 2$, the corresponding Artin group as follows:

$$A_\Gamma = \langle V(\Gamma) \mid \underbrace{aba\cdots}_{m_{ab}} = \underbrace{bab\cdots}_{m_{ab}} \text{ for } a, b \text{ connected by an edge of } \Gamma\rangle.$$

Note that by adding the relation $a^2 = 1$ for every vertex $a \in V(\Gamma)$, one obtains the corresponding Coxeter group W_Γ. Unlike their Coxeter relatives however, the structure and geometry of Artin groups are still mysterious in general. Substantial progress has been made in recent years for several classes of Artin groups (two-dimensional, FC type, etc.) In particular, the results obtained so far confirm the general belief that Artin groups enjoy many of the algebraic, algorithmic, and geometric properties that Coxeter groups possess.

The talk started with a survey of various results on the geometry of Artin groups, focusing in particular on the hyperbolic features of these groups. It is conjectured that for an irreducible Artin group A_Γ, its central quotient $A_\Gamma/Z(A_\Gamma)$ is acylindrically hyperbolic, and this has indeed been verified for most standard classes of Artin groups (see for instance [3, 4, 11]). However, acylindrical hyperbolicity is a notion that does not offer control over the coarse geometry of the whole group. A much finer notion, with striking consequences, is the notion of *hierarchically hyperbolic group* introduced by Behrstock-Hagen-Sisto to provide a unified framework generalising the geometry of mapping class groups and of most CAT(0) cubical groups [2]. Right-angled Artin groups and braid groups are known to be hierarchically hyperbolic, and Calvez-Wiest have asked whether all Artin groups are hierarchically hyperbolic [5].

In this talk, I presented recent work on this question, in collaboration with Hagen and Sisto [7]. Recall that an Artin group A_Γ is said to be two-dimensional if for every triangle a, b, c contained in Γ, we have $\frac{1}{m_{ab}} + \frac{1}{m_{bc}} + \frac{1}{m_{ac}} \leq 1$, and is said to be of hyperbolic type if the associated Coxeter group W_Γ is hyperbolic. These groups contain in particular all Artin groups of extra-large type, and they were shown to be acylindrically hyperbolic by Martin–Przytycki [9]. We proved the following:

Theorem 9 ([7])**.** *Two-dimensional Artin groups of hyperbolic type are hierarchically hyperbolic. In particular, they have finite asymptotic dimension, have uniform exponential growth, and are semi-hyperbolic.*

Our proof uses a recent combinatorial approach to hierarchical hyperbolicity due to Behrstock-Hagen-Martin-Sisto [1], which proves the hierarchical hyperbolicity of a group by means of its action on a suitable hyperbolic complex with a fine control of its local geometry. We construct such a complex out of the *commutation graph* of A_Γ, which encodes the patterns of intersections of the maximal subgroups of A_Γ that are virtual direct products. This commutation graph turns out to be a natural object that generalises to all Artin groups the extension graph of Kim–Koberda of right-angled Artin groups [8]. This graph is also (equivariantly) quasi-isometric to the coned-off Deligne complex introduced by Przytycki–Martin [9] and is (equivariantly) isomorphic to the graph of proper irreducible parabolic subgroups of finite type of A_Γ. The latter was introduced by Cumplido–Gebhardt–González-Meneses–Wiest in the finite type case [6], and by Morris-Wright in the FC-type case [10]. In both cases, these graphs are conjectured to be an analogue of the curve complex. We generalise and confirm this conjecture in the two-dimensional case by showing the following:

Corollary 10 ([7])**.** *Every two-dimensional Artin group of hyperbolic type A_Γ on at least three generators admits a hierarchically hyperbolic group structure such that its maximal hyperbolic space is the graph of proper irreducible parabolic subgroups of finite type of A_Γ.*

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Coarse-median preserving automorphisms

Elia Fioravanti

If Γ is a finite simplicial graph, the right-angled Artin group $\mathcal{A}_\Gamma$ is defined by the presentation where generators are vertices of Γ and each edge of Γ imposes that the corresponding generators commute. In the extremal cases where Γ is discrete or complete, $\mathcal{A}_\Gamma$ becomes, respectively, a free group F_n or a free abelian group $\mathbb{Z}^n$.

The groups $\mathrm{Out}(F_n)$ and $\mathrm{Out}(\mathbb{Z}^n) \simeq GL_n\mathbb{Z}$ have attracted a tremendous amount of research over the last century: the former first in the context of combinatorial group theory and later for its analogies with mapping class groups of compact surfaces; the latter as a special instance of an arithmetic group. Despite this, much is still unknown about them.

Comparatively, automorphisms of general right-angled Artin groups (RAAGs) have long been overlooked. Only in recent years have they started drawing a significant amount of attention, as they can be viewed as an interpolation between the extremal cases of $\mathrm{Out}(F_n)$ and $\mathrm{Out}(\mathbb{Z}^n)$.

