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Geometric Methods of Complex Analysis (hybrid meeting)

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Abstract. The purpose of this workshop was to discuss recent results in Several Complex Variables, Complex Geometry and Complex Dynamical Systems with a special focus on the exchange of ideas and methods among these areas. The main topics of the workshop included Holomorphic Dynamics and Nevanlinna's Theory; L^2-methods and Cohomologies; Plurisubharmonic Functions and Pluripotential Theory; Geometric Questions of Complex Analysis.

Mathematics Subject Classification (2010): 32xx, 53xx, 14xx, 37xx.

Introduction by the Organizers

The workshop *Geometric Methods of Complex Analysis* this time, in view of the corona pandemy, took place as a hybrid meeting. It has attracted 53 researchers from 14 countries. Both, leading experts in the field and young researchers (including one Ph. D. student and three postdocs) were represented in the meeting. There was 10 female researchers among the participants of the workshop. A rather wide spectrum of topics related to Complex Analysis (and this was one of the aims of the workshop) was covered by the talks and following after them informal discussions. All 19 lectures presented on the meeting can be conditionally divided into the following groups.

Holomorphic Dynamics and Nevanlinna's Theory were represented by the talks of T.-C. Dihn and N. Sibony. Dihn discussed the theory of random walks on the group $\mathrm{G} = \mathrm{SL}_2(\mathbb{C})$ acting by linear transformations on the complex projective line $\mathbb{P}^1$. In this setting new versions of the Local Limit Theorems for the norm cocycle and for the random matrices under the optimal moment conditions were established. Sibony presented some analogies between holomorphic dynamical systems and equidistribution problems in Nevanlinna's Theory. He explained that under some natural assumptions on a non-degenerate holomorphic map from an open complex manifold with a good p.s.h. exhaustion function to a compact complex manifold the Valiron defect is zero except for a pluri-polar set of parameters.

L^2-methods and Cohomologies were represented by the talks of X. Zhou, J.-P. Demailly, X. Wang, B.-Y. Chen, T. Ohsawa and N. Tardini. Zhou gave some generalizations of Siu's lemma related to multiplier ideal sheaves and explained their applications in some problems related to L^2 extension in Kähler case, comparison between singular metrics on the twisted relative pluricanonical bundles, subadditivity of the generalized Kodaira-Iitaka's dimensions and extension of cohomology classes. Demailly presented new results on the existence of differential equations that strongly restrain the locus of entire curves in the general context of foliated or directed varieties, under appropriate semistability conditions. These results are closely related to the famous Green-Griffiths-Lang conjecture. Wang explained how to obtain an explicit estimate of the Bergman kernel for positive line bundles. Chen presented results that generalize classical Bergman theory to the case of L^p-spaces. Ohsawa explained how Hörmander's method of comparing L^2 cohomology groups with respect to different weights can be revisited and refined to deduce some extension theorems and approximation theorems of new type. Tardini showed that on a compact 4-dimensional almost-complex manifold X the Hodge number $h^{1,1}_{\overline{\partial}}(X)$ is a topological invariant for locally conformally Kähler and globally conformally Kähler metrics.

Plurisubharmonic Functions and Pluripotential Theory were represented by the talks of E. Di Nezza, E. Wulcan, V. Guedj, D. Witt Nyström, S. Kolodziej, L. Lempert, B. Stensønes and Z. Błocki. Di Nezza presented results on the behavior of the Monge-Ampère measures on contact sets. These results were motivated by the study of geodesics in the space of Kähler metrics and the transcendental holomorphic Morse inequalities on projective manifolds. Wulcan explained how to define a Monge-Ampère operator with nice continuity properties for a large class of (in particular not locally bounded) plurisubharmonic functions. These results extend in a natural way earlier results of Demailly, Cegrell and Błocki. Guedj presented a systematic study of quasi-plurisubharmonic potentials whose Monge-Ampère measures have finite entropy. He explained that these potentials belong to the finite energy class $\mathcal{E}^{\frac{n}{n-1}}$, where n denotes the complex dimension, and gave examples showing that this critical exponent is sharp. Witt Nyström showed how the Hele-Shaw flows can be used to produce examples of the solutions to the Dirichlet problem for the homogeneous complex Monge-Ampère equation which fail to have C^2-regularity. Kolodziej presented results on the existence of a continuous quasi-plurisubharmonic solution to Monge-Ampère equations with very general right hand side on a compact Hermitian manifold. As a consequence, he gave a characterization of measures admitting Hölder continuous quasi-plurisubharmonic potential, inspired from the work of Dinh-Nguyen. Lempert explained how to generalize earlier results of Darvas on geodesics in the space of relative Kähler potentials to Lagrangians that are fiberwise convex, continuous, and invariant under parallel translation. Stensønes presented results on domains in $\mathbb{C}^n$, $n \geq 3$. with real analytic boundary which can be bumped to type. In particular, she explained that if $\Omega \subset\subset \mathbb{C}^n$ is pseudoconvex and can be bumped to type at every boundary point, then we get supnorm estimates for $\overline{\partial}$. She also presented results in the case of dimension $n = 3$, which asserts that Ω can be bumped to type. Błocki discussed results and problems related to interior regularity of solutions of the Dirichlet problem for the real and complex Monge-Ampère equation for domains which are not necessarily strongly pseudoconvex (namely, for non-smooth bounded admissible domains). He has presented some partial results in this direction as well as some recent work in progress.

Geometric Questions of Complex Analysis were represented by the talks of F. Forstnerič, B. Jöricke and A. Zimmer. Forstnerič explained how to establish precise estimates of derivatives and the rate of growth of conformal harmonic maps from hyperbolic conformal surfaces into the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$ for any $n \geq 3$. Such maps parameterize minimal surfaces, objects of high interest in geometry. As a special case of his main result one gets a generalization of the *Schwarz–Pick lemma*, due to H. A. Schwarz (1869), H. Poincaré (1884), and G. A. Pick (1915), to a much larger class of maps. Jöricke discussed the question of the restricted validity of Gromov's Oka principle and obstructions to this principle in case the target is not a Gromov-Oka manifold. She also explained how to get upper and lower bounds (differing by multiplicative constants) of the conformal modules of conjugacy classes of elements of $\pi_1(\mathbb{C} \setminus \{-1, 1\}, 0)$ introduced by Gromov, by quantities that are expressed in terms of certain representing words and are easy to compute. Zimmer defined a new notion of a domain $\Omega \subset \mathbb{C}^d$ with *bounded intrinsic geometry.* He then explained how to generalize a classical result of Fu-Straube on compactness of the $\bar{\partial}$-Nuemann operator N_q on $(0, q)$-forms for bounded convex domains to the case of domains with *bounded intrinsic geometry.* Zimmer also presented a generalization to the defined by him class of domains of Li's theorem which characterizes the symbols in $L^2(\Omega)$ for which the associated Hankel operator is compact.

Workshop (hybrid meeting): Geometric Methods of Complex Analysis

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Abstracts

Siu's lemma: generalizations and applications

Xiangyu Zhou
(joint work with Langfeng Zhu)

Siu's lemma related to multiplier ideal sheaves (m.i.s.), which is coined by Phong and Sturm, plays an important role in Siu's work on some open problems in complex algebraic geometry. In this talk, we'll present some generalizations and their applications in some problems related to L^2 extension in Kähler case, comparison between singular metrics on the twisted relative pluricanonical bundles, subadditivity of the generalized Kodaira-Iitaka's dimensions with m.i.s., and extension of cohomology classes. This is joint work with Langfeng Zhu.

Siu's lemma [6]***:*** *Let $\varphi(z)$ be a nonpositive plurisubharmonic function on $\mathbb{B}^1_r \times \mathbb{B}^{n-1}_r$ such that*

$$I_\varphi := \int_{(z_2,\cdots,z_n)\in\mathbb{B}^{n-1}_r} e^{-\varphi(0,z_2,\cdots,z_n)} d\lambda_{n-1} < +\infty,$$

Assume that $r_1 \in (0,r)$. Then there exists a positive number C independent of φ, such that

$$\varliminf_{z_1\to 0} \int_{(z_2,\cdots,z_n)\in\mathbb{B}^{n-1}_{r_1}} e^{-\varphi(z_1,z_2,\cdots,z_n)} d\lambda_{n-1} \leq C I_\varphi.$$

Using Ohsawa-Takegoshi L^2 extension theorem movably which is a key idea in the proof of the Demailly's strong openness conjecture ([4]), Zhou and Zhu prove **generalized Siu's lemma with trivial m.i.s** [7]**:**

Let $\varphi(z',z'')$ be a plurisubharmonic function, $P(z',z'')$ be a nonnegative continuous function on $\mathbb{B}^1_r \times \mathbb{B}^{n-1}_r$.

$$\begin{aligned}&\lim_{\varepsilon\to 0^+} \frac{1}{\mu(\mathbb{B}^1_\varepsilon)} \int_{\mathbb{B}^1_\varepsilon \times \mathbb{B}^{n-1}_r} P(z',z'') e^{-\varphi(z',z'')} d\lambda_n \\ = & \int_{z''\in\mathbb{B}^{n-1}_r} P(0,z'') e^{-\varphi(0,z'')} d\lambda_{n-1}\end{aligned}$$

The above implies Siu's Lemma with $C = 1$.

Using the strong openness property of multiplier ideal sheaves ([4]), Zhou and Zhu prove: **generalized Siu's lemma with nontrivial m.i.s.**

Assume that

$$I_{f,\varphi} := \int_{z''\in\mathbb{B}^{n-m}_r} |f(z'')|^2 e^{-\varphi(0,z'')} d\lambda_{n-m} < +\infty$$

Assume that $\varepsilon, r_1, r_2 \in (0,r)$ and $r_1 < r_2$.

Then there exists a holomorphic function $F(z',z'')$ on $\mathbb{B}^m_{r_2} \times \mathbb{B}^{n-m}_{r_2}$ $(1 \leq m \leq n)$ such that $F(0,z'') = f(z'')$ on $\mathbb{B}^{n-m}_{r_2}$,

$$\int_{\mathbb{B}^m_{r_2}\times\mathbb{B}^{n-m}_{r_2}} |F(z',z'')|^2 e^{-\varphi(z',z'')} d\lambda_n < +\infty,$$

and

$$\lim_{\varepsilon\to 0^+} \frac{1}{\lambda(\mathbb{B}^m_\varepsilon)} \int_{\mathbb{B}^m_\varepsilon\times\mathbb{B}^{n-m}_{r_1}} P(z',z'')|F(z',z'')|^2 e^{-\varphi(z',z'')} d\lambda_n$$
$$= \int_{z''\in\mathbb{B}^{n-m}_{r_1}} P(0,z'')|f(z'')|^2 e^{-\varphi(0,z'')} d\lambda_{n-m}$$

Using the iteration method by Berndtsson-Paun, Zhou-Zhu establish **generalized Siu's lemma with nontrivial L^p m.i.s.**.

Let $p\in(0,2]$. Let $\varphi(z',z'')$ be a psh function on $\mathbb{B}^m_r\times\mathbb{B}^{n-m}_r$, $M(z')$ be a bounded nonnegative measurable function on $\mathbb{C}^m$ with compact support, and $f(z'')$ be a holomorphic function on $\mathbb{B}^{n-m}_r$ satisfying

$$\int_{z''\in\mathbb{B}^{n-m}_r} |f(z'')|^p e^{-\varphi(0,z'')} d\lambda_{n-m} < +\infty.$$

Theorem ([8]). *Assume that $r_1,r_2,r_3\in(0,r)$ and $r_1<r_2<r_3$. Let β be a positive number such that*

$$I_\beta := \int_{z''\in\mathbb{B}^{n-m}_{r_3}} |f(z'')|^p e^{-(1+\beta)\varphi(0,z'')} d\lambda_{n-m} < +\infty$$

(the existence of β is guaranteed by strong openness of the multiplier ideal sheaves), and $\alpha\in(1-\frac{p}{2m}\beta,1)$ be a nonnegative number. Then there exists a holomorphic function $F(z',z'')$ on $\mathbb{B}^m_r\times\mathbb{B}^{n-m}_{r_3}$ such that $F(0,z'')=f(z'')$ on $\mathbb{B}^{n-m}_{r_3}$,

$$\int_{(z',z'')\in\mathbb{B}^m_r\times\mathbb{B}^{n-m}_{r_3}} \frac{|F(z',z'')|^p e^{-(1+\beta)\varphi(z',z'')}}{|z'|^{2m\alpha}} d\lambda_n < +\infty$$

and

$$\lim_{\varepsilon\to 0^+} \int_{(z',z'')\in\mathbb{B}^m_r\times\mathbb{B}^{n-m}_{r_1}} \frac{1}{\varepsilon^{2m}} M\big(\frac{z'}{\varepsilon}\big) P(z',z'')|F(z',z'')|^p e^{-\varphi(z',z'')} d\lambda_n$$
$$= \int_{z'\in\mathbb{C}^m} M(z')d\lambda_m \int_{z''\in\mathbb{B}^{n-m}_{r_1}} P(0,z'')|f(z'')|^p e^{-\varphi(0,z'')} d\lambda_{n-m}.$$

Among applications, one can obtain the pseudoeffectivity of the twisted relative pluricanonical bundles in the Kähler case ([1], [5], [8]); prove a comparison conjecture between two singular metrics on the twisted relative pluricanonical bundles posed in [1] and [5] ([8]); introduce the generalized Kodaira-Iitaka's dimensions with m.i.s. and obtain the subadditivity of these dimensions for Kähler fibration ([9]); solve a problem on the extension of cohomology classes posed in [2] by developing Demailly's technique on regularization of the closed positive $(1,1)$ currents in [3] and then obtain a new injectivity theorem which unifies the previous ones ([10]).

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Entire curves in complex projective varieties and differential equations

Jean-Pierre Demailly

For a given complex projective variety, the existence of entire curves is strongly constrained by the positivity properties of the cotangent bundle. The Green-Griffiths-Lang conjecture stipulates that entire curves drawn on a variety of general type should all be contained in a proper algebraic subvariety. We present here new results on the existence of differential equations that strongly restrain the locus of entire curves in the general context of foliated or directed varieties, under appropriate semistability conditions.

Let X be a complex projective manifold, $\dim_{\mathbb{C}} X = n$. Our goal is to study the existence and distribution of entire curves, i.e. non constant holomorphic curves $f : \mathbb{C} \to X$, in view of the following fundamental conjecture.

Conjecture (Green-Griffiths-Lang). *Assume that X is of general type, namely that $\kappa(X) = n = \dim X$ where $\kappa(X) := \limsup_{m\to+\infty} \log h^0(X, K_X^{\otimes m})/\log m$. Then there exists an algebraic subvariety $Y \subsetneq X$ containing all entire curves $f : \mathbb{C} \to X$.*

In case X is defined over a number field $\mathbb{K}_0$, the smallest algebraic subvariety Y as above is expected to coincide with the *Mordell locus* of points where the $\mathbb{K}$-rational points accumulate, for all number fields $\mathbb{K}$ containing $\mathbb{K}_0$. We concentrate here on the geometric problem of studying the locus of entire curves. It is convenient to work in the category of directed varieties:

- objects are pairs (X,V) where X is a complex manifold and $V \subset T_X$ a linear subspace of T_X, defined by a coherent subsheaf $\mathcal{O}(V)$;
- arrows $\psi : (X,V) \to (Y,W)$ are holomorphic maps $X \to Y$ such that $d\psi(V) \subset W$.

The absolute case refers to the case of pairs (X,T_X) where $V = T_X$, but the integrable situation $[\mathcal{O}(V),\mathcal{O}(V)] \subset \mathcal{O}(V)$ corresponds to the interesting case of holomorphic (possibly singular) foliations. An important object attached to a directed manifold (X,V) is its *canonical sheaf*, which is simply defined to be $K_V = \det(V^*)$ when V is a subbundle, and is taken to be the rank 1 sheaf ${}^b\mathcal{K}_V$ of sections of $\det V^*$ that are locally bounded with respect to a smooth ambient metric on T_X in the singular case.

