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Homogenization Theory: Periodic and Beyond (online meeting)

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Abstract. The objective of the workshop has been to review the latest developments in homogenization theory for a large category of equations and settings arising in the modeling of solid, fluids, wave propagation, heterogeneous media, etc. The topics approached have covered periodic and non-periodic deterministic homogenization, stochastic homogenization, regularity theory, derivation of wall laws and detailed study of boundary layers,...

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Introduction by the Organizers

The field of multi-scale problems is ubiquitous in contemporary applied science and has already been well explored from the mathematical standpoint. The theoretical side of the field is homogenization theory. The story began with periodic homogenization in the 1960s (see the celebrated monographs by Bensoussan-Lions-Papanicolaou and Jikov-Kozlov-Oleinik) and proceeded with stochastic homogenization in the 1980s, a field that has witnessed tremendous progress lately, with contributions e.g. by Caffarelli and Souganidis, Otto, Armstrong and their respective collaborators. Further generality can be achieved, with general homogenization theory, à la De Giorgi or à la Tartar. In the latter theory, not much is assumed on the setting but the results provided, based on compactness arguments, are not quantitatively informative: typically, existence of a homogenized limit, without saying anything significant in terms of uniqueness, characterization of the limit,

computability of this limit, rates of convergence to the limit and so on and so forth.

Although well documented, periodicity is still too much an idealized assumption for the many practically relevant media (or, in more specific contexts, materials), while the stochastic setting, although theoretically attractive, may be prohibitively expensive computationally. Not even bringing up the case of general homogenization theory. The future of homogenization theory lies somewhere in the gap looming between the periodic setting, the stochastic setting, and the fully general setting. The objective of the workshop has been to review the latest developments in homogenization theory for a large category of equations and settings.

Similar challenges emerge within the theory of boundary layers in fluid mechanics. After the seminal works of Achdou, Pironneau and Valentin in the 1990s, several authors have investigated the behavior of a viscous fluid near a rough wall, starting with the case when the roughness is periodic, and then investigating more general, and possibly more physical cases: quasi-periodic or random roughness, and even roughness profiles without any structural assumption. But the boundary layer model that has attracted the most attention, both in the physical and the mathematical community, is certainly the Prandtl boundary layer, which describes the behavior of an incompressible fluid with small viscosity close to a flat wall. The study of this model raises issues that are connected to, but slightly different from the topics raised above, such as singularity formation and instabilities.

Additionally, under the impulsion of De Lellis and Székelyhidi, and pondering on the works of Nash, Scheffer and Shnirelman, convex integration techniques have had a considerable impact over the mathematical fluid dynamics community over the past ten years. These techniques are closely related to homogenization theory: the idea is to build a solution of an equation (say, the Euler system) in which arbitrarily small scales exist. These small scales add up to create a macroscopic behavior, such as the creation of a non-zero energy. Their effect also results in the creation of singularities within the solution. As a consequence, one of the purposes of the workshop was to highlight the links between homogenization theory, convex integration, and singularity formation.

The participants of the workshop brought expertise in all areas of mathematics that are necessary to address the aforementioned challenges. In particular, analysts and probabilists with the necessary expertise to make progress in these areas attended the workshop. The comparison of approaches and settings generated intense discussion among the participants.

Workshop (online meeting): Homogenization Theory: Periodic and Beyond

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Abstracts

Homogenization of zero order convolution type operators. Periodic and beyond

Andrey Piatnitski
(joint work with Andrea Braides, Elena Zhizhina)

In this presentation we deal with zero order convolution type non-local operator A in $L^2(\mathbb{R}^d)$, $d \geq 1$, with periodic or random statistically homogeneous coefficients. More precisely, we consider operators of the form

$$Au(x) = \int_{\mathbb{R}^d} \Lambda(x,y)a(x-y)\big(u(y)-u(x)\big)dy, \quad u \in L^2(\mathbb{R}^d)$$

where the convolution kernel $a = a(z)$ is a deterministic non-negative integrable function, $a : \mathbb{R}^d \mapsto \mathbb{R}^+$, $\Lambda(x,y)$ is a periodic or stationary random field that satisfies the uniform ellipticity conditions. This field represents the local characteristics of the medium. Under the above assumptions A is a bounded operator in $L^2(\mathbb{R}^d)$.

The corresponding evolution equation describes a jump Markov dynamics in a periodic or random stationary environment.

When studying the large time behaviour of the said Markov evolution it is natural to make the diffusive scaling of spatial and temporal variables: given a small parameter $\varepsilon > 0$ we introduce new variables

$$x \longrightarrow \varepsilon x, \qquad t \longrightarrow \varepsilon^2 t.$$

The generator of the scaled dynamics takes the form

$$(A^\varepsilon u)(x) \;=\; \frac{1}{\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\Lambda\Big(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\Big)\big(u(y)-u(x)\big)dy. \tag{1}$$

The presentation focuses on the homogenization problem for this family of operators, as $\varepsilon \to 0$. Our goal is to formulate the homogenization result and to describe the properties of the limit problem.

The problems of this type appear in various ecological and population dynamics models as well as in computer vision. Some upscaling problems in porous media also lead to homogenization of non-local convolution type operators.

Previously, homogenization results for Levy type non-local operators have been obtained in [1], [3], [2] and some other works. Further progress in this direction was achieved in [4] and [5].

Here we consider zero order convolution type operators. Homogenization problems for these operators were studied in [6], [7] and [8] .

Assumptions. We assume that the following conditions are fulfilled:

(C1.) Integrability: $a(z) \geq 0$; $\qquad a(z) \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.

(C2.) Symmetry:

$$a(z) = a(-z) \quad \text{for all } z \in \mathbb{R}^d,$$
$$\Lambda(\xi,\eta) = \lambda(\eta,\xi) \quad \text{for all } \xi \text{ and } \eta \in \mathbb{R}^d.$$

(C3.) Normalization and second moments:

$$\int_{\mathbb{R}^d} a(x)dx = 1, \quad \int_{\mathbb{R}^d} |x|^2 a(x)dx < \infty.$$

(C4.) Uniform ellipticity: $0 < \Lambda^- \le \Lambda(\xi, \eta) \le \Lambda^+$.

In the periodic case we also suppose

(P1.) Periodicity. $\Lambda(\xi, \eta)$ is a periodic function with period $(0, 1]^d$ in ξ and η.

In the stochastic case we suppose

(S1.) Stationarity. $\Lambda(\xi, \eta)$ is a statistically homogeneous ergodic function of $(\xi, \eta) \in \mathbb{R}^{2d}$.

Homogenization. Given a function $f \in L^2(\mathbb{R}^d)$ and $\lambda > 0$, consider a problem

$$-A^\varepsilon u(x) + \lambda u(x) \;=\; f(x) \qquad \text{in } \mathbb{R}^d. \tag{2}$$

Lemma. *For each $\varepsilon > 0$ and for any $\lambda > 0$ and $f \in L^2(\mathbb{R}^d)$ this problem has a unique solution $u^\varepsilon \in L^2(\mathbb{R}^d)$.*

Definition. *We say that the family A^ε admits homogenization, as $\varepsilon \to 0$, if there exists an operator A^0 in $L^2(\mathbb{R}^d)$ such that the problem*

$$-A^0 u(x) + \lambda u(x) \;=\; f(x) \qquad \text{in } \mathbb{R}^d. \tag{3}$$

has a unique solution u^0, and

$$u^\varepsilon \longrightarrow u^0 \qquad \text{in } L^2(\mathbb{R}^d), \quad \text{as } \varepsilon \to 0.$$

A^0 is called the effective operator.

In the periodic case the following homogenization result holds:

Theorem. *Under assumption $(C1.)$–$(C4.)$ and $(P1.)$ problem (2) admits homogenization. The effective operator A^0 is*

$$A^0 v = \operatorname{div}\bigl(a^0 \nabla v\bigr)$$

with a positive definite symmetric constant matrix a^0.

In the stochastic case we have

Theorem. *Under assumption $(C1.)$–$(C4.)$ and $(S1.)$ problem (2) almost surely admits homogenization. The effective operator A^0 is given by*

$$A^0 v = \operatorname{div}\bigl(a^0 \nabla v\bigr)$$

with a positive definite symmetric deyterministic constant matrix a^0.

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Quantitative homogenization in nonlinear elasticity: periodic composites and random laminates

Stefan Neukamm
(joint work with Mathias Schäffner, Mario Varga)

Homogenization of nonlinear composites. A standard model for a microheterogeneous, nonlinearly elastic material is given by the energy functional $\mathcal{E}_\varepsilon(u) := \int_O W(\frac{x}{\varepsilon}, \nabla u)\,dx$. Here, $O \subseteq \mathbb{R}^d$ denotes the reference domain occupied by the elastic material and $u : O \to \mathbb{R}^d$ its deformation. The material properties are encoded in the stored energy function $W : \mathbb{R}^d \times \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{+\infty\}$, which for a non-degenerate, single-well material satisfies (for all $x \in \mathbb{R}^d$) the following properties:

($A1$) $W(x,F) = W(x,RF)\ \forall F \in \mathbb{R}^{d\times d}$, $R \in SO(d)$ (frame indifference),
($A2$) $W(x,Id) = \min W(x,\cdot) = 0$ (stress-free reference configuration),
($A3$) $W(x,F) \geq \operatorname{dist}^2(F, SO(d))\ \forall F \in \mathbb{R}^{d\times d}$ (non-degeneracy).

We are interested in the macroscopic behaviour of composites with a *periodic* or *random* microstructure. Thus, we assume that for all $F \in \mathbb{R}^{d\times d}$ the map $y \mapsto W(y,F)$ is either a periodic function or a random field with a stationary and ergodic distribution and study the homogenization limit $\varepsilon \to 0$ of the non-convex functional $\mathcal{E}_\varepsilon$ in the framework of Γ-convergence.

The first rigorous result on homogenization for hyperelastic materials have been obtained in [13, 3] for (quasi)periodic materials: Under additional growth conditions on W (in particular, standard p-growth with $1 < p < \infty$), it is shown that $\mathcal{E}_\varepsilon$ Γ-converges to an energy functional of the form $u \mapsto \int_O W_{\mathrm{hom}}(\nabla u)\,dx$ with a

homogenized stored energy function given by the *multi-cell formula*

$$
\begin{aligned}
W_{\mathrm{hom}}(F) &:= \lim_{L\to\infty} W_{\mathrm{hom},L}(F),\\
W_{\mathrm{hom},L}(F) &:= \inf_{\varphi\in W^{1,p}_{\mathrm{per}}(\square_L)} L^{-d}\int_{\square_L} W(x, F+\nabla\varphi)\,dx,
\end{aligned}
\tag{1}
$$

where $\square_L := [-\frac{L}{2},\frac{L}{2})^d$ and $W^{1,p}_{\mathrm{per}}(\square_L)$ denotes the space of L-periodic Sobolev functions in $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$. A similar homogenization result and formula (1) are also valid in the random case, see [5, 4, 11].

Effective stress-strain relation. In mechanics one is especially interested in the *first Piola-Kirchhoff stress tensor* and the *tangent modulus*, i.e., the Jacobian $DW_{\mathrm{hom}}(F)$ and the Hessian $D^2W_{\mathrm{hom}}(F)$ of W_{hom} at a deformation F. Therefore, it is natural to investigate the following questions:

(Q1) Is W_{hom} twice continuously differentiable?
(Q2) If W_{hom} is C^2 at F, how can $DW_{\mathrm{hom}}(F)$ and $D^2W_{\mathrm{hom}}(F)$ be computed or approximated?

In the following we first briefly recall classical results regarding these questions in the convex case. We then discuss recent results that apply to geometrically nonlinear composites with periodic or random laminate microstructure. Finally, we announce some new error estimates regarding *representative volume element* (RVE) approximations for DW_{hom} and D^2W_{hom} in the case of random laminates.

Convex versus non-convex. It turns out that problems (Q1) and (Q2) are only well-understood in the case of convex integrands with quadratic growth. Indeed, as shown in [13], if $W(x,F)$ is periodic in x and *convex* in F, the multi-cell formula reduces to a *one-cell formula* that can be represented with help of a *corrector*, i.e., $W_{\mathrm{hom}}(F) = W_{\mathrm{hom},1}(F) = \int_\square W(y, F+\nabla\varphi_F)\,dy$ for all $F\in\mathbb{R}^{d\times d}$ and a corrector φ_F, which solves the minimization problem in the definition of $W_{\mathrm{hom},1}(F)$. Furthermore, if one additionally assumes that $W(x,F)$ is C^2 in F and satisfies a quadratic growth condition in F, then W_{hom} is C^2 as can be seen by a soft argument that exploits the *corrector representation* of W_{hom}, see [12, Theorem 5.4] and note that the argument extends to the case of a random material.

These results are not applicable to nonlinear elasticity where the stored energy function is necessarily non-convex. In fact, Müller [13, Theorem 4.3] provides a counterexample in form of a laminate material that features a buckling instability under compressive loading; in particular, one has $W_{\mathrm{hom}}(F) < W_{\mathrm{hom},L}(F)$ for some $F\notin SO(d)$ and all $L\in\mathbb{N}$. This shows that one cannot expect a one-cell formula nor a corrector representation to hold in general. However, for single-well materials, close to the non-degenerate, natural state $F\in SO(d)$ a better behaviour can be expected. Indeed, in [14, 10] we showed (with help of the geometric rigidity estimate from [1]) that W_{hom} admits a quadratic expansion at $F\in SO(d)$. More precisely, we proved that $W_{\mathrm{hom}}(Id+G) = Q_{\mathrm{hom}}(G) + o(|G|^2)$, where $Q_{\mathrm{hom}}(G) := \inf_{\varphi\in H^1_{\mathrm{per}}(\square)}\int_\square Q(x, G+\nabla\varphi)\,dx$, and $Q(x,G) := \frac{1}{2}D^2W(x,Id)[G,G]$ denotes the quadratic term in the expansion of W at identity. We established this result for

periodic and random composites under the assumption that, next to (A1) – (A3), the stored energy function $W(x,\cdot)$ admits a quadratic expansion at identity.

Differentiability close to $SO(d)$. To establish a similar expansion for $F \notin SO(d)$ is nontrivial — and in view of Müller's counterexample not always possible. Therefore, in [15, 16, 17] we focus on the case of small-strains, i.e., when F is close to the set of rotations. In particular, in [15, 16] we consider a periodic composite material with a regular microstructure, say, a matrix material with smooth, possibly touching inclusions. Next to (A1) – (A3) we assumed that $W(x,\cdot)$ is C^3 in a neighbourhood of $SO(d)$ and we show that in this case, $W_{\rm hom}$ is C^2 in a neighbourhood of rotations. More precisely, we prove that for some $\rho > 0$,

- $W_{\rm hom} \in C^2(U_\rho)$ where $U_\rho := \big\{F \in \mathbb{R}^{d\times d} : \operatorname{dist}(F, SO(d)) < \rho\big\}$.
- For all $F \in U_\rho$, $W_{\rm hom}(F)$ and its derivatives can be represented by single-cell formulas that invoke corrector functions. In particular, we obtain for the tangent modulus at $F \in U_\rho$ the identity

$$D^2W_{\rm hom}(F)H \cdot G = \int_\square D^2W(x, F + \nabla\varphi_F)(H + \nabla\varphi_{F,H}) \cdot G\, dx$$

  where the corrector $\varphi_F \in W^{1,\infty}_{\rm per}(\square)$ and the linearized corrector $\varphi_{F,H} \in W^{1,\infty}_{\rm per}(\square)$ are solutions to the nonlinear corrector equation $-\nabla\cdot DW(x, F+\nabla\varphi_F) = 0$ in $\mathbb{R}^d$, and the linearized corrector equation $-\nabla \cdot D^2W(x, F + \nabla\varphi_F)\big(H + \nabla\varphi_{F,H}\big) = 0$ in $\mathbb{R}^d$, respectively.

The proof of this result is based on two major ingredients:
1. Inspired by [6] we construct in [15, Corollary 2.3] a *matching convex lower bound* for W: There exists a strongly convex integrand V with quadratic growth such that (for some $\mu > 0$ only depending on W),

$$\begin{aligned} W(x,F) + \mu \det F &\geq V(x,F) \qquad \text{for all } F \in \mathbb{R}^{d\times d},\\ W(x,F) + \mu \det F &= V(x,F) \qquad \text{for all } F \in U_\rho. \end{aligned}$$

With help of V, convex homogenization and the fact that $F \mapsto \det F$ is a Null-Lagrangian, we obtain the lower bound $W_{\rm hom}(F)+\mu \det F \geq V_{\rm hom}(F) = \int_\square V(x, F +\nabla\varphi_F)$ with a convex corrector φ_F given as the unique, periodic, mean-free solution to the corrector problem $-\nabla \cdot DV(x, F + \nabla\varphi_F) = 0$, which is by construction of V a monotone, uniformly elliptic system.
2. By the *regularity theory* for uniformly elliptic, monotone systems with piecewise constant, periodic coefficients, we establish a Lipschitz estimate for the convex corrector of the form

$$\|\operatorname{dist}(F + \nabla\varphi_F, SO(d))\|_{L^\infty(\square)} \leq C \operatorname{dist}(F, SO(d)), \tag{2}$$

where for $d > 2$ we require the right-hand side to satisfy a smallness condition, see [16, Corollary 1]. Combined with the matching property of V, we obtain for F sufficiently close to $SO(d)$ the sought for corrector representation $W_{\rm hom}(F) = W_{\rm hom,1}(F) = \int_\square W(x, F + \nabla\varphi_F)\, dx$.