There are still many basic gaps in our understanding of automorphisms of RAAGs. In this work, we address those relating to the following question:

Question 11. Given $\varphi \in \mathrm{Aut}(\mathcal{A}_\Gamma)$, what can be said on the subgroup $\mathrm{Fix}\,\varphi \leq \mathcal{A}_\Gamma$?

This problem received a lot of attention from the late 70s to early 90s of the last century in the special case of $\mathcal{A}_\Gamma = F_n$. First, Dyer and Scott showed that $\mathrm{Fix}\,\varphi$ is always a free factor if φ has finite order [8]. Then Gersten proved that $\mathrm{Fix}\,\varphi$ is finitely generated for arbitrary $\varphi \in \mathrm{Aut}(F_n)$ [10]; several alternative proofs soon followed, e.g. [7, 11]. Finally, extending to maps between graphs various classical ideas from Nielsen–Thurston theory, Bestvina and Handel provided a complete description [2], thus solving the so known Scott Conjecture:

"for every $\varphi \in \mathrm{Aut}(F_n)$, the subgroup Fix φ *is generated by at most n elements."*

Trying to extend these results to more general right-angled Artin groups may seem hopeless. One is immediately faced with the fundamental difficulties brought on by the lack of hyperbolicity. In addition, free groups are *locally quasi-convex*: every finitely generated subgroup is undistorted and itself free.

By contrast, subgroups of general right-angled Artin groups are known to be a receptacle for groups with all sorts of unpleasant behaviours, as demonstrated by Bestvina-Brady groups [1]. These can indeed occur in relation to fixed subgroups of automorphisms, as the next example shows.

Example 12. Let $\mathcal{A}_\Gamma$ be the right-angled Artin group defined by an arbitrary finite simplicial graph Γ. Let $\alpha\colon \mathcal{A}_\Gamma \to \mathbb{Z}$ be the homomorphism that maps all standard generators $v \in \Gamma^{(0)}$ to the same generator of $\mathbb{Z}$. Recall that $\ker \alpha$ is the Bestvina–Brady group $BB_\Gamma \leq \mathcal{A}_\Gamma$ [1].

Consider the slightly larger right-angled Artin group $G = \mathcal{A}_\Gamma \times \mathbb{Z}$. We can define an automorphism $\varphi_0 \in \mathrm{Aut}(G)$ as follows. Let $\mathbb{Z} = \langle z \rangle$ and represent elements of G as pairs (g, z^n) with $g \in \mathcal{A}_\Gamma$. Then set:

$$\varphi_0(g, z^n) := (g, z^{n+\alpha(g)}).$$

Observing that Fix $\varphi_0 = BB_\Gamma \times \mathbb{Z}$, we deduce that:

(1) Fix φ_0 is finitely generated if and only if Γ is connected. In particular, Fix φ_0 is not finitely generated for $G = F_2 \times \mathbb{Z}$.
(2) If Γ is connected and not a join, then Fix φ_0 is distorted in G [13].
(3) If the flag complex L_Γ associated to Γ is not contractible, then Fix φ_0 does not admit a classifying space with finitely many cells [1]. In particular, Fix φ_0 is not cocompactly cubulated.

Remark 13. In the previous example, the fact that the ambient group G splits as a product is not really relevant to the construction. Indeed, we can always embed G as a parabolic subgroup of a larger, irreducible right-angled Artin group G' so that $\varphi_0 \in \mathrm{Aut}(G)$ extends to an automorphism $\varphi_0' \in \mathrm{Aut}(G')$ with $\varphi_0'(G) = G$ and Fix $\varphi_0' =$ Fix φ_0.

One might find unpleasant that the resulting automorphism $\varphi_0' \in \mathrm{Aut}(G')$ is not "irreducible": it leaves invariant the parabolic subgroup $G \leq G'$. However, when $\mathcal{A}_\Gamma$ is neither free nor free abelian, "irreducible" automorphisms of $\mathcal{A}_\Gamma$ do not actually exist: there always exists a full subgraph $\Delta \subseteq \Gamma$ such that, for every $\varphi \in \mathrm{Aut}(\mathcal{A}_\Gamma)$, some power of φ takes $\mathcal{A}_\Delta \leq \mathcal{A}_\Gamma$ to a conjugate [4, Proposition 3.2].

As it turns out, many automorphisms of right-angled Artin groups are much better behaved and exhibit properties closer to those of automorphisms of F_n. These are the titular *coarse-median preserving* automorphisms.

Recall that every right-angled Artin group $\mathcal{A}_\Gamma$ is equipped with a *coarse-median operator* $\mu\colon \mathcal{A}_\Gamma^3 \to \mathcal{A}_\Gamma$. This is true more generally of every cocompactly cubulated group: given a proper cocompact action on a CAT(0) cube complex $G \curvearrowright X$, the classical median operator $m\colon X^3 \to X$ [6] can be pulled back to a map $\mu\colon G^3 \to G$ via any G–equivariant quasi-isometry $G \to X$.