Definition. *We say that (X,V) is of* general type *if ${}^b\mathcal{K}_V$ is big, i.e. if the space $H^0(X,({}^b\mathcal{K}_V)^{\otimes m})$ provides a generic embedding of X for a suitable $m \gg 1$, possibly after taking a suitable blow-up of X.*

Generalized Green-Griffiths-Lang conjecture (GGL conjecture). *If (X,V) is directed manifold of general type, i.e. if $\mathcal{K}_V^{[\bullet]}$ is big, then there exists an algebraic locus $Y \subsetneq X$ such that for every $f : (\mathbb{C},T_{\mathbb{C}}) \to (X,V)$, one has $f(\mathbb{C}) \subset Y$.*

The generalized GGL conjecture is an elementary consequence of the Ahlfors-Schwarz lemma when $r = \operatorname{rank} V = 1$. In fact, the function $t \mapsto \log\|f'(t)\|_{V,h}$ is strictly subharmonic if $r = 1$ and (V^*,h^*) big. The conjecture is possibly too optimistic: it might be safer to add a suitable (semi-) stability condition on V. The basic strategy is to how that entire curves $f : (\mathbb{C},T_{\mathbb{C}}) \to (X,V)$ must satisfy nontrivial algebraic differential equations $P(f\,;\,f',f'',\ldots,f^{(k)}) = 0$ with operators of the form

$$P(f_{[k]}) = P(f\,;\,f',f'',\ldots,f^{(k)}) = \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\ f'(t)^{\alpha_1} f''(t)^{\alpha_2}\ldots f^{(k)}(t)^{\alpha_k}.$$

Such operators of homogeneous degree $m = |\alpha_1|+2|\alpha_2|+\ldots+k|\alpha_k|$ define a sheaf denoted $E^{\rm GG}_{k,m}V^*$. If we let $J_k^{\rm nc}V \subset J_kV$ be the set of non constant k-jets of curves, the *Green-Griffiths bundle* is the projectivized bundle $X_k^{\rm GG} = J_k^{\rm nc}V/\mathbb{C}^*$ obtained as a quotient by the natural weighted $\mathbb{C}^*$ action, and we have the direct image formula

$$E^{\rm GG}_{k,m}V^* = (\pi_k)_*\mathcal{O}_{X_k^{\rm GG}}(m).$$

where $\mathcal{O}_{X_k^{\rm GG}}(1)$ is the associated tautological rank 1 sheaf and $\pi_k : X_k^{\rm GG} \to X$ the natural projection. Given a real $(1,1)$-form θ on X, the q-index set of θ is defined to be $X(\theta,q) = \{x \in X \mid \theta(x)$ has signature $(n-q,q)\}$, and we also set $X(\theta,\le q) = \bigcup_{0\le j\le q} X(\theta,j)$. An application of holomorphic Morse inequalities to suitable Finsler metrics on $\mathcal{O}_{X_k^{\rm GG}}(1)$ yields the following fundamental estimates.

Main cohomology estimates (D-, 2010). *Let (X,V) be a directed manifold, A an ample $\mathbb{Q}$-line bundle over X, (V,h) and (A,h_A) hermitian structures such that $\Theta_{A,h_A} > 0$. Define*

$$L_k = \mathcal{O}_{X_k^{\rm GG}}(1) \otimes \pi_k^*\mathcal{O}\Big(-\tfrac{1}{kr}\Big(1+\tfrac{1}{2}+\ldots+\tfrac{1}{k}\Big)A\Big),$$

$$\eta = \Theta_{\det V^*, \det h^*} - \Theta_{A,h_A}.$$

Then all $m \gg k \gg 1$ such that m is sufficiently divisible and for all $q \geq 0$ we have upper and lower bounds

$$h^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) \leq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n!\,(k!)^r}\left(\int_{X(\eta,q)}(-1)^q\eta^n + \frac{C}{\log k}\right),$$

$$h^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) \geq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n!\,(k!)^r}\left(\int_{X(\eta,\{q,\,q\pm1\})}(-1)^q\eta^n - \frac{C}{\log k}\right).$$

The case $q = 0$ is the most useful one, as it gives estimates for the number of holomorphic sections in

$$H^0(X_k^{\mathrm{GG}}, L_{k,\varepsilon}^{\otimes m}) \simeq H^0(X, E_{k,m}^{\mathrm{GG}} V^* \otimes \mathcal{O}(-m\delta_k\varepsilon A))$$

for $m \gg k \gg 1$ and $\varepsilon \in \mathbb{Q}_{>0}$ small. By the fundamental vanishing theorem due to [Green-Griffiths 1979], [Demailly 1995] and [Siu-Yeung 1996]), all global sections P of the above group provide a differential equation $P(f_{[k]}) \equiv 0$ for the entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$. The hardest part consists in investigating the base locus

$$Z = \bigcap_{m\in\mathbb{N}^*} \bigcap_{\sigma\in H^0(X_k^{\mathrm{GG}}, L_{k,\varepsilon}^{\otimes m})} \sigma^{-1}(0) \ \subset\ X_k^{\mathrm{GG}}.$$

and trying to show that $\pi_k(Z) \subsetneq X$. For this, one would like to construct nonzero section of $H^0(Z, L_k^{\otimes m})$, so as to get new differential equations that reduce further the base locus. Very recently (April 2021), we succeeded to get such sections when $Z \subset X_k^{\mathrm{GG}}$ is a component of a complete intersection of irreducible hypersurfaces

$$\bigcap_{1\leq j\leq \ell} \big\{k\text{-jets } f_{[k]} \in X_k^{\mathrm{GG}}\,;\, P_j(f) = 0\big\},\ P_j \in H^0(X, E_{s_j,m_j}^{\mathrm{GG}} V^* \otimes G_j)$$

with $k \geq k_0$, $\mathrm{ord}(P_j) = s_j$, $1 \leq s_1 < \cdots < s_\ell \leq k$, $\sum_{1\leq j\leq \ell} \frac{1}{s_j} \leq \delta \log k$, and $G_j \in \mathrm{Pic}(X)$ (agin using holomorphic Morse inequalities). Unfortunately, this is yet insufficient to prove the GGL conjecture.

A promising approach consists in the use of the tower of Semple bundles (X_k, V_k), where $X_k = P(V_{k-1})$, $\dim X_k = n + k(r-1)$, $\mathrm{rank}\, V_k = r$,

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X,$$

and V_k is defined inductively by

$$V_{k,(x,[v])} = \big\{\xi \in T_{X_k,(x,[v])}\,;\, d\pi_{k,k-1}(\xi) \in \mathbb{C}v \subset V_{k-1} \subset T_{X_{k-1},x}\big\}.$$

Every curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ admits a k-jet lifting $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$. Far any irreducible algebraic subset Z of X_k, one also gets an *induced directed structure* $(Z, W) \hookrightarrow (X_k, V_k)$ by taking the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure of $T_{Z'} \cap V_k$ taken on a suitable Zariski open set $Z' \subset Z_{\mathrm{reg}}$ where the intersection has constant rank and is a subbundle of $T_{Z'}$. Using this technology and the existence of suitable "tautological morphisms", we recently proved

Theorem (D-, 2021). *Let (X, V) be a directed variety. Assume that ${}^b\Lambda^p V^*$ is* strongly big *for some $p \le r = \operatorname{rank} V$, in the sense that for $A \in \operatorname{Pic}(X)$ ample, the symmetric powers $S^m({}^b\Lambda^p V^*) \otimes \mathcal{O}(-A)$ are generated by their sections over a Zariski open set of X, for $m \gg 1$.*

- *If $p = 1$, (X, V) satisfies the generalized GGL conjecture (well known fact!)*
- *If $p \ge 2$, there exists a subvariety $Y \subsetneq X$ and an induced directed variety $(Z, W) \subset (X_k, V_k)$ with $\operatorname{rank} W \le p - 1$, such that all entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ satisfy either $f(\mathbb{C}) \subset Y$ or $f_{[k]}(\mathbb{C}) \subset Z$.*
- *In particular, if $p = 2$, all entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ are either contained in $Y \subsetneq X$, or they are tangent to a* rank 1 foliation *on a subvariety $Z \subset X_k$. This implies that the leaves are parametrized by a finite dimensional space.*

The above results also give rise to logarithmic and orbifold directed versions. Let $\Delta = \sum \Delta_j$ be a reduced normal crossing divisor in X. We want to study entire curves $f : \mathbb{C} \to X \setminus \Delta$ drawn in the complement of Δ. At a point where $\Delta = \{z_1 \ldots z_p = 0\}$ one defines the *logarithmic cotangent sheaf* $T^*_{X\langle\Delta\rangle}$ to be generated by the 1-forms $dz_1/z_1, ..., dz_p/z_p, dz_{p+1}, ..., dz_n$.

Logarithmic statement (D-, 2021). *If $\Lambda^2 T^*_{X\langle\Delta\rangle}$ is strongly big on X, there exists a subvariety $Y \subsetneq X$ and a rank 1 foliation $\mathcal{F}$ on some k-jet bundle X_k, such that all entire curves $f : \mathbb{C} \to X \setminus \Delta$ are contained in Y or tangent to $\mathcal{F}$.*

For the orbifold case, we refer the reader to our forthcoming work in collaboration with F. Campana, L. Darondeau & E. Rousseau. In all cases, proving the GGL conjecture with optimal positivity conditions (i.e. only assuming bigness of the logarithmic/orbifold canonical sheaf) seems to require a better understanding of *stability properties* of the cotangent bundle.

An explicit estimate of the Bergman kernel for positive line bundles

Xu Wang

Let $(L, e^{-\phi})$ be a positive line bundle over an n-dimensional complex manifold X. Let m be a positive integer. Let K_X be the canonical line bundle over X. We call

$$\mathrm{K}_{m\phi}(x) := \sup_{u \in H^0(X, K_X + mL)} \frac{u(x) \wedge \overline{u(x)}\, e^{-m\phi(x)}}{\int_X u \wedge \bar{u}\, e^{-m\phi}}, \tag{1}$$

the *Bergman kernel forms* and

$$\mathrm{B}_{m\phi}(x) := \sup_{u \in H^0(X, mL)} \frac{|u(x)|^2 e^{-m\phi(x)}}{\int_X |u|^2 e^{-m\phi}\, \mathrm{MA}_{m\phi}}, \qquad \mathrm{MA}_{m\phi} := \frac{(i\partial\overline{\partial}(m\phi))^n}{n!}, \tag{2}$$

the *Bergman kernel functions*. In [11] Tian proved that if X is compact then

$$\lim_{m\to\infty} \frac{\mathrm{K}_{m\phi}}{\mathrm{MA}_{m\phi}} = \lim_{m\to\infty} \mathrm{B}_{m\phi} = \frac{1}{(2\pi)^n}. \tag{3}$$

Effective lower bound estimate (with Ricci curvature, diameter and volume assumptions) for $\mathrm{B}_{m\phi}$ is known as Tian's *partial C^0-estimate* [12]. The first general result is obtained by Donaldson–Sun [5] using proof by contradiction. Our main results are the followings:

Theorem A. *Let $(L, e^{-\phi})$ be a positive line bundle over a compact Riemann surface X. Put $\omega := \mathrm{MA}_\phi = i\partial\overline{\partial}\phi$. Denote by $\mathrm{Ric}\,\omega := i\overline{\partial}\partial \log \omega$ the Ricci form of ω. Assume that*

$$\mathrm{Ric}\,\omega \le \omega, \quad L_0 \ge 2\pi,$$

where L_0 denotes the infimum of the length of closed geodesics in X, then we have $\mathrm{K}_\phi/\mathrm{MA}_\phi \ge \frac{1}{8\pi}$.

Theorem B. *Let $(L, e^{-\phi})$ be a positive line bundle over a compact Riemann surface X. If*

$$-\omega/2 \le \mathrm{Ric}\,\omega \le \omega/2, \quad L_0 \ge 2\pi\sqrt{2},$$

then $\mathrm{B}_\phi \ge \frac{1}{16\pi}$.

Remark. *In case $X = \mathbb{P}^1$ and $\omega = 2 \cdot i\partial\overline{\partial} \log(1 + |z|^2)$ we have*

$$\mathrm{Ric}\,\omega = \omega, \quad L_0 = 2\pi,$$

a direct computation gives $L = -K_X$ and $\mathrm{K}_\phi/\mathrm{MA}_\phi = \frac{1}{4\pi}$. We do not know whether

$$\mathrm{K}_\phi/\mathrm{MA}_\phi \ge \frac{1}{4\pi}$$

is always true with the assumptions in Theorem A. On the other hand, Theorem A implies $K_{m\phi}/\mathrm{MA}_{m\phi} \ge 1/(8\pi)$ for every positive integer m. This is also near optimal since by (3)

$$\lim_{m\to\infty} K_{m\phi}/\mathrm{MA}_{m\phi} = 1/(2\pi).$$

In case $\mathrm{Ric}\,\omega \le 0$, $L_0/2$ is equal to the injectivity radius. For example if $X = \mathbb{C}/\Gamma$ is a torus and $\omega = i\partial\overline{\partial}(|z|^2/2)$ then $\mathrm{Ric}\,\omega = 0$ and $L_0 = \inf_{0\neq\gamma\in\Gamma} |\gamma|$.

In the first version of this paper, a weaker version of the above theorems is proved using an Ohsawa–Takegoshi type theorem, a variant of the Blocki–Zwonek estimate [2] and the isoperimetric inequality. Later we find that one may use the Hessian comparison theorem to simplify the proof and generalize the above theorems to the followings higher dimensional cases.

Theorem An. *Let $(L, e^{-\phi})$ be a positive line bundle over an n-dimensional compact complex manifold X. Assume that the sectional curvature of $\omega := i\partial\overline{\partial}\phi$ is bounded above by $1/(4n)$ and $L_0 \ge 2\pi\sqrt{n}$ then $\mathrm{K}_\phi/\mathrm{MA}_\phi \ge \frac{1}{2}\frac{1}{(4\pi n)^n}$.*

Theorem Bn. *Let $(L, e^{-\phi})$ be a positive line bundle over an n-dimensional compact manifold X. Assume that the sectional curvature of $\omega := i\partial\overline{\partial}\phi$ is bounded above by $1/(8n)$, $L_0 \ge 2\pi\sqrt{2n}$ and $\mathrm{Ric}\,\omega \ge -\omega/2$. Then $\mathrm{B}_\phi \ge \frac{1}{2}\frac{1}{(8\pi n)^n}$.*

Remark. *Since the Ricci curvature is certain sums of sectional curvatures, the curvature assumptions in Theorem Bn also imply a lower bound of the sectional curvature. Hence one may use* [6, Corollary 2.3.2] *to find a lower bound of L_0 in terms of the lower bound of the volume and the upper bound of the diameter. Thus, except the upper bound of the sectional curvature, the assumptions in Theorem Bn follow from the standard assumptions in Tian's partial C^0-estimate (for results on Tian's partial C^0-estimate, see* [1, 3, 4, 7, 8, 9, 10, 13, 14, 15]*, etc). Our main contribution is the explicit constant in the estimate. Moreover, our estimate implies that*

$$(\star) \qquad \mathrm{B}_{m\phi} \geq \frac{1}{2}\frac{1}{(8\pi n)^n}$$

for all positive integers m. From the last section in [5]*, it seems that for general positive line bundles over higher dimensional manifolds, the Ricci curvature assumptions might not enough to derive* $(\star)$.

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Monge-Ampère measures on contact sets

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(joint work with Stefano Trapani)

Let (X,ω) be a compact Kähler manifold of dimension n and fix θ a smooth closed real $(1,1)$-form. A function $\varphi : X \to \mathbb{R}\cup\{-\infty\}$ is called *quasi-plurisubharmonic* (qpsh for short) if locally $\varphi = \rho+u$, where ρ is smooth and u is a plurisubharmonic function. We say that φ is θ-plurisubharmonic (θ-psh for short) if it is quasi-plurisubharmonic and $\theta_\varphi := \theta + i\partial\bar\partial\varphi \geq 0$ in the weak sense of currents on X. We let $\mathrm{PSH}(X,\theta)$ denote the space of all θ-psh functions on X. The class $\{\theta\}$ is *pseudoeffective* if $\mathrm{PSH}(X,\theta)\neq\emptyset$ and it is *big* if there exists $\psi\in\mathrm{PSH}(X,\theta)$ such that $\theta + i\partial\bar\partial\psi \geq \varepsilon\omega$ for some $\varepsilon>0$, or equivalently if $\mathrm{PSH}(X,\theta-\varepsilon\omega)\neq\emptyset$.