The corrector representation for $W_{\rm hom}$ for deformations F close to $SO(d)$ is the starting point to analyze properties of $W_{\rm hom}$ and minimizers of $\mathcal{E}_\varepsilon$; next to

regularity of $W_{\rm hom}$, in [15, 16], we prove error estimates for the nonlinear two-scale expansion and we establish Lipschitz estimates that are uniform in $0 < \varepsilon \ll 1$ for minimizers of $\mathcal{E}_\varepsilon$ (subject to periodic boundary conditions).

Periodic versus random. Although the construction of the matching convex lower bound V verbatimly extends to the random case, the analysis in [15, 16] is restricted to the periodic case, since for the regularity estimate periodicity is critical: In [16] we obtain the Lipschitz estimate for the convex corrector by combining a small-scale Lipschitz estimate (cf. [16, Theorem 4]) of the form $\|\nabla\varphi_F\|_{L^\infty(\square_{1/2})} \leq C\|\nabla\varphi_F\|_{L^2(\square)}$ with the energy estimate $\|\nabla\varphi_F\|_{L^2(\square)} \leq C{\rm dist}(F, SO(d))$. While the latter is standard in the periodic case, in the random case such an estimate only holds in the modified form of a *large scale* L^2-estimate: $L^{-\frac{d}{2}}\|\nabla\varphi_F\|_{L^2(\square_L)} \leq C{\rm dist}(F, SO(d))$ for all L larger than a random minimal radius; we refer to [2] where such an estimate has been established for monotone systems. In conclusion, in the random case we obtain the following estimate for the random homogenization corrector,

$$\|{\rm dist}(F + \nabla\varphi_F, SO(d))\|_{L^\infty(B_1(x))} \leq \mathcal{C}(x){\rm dist}(F, SO(d)),$$

with a constant $\mathcal{C}(x)$ that is a stationary random field with stretched exponential moments. This Lipschitz estimate is not global, which prevents us to follow the strategy of [16] to establish a corrector representation for $W_{\rm hom}(F)$. In fact, as in the linear case, we do not expect a global Lipschitz estimate to hold for general random materials.

Random laminates and quantitative RVE-approximation. For random laminates, as considered in [17], the situation is better, since then the corrector problems for φ_F and the linearized corrector $\varphi_{F,G}$, simplify to ordinary differential equations. This allows us to retrieve global Lipschitz estimates by appealing to ODE-arguments. In [17] we consider the following model of a (parametrized) random laminate material: Let $\Omega := \{\omega : \mathbb{R} \to \mathbb{R}^N\}$ denote a probability space of parametrizations equipped with a probability measure $\mathbb{P}$ that is stationary and ergodic w.r.t. shifts $\omega \mapsto \omega(\cdot + t)$, $t \in \mathbb{R}$. We assume that $W(x, F) = W_0(\omega(x_d), F)$, where x_d denotes the dth coordinate of $x \in \mathbb{R}^d$, and $W_0 : \mathbb{R}^N \times \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{+\infty\}$ denotes a stored energy function satisfying (A1) – (A3). As a first result we prove a corrector representation of the form

$$W_{\rm hom}(F) = \mathbb{E}\Big[\int_\square W(x, F + \nabla\varphi_F)\, dx\Big], \tag{3}$$

for F sufficiently close to $SO(d)$, and establish similar representations for the stress tensor $DW_{\rm hom}$ and the tangent modulus $D^2W_{\rm hom}$. In (3) the corrector is defined by a corrector problem on an infinite domain and thus requires approximation in practice. The *representative volume element* (RVE) method is a well-established procedure for this purpose. In this method, the corrector problem is considered on a finite domain $\square_L$ (with $L \gg 1$) together with suitable boundary conditions. It is an ongoing discussion in the computational mechanics community how to choose the size of the RVE, e.g. see [19, 18]. The first convergence results with optimal

scaling in L for the RVE-approximation have been obtained in [8, 9, 7] for linear elliptic equations and systems. In particular, in [7] periodic-RVEs are introduced and analyzed. In a periodic-RVE approximation the original distribution $\mathbb{P}$ is approximated by a distribution $\mathbb{P}_L$ that is supported on L-periodic parametrizations ω, see [2, Section 2.3]. Together with J. Fischer we recently obtained the first optimal result for periodic-RVEs in the case of monotone, uniformly elliptic systems with quadratic growth, see [2].

In the following we discuss the result in [17], where we establish the first optimal periodic-RVE estimate in the case of nonlinearly elastic, random laminates. We addditionally make the following assumption:

- *Existence of a L-periodic approximation*: There exists a shift-invariant probability measure $\mathbb{P}_L$ that concentrates on L-periodic fields in Ω and that recovers the distribution of $\mathbb{P}$ in $\square_{L/2}$. With the latter we mean the following: If ω and ω_L are random fields with distribution $\mathbb{P}$ and $\mathbb{P}_L$, respectively, then $\omega|_{\square_{L/2}} = \omega_L|_{\square_{L/2}}$ in distribution.
- *Fast decorrelations*: $(\Omega, \mathbb{P})$ and $(\Omega, \mathbb{P}_L)$ feature *spectral gap estimates* (similar to [2, Definition 16]) that encode fast decorrelations on scales ≥ 1.

Moreover, we require $W_0(\omega, F)$ to be sufficiently regular in ω and F, and consider deformations F that are sufficiently close to $SO(d)$. Our main result then yields estimates on the error of the RVE-approximation of $W_{\text{hom}}(F)$, the stress tensor DW_{hom} and the tangent modulus D^2W_{hom}. In particular, for $W_{\text{hom},L}$ we get the following estimates on the *random fluctuations* and on the *systematic error*:

$$|W_{\text{hom},L}(F) - \mathbb{E}_L[W_{\text{hom},L}(F)]| \leq \mathcal{C}\,\text{dist}^2(F, SO(d))L^{-\frac{1}{2}},$$
$$\Big|\mathbb{E}_L\big[W_{\text{hom},L}(\cdot, F)\big] - W_{\text{hom}}(F)\Big| \leq C\,\text{dist}^2(F, SO(d))\frac{\ln L}{L}.$$

Above, $\mathcal{C}$ denotes a random constant with $\mathbb{E}_L\big[\exp(\frac{1}{C}\mathcal{C})\big] \leq 2$, and C denotes a deterministic constant. These estimates are optimal w.r.t. scaling in L and w.r.t. the integrability of $\mathcal{C}$.

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Homogenization of ferromagnetic energies on Poisson random sets in the plane

Andrea Braides
(joint work with Andrey Piatnitski)

A *Poisson random set* $\mathcal{N}$ with intensity λ is a subset of $\mathbb{R}^2$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ equipped with a dynamical system $T_x : \Omega \mapsto \Omega$, $x \in \mathbb{R}^2$, such that for any bounded Borel set B and any $x \in \mathbb{R}^2$, we have $\#\big((B+x) \cap \mathcal{N}\big)(\omega) = \#\big(B \cap \mathcal{N}\big)(T_x\omega)$. We suppose that T_x is a group of measurable measure-preserving transformations in Ω and is ergodic. A Poisson random set is characterized by the properties:

- for any bounded Borel set $B \subset \mathbb{R}^2$, the *number of points in* $B \cap \mathcal{N}$ *has a Poisson law with parameter* $\lambda|B|$

$$\mathbf{P}(\{\#(B \cap \mathcal{N}) = n\}) = e^{-\lambda|B|}\frac{(\lambda|B|)^n}{n!};$$

- for any collection of bounded disjoint Borel subsets in $\mathbb{R}^2$ the *random variables defined as the number of points of $\mathcal{N}$ in these subsets are independent.*

  Contrary to stochastic lattices considered e.g. in [3, 1, 2],
- $\mathcal{N}$ *is not "regular"*: we have pairs of points of $\mathcal{N}$ arbitrarily close, and we have squares of arbitrary size not containing points of $\mathcal{N}$;
- $\mathcal{N}$ *is isotropic* since the properties of Poisson random sets are invariant under (translations and) rotations.

We define the energy in terms of *nearest neighbours* in $\mathcal{N}$, which are pairs (denoted by $\langle i, j\rangle$) such that the related *Voronoi cells*

$$C_i = \{x \in \mathbb{R}^2 : \|x - i\| \le \|x - j\| \text{ for all } j \in \mathcal{N}\}$$

have a common edge. The (nearest-neighbour) *ferromagnetic energy on the Poisson random set* is defined for $u : \mathcal{N} \to \{0, 1\}$ as

$$E(u) = \sum_{\langle i,j\rangle} |u_i - u_j|.$$

We introduce a small scaling parameter $\varepsilon > 0$ and define the scaled energies

$$E_\varepsilon(u) = \sum_{\langle i,j\rangle} \varepsilon|u_i - u_j|$$

for $u : \varepsilon\mathcal{N} \to \{0, 1\}$, where $u_i = u(\varepsilon i)$. The overall properties of $\mathcal{N}$ will be described by computing a suitable Γ-limit of E_ε.

If $u : \varepsilon\mathcal{N} \to \{0, 1\}$, we define $V_\varepsilon(u)$ as the union of the scaled Voronoi cells εC_i where $u_i = 1$. Note that, contrary to ferromagnetic energies on stochastic lattices, we cannot estimate $\operatorname{Per} V_\varepsilon(u)$ in terms of $E_\varepsilon(u)$. However, we may estimate the perimeter of sets $V_\varepsilon(u)$ containing 'many' cells as follows, thanks to a Percolation lemma by Pimentel [8].

Lemma. *Let u^ε be such that $\sup_\varepsilon E_\varepsilon(u^\varepsilon) < +\infty$. Then we can write*

$$V_\varepsilon(u^\varepsilon) = (A_\varepsilon \cup B'_\varepsilon) \setminus B''_\varepsilon,$$

where $|B'_\varepsilon| + |B''_\varepsilon| \to 0$, and the sets A_ε have equibounded perimeter. Hence, the family $V_\varepsilon(u^\varepsilon)$ is precompact with respect to local L^1-convergence of the corresponding characteristic functions, and each each cluster point A of the family $V_\varepsilon(u^\varepsilon)$ is a set of finite perimeter.

The lemma above justifies the definition of a *discrete-to-continuum convergence* $u^\varepsilon \to A$ as the convergence $V_\varepsilon(u^\varepsilon) \to A$ locally in L^1. In order to prove a Γ-convergence result with respect to this convergence, it is necessary to better describe the geometry of "regular" Voronoi cells as follows.

Given $\alpha > 0$, we define the set of α*-regular points* of $\mathcal{N}$ as

$$\Big\{i \in \mathcal{N} : C_i \text{ contains a ball of radius } \alpha, \operatorname{diam} C_i \le \frac{1}{\alpha}, \#\text{edges} \le \frac{1}{\alpha}\Big\}.$$

Using Bernoulli percolation results as in [4], it can be proved that the subset of the Delaunay triangulation $\mathcal{D}$ (the network joining nearest neighbours) of $\mathcal{N}$ with α-regular endpoints contains an *infinite connected component* $\mathcal{D}_\alpha$ whose complement

is composed of isolated bounded sets. Furthermore $\mathcal{D}_\alpha$ is 'regular', in the sense that there exists τ_α such that each two points $x, y \in \mathcal{D}_\alpha$ are connected by a path with length not more than $\tau_\alpha \|x-y\|$. As a consequence, we immediately have the finiteness of the Γ-limsup of E_ε on sets of finite perimeter. Another consequence (with more refined properties of 'uniform regularity' of $\mathcal{D}_\alpha$) is that we may use paths in $\mathcal{D}_\alpha$ in "discrete area-formula-type arguments" (in particular to match boundary conditions). The possibility to match boundary conditions, and the invariance by rotations, give a candidate formula for the surface tension:

$$\tau = \lim_{t\to+\infty} \frac{1}{t} \min\big\{\#\big(\text{segments of a path in } \mathcal{D} \text{ 'almost' joining } (0,0) \text{ and } (t,0)\big)\big\}.$$

A subadditive argument allows to show that this limit exists a.s. and is deterministic. This formula allows to construct a matching upper bound. Finally, after a scaling argument which shows that $\tau = \tau_0\sqrt{\lambda}$, we have the following result, which shows that ferromagnetic energies on $\mathcal{N}$ overall behaviour is almost surely isotropic and characterized by a universal constant τ_0.

Theorem. [5] *Almost surely the functionals E_ε Γ-converge to $\tau_0\sqrt{\lambda}\,\mathcal{H}^1(\partial^* A)$ with respect to the local L^1 convergence of $V_\varepsilon(u^\varepsilon)$ to A.*

It is worth noting that we can use the properties of $\mathcal{D}_\alpha$ also to prove that *for $R > 0$ large enough* (corresponding to α small enough) the energies

$$E_\varepsilon^R(u) = \sum_{\|i-j\|\le R} \varepsilon |u_i - u_j|$$

almost surely Γ-converge to an isotropic energy $\tau_R\sqrt{\lambda}\,\mathcal{H}^1(\partial^* A)$.

This result can be compared with results in variational approaches to Data Science where $R = R_\varepsilon \to +\infty$ [7]. Note that the energies E_ε^R cannot be directly compared with the nearest-neighbour energies E_ε. Note moreover that in the analog deterministic case the limit perimeter is not isotropic (more precisely, it is *crystalline*) [6].

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High order periodic homogenization in presence of boundaries and interfaces

Sonia Fliss

(joint work with Clément Beneteau, Xavier Claeys, Valentin Vinoles)

The first motivation of this work concerns the modelling of electromagnetic wave propagation in presence of so-called metamaterials. Recent discoveries have shown the possibility of producing weakly dissipative electromagnetic materials whose effective dielectric and magnetic constants have negative real parts. These "metamaterials", of complex multiscale structure, lead to extraordinary phenomena as regards the propagation of electromagnetic waves (negative refraction, resonance of "wavelength" cavities, etc.) and thus arouse great interest in view of many potential applications (super lenses, stealth coating, miniaturization of antennas, ...). The structure of these media presenting several scales of very different size, it is very expensive or even impossible to simulate the wave propagation in these media taking into account all their complexity. An attractive alternative is to model the metamaterial by a homogeneous material, with physical constants of negative real part. This approach is now widely used by physicists and is the subject of active mathematical research in the homogenization community. Thus, one can find in the literature that for certain periodic media whose structure has resonance mechanisms (being related to the geometry via Helmholtz resonators for example or to the characteristics of the materials), the dielectric permittivity (see e.g. [1]) or magnetic permeability (see e.g. [2]) or even both [3] can become negative for certain frequency ranges. Specific techniques of homogenization (reiterated homogenization techniques for instance) have to be introduced in order to take into account the resonances phenomena and the multi-scale effects. Convergence (of two-scale type) results have also been proved. However, it is well known that classical homogenization process poorly takes into account boundaries or interfaces. This is particularly unfortunate when considering negative materials, because important phenomena arise precisely at their surface (plasmonic waves for instance). Even worse than being imprecise, the effective model can be completely false. Indeed, when we consider an interface between a dielectric and a metamaterial and that the permittivity and/or permeability contrast is equal to -1, it appears at the interface an accumulation of energy that is not compatible with the usual mathematical/physical framework [4, 5]. It seems that these difficulties are due to an insufficiently fine asymptotic description of the propagation phenomena in the vicinity of the interfaces. This is why we want to revisit the asymptotic process in order to propose a new homogenized model that is simple to implement and more accurate near the boundaries or the interfaces.

The second motivation concerns the long time behaviour of the solution of the time-dependent wave equation in periodic media in the homogenization regime. It is well-known that the classical homogenized time-dependent wave equation is less accurate for long times since it does not capture the long time dispersion of the exact solution [7, 8]. In several works, high order homogenized models involving

differential operators of high order (at least 4), are proposed to approximate the exact solution for long times (see for instance [9, 10, 11]). For now, only infinite domains were considered. Dealing with boundaries and proposing boundary conditions for these models were open questions.

Of course, this subject is linked to the presence of boundary layers which appear when considering asymptotic model near boundaries or interfaces. It has already been pointed out for instance in [6] and studied and analyzed in [12, 13, 14, 15, 16, 18] for elliptic systems with Dirichlet and Neumann conditions and in a more general setting in [19]. Concerning transmission problems, very few results are available [12, 17].

In this work, we propose an enriched asymptotic expansion which enables to derive high order effective models at any order. For now, we have treated and analyzed the case of simple geometries : for instance the transmission problem between two periodic half-spaces with the limitation that the whole medium has to be periodic in the direction of the interface. We have derived a high order approximate model which consists in replacing the PDE with periodic coefficients by high order homogenized PDE with transmission conditions which are not classical. For the model at order 1, the homogenized PDEs are the same than the classical ones, but the transmission conditions are modified. These conditions are more precise than the classical transmission conditions (corresponding to the continuity of the solution and its normal derivative across the interface) but less standard since they involve differential operators along the interface. More precisely, the obtained conditions involve Laplace-Beltrami operators at the interface and require to solve cell problems in periodicity cell (as in classical homogenization) and in infinite strips (to take into account the phenomena near the interface). We establish well posedness for the approximate model. An error analysis confirms this accuracy and numerical results illustrate the efficiency and the accuracy of these new conditions. The extension of such approach to more general geometries is under progress.