Coarse median operators can actually be defined more generally, even in the absence of an action on a cube complex. This was done by Bowditch [3], who showed that all Gromov-hyperbolic groups and mapping class groups of compact surfaces can be endowed with a coarse median operator. Such pairs (G, μ) are known as *coarse median groups.*

Definition 14. Let (G, μ) be a coarse median group. An automorphism $\varphi \in \mathrm{Aut}(G)$ is *coarse-median preserving* if there exists a constant $C \geq 0$ such that:

$$\varphi(\mu(x, y, z)) \approx_C \mu(\varphi(x), \varphi(y), \varphi(z)),\ \forall x, y, z \in G,$$

with respect to some fixed word metric on G. Coarse-median preserving automorphisms form a subgroup $\mathrm{Aut}_{\mathrm{cmp}}(G) \leq \mathrm{Aut}(G)$.

For many classical groups, coarse-median preserving automorphisms are plentiful. In particular, every automorphism of F_n is coarse-median preserving.

Proposition 15 ([9])**.**

(1) If G is hyperbolic, then $\mathrm{Aut}_{\mathrm{cmp}}(G) = \mathrm{Aut}(G)$ *(this follows from* [12]*).*
(2) If G is a right-angled Coxeter group, then $\mathrm{Aut}_{\mathrm{cmp}}(G) = \mathrm{Aut}(G)$
(3) If G is a right-angled Artin group, then $\mathrm{Aut}_{\mathrm{cmp}}(G) = U(G)$*, where* $U(G) \leq \mathrm{Aut}(G)$ *is the subgroup of "untwisted" automorphisms introduced in* [5].

For coarse-median preserving automorphisms of cocompactly cubulated groups, we show that all the bad behaviours described in Example 12 do not actually occur.

Theorem 16 ([9])**.** *Let G be a cocompactly cubulated group, equipped with the induced coarse median operator. If $\varphi \in \mathrm{Aut}_{\mathrm{cmp}}(G)$, then:*

(1) Fix φ *is finitely generated and undistorted in G;*
(2) Fix φ *admits a cocompact cubulation.*

In general, the cubulation of Fix φ only arises from a median subalgebra of the cubulation of G, and it cannot be realised as a *convex* subcomplex thereof. This can be observed for the automorphism of $\mathbb{Z}^2$ that swaps the standard generators.

However, for automorphisms of right-angled Artin or Coxeter groups, a stronger result holds up to replacing φ with a power:

Theorem 17 ([9])**.** *If G is a right-angled Artin/Coxeter group, there exists a finite-index subgroup* $\mathrm{Aut}^0_{\mathrm{cmp}}(G) \leq \mathrm{Aut}_{\mathrm{cmp}}(G)$ *such that, for every* $\varphi \in \mathrm{Aut}^0_{\mathrm{cmp}}(G)$*:*

(1) Fix φ *is quasi-convex in G with respect to the standard word metric;*
(2) Fix φ *is a special group in the sense of Haglund–Wise.*

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Helly groups and relative hyperbolicity

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(joint work with Damian Osajda)

Let Γ be a connected simplicial graph. We say Γ is *Helly* (respectively ξ*-coarse Helly*, for some fixed constant $\xi \geq 0$) if given any collection of pairwise intersecting metric balls $\{B_{r_i}(x_i) \mid i \in \mathcal{I}\}$ we have $\bigcap_{i\in\mathcal{I}} B_{r_i}(x_i) \neq \varnothing$ (respectively $\bigcap_{i\in\mathcal{I}} B_{r_i+\xi}(x_i) \neq \varnothing$); we say Γ is *coarse Helly* if it is ξ-coarse Helly for some $\xi \geq 0$. Given two vertices $x, y \in \Gamma$, an *interval* $[x, y]$ is the set of vertices $u \in \Gamma$ such that $d_\Gamma(x, y) = d_\Gamma(x, u) + d_\Gamma(u, y)$, and given $\beta \geq 1$, we say Γ has β*-stable intervals* if for any vertices $x, y, z \in \Gamma$ with $d_\Gamma(y, z) = 1$, the intervals $[x, y]$ and $[x, z]$ are Hausdorff distance $\leq \beta$ apart.

A finitely generated group is said to be *Helly* (respectively *coarse Helly*) if it acts geometrically on a Helly (respectively coarse Helly) graph. The study of Helly groups was initiated in [1]. In there, it was shown that many classes of 'non-positively curved' groups, such as hyperbolic, cocompactly cubulated, and graphical C(4)–T(4) small cancellation groups, are Helly. The classes of Helly and coarse Helly groups has gathered substantial attention recently: see [3], [4].

It was also shown in [1] that the class of Helly groups is stable under many group-theoretic constructions, such as amalgamated free products and HNN extensions over finite subgroups, graph products, and quotients by finite normal subgroups. A number of open questions on stability under other constructions have been posed;

in particular, the question on whether a group hyperbolic relative to a collection of Helly groups is itself Helly was asked. We answer this question in [5].