When θ is non-Kähler, elements of $\mathrm{PSH}(X,\theta)$ can be quite singular, and we distinguish the potential with the smallest singularity type in the following manner:

$$V_\theta := \sup\{u\in\mathrm{PSH}(X,\theta) \text{ such that } u\leq 0\}.$$

Given $\varphi\in\mathrm{PSH}(X,\theta)$, following the construction of Bedford-Taylor [1] in the local setting, it has been shown in [5] that the sequence of positive measures

$$\mathbf{1}_{\{\varphi>V_\theta-k\}}\theta^n_{\max(\varphi,V_\theta-k)} \tag{1}$$

has total mass (uniformly) bounded from above and is non-decreasing in $k\in\mathbb{R}$, hence converges weakly to the so called *non-pluripolar Monge-Ampère measure* θ^n_φ. As a consequence of Bedford-Taylor theory it can be seen that the measures in (1) all have total mass less than $\int_X \theta^n_{V_\theta}$, in particular, after letting $k\to\infty$

$$\int_X \theta^n_\varphi \leq \int_X \theta^n_{V_\theta}.$$

In the following we are going to work with some well known envelope constructions:

$$P_\theta(f),\ P_\theta[\varphi](f).$$

Given a function f on X bounded from above, we consider the "rooftop envelopes"

$$P_\theta(f) := (\sup\{v\in\mathrm{PSH}(X,\theta),\ v\leq f)\})^*.$$

Then, given a θ-psh function φ, the above procedure allows us to introduce

$$P_\theta[\varphi](f) := \Big(\lim_{C\to+\infty} P_\theta(\min(\varphi+C,f))\Big)^*.$$

Note that by definition we have $P_\theta[\varphi](f) = P_\theta[\varphi](P_\theta(f))$, and that when $f=0$, $P_\theta(0)=V_\theta$.

The questions we are interesting in concerns the regularity of such envelopes and what kind of information one can get on the non-pluripolar Monge-Ampère measures $\theta^n_{P_\theta(f)}$ and $\theta^n_{P_\theta[\varphi](f)}$.

The geometric motivations we can mention are, among others, the study of geodesics in the space of Kähler metrics and the transcendental holomorphic Morse inequalities on projective manifolds.

The study of such envelopes has lead to several works. We start by summarizing them in the case of a smooth barrier function f.
The first result to mention is [3], where the author proves that in the case $\theta \in c_1(L)$ where L is a big line bundle over X, the envelope $P_\theta(f)$ is $C^{1,1}$ on $\mathrm{Amp}(\{\theta\})$ and moreover

$$\theta^n_{P_\theta(f)} = \mathbf{1}_{\{P_\theta(f)=f\}}\theta^n_f. \tag{2}$$

After [2], people started to work on possible generalizations of the above results in the case of a pseudoeffective class $\{\theta\}$, that does not necessarily represent the first Chern class of a line bundle. Assume that $\{\theta\}$ is big and nef, Berman [4], using PDE methods, proved that the envelope $P_\theta(f)$ is in $C^{1,\alpha}$ on $\mathrm{Amp}(\{\theta\})$ for any $\alpha \in (0,1)$ and that the identity in (2) holds. The optimal regularity $C^{1,1}$ in the Kähler case was then proved independently by [9] and [7], while the big and nef case was settled in [6].

The result obtained in collaboration with Stefano Trapani answers to the question about the behavior of the Monge-Ampère measures on contact sets and it states as follows:

Theorem (Di Nezza-Trapani [8]). *Let θ be smooth closed real $(1,1)$-form on X such that the cohomology class $\{\theta\}$ is pseudoeffective. Let φ be a θ-plurisubharmonic function and $f \in C^{1,\alpha}(X)$ for any $\alpha \in (0,1)$. Assume $\varphi \le f$. Then the non-pluripolar product θ^n_φ satisfies the equality*

$$\mathbf{1}_{\{\varphi=f\}}\theta^n_\varphi = \mathbf{1}_{\{\varphi=f\}}\theta^n_f. \tag{3}$$

As (almost an immediate) corollary we get:

Corollary. *Let $\varphi \in \mathrm{PSH}(X,\theta)$ and $f \in C^{1,\alpha}(X)$ for any $\alpha \in (0,1)$, be such that $\varphi \le f$. We have:*

i) $\theta^n_{P_\theta(f)} = \mathbf{1}_{\{P_\theta(f)=f\}}\theta^n_f$.
ii) $\theta^n_{P[\varphi](f)} = \mathbf{1}_{\{P[\varphi](f)=f\}}\theta^n_f$.

We conclude by observing that the above Theorem can not hold when the barrier function f is singular. The following counterexample shows indeed that (3) does not hold when f is merely continuous.

Let $\mathbb{B} \subset X$ be a small open ball and let ρ be a smooth potential such that $\omega = dd^c\rho$ in a neighborhood of $\overline{\mathbb{B}}$. We solve the Dirichlet problem

$$(dd^c(\rho+v))^n = 0 \quad \text{in } \mathbb{B}, \qquad v|_{\partial\mathbb{B}} = 0.$$

Since the boundary data is continuous, one can guarantee the existence of a continuous solution $v \ge 0$ which is ω-psh in $\mathbb{B}$. We then define

$$f := \begin{cases} v & \text{in } \mathbb{B} \\ 0 & \text{in } X \setminus \mathbb{B}. \end{cases}$$

By construction $f \ge 0$ is a continuous function and, since $\max(v,0) = v$, one can infer that f is also ω-psh. On the other hand we observe that

$$\int_{X\setminus\mathbb{B}} \omega^n_f = \int_X \omega^n_f = \int_X \omega^n > \int_{X\setminus\mathbb{B}} \omega^n.$$

Since $\{f = 0\} \subseteq X \setminus \mathbb{B}$, we then deduce that the two measures $\mathbf{1}_{\{f=0\}}\omega^n$ and $\mathbf{1}_{\{f=0\}}\omega_f^n$ can not coincide.

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Random walks on $\mathbf{SL_2}(\mathbb{C})$

Tien-Cuong Dinh

(joint work with Lucas Kaufmann, Hao Wu)

The theory of random walks on Lie groups is a classical and well developed topic. Initiated from the work of Furstenberg and Kesten in the 1960s, it was later developed by Guivarc'h, Kifer, Le Page, Raugi, Margulis, Goldsheid and others [3]. Even after important progress in the last decades, there are questions whose answers were found only recently and others remain unsolved, see e.g. [1]. This topic is still very active and a modern overview can be found in the book [2]. Here, we focus on the group $G = \mathrm{SL}_2(\mathbb{C})$ acting by linear transformations on the complex projective line $\mathbb{P}^1$.

The general problem can be described as follows. Let μ be a probability measure on G. Then, μ induces a random walk on G: for $n \geq 1$ we let $S_n := g_n \cdots g_1$ where the g_j's are independent random elements of G with law given by μ. One also has an induced random walk on $\mathbb{P}^1$: for a point $x \in \mathbb{P}^1$ we look at its trajectory under S_n, i.e. $S_n \cdot x = g_n \cdots g_1 \cdot x$. The general goal is to describe the asymptotic behaviour of these random walks and study questions such as the growth and distribution of the norm of S_n or of its coefficients, etc.

A remarkable fact is that these random processes satisfy analogues of many of the classical limit theorems for sums of i.i.d. random variables. More precisely, consider the *norm cocycle* defined by

$$\sigma(g,x) = \sigma_g(x) := \log \frac{\|gv\|}{\|v\|}, \quad \text{for } v \in \mathbb{C}^2 \setminus \{0\}, x = [v] \in \mathbb{P}^1 \text{ and } g \in G.$$

Notice that $\|\sigma_g\|_\infty = \log\|g\|$, where $\|g\|$ is denotes operator norm of g.

When μ has a finite first moment, that is, $\int_G \log\|g\|\,\mathrm{d}\mu(g) < \infty$, its *(upper) Lyapunov exponent* is the finite number defined by

$$\gamma := \lim_{n\to\infty} \frac{1}{n}\mathbf{E}\big(\log\|S_n\|\big) = \lim_{n\to\infty} \frac{1}{n}\int \log\|g_n\cdots g_1\|\,\mathrm{d}\mu(g_1)\cdots\mathrm{d}\mu(g_n).$$

A fundamental result of Furstenberg-Kesten (the analogue of the Law of Large Numbers) tells us that we actually have that $\frac{1}{n}\log\|S_n\|$ converges to γ almost surely as $n\to\infty$.

In order to obtain finer limit theorems, some conditions on μ need to be imposed. We need the support of μ to be sufficiently rich and, as in the case of sums of i.i.d.'s, the random variables must verify some moment conditions.

We say that μ is **non-elementary** if its support does not preserve a finite subset of $\mathbb{P}^1$ and if the semi-group it generates is not relatively compact in G. It is an important result of Furstenberg that a non-elementary measure admits a unique **stationary measure**, also called the Furstenberg measure. This is the only probability measure ν on $\mathbb{P}^1$ such that

$$\int_G g_*\nu\,\mathrm{d}\mu(g) = \nu.$$

To date, many limit theorems are know both for the sequences $\sigma(S_n,x)$ and $\log\|S_n\|$. However, most of them require the strong condition of *finite exponential moment*, that is, $\int_G \|g\|^\alpha\,\mathrm{d}\mu(g) < \infty$ for some $\alpha > 0$. These results include the Central Limit Theorem (CLT), the Law of Iterated Logarithm (LIL), the Local Limit Theorem (LLT), etc. See [2] for a complete treatment in the more general setting of reductive groups. These results rely on the fundamental work of Le Page [9], which studies in detail the Markov operator and its perturbations for measures with finite exponential moment.

However, it turns out that the above exponential moment condition is too strong, since the classical versions of the aforementioned limit theorems all hold under much weaker conditions. A recent breakthrough by Benoist-Quint lead to the proof of the optimal version the CLT under the second moment condition $\int_G \log^2\|g\|\mathrm{d}\mu(g) < \infty$, see [1]. See also [5] for other results under low moment conditions based on their techniques. However, this method is difficult to tackle the LLT and the other problems we study.

Up to now, the approach using the Markov operator and its perturbations has not been successful when the moment conditions are weak. The main difficulty is to study the spectral properties of these operators without the exponential moment hypothesis. A recent progress was made in [7], where we were able to obtain a spectral gap theorem for the Markov operator under the first moment condition. This allowed us to give an independent proof of the CLT under the second moment condition in the $\mathrm{SL}_2(\mathbb{C})$ case, along the lines of [9]. The main idea of [7] is to replace the random process induced by μ by a generalized (deterministic) self-map of $\mathbb{P}^1$. From there, we were able to use techniques from Complex Dynamics, most notably the ones developed in [6].

In this work we continue to develop the ideas of [7] and apply them to prove some new results about $\mathrm{SL}_2(\mathbb{C})$ random walks under low moment conditions.

Local Limit Theorems (LLTs). Our first main result is the following LLT for the norm cocycle under the optimal moment condition. When μ has a finite exponential moment this result is known since Le Page [9].

Theorem 1 (LLT for the norm cocycle)**.** *Let μ be a non-elementary probability measure on $G := \mathrm{SL}_2(\mathbb{C})$ with a finite second moment. Let γ be its Lyapunov exponent and ν be the corresponding stationary measure. Then the norm cocycle satisfies the LLT.*

More precisely, let $S_n = g_n \cdots g_1$ where the g_i are i.i.d. with law μ and set $p(t) := (2\pi)^{-1/2} \exp(-t^2/2)$. Then, there exists a number $a > 0$ such that for every continuous function f with compact support on $\mathbb{R} \times \mathbb{P}^1$ we have

$$\sup_{(t,x)\in\mathbb{R}\times\mathbb{P}^1} \Big| \sqrt{n}\, a\, \mathbf{E}\Big(f\big(t + \sigma(S_n, x) - n\gamma, S_n \cdot x\big)\Big) - p\big(\frac{t}{a\sqrt{n}}\big) \int_{\mathbb{R}\times\mathbb{P}^1} f(s,y)\,\mathrm{d}s\,\mathrm{d}\nu(y)\Big| \to 0, \ \text{for } n \to \infty.$$

Next, we obtain an analogous LLT for the coefficients of the random matrices. It is worth noting that, even in the case of finite exponential moment, this property has been proved only recently by Grama-Quint-Xiao [8].

Theorem 2 (LLT for matrix coefficients)**.** *Let μ be a non-elementary probability measure on $G = \mathrm{SL}_2(\mathbb{C})$ with a finite 3-moment and let γ be its Lyapunov exponent. Then the coefficients of S_n satisfies the LLT: for any $b_1 < b_2$ in $\mathbb{R}$ and uniformly on $u, v \in \mathbb{C}^2 \setminus \{0\}$, we have*

$$\lim_{n\to\infty} \sqrt{2\pi n}\, a\, \mathbf{P}\Big(\log \frac{|\langle S_n u, v\rangle|}{\|u\|\|v\|} - n\gamma \in [b_1, b_2]\Big) = b_2 - b_1.$$

Fourier coefficients. Assume now that μ is non-elementary and it is supported by $\mathrm{SL}_2(\mathbb{R})$, that is, the random matrices have real coefficients. In this case, the action on $\mathbb{P}^1$ preserves the real projective line $\mathbb{RP}^1$, which via a stereographic projection is naturally identified with a circle $\mathbf{C}$ parametrized by $0 \leq \theta < 2\pi$. Then, the stationary measure ν is supported by $\mathbf{C}$. We define the Fourier transform of ν by

$$\widehat{\nu}(k) := \int_{\mathbf{C}} e^{ik\theta} \mathrm{d}\nu(\theta) \quad \text{for} \quad k \in \mathbb{R}.$$

An important class of probability measures, called *Rajchman measures*, are those for which the Fourier transform vanishes at infinity. This is a fine property of the measure that is related to its regularity, but also to some non-arithmeticity of its support. For instance, it is known that any measure on the standard middle-thirds Cantor set (no matter how regular) is not a Rajchman measure. Rajchman property is closely related to sets of uniqueness for trigonometric series.

Recently, Li showed that stationary measures are Rajchman measures under a finite exponential moment condition, see [10] and also [4, 11] for some arithmetic and geometric settings. Here, we generalize this result by relaxing the moment condition.

Theorem 3. *Let μ be a non-elementary probability measure on $\mathrm{SL}_2(\mathbb{R})$ with a finite second moment. Then the associated stationary measure ν on $\mathbf{C} \subset \mathbb{P}^1$ is a Rajchman measure.*

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Schwarz–Pick lemma for harmonic maps which are conformal at a point

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(joint work with David Kalaj)

The original preprint is available at `https://arxiv.org/abs/2102.12403`.

We establish precise estimates of derivatives and the rate of growth of conformal harmonic maps from hyperbolic conformal surfaces into the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$ for any $n \geq 3$. Such maps parameterize minimal surfaces, objects of high interest in geometry.

The following special case of our main result generalizes the *Schwarz–Pick lemma*, due to H. A. Schwarz [15, Bd. II, p. 108] (1869), H. Poincaré [13] (1884), and G. A. Pick [12] (1915), to a much larger class of maps. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc.

Theorem 1. *If $f : \mathbb{D} \to \mathbb{D}$ is a harmonic map which is conformal at a point $z \in \mathbb{D}$, then at this point we have that*

$$\|df_z\| \le \frac{1-|f(z)|^2}{1-|z|^2}, \tag{1}$$

with equality if and only if f is a conformal diffeomorphism of the disc $\mathbb{D}$.