Let us mention that the model of order 2 that we propose is particularly relevant when one is interested in the long time behaviour of the solution of the time-dependent wave equation. Our approach enables to propose appropriate and accurate boundary conditions for the high order homogenized models involving differential operators of order 4. The analysis of such model and its implementation is under investigation.

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Large-scale regularity for fluids over rough boundaries

Christophe Prange
(joint work with Mitsuo Higaki, Carlos Kenig and Jinping Zhuge)

This talk is concerned with the study of the large-scale regularity of steady fluids over rough boundaries. Roughness is ubiquitous in nature (seafloor, forests and cities at the surface of the Earth, scales of sharks) and industry (fluids in engineered microchannels). The analysis of the influence of roughness on physical phenomena (Ekman pumping in the oceans, drag reduction in microfluidics) covers a wide range of scales. In fluid mechanics, a widespread idea is that roughness favors slip. However there seems to be no consensus about the effect of roughness on the onset of turbulence, some studies suggesting that roughness lowers the critical Reynolds

number for the transition from laminar to turbulent, others indicating in certain regimes a stabilizing effect of inhomogeneities of the fluid domain.

From a mathematical perspective, a lot of effort has been devoted to deriving wall laws for fluids over highly oscillating smooth boundaries. In a rather general random stationary and ergodic setting, a no-slip boundary condition on the rough boundary can be replaced by a Navier-slip condition on a fictitious flat boundary. Moreover, it is possible to prove optimal quantitative error estimates under certain structure assumptions on the roughness, such as periodicity [7], almost periodicity with non-resonance [5], or random stationarity with decorrelation assumptions that entail quantitative ergodicity [4].

In our work, we study the influence of roughness on solutions to elliptic or fluid equations from the point of view of regularity theory. We have two leitmotive: (i) identify building blocks that describe the local behavior of solutions, (ii) prove local error estimates at mesoscopic scales. Our current work with Mitsuo Higaki and Jinping Zhuge on the stationary Navier-Stokes equations builds upon earlier work with Carlos Kenig on uniform regularity for elliptic equations in bumpy domains [8, 9].

In our view, there are two related aspects of roughness: (i) bumpiness, i.e. domains with a highly oscillating boundary, typically $x_3 = \varepsilon\eta(x'/\varepsilon)$, (ii) lack of regularity of the boundary, which is Lipschitz or fractal. Our main goal is to decouple the large-scale regularity of the solutions from the small-scale properties of the boundary, where singularities prevent the solutions from being regular. Hence, we prove that on large scales solutions have improved Lipschitz or $C^{1,\mu}$ regularity. We develop tools that enable to handle singular domains. We prove that large-scale Lipschitz regularity can be proved without relying on structure assumptions on the microscopic oscillations, while higher-order regularity requires certain structure assumptions that ensure quantitative ergodicity properties for the boundary layers.

In a joint work with Mitsuo Higaki [6], we build up a compactness scheme with boundary layers inspired from [9] and the seminal work [1]. This enables to prove that solutions to the stationary Navier-Stokes equations in a bumpy domain $B^\varepsilon_{1,+}(0) \subset \mathbb{R}^3$ with a Lipschitz and ε-periodic boundary are well approximated at mesoscopic scales by Navier polynomials P^ε_N:

$$(C^{1,\mu}) \qquad \sup_{r\in(\varepsilon,\frac{1}{2})} r^{-3} \int_{B^\varepsilon_{r,+}(0)} |u^\varepsilon(x) - c(r)P^\varepsilon_N(x)|^2 \leq C(M)(r^{2+2\mu} + \varepsilon^3 r^{-1}).$$

This is a large-scale $C^{1,\mu}$ estimate. It holds outside of the perturbative regime because there is no restriction on the size $M := \int_{B^\varepsilon_{1,+}(0)} |u^\varepsilon|^2$ of the solutions. The Navier polynomials are shear flow solutions to the Navier-Stokes equations in a flat domain with a Navier boundary condition. Hence, estimate $(C^{1,\mu})$ also reads as a local justification of the Navier wall law.

In an ongoing work with Mitsuo Higaki and Jinping Zhuge, we use a quantitative scheme inspired from the Schauder theory and the works [3, 2], to prove a large-scale Lipschitz estimate for solutions to the stationary Navier-Stokes equations in

genuinely rough domains, namely John domains. This class of domains includes for instance certain fractals. We are able to circumvent the use of boundary layers thanks to the quantitative scheme.

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A periodic homogenization problem with defects rare at infinity

Rémi Goudey

We consider a homogenization problem for a second order elliptic equation in divergence form with a certain class of oscillating coefficients :

$$\begin{cases} -\operatorname{div}(a(x/\varepsilon)\nabla u^\varepsilon) = f & \text{in } \Omega, \\ u^\varepsilon(x) = 0 & \text{in } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^d$ ($d \geq 1$) sufficiently regular and f is a function in $L^2(\Omega)$. The class of (matrix-valued) coefficients a considered is that of the form

$$a_{per} + \tilde{a} \tag{2}$$

which describes a periodic geometry encoded in the coefficient a_{per} and perturbed by a coefficient $\tilde{a}$ that represents a non-local perturbation (a "defect") that, although it does not vanish at infinity, becomes rare at infinity. More specifically, we consider coefficients $\tilde{a}$ that locally behave like $L^2(\mathbb{R}^d)$ functions in the neighborhood of a set of points localized at an exponentially increasing distance far from the origin. Formally, the coefficient $\tilde{a}$ is an infinite sum of localized perturbations, increasingly distant from one another. A prototypical one-dimensional example of such a defect reads as $\sum_{k\in\mathbb{Z}} \phi(x - \operatorname{sign}(k)2^{|k|})$ for some fixed $\phi \in \mathcal{D}(\mathbb{R})$, where $|k|$ denotes the absolute value of k and $\operatorname{sign}(k)$ denotes its sign.

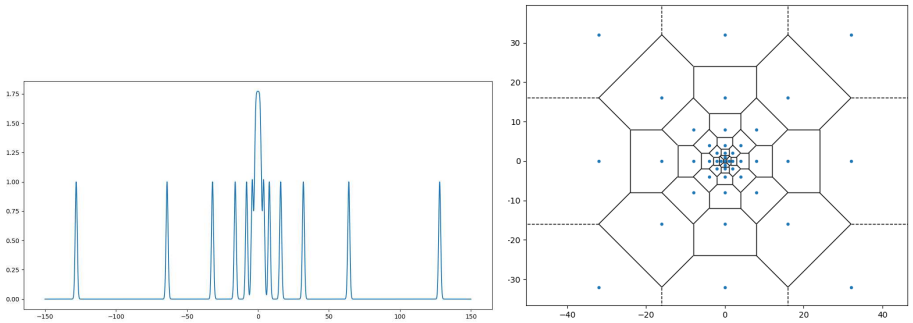

FIGURE 1. Left: Prototype perturbation in dimension $d = 1$. Right: Example of points localized at an exponentially increasing distance in dimension $d = 2$.

The purpose here is to extend the well-known results of the periodic case (that is when $\tilde{a} = 0$) to the setting of the perturbed problem (1)-(2). The main difficulty is that the corrector equation

$$-\operatorname{div}\left(\left(a_{per} + \tilde{a}\right)\left(\nabla w_p + p\right)\right) = 0,$$

(formally obtained by a two-scale expansion), defined on the whole space $\mathbb{R}^d$, cannot be reduced to an equation posed on a bounded domain, as is the case in periodic context in particular, which prevents us from using classical techniques.

The present work follows up on some previous works [1, 2, 3, 4] where the authors have developed an homogenization theory in the case where $\tilde{a} \in L^p(\mathbb{R}^d)$ for $p \in]1, \infty[$. The existence and uniqueness (again up to an additive constant) of a corrector, the gradient of which shares the same structure "periodic + L^p" as the coefficient a, is established. Convergence rates are also made precise.

In our case, by introducing a suitable functional setting to describe the class of defects we consider, we also establish the existence of a corrector (unique up to the addition of a constant), and such that its gradient has the structure (2) of the diffusion coefficient : it can be decomposed as a sum of the gradient of a periodic corrector and a gradient that becomes rare at infinity. Using this adapted corrector, we therefore identify the homogenized limit of u^ε and we also make precise the convergence rates.

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Micro-to-Macro Transition: From Physics/Materials to Living Systems

Leonid Berlyand

In this talk we present a review of our work on PDE models of active matter with the focus on micro-to-macro transition. We emphasize the differences between studying this transition in living systems such as active matter and in the classical physical systems. The key example here is a striking difference in the effective viscosity which drastically decreases in active suspensions versus the well known increase in passive suspensions [1].

The talk consists of the three parts.

In part one we present our results on cell motility and active gels. Here we consider phenomenological macroscopic phase-field model that describes average orientation of the actin filament network in the cytoskeleton [4]. We also discuss a "macro-homogenization" model for the collective motion of many cells. Next we introduce a free boundary Hele-Shaw-Keller-Segel PDE model of keratocyte motility, where we study stability of traveling wave solutions that describe persistent motion of such cells [6, 9, 10]. We explore similarities between the asymptotic techniques in this analysis and classical two-scale homogenization techniques.

In part two we study bacteria swimming in Newtonian fluids. We introduce a hierarchy of stochastic models and use homogenization techniques to obtain an explicit formula for the effective viscosity which generalizes the famous Einstein's formula in dilute [1, 2] and semi-dilute suspensions [3].

In part three we study artificial microswimmers, e.g. biomimetic Janus particles in a fluid flow [5]. We employ boundary layer techniques to analytically capture the key features of active swimmers such as rheotaxis (traveling upstream) and bordertaxis (swimming toward and along the walls) [7]. We briefly discuss the most recent work on active swimmers in anisotropic fluids such as liquid crystals and mucus.

We conclude with summarizing the key challenges due to activity such as the multitude of models in living systems versus few well established models in classical physics/materials problems and the non applicability of classical techniques (such as variational approaches) to active systems leading to the need in development of the new techniques, particularly in the stochastic homogenization. Finally we briefly mention open "micro-homogenization" problems such as the derivation of phenomenological macroscopic models in living systems analogous to e.g., the atomistic-to-continuum transition in solid state physics.

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Power exchange and onset of energy equipartition among surface and body waves

Liliana Borcea

(joint work with Josselin Garnier, Knut Sølna)

Quantifying the exchange of energy among the modes (components) of a wave field propagating in a heterogeneous (random) medium is of interest in seismology, imaging and communications through the turbulent atmosphere, ocean acoustics, ultrasonic inspection of aging structures, manipulation of waves and so on. Mathematically, the exchange is described by the radiative transfer equation that was introduced in a phenomenological way in [1] for wave propagation in open environments. This equation holds for a scaling regime with weak fluctuations of the properties of the medium (e.g. the wave speed) on a length scale (correlation length) that is comparable to the wavelength, and for propagation over many wavelengths. It describes the evolution of the mean wave mode powers and it has been mathematically justified in open environments (see for example [2, 5, 6, 9, 10]). Boundaries complicate the problem, although progress has been made in studying wave propagation on rough surfaces [3], deriving effective boundary conditions for reflection at rough surfaces [4] and quantifying the energy loss to radiation modes [8]. What is not understood is the back and forth transfer of power among surface and body waves. This question motivates our study presented in this talk.

We propose a model of wave propagation that is simple enough to allow an in-depth and explicit characterization of the power exchange between body and surface waves and at the same time it is flexible to allow control of the number of surface and body waves. The model consists of a two-dimensional acoustic waveguide with perfectly reflecting boundaries, which contains a thin layer of depth d filled with a random medium, adjacent to the top boundary. We call this the "surface layer". The remaining part of the domain, with depth D is filled with a homogeneous medium. By taking a mean index of refraction n that is larger in the surface layer than below it, where it equals 1, we ensure that there are N_s wave modes that are trapped in the surface layer and thus mimic surface waves. This N_s is directly proportional to kd and $\sqrt{n^2-1}$, where k is the wavenumber. Thus, by changing d and/or n we can vary the number of monochromatic surface waves with wavelength $2\pi/k$. The waveguide also supports N_b body waves, which are propagating modes that are large throughout the cross-section of the waveguide. This N_b is directly proportional to kD. Finally, there is an infinite number of evanescent modes.

The medium in the surface layer is modeled by random fluctuations of the squared index of refraction. Its role is to couple the wave modes and thus lead to transfer of power. We take a mean zero random process that is mixing, with correlation length ℓ satisfying $k\ell = O(1)$, and has smooth covariance. The amplitude of the fluctuations is small, as modeled by the asymptotic parameter $\epsilon \ll 1$. The radiative transfer regime describes the energy exchange among the wave modes at distance of propagation of order $1/(k\epsilon^2)$.

We analyze the mode coupling in the asymptotic limit $\epsilon \to 0$, using the diffusion approximation theorem, as in [7, Chapter 20]. Due to the assumed smoothness of the covariance of the random fluctuation in the surface layer we prove that the backward going waves can be neglected i.e., we can use the forward scattering approximation. We also show that the coupling of the forward going modes and the evanescent modes results in some anomalous phases but it does not affect the forward going mode powers. The radiative transfer equation describes the evolution of the mean powers of the $N = N_s + N_b$ forward going modes. They take a simple algebraic form, and the mean mode powers are given in terms of the exponential of a transfer matrix Γ. The expression of this matrix is explicit and depends on the covariance of the random fluctuations of the index of refraction. We are interested in the case of a deep waveguide, in the sense $kD \gg 1$, which supports $N_b \gg 1$ body waves. The depth d of the surface layer is chosen so that there is a finite number $N_s > 1$ of surface waves.

The analysis of the mode power exchange amounts to carrying out an explicit spectral decomposition of Γ, which is a symmetric, irreducible, negative semidefinite matrix with rows summing to zero. The Perron-Frobenius theory guarantees that the leading eigenvalue of Γ, which equals 0, is simple and corresponds to the eigenvector $\mathbf{1}_N$ with all entries equal to 1. Therefore, as the distance of propagation grows, the vector of N mean mode powers becomes aligned to $\mathbf{1}_N$ and the system is in equipartition. The main result is the description of the transition to

equipartition. We show that due to the back and forth transfer of power among the surface and body waves the onset of equipartition is multiscale. Specifically, starting from the initial mode power distribution determined by the wave source excitation, there is a quick exchange of power among the surface modes, which results in a metastable distribution among these modes that evolves slowly due to the interaction with the body waves. The power exchange among the body waves occurs slowly and only after a very long distance, quantified precisely by the analysis, it drives the whole system to equipartition.

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Mathematical Analysis of the Effective Viscosity of Suspensions

David Gerard-Varet

We review recent mathematical results on the correction to Einstein's formula for the effective viscosity of dilute suspensions. This correction implies to consider pair interactions of particles, which yields interesting problems related to mean-field limits and stochastic homogenization. Connection to recent mathematical works on Coulomb gases is discussed as well. The talk is based on joint works with M. Hillairet, A. Mecherbet, R. Höfer.

Homogenization of a reaction-diffusion-advection problem in an evolving micro-domain and including nonlinear boundary conditions.

Maria Neuss-Radu
(joint work with Markus Gahn, Iuliu Sorin Pop)

We consider a reaction-diffusion-advection equation in a heterogeneous medium with evolving microstructure. We assume that the evolution of the micro-domain $\Omega_\epsilon(t)$ is known *a priori* and that there is a transformation between a fixed periodically perforated domain Ω_ϵ and the time-dependent domain $\Omega_\epsilon(t)$. The parameter ϵ describes the ratio between the size of the whole domain Ω_ϵ and the periodicity of Ω_ϵ, see Figure 1.

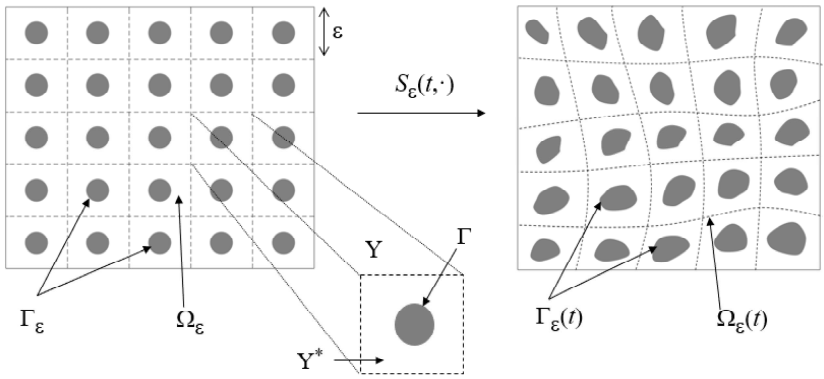

Figure 1. The fixed domain Ω_ϵ and the time-dependent domain $\Omega_\epsilon(t)$ obtained through the mapping $S_\epsilon(t,\cdot)$.