In order to discuss relative hyperbolicity, let G be a finitely generated group (with a finite generating set X), let $H_1, \ldots, H_m < G$ be subgroups, and set $\mathcal{H} := \bigcup_{j=1}^{m} H_j \subseteq G$. We then say G is *hyperbolic relative to* $H_1, \ldots, H_m$ if the Cayley graph $\mathrm{Cay}(G, X \cup \mathcal{H})$ is hyperbolic and satisfies the *bounded coset penetration* property: the latter is a condition refining the geometric and combinatorial structure of $\mathrm{Cay}(G, X \cup \mathcal{H})$ in terms of the locally finite graph $\mathrm{Cay}(G, X)$. Relative hyperbolicity is a widely studied construction, and many properties of hyperbolic groups pass via relative hyperbolicity: that is, if G is hyperbolic relative to $H_1, \ldots, H_m$ and each H_j satisfies a certain property (that is satisfied by all hyperbolic groups), then so does G.

For a group G is hyperbolic relative to $H_1, \ldots, H_m$, where each H_j acts on a graph Γ_j geometrically, in [5] we construct a graph $\Gamma(N)$ with a geometric G-action. We show that geodesics in $\Gamma(N)$ can be transformed into 'nice' paths in $\mathrm{Cay}(G, X \cup \mathcal{H})$. This allows us to deduce that if each Γ_j is coarse Helly (respectively has β-stable intervals), then so is (respectively does) $\Gamma(N)$. We then invoke a result from [1], saying that a group is Helly if and only if it acts geometrically on a coarse Helly graph with β-stable intervals. Consequently, we prove the following.

Theorem 18 ([5], Theorems 1.1 & 1.2)**.** *Let G be a finitely generated group that is hyperbolic relative to a collection of subgroups $H_1, \ldots, H_m \leq G$. If each H_j is Helly (respectively, coarse Helly), then so is G.*

In the other direction, we study 'strongly quasiconvex' subgroups of Helly and coarse Helly groups. Given a finitely generated group G with a finite generating set X, we say a subset $A \subseteq G$ is *strongly quasiconvex* in G if for any $\lambda \geq 1$ and $c \geq 0$, there exists $K \geq 0$ such that any (λ, c)-quasigeodesic in $\mathrm{Cay}(G, X)$ with endpoints in A belongs to the K-neighbourhood of A; this property does not depend on the choice of a generating set X. In [5], we show the following.

Theorem 19 ([5], Theorem 1.4)**.** *Let H be a strongly quasiconvex subgroup of a finitely generated group G. If G is Helly (respectively, coarse Helly), then so is H.*

If G is hyperbolic relative to $H_1, \ldots, H_m$, then each H_j is strongly quasiconvex in G [2], implying that if G is (coarse) Helly then so are $H_1, \ldots, H_m$.

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Action of the Cremona group on a CAT(0) cube complex

Anne Lonjou
(joint work with Christian Urech)

The Cremona group (of rank 2), denoted by $\mathrm{Bir}(\mathbb{P}^2)$, is the group of *birational transformations* of the projective plane (isomorphisms between two open dense subsets of the projective plane). This group has been introduced by L. Cremona in 1863-1865 [1]. Even if this group comes from algebraic geometry, tools from geometric group theory have been powerful to study it. For instance, its action on a infinite dimensional hyperbolic space has been used to prove that this group is not a simple group over any field ([2], [3]).

More precisely a Cremona transformation f has the following form:

$$\begin{array}{cccc} f: & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2 \\ & [x:y:z] & \mapsto & [f_0(x,y,z):f_1(x,y,z):f_2(x,y,z)] \end{array}$$

where $f_0, f_1, f_2 \in k[x,y,z]$ are homogeneous polynomials of the same degree without common factor. By a theorem of Zariski, any birational transformation f between surfaces can be decomposed has the composition of blow-ups and then blow-downs. If we consider a minimal one, the points blown-up in this compositions are called *base-points* of f. For instance, the base-points of the standard quadratic involution

$$\begin{array}{cccc} \sigma: & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2 \\ & [x:y:z] & \mapsto & [yz:xz:xy] \end{array}$$

are the points $[0:0:1], [1:0:0], [0:1:0]$. Note, that base-points of a Cremona transformation do not always lie in $\mathbb{P}^2$, but can also lie in surfaces obtained by blowing-up $\mathbb{P}^2$.

In this talk, based on a joint work with Christian Urech [4], we explain our constrution of a CAT(0) cube complex, called "blow-up complex" and the action of the Cremona group on it. This construction gives a new interesting geometric space for the Cremona group of rank 2. Moreover, it has been also a step towards the construction of CAT(0) cube complexes for Cremona groups of higher rank.

The construction of the blow-up complex is the following.