The classical Schwarz–Pick lemma gives the same conclusion provided the map f is holomorphic or antiholomorphic, which means that *it is conformal at every noncritical point*. This is the most fundamental rigidity result in complex analysis, leading to the notion of Kobayashi hyperbolic manifolds and providing a connection to complex differential geometry via the Ahlfors lemma and its generalizations.

Given a differentiable map $f = (f_1, \ldots, f_n) : \mathbb{D} \to \mathbb{R}^n$, we denote by f_x and f_y its partial derivatives with respect to x and y, where $z = x + \imath y \in \mathbb{D}$. The gradient $\nabla f = (f_x, f_y)$ is an $n \times 2$ matrix representing the differential df. The map f is said to be *conformal* at $z \in \mathbb{D}$ if the differential df_z preserves angles, which holds if and only if

$$|f_x(z)| = |f_y(z)| > 0 \quad \text{and} \quad f_x(z) \cdot f_y(z) = 0.$$

Here, the dot stands for the Euclidean inner product on $\mathbb{R}^n$, and $|\mathbf{x}|$ is the Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^n$. We allow maps to have branch points; the estimates that we shall present are trivially fulfilled at such points. We denote by $|\nabla f|$ the Euclidean norm of the gradient:

$$|\nabla f(z)|^2 = |f_x(z)|^2 + |f_y(z)|^2, \qquad z \in \mathbb{D}.$$

If f is conformal at z, then clearly $\|df_z\| = \sqrt{2}^{-1} |\nabla f(z)| = |f_x(z)| = |f_y(z)|$. The map $f = (f_1, \ldots, f_n) : \mathbb{D} \to \mathbb{R}^n$ is harmonic if and only if every component f_k is a harmonic function on $\mathbb{D}$, meaning that the Laplacian $\Delta f_k = \frac{\partial^2 f_k}{\partial x^2} + \frac{\partial^2 f_k}{\partial y^2}$ vanishes identically.

We denote by $\mathbb{B}^n$ the unit ball of $\mathbb{R}^n$. Our main result is the following.

Theorem 2. *Let $f : \mathbb{D} \to \mathbb{B}^n$ for $n \ge 2$ be a harmonic map. If f is conformal at a point $z \in \mathbb{D}$ and $\theta \in [0, \pi/2]$ denotes the angle between the vector $f(z)$ and the plane $\Lambda = df_z(\mathbb{R}^2) \subset \mathbb{R}^n$, then*

$$\|df_z\| = \frac{1}{\sqrt{2}} |\nabla f(z)| \ \le \ \frac{1-|f(z)|^2}{1-|z|^2} \frac{1}{\sqrt{1-|f(z)|^2 \sin^2\theta}}, \tag{2}$$

with equality if and only if f is a conformal diffeomorphism of $\mathbb{D}$ onto the affine disc $(f(z) + \Lambda) \cap \mathbb{B}^n$. Without assuming that f is conformal, we have that

$$\frac{1}{\sqrt{2}} |\nabla f(z)| \ \le \ \frac{\sqrt{1-|f(z)|^2}}{1-|z|^2}, \qquad z \in \mathbb{D}. \tag{3}$$

Equality holds in (3) for some $z_0 \in \mathbb{D}$ if $f(z_0)$ is orthogonal to the 2-plane $\Lambda = df_{z_0}(\mathbb{R}^2)$ and f is a conformal diffeomorphism onto the affine disc $(f(z_0) + \Lambda) \cap \mathbb{B}^n$.

In dimension $n = 2$ we necessarily have $\theta = 0$, so (1) is a special case of (2). Without assuming that f is conformal at z or that $f(z) = 0$, the inequality (2) fails for some harmonic diffeomorphisms of the disc. The first and the main part of Theorem 2 is proved by exploring a connection to Lempert's seminal work [9] from 1981 on complex geodesics of the Kobayashi metric on bounded convex domains in $\mathbb{C}^n$. The inequality (3) is obtained from the L^2-estimate of the map.

It is classical that a conformal map $f : \mathbb{D} \to \mathbb{B}^n$ ($n \geq 3$) parameterizes a minimal surface if and only if it is harmonic. The inequality (2) can be interpreted as the distance-decreasing property of conformal harmonic maps $\mathbb{D} \to \mathbb{B}^n$ with respect to the *Poincaré metric* $\mathcal{P}_{\mathbb{D}}$ on the disc$\mathbb{D}$ and the *Cayley–Klein metric*[1] on the ball $\mathbb{B}^n$, where the latter is defined for any point $\mathbf{x} \in \mathbb{B}^n$ and tangent vector $\mathbf{v} \in \mathbb{R}^n$ by

$$\mathcal{CK}(\mathbf{x},\mathbf{v})^2 = \frac{(1-|\mathbf{x}|^2)|\mathbf{v}|^2 + |\mathbf{x}\cdot\mathbf{v}|^2}{(1-|\mathbf{x}|^2)^2} = \frac{|\mathbf{v}|^2}{1-|\mathbf{x}|^2} + \frac{|\mathbf{x}\cdot\mathbf{v}|^2}{(1-|\mathbf{x}|^2)^2}.$$

Corollary 3. *If $f : \mathbb{D} \to \mathbb{B}^n$ is a conformal harmonic map then*

$$\mathcal{CK}\big(f(z), df_z(\xi)\big) \leq \frac{|\xi|}{1-|z|^2} = \mathcal{P}_{\mathbb{D}}(z,\xi), \qquad z \in \mathbb{D},\ \xi \in \mathbb{R}^2,$$

with equality for some $z \in \mathbb{D}$ and $\xi \in \mathbb{R}^2 \setminus \{0\}$ if and only if f is a conformal diffeomorphism onto the affine disc $\Sigma = (f(z) + df_z(\mathbb{R}^2)) \cap \mathbb{B}^n$ and the vector $df_z(\xi)$ is tangent to the diameter of Σ through the point $f(z)$.

This shows in particular that every *linear* conformal embedding $f : \mathbb{D} \to \Sigma$ onto an affine disc in $\mathbb{B}^n$ is geodesic on each diameter $(-1,+1) \ni r \mapsto f(re^{\imath t}) \in \Sigma$ for every fixed $t \in \mathbb{R}$. However, distances between points of different rays strictly decrease from the Poincaré metric on $\mathbb{D}$ to the Cayley–Klein metric on the disc $\Sigma \subset \mathbb{B}^n$.

Our results also hold if the disc $\mathbb{D}$ is replaced by an arbitrary hyperbolic conformal surface M, i.e., one whose universal conformal covering space is the disc $\mathbb{D}$. Such M is endowed with the Poincaré metric $\mathcal{P}_M$, which is defined by the condition that the universal conformal covering map $\mathbb{D} \to M$ is an isometry in the Poincaré metrics on the respective surfaces.

Our results provide foundations of hyperbolicity theory for domains in $\mathbb{R}^n$ ($n \geq 3$) in terms of conformal minimal surfaces that they contain, in analogy to Kobayashi's approach to hyperbolicity of complex manifolds; see [6, 7, 8].

[1]The Cayley–Klein model (also called the Beltrami–Klein model) of hyperbolic geometry was introduced by Arthur Cayley [2] (1859) and Eugenio Beltrami [1] (1968), and was developed by Felix Klein [4, 5] (1871, 1873). The underlying space is the n-dimensional unit ball, and geodesics are straight line segments with ideal endpoints on the boundary sphere. See Ratcliffe [14] for a modern treatment. This is a special case of the Hilbert metric on convex domains in $\mathbb{R}^n$, introduced by David Hilbert in 1895 [3]. I am indebted to László Lempert who told me about this metric; see also his papers [10, 11].

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An extended Monge-Ampère operator

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(joint work with Mats Andersson, David Witt Nyström)

This is joint work in progress with Mats Andersson and David Witt Nyström. The overall goal is to define a Monge-Ampère operator with nice continuity properties for a large class of plurisubharmonic (psh) functions.

Let Ω be a domain in $\mathbf{C}^n$, and let u be a psh function on Ω. If u is assumed to be C^2, then dd^cu is a positive form. The associated Monge-Ampère measure, defined as the top wedge power of this form with itself plays a leading role in pluripotential theory, similar to role played by the Laplacian in ordinary potential theory.

In the 80's Bedford and Taylor [4, 5] defined Monge-Ampère products for locally bounded psh functions. Assume that T is a closed positive current and that u is a locally bounded psh function. Then uT is a well-defined current, and $dd^cu \wedge T := dd^c(uT)$ is again positive and closed. Thus one can recursively define closed positive currents

$$(dd^c u)^p = dd^c(u(dd^c u)^{p-1}).$$

The Monge-Ampère operators $u \mapsto (dd^c u)^p$ have some essential continuity properties. In particular, if u_ℓ is a sequence of psh functions decreasing to u, then $(dd^c u_\ell)^p$ converges weakly to $(dd^c u)^p$.

We are interested in the situation when u is not locally bounded. Demailly showed that it is possible to extend the Bedford-Taylor Monge-Ampère operators to psh functions that are bounded outside small sets, see, e.g., [11]. Moreover, Błocki, [7], and Cegrell, [10], characterized the largest class $\mathcal{D}(\Omega)$ of psh functions on which there is a Monge-Ampère operator $u \mapsto (dd^c u)^n$ that is continuous under decreasing sequences.

To handle more singular psh functions Bedford and Taylor [6] introduced the notion of non-pluripolar Monge-Ampère products

$$\langle dd^c u\rangle^p = \lim_{\ell\to\infty} 1_{\{u>-\ell\}}(dd^c \max(u,-\ell))^p. \tag{1}$$

If the limit is locally finite, then $\langle dd^c u\rangle^p$ is a positive closed (p,p)-current see [9]. However, the Monge-Ampère operators $u \mapsto \langle dd^c u\rangle^p$ are far from being continuous under decreasing sequences in general.

We introduce a new class of psh functions.

Definition 1. Let Ω be a domain in $\mathbf{C}^n$. We say that a psh function u on Ω is in $\mathcal{G}(\Omega)$ if, for each $p < n$, $\langle dd^c u\rangle^p$ is locally finite and u is locally integrable against $\langle dd^c u\rangle^p$.

If $u \in \mathcal{G}(\Omega)$ and $p \leq n-1$, then $u\langle dd^c u\rangle^p$ is a well-defined current and thus by mimicking the original construction by Bedford-Taylor we can define generalized Monge-Ampère products.

Definition 2. Given $u \in \mathcal{G}(\Omega)$ for $p = 1,\dots,n$ we define

$$[dd^c u]^p = dd^c(u\langle dd^c u\rangle^{p-1})$$

and

$$S_p(u) = [dd^c u]^p - \langle dd^c u\rangle^p.$$

Using basic properties of the non-pluripolar Monge-Ampère operator it is easily verified that these currents are closed positive currents. In particular $[dd^c u]^n$ is a positive measure that dominates the non-pluripolar Monge-Ampère measure $\langle dd^c u\rangle^n$.

These definitions are inspired by the construction of generalized Monge-Ampère products in [1, 2]. From [2] it follows that psh functions with analytic singularities (i.e., psh functions that are locally of the form $c\log|f|^2 + b$, where $c > 0$, f is a tuple of holomorphic functions, and b is locally bounded) are in $\mathcal{G}(\Omega)$. In particular, there are functions in $\mathcal{G}(\Omega)$ that are not in $\mathcal{D}(\Omega)$.

Given a psh function u, note that $u_\ell := \max(u,-\ell)$ is a natural sequence of locally bounded psh-functions decreasing to u, cf. (1).

Theorem 3. *Assume that $u \in \mathcal{G}(\Omega)$. Then*

$$\big(dd^c \max(u, -\ell)\big)^p \to [dd^c u]^p, \quad \ell \to \infty.$$

More generally, let $\chi_\ell : \mathbf{R} \to \mathbf{R}$ be a sequence of nondecreasing convex functions, bounded from below, that decreases to t as $\ell \to \infty$, and let $u_\ell = \chi_\ell \circ u$. Then

$$(dd^c u_\ell)^p \to [dd^c u]^p, \quad \ell \to \infty.$$

For u with analytic singularities, this was proved in [3].

Since there are functions in $\mathcal{G}(\Omega)$ that are not in $\mathcal{D}(\Omega)$ we cannot expect continuity for all decreasing sequences. Our next result is a variant of Theorem 3 that illustrates the failure of continuity. Let v be a locally bounded psh-funtion on Ω. Then $u_\ell := \max(u, v - \ell)$ is a sequence of locally bounded psh functions decreasing to u.

Theorem 4. *Assume that $u \in \mathcal{G}(\Omega)$ and that v is a smooth psh function on Ω. Then*

$$\big(dd^c \max(u, v - \ell)\big)^p \to [dd^c u]^p + \sum_{j=1}^{p-1} (dd^c v)^{p-j} \wedge S_j(u), \quad \ell \to \infty.$$

More generally, let $\chi_\ell : \mathbf{R} \to \mathbf{R}$ be a sequence of nondecreasing convex functions, bounded from below, that decreases to t as $\ell \to \infty$, and let $u_\ell = \chi_\ell \circ (u - v) + v$. Then

$$(dd^c u_\ell)^p \to [dd^c u]^p + \sum_{j=1}^{p-1} (dd^c v)^{p-j} \wedge S_j(u), \quad \ell \to \infty.$$

Note in particular that the lower degree products $[dd^c u]^p$ comes into play. It follows from Theorem 4 that if $S_p(u) \neq 0$ for some $p < n$, then $u \notin \mathcal{D}(\Omega)$.

Let us turn to the global setting. Assume that (X, ω) is a compact Kähler manifold of dimension n, and that φ is an ω-psh function on X, i.e. $\varphi + h$ is psh if h is a local dd^c-potential for ω. We say that an ω-psh function φ is in $\mathcal{G}(X, \omega)$ if $\varphi + h \in \mathcal{G}(\Omega)$, where $dd^c h = \omega$ in Ω. Then there are well-defined closed positive currents $[dd^c \varphi + \omega]^p$, locally defined as $[dd^c(\varphi + h)]^p$, and corresponding currents $S_p^\omega(\varphi)$. We have the following global version of Theorem 4.

Theorem 5. *Assume that $\varphi \in \mathcal{G}(X, \omega)$. Then*

$$\big(dd^c \max(\varphi, -\ell) + \omega\big)^p \to [dd^c \varphi + \omega]^p + \sum_{j=1}^{p-1} \omega^{p-j} \wedge S_j^\omega(\varphi), \quad \ell \to \infty.$$

More generally, let $\chi_\ell : \mathbf{R} \to \mathbf{R}$ be a sequence of nondecreasing convex functions, bounded from below, that decreases to t as $\ell \to \infty$, and let $\varphi_\ell = \chi_\ell \circ \varphi$. Then

$$(dd^c \varphi_\ell + \omega)^p \to [dd^c \varphi + \omega]^p + \sum_{j=1}^{p-1} \omega^{p-j} \wedge S_j^\omega(\varphi), \quad \ell \to \infty.$$

In the situation of analytic singularities this was proved in [8]. From Theorem 5 we get the following mass formula. In fact, this is also an immediate cohomological consequence of the definition of $[dd^c\varphi + \omega]^p$.

Theorem 6. *Let (X, ω) be a compact Kähler manifold X of dimension n. Assume that $\varphi \in \mathcal{G}(X, \omega)$. Then*

$$\int_X \langle dd^c\varphi + \omega\rangle^n + \sum_{p=1}^{n} \int_X S_p^\omega(\varphi) \wedge \omega^{n-p} = \int_X \omega^n.$$

In particular $\langle dd^c\varphi + \omega\rangle^n$ has full mass if and only if $S_p^\omega(\varphi) = 0$ for all p.

Similar to the the local case, ω-psh functions with analytic singularities are in $\mathcal{G}(X, \omega)$. Moreover, $\mathcal{D}(X, \omega) \subset \mathcal{G}(X, \omega)$, where φ is in $\mathcal{D}(X, \omega)$ if $\varphi + h \in \mathcal{D}(\Omega)$. The definition of $\mathcal{G}(X, \omega)$ can be understood in terms of energy. In particular, it follows that the class of ω-psh functions with finite energy, $\mathcal{E}_1(X, \omega)$, is contained in $\mathcal{G}(X, \omega)$, which in turn implies that $\mathcal{G}(X, \omega)$ has certain convexity properties and thus a quite rich structure.