In the domain $\Omega_\epsilon(t)$ we consider the following reaction-diffusion-advection problem

$$
\begin{aligned}
\partial_t \tilde{u}_\epsilon - \nabla\cdot\left(\epsilon^2 D\nabla\tilde{u}_\epsilon - \epsilon\tilde{q}_\epsilon\tilde{u}_\epsilon\right) &= f(\tilde{u}_\epsilon) && \text{in } \bigcup_{t\in(0,T)}\{t\}\times\Omega_\epsilon(t)\\
-\epsilon^2 D\nabla\tilde{u}_\epsilon\cdot\nu &= -\epsilon g(\tilde{u}_\epsilon) && \text{on } \bigcup_{t\in(0,T)}\{t\}\times\Gamma_\epsilon(t),\\
-\epsilon^2 D\nabla\tilde{u}_\epsilon\cdot\nu &= 0 && \text{on } \bigcup_{t\in(0,T)}\{t\}\times\partial\Omega_\epsilon(t)\setminus\Gamma_\epsilon(t),\\
\tilde{u}_\epsilon(0) &= \tilde{u}^0_\epsilon && \text{in } \Omega_\epsilon(0).
\end{aligned}
\tag{1}
$$

Here, ν denotes the outer unit normal with respect to $\Omega_\epsilon(t)$, $\epsilon^2 D$ denotes the diffusion-tensor, and $\epsilon\tilde{q}_\epsilon : \overline{\Omega_\epsilon}\to\mathbb{R}^n$ is a material velocity with the property that at the moving surface $\partial\Omega_\epsilon(t)$ it is equal to the evolution of the surface, i.e. we have $\epsilon\tilde{q}_\epsilon(t,\cdot)\cdot\nu = \partial_t S_\epsilon\big(t, S_\epsilon(t,\cdot)^{-1}\big)\cdot\nu$. The functions f respectively ϵg describe reaction kinetics in the bulk domain $\Omega_\epsilon(t)$ respectively at the surface $\Gamma_\epsilon(t)$.

The aim of our contribution is the derivation of a macroscopic model, the solution of which approximates the solution of the microscopic model. This is done by using rigorous multi-scale techniques like the two-scale convergence and the unfolding method. The main challenge is to pass to the limit in the nonlinear terms. Therefore, we prove strong two-scale compactness results just based on estimates for the solution of the micro-model.

We make the following assumptions on the mapping S_ϵ:

(1) $S_\epsilon \in C^1([0,T]\times\overline{\Omega_\epsilon})^n$ with $J_\epsilon := \det\big(\nabla S_\epsilon\big)$ and

$$\frac{1}{\epsilon}\|\partial_t S_\epsilon\|_{C^0([0,T]\times\overline{\Omega_\epsilon})} + \|S_\epsilon\|_{C^0([0,T]\times\overline{\Omega_\epsilon})} + \|\nabla S_\epsilon\|_{C^0([0,T]\times\overline{\Omega_\epsilon})} \le C.$$

(2) We have $J_\epsilon \in C^1([0,T]\times\overline{\Omega_\epsilon})$ and there exist constants $c_0, C_0 > 0$ independently of ϵ, such that

$$c_0 \le J_\epsilon \le C_0,$$
$$\|\partial_t J_\epsilon\|_{L^2((0,T),\mathcal{H}'_\epsilon)} + \epsilon\|\nabla J_\epsilon\|_{L^\infty((0,T)\times\Omega_\epsilon)} \le C.$$

(3) There exists $S_0 \in C^0\big(\overline{\Omega}, C^1\big([0,T]\times\overline{Y^*}\big)\big)^n$ such that $S_0(t,x,\cdot_y)$ is Y-periodic and $S_0(t,x,\cdot) : Y^* \to Y(t,x) := \mathrm{R}(\mathrm{S_0(t,x,\cdot)})$ (the range of $S_0(t,x,\cdot_y)$) is a C^1-diffeomorphism and

$$\begin{aligned}
S_\epsilon(t,x) &\to x && \text{strongly in the two-scale sense,}\\
\nabla S_\epsilon(t,x) &\to \nabla_y S_0(t,x,y) && \text{strongly in the two-scale sense,}\\
\nabla S_\epsilon^{-1}(t,x) &\to \nabla_y S_0^{-1}(t,x,y) && \text{strongly in the two-scale sense,}\\
\epsilon^{-1}\partial_t S_\epsilon(t,x,y) &\to \partial_t S_0(t,x,y) && \text{strongly in the two-scale sense.}
\end{aligned}$$

Especially, it holds that $J_\epsilon \to J_0 := \det\nabla_y S_0$ strongly in the two-scale sense. Furthermore, we assume that appropriate estimates for the differences between the shifted function and the function itself, as well as for its gradient hold.

Using the mapping S_ϵ, we transform the problem in (1) to the fixed domain Ω_ϵ. Let us define $u_\epsilon : (0,T)\times\Omega_\epsilon \to \mathbb{R}$, $\quad u_\epsilon(t,x) := \tilde{u}_\epsilon(t, S_\epsilon(t,x))$ and for $(t,x) \in (0,T)\times\Omega_\epsilon$

$$\begin{aligned}
D_\epsilon(t,x) &:= \nabla S_\epsilon(t,x)^{-1} D \nabla S_\epsilon(t,x)^{-T},\\
q_\epsilon(t,x) &:= \nabla S_\epsilon(t,x)^{-1}\tilde{q}_\epsilon(t, S_\epsilon(t,x)),\\
v_\epsilon(t,x) &:= \nabla S_\epsilon(t,x)^{-1}\partial_t S_\epsilon(t,x).
\end{aligned}$$

Then, on the fixed domain Ω_ϵ we obtain the transformed problem for u_ϵ :

$$\begin{aligned}
\partial_t\big(J_\epsilon u_\epsilon\big) - \nabla\cdot\big(\epsilon^2 J_\epsilon D_\epsilon\nabla u_\epsilon - \epsilon J_\epsilon q_\epsilon u_\epsilon + J_\epsilon v_\epsilon u_\epsilon\big) &= J_\epsilon f(u_\epsilon) && \text{in } (0,T)\times\Omega_\epsilon,\\
-\epsilon^2 J_\epsilon D_\epsilon\nabla u_\epsilon\cdot\nu &= -\epsilon J_\epsilon g(u_\epsilon) && \text{on } (0,T)\times\Gamma_\epsilon,\\
-\epsilon^2 J_\epsilon D_\epsilon\nabla u_\epsilon\cdot\nu &= 0 && \text{on } (0,T)\times\partial\Omega,\\
u_\epsilon(0) &= u_\epsilon^0 && \text{in } \Omega_\epsilon.
\end{aligned}$$

To derive convergence results for the the sequence of solutions u_ϵ, we start by showing *a priori* estimates, especially estimates for the differences between the

shifted solution and the solution itself. For this purpose, we introduce the space $\mathcal{H}_\epsilon$ with the weighted Sobolev-norm

$$\|u_\epsilon\|_{\mathcal{H}_\epsilon} = \|u_\epsilon\|_{L^2(\Omega_\epsilon)} + \epsilon^2 \|\nabla u_\epsilon\|_{L^2(\Omega_\epsilon)}$$

adapted to the slow diffusion scaling (of order ϵ^2). This function space is the basis for our weak and strong two-scale compactness results. Based on these *a priori* estimates, we show a two-scale compactness result for the generalized time-derivative and a general strong two-scale compactness result of Kolmogorov-Simon-type. These compactness results allow us to pass to the limit $\epsilon \to 0$ in the microscopic model and to obtain the following macroscopic problem in an evolving macroscopic domain. With $Y(t,x) := S_0(t,x,Y^*)$ and $\Gamma(t,x) := \partial Y(t,x) \setminus \partial Y$, we have:

$$\begin{aligned}
\partial_t \tilde{u}_0 - \nabla_y \cdot (D\nabla_y \tilde{u}_0 - \tilde{q}_0 \tilde{u}_0) &= f(\tilde{u_0}) && \text{in} \bigcup_{(t,x)\in(0,T)\times\Omega} \{(t,x)\} \times Y(t,x), \\
-D\nabla_y \tilde{u}_0 \cdot \nu &= -g(\tilde{u}_0) && \text{on} \bigcup_{(t,x)\in(0,T)\times\Omega} \{(t,x)\} \times \Gamma(t,x), \\
\tilde{u}_0(0) &= \tilde{u}^0 && \text{in} \bigcup_{x\in\Omega} \{x\} \times Y(0,x), \\
\tilde{u}_0 &\text{ is } Y\text{-periodic.}
\end{aligned} \tag{2}$$

In the homogenized model (2), we obtain again a reaction-diffusion-advection equation, which depends on a macro-variable $x \in \Omega$ and a micro-variable $y \in Y(t,x)$. Here, for every time $t \in [0,T]$ and in every macroscopic point $x \in \Omega$, $Y(t,x)$ denotes an evolving reference element. Hence, the macro-variable only acts as a parameter and we have to solve a parabolic problem in every point $x \in \Omega$ on the evolving cell $Y(t,x)$. In applications, usually, the evolution of the micro-domain depends on unknowns of the problem, leading to free boundary problems. Here, we make the strong simplification that the evolution is a priori known. Hence, our paper is a first step in the treatment of more complex applications.

While rigorous homogenization results for nonlinear problems with an a priori known evolving microstructure seem to be missing, there are a variety of results for linear problems, as can be seen e.g. in the references below.

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A fixed-point approach to Claussius-Mossotti's formulas

Jules Pertinand

Consider the conductivity $a(L\mathcal{P},\cdot)$ of a two phase material with compactly supported inclusions randomly distributed in an homogeneous background along a point process $\mathcal{P}=\{x\}_{x\in\mathcal{P}}$ rescaled by a factor L. Typically,

$$a(L\mathcal{P},\cdot):y\mapsto \alpha\mathrm{Id}+\sum_{x\in\mathcal{P}}\beta\mathbf{1}_{B_1(Lx)}(y)\mathrm{Id}\ .$$

If $\mathcal{P}$ is stationary and ergodic, homogenization theory yields an associated homogenized coefficient $\overline{a}(L\mathcal{P})$. This quantity depends on the full law of $\mathcal{P}$ making it computationally expensive to evaluate. At leading order, the point process is characterized by its intensity $\theta(\mathcal{P})$, the average number of point per unit of volume and one can hope to accurately describe the homogenised coefficient using only the intensity. Indeed, Clausius-Mossotti formula gives an approximation of $\overline{a}(L\mathcal{P})$ as a power series in $\theta(L\mathcal{P})=L^{-d}\theta(\mathcal{P})\ll 1$ in the dilute regime $L\gg 1$. Such expansions for other models have been investigated, for instance in [1] or [2]. For this model, what can we say about the map $L^{-1}\mapsto\overline{a}(L\mathcal{P})$ for $L\gg 1$?

The following is established in [3].

Theorem 1. *Let $\mathcal{P}$ be a hardcore stationary ergodic point process with arbitrarily small α-mixing. The map $L^{-1}\mapsto\overline{a}(L\mathcal{P})$ admit a Taylor expansion at 0 up to any order. More precisely, there exists $(\mathcal{A}^{(i)})_{i\in\mathbb{N}}\in\left(\mathbb{R}^{d\times d}\right)^{\mathbb{N}}$ such that for all $n\geq 0$, we have*

$$\overline{a}(L\mathcal{P})=\alpha\mathrm{Id}+\sum_{i=0}^{n}\mathcal{A}^{(i)}L^{-(i+d)}+O\left(L^{-(n+1^-)}\right)\ . \tag{1}$$

Inspired by [4], the proof relies on a comparison between the standard corrector $\nabla\varphi(L\mathcal{P},\cdot)$ associated to $a(L\mathcal{P},\cdot)$ and the *single inclusion problem* $\nabla\varphi^\circ$ associated to $a^\circ=\alpha\mathrm{Id}+\beta\mathbf{1}_{B_1}\mathrm{Id}$ via $-\nabla\cdot a^\circ(\nabla\varphi^\circ+e)=0$ on $\mathbb{R}^d$. Since $L\mathcal{P}$ is hard-core and dilute, one expects for all $x\in\mathcal{P}$ and $y\in B_1(Lx)$ that $\nabla\varphi(L\mathcal{P},y)\simeq\nabla\varphi^\circ(y-Lx)$. This statement can be made quantitative noting that the difference $\varphi-\varphi^\circ$ solves an elliptic equation with source terms located far from $B_1(Lx)$ but depending on $\nabla\varphi(L\mathcal{P},\cdot)$. It is then possible to rewrite this PDE as a fixed point for the corrector $\nabla\varphi+e=(\mathrm{Id}-K^L)^{-1}(\nabla\varphi^\circ+e)$ with an explicit operator K^L involving ∇^2G (the second gradient of the Green function for $-\Delta$ on the whole space). Thanks to the homogeneity of G and using a Taylor expansion, it is possible to get a full expansion in L^{-1} of K^L and then deduce one for $\nabla\varphi$.

A key object to define K^L is the formal sum $\sum_{x\in\mathcal{P}^\circ\setminus\{0\}}\nabla^2G(x)$ with $\mathcal{P}^\circ$ a random hardcore point process obtained by conditioning $\mathcal{P}$ to have an inclusion at the origin. Since ∇^2G is not summable, this object is not necessarily well defined for a given realisation. To circumvent this issue, we introduce a massive approximation adding $\frac{1}{T}$ to the operator $-\nabla\cdot a\nabla$ in order to screen the sources at scale $>\sqrt{T}$. As in (1), one can obtain a Taylor expansion of $\overline{a}_T(L\mathcal{P})$ (the massive approximation of $\overline{a}(L\mathcal{P})$). Although, it is not possible to directly pass to the limit

$T \to \infty$ since the bounds on coefficients and on the remainder logarithmically blow up in T. This is where mixing comes into play. Using [5], arbitrarily small algebraic α-mixing yields a small algebraic rate of convergence of $\overline{a}_T(L\mathcal{P})$ to $\overline{a}(L\mathcal{P})$. Using a simple optimization procedure in L and T following a method used in [6], this is enough to compensate the logarithmic divergence of our bounds and obtain the Taylor expansion (1).

In particular, in this random case, the approach bypasses the need for an explicit renormalisation of the sum. Note that for periodic $\mathcal{P}$, one can indeed pass to the limit in $\lim_{T\to\infty} \sum_{x\in\mathcal{P}^\circ\setminus\{0\}} \nabla^2 G_T(x)$. In the periodic case, we recover the expected analyticity of the map $L^{-1} \mapsto \overline{a}(L\mathcal{P})$ at 0 as mentioned in [4] and [7].

The techniques presented here mostly rely on energy estimates and could be extended in various directions to treat non constant background (periodic or, to some extent, random) and other equations (cf [6] for Stokes equation and the effective viscosity).

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The Batchelor power spectrum of passive scalar turbulence at fixed Reynolds number

Jacob Bedrossian

(joint work with Alex Blumenthal, Sam Punshon-Smith)

The talk focuses on the sequence of works [10, 12, 11, 1], which studies (A) the chaotic mixing of the Lagrangian flow map (defined below) associated to the stochastically forced Navier-Stokes equations in $\mathbb{T}^d$ and (B) the passive scalar turbulence problem at fixed Reynolds number.

We use the following mathematical model for passive scalar turbulence at fixed Reynolds number on $\mathbb{T}^2$: stochastically forced Navier-Stokes and a stochastically

forced scalar field

$$\partial_t u_t + u_t \cdot \nabla u_t - \nabla p_t = \nu \Delta u_t + \dot{W}_t \tag{1}$$

$$\nabla \cdot u_t = 0 \tag{2}$$

$$\partial_t g_t + u_t \cdot \nabla g_t = \kappa \Delta g_t + \dot{\xi}_t, \tag{3}$$

where W, ξ are (mean-zero) white-in-time, sufficiently regular in space Gaussian forcing. The field W_t also must satisfy a non-degeneracy condition: (A) W_t is diagonalizable with respect to the basis for the Stokes eigenfunctions in the sense that

$$W_t = \sum_k q_k e_k(x) W_t^{(k)}, \tag{4}$$

where $\{e_k\}_k$ constitute a basis for the real-valued, divergence free vector fields and the $W_t^{(k)}$ are independent, 1d Brownian motions on a common stochastic basis and (B) the coefficients must satisfy: $\exists \alpha > 5$ such that

$$|q_k| \approx |k|^{-\alpha}. \tag{5}$$

Extensions to 3d exist for all of the results discussed provided one uses hyper-viscosity $\nu \Delta u_t \mapsto -\nu(\Delta)^2 u_t$. Regarding the passive scalar, the final result of our analysis, carried out in [1], is that in the limit of $\kappa \to 0$ (with ν fixed – hence *fixed Reynolds number*), we show that Batchelor's 1959 prediction for the cumulative power spectrum holds:

Theorem 1 (JB/Blumenthal/Punshon-Smith '19 [1])**.** *For all $\nu, \{q_k\}$,..., there exists an N_0 and C_0 (both independent of κ) such that the following holds for every initial $g_0 \in L^2$ and $\mathbf{P}$-a.e. ω, and all $N_0 < N < \kappa^{-1/2}$:*

$$\frac{1}{C_0} \log N < \lim_{T \to \infty} \frac{1}{T} \sum_{|k| \leq N} \int_0^T |\hat{g}(t,k)|^2 \, dt < C_0 \log N, \tag{6}$$

where $\hat{g}$ denotes the Fourier transform.