- <u>The vertices</u> $[(S,\varphi)]$ are equivalent classes of marked projective regular rational surfaces where two marked surfaces (S,φ) and (S',φ') are equivalent if $\varphi'^{-1}\varphi$ is an isomorphism.
- <u>Edges :</u> There is an edge oriented from $[(S,\varphi)]$ to $[(T,\psi)]$ if and only if $\varphi^{-1}\psi$ is the blow-up of a closed point of T.
- <u>Cubes :</u> There is a n-cube between the vertices $[(S_1,\varphi_1)],\dots,[(S_{2^n},\varphi_{2^n})]$, if there exists $1 \leq r \leq 2^n$ such that for any $1 \leq j \leq 2^n$:

– there exist n distinct closed points $p_1, \dots, p_n$ in S_r,
– $\varphi_r^{-1}\varphi_j : S_j \to S_r$ is the blow-up of $E \subset \{p_1, \dots, p_n\}$.

This cube complex is sadly not locally compact (when the field is infinite), of infinite dimension but fortunately it is oriented.

Theorem 20 ([4]). *The blow-up complex is a CAT(0) cube complex.*

The Cremona group acts on this complex by acting on the marking of the vertices: for any $f \in \mathrm{Bir}(\mathbb{P}^2)$ and any vertex $[(S, \varphi)]$ of the blow-up complex:

$$f \bullet [(S, \varphi)] = [(S, f\varphi)].$$

This cube complex is really natural and gives a geometric interpretation of several notions of the Cremona group. For instance, the number of base-points of a Cremona transformation f is half of the distance between the vertices $[(\mathbb{P}^2, id)]$ and its image by f. The Cremona transformations which are conjugate to an automorphism of a surface (called *regularisable*) are the elements which are elliptic for this action.

An open question for the Cremona group is the following:

Question 21. Consider a finitely generated subgroup G of the Cremona group such that each of its elements is regularizable. Does it imply that G **is regularizable**; meaning that it is a subgroup of the automorphism group of a surface?

A reformulation of this question in terms of the action of the Cremona group on the blow-up complex is the following. Consider a finitely generated subgroup G of the Cremona group which is locally elliptic for the action on the blow-up complex. **Does it implies that the action of G is (globally) elliptic ?**

Locally elliptic actions of finitely generated groups on CAT(0) cube complex of finite dimension are elliptic [6]. On the other hand, if the hypothesis on the dimension of the cube complex is removed this statement is not true anymore [5]. Nevertheless, as the cube complex has a height on the vertices and that we understand reasonably well the action we expect to answer positively to it in the case of the Cremona group.

Notice that the hypothesis that the subgroup is finitely generated is mandatory. The subgroup of the Cremona group consisting of the elements $\{(x, y + x^n) \mid n \in \mathbb{N}^*\}$ is locally elliptic but not globally elliptic.

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Fixed points for group actions on 2-dimensional buildings

Anne Thomas
(joint work with Jeroen Schillewaert, Koen Struyve)

This direction of research originates with Serre, who proved in his book on trees [7] that if a finitely generated group G acts without inversions on a simplicial tree T, and every element of G fixes a point of T, then G has a global fixed point in T. Morgan and Shalen [4] extended this result to $\mathbb{R}$-trees.

Our main theorem says: if G is a finitely generated group of automorphisms of a (possibly nondiscrete) affine building X of type $\tilde{A}_2$ or $\tilde{C}_2$, and every element of G fixes a point of X, then G fixes a point of X. This theorem overlaps with a recent result of Norin, Osajda and Przytycki [5], who considered CAT(0) triangular complexes, and under certain conditions on the complex or the group action, obtained the same conclusion. In particular, the work of [5] applies to discrete buildings of types $\tilde{A}_2$, $\tilde{C}_2$ and $\tilde{G}_2$, while our result holds for some non-discrete buildings, and we believe that our method fails in type $\tilde{G}_2$.

There are several interesting corollaries of our main result, including:

(1) Removing the hypothesis of finite generation, we can use a theorem of Caprace and Lytchak [1] to prove that if a group G acts on a complete affine building X of type $\tilde{A}_2$ or $\tilde{C}_2$, and every element of G fixes a point of X, then G has a fixed point in the bordification $\overline{X} = X \cup \partial X$.
(2) As shown by Parreau [6], every automorphism of a complete affine building is either elliptic (i.e. fixes a point) or hyperbolic (and hence has has infinite order). It follows from this and our main theorem that if a finitely generated group G acts without a global fixed point on a complete affine building of type $\tilde{A}_2$ or $\tilde{C}_2$, the group G contains a hyperbolic element, hence $\mathbb{Z} < G$. This can be seen as a first small step towards proving a Tits Alternative in this setting.
(3) Since finite order automorphisms of affine buildings are elliptic, we can conclude that the action of any finitely generated infinite torsion group G on a building of type $\tilde{A}_2$ or $\tilde{C}_2$ has a global fixed point.

