Mathematisches Forschungsinstitut Oberwolfach

Report No. 28/2021

DOI: 10.4171/OWR/2021/28

Enumerative Geometry of Surfaces (hybrid meeting)

Organized by
Gaëtan Borot, Berlin
Samuel Grushevsky, Stony Brook
Martin Möller, Frankfurt

13 June – 19 June 2021

Abstract. The recent developments in hyperbolic geometry and flat geometry in real dimension 2 formed the core of the workshop, with an emphasis on enumerative aspects. A particularly important role in this regard was played by intersection-theoretic techniques on $\overline{\mathcal{M}}_{g,n}$, the geometry of the strata of differentials, the geometry of Hurwitz spaces, topological recursion techniques, and large genus asymptotics. The workshop included an exploration of relations with similar problems in complex dimension 2, tropical techniques for enumerative problems, and relations to mathematical physics.

Mathematics Subject Classification (2010): 32G15, 14H10, 30F30.

Introduction by the Organizers

The workshop *Enumerative Geometry of Surfaces* was aimed at bringing together the communities working on geometric and counting problems related to two important types of metrics on Riemann surfaces (aka complex algebraic curves). On the one hand, there are hyperbolic metrics. On the other hand, the choice of a holomorphic one-form on a Riemann surface determines a flat metric with conical singularities at the zeroes of the differential, and the dynamics of the $\mathrm{SL}_2(\mathbb{R})$-action on the moduli of flat surfaces has motivated various counting problems, the first of them being the determination of Masur-Veech volumes. Many experts on either the flat or hyperbolic (or both) geometry viewpoint were present at this workshop (in person or remotely, thanks to the hybrid format). The program was comprised of 17 research talks by participants (50min + questions), to which were added 2 evening talks of the same format by distinguished guests Maxim Kontsevich and

Amol Aggarwal. The diverse techniques (hyperbolic geometry, differential geometry and topology, algebraic geometry and intersection theory, tropical geometry, dynamics, combinatorics) relevant to such questions were represented. Most talks addressed the above core areas, and a few of them outreached towards adjacent topics. The majority of results presented by the lecturers date from the last two years, or concerned ongoing work, showing how fast the boundary of knowledge has moved. Although the main focus was on enumerative questions in complex dimension 1/real dimension 2, the richer but less understood world of complex dimension 2 was also discussed in some talks.

The moduli space of algebraic curves (or Riemann surfaces) of genus g is one of the classical object of study in modern algebraic geometry. Solutions of many enumerative geometry questions about Riemann surfaces can be expressed in terms of intersection theory on the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of curves. Fundamental examples of this fact are the famous proof of Witten's conjecture on intersection numbers of ψ-classes, first by Kontsevich (using graph counting ideas from mathematical physics), then by Mirzakhani (using the ideas from the hyperbolic geometry of surfaces), as well as the discovery by Ekedahl–Lando–Shapiro–Vainshtein of an intersection-theoretic formula for the number of simple branched covers of the Riemann sphere. Even three decades after Kontsevich's proof, basic asymptotic questions regarding the behavior of intersection numbers on $\overline{\mathcal{M}}_{g,n}$ in large genus, which has a direct relevance in dynamics, were only recently solved or are just about to be solved, as *Aggarwal* reported in his talk.

In recent years the intersection theory on suitable compactifications of moduli spaces of flat surfaces has revealed striking parallels to that on $\overline{\mathcal{M}}_{g,n}$, on the one hand, and to the geometry of the double ramification cycle on the other hand. This, in turn, connects to counting problems arising from Hurwitz spaces, i.e. spaces of branched covers of the Riemann sphere. Core questions of flat geometry were addressed by *Goujard* on Masur-Veech volumes for all strata quadratic differentials, and in terms of compactifications and evaluation of intersection numbers — by *Nguyen* and *Schmitt*. More precisely, the talk of *Nguyen* described the incidence variety compactification of the moduli of k-differentials with prescribed orders of zeroes and poles, and in this context obtained a formula for their volume as an intersection number of divisors.In his talk, *Schmitt* explained that such intersection numbers can be compared (including boundary contributions) to the top intersection of ψ-classes on the double ramification locus in $\overline{\mathcal{M}}_{g,n}$. For the latter, he gave an efficient formula whose flavor is similar to those obtained by Okounkov-Pandharipande in the Gromov-Witten theory of $\mathbb{P}^1$ about 20 years ago; he also conjectured a formula of the same kind for refined counts taking into account the parity of spin structures which appear for k odd. Besides, *Goujard* presented her ongoing work leading to a combinatorial formula (in terms of ribbon graph counts previously studied by Kontsevich in his 1991 work on Witten's conjecture) for the Masur–Veech volume of an arbitrary stratum of the moduli of quadratic differentials.