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Finite entropy vs finite energy

Vincent Guedj

(joint work with Eleonora Di Nezza, Chinh H. Lu)

Probability measures with either finite energy [4] or finite entropy [3] have played an important role in recent developments in Kähler geometry (see [5, 6, 7, 9, 3] and the references therein).

Indeed the search for Kähler-Einstein metrics on Fano manifolds boils down to maximizing the Ding functional whose leading term is a Monge-Ampère energy, while a constant scalar curvature Kähler metric minimizes the Mabuchi functional, whose leading term is an entropy. The purpose of this talk is to systematically compare these two notions.

Let (X, ω) be a compact Kähler manifold of complex dimension $n \geq 1$, normalized so that $\mathrm{Vol}_\omega(X) := \int_X \omega^n = 1$. We consider $\mu = f\omega^n$, $0 \leq f$, a probability measure with finite entropy

$$0 \leq \mathrm{Ent}_{\omega^n}(\mu) := \int_X f \log f \; \omega^n < +\infty.$$

Since μ is absolutely continuous with respect to the volume form ω^n, it is in particular "non-pluripolar" hence it follows from [10, 8] that there exists a unique full mass potential $\varphi \in \mathcal{E}(X, \omega)$ such that $\sup_X \varphi = 0$ and

$$(\omega + dd^c\varphi)^n = \mu.$$

Here $d = \partial + \overline{\partial}$ and $d^c = \frac{i}{2\pi}\left(\partial - \overline{\partial}\right)$ are real operators so that $dd^c = \frac{i}{\pi}\partial\overline{\partial}$, and $\mathcal{E}(X, \omega)$ denotes the set of ω-plurisubharmonic functions φ whose non pluripolar Monge-Ampère measure $(\omega + dd^c\varphi)^n$ is a probability measure. We consider, for $p > 0$,

$$\mathcal{E}^p(X, \omega) := \{\varphi \in \mathcal{E}(X, \omega) \mid E_p(\varphi) < +\infty\},$$

where $E_p(\varphi) := \int_X |\varphi|^p (\omega + dd^c\varphi)^n$.

It has been observed in [3] that

$$\mathrm{Ent}(X, \omega) \subset \mathcal{E}^1(X, \omega),$$

where $\mathrm{Ent}(X, \omega)$ is the set of ω-psh functions whose Monge-Ampère measure has finite entropy. However all computable examples suggest that φ actually belongs to a higher energy class $\mathcal{E}^p(X, \omega)$ for some $p > 1$ depending on the dimension. We confirm this experimental observation by showing the following:

Theorem A. *Let* $\mu = (\omega + dd^c\varphi)^n = f\omega^n$ *be a probability measure with finite entropy* $\mathrm{Ent}_{\omega^n}(\mu) = \int_X f \log f\omega^n < +\infty$. *Then*

$$\varphi \in \mathcal{E}^{\frac{n}{n-1}}(X, \omega).$$

Moreover the inclusion $\mathrm{Ent}(X, \omega) \hookrightarrow \mathcal{E}^p(X, \omega)$ *is compact for any* $p < \frac{n}{n-1}$.

This exponent is sharp when $n \geq 2$. *If* $n = 1$ *then* φ *is continuous, hence it belongs to* $\mathcal{E}^p(X, \omega)$ *for all* $p > 0$.

The case of Riemann surfaces deserves a special treatment: finite entropy potentials turn out to be bounded (and even continuous), but this is no longer the case in higher dimension. The proof of Theorem A relies on a Moser-Trudinger inequality which provides a strong integrability property of finite energy potentials. This is the content of our second main result:

Theorem B. *Fix $p > 0$. There exist positive constants $c, C > 0$ depending on X, ω, n, p such that, for all $\varphi \in \mathcal{E}^p(X,\omega)$ with $\sup_X \varphi = -1$,*

$$\int_X \exp\left(c|E_p(\varphi)|^{-1/n}|\varphi|^{1+\frac{p}{n}}\right)\omega^n \leq C.$$

Theorem B is an interesting variant of Trudinger's inequality on compact Kähler manifolds. The case $p = 1$ settles a conjecture of Aubin (called Hypothèse fondamentale [1]) which is motivated by the search for Kähler-Einstein metrics on Fano manifolds. The conjecture was previously proved by Berman-Berndtsson [2] under the assumption that the cohomology class of ω is the first Chern class of an ample holomorphic line bundle.

We also establish local versions of these results, valid in any bounded hyperconvex domain of $\mathbb{C}^n$.

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On the $p-$Bergman theory

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(joint work with Liyou Zhang)

In this work we attempt to develop a general $p-$Bergman theory. The Bergman theory makes essential use of complete orthonormal bases in the Bergman space, which are not available in the L^p case. On the other hand, there is some similarity between the Bergman theory and the spectrum theory of the Laplacian; one can also expect that the spectrum theory of the *nonlinear* $p-$Laplacian would play a role on the $p-$Bergman theory for $p \neq 2$.

For a bounded domain $\Omega \subset \mathbb{C}^n$, we define $A^p(\Omega)$ to be the $p-$Bergman space of L^p holomorphic functions on Ω. Consider the following minimizing problem:

$$m_p(z) := \inf \{\|f\|_p : f \in A^p(\Omega), f(z) = 1\}.$$

It is easy to see that there exists at least one minimizer for $p > 0$ and exactly one minimizer, say $m_p(\cdot, z)$, for $p \geq 1$. We define the $p-$Bergman kernel $K_p(z)$ $(p > 0)$ and the off-diagonal $p-$Bergman kernel $K_p(z, w)$ $(p \geq 1)$ by

$$K_p(z) := m_p(z)^{-p} \quad \text{and} \quad K_p(z, w) := m_p(z, w)K_p(z).$$

Clearly, one has $K_p(z) = K_p(z, z)$ and $K_2(z, w)$ is the standard Bergman kernel. The $p-$Bergman metric is given by

$$B_p(z; X) := K_p(z)^{-\frac{1}{p}} \cdot \sup\{|Xf(z)| : f \in A^p(\Omega), f(z) = 0, \|f\|_p = 1\}.$$

Note that $B_2(z; X)$ is the standard Bergman metric.

Uniqueness of the minimizer yields the following

Proposition 1. We have

(1) Let $F : \Omega_1 \to \Omega_2$ be a biholomorphic mapping between bounded simply-connected domains. Then

$$K_{\Omega_1,p}(z, w) = K_{\Omega_2,p}(F(z), F(w))J_F(z)^{\frac{2}{p}} J_F(w)^{1-\frac{2}{p}} \overline{J_F(w)}.$$

(2) Let Ω' and Ω'' be bounded domains in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. Set $\Omega = \Omega' \times \Omega''$ and $z = (z', z'')$. Then we have

$$K_{\Omega,p}(z, w) = K_{\Omega',p}(z', w') \cdot K_{\Omega'',p}(z'', w'').$$

(3) $K_p(z, w)$ is continuous in (z, w).

(4) If Ω is simply-connected and $m_p(\cdot, z)$ is zero-free, then $K_s(z) = K_p(z)$ for $s \geq p$ and $m_s(\cdot, z) = m_p(\cdot, z)^{p/s}$ for $s \geq p \geq 1$.

Remark. *By (4) we see that if Ω is a simply-connected Lu Qi-keng domain, i.e., $K_2(z, w)$ is zero-free on $\Omega \times \Omega$, then $K_p(z, w) = K_2(z, w)^{2/p} K_2(w)^{2/p-1}$ for any $z, w \in \Omega$ and $p \geq 2$.*

As an application of the transformation formula, we have

Proposition 2. Let $\Omega \subset \mathbb{C}^n$ be a bounded simply-connected domain and $G \subset \mathrm{Aut}(\Omega)$ a properly discontinuous group. Let $L(G)$ denote the limit set of G. Then

(1) For any p and any $w \in L(G)$, there exists $f \in A^p(\Omega)$ such that

$$\limsup_{z \to w} |f(z)| = \infty.$$

(2) For any neighborhood U of $w \in L(G)$, the Hausdorff dimension of $\partial\Omega \cap U$ is no less than $2n - 1$.

The calculus of variations yields

$$f(z) = \int_\Omega |m_p(\cdot, z)|^{p-2} \overline{K_p(\cdot, z)}\, f, \quad \forall\, f \in A^p(\Omega),$$

which in turn implies that

Proposition 3. (1) $|K_p(z,w)| \le K_p(z)^{\frac{1}{p}}\, K_p(w)^{\frac{1}{q}}$ with equalities hold if and only if $z=w$, where $1/p+1/q=1$.

(2) $\mathrm{Re}\,\{K_p(z,w)+K_p(w,z)\} \le K_p(z)+K_p(w)$, with equalities hold if and only if $z=w$.

The nonlinear fact $|m_p(\cdot,z)|^{p-2}$ in the reproducing formula causes the real difficulty for applications. With the help of techniques from nonlinear analysis of the $p-$Laplacian, we are able to show the following regularity result.

Theorem 4. (1) *For any $p\ge 2$ and any compact set $S\subset\Omega$, there exists a constant $C>0$ such that*

$$|K_p(z,w)-K_p(z,w')| \le C|w-w'|^{\frac{1}{2}}.$$

(2) *For any $1<p<2$ and any compact set $S\subset\Omega$, there exists a constant $C>0$ such that*

$$|K_p(z,w)-K_p(z,w')| \le C|w-w'|^{\frac{1}{p}}.$$

(3) *Let $S_w := \{K_1(\cdot,w)=0\}$. For every open set $w\in U \subset\subset \Omega\backslash S_w$, there exists a constant $C>0$ such that*

$$|K_1(z,w)-K_1(z,w')| \le C|w-w'|^{\frac{1}{2}}, \quad \forall\, z,w'\in U.$$

For a real-valued upper semicontinuous function u defined on a domain $\Omega\subset\mathbb{C}^n$, we define the *generalized Levi form* of u by

$$i\partial\bar\partial u(z;X) := \liminf_{r\to 0+} \frac{1}{r^2}\left\{\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta}X)d\theta - u(z)\right\}.$$

It is well-known that $i\partial\bar\partial \log K_2(z;X) = B_2(z;X)^2$. Using the second variation, we are able to show the following

Theorem 5. (1) $i\partial\bar\partial\log K_p(z;X) \ge \frac{p}{2(p-1)}\, B_p(z;X)^2$ *for $p\ge 2$.*

(2) $i\partial\bar\partial\log K_p(z;X) \ge \frac{p}{2}\, C(z;X)^2$ *for $p\le 2$, where $C(z;X)$ denotes the Carathéodory metric.*

Remark. *In particular, $\log K_p(z)$ is a Lipschitz continuous strictly psh function, so that the minimal set of $K_p(z)$ defined by $\{z\in\Omega : K_p(z)=\inf_{\zeta\in\Omega} K_p(\zeta)\}$ is either empty or a totally real subset of Ω.*

Concerning the asymptotic behavior as $p\to\infty$, we have

Proposition 6. (1) $\lim_{p\to\infty} m_p(z,w)=1$.

(2) $\lim_{p\to\infty} B_p(z;X) = C(z;X)$.

We also have the following stability result.

Theorem 7. (1) $\lim_{s\to p-} K_s(z,w) = K_p(z,w)$ *for $p>1$.*

(2) $\lim_{s\to p+} K_s(z,w)$ *exists. Moreover, if $A^{p'}(\Omega)$ lies dense in $A^p(\Omega)$ for some $p'>p$, then $K_p(z,w)=\lim_{s\to p+} K_s(z,w)$.*

Corollary 8. *If Ω has positive hyperconvexity index, i.e., there exists a negative continuous psh function ρ with $-\rho \lesssim \delta^\alpha$ for some $\alpha > 0$, where δ denotes the boundary distance, then*

$$K_2(z,w) = \lim_{p\to 2} K_p(z,w).$$

On the other hand, we have

Proposition 9. Let $\Omega = D\backslash S$ where D is a bounded domain in $\mathbb{C}$ and S is a compact set in D which has positive 2−capacity but zero p−capacity for every $p < 2$. Then

$$K_2(z) > \lim_{p\to 2+} K_p(z).$$

Recall that the p−capacity of S is given by $\mathrm{Cap}_p(S) := \inf_\phi \int_{\mathbb{C}} |\nabla\phi|^p$ where the infimum is taken over all $\phi \in C_0^\infty(\mathbb{C})$ such that $\phi \geq 1$ on S. The condition of Proposition 0.9 is satisfied for instance, if the h−Hausdorff measure $\Lambda_h(S)$ of S is positive and finite where $h(t) = (\log 1/t)^{-\alpha}$ for some $\alpha > 1$.

Finally, we compare $K_p(z)$ with $K_2(z)$ as follows.

Theorem 10. *Let Ω be a bounded pseudoconvex domain with C^2−boundary.*

(1) *There exist constants $\gamma, C > 0$ such that the following inequalities hold near $\partial\Omega$:*

$$\begin{aligned} K_p(z)^{\frac{1}{p}}/K_2(z)^{\frac{1}{2}} &\leq C\,\delta(z)^{\frac{1}{2}-\frac{1}{p}}|\log\delta(z)|^{\frac{n(p-2)}{2p\gamma}}, \quad p \geq 2,\\ K_p(z)^{\frac{1}{p}}/K_2(z)^{\frac{1}{2}} &\geq C^{-1}\,\delta(z)^{\frac{1}{2}-\frac{1}{p}}|\log\delta(z)|^{-\frac{(n+\gamma)(p-2)}{2p\gamma}}, \quad p \leq 2.\end{aligned}$$

(2) *For every $2 \leq p < 2 + \frac{2}{n}$ there exists a constant $C = C_{p,\Omega} > 0$ such that the following inequality holds near $\partial\Omega$:*

$$K_p(z)^{\frac{1}{p}}/K_2(z)^{\frac{1}{2}} \geq C^{-1}\,\delta(z)^{\frac{(n+1)(p-2)}{2p}}|\log\delta(z)|^{-\frac{(n+1)(p-2)}{2p\gamma}}.$$

Corollary 11. *Let Ω be a bounded pseudoconvex domain with C^2−boundary. For every $2 \leq p < 2 + \frac{2}{n}$, $K_p(z)$ is an exhaustion function on Ω.*

Singular solutions to the homogeneous Monge-Ampère equation via the Hele-Shaw flow

David Witt Nyström
(joint work with Julius Ross)

This is based on joint work/work in progress with Julius Ross.

The Dirichlet problem for the homogeneous Monge-Ampère equation. Recall that if u is psh and locally bounded, then by the work of Bedford-Taylor [3] the Monge-Ampère $MA(u)$ of u is a welldefined positive measure which equals $(dd^c u)^n$ when u is C^2. The Dirichlet problem for the homogeneous complex Monge-Ampère equation (here HMAE for short) asks us to find a psh function u such that $MA(u) = 0$, and such that $u = f$ on the boundary of the given domain. Two particular settings of this Dirichlet problem have been the focus of much research, and I will discuss them both in turn.

The local setting. Let $D \subseteq \mathbb{C}^n$ be a smoothly bounded strictly pseudoconvex domain, and let f be a smooth function on ∂D. Then the Dirichlet problem for the HMAE asks us to find $u \in PSH(D) \cap C(\bar{D})$ such that $MA(u) = 0$ on D, while $u_{|\partial D} = f$. The existence and uniqueness of a solution u was established by Bedford-Taylor [2]. It was later showed by Krylov that the solution lies in $C^{1,1}(\bar{D})$ (see e.g. [8] and references therein).

That Krylov's regularity result is sharp is shown by the following exaple due to Gamelin-Sibony [7].

Example 1. We consider the unit ball $\mathbb{B} \subseteq \mathbb{C}^2$, and let

$$f(z,w) := \begin{cases} (|z|^2 - 1/2)^2, & \text{for } (z,w) \in \partial\mathbb{B},\ |z|^2 \geq 1/2, \\ (|w|^2 - 1/2)^2, & \text{for } (z,w) \in \partial\mathbb{B},\ |w|^2 > 1/2. \end{cases}$$

Then it can be easily show that $u(z,w) := (\max(|z|^2 - 1/2, |w|^2 - 1/2, 0))^2$ solves the Dirichlet problem, and one notes that u is not twice differentiable along the hypersurfaces $|z|^2 = 1/2$ and $|w|^2 = 1/2$.