To prove our main theorem, we first carry out several reductions, to show that it suffices to consider X a complete $\mathbb{R}$-building in which each point is a special vertex, and G type-preserving. For completeness of X, we use the ultrapower of X and theorems of Kleiner and Leeb [3] and Struyve [8]. A key lemma then shows that the distance between certain fixed sets in X is actually realised by points in X. The proof of this lemma combines a theorem for finite-dimensioanl complete CAT(0) spaces from [1] with properties of 2-dimensional affine buildings, and results of Culler and Morgan [2] for the panel tree (this is a tree sitting "at the boundary" of a 2-dimensional affine building). We then establish that if G has two proper finitely generated subgroups whose fixed sets are disjoint, then G contains a hyperbolic element (using induction on the number of generators, it is now easy to prove the main theorem). We construct a hyperbolic element in G by combining standard properties of complete CAT(0) spaces with very specific arguments for affine buildings of types $\tilde{A}_2$ and $\tilde{C}_2$ and their vertex links, which are spherical buildings.

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Boundaries-at-infinity from combings

Alexander Engel
(joint work with Christopher Wulff)

Many non-positively curved spaces admit 'nice' compactifications. Examples of the respective boundaries that one attaches to the original spaces to compactify them are the Gromov boundary of a hyperbolic metric space and the visual boundary of a CAT(0)-space. Further, if a group acts on these spaces, then the action extends to these boundaries. In joint work with Christopher Wulff we developed a general principle of constructing such compactifications.

Assume that we are given the following situation: We have a 'nice' (technically, an ANR) contractible metric space and a group G that acts freely[1], cocompactly and isometrically on it. Examples of the situation and to which our theory will be applicable are the following:

[1]It is possible to generalize our construction to proper actions. But to keep the presentation simple we restrict here to free actions.

group	*space*
CAT(0)-group	CAT(0)-space on which it acts
systolic group	systolic space on which it acts
injective group	injective space on which it acts
hyperbolic group	a large enough Rips complex of it

Our goal is to construct a compactification $\bar{X}$ of X with the following properties:

- $\bar{X}$ is contractible
- the group action extends continuously to $\bar{X}$
- continuous functions on $\bar{X}$ have vanishing variation on X[2]
- $\bar{X}$ is an ANR
- $\partial X := \bar{X} \setminus X$ is a $\mathcal{Z}$-boundary, i.e. can be instantly homotoped into X

If we can solve this problem, i.e. construct such a compactification for the given space X with a given action by G, then the following holds (among many other):

- $\dim(\partial X) = \mathrm{cd}(G) - 1$ [Bestvina–Mess]
- $\dim(\partial X) \leq \text{as-dim}(G)$ [E.–Wulff]
- assembly maps for G are split-injective [*many people*]

Note that the last point implies the (strong) Novikov conjecture for G.

Our construction principle is based on combings. A *combing* on X with base point $p \in X$ associates to every $x \in X$ a discrete path $\sigma_n(x)$ from p to x with:

- $d(\sigma_n(x), \sigma_{n+1}(x)) \leq C$ for all $x \in X$ and $n \in \mathbb{N}$
- $d(\sigma_n(x), \sigma_n(y)) \leq C \cdot d(x, y)$ for all $x, y \in X$ and $n \in \mathbb{N}$

We introduce the following properties of a combing σ:

- σ is *coherent* if there exists $C > 0$ such that $d(\sigma_m(\sigma_n(x)), \sigma_m(x)) < C$ for all $x \in X$ and $m \leq n \in \mathbb{N}$.
- σ is *expanding* if there exists $C > 0$ such that for all $D > 0$ and all $n \in \mathbb{N}$ there is a compact subset $K_{D,n} \subset X$ with $\sigma_n(B_D(x)) \subset B_C(\sigma_n(x))$, where $B_{-}(x)$ denotes the metric ball around x.
- σ is *coarsely equivariant* if for every $g \in G$ there exists $C(g) > 0$ such that $d(g.\sigma_n(x), \sigma_n(g.x)) < C(g)$ for all $x \in X$ and all $n \in \mathbb{N}$.

<u>The main result is as follows:</u> Every combing on X which is coherent, expanding and coarsely equivariant gives a compactification as described above, see [1].

Examples of groups to which the main result applies are: hyperbolic, CAT(0), systolic, injective and hierarchically hyperbolic groups.

The main idea in the proof is to find a convenient definition of the compactification. We define a commutative C^*-algebra $C_\sigma(X)$ consisting of all the bounded, continuous functions $f\colon X \to \mathbb{C}$ with:

- f has vanishing variation
- $\sigma_n^*(f) \xrightarrow{n\to\infty} f$ in sup-norm

[2]A continuous function $f\colon \bar{X} \to \mathbb{C}$ has *vanishing variation* on X, if for any $R > 0$ and $\varepsilon > 0$ there is a compact subset $K \subset X$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in X \setminus K$ with $d(x, y) < R$.

The compactification $\bar{X}$ is then the Gelfand dual of $C_\sigma(X)$.

Interestingly, there seems to be a dichotomy between combings from automatic structures and expanding combings. To my knowledge, hyperbolic groups are the only ones admitting combings which are both automatic and expanding.

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Abstract group actions of locally compact groups on CAT(0) spaces

Olga Varghese
(joint work with Philip Möller)

Given two objects A and B and a map $f\colon A \to B$ between them, it is common to ask whether some or all of the structure of the objects is respected by the map f. When A and B are groups, then the question is whether f is a group homomorphism and when A and B are topological spaces, one can ask whether the map f is continuous. When the objects A and B possess more structure, for instance if A and B are topological groups, one can ask whether preservation of the group structure implies preservation of the topological structure.