This theorem shows the creation of essentially all sufficiently small scales, as is expected for any "turbulent" system.

The fundamental driver behind Theorem 1 turns out to be *(uniform, almost-sure) exponential mixing*, proved in three steps carried out in [10, 12, 11]. Let f_t solve the following initial value problem for the passive scalar

$$\partial_t f_t + u_t \cdot \nabla f_t = \kappa \Delta f_t \tag{7}$$

$$f_0 = f. \tag{8}$$

We say a scalar is exponentially mixing if $\exists c > 0$ such that (assuming f is mean-zero)

$$||f_t||_{H^{-1}} \lesssim e^{-ct} ||f||_{H^1}. \tag{9}$$

To understand this, if $\kappa = 0$, recall that $||f_t||_{L^2} = ||f||_{L^2}$ for all t, and hence by the Fourier characterization of the H^{-1} space, exponential decay is *only* possible if *all* of the 'energy' of the scalar is shifting to higher frequencies exponentially

fast. Many mathematical works in the literature study mixing, for example see [2, 13, 17, 6, 7, 5, 16, 14, 8, 3, 19, 4] and related references. The following theorem (proved in [10, 12, 11]), shows that velocity fields arising from stochastically forced Navier-Stokes and a variety of other fluid-like models, this happens *almost-surely* and *uniformly* in κ.

Theorem 2 (JB/Blumenthal/Punshon-Smith '19 [10, 12, 11])**.** *For 2D Navier-Stokes (& 3D hyper-viscous NSE, and 'toy models'...), $\forall \nu > 0$, $\forall p \geq 1$, there exists a* ***deterministic*** *$\gamma = \gamma(\nu, p) > 0$ (independent of κ) and $\forall \kappa \in [0, 1]$ a random constant $D = D(\kappa, \omega, u_0, p)$ such that*

$$||f_t||_{H^{-1}} \leq D(\omega, u_0) e^{-\gamma t} ||f||_{H^1} ,$$

and $\forall \eta$ we have the κ-independent moment bound

$$\mathbf{E} D^p(\cdot, u_0) \lesssim_\eta (1 + ||u_0||_{\mathbf{H}})^{p\beta} \exp\left(\eta ||u_0||_{H^1}^2\right).$$

Fundamentally, this theorem is about the chaotic mixing of the Lagrangian flow map, that is, the diffeomorphism $\phi^t : \mathbb{T}^d \to \mathbb{T}^d$ given by the family of random ODEs

$$\partial_t \phi^t(x) = u_t(\phi^t(x)) \tag{10}$$

$$\phi^0(x) = x. \tag{11}$$

If $\kappa = 0$, by the duality characterization of the H^{-1} space, we observe that exponential mixing is equivalent to the assertion that for all f, g mean zero, there holds

$$\left| \int_{\mathbb{T}^d} f(x) g(\phi^t(x)) dx \right| \lesssim e^{-ct} ||f||_{H^1} ||g||_{H^1} , \tag{12}$$

and so we see the notion of mixing in fluid mechanics is a quantitative variation of the notion of strong mixing in ergodic theory. Theorem 2 is qualitatively optimal in various senses; see [12, 11] for details, but essentially, for $t \lesssim |\log \kappa|$ almost the reverse inequality holds: i.e. there is a deterministic γ' (independent of κ etc) and a random constant D' satisfying analogous moment estimates such that

$$||f_t||_{H^{-1}} > D'(\omega, u_0) e^{-\gamma' t} ||f||_{H^{-1}} . \tag{13}$$

The fields studied in [11] are the first examples (stochastic or deterministic) of uniform-in-κ exponentially mixing fields to our knowledge.

The main theme of the proof of Theorem 2 is the introduction of many ideas from the field of random dynamical systems. In the first work, [10] we proved *Lagrangian chaos*, which is the theorem

Theorem 3 (JB/Blumenthal/Punshon-Smith '18 [10])**.** *For 2D Navier-Stokes (also 3D hyper-viscous NSE, Stokes, some other models...), $\forall \nu > 0$, $\exists$* ***deterministic*** *$\lambda > 0$ such that*

$$\lim_{t \to \infty} \frac{1}{t} \log |D_x \phi_t(x)| = \lambda > 0 \qquad \textbf{\textit{almost surely}}$$

for ***all*** *initial $x \in \mathbb{T}^d$ and* ***all*** *$u_0 \in \mathbf{H}$*

The quantity λ is called the *Lyapunov exponent* Proving the existence of such quantity is not the part that is most challenging (though it is not so easy and to our knowledge, it had not previously been shown rigorously). The much more difficult and subtle aspect of the result however, is proving that Lyapunov exponent is strictly positive. For this we adapt ideas of à la Furstenberg, building off of ideas of classical works in the random dynamics literature, specifically [21, 20, 15], to show that if the Lyapunov exponent were zero (it cannot be negative by incompressibility), then there must exist a certain set of almost-sure degeneracies in the dynamics. A suitable approximate control argument is used to rule out these degeneracies, i.e. that the fluid and the Lagrangian flow map can simultaneously reach a wide enough range of configurations. The infinite dimensional setting as well as the unboundedness of the velocity fields both produce difficulties that require a number of new ideas to treat.

To upgrade Lagrangian chaos to exponential mixing at $\kappa = 0$ in [12], we use a two-point process approach, building off of ideas for stochastic (Markov) diffeomorphisms on manifolds, namely [9, 18]. Essentially the idea is that by a Borel-Cantelli argument, almost-sure exponential mixing can be deduced from the exponential ergodicity of a suitable Markov process, that is if we consider the Markov process (u_t, x_t, y_t) with

$$x_t = \phi^t(x), \quad y_t = \phi^t(y) \tag{14}$$

then there is some positive function $\mathcal{V}$ such that for all average-zero observables φ and all initial u, x, y with $x \neq y$ there holds

$$|\mathbf{E}_{u,x,y}\varphi(u_t, x_t, y_t)| \lesssim \mathcal{V}(u, x, y)e^{-\mu t}. \tag{15}$$

In order to make this proof work, we need to construct a *drift condition*, which is a way of saying that the two-point process is ejected exponentially fast from degenerate parts of phase space whenever it happens to venture there. That is, we need to show

$$\mathbf{E}_{u,x,y}\mathcal{V}(u_t, x_t, y_t) \leq \mathcal{V}(u, x, y)e^{-\zeta t} + C, \tag{16}$$

where $\mathcal{V} \to \infty$ if $u \to \infty$ or $x \to y$. As it turns out, this is the difficult part of the theorem. The proof is a bit technical, but it amounts to studying what is essentially a large deviations principle on the Lyapunov exponent, basically showing (more or less) that the asymptotic exponential growth rate is realized exponentially fast with high probability (and exponential tails on the distribution). The proof uses [10] as a black-box lemma. Again, due to the unbounded, infinite dimensional setting, carrying out this procedure requires the development of a lot of new ideas. To treat the $\kappa > 0$ case in [1], let it just suffice to say that it is not trivial, due to the fact that the $\kappa\Delta$ is a singular perturbation. The key idea is to use the stochastic Lagrangian trajectories

$$x_t = \phi^t(x) + \sqrt{\kappa}d\tilde{W}_t^{(1)}, \quad y_t = \phi^t(y) + \sqrt{\kappa}d\tilde{W}_t^{(2)} \tag{17}$$

and then carry out the proof of mixing in a κ-uniform way. This is problematic because the perturbation is truly singular. However, the proof of [12] is based on

spectral theory for certain Markov and Feynman-Kac semigroups, and it turns out that we can pass to the $\kappa \to 0$ limit in the dominant eigenvalue at least.

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Homogenization of Interface Motions in the Parabolic Scaling

Peter S. Morfe

We consider the homogenization of interface motions in periodic media in the parabolic scaling regime.

Phase transitions in periodic media. Given a $\mathbb{Z}^d$-periodic uniformly elliptic matrix field a and a double-well potential W, it is a natural question to ask what is the behavior of the solutions $(u^\epsilon)_{\epsilon>0}$, as $\epsilon \to 0^+$, of the following phase transitions problem in $\mathbb{R}^d$:

$$\begin{cases} u^\epsilon_t - \operatorname{div}(a(\epsilon^{-1}x)Du^\epsilon) + \epsilon^{-2}W'(u^\epsilon) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u^\epsilon = u_0 & \text{on } \mathbb{R}^d \times \{0\}. \end{cases} \tag{1}$$

Here, for definiteness, assume that W has unique global minima at 1 and -1 and the solutions $(u^\epsilon)_{\epsilon>0}$ take values in $[-1,1]$.

In [7], I provide an answer in the setting of *laminar media*, that is, when a depends on fewer than d of the variables. Roughly speaking, [7] shows that if a is laminar and the zero level set $\{u_0 = 0\}$ is a graph that crosses the laminations, then

$$u^\epsilon(\cdot,t) \to 1 \quad \text{in } E_t, \quad u^\epsilon(\cdot,t) \to -1 \quad \text{in } \mathbb{R}^d \setminus \overline{E_t},$$

where the family of open sets $(E_t)_{t\geq 0}$ moves with normal velocity

$$V_{\partial E_t} = \tilde{M}(\nu_{\partial E_t})^{-1}\operatorname{tr}(D^2\tilde{\varphi}(\nu_{\partial E_t})A_{\partial E_t}).$$

Above $\nu_{\partial E_t}$ is the normal vector of ∂E_t and $A_{\partial E_t}$, its second fundamental form; $\tilde{M}$, the *mobility*, describes the average rate of energy dissipation; and $\tilde{\varphi}$ is the *surface tension* as obtained in [1].

The proof of the theorem is an adaptation of the original approach of Barles and Souganidis [3]. The difficulty here is the existence of so-called *pulsating standing waves* used in the asymptotic expansion. The advantage of laminar media is that, in directions that cross the laminations, it is always possible to find smooth pulsating standing waves. Hence, when the initial interface is a graph, there is some room to build sub- and supersolutions.

In the setting of anisotropic curvature flows, similar results have been obtained by Barles, Cesaroni, and Novaga [2] and Novaga and Valdinoci [9].

Level set PDE in non-divergence form. Given a positive $\mathbb{Z}^d$-periodic function m and a $\mathbb{Z}^d$-periodic uniformly elliptic matrix field a, consider the problem

$$\begin{cases} m(\epsilon^{-1}x, \widehat{Du^\epsilon})u^\epsilon_t - \operatorname{tr}\left(A(\epsilon^{-1}x, \widehat{Du^\epsilon})D^2u^\epsilon\right) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u^\epsilon = u_0 & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \tag{2}$$

where $A(y,e) = (\mathrm{Id} - e \otimes e)a(y,e)(\mathrm{Id} - e \otimes e)$.

The main result of [8] is that (2) homogenizes and the effective behavior is described by a similar PDE with possibly discontinuous coefficients:

Theorem 1. [8, Theorem 2] *There are functions $\bar{m} : S^{d-1} \setminus \mathbb{RZ}^d \to (0,\infty)$ and $\bar{a} : S^{d-1} \setminus \mathbb{RZ}^d \to Sym(d)$ such that, for each $u_0 \in UC(\mathbb{R}^d)$,*

(i) *There is a unique viscosity solution* $\bar{u} \in C(\mathbb{R}^d \times [0,\infty))$ *of the effective equation*

$$\begin{cases} \bar{m}(\widehat{D\bar{u}})\bar{u}_t - tr\left(\bar{A}(\widehat{D\bar{u}})D^2\bar{u}\right) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ \bar{u} = u_0 & \text{on } \mathbb{R}^d \times \{0\}, \end{cases} \tag{3}$$

where $\bar{A}(e) = (Id - e \otimes e)\bar{a}(e)(Id - e \otimes e)$.

(ii) $u^\epsilon \to \bar{u}$ *locally uniformly in* $\mathbb{R}^d \times [0,\infty)$.

It is shown in [8] that the discontinuity of $\bar{m}$ and $\bar{a}$ is generic. To overcome this, the proof of Theorem 1 employs ideas originally developed by Feldman and Kim [6] in the study of oscillating boundary value problems.

Other examples of interface motions or front propagation problems where the homogenized coefficients exhibit discontinuities can be found in the work of Cesaroni, Novaga, and Valdinoci [5] and Cesaroni, Dirr, and Novaga [4].

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Optimal-order estimates in stochastic homogenization: Beyond linear equations and smooth data

Julian Fischer

(joint work with Stefan Neukamm, Peter Bella, Marc Josien, Claudia Raithel)

For linear elliptic PDEs

$$-\nabla \cdot (a_\varepsilon \nabla u_\varepsilon) = f \tag{1}$$

with a random coefficient field a_ε with short correlation length $\varepsilon > 0$ and sufficiently smooth data, optimal rates of convergence to the solution of a homogenized problem have been derived in the last decade [10, 11, 12, 3].

The quantitative homogenization theory of *nonlinear* random PDEs, on the other hand, had been limited to at most *small algebraic rates* of convergence like $\varepsilon^{1/8}$, $\varepsilon^{1/90}$, or even merely ε^{δ}, depending on the PDE [8, 5, 7].

In the recent work [1], we establish *optimal-order homogenization error estimates* for *nonlinear elliptic PDEs* with random material law in the case of monotone nonlinearities: We consider the PDE

$$-\nabla \cdot \big(A_\varepsilon(x, \nabla u_\varepsilon)\big) = f \tag{2}$$

and assume that at each point $x \in \mathbb{R}^d$ the material law $A_\varepsilon(x, \xi)$ is determined by a random field $\omega_\varepsilon(x)$ according to $A_\varepsilon(x, \xi) := A(\omega_\varepsilon(x), \xi)$. Here, the random field ω_ε is assumed to be stationary (i. e. have spatially homogeneous statistics) and feature fast decorrelation on scales larger than a microscale $\varepsilon > 0$, quantified by a spectral gap inequality. Under natural 2-growth-type assumptions on the monotone operators A – a case that in particular comprises linear elliptic PDEs as a special case – , we prove that the solution to the problem (2) may be approximated by a homogenized effective equation of the form

$$-\nabla \cdot \big(A_{\mathrm{hom}}(\nabla u_{\mathrm{hom}})\big) = f, \tag{3}$$

up to an error of the order

$$||u_\varepsilon - u_{\mathrm{hom}}||_{L^p} \leq \begin{cases} \mathcal{C}\varepsilon^{1/2} & \text{for } d = 1, \\ \mathcal{C}\varepsilon |\log \varepsilon|^{1/2} & \text{for } d = 2, \\ \mathcal{C}\varepsilon & \text{for } d \geq 3. \end{cases}$$

Here, $\mathcal{C}$ is a random prefactor satisfying a stretched exponential moment bound of the form $\mathbb{E}[\mathcal{C}^\delta / C(d, \mathrm{data})] \leq 2$ for some $\delta > 0$. To obtain the same rates of convergence in the case of nonlinear monotone systems, we additionally require a small-scale regularity condition, satisfied for instance in $d \leq 2$ or for systems with Uhlenbeck structure.

This result in [1] may be seen as the optimal quantitative counterpart in the case of 2-growth to the qualitative stochastic homogenization theory for convex integral functionals developed by Dal Maso and Modica [9]. Note that we are aware of an alternative approach to this problem, being currently in development by Armstrong, Kuusi, and coworkers [4].

In stochastic homogenization, effective material laws like A_{hom} in (3) must usually be determined by the method of representative volumes: A small sample of the random medium is chosen – say, a realization $A_\varepsilon|_{[-\frac{L}{2}, \frac{L}{2}]^d}$ on a box of side length L, with typically $\varepsilon \ll L \ll 1$ – and the effective material law A_{hom} is approximated by "probing" the behavior of the material sample (the "representative volume"). In [1] we derive an optimal-order error estimate for the approximation of the effective material law A_{hom} by the method of representative volumes: We prove that when using a suitably periodized representative volume $A_\varepsilon|_{[-\frac{L}{2}, \frac{L}{2}]^d}$, the approximation $A_\varepsilon^{\mathrm{RVE}, L}$ of the effective material law A_{hom} is of accuracy

$$\big|A_\varepsilon^{\mathrm{RVE}, L}(\xi) - A_{\mathrm{hom}}(\xi)\big| \leq \mathcal{C}|\xi|(1 + |\xi|^C)\left(\frac{\varepsilon}{L}\right)^{d/2}$$

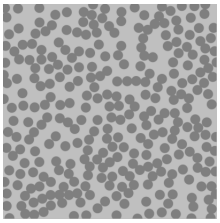
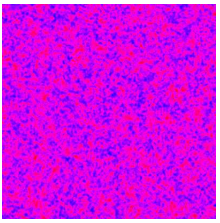

FIGURE 1. Two examples of random fields. Left: Random non-overlapping spherical inclusions. Right: A Gaussian random field with short correlation length.

for dimensions $d \leq 7$ (the random variable $\mathcal{C}$ again satisfying stretched exponential stochastic moment bounds). In expectation, the approximation is of higher order in the sense

$$\left|\mathbb{E}[A_\varepsilon^{\mathrm{RVE},L}(\xi)] - A_{\mathrm{hom}}(\xi)\right| \leq C|\xi|(1+|\xi|^C)\left(\frac{\varepsilon}{L}\right)^d \left(\log \tfrac{L}{\varepsilon}\right)^C.$$

Both convergence rates basically coincide with the rate in the linear elliptic setting [10, 11] and are therefore optimal (up to logarithmic factors).