The global setting. Here we let (X,ω) be a compact Kähler manifold, and Σ a Riemann surface with boundary. In this talk we will only consider the cases when Σ is either the unit disc $\mathbb{D} := \{\tau : |\tau| < 1\}$ or the unit strip $S := \{t + is : 0 < t < 1\}$. We also let $f \in C^\infty(X \times \partial\Sigma)$, and we also assume that $f(\cdot,\tau)$ is ω-psh (i.e. $\omega + dd^c_z f(z,\tau) \geq 0$) for each $\tau \in \partial\Sigma$.

Recall that u is said to be $\pi_X^*\omega$-psh if, given a local potential h of $\pi_X^*\omega$, $u + h$ is psh, and that if u is locally bounded, the Monge-Ampère $MA_{\pi_X^*\omega}(u)$ of u is the positive measure locally defined as $MA(u+h)$.
The Dirichlet problem for the HMAE in this setting now asks us to find $u \in PSH(X\times\Sigma, \pi_X^*\omega) \cap C(X\times\bar{\Sigma})$ such that $MA_{\pi_X^*\omega}(u) = 0$ on $X\times\Sigma$, while $u_{|X\Sigma} = f$.

One should note that when $\Sigma = S$ and $f(z, is) = u_0(z)$ and $f(z, 1+is) = u_1(z)$, then $u_t(z) := u(z,t)$ defines a so called weak geodesic in the space of Kähler metrics cohomologous to ω (or really in its completion) (see e.g. [8]).

The existence and uniqueness of a solution u was established by Chen [5]. Chen with complements by Blocki established the almost $C^{1,1}$ regularity of u [4], while finally the full $C^{1,1}$ regularity up to the boundary was shown by Chu-Tosatti-Weinkove [6].

Lempert-Vivas proved in [11] that in the case of $\Sigma = S$ this is optimal, by showing that there are examples of geodesics u which do not belong to $C^2(X \times \bar{S})$. In comparison with the easy example of Gamelin-Sibony, their example is very unexplicit. For instance one cannot see where it fails to be C^2, if it is on the boundary or in the interior, or both.

In [13] Ross and I constructed a solution on $\mathbb{P}^1 \times \bar{\mathbb{D}}$ which also fails to be C^2. This solution was more explicit, and could be seen to be not twice differentiable along a certain curve on the boundary. However, the behaviour of that solution in the interior was still not easily understood.

The goal of the talk was to explain how one can construct a more explicit solution where the failure of C^2 happen also in the interior. The construction rests on the same foundation as the earlier example, namely the connection between a certain class of solutions and the so called Hele-Shaw flow.

The Hele-Shaw flow. The Hele-Shaw flow models the propagation of a viscous fluid trapped in a thin layer, as in a Hele-Shaw cell, where two glass plates have a small gap between them, and fluid is pumped into this gap through a small hole. The thinness of the layer makes the situation essentially two-dimensional, and the region the fluid uccupies at a time t is represented by a domain Ω_t in $\mathbb{C}$. We place the point of injection at the the origin. The fluid will then spread (approximately) according to Darcy's Law, which in this setting postulates that the normal velocity V_t of the boundary of the fluid domain Ω_t should equal minus the gradient of the pressure p_t in the fluid. If the pressure is assumed to be zero on the boundary, and harmonic away from the injection point, it follows that $-p_t(z) = G_{\Omega_t}(z, 0)$, where $G_{\Omega_t}(z, 0)$ denotes the Greens function of the domain Ω_t with a logarithmic pole at 0.

If one instead assumes that the medium between the plates have varying permeability κ, then the equation becomes

$$V_t = -\kappa \nabla p_t.$$

This setting was first properly inverstigated by Hedenmalm-Shimorin [10].

This is the classical formulation of the Hele-Shaw flow, but there is also a weak formulation which goes back to Gustafsson [9]. Let ϕ be a subharmonic function on $\mathbb{C}$ such that

$$dd^c\phi = \frac{d\lambda}{\kappa},$$

where $d\lambda$ denotes the Lebesgue measure. For $t \geq 0$ we let

$$\phi_t := \sup\{\psi \leq \phi : \psi \in SH(\mathbb{C}), \nu_0(\psi) \geq t\},$$

where $\nu_0(\psi)$ denotes the Lelong number of ψ at 0. We also let $\Omega_t := \{\phi_t < \phi\}$. The increasing family of domains Ω_t is called the (weak) Hele-Shaw flow with respect to ϕ (or $dd^\phi = d\lambda/\kappa$). It can be shown that $dd^c\phi_t = 1_{\{\Omega_t^c\}} dd^c\phi$, so (ϕ, Ω_t) and ϕ_t contains the same information. Thus the family ϕ_t is also called the weak Hele-Shaw flow of ϕ. It can be shown that this indeed is the correct weak notion of the Hele-Shaw flow, but we will not explain that here.

Connection to the HMAE. In [12] Ross and I established the following connection between the Hele-Shaw flow and a class of solutions to the HMAE.

Theorem 2. *If $\phi \in SH(\mathbb{C})$ and $\phi - \ln(1 + |z|^2)$ is bounded, and ϕ_t denotes its weak Hele-Shaw flow, then*

$$u(z, \tau) := \sup_{0 \leq t \leq 1} (\phi_t(\tau z) - t \ln |\tau|^2)$$

solves the HMAE on $\mathbb{C} \times \mathbb{D}$ with the boundary condition $f(z, \tau) := \phi(\tau z)$. Using the Fubini-Study form $\omega_F S$ on $\mathbb{P}^1$, u can be extended to $\mathbb{P}^1 \times \mathbb{D}$.

We also recall the following definition, which is fundamental for the analysis of solutions to the HMAE.

Definition 3. If u solves the HMAE in some domain D, and $h : \mathbb{D} \to D$ is a holomorphic disc, then h is called a harmonic disc if $u \circ h$ is harmonic.

Solutions u of the HMAE that are regular enough locally give rise to foliations of D by harmonic discs. Importantly, if you have a holomorphic fibration which is transverse to such a foliation, the corresponding flow between the fibers will preserve the form $dd^c u$ (see [1]).

In our setting there is a clear correspondence between proper harmonic discs and simply connected Hele-Shaw doamins Ω_t, expressed in the following theorem.

Theorem 4. *The graph of the holomorphic function $g : \mathbb{D} \to \mathbb{P}^1$ is a harmonic disc iff 1): $g = 0$, or 2): $g(\tau) = w/\tau$ for some $w \in \Omega_1^c$, or 3): $\tau \mapsto \tau g(\tau)$ is a Riemann mapping for a simply connected Hele-Shaw domain Ω_t such that* $0 \mapsto 0$.

How to find $u \notin C^2(\mathbb{P}^1 \times \mathbb{D})$? The idea is to find ϕ such that its Hele-Shaw domains Ω_t all are simple connected. Then let $f_1 : \mathbb{D} \to \Omega_1$ be a Riemann mapping such that $f_1(0) = 0$. Thanks to Theorem 4 we will get at foliation of $U \subseteq \mathbb{P}^1 \times \mathbb{D}$ by harmonic discs, where U intersected with the fiber over τ is $f_1(\tau\mathbb{D})/\tau$. The fact mentioned above that the flow along a foliation by harmonic discs transverse to a fibration preserves the form $dd^c\phi$ then shows that u fails to be C^2 along the boundary of U.

The question remains how to find such a ϕ. Here one can use the fact that any nice increasing family of domains Ω_t actually gives rise to a ϕ by the formula

$$dd^c\phi = \frac{|\nabla p_t| d\lambda}{|V_t|}.$$

The problem is to make sure that the resulting ϕ is smooth, but this seems to work, at least if we allow $dd^c\phi$ to be zero at the singular points of Ω_1^c.

One should note that the construction is closely related to the example in [14], but there only the disc corresponding to Ω_1 was used.

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New continuous solutions to Monge-Ampère equations on Hermitian manifolds

Sławomir Kołodziej
(joint work with Cuong Ngoc Nguyen)

We prove the existence of a continuous quasi-plurisubharmonic solution to Monge-Ampère equations with very general the right hand side on a compact Hermitian manifold. These are measures dominated by capacity, in particular, moderate measures studied by Dinh-Nguyen-Sibony. As a consequence, we give a characterization of measures admitting Hölder continuous quasi-plurisubharmonic potential, inspired from the work of Dinh-Nguyen.

The setting is as follows: a compact Hermitian manifold (X,ω) of dimension n. The fundamental form ω is given in local coordinates by

$$\omega = \frac{i}{2}\sum_{k,j} g_{k\bar{j}} dz^k \wedge d\bar{z}^j.$$

The matrix $(g_{k\bar{j}})$ is positive definite and Hermitian symmetric. Such a form exists on any complex manifold, which is not the case for Kähler forms $d\omega = 0$. An upper semicontinuous function u on X is called $\omega-$psh if $dd^c u + \omega \geq 0$ (as currents). Then we write $u \in PSH(\omega)$.

Consider a positive Radon measure μ with finite total mass on X. To solve the complex Monge-Ampère equation for μ we need to find an $\omega-$psh function u and a positive constant c such that

$$(\omega + dd^c u)^n = c\,\mu.$$

For smooth, strictly positive measures Cherrier (1987) obtained smooth solutions of the M-A equation assuming either $n = 2$ or that ω is balanced $(d(\omega^{n-1}) = 0)$. Tosatti-Weinkove (2010) got the statement without those extra assumptions.

Dinew-Kołodziej. and Kołodziej-Nguyen '12-'16: The existence of continuous solutions for $f \in L^p,\ p > 1$.

Stability and Hölder continuity for $f \in L^p,\ p > 1$ by Kołodziej-Nguyen '18 and Lu-Phung-To '20. In the latter the Hölder exponent is improved, matching the one for Kähler manifolds.

Kołodziej-Nguyen '19 proved that Hölder continuous subsolution implies the existence of such solution. Moreover the property that a measure is the M-A mass of a Hölder continuous function is a local property.

To formulate the results we need the Bedford-Taylor capacity: For a Borel subset E of X

$$cap_\omega(E) := \sup \left\{ \int_E (\omega + dd^c u)^n : u \in PSH(\omega), 0 \le u \le 1 \right\}.$$

We shall distinguish classes of measures dominated by the capacity in a suitable way. Let $h : \mathbb{R}_+ \to (0, \infty)$ be an increasing function such that

$$\int_1^\infty \frac{1}{x[h(x)]^{\frac{1}{n}}}\, dx < +\infty.$$

In particular, $\lim_{x\to\infty} h(x) = +\infty$. Such a function h is called *admissible.* In what follows h is always admissible. Set $F_h(x) = \frac{x}{h(x^{-\frac{1}{n}})}$.

Measures dominated by capacity. Let μ be a positive Radon measure satisfying

$$\mu(E) \le F_h(cap_\omega(E)),$$

for any Borel set $E \subset X$ and some F_h. Denote by $\mathcal{F}(X, h)$ the set of all measures satisfying the above inequality for some admissible h.

Here are results for continuous solutions.

Theorem 1. *Let $\mu \in \mathcal{F}(X, h)$ be such that $\mu(X) > 0$. Then, there exists a continuous ω-psh function u and a constant $c > 0$ solving the equation*

$$(\omega + dd^c u)^n = c\, \mu.$$

If $\mu = 0$ then there are no bounded solutions.

Theorem 2. *Let $\mu \in \mathcal{F}(X, h)$, $\mu(X) > 0$ and $\lambda > 0$. Then, there exists a unique continuous ω-psh solution v to*

$$(\omega + dd^c v)^n = e^{\lambda v} \mu.$$

Examples of measures satisfying the assumptions:

- measures with densities in $L^p, p > 1$, or even broader Orlicz spaces,
- smooth forms on smooth hypersurfaces or totally real submanifolds, (those can be shown to belong to one of the classes $\mathcal{H}(\tau)$ which are unions (over $C > 0$) of $\mathcal{F}(X, h_1)$ with $h_1(x) = Cx^{n\tau}$ and fixed $\tau > 0$)

- Borel measures locally dominated by Monge-Ampère measures of continuous plurisubharmonic functions whose moduli of continuity $\eta(t)$ satisfy the Dini-type condition
$$\int_0^1 \frac{[\eta(t)]^{\frac{1}{n}}}{t|\log t|} dt < +\infty.$$
- moderate measures (Dinh-Nguyen-Sibony) - the union over $C > 0, \alpha > 0$ of $\mathcal{F}(X, h_2)$ with $h_2(x) = Ce^{\alpha x}$ for $C, \alpha > 0$.

The next result extends a theorem of Dinh and Nguyen on Hölder continuous solutions for Kähler manifolds.

Theorem 3. *A positive Radon measure μ belongs to $\mathcal{H}(\tau)$ and μ is Hölder continuous on $PSH_{(0,1)}(\omega)$ if and only if there exists a Hölder continuous ω-psh function u and a constant $c > 0$ such that*
$$(\omega + dd^c u)^n = c\,\mu.$$

This is used to generalize to Hermitian manifolds recent results of Pham and Vu.

The principle of least action in the space of Kähler potentials

László Lempert

Let (X, ω) be a compact Kähler manifold and
$$\mathcal{H} = \mathcal{H}_\omega = \{u \in C^\infty(X) : \omega + i\partial\overline{\partial}u = \omega_u > 0\}$$
its space of relative Kähler potentials. Here $C^\infty(X)$ refers to the Fréchet space of real valued smooth functions on X. The space $\mathcal{H}$, as an open subset of a Fréchet space, inherits a Féchet manifold structure, whose tangent bundle has a canonical trivialization $T\mathcal{H} \approx \mathcal{H} \times C^\infty(X)$. In the 1980–90s Mabuchi and Semmes independently and with different motivations introduced a torsion free connection ∇ on $T\mathcal{H}$.

One way to explain ∇ is through its parallel transport. We will use dot $\dot{}$ to denote derivative of a function of one real variable, and grad_v to indicate gradient of a function $X \to \mathbb{R}$ with respect to the Kähler metric of ω_v. Let $u : [a, b] \to \mathcal{H}$ be a smooth path. By integrating the time dependent vector field $(-1/2)\ \mathrm{grad}_{u(t)}\dot{u}(t)$ on X we obtain a smooth family of diffeomorphisms $\varphi(t) : X \to X$. In fact $\varphi(t) : (X, \omega_{u(0)}) \to (X, \omega_{u(t)})$ is symplectomorphic. The parallel translate of $\xi \in T_{u(t)}\mathcal{H} \approx C^\infty(X)$ to $u(0)$ along the path u is then
$$\xi \circ \varphi(t) \in C^\infty(X) \approx T_{u(0)}\mathcal{H}.$$
Given a smooth path $u : [a, b] \to \mathcal{H}$, a vector field
$$\xi : [a, b] \ni t \mapsto \xi(t) \in T_{u(t)}\mathcal{H}$$
along it is parallel if invariant under parallel translation; and the path is a geodesic for ∇ if its velocity vector field $\dot{u}(t)$ is parallel.

Mabuchi, Semmes and Darvas introduced various Riemannian and Finsler metrics that are compatible with the connection. Darvas found that (with slight simplification) geodesics of ∇ are the shortest paths between their endpoints, when length of a path is computed using any of his metrics. The talk generalized Darvas's result to all parallel Finsler metrics, and in fact even beyond, to Lagrangians $L : T\mathcal{H} \to \mathbb{R}$ that are fiberwise convex and continuous, and invariant under parallel translation. From now on L is such a Lagrangian.

Theorem 1 (Principle of least action)**.** *If $v : [0, T] \to \mathcal{H}$ is a geodesic for ∇, then $u = v$ minimizes action*

$$\int_0^T L(\dot{u}(t))dt$$

among all piecewise C^1 paths $u : [0, T] \to \mathcal{H}$ with $u(0) = v(0)$, $u(T) = v(T)$.