We want to understand the ways in which topological groups can act on spaces of non-positive curvature with the focus on automatic continuity. The main idea of automatic continuity is to establish conditions on topological groups G and H under which an abstract group homomorphism $\varphi\colon G \to H$ is necessarily continuous. There are several results in this direction in the literature, see [1], [2], [4], [5], [6], [7]. Here, the group G will be a locally compact Hausdorff group while H will be the isometry group of a CAT(0) space equipped with the discrete topology.

One of the powerful theorems in this direction is due to Dudley [5]. He proved that any abstract group homomorphism from a locally compact Hausdorff group into a free group is continuous. By the Nielsen-Schreier-Serre Theorem, a group is free if and only if it acts freely on a tree [8, I §3.3 Theorem 4]. Hence, the automatic continuity result by Dudley translates into geometric group theory as follows: Any abstract action of a locally compact Hausdorff group on a simplicial tree that is via hyperbolic isometries is continuous.

Inspired by this result we started to investigate actions of locally compact Hausdorff groups on a higher dimensional generalization of simplicial trees and we prove the following result.

Main Theorem 1. *Let $\Phi\colon L \to \mathrm{Isom}(X)$ be an abstract group action of a locally compact Hausdorff group L on a complete* CAT(0) *space X of finite flat rank.*

(1) If L is almost connected (i.e. L/L° is compact) and
  - *(i) the action is semi-simple,*
  - *(ii) the infimum of the translation lengths of hyperbolic isometries is positive,*

(iii) any finitely generated subgroup of L which acts on X via elliptic isometries has a global fixed point,

(iv) any subfamily of $\{\mathrm{Fix}(\Phi(l)) \mid l \in L\}$ with the finite intersection property has a non-empty intersection.

Then Φ has a global fixed point.

(2) If L is totally disconnected and

(v) the poset $\{\mathrm{Fix}(\Phi(K)) \neq \emptyset \mid K \subseteq L$ compact open subgroup$\}$ is non-empty and has a maximal element,

then Φ is continuous or Φ preserves a non-empty proper fixed point set $\mathrm{Fix}(\Phi(K'))$ of a compact open subgroup $K' \subseteq L$.

(3) In particular, if Φ satisfies properties (i)-(iv) and any subfamily of the poset $\{\mathrm{Fix}(\Phi(H)) \mid H \subseteq L$ closed subgroup$\}$ has a maximal element, then Φ is continuous or Φ preserves a non-empty proper fixed point set $\mathrm{Fix}(\Phi(H'))$ of a closed subgroup $H' \subseteq L$.

As an application we obtain the following result.

Corollary 22. *Any abstract group homomorphism $\varphi\colon L \to G$ from a locally compact Hausdorff group L into a* CAT(0) *group G whose torsion groups are finite is continuous unless the image $\varphi(L)$ is contained in the normalizer of a finite non-trivial subgroup of G. In particular, any abstract group homomorphism from a locally compact group into a right-angled Artin group or a limit group is continuous.*

Structure of the proof: For the first part of the Main theorem we show that any action of an abelian group without epimorphisms to $\mathbb{Z}$ on any CAT(0) space with finite flat rank has to be via elliptic isometries. Applying CAT(0) geometry and Iwasawa's Structure Theorem of connected locally compact groups it follows that any action of an almost connected locally compact group on a complete CAT(0) space of finite flat rank has a global fixed point. The second statement follows with an application of a theorem by van Dantzig. We obtain the third result of the Main theorem by combining the first and second statement.

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Crystallographic Helly groups

Nima Hoda

We are concerned with virtually abelian groups, including crystallographic groups, and *Helly groups*, i.e., those acting properly and cocompactly on Helly graphs [4]. Our goal is to give a characterization of those virtually abelian groups that are Helly. Along the way we will give an overview of crystallographic groups and discover properties of asymptotic cones of Helly graphs. Though our methods will differ, our characterization of virtually abelian Helly groups will coincide with Hagen's characterization of cocompactly cubulated virtually abelian groups [6] thus we will also see that these two classes are the same.

1. Helly graphs, hyperconvex spaces and groups

Recall that a graph is *Helly* if any pairwise intersecting family of metric balls of its vertex set Γ^0 has nonempty total intersection. In order to prove our main result we will also need to consider hyperconvex spaces. A geodesic metric space is *(countably) hyperconvex* if every pairwise intersecting (countable) family of closed metric balls has nonempty total intersection. Recall that a metric space is hyperconvex if and only if it is injective. For introductions to Helly graphs and injective metric spaces see, for example, Chalopin et al. [4] and Lang [8], respectively.

The proof of our main result relies on the following theorem of Nachbin.