Returning to the case of linear elliptic PDEs but going beyond the homogenization problem for smooth data, in [2] we analyze the properties of the boundary layer corrector $\theta_i^{\mathbb{H}^{\mathrm{n}}_+}$ for half-spaces $\mathbb{H}^{\mathrm{n}}_+ := \{x \in \mathbb{R}^d : x \cdot \mathrm{n} > 0\}$, determined as the solution to the fluctuating Dirichlet problem

$$\begin{aligned} -\nabla \cdot \left(a_\varepsilon \nabla \theta_i^{\mathbb{H}^{\mathrm{n}}_+}\right) &= 0 && \text{in } \mathbb{H}^{\mathrm{n}}_+, \\ \theta_i^{\mathbb{H}^{\mathrm{n}}_+} &= \phi_i && \text{on } \partial\mathbb{H}^{\mathrm{n}}_+. \end{aligned}$$

Here, ϕ_i denotes the homogenization corrector on the whole space $\mathbb{R}^d$. Note that for $d \geq 3$, ϕ_i typically takes values of the order ε and fluctuates rapidly on the scale ε; hence, one may expect nontrivial nonlinear interactions between the fluctuating coefficient field a_ε and the fluctuating boundary data ϕ_i. In [2], in the case of three or more dimensions $d \geq 3$ and under standard (fast decorrelation) assumptions on the random coefficient field a_ε we prove the optimal-order decay estimate

$$\left|\nabla \theta_i^{\mathbb{H}^{\mathrm{n}}_+}(x)\right| \leq \mathcal{C}(a_\varepsilon, x)\left(\frac{\varepsilon}{\varepsilon + \operatorname{dist}(x, \partial\mathbb{H}^{\mathrm{n}}_+)}\right)^{d/2} \tag{4}$$

where the random prefactor $\mathcal{C}$ is again subject to stretched exponential moment bounds of the form $\mathbb{E}[\exp(\mathcal{C}^\delta/C)] \leq 2$. For the stochastically averaged gradient $\mathbb{E}\big[\nabla \theta_i^{\mathbb{H}^{\mathrm{n}}_+}(x)\big]$, we even establish the higher rate of decay $\big(\varepsilon/\operatorname{dist}(x, \partial\mathbb{H}^{\mathrm{n}}_+)\big)^{d/2+1}$. Our result has implications for the rate of convergence of the method of representative volumes with Dirichlet boundary conditions.

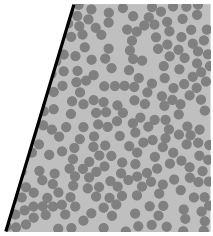

FIGURE 2. An illustration of the homogenization problem on the half-space $\mathbb{H}^n_+$.

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A general deterministic approach to homogenization theory

JEAN LOUIS WOUKENG

The homogenization theory offers a rigorous mathematical framework permeating the modelling and analysis of composites in various environments. It is concerned with the description of macroscopic properties of heterogeneous materials immersed in various environments in terms of their microscopic properties. These heterogeneous materials have microscopic features which are spatially oscillatory with the frequencies of oscillations depending on the distribution of constituents

of underlying materials. This renders the direct numerical treatment of models of composites very difficult and in most cases impractical. Indeed, meshes in numerical schemes for approximating computational domains occupied by a composite must be very fine to capture microscopic behaviours. Homogenization seeks to mitigate this problem by approximating the underlying composite by a homogeneous one readily amenable to numerical computations. One of the key challenges is therefore to control the error made by approximating the original heterogeneous phenomenon by the homogeneous one. This falls within the scope of *quantitative homogenization* theory. Quantitative homogenization beyond the periodic setting is still at its early stage. Its study poses new challenges among which, the resolution of the corrector equation (in the sense of the distributions) in the locally uniform spaces. This is the starting point for the quantitative theory of deterministic homogenization beyond the periodic framework. As a result, we have to design a numerical strategy to approximate the homogenized coefficients and find a rate of convergence in order to implement, simulate and solve numerically concrete non-periodic homogenization problems.

A typical homogenization problem for linear elliptic equations under divergence form consists in replacing an equation with oscillatory coefficients A_ε of scale $0 < \varepsilon << 1$, $-\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f$, by an equation with constant coefficients $\widehat{A}$, $-\operatorname{div}(\widehat{A}\nabla u_0) = f$, where $\widehat{A}$ (the *homogenized* coefficient) depends only on A_ε (not on f) and has constant entries. Such a theory is well developed (both qualitatively and quantitatively as well) for $A_\varepsilon = A(\frac{\cdot}{\varepsilon})$ when A is periodic. When A is no more periodic but satisfies some structural assumptions that are deterministic (they may vary from the periodicity, the almost periodicity to more complicated ones) the qualitative theory has already been addressed. In that case, the homogenized coefficient $\widehat{A}$ has the form $\widehat{A} = \langle A(I + \nabla\chi)\rangle$ where $\langle v\rangle = \lim_{R\to\infty} \fint_{(-R,R)^d} v(y)dy$ (with $\fint_{(-R,R)^d} = (2R)^{-d}\int_{(-R,R)^d}$) denotes the mean value, I the square identity $d \times d$ matrix, and $\chi = (\chi_j)_{1\le j\le d}$ is the solution of the so-called *corrector problem*

$$-\nabla \cdot (A(e_j + \nabla\chi_j)) = 0 \text{ in } \mathbb{R}^d \tag{1.1}$$

in which e_j is the j th vector of the canonical basis of $\mathbb{R}^d$. In contrast with the qualitative theory, the quantitative theory is in its early stage. Indeed with the help of the function χ determined by (1.1), we may define the approximation $v_\varepsilon(x) = u_0(x) + \varepsilon\chi(\frac{x}{\varepsilon})\nabla u_0(x)$ of u_ε and show that $u_\varepsilon - v_\varepsilon \to 0$ in a suitable sense. Keeping also in mind that the homogenized coefficient $\widehat{A}$ is defined in a asymptotic way, two questions readily come in mind: **1**) How large should we choose the real number R in order to get a more convenient approximation of $\widehat{A}$ in terms of an average over $(-R, R)^d$? **2**) How small should we choose ε so that the approximation of u_ε by v_ε is as accurate as possible? These challenging questions are at the heart of the current work in deterministic homogenization theory beyond periodicity. They are motivated by the numerical implementation and simulation of results in multiscale analysis beyond periodicity. Currently, many authors are working on how to address these questions in the general deterministic theory including the almost periodic behaviour, the asymptotic periodic and the asymptotic almost

periodic behaviours, and some others. Namely the current main challenges are twofold:

- **A**1 Study the approximation of homogenized coefficients and the rates of convergence by establishing:
  - (i) regularity results at large scales under minimal assumptions on the coefficients of the problem under consideration;
  - (ii) optimal bounds on the growth of the corrector function based on assumptions on the coefficients.
- **A**2 Design numerical method to approximate the solution of the homogenized models in order to perform simulation to confirm and validate the analytical study.

It seems convenient to precise the motivation of **A1** and **A2**. In [6, 9], we have shown that the distributional solution of problem (1.1) belongs to a Sobolev-Besicovitch type space that is needed for **A**1 and **A**2. However this remains a very challenging issue. As an illustration, there are very important results dealing only with the existence of solutions of PDE in locally uniform spaces; see, e.g., [10]. With this in mind, the next step consists in finding a suitable approximation scheme for the homogenized matrix $\widehat{A}$. This does not matter in the periodic framework since under the periodicity assumption, the corrector problem is posed on a bounded domain (namely the periodic cell $Y = [0,1)^d$) as in that case, the solution χ_j is Y-periodic. A huge contrast between the periodic setting and the general deterministic setting is that in the latter, the corrector problem is posed on the whole space $\mathbb{R}^d$, and cannot be reduced to a problem on a bounded domain. As a result, the solution of the corrector problem (1.1) (and hence the homogenized matrix which depends on this solution) can not be computed directly. Therefore, as in the random setting [5], truncations of (1.1) must be considered, particularly on large domains $(-R,R)^d$ with appropriate boundary conditions, and the homogenized coefficients will therefore be captured in the asymptotic regime. This falls within the scope of **A**1, and has been addressed in [9] for the model problem above. After having considered this problem, the next natural step is to evaluate the rate of convergence of the approximation in terms of the truncation size, that is, to find a rate of convergence for the approximation of both $\widehat{A}$ and u_ε. This problem has been firstly addressed in the periodic framework by Avellaneda and Lin [1], and in the random setting (that is, for second order linear elliptic equations with random coefficients) by Yurinskii [13], Bourgeat and Piatnitski [5] (see also a recent series of works by Gloria and Otto [7, 8] for the random discrete linear elliptic equations). Although the qualitative deterministic homogenization theory can be seen as a special case of random homogenization theory, we can not expect to use this random formulation to address the issues of rate of convergence in the deterministic setting. Indeed, in the random framework, the rate of convergence relies systematically on the *uniform mixing* property (see e.g. [5, 13]) of the coefficients of the equation. Indeed, as proved in [4], the almost periodic setting does not satisfy the uniform mixing property. As a result, we can not use the random framework to address the issue in the general deterministic

setting. We therefore need to elaborate a new framework for solving the problem considered. The very first work that uses the general almost periodicity assumption is a recent paper by Shen [11]; see also [12]. In [6, 9] we have addressed these issues in some general way including the almost periodic setting, the asymptotic periodic and asymptotic almost periodic settings.

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Homogenization of Poisson equation and Stokes system in some non-periodically perforated domains

Sylvain Wolf

(joint work with Xavier Blanc)

We study the homogenization of Poisson equation and Stokes system in a class of non-periodically perforated domains. The size of the perforations is denoted by ε. This microscale corresponds also to the distance between two neighbouring holes.

We consider the PDEs

$$(1) \qquad \begin{cases} -\Delta u_\varepsilon = f & \text{in} \quad \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega_\varepsilon \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta U_\varepsilon + \nabla P_\varepsilon = F & \text{in} \quad \Omega_\varepsilon \\ \operatorname{div}(U_\varepsilon) = 0 & \\ U_\varepsilon = 0 & \text{on} \quad \partial\Omega_\varepsilon, \end{cases}$$

where Ω_ε is the perforated domain. The homogenization of the equations (1) has been extensively studied when the holes are periodically distributed in space (see e.g. [1, 6] for Poisson equation and [2, 7] for Stokes system, see also [5] for a non-periodic extension). Our aim is to extend the results of the periodic case to local perturbations of this setting.

We describe in this paragraph the non-periodic setting considered in the works [3, 4]. We fix $d \geq 2$ and we denote by Q the unit cube of $\mathbb{R}^d$. If $k \in \mathbb{Z}^d$, we write $Q_k := Q + k$. We fix $\mathcal{O}_0^{\rm per} \subset\subset Q$ and we define $\mathcal{O}_k^{\rm per} := \mathcal{O}_0^{\rm per} + k$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. The periodic perforated domain is defined by $\Omega_\varepsilon^{\rm per} := \Omega \setminus \varepsilon \bigcup_{k \in Y_\varepsilon} \mathcal{O}_k^{\rm per}$, where $Y_\varepsilon := \{k \in \mathbb{Z}^d,\ \varepsilon Q_k \subset \Omega\}$. We fix a periodic distribution of holes $(\mathcal{O}_k^{\rm per})_{k\in\mathbb{Z}^d}$ and we define the non-periodic perforations by the properties **(A1)-(A2)** below:

(A1) For all $k \in \mathbb{Z}^d$, $\mathcal{O}_k \subset\subset Q_k$ and $Q_k \setminus \overline{\mathcal{O}_k}$ is connected;

(A2) There exists a sequence $(\alpha_k)_{k\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$ such that

$$\mathcal{O}_k^{\rm per,-}(\alpha_k) \subset \mathcal{O}_k \subset \mathcal{O}_k^{\rm per,+}(\alpha_k)$$

where for $\alpha > 0$, $\mathcal{O}_k^{\rm per,+}(\alpha) := \{x \in \mathbb{R}^d \text{ s.t. } \mathrm{d}(x, \mathcal{O}_k^{\rm per}) < \alpha\}$ and $\mathcal{O}_k^{\rm per,-}(\alpha) := \{x \in \mathcal{O}_k^{\rm per} \text{ s.t. } \mathrm{d}(x, \partial\mathcal{O}_k^{\rm per}) > \alpha\}$.

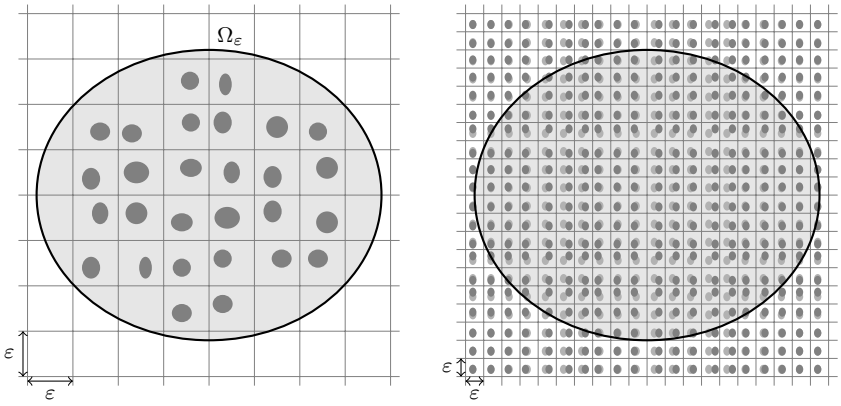

FIGURE 1. Example of domains Ω_ε for large ε (left) and small ε (right).

Let $\mathcal{O} := \bigcup_{k\in\mathbb{Z}^d} \mathcal{O}_k$ be the set of non-periodic perforations. We define $\Omega_\varepsilon := \Omega \setminus \varepsilon \bigcup_{k\in Y_\varepsilon} \mathcal{O}_k$ (see Figure 1) and we study (1) in Ω_ε. The two-scale expansions

of u_ε and $(U_\varepsilon, P_\varepsilon)$ as $\varepsilon \to 0$ yield the following corrector problems

$$(2) \qquad \begin{cases} -\Delta w = 1 \\ w_{|\partial\mathcal{O}} = 0 \end{cases} \quad \text{and} \quad \begin{cases} -\Delta W_j + \nabla P_j = e_j \\ \text{div} W_j = 0 \qquad j = 1, ..., d \\ W_{j|\partial\mathcal{O}} = 0, \end{cases}$$

which are posed in $\mathbb{R}^d \setminus \overline{\mathcal{O}}$. We first prove that the PDEs (2) are solvable. For Poisson equation, we then obtain convergence rates of u_ε to its two scale expansion in $W^{1,p}-$norms for all $1 < p < \infty$. For Stokes system, we obtain convergence rates in H^2-norms for the three dimensional case when we impose additional conditions on the force field F. We finally discuss the optimality of these convergence rates.

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From adherence to slip in nanofluidics: a mathematical justification based on a drop of viscosity

Matthieu Bonnivard

(joint work with Julien Olivier)

The no-slip boundary condition is the assumption that the layer of liquid next to a solid surface moves with the same velocity as the surface. No-slip cannot be proved based on hydrodynamic considerations, but has been observed in numerous macroscopic experiments. Hence, no-slip is the most commonly used BC for macroscopic flows. However, viscous flows in confined domains (such as carbon nanotubes) exhibit slip [1, 2]. The origin of this slip is the subject of current debate in the physics community [3, 4]. From a mathematical perspective, one successful strategy initiated in the late 1990's [5, 6, 7] to explain the occurrence of adherence or partial slip on solid walls, consists in modeling micro-asperities on the surface and analyzing their effect on the flow by an homogenisation process, imposing only a mild non penetration boundary condition on the rugous wall, *i.e.* that the normal component of the fluid velocity vanishes. This so-called "rugosity effect" has been studied quite extensively in the last decades, which has led to a

rather complete description of the asymptotic effect of rough patterns on viscous flow [8, 9, 10].

In this talk, we investigate a completely different interpretation of the apparent slip length measured in nanoscopic devices, proposed by Myers in [11], where the author postulates that the source of this slip arises from a "depletion layer with reduced viscosity near the wall". This hypothesis is supported by experimental evidence [12] and Molecular Dynamics simulations [13] bringing out that the viscosity drops near the wall of the nanotube.

The problem we consider is strongly related to the so-called reinforcement problems introduced by Sanchez-Palencia in [14], where an elastic medium is reinforced by the adjunction of a thin layer of very strong material. From a mathematical point of view, such models give rise to singular perturbation problems, where the modulus of ellipticity of the operator tends to zero in the thin layer of extra material, as the layer shrinks. Brézis, Caffarelli and Friedman solved the interior and boundary reinforcement problems for elliptic equations, in the case of Dirichlet boundary conditions on a C^2 boundary and using strong solutions in [15]. A few years later, geometric measure theory and Gamma-convergence were successfully applied to boundary reinforcement problems (see for instance Acerbi and Buttazzo [16]). In a recent preprint [17], we have proposed a different approach based on a rescaling of the solution in the depletion layer, in the spirit of the unfolding method [18, 19], and on the construction of a relevant sequence of test functions that are able to capture the asymptotic behaviour of the problem in the boundary layer associated with the region of low viscosity.