The next result is about how least action varies as one moves along geodesics; it is a manifestation of seminegative curvature. Fix $T > 0$. If $w, w' \in \mathcal{H}$, the least action $\mathcal{L}_T(w, w')$ between them is the infimum of the actions $\int_0^T L(\dot{u}(t))dt$ over all piecewise C^1 paths $u : [0, T] \to \mathcal{H}$ connecting w with w'. It is not obvious, but $\mathcal{L}_T(w, w')$ is finite.

Theorem 2. *If $u, v : [a, b] \to \mathcal{H}$ are geodesics for ∇, then the function $\mathcal{L}_T(u, v) : [a, b] \to \mathbb{R}$ is convex.*

We also prove a converse to the Principle of least action: under certain conditions, *only* geodesics minimize action. When L defines Mabuchi's metric, Calabi, Chen, and Darvas already proved this, even for paths more general than what our theorem covers.

Theorem 3. *Suppose $u : [a, b] \to \mathcal{H}$ is piecewise C^1, $v : [a, b] \to \mathcal{H}$ is a geodesic for ∇ connecting $u(a)$ and $u(b)$, and $\int_a^b L(\dot{u}(t))\, dt = \int_a^b L(\dot{v}(t))\, dt$. If L is strictly convex in the sense that for all $w \in \mathcal{H}$, $\xi, \eta \in T_w\mathcal{H}$*

$$L\Big(\frac{\xi + \eta}{2}\Big) < \frac{L(\xi) + L(\eta)}{2} \quad \textit{unless } \xi = \eta,$$

then $u = v$.

In fact, versions of the above theorems hold for a certain generalization of geodesics, called weak geodesics, in the space of bounded ω–plurisubharmonic functions, rather than in $\mathcal{H}$.

Towards unconventional L^2 extension

Takeo Ohsawa

Hörmander's method of comparing L^2 cohomology groups with respect to different weights was revisited and refined to deduce some extension theorems and approximation theorems of new type in [3]. For instance the following have been shown.

Theorem 1. *If M is a weakly 1-complete manifold, A is an effective divisor on M with compact support such that $[A]|_{|A|} \geq 0$ and $E|_{A\cup\overline{\Omega}} > 0$ for some open set $\Omega \subset M$ with compact complement, then $\exists \mu_0 \in \mathbb{N}$ such that $H^{0,0}(M, K_M \otimes E \otimes [A]^\mu) \twoheadrightarrow H^{0,0}(A, K_M \otimes E \otimes [A]^\mu)$ for all $\mu \geq \mu_0$. Here K_M denotes the canonical bundle of M.*

Theorem 2. *In the situation of Theorem 1, assume that A is pseudoconcave of order >1*[1]*, then*

$$H^{0,q}(M, K_M \otimes E \otimes [A]^\mu) \cong H^{0,q}(M \setminus |A|, K_M \otimes E), \quad q \geq 1$$

hold for sufficiently large μ and the set of meromorphic sections of $K_M \otimes E$ with poles (at most) along $|A|$ is dense in $H^{0,0}(M \setminus |A|, K_M \otimes E)$.

Corollary. *Let S be a connected compact complex surface, let $C \subset S$ be a smooth complex curve of finite type in the sense of Ueda (cf. [5]) and let $L \to S$ be a holomorphic line bundle such that $L|_C$ is positive. Then, for sufficiently large μ, one can find holomorphic sections of $K_S \otimes L^3 \otimes [C]^\mu$ whose ratio embeds a neighborhood of C to $\mathbb{CP}^5$.*

[3] is a natural continuation of [1, 2] and followed by [4], where the following is proved.

Theorem 3. (to appear in Pure and Appl. Math. Quart.) *Let $\Omega \Subset M$ be a pseudoconvex domain with C^2-smooth boundary. Assume that M admits a Kähler metric and $K_M|_{\partial\Omega} < 0$. Then there exists a holomorphic map with connected fibers from Ω to $\mathbb{C}^N$ for some $N \in \mathbb{N}$ which is proper onto the image.*

Open questions which seem to be solvable by a similar method were discussed in the talk.

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[1] A is said to be pseudoconcave of order a if $\exists$ a fiber metric of $[A]$ and a neighborhood $U \supset |A|$ s.t. $|s|^{-a}$ is p.s.h. on $U \setminus |A|$ for a canonical section $s : U \to [A]$.

Harmonic forms on almost-Hermitian 4-dimensional manifolds

Nicoletta Tardini
(joint work with Adriano Tomassini)

Let X be a $2n$-dimensional smooth manifold endowed with an almost-complex structure J, namely a $(1,1)$-tensor such that $J^2 = -\mathrm{Id}$. Then, the exterior derivative d splits as the sum of four operators. More precisely, if $A^{p,q}(X)$ denotes the space of (p,q)-forms on X, one has that

$$d : A^{p,q}(X) \to A^{p+2,q-1}(X) \oplus A^{p+1,q}(X) \oplus A^{p,q+1}(X) \oplus A^{p-1,q+2}(X)$$

$$d = \mu + \partial + \overline{\partial} + \bar{\mu}\,.$$

Therefore, in the non-integrable case, the vanishing of d^2 does not imply $\overline{\partial}^2 = 0$ and so the Dolbeault cohomology

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) := \frac{\mathrm{Ker}\,\overline{\partial}}{\mathrm{Im}\,\overline{\partial}}$$

is not well defined. More precisely, the Dolbeault cohomology is well defined if and only if J is integrable.

On a compact complex manifold an effective way to compute the Dolbeault cohomology is to fix an Hermitian metric and compute the associated space of harmonic forms.

If J is non integrable, even though the Dolbeault cohomology is not well defined we can still define the space of harmonic forms. Indeed, if we fix an Hermitian metric g on an almost-complex manifold (X, J) the operator $\Delta_{\overline{\partial}} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is elliptic. When J is integrable and X is compact, $\mathcal{H}^{\bullet,\bullet}_{\overline{\partial}}(X) := \mathrm{Ker}\,\Delta_{\overline{\partial}}$ is isomorphic to the Dolbeault cohomology and so its dimension, denoted by $h^{\bullet,\bullet}_{\overline{\partial}}(X)$, is a holomorphic invariant. When J is non integrable $\mathcal{H}^{\bullet,\bullet}_{\overline{\partial}}(X)$ a priori depends on the metric. This motivated Kodaira and Spencer to raise a question, listed as Problem 20 in a paper by Hirzebruch [1], regarding whether, on compact almost-complex manifolds, the Hodge numbers $h^{\bullet,\bullet}_{\overline{\partial}}(X)$ depend on the choice of the Hermitian metric.

In [2] Holt and Zhang answered positively to this question, giving an explicit construction on the Kodaira-Thurston manifold of an almost-complex structure with $h^{0,1}_{\overline{\partial}}(X)$ varying with different choices of Hermitian metrics.

However, if the Hermitian metric is almost-Kähler and $2n = 4$, in [2, Proposition 6.1] it was shown that $h^{1,1}_{\overline{\partial}} = b_- + 1$, where b_- denotes the dimension of the space of the anti-self-dual harmonic 2-forms. So, in such a case, $h^{1,1}_{\overline{\partial}}$ depends only on the topology of X.

In [2, Question 6.2] the authors asked the following

Question. *Let (M, J) be a compact almost-complex 4-dimensional manifold which admit an almost-Kähler structure. Does it have a non almost-Kähler Hermitian metric such that*

$$h^{1,1}_{\overline{\partial}} \neq b_- + 1?$$

We studied this problem. First of all, in [3] it is shown that on 4-dimensional manifolds $h^{1,1}_{\overline{\partial}}$ is a conformal invariant and so Holt and Zhang's result holds also for globally conformally Kähler metrics.

So, using the existence (and uniqueness up to omotheties) of a Gauduchon metric in every conformal class, namely an Hermitian metric ω such that $dd^c\omega^{n-1} = 0$, we proved the following

Theorem *Let* (X^4, J) *be a compact almost-complex manifold of dimension* 4*, then, with respect to a (strictly) locally conformally Kähler metric,*

$$h^{1,1}_{\overline{\partial}} = b_-.$$

Here, by (*strictly*) *locally conformally Kähler metric* we mean an Hermitian metric ω, such that

$$d\omega = \theta \wedge \omega$$

with θ a d-closed, non d-exact, differential 1-form.

In particular, for locally conformally Kähler and globally conformally Kähler metrics on compact 4-dimensional almost-complex manifolds, $h^{1,1}_{\overline{\partial}}$ is a topological invariant.

Notice that by [4], (strictly) locally conformally Kähler and globally conformally Kähler metrics cannot coexist on compact complex manifolds. However, this is not the case on compact almost-complex manifolds. Indeed, we construct explicitly a family of almost-complex structures J_a, with $a \in \mathbb{R} \setminus \{0\}$, $a^2 < 1$, on the Kodaira-Thurston manifold X that admit both almost-Kähler and (strictly) locally conformally Kähler metrics. More precisely, (X, J_a) is a compact almost-complex 4-dimensional manifold which admit an almost-Kähler metric $\tilde{\omega}_a$ and a non almost-Kähler Hermitian metric ω_a such that

$$h^{1,1}_{\overline{\partial},\omega_a} = b_-\,.$$

Hence, this example answers affirmatively to [2, Question 6.2] in the case of the Kodaira-Thurston manifold endowed with the 1-parameter family of almost-complex structures J_a.

Moreover, this answers to Kodaira and Spencer's question, showing that also the Hodge number $h^{1,1}_{\overline{\partial}}$ depeends on the choice of the Hermitian metric and not just on the almost-complex structure.

Since on compact complex surfaces $h^{1,1}_{\overline{\partial}}$ is either $b_- + 1$ (in the Kähler case) or b_- (in the non Kähler case) we ask the following

Question. *Is there an example of a compact almost-complex* 4*-dimensional manifold admitting an Hermitian metric* ω *such that*

$$h^{1,1}_{\overline{\partial}} \neq b_- \quad \textit{and} \quad h^{1,1}_{\overline{\partial}} \neq b_- + 1?$$

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Mappings from open Riemann surfaces to the twice punctured complex plane and the restricted validity of Gromov's Oka principle

Burglind Jöricke

According to Gromov the Oka principle holds for holomorphic mappings from a complex manifold X to a complex manifold Y if each continuous mapping $X \to Y$ is homotopic to a holomorphic mapping. Giving sufficient conditions on the target Y for the validity of the Oka principle for holomorphic mappings from any Stein manifold to Y, he initiated a line of interesting and fruitful research.

On the other hand he mentions mappings from annuli to the twice punctured complex plane as simplest example for which this Oka principle fails and draws attention to the fact that mappings from annuli play a crucial role for understanding the "rigidity" of the target Y in case the Oka principle fails for mappings from some Stein manifolds to Y.

We address the question of the restricted validity of Gromov's Oka principle and obstructions to this principle in case the target is not a Gromov-Oka manifold. We restrict ourselves to the case of the twice punctured complex plane as target.

Obstructions for Gromov's Oka principle are based on the relation between conformal invariants of the source and the target. Conformal invariants are built on Ahlfors' conformal module of an annulus.

For a complex manifold $\mathcal{X}$ and a free homotopy class $\hat{e}$ of loops in $\mathcal{X}$ (equivalently, for a conjugacy class of elements of the fundamental group of $\mathcal{X}$) we put $\mathcal{M}(\hat{e}) = \sup m(A)$ where the supremum ranges over all annuli $A = \{0 < r < |z| < R < \infty\}$ that admit a holomorphic mapping to $\mathcal{X}$ whose restriction to $\{|z| = \sqrt{rR}\}$ represents $\hat{e}$. This invariant was introduced by Gromov for targets that are not Gromov-Oka manifolds. It is supposed to capture certain "conformal rigidity' of such a manifold.

We give upper and lower bounds (differing by multiplicative constants) of the conformal modules of conjugacy classes of elements of $\pi_1(\mathbb{C}\setminus\{-1,1\},0)$ introduced by Gromov, by quantities that are expressed in terms of certain representing words and are easy to compute.

A mapping $f : X \to \mathbb{C} \setminus \{-1,1\}$ is called reducible if it is homotopic (as a mapping to the twice punctured plane) to a mapping with image contained in a punctured disc. There are infinitely many homotopy classes of reducible mappings from a finite open Riemann surface with non-trivial fundamental group to $\mathbb{C} \setminus \{-1,1\}$, and if the Riemann surface has only thick ends, each reducible homotopy class contains a holomorphic map.

On the other hand the estimates of Gromov's invariants imply for instance, that for X_ε being the ε-neighbourhood of a skeleton of a torus with a hole, the number of irreducible holomorphic mappings $X_\varepsilon \to \mathbb{C} \setminus \{-1,1\}$ up to homotopy grows exponentially in $\frac{1}{\varepsilon}$. This is a statement on the restricted validity of Gromov's Oka principle.

Further, we will say that a continuous mapping f from a finite open Riemann surface X to the twice punctured complex plane has the Gromov-Oka property if for each orientation preserving homeomorphism $\omega : X \to \omega(X)$ onto a Riemann surface $\omega(X)$ with only thick ends the mapping $f \circ \omega^{-1}$ is homotopic to a holomorphic mapping. For finite open Riemann surfaces we show the existence of finitely many embedded annuli in X, such that f has the Gromov-Oka property iff its restriction to each of the annuli has this property, and describe all mappings with the Gromov-Oka property. The mappings with the Gromov-Oka property are the reducible mappings, and in case X is the two-sphere with at least three holes there are also irreducible homotopy classes of mappings with the Gromov-Oka property. Each consists of mappings with the following property. For each orientation preserving homeomorphism $\omega : X \to \omega(X)$ onto a Riemann surface $\omega(X)$ (maybe, of first kind) $f \circ \omega^{-1}$ is homotopic to a holomorphic mapping that extends to a conformal mapping $\mathbb{P}^1 \to \mathbb{P}^1$ that maps three points in different holes to $-1, 1$ and ∞, respectively (in some order depending on the class).

Analytic problems on domains with bounded intrinsic geometry

Andrew Zimmer

The aim of this talk is to summarize some of the results from [3] and [4] concerning analytic problems on domains satisfying a certain bounded geometry condition.

In [3] we introduced the following class of domains.

Definition 1. [3, Definition 1.1] A domain $\Omega \subset \mathbb{C}^d$ has *bounded intrinsic geometry* if there exists a complete Kähler metric g on Ω such that

- the metric g has bounded sectional curvature and positive injectivity radius,
- there exists a $\mathcal{C}^2$ function $\lambda : \Omega \to \mathbb{R}$ such that the Levi form of λ is uniformly bi-Lipschitz to g and $\|\partial\lambda\|_g$ is bounded on Ω.

Many families of domains have bounded intrinsic geometry such as

(1) strongly pseudoconvex domains,
(2) finite type domains in $\mathbb{C}^2$,
(3) convex domains or more generally $\mathbb{C}$-convex domains which are Kobayashi hyperbolic (with no boundary regularity assumptions),
(4) simply connected domains which have a complete Kähler metric with pinched negative sectional curvature,
(5) bounded homogeneous domains, and
(6) the Bers embeddings of the Teichmüller space of hyperbolic surfaces of genus g with n punctures.

Further, by definition, any domain biholomorphic to one of the domains listed above also has bounded intrinsic geometry.

In [3] we considered the problem of characterizing when the $\bar{\partial}$-Nuemann operator N_q on $(0, q)$-forms is compact. A classical result of Fu-Straube [1] states that for bounded convex domains, the operator N_q is compact if and only if the boundary does not contain any q-dimensional analytic varieties. In [3], we extended their result as follows.