In the talk of *Lewański*, we also heard about the refined count of branched covers of the Riemann sphere by spin parity, which are called spin Hurwitz numbers. He explained that, very much like the usual Hurwitz numbers, the spin Hurwitz numbers can be computed in three ways: intersection theory on $\overline{\mathcal{M}}_{g,n}$ using notably Chiodo classes (now twisted by Witten's 2-spin class), the semi-infinite wedge and integrability (now with neutral fermions and BKP hierarchy instead of charged fermions and KP), and topological recursion (now involving $\mathbb{Z}_2$-equivariant spectral curves). In particular, he showcased the appearance of double Hodge integrals in this problem, and mentioned expected but conjectural relations to the Gromov-Witten theory of complex surfaces. The talk of *Oberdieck* explained how to approach curve counting in complex surfaces via the Gromov-Witten theory of their Hilbert scheme, and proposed fundamental conjectures that should govern this computation for K3 surfaces: quasi-Jacobi modularity, a relation between multicover counts and Hecke operators, and a holomorphic anomaly equation.

Farkas presented recent developments in counting of maps of fixed degree d from a curve of genus g to $\mathbb{P}^r$ with marked points on the source being sent to marked points on the target. When $r = 1$, this relates to the geometry of Hurwitz spaces and the computation of Tevelev degrees. For general r, Farkas gave an interpretation of this problem in terms of intersection theory on the Grassmannian, which led to a formula for a restricted range of (g, d) in terms of Schubert calculus.

The talks of *Cavalieri* and *Markwig* highlighted the tropical aspects of curve counting problems, which is an important trend of modern research. Although it has not yet been connected to the available corpus of knowledge on the aforementioned questions, we can still notice common themes and the possibility of future connections. Starting from the classical problem of counting bitangents of plane quartics in complex or real algebraic geometry, *Markwig* presented a tropical analogue of this problem which led her to formulate an arithmetic count (taking values in the Grothendieck ring of quadratic forms on a given base field) which is capable of retrieving simultaneously the signed counts over the reals and the usual (complex) count. In his talk, *Cavalieri* reported on the construction ψ-classes on the tropical compactified moduli space of curves, which contains more information than the usual ψ-classes on $\overline{\mathcal{M}}_{1,1}$, but whose intersection theory in adequate situations retrieves the usual intersection theory. He exemplified this with the tropical analogue of the computation of $\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}$. The construction of tropical analogues of the other natural classes on the moduli space of curves, and their relevance to enumerative geometry questions, which is well-documented in the case of $\overline{\mathcal{M}}_{g,n}$, is an interesting question for the future. One can note that, when lifted to the Teichmüller space, the tropical compactification seems to sit in between the augmented Teichmüller space (whose quotient by the mapping class group gives the Deligne-Mumford compactification) and the (much bigger) Thurston compactification using measured foliations.

In Thurston compactification, projectivized measured foliations on real surfaces are used to describe degenerations of hyperbolic surfaces. In his talk, *Filip* started with this point of view to propose an analogue of measured foliations on those K3 surfaces whose automorphism group is a lattice in $SO_{1,\rho-1}(\mathbb{R})$. This is done by associating a closed positive $(1,1)$-current to a class in the boundary of the ample cone. For certain aspects this is related to the degeneration of the (unique) Ricci-flat metric in a given cohomology class that one lets converge to the boundary of the ample cone.

A second important direction covered by the workshop was the hyperbolic geometry of surfaces, and the many developments building upon the works of Mirzakhani. Although the emphasis of this workshop was not on dynamics, many of these questions, for instance about the relative frequency of various topological types of curves or the first eigenvalue λ_1 of the Laplacian, have a direct relevance for the dynamics on Teichmüller space. The Selberg trace formula was mentioned in several talks. In relation with a special case of Selberg's $\frac{1}{4}$-conjecture, an important open problem is showing that λ_1 should be close fo $\frac{1}{4}$ (the value for the hyperbolic upper half-plane) for a typical hyperbolic surface of large genus g chosen randomly with respect to the Weil-Petersson metric on the moduli space of curves. *Wright* reported important recent advance on this problem, showing that for any $\epsilon > 0$, the probability of $\lambda_1 > \frac{3}{16} - \epsilon$ converges to 0 when $g \to \infty$. The method hinges on a clever use of Selberg trace formula. *Petri*'s talk addressed extremal questions about λ_1 and its multiplicity m_1, the systole and the kissing number on hyperbolic manifolds, and reported some results in any dimension for the kissing number and proved with help of the Selberg trace formula that the maximum of m_1 is realized by the Klein quartic. *Andersen* presented the formalism of geometric recursion as a far-reaching generalization and abstraction of Mirzakhani-McShane identities, allowing a construction of natural functions (having e.g. an enumerative meaning) on the moduli space of curves by applying the cut-and-paste recursively. The integrals of these functions against the Weil-Petersson measure automatically satisfy a topological recursion. He described an ongoing work providing a version of Mirzakhani's identity for surfaces with boundary and corners, opening the way towards a geometric recursion for statistics of closed curves and (by means of the Selberg trace formula) statistics of eigenvalues of the Laplacian.