In this talk, we treat the case of a three-dimensional Stokes system. Starting from the natural energy bound associated with the problem, we obtain compactness on the rescaled velocity and pressure fields in the low viscosity layer by adapting arguments from [20], and derive the Reynolds equation for the rescaled pressure. Then, we use a sequence of well-adapted test functions, whose behaviour in the vicinity of the wall is also determined by a Reynolds equation. This procedure allows us to determine the boundary condition satisfied by the limit velocity field, which depends, as expected, on the ratio between the value of the viscosity and the thickness of the depletion layer. In this presentation, we focus on the critical case, where the previous ratio converges to a positive constant. In this setting, the boundary condition satisfied by the effective velocity field is a Navier condition that we are able to express as a function of the geometry of the depletion layer. This result shows that introducing in a viscous flow model a drop of viscosity in the vicinity of a solid wall, may provide a mathematical justification of the emergence of slip.

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Global existence for the 2D Kuramoto-Sivashinsky equation by mixing

Anna L. Mazzucato

We consider the Kuramoto-Sivashinsky equation (KSE) on a two-dimensional torus. We represent the torus by a periodic square with side lengths L_1, L_2 and denote it by $\mathbb{T}^2$ in what follows.

This equation is a model for long-wave instability in dissipative system and has been used for flame-front propagation in combustion, for example. It is a

hyperdiffusion non-linear equation for a scalar potential ϕ:

$$\partial_t \phi + \Delta^2 \phi + \Delta \phi + |\nabla \phi|^2 = 0. \tag{1}$$

While there is an extensive literature on the KSE in one space dimension, much less is known in two and higher dimensions (we refer to [2, 3, 4] and references therein for further discussion). In fact, the issue of global existence of solutions is still essentially open, as there are no known *a priori* estimates available in this case. The presence of a biharmonic operator precludes the use of the maximum principle, while the form of the non-linearity does not lead to a global Grönwall's estimate in L^p. Furthermore, there are exponentially growing modes for the linearized operator $\partial_t + \Delta^2 + \Delta$, when L_1 or $L_2 \geq 2\pi$.

Recently, using a mild formulation and dynamical system arguments, David Ambrose and the author have proved global existence of solutions to (1) with data that have a gradient in the Wiener algebra with sufficiently small norm, first in the absence of growing modes [1] and then in the presence of one growing mode in each direction [2]. However, no results to date are available with an arbitrary number of modes, even for small data. Since the instability arises at large scale, to control it one seeks to incorporate a mechanism that leads to an efficient transfer of energy from large to small scales, where hyperdiffusion dissipates it. Such a mechanism should give a global control on the L^2-norm of the solution, which in turn would allow to continue the local solution to a global one. Under some conditions, mechanism of this type is provided by adding advection to the KSE:

$$\partial_t \phi + \Delta^2 \phi + \Delta \phi + |\nabla \phi|^2 + v \cdot \nabla \phi = 0, \tag{2}$$

where v is a given, divergence-free vector field, assumed Lipschitz continuous in space uniformly in time. We will refer to this modification as the *advective* KSE (AKSE), which has been used in the literature to model flame propagation in premixed combustion. In fact, the interplay between advection and (hyper)-diffusion can lead to enhanced dissipation.

We will measure this enhancement by means of the concept of *dissipation time* τ^*. We let $\mathcal{S}_{s,t}$, $0 \leq s \leq t$, be the evolution system associated to the advection-hyperdiffusion equation:

$$\partial_t f + \Delta^2 + v \cdot \nabla f = 0.$$

The number $0 < \tau^* < \infty$ given by

$$\tau^* = \inf \left\{ t \geq 0 \ \mid \ \|\mathcal{S}_{s,s+t}\|_{L^2 \to L^2} \leq \frac{1}{2}, \text{ for all } s \geq 0 \right\}, \tag{3}$$

is called the *dissipation time* associated to the system $\mathcal{S}_{s,t}$, $0 \leq s \leq t$.
With slight abuse of notation, we will also refer to τ^* as the dissipation time of (the flow of) v, $\tau^*(v)$. Addition of a non-trivial advection term will not increase the dissipation time and it may decrease it. We introduce a parameter $A > 0$, which represents the amplitude of the flow, that is, we replace v by Av. A flow will be called *relaxation enhancing*, if $\tau^*(AV) \to 0$ for $A \to \infty$. An example is given by a (weekly) mixing, sufficiently regular ($C^2$is enough) flow. Informally, we will say that the flow Φ of v is mixing if $g(\Phi^{-1}(\cdot, t))$ converges weakly to zero in L^2 as $t \to t_{\text{mix}}$ for some $0 < t_{\text{mix}} \leq \infty$ for g in a dense subset of $L^2(\mathbb{T}^2)$. For

more examples of relaxation-enhancing flows and durther discussion of enhanced dissipation, we refer to [4].

It was shown in [4] that, if the dissipation time v is small enough, then an *a priori* uniform bound on the L^2-norm of a weak solution holds. The smallness of the dissipation time is determined by the size of the periodic square and by the size of the initial data. As a consequence, one obtains a global existence result:

Let $\phi_0 \in L^2(\mathbb{T}^2)$. There exists $\tau_0^* > 0$ such that, if the dissipation time τ^* of v satisfies $0 < \tau^* < \tau_0^*$, then there exists a unique weak solution to (2) with initial data ϕ_0 on any time interval $[0, T]$ for all $0 < T < \infty$.

Enhanced dissipation holds even if the flow of v is not mixing. In particular, it holds for steady *shear flows*, for which the advecting velocity $v(x, y) = (u(y), 0)$, under some conditions on u. In [3], a global existence result for AKSE with advection by a shear flow was obtained, if u satisfies a certain condition that implies suitable decay rates in time for $S_{s,t}$. Such condition is satisfied, for instance, by $u(y) = \sin(y)^m$, $m \geq 1$. Decay rates hold by hypocoercivity for shear flows with finitely-many critical points. In [3], pseudo-spectral estimates were employed that lead to optimal rates of decay. Fast decay rates for $S_{s,t}$ replace here the smallness of the dissipation time, but are valid only on the orthogonal complement to the kernel of the advection operator $v \cdot \nabla$. However, transfer of energy to high-frequency modes still holds in this case, as all modes are coupled by the nonlinearity.

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Sedimentation of random suspensions in Stokes fluids

Mitia Duerinckx

(joint work with Antoine Gloria)

We consider a system of rigid particles settling under gravity g in a 3D Stokes fluid, and we study the collective bulk sedimentation process. More precisely, we focus for simplicity on a quasi-static statistical description: given a translation-invariant statistical ensemble of particle positions $\{x_{n,L}\}_n$ in a large tank of size L, the Stokes equations allow to compute the corresponding velocities $\{V_{n,L}\}_n$, cf. (1)&(2) below, and we then investigate

- the mean settling speed $\bar{V}_L := \frac{g}{|g|} \cdot \mathbb{E}[V_{1,L}]$,
- velocity fluctuations $\sigma_L^2 := \mathrm{Var}[V_{1,L}]$,

in the large-volume limit $L \uparrow \infty$ with fixed particle volume fraction. Particles interact via the fluid flow that they generate, and the difficulty of the problem is related to the very nature of hydrodynamic interactions.

(I) Interactions are *long-range*: the flow disturbance at x due to a particle at y decays only as $O(|x-y|^{-1})$ as $|x-y| \uparrow \infty$. This leads to various summability issues.
(II) Interactions are *multi-body*: the force on each particle cannot be decomposed as a sum of contributions of each other particle, but it depends on all particles and their microstructure at once.

Previous rigorous contributions on the topic, e.g. [11, 13, 10], as well as most physicists' formal calculations, e.g. [9] and references therein, focus on the dilute regime: particle interactions can then be replaced by long-range pairwise interactions in form of a point-force approximation via the Stokeslet, so that difficulty (II) above is fully lifted. In the present talk, in contrast, we focus on the rigorous treatment of multi-body hydrodynamic interactions away from dilute regime, using tools from homogenization theory.

We start by describing predictions from the physics literature. The mean settling speed $\bar{V}_L$ was first analyzed by Batchelor [2] and was formally shown to be uniformly bounded in the large-volume limit, $\bar{V}_L = O(1)$. As long-range particle contributions are not summable, Batchelor had to use a suitable renormalization. The analysis of velocity fluctuations is more subtle: a celebrated calculation by Caflisch and Luke [4] showed that, for a Poisson ensemble of particle positions, velocity fluctuations should diverge linearly in the size of the tank, $\sigma_L^2 = O(L)$, which contradicts intuition, and Koch and Shaqfeh [12] later showed that fluctuations are bounded if and only if the ensemble of particle positions is hyperuniform. In experiments, it is observed that, after initial mixing of the suspension, fluctuations follow the Caflisch–Luke divergent prediction, but these large-scale fluctuations are transient: fluctuations become bounded in a steady-state plateau regime before the arrival of the upper sedimentation front. A consensus is still lacking on how to explain this plateau regime: the arisal of hyperuniformity has been invalidated by some experimentalists, and other screening mechanisms have been proposed — taking into account other effects such as inertia, density stratification, or side walls, see e.g. [3, 9].

Let us introduce the fluid equations describing the system. The large tank is chosen for simplicity as the cube $Q_L = [-\frac{L}{2}, \frac{L}{2}]^3$ with periodic boundary conditions. We denote by $\{B_{n,L}\}_n$ the ensemble of particles centered at the points $\{x_{n,L}\}_n$, with volume fraction $\lambda_L = L^{-d} \sum_n |B_{n,L}|$, we denote by ϕ_L the velocity field of the Stokes fluid, and we assume no-slip conditions at particle boundaries. The fluid

equations then read as follows for $\phi_L \in H^1(Q_L)^3/\mathbb{R}^3$,

$$(1)\qquad \begin{cases} -\triangle\phi_L + \nabla\Pi_L = -\alpha_L g, & \text{in } Q_L \setminus \cup_n B_{n,L}, \\ \operatorname{div}(\phi_L) = 0, & \text{in } Q_L \setminus \cup_n B_{n,L}, \\ D(\phi_L) = 0, & \text{in } \cup_n B_{n,L}, \\ g|B_{n,L}| + \int_{\partial B_{n,L}} \sigma(\phi_L,\Pi_L)\nu = 0, & \forall n, \\ \int_{\partial B_{n,L}} (x - x_{n,L}) \times \sigma(\phi_L,\Pi_L)\nu = 0, & \forall n, \end{cases}$$

where $\alpha_L g = \lambda_L(1-\lambda_L)^{-1} g$ is the constant backflow, where $D(\phi_L)$ stands for the symmetric gradient, and where $\sigma(\phi_L,\Pi_L) = 2D(\phi_L) - \Pi_L \operatorname{Id}$ is the Cauchy stress tensor. Note that the condition $D(\phi_L) = 0$ in $\cup_n B_{n,L}$ expresses the rigidity of particles, and the last two conditions in (1) are the balance of forces and torques at particle boundaries. Particle velocities are then computed as

$$(2)\qquad V_{n,L} = |B_{n,L}|^{-1} \int_{B_{n,L}} \phi_L.$$

In a non-perturbative non-dilute setting, $\lambda_L = O(1)$, we establish the following first rigorous version of physicists' predictions [2, 4, 12].

Main Theorem (see [6, Theorems 1&2] for a precise statement)**.**

(i) *If $\{B_{n,L}\}_n$ has integrable correlations, then we have $\bar{V}_L = O(1)$.*
(ii) *Under an additional mixing assumption, we have $\sigma_L^2 = O(L)$.*
(iii) *Under an additional hyperuniformity assumption, we have $\sigma_L^2 = O(1)$.*

While the fluid equation (1) looks highly intricate, its structure is simplified by the following observation [13]: we can write ϕ_L as $\phi_L = \pi_L \phi_L^\circ$ where

- ϕ_L° captures the linear response as if particles only had pairwise interactions via the fluid: it is the solution of
$$-\triangle\phi_L^\circ + \nabla\Pi_L^\circ = (1-\lambda_L)^{-1}(1_{\cup_n B_{n,L}} - \lambda_L)g, \qquad \operatorname{div}(\phi_L^\circ), \qquad \text{in } Q_L,$$
where gravity is viewed as simply creating internal forces in the fluid at the location of the different particles.
- π_L captures the multi-body character of hydrodynamic interactions: it is the orthogonal projection of $\dot{H}^1_{\operatorname{div}}(Q_L)$ onto its subspace $\{\phi \in \dot{H}^1_{\operatorname{div}}(Q_L) : D(\phi) = 0 \text{ in } \cup_n B_{n,L}\}$.

Noting that the energy identity for (1) reads $\mathbb{E}[|\nabla\phi_L|^2] = \alpha_L |g| \bar{V}_L$, the projection property gives $\alpha_L |g| \bar{V}_L \le \mathbb{E}[|\nabla\phi_L^\circ|^2]$. The estimation of the mean settling speed is therefore reduced to that for the linear response, for which stochastic cancellations are explicit and easily yield the conclusion (i).

The proof of (ii)&(iii) for velocity fluctuations is more involved. We note that $\sigma_L^2 \sim \mathbb{E}[|\phi_L|^2]$, and by integration we can reduce to unraveling the fluctuation scaling of large-scale averages of the gradient $\nabla\phi_L$. While the fluctuation scaling is clear for the linear response $\nabla\phi_L^\circ$, it remains to study how it is transformed by the projection π_L. For that, we take inspiration from the recent quantitative theory of homogenization developed for divergence-form elliptic problems, e.g. [1, 8]. The relation with the present fluid problem with rigid inclusions is explained by the following simple observation.

Lemma. *Consider a Stokes fluid in Q_L with droplets $\{B_{n,L}\}_n$ of another fluid with shear viscosity κ,*

$$\begin{cases} -\operatorname{div}\big(2(1+(\kappa-1)1_{\cup_n B_{n,L}})D(\phi_L^\kappa)\big)+\nabla\Pi_L^\kappa \;=\; (1-\lambda_L)^{-1}(1_{\cup_n B_{n,L}}-\lambda_L)g,\\ \operatorname{div}(\phi_L^\kappa)=0. \end{cases}$$

Then $\nabla\phi_L^\kappa \rightharpoonup \nabla\phi_L$ in $L^2(Q_L)$ as $\kappa\uparrow\infty$.

Using this analogy, we manage to adapt the theory available for divergence-form elliptic problems [1, 8] to our incompressible fluid setting with rigid inclusions. We first show in [5] that the projection π_L on large scales amounts to replacing the fluid viscosity by an effective value. Making this homogenization result quantitative then allows us to derive in [7] a large-scale regularity result for π_L, which happens to be the key technical ingredient for the proof of (ii)&(iii).

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Boundary layer behavior near rough coasts: a wind-driven ocean circulation model

Gabriela Lopez-Ruiz

This talk is concerned with boundary layer formation near rough coasts in the case of the homogeneous wind-driven circulation model, also know, as quasi-geostrophic model. In numerous real-life phenomena not all physical processes are influential at all scales of magnitude. Some are negligible at certain scales and important at

others, and ocean circulation is not an exception. We are interested in the effect of roughness for large-scale oceanic motion since it has been recognized that the small irregularities of the solid surface can alter deeply various aspects of the fluid dynamics.

The quasi-geostrophic system is a well-known model in meteorology, mainly in the study of oceanic circulation. This model describes the external force applied by the wind on the free surface of the ocean. In other to simplify as possible the presentation, we focus our attention on a linear and stationary version of this problem to describe the stream function's behavior. In a simply connected domain $\Omega \subset \mathbb{R}^2$, the system reads

$$\left\{ \begin{array}{rcl} \varepsilon^{-3}\partial_x \Psi - \Delta^2 \Psi = & \varepsilon^{-3} \operatorname{curl} \tau & \text{in } \Omega^\varepsilon \\ \Psi|_{\partial\Omega^\varepsilon} & = & \frac{\partial \Psi}{\partial n}|_{\partial\Omega^\varepsilon} = 0, \end{array} \right. \tag{1}$$

Here, τ denotes the stress tensor resulting from the action of the wind, ε is a small parameter linked to the Rossby number and is proportionally inverse to the beta-plane parameter in the linearization of the Coriolis force ($\beta = \varepsilon^{-3}$).

When $\varepsilon \to 0$, the behavior in the boundary layer is very singular. Mathematically, these problems are often tackled by a multi-scale approach in the limit of small ε. The goal is to construct an approximate solution that is close to the one of the original problem when $\epsilon \to 0$ with coefficients depending on the global variables t, x, y, and the fast variables $Y = Y(y, \varepsilon)$, $X = X(x, y, \varepsilon)$

$$\Psi^\varepsilon_{\text{app}}(t, x, y) \sim \sum_{k=0}^{\infty} \varepsilon^k \left(\Psi^k_{int}(t, x, y) + \Psi^k_{bl}(t, y, X, Y) \right), \tag{2}$$

where $\Psi^k_{int}(t, x, y)$ correspond to the interior terms, while Ψ^k_{bl} refer to the corrector terms in the boundary layer.