Theorem 2 (Z. [3]). *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded domain with bounded intrinsic geometry and g_Ω is the Bergman metric on Ω. Then the following are equivalent:*

(1) *N_q is compact.*

(2) *If $g_{\Omega,z}$ is identified with the d-by-d matrix $\left[g_{\Omega,z}(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})\right]$, then*

$$\lim_{z \to \partial\Omega} (\text{the } q^{th}\text{-smallest singular value of } g_{\Omega,z}) = \infty.$$

If, in addition, $\partial\Omega$ is $\mathcal{C}^0$, then the above conditions are equivalent to:

(3) *$\partial\Omega$ does not contain any q-dimensional analytic varieties.*

In [4] we considered the problem of characterizing the L^2 symbols where the associated Hankel operator is compact. More precisely, given a pseudoconvex domain $\Omega \subset \mathbb{C}^d$ let $A^2(\Omega) \subset L^2(\Omega)$ denote the subspace of holomorphic square integrable (with respect to the Lebesgue measure μ) functions on Ω. Then let $P_\Omega : L^2(\Omega) \to A^2(\Omega)$ denote the Bergman projection, i.e. the orthogonal projection of $L^2(\Omega)$ onto $A^2(\Omega)$. Finally, given $\phi \in L^2(\Omega)$, the associated *Hankel operator* H_ϕ has domain

$$\operatorname{dom}(H_\phi) = \left\{ f \in A^2(\Omega) : \phi \cdot f \in L^2(\Omega) \right\}$$

and is defined by

$$H_\phi(f) = (\operatorname{id} - P_\Omega)(\phi \cdot f) = \phi \cdot f - P_\Omega(\phi \cdot f).$$

For strongly pseudoconvex domains, Li [2] characterized the symbols in $L^2(\Omega)$ for which the associated Hankel operator is compact in terms of how well the symbol is approximated by holomorphic functions on each open metric ball $\mathbb{B}_\Omega(\zeta, r)$ in the Bergman metric.

Theorem 3 (Li [2]). *Suppose $\Omega \subset \mathbb{C}^d$ is a strongly pseudoconvex domain and $\phi \in L^2(\Omega)$. Then the following are equivalent:*

(1) *H_ϕ extends to a compact operator on $A^2(\Omega)$,*

(2) *for some $r > 0$*

$$\lim_{\zeta \to \partial\Omega} \inf \left\{ \frac{1}{\mu(\mathbb{B}_\Omega(\zeta, r))} \int_{\mathbb{B}_\Omega(\zeta,r)} |\phi - h|^2 \, d\mu : h \in \operatorname{Hol}\Big(\mathbb{B}_\Omega(\zeta, r)\Big) \right\} = 0.$$

For strongly pseudoconvex domains, one can show that on each sufficiently small metric ball $\mathbb{B}_\Omega(\zeta, r)$ the volume form $\frac{1}{\mu(\mathbb{B}_\Omega(\zeta,r))} d\mu$ is uniformly equivalent to the volume form dV_Ω induced by the Bergman metric. Also, for strongly pseudoconvex domains one can show that H_ϕ is densely defined for any $\phi \in L^2(\Omega)$. For a general pseudoconvex domain, we let $\mathcal{S}(\Omega) \subset L^2(\Omega)$ denote the symbols ϕ where H_ϕ is densely defined.

With these remarks in mind, we extend Li's theorem to general domains with bounded intrinsic geometry as follows.

Theorem 4 (Z. [4]). *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded domain with bounded intrinsic geometry and $\phi \in \mathcal{S}(\Omega)$. Then the following are equivalent:*

(1) *H_ϕ extends to a compact operator on $A^2(\Omega)$,*
(2) *for some $r > 0$*

$$\lim_{\zeta \to \partial\Omega} \inf \left\{ \int_{\mathbb{B}_\Omega(\zeta,r)} |\phi - h|^2 \, dV_\Omega : h \in \mathrm{Hol}\Big(\mathbb{B}_\Omega(\zeta,r)\Big) \right\} = 0.$$

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Finite type domains and plurisubharmonic polynomials

Berit Stensønes

Problem. *Let $\Omega \subset \mathbb{C}^n$ and f be a bounded $(0,1)-$form such that $\overline{\partial} f = 0$. Can we find a function u such that*

$$\overline{\partial} u = f$$

and

$$\|u\|_\infty \leq C(\Omega)\|f\|_\infty$$

where $C(\Omega)$ only depends on the domain.

If Ω is strongly pseudoconvex, then there is a positive answer (see Henkin, Ramirez... 1970's). Sibony found a smoothly bounded pseudoconvex counterexample in $\mathbb{C}^3$. We shall restrict our attention to pseudoconvex domains of finite type.

If $\Omega \subset \mathbb{C}^2$ the answer is positive. This is due to Fornæss, Fefferman-Kohn and Range. So the task is to study the case when $\Omega \subset \mathbb{C}^n$, $n \geq 3$. The following is joint work with Dusty Grundmeier and Lars Simon.

Let $\Omega \subset\subset \mathbb{C}^n$ pseudoconvex with real analytic boundary, then given a point $p \in \partial\Omega$, we can find local coordinates $(z_1, \ldots, z_n)$ such that:
(1) $p = 0$
(2) Locally

$$\Omega = \{\Re(z_1) + r(\tilde{z}, \overline{\tilde{z}}) + \mathcal{O}(z_1\tilde{z}, \|z_1\|^2) < 0\}$$

where $\tilde{z} = (z_2, \ldots, z_n)$.

Definition. *Ω can be bumped to type at 0 if there exists another pseudoconvex domain Ω^* such that $\overline{\Omega} \setminus \{0\} \subset \Omega^*$, $0 \in \partial\Omega^*$ and we can find a smooth function Φ such that*
(1) $\Phi = z_1 - AF(z_2, \dots, z_n)$ where $\{z; \Phi(z) = 0\} \subset \mathbb{C}^n \setminus \overline{\Omega}^*$ except for 0
(2) If $z = (z_1, \dots, z_n) \in \partial\Omega$ then dist $(z, \partial\Omega^*) \geq c|\Phi(z)|$.

Kohn-Nirenberg example:

$$\begin{aligned} \Omega &= \{\Re(z_1) + |z_2|^8 + \frac{15}{7}|z_2|^2\Re(z_2^6) < 0\} \\ &= \{\Re(z_1) + r(z_2, \overline{z}_2) < 0\}. \end{aligned}$$

Now, r will be both positive and negative in different directions. Choose $A > 0$ large and $\Phi = z_1 - A|z_2|^8$. We have that $\frac{\partial^2 r}{\partial z \overline{\partial} z} \geq |z_2|^6$. Let

$$r'(z) = (1 - \frac{1}{32})|z|^8 + \frac{15}{7}|z_2|^2\Re(z_2^6).$$

Then r' is subharmonic so

$$\Omega^* = \{\Re(z_1) + r'(z_2, \overline{z}_2) < 0\}$$

is pseudoconvex. Let $(z_1, z_2) \in \partial\Omega$, then

$$\text{dist}((z_1, z_2), \partial\Omega^*) \sim |z_1| + |z|^8.$$

Further if $\Phi = 0$ then $|z_1| = A|z_2|^8$ so

$$\Re(z_1) + r'(z_2) = A|z_2|^8 + r'(z_2) > 0.$$

Theorem. *If $\Omega \subset\subset \mathbb{C}^n$ is pseudoconvex with real analytic boundary and Ω can be bumped to type at every boundary point then we get supnorm estimates for $\overline{\partial}$*

Theorem. *If $n = 3$, then Ω can be bumped to type.*

In order to prove bumping to type we need to study plurisubharmonic polynomials. We look at

$$\{\Re(z_1) + r(\tilde{z}, \overline{\tilde{z}}) + \mathcal{O} < 0\}$$

To bump we have to study $r(\tilde{z}, \overline{\tilde{z}})$

$$r = \sum_{j=2k} P_j.$$

The lowest degree is even since Ω is pseudoconvex, moreover P_{2k} is plurisubharmonic and homogeneous.

Question. *Is there a positive function G such that $G \geq |P_{2k}|$ and there is another homogeneous plurisubharmonic P'_{2k} such that*

$$P'_{2k} \leq P_{2k} - \epsilon G$$

If yes, we get similar estimates for weighted homogeneous polynomials.

If P_{2k} vanishes on a complex line L, we need to look at the higher order terms of r.

The main difficulty is when there are complex structures in Γ, the set where the Levi-form is degenerate, along which P_{2k} is pluriharmonic.

This only appears when our domain is in $\mathbb{C}^n$, $n \geq 3$, and P_{2k} is a function of two or more complex variables.

In a joint work with Gautam Bharali we were able to give a positive answer to Question 1 when P_{2k} is a polynomial in 2 complex variables.

We have several results on homogeneous plurisubharmonic, nonpluriharmonic polynomials $P_{2k} : \mathbb{C}^2 \to \mathbb{R}$.

(1) P_{2k} can only vanish on finitely many complex lines through 0. If P_{2k} is not plurisubharmonic this is false: Example $|z|^4 - |w|^4$. Also, this result does not hold if P_{2k} depends on 3 or more variables. But Lars Simon has been able to prove: If $P_{2k} : \mathbb{C}^n \to \mathbb{R}$ is as above, then it is pluriharmonic along at most finitely many hyperplanes through 0.

Challenge: Describe the family of complex lines along which P_{2k} is harmonic.

When we were looking at how P_{2k} vanish along such a complex line we stumbled on the following result:

(2) If $P_{2k}(z, \overline{z}, w, \overline{w})$ is homogeneous in $z, \overline{z}$ and $w, \overline{w}$ separately, then $P_{2k} = s(z^\alpha w^\beta)$ where s is subharmonic.

What if P_{2k} is a polynomial of m variables and P_{2k} is homogeneous in ℓ variables can we find a map $F : \mathbb{C}^m \to \mathbb{C}^{m-\ell}$ and a plurisubharmonic polynomial $Q : \mathbb{C}^{m-\ell} \to \mathbb{R}$ such that $P_{2k} = Q \circ F$?

(3) If P_{2k} is harmonic along $\Sigma_c = \{g(z, w) = c\}$ for an open set of values c for some holomorphic g, then again

$$P_{2k} = s(f(z, w))$$

where s is subharmonic and f is holomorphic.

Question A. *P_{2k} as above. Assume there exists holomorphic map*

$$G : \mathbb{C}^n \to \mathbb{C}^m, 1 \leq m \leq n - 1,$$

nonsingular on an open set $U \subset \mathbb{C}^n$. Assume that P_{2k} is harmonic along every level set of $G_{|U}$. Does this mean that there is a plurisubharmonic $Q : \mathbb{C}^m \to \mathbb{R}$ and holomorphic polynomials $F_1, \ldots, F_m : \mathbb{C}^n \to \mathbb{C}$ such that $P = Q \circ (F_1, \ldots, F_m)$.

Nevanlinna's theory, equidistribution of preimages

Nessim Sibony

The goal is to draw some analogies between holomorphic dynamical systems and equidistribution problems in Nevanlinna's Theory. A sample of the questions considered is as follows.

Let $f : (\mathcal{Y}, \sigma) \to (\mathcal{N}, \omega)$, be a non-degenerate holomorphic map, from an open complex manifold $\mathcal{Y}$ of dimension m, with a good p.s.h. exhaustion function σ, and ω is a Kähler form on $\mathcal{N}$, a compact complex manifold of dimension n.

A basic question in Nevanlinna's theory is to study the distribution of preimages under the map f, of subvarieties $\mathcal{D}_a$ of codimension p, $\mathcal{D}_a \subset \mathcal{N}$, which are in the same cohomology class.

When $p = 1$, we are given a large family of hypersurfaces $\mathcal{D}_a$, parametrized by a complex manifold V. We assume they are in the cohomology class of ω.

Large means that it satisfies the following condition

$$\bigcap_{a \in V} \mathcal{D}_a = \emptyset.$$

Or large enough with respect to $f : \mathcal{Y} \to \mathcal{N}$. i.e.

$$\bigcap_{a \in V} \mathcal{D}_a \bigcap f(\mathcal{Y}) = \emptyset.$$

As usual we define

$$\begin{aligned}
T_f(\omega, r) &:= \int f^*(\omega) \wedge (dd^c \log \sigma)^{m-1} \log^+ \frac{r}{\sigma} \\
N(\mathcal{D}_a, r) &:= \int f^*(\mathcal{D}_a) \wedge (dd^c \log \sigma)^{m-1} \log^+ \frac{r}{\sigma} \\
m(a, r) &:= \int f^*(u_a) d^c \log \sigma|_{\S(r)} \wedge (dd^c \log \sigma)^{m-1} \\
d(a, r) &:= \int_{\mathbb{B}(r)} f^*(u_a)(dd^c \log \sigma)^m.
\end{aligned}$$

The Valiron defect for the hypersurface $\mathcal{D}_a$ is defined as

$$\Delta(\mathcal{D}_a) = 1 - \liminf_{r \to R} \frac{N(\mathcal{D}_a, r)}{T(r)} = \limsup_{r \to R} \frac{m(a, r)}{T(r)}.$$

We show that the Valiron defect is zero except on a pluri-polar set of parameters, provided $T_f(\omega, r) \to \infty$, as $r \to R$ and $(dd^c \log \sigma)^m$ is compactly supported.

This shows that there is an optimal estimate for the growth of the preimage except for a "small" set of parameters.

We show that except on a a pluripolar set of parameters a,

$$\frac{1}{T_f(\omega, r)} \big(f^* \omega - f^*(\mathcal{D}_a) \big) \log^+ \frac{r}{\sigma}$$

converges to zero, as currents. This shows that the preimages distribute according to the same pattern, except for a negligible set of parameters.

I will discuss also the case of points ,i.e. $p = n$. The notion of "small" set has to be modified. It depends on the type of singularities, for the kernel solving the dd^c-equation on $\mathcal{N}$, for currents of bi-degree (p, p), cohomologous to zero.

Under a hypothesis on the various characteristics, we obtain similar equidistribution results, for $p = n$.

Interior Regularity of the Complex Monge-Ampère Equation in Non-smooth Domains

Zbigniew Błocki

The following result generelized the Bedford-Taylor [1] solution of the Dirichlet problem for the complex Monge-Ampère equation from strongly pseudoconvex to more general domains:

Theorem. ([2]) *Assume that Ω is a bounded hyperconvex domain in $\mathbb{C}^n$. Then for any $f \in C(\overline{\Omega})$, $f \geq 0$, there exists unique $u \in PSH(\Omega) \cap C(\overline{\Omega})$ such that $u = 0$ on $\partial\Omega$ and $(dd^c u)^n = f\, d\lambda$ in Ω.*

The talk is a survey on results related to the following problem which remains still open:

Main Conjecture. If $f \in C^\infty(\Omega)$, $f > 0$, then $u \in C^\infty(\Omega)$.

Thanks to the estimate of Pogorelov [5] it is known that the analogous problem for the real Monge-Ampère equation and bounded convex domains has an affirmative solution. Essentially the only known case of a non-smooth domain in $\mathbb{C}^n$ where the answer is positive is a polydisc (see [3]). Cheng-Yau proved the following version of the Pogorelov estimate in the complex case:

Theorem. ([4]) *Assume that $u \in PSH \cap C^4(\Omega)$ is such that $u = 0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{C}^n$. If $\det(u_{i\bar{j}}) = f > 0$ then*

$$(-u)^{2n-1}\Delta u \leq C,$$

where C depends only on f, diam Ω, and on the upper bound for $\sup_\Omega u^{i\bar{j}} u_i u_{\bar{j}}$.

It would be therefore enough to prove an interior estimate for $u^{i\bar{j}} u_i u_{\bar{j}}$. A possible step in this direction is the following estimate:

Theorem. *Assume that $u \in PSH \cap C^4$, $u < 0$, and $\det(u_{i\bar{j}}) = f > 0$. Then for $w := u^{i\bar{j}} u_i u_{\bar{j}} - u$ we have*

$$u^{p\bar{q}}(\log w)_{p\bar{q}} \geq -\frac{1}{w}\sum_{i,j} \frac{u_i u_{\bar{j}} (\log f)_{\bar{i}j}}{u_{i\bar{i}} u_{j\bar{j}}}.$$

In particular, if $f \equiv 1$ then $u^{p\bar{q}}(\log w)_{p\bar{q}} \geq 0$.

One of plausible conjectures is that for convex domains in $\mathbb{C}^n$ the expression $u^{i\bar{j}} u_i u_{\bar{j}}$ is globally bounded in Ω which would settle the main conjecture for convex domains.

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