Theorem 23 (Nachbin [9])**.** *Let $|\cdot|$ be a norm on $\mathbb{R}^n$. Then $(\mathbb{R}^n, |\cdot|)$ is hyperconvex iff $|\cdot|$ is an L^∞-norm, i.e., some element of $\mathrm{GL}_n(\mathbb{R})$ sends $|\cdot|$ to $|\cdot|_\infty$.*

2. Crystallographic groups

In order to give the statement of our characterization, we will need to recall some of the theory of crystallographic groups. Recall that a *crystallographic group* is a group acting faithfully, properly and cocompactly by isometries on Euclidean space $\mathbb{E}^n$ of some dimension n and that there is a split short exact sequence

$$1 \to \mathrm{Trans}(n) \to \mathrm{Isom}(\mathbb{E}^n) \to O(n) \to 1$$

where $\mathrm{Trans}(n) = \mathbb{R}^n$ is the subgroup of translations and $\mathrm{Isom}(\mathbb{E}^n) \to O(n)$ is the action of $\mathrm{Isom}(\mathbb{E}^n)$ on the sphere at infinity. This splits so we can represent $\mathrm{Isom}(\mathbb{E}^n)$ as a semidirect product.

$$\mathrm{Isom}(\mathbb{E}^n) = \mathbb{R}^n \rtimes O(n)$$

The following theorem is essential to the study of crystallographic groups.

Theorem 24 ([1, 2])**.** *Let $G < \mathrm{Isom}(\mathbb{E}^n)$ be a crystallographic group. Then the following statements hold.*

- *The image of G in $O(n)$ is finite.*
- *The intersection $G \cap \mathrm{Trans}(n)$ is a lattice in $\mathrm{Trans}(n) = \mathbb{R}^n$.*

Consequently, we have a short exact sequence

$$1 \to \mathbb{Z}^n \to G \to P \to 1$$

with P finite.

More generally, if G is any finitely generated virtually abelian group then there is a short exact sequence

$$1 \to \mathbb{Z}^n \to G \to P \to 1$$

with P finite. We can show that there exists a canonical morphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & G & \longrightarrow & P & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle 1_P} & & \\ 1 & \longrightarrow & \mathbb{R}^n & \longrightarrow & G_{\mathbb{R}} & \longrightarrow & P & \longrightarrow & 1 \end{array}$$

(specifically $G_{\mathbb{R}} = \mathbb{R}^n \rtimes G/\{(z^{-1}, z) : z \in \mathbb{Z}^n\}$). Since P is finite, we have $H^2(P, \mathbb{R}^n) = 0$ so that $G_{\mathbb{R}} = \mathbb{R}^n \rtimes P$. Thus we have

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{R}^n & \longrightarrow & G_{\mathbb{R}} & \longrightarrow & P & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle \psi} & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \mathbb{R}^n & \longrightarrow & \mathrm{Isom}(\mathbb{E}^n) & \longrightarrow & O(n) & \longrightarrow & 1 \end{array}$$

where $\psi \in \mathrm{GL}_n(\mathbb{R})$ conjugates $P' = \mathrm{im}\big(P \to \mathrm{GL}_n(\mathbb{R})\big)$ into $O(n)$. With this we have essentially proven the following theorem of Zassenhaus.

Theorem 25 (Zassenhaus [10])**.** *If G is a finitely generated virtually abelian group then G acts properly and cocompactly on $\mathbb{E}^n$ by isometries. The action is faithful (i.e. G is crystallographic) iff $P \to \mathrm{GL}_n(\mathbb{R})$ is injective.*

The following definition is essential for the statement of our main result.

Definition 26. The group $P' < \mathrm{GL}_n(\mathbb{R})$ is the *point group* of G. This is well-defined for a finitely generated virtually abelian group up to conjugation by elements of $\mathrm{GL}_n(\mathbb{R})$.

We can now state our main result characterizing virtually abelian Helly groups.

Theorem 27 (H. [7])**.** *Let G be a virtually abelian group. If G is Helly then its point group preserves an L^∞-norm on $\mathbb{R}^n$. Consequently, the group G acts properly and cocompactly by isometries on $\big(\mathbb{R}^n, |\cdot|_\infty\big)$.*

The following corollary gives the first example of a systolic group that is not Helly, thus answering a question of Chalopin et al. [4].

Corollary 28. *The 3-3-3 Coxeter group is not Helly.*

The converse of the theorem also holds since any group acting properly and cocompactly on $\big(\mathbb{R}^n, |\cdot|_\infty\big)$ preserves a standard cubulation of $\mathbb{R}^n$. This can be seen directly but is also a consequence of Hagen's work characterizing cocompactly cubulated virtually abelian groups [6].

Corollary 29. *Let G be a virtually abelian group. Then G is Helly iff G is cocompactly cubulated.*

The proof of Theorem 27 relies on the following theorem about asymptotic cones of Helly graphs that may be of independent interest.

Theorem 30 (H. [7])**.** *Let Γ be a Helly graph and let $v_0 \in \Gamma^0$. Then* $\mathrm{AsCone}(\Gamma, v_0)$ *is countably hyperconvex.*

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