Arana-Herrera described power saving error terms for the asymptotic growth of the number of filling curves of length $L \to \infty$ on a fixed hyperbolic surface. *Erlandsson* studied the rigidity of hyperbolic metrics having conical points of angle $> 2\pi$, i.e. asking whether the "endpoints" of infinite geodesics (in the universal cover) determine the metric, and gave a positive answer (up to orbifold branched covers) to this question. In the rigid situation this gives for instance a characterization of the metric of hyperbolic polygons in terms of their billiard dynamics.

In a beautiful combination of differential-geometric and algebro-geometric tools, *Norbury* related the volume of the moduli space of super Riemann surfaces to intersection theory of the Θ-class (coming from the geometry of 2-spin curves) and the Weil-Petersson class, up to foundational unresolved questions. He proposed a generalization of Witten's conjecture, namely that the generating series of these volumes should be a tau function of the KdV hierarchy, and should furthermore satisfy the topological recursion (i.e. obey Virasoro constraints), the equivalence of the two characterizations being a theorem. The topological recursion is expected to come from the integration of the super Mirzakhani identity found by Stanford and Witten, by a mechanism similar to the one mentioned in Andersen's talk.

Three talks have put forward the relation of the core areas of the workshop to currently hot topics in mathematical physics. *Zvonkine* showed how to reformulate the study of Laughlin states (that provide a model for certain types of quantum Hall effects) as the study of the space of sections of a line bundle on the n'th symmetric power of a complex curve. Classical algebro-geometric techniques can then be applied to answer (in terms of characteristic classes) physics questions such as determining the number of states, the behavior of the system under change of the magnetic field and/or the underlying metric surface. Flat geometry and more specifically the geometry of trajectories of a differential is relevant in the study of asymptotics of solutions of ODEs (aka WKB expansions in physics) and their Stokes phenomena. This in turn plays an important role in the determination of BPS invariants in gauge theory. Motivated by applications to WKB, *Korotkin* related the Goldman and the Kostant-Kirillov symplectic structure on character varieties of complex curves, and showed how to express them in terms of Fock-Goncharov coordinates depending on a graph on the curve (which in the WKB context is determined by the geometry of trajectories of the underlying differential). He also emphasized the role of tau functions as canonical transformations, described their transformation under the change of graphs (so-called mutations in the language of cluster algebras). Such transformations are important to describe wall-crossing phenomena in the asymptotics of solutions of families of ODEs. In a different vein, *Kontsevich* sketched an algorithmically effective approach to describe the wall-crossing phenomena for abelian differentials, which in concrete terms means the determination of a Novikov ring generating series of (families of topologically equivalent) trajectories of abelian differentials. He motivated his proposal by classical facts from topology and Floer theory.

The talks led to lively interaction among the participants, even more intensely among the 17 participants present in Oberwolfach. They unanimously appreciated the efficiency of in-person mathematical discussions with long-missed colleagues after a long period of online-only conferences and too-formal scheduled-only conversations during the pandemic.

Workshop (hybrid meeting): Enumerative Geometry of Surfaces

Table of Contents

Abstracts

A new spin for Hurwitz numbers and Chiodo classes

Danilo Lewański
(joint work with Alessandro Giacchetto, Reinier Kramer)

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable curves of genus g with n marked points. Chiodo classes can be thought as elements of a certain collection[1]

$$\left\{ \Omega^{[x]}_{g,n}(r,s;a_1,\ldots,a_n) \in H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n})[x] \right\}_{2g-2+n>0,\ \sum a_i \equiv (2g-2+n)s \mod r}$$

of cohomology classes arising from the moduli space of spin structures and related to each other (they form a cohomological field theory). These classes lie inside well-behaved subrings of the cohomology called tautological, and their explicit expression in terms of stable graphs decorated with ψ- and κ-classes is known.

Especially in the very last few years, applications of these classes blossomed in the literature. For example, many kinds of Hurwitz numbers can be expressed as integrals of Chiodo classes against ψ-classes, just as Norbury Θ-classes, Masur-Veech volumes, the Euler characteristic of open moduli spaces of curves.