Desjardins and E. Grenier proved this model determined the behavior of a 2D fluid [5] and then, performed a complete boundary layer analysis when

$$\Omega^\varepsilon = \{ \chi_w(y) \le x \le \chi_e(y),\ y \in [y_{min}, y_{max}] \},$$

where χ_w and χ_e are smooth functions. Their results were later generalized by D. Bresch and D. Gérard-Varet [2] for the case when rough boundaries are added to domain, i.e.,

$$\Omega^\varepsilon = \left\{ (x, y) \in \mathbb{R}^2 : \chi_w(y) - \varepsilon\gamma_w(\varepsilon^{-1}y) < x < \chi_e(y) + \varepsilon\gamma_e(\varepsilon^{-1}y),\ y_{min} \le y \le y_{max} \right\}.$$

In this case, the functions describing the roughness γ_e and γ_w were considered to be periodic and the irregularity of size ε. The periodicity hypotheses is quite usual in mechanics since they simplify the physical phenomena while remaining true for most relevant cases. The assumption on the size of the characteristic roughness is purely mathematical. With the added roughness, the boundary layer are described by elliptic partial differential equations: non-linear and quasi-linear, in the western domain and linear, in the East. A natural extension of this work is dropping the periodicity assumption and considering γ_e, γ_w to be regular arbitrary functions. This makes more sense from a physics stand-point since the geometry of the coast is not meant to follow a particular spatial pattern. As a consequence, the boundary layer domains are now infinite, thus, we look for the

solutions Kato spaces H^s_{uloc}. The analysis becomes more involved than the periodic case, as shown in [1, 6, 4, 3]. Moreover, the absence of compactness both in the tangential and transverse variables and the presence of singularities at low frequencies for the eastern boundary layer functions make proving convergence in a deterministic setting extremely difficult. We therefore use the ergodic theorem to specify the behavior of the solution of the eastern boundary layer far from the boundary and later, to find the energy estimates in the analysis of the quality of the approximation.

To simplify the presentation even further, we consider χ_w, χ_e to be constant functions. For example, assume $\chi_w(y) = C_1$ and $\chi_e(y) = C_2$, for $y \in [y_{min}, y_{max}]$.

The first approximation yields

$$\begin{aligned} \Psi_{int}(t,x,y) &= -\int_x^{C_2} \operatorname{curl} \tau(t,x',y)dx' \quad \text{in} \quad \Omega \\ &= 0 \qquad \text{in} \quad \Omega^\varepsilon \setminus \Omega. \end{aligned}$$

The singular nature of the equation driving the eastern boundary layer's behavior influence our choice of boundary condition for the interior profile and in turn, determines the solution of the eastern boundary layer

$$\begin{cases} -\partial_X \Psi^0_e - \Delta^2_e \Psi^0_e &= 0, \quad \text{in} \quad \omega^+_e \cup \omega^-_e \\ \left[\partial^k_X \Psi^0_e\right]\big|_{X=0} &= 0, \quad k = 0, \dots, 3, \\ \Psi^0_e\big|_{X=-\gamma_e(Y)} = \dfrac{\partial \Psi^0_e}{\partial n_e}\big|_{X=-\gamma_e(Y)} &= 0, \\ \Psi^0_e \longrightarrow 0 \qquad & \text{when} \quad X \to \infty, \end{cases}$$

is equal to zero. The western boundary layer, on the other hand, satisfies

$$\begin{cases} \partial_X \Psi^0_w - \Delta^2_w \Psi^0_w &= 0, \quad \text{in} \quad \omega^+_w \cup \omega^-_w \\ \left[\Psi^0_w\right]\big|_{X=0} &= \phi, \\ \left[\partial^k_X \Psi^0_w\right]\big|_{X=0} &= 0, \quad k = 1, \dots, 3, \\ \Psi^0_w\big|_{X=-\gamma_w(Y)} = \dfrac{\partial \Psi^0_w}{\partial n_w}\big|_{X=-\gamma_w(Y)} &= 0, \\ \Psi^0_w \longrightarrow 0 \qquad & \text{when} \quad X \to \infty, \end{cases} \tag{3}$$

where the western boundary layer domain $\omega_w = \omega^+_w \cup \sigma_w \cup \omega^-_w$ is defined by

$$\begin{aligned} \omega^+_w &= \{X > 0, \quad Y \in \mathbb{R}\}, \qquad \sigma_w = \{X = 0, \quad Y \in \mathbb{R}\} \\ \omega^-_w &= \{-\gamma_w(Y) < X < 0, \quad Y \in \mathbb{R}\}. \end{aligned}$$

The eastern boundary layer domain ω_e can be described similarly.

Solving problem (3) involves the idea introduced by [6] which draws inspiration from the works of Ladyženskaya and Solonnikov [7]. In particular, we impose a so-called transparent boundary condition when the variable in the normal direction is equal to a certain value $M > 0$. Then, use a priori estimates corresponding to the solution of the problem in the half-space to define Poincaré-Steklov type operators which are later used to solve an equivalent problem. This leads ultimately to

showing (3) has a unique solution in H^2_{uloc} decaying exponentially far from the boundary.

The next profiles in (2) are built inductively. At order ε^n the interior profile satisfies $\Psi^n_{int} = C^n(t,y) - \int_x^{\chi_e(y)} F_n \, dx$, where F_n depends on Ψ^j_{int}, $j \leq n-1$. The value of $C^n(t,y)$ will be specified later. We prove that higher profiles of the western boundary layer meeting the conditions

$$\begin{cases} \partial_X \Psi^n_w - \Delta^2_w \Psi^n_w &= 0, \quad \text{in } \omega^-_w \cup \omega^+_w, \\ \left[\partial^k_X \Psi^n_w\right]\big|_{\sigma_w} &= -\left[\partial_x \Psi^{n-k}_{int}\right]\big|_{x=\chi_w(y)} - \left[\partial^k_X \Psi^n_e\right]\big|_{\sigma_w}, \quad k = 0, \dots, 3, \\ \Psi^n_w\big|_{X=-\gamma_w(Y)} = 0, & \quad \frac{\partial \Psi^n_w}{\partial n_w}\big|_{X=-\gamma_w(Y)} = 0, \end{cases}$$

are well-posed and uniformly bounded using the same reasoning. Conversely, the eastern boundary layer Ψ^n_e, $n \geq 1$,

$$\begin{aligned} -\partial_X \Psi^n_e - \Delta^2_e \Psi^n_e = 0, \quad \text{in} \quad \omega^-_e \cup \omega^+_e \\ \left[\Psi^n_e\right]\big|_{\sigma_e} = C^n(t,y), \\ \left[\partial^k_X \Psi^n_e\right]\big|_{\sigma_e} = g_k, \ k = 1, \dots, 3, \\ \Psi^n_e\big|_{X=-\gamma_e(Y)} = \frac{\partial \Psi^n_e}{\partial n_e}\big|_{X=-\gamma_e(Y)} = 0, \end{aligned} \tag{4}$$

where $g_k \in W^{2-k,\infty}(\mathbb{R})$, presents singularities at low frequencies far from the eastern boundary. Consequently, they impact the behavior of the western boundary layer function mainly through the jump at the interface with the interior domain. In this context we distinguish three components of Ψ^n_e with different asymptotic behaviors far from the boundary: one, decaying exponentially; another, decaying to zero in a slower manner with polynomial weight and the last element, converging thanks to the addition of probabilistic information (ergodic properties).

The core of the talk is consecrated to showing:

Theorem (LR, 2021)**.** *Let $\varepsilon > 0$ and (P, Π, μ) be a probability space where μ is a the probability measure preserved by the translation group (τ_Y) acting on P. Assume that for $m \in P$: $\omega_e(m) = \left\{(X,Y) \in \mathbb{R}^2 : \ X < \varepsilon \gamma_e(m,Y)\right\}$ and γ_e an ergodic stationary random process, K-Lipschitz almost surely, for some $K > 0$. Then, there exist a unique measurable map $C^n(t,y)$ such that problem* (4) *has a unique solution $\Psi^n_e = \Psi^n_{\text{exp}} + \Psi^n_{\text{alg}} + \Psi^n_{\text{erg}}$ where*

1. $\left\|\Psi^n_{\text{erg}}\right\|_{L^q(\omega^+_e)} \xrightarrow[X \to +\infty]{} 0$, *locally uniformly in Y, almost surely and in $L^q(P)$ for all finite q,*
2. *there exist constants $\delta, C > 0$ such that*

$$\|e^{\delta X} \Psi^n_{\text{exp}}\|_{L^\infty(\omega^+_e)} \leq C \left(\sum_{k=1}^{3} \|g_k\|_{W^{1,\infty}} + \|C\|_{W^{2,\infty}} \right),$$

(3) *there exists a constant* $C > 0$ *such that*

$$\|(1+X)^{-1/4}\Psi^n_{\mathrm{alg}}\|_{L^\infty(\omega_e^+)} \le C\left(\sum_{k=1}^{3}\|g_k\|_{W^{2,\infty}} + \|C\|_{W^{2,\infty}}\right).$$

Moreover, Ψ_e *satisfies*

$$\|\Psi^n_e\|_{H^2_{\mathrm{uloc}}(\omega_e)} < +\infty, \quad \textit{almost surely.}$$

This result is part of the speaker's research paper that is currently in preparation.

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Upscaling reactive flow and transport in evolving porous media

Nadja Ray

(joint work with Peter Frolkovič, Stephan Gärttner, Peter Knabner, Jens Oberlander, Raphael Schulz)

We consider a pore-scale model for reactive flow and transport in an evolving porous medium. The transport of a concentration c_ε is given by

$$\begin{aligned} \partial_t c_\varepsilon - \nabla\cdot(-v_\varepsilon c_\varepsilon + D_m\nabla c_\varepsilon) &= 0, \quad x\in\Omega_\varepsilon(t), \quad t\in(0,T),\\ (-v_\varepsilon c_\varepsilon + D_m\nabla c_\varepsilon)\cdot\nu_\varepsilon - \varepsilon\alpha f(c_\varepsilon,\rho)(c_\varepsilon-\rho) &= 0, \quad x\in\Gamma_\varepsilon(t), \quad t\in(0,T) \end{aligned}$$

with molecular diffusion D_m, (given) advective velocity v_ε, heterogeneous reaction $f(c_\varepsilon,\rho)$, mineral density ρ, and $\alpha = 1/\rho$.

The advective velocity v_ε and pressure p_ε are determined by the Stokes equations

$$\begin{aligned} \mu\Delta v_\varepsilon - \nabla p_\varepsilon &= 0, && x\in\Omega_\varepsilon(t), && t\in(0,T),\\ \nabla\cdot v_\varepsilon &= 0, && x\in\Omega_\varepsilon(t), && t\in(0,T),\\ v_\varepsilon &= v_{n,\varepsilon}\nu_\varepsilon, && x\in\Gamma_\varepsilon(t), && t\in(0,T) \end{aligned}$$

with dynamic viscosity μ and interface normal velocity $v_{n,\varepsilon}\sim f(c_\varepsilon,\rho)$.

The evolution of the microstructure is described by the levelset equation for the level-set L_ε

$$\partial_t L_\varepsilon + v_{n,\varepsilon}|\nabla L_\varepsilon| = 0 \qquad\qquad x \in \Omega$$

with normal velocity of the interface $v_{n,\varepsilon} \sim f(c_\varepsilon, \rho)$.

Applying formal two-scale asymptotic expansion in a level-set framework, an effective micro-macro model is derived [5] resulting in flow and transport equations on the macroscopic scale. These include effective hydrodynamic parameters calculated from representative unit cells. The transport equation for the macroscopic concentration c_0 is given by

$$\partial_t(\theta c_0) + \nabla_x \cdot (\bar{v}_0 c_0) - \nabla_x \cdot (D\nabla_x c_0) = -\sigma f(c_0, \rho) \qquad x \in \Omega,\ t \in (0,T)$$

with time- and space-dependent porosity $\theta = |Y_{l,0}(t,x)|/|Y|$, specific surface $\sigma = |\Gamma_0(t,x)|/|Y|$, and diffusion tensor $D_{ij} := \int_{Y_{l,0}(t,x)} D_m\,(\partial_{y_i}\zeta_j + \delta_{ij})\ dy$.

The velocity field $\bar{v}_0$ and pressure p_0 are calculated from Darcy's equation

$$\begin{aligned} \bar{v}_0 &= -\frac{K}{\mu}\nabla p_0, && x \in \Omega,\ t \in (0,T), \\ \nabla \cdot \bar{v}_0 &= \partial_t \theta && x \in \Omega,\ t \in (0,T) \end{aligned}$$

with dynamic viscosity μ and time- and space-dependent permeability tensor $K_{i,j} := \int_{Y_{l,0}(t,x)} \omega_j^i\ dy$.

The auxiliary cell problems are defined on evolving microscopic geometries. For the diffusion tensor they read

$$\begin{aligned} -\nabla_y \cdot (\nabla_y \zeta_j) &= 0 && y \in Y_{l,0}(t,x), \\ \nabla_y \zeta_j \cdot \nu_0 &= -e_j \cdot \nu_0 && y \in \Gamma_0(t,x), \end{aligned}$$

$$\zeta_j \text{ periodic in } y, \quad \int_{Y_{l,0}(t,x)} \zeta_j\ dy = 0$$

and for the permeability tensor

$$\begin{aligned} -\Delta_y \omega_j + \nabla_y \pi_j &= e_j && y \in Y_{l,0}(t,x), \\ \nabla_y \cdot \omega_j &= 0 && y \in Y_{l,0}(t,x), \\ \omega_j &= 0 && y \in \Gamma_0(t,x), \end{aligned}$$

$$\omega_j, \pi_j \text{ periodic in } y, \quad \int_{Y_{l,0}(t,x)} \pi_j\ dy = 0.$$

Finally, the macroscopic solutes' concentrations alter the unit cells' geometrical structure by triggering dissolution or precipitation processes:

$$\partial_t L_0 + v_{n,0}|\nabla_y L_0| = 0 \qquad (y,x) \in Y \times \Omega,\ t \in (0,T)$$

with $Y_{l,0}(t,x) := \{y : L_0(t,x,y) < 0\}$, $\Gamma_0(t,x) := \{y : L_0(t,x,y) = 0\}$, and $v_{n,0} \sim f(c_0, \rho)$.

We emphasize the potential degeneration of the hydrodynamic parameters with examples taken from [4]. Moreover, we numerically investigate the dissolution of an array of dolomite grains and compare the results of the effective model with a pore

scale model, cf. [3]. Following [2], we discuss the unique solvability of a regularized, but degenerate subproblem adapting the ideas from [6]. To this end, we assume that instead of the level-set equation the time- and space-dependent porosity field is prescribed and all further hydrodynamic parameters are given functions in terms of the porosity. Finally, we extend the modeling and numerics to the situation of two competing dissolving/precipitating mineral phases as outlined in [1]. In doing so, the applicability of machine learning techniques is evaluated.

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Wave Propagation in Random Media: Beyond Gaussian Statistics

Josselin Garnier

In this talk we consider wave propagation in random media. We consider both random open media and random open waveguides. In both cases an asymptotic analysis based on a separation of scales technique [3] makes it possible to compute the statistics of the wave field.

In random open media the wave field can be characterized in the random paraxial regime. In this regime the wavelength is smaller than the correlation length of the medium and the beam radius, which are themselves smaller than the typical propagation distance [6]. The mean or coherent wave then decays exponentially with the propagation distance and the mean Wigner transform (the partial Fourier transform of the two-point covariance function of the wave field) satisfies a radiative transfer equation. The fourth-order moment analysis [7] also reveals that the statistics of the wave field behaves as a Gaussian process, in the sense that the fourth-order moments satisfy the Isserlis formula and the scintillation index (the relative variance of the intensity) goes to one for large propagation distances.

In random open waveguides the wave field can be expanded on the complete set of the modes of the unperturbed, ideal waveguide. When random perturbations affect the index of refraction within the core of the waveguide or the geometry of the core boundary, the mean guided mode amplitudes decay exponentially and the mean guided mode powers satisfy a coupled mode equation, which can be interpreted as a discrete form of the radiative transfer equation [1, 5]. The coupling between guided and radiating modes also induces effective losses for the mean

guided mode powers through an irreversible transfer of power towards the radiating modes [4]. The fourth-order moment analysis also reveals that the fluctuations of the guided mode powers grow exponentially with the propagation distance [4]. This is, therefore, in contrast with the situation in open random medium.

The comparison of the two results makes it possible to discuss the physical conjecture that claims that the wave field acquires Gaussian statistics when it propagates over a large distance in a weakly randomly scattering medium. Under certain circumstances, the conjecture turns out to be true, and then the picture is clear and simple: the wave field has Gaussian statistics, it is fully characterized by its mean that is zero and by its covariance function or equivalently by its mean Wigner transform that satisfies a radiative transfer equation. Under other circumstances, however, the conjecture may be wrong. This observation opens new perspectives and new challenges as the situation requires a deeper analysis, that could have impact in communication and imaging [2].

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