Let $r = 2s$ be an even positive integer. Spin Hurwitz numbers $h^{r,\theta}_{g,\mu}$ are related to spin curves (curves C together with a theta characteristic N of their canonical $N^{\otimes 2} \cong K_C$) and are defined as

$$h^{r,\theta}_{g,\mu} = \sum_{f \in H_r} \frac{(-1)^{\mathrm{Arf}(f)}}{|\operatorname{Aut}(f)|}$$

where f runs over connected genus g branched covers of the Riemann sphere with the prescribed ramification μ over zero (partition which must have only odd parts) and all other ramifications are given by a spin analogue of the $(r+1)$-completed cycles (in any case equal to $(r+1)$ cover sheets meeting at the branch point plus a particular linear combination of lower order terms). The Arf invariant is defined by pulling back the only spin structure of the Riemann sphere and tensoring it with the ramification divisor R_f

$$\mathrm{Arf}(f) = h_0(C, (f^*\mathcal{O}(-1)) \otimes \mathcal{O}_C(R_f/2))$$

The motivation for studying these numbers arises from different areas of mathematics:

1. **Integrability:** It is known that usual Hurwitz numbers with completed cycles $h^r_{g,\mu}$ satisfy an integrability of type KP, in the sense that their partition function is a τ-function of the KP integrable hierarchy. It has also been proved that the numbers $h^{r,\theta}_{g,\mu}$ are integrable, but of type BKP.

[1] here all parameters are integers with the exception of the degree formal variable x, and $r > 0$

2. **Topological recursion perspective:** Topological recursion in the sense of Eynard and Orantin is a universal recursive procedure that generates solutions of enumerative geometric problems from the input of a so-called "spectral curve". It has been proved that $h^r_{g,\mu}$ are generated via topological recursion from a certain known spectral curve. It is a natural question whether $h^{r,\theta}_{g,\mu}$ also can be computed via topological recursion, and which feature should the corresponding spectral curve have, also possibly in relation with the different type of integrability.
3. **Gromov-Witten theory perspective:** usual Hurwitz numbers with completed cycles $h^r_{g,\mu}$ are the key Hurwitz counterpart of the celebrated Gromov-Witten/Hurwitz correspondence for curve targets of Okounkov-Pandharipande. Conjecturally, there exist a spin enriched version of the GW/H correspondence involving spin curves as targets and $h^{r,\theta}_{g,\mu}$ as Hurwitz numbers. Moreover, it has been proved that the Gromov-Witten theory of Kähler surfaces with smooth canonical divisor is completely determined by its embedded spin curves.
4. **Cohomology of $\overline{\mathcal{M}}_{g,n}$:** The main advantage that topological recursion provides to algebraic geometry is a cohomological representation over the moduli space of curves of the numbers it produces (i.e., given a spectral curve, it spits out infinitely many numbers $N_{g,\mu}$ together with a recipe to build the statement of the form "$N_{g,\mu}$ is equal to an explicit linear combination of integrals $\int_{\overline{\mathcal{M}}_{g,n=\ell(\mu)}} C_{g,n}\psi_1^{d_1}\cdots\psi_n^{d_n}$, where the cohomology class $C_{g,n}$ can be determined from the initial data of the spectral curve).

Apart from proving the spin GW/H correspondence, investigating the points above is what we do.

Firstly, together with the coauthors A. Giacchetto and R. Kramer, we propose a conjectural spectral curve for $h^{r,\theta}_{g,\mu}$, and we test the conjecture numerically.

The conjecture has recently been proved by A. Alexandrov and S. Shadrin in [1].

Conjecture 1 ([2], Theorem [1]). *For $s \geq 1$ and for $r = 2s$, the spectral curve given by*

(1)
$$x(z) = \log z - z^r, \qquad y(z) = z\,, \qquad B(z_1,z_2) = \frac{1}{2}\left(\frac{dz_1\,dz_2}{(z_1-z_2)^2} + \frac{dz_1\,dz_2}{(z_1+z_2)^2}\right).$$

produces via Eynard-Orantin topological recursion correlators $\omega_{g,n}$ such that

$$\int^{x_1}\cdots\int^{x_n}\omega_{g,n} = \sum_{\mu\ odd} h^{r,\theta}_{g,\mu}\prod_{i=1}^{n} e^{\mu_i x_i}$$

We then apply the recipe of topological recursion to build an explicit cohomological representation of $h^{r,\theta}_{g;\mu}$ over the moduli space of curves. The answer arises at first as sum over decorated stable graphs, which can then be interpreted as the product of Chiodo's class with Witten's 2-spin class.

Theorem 2. *The conjecture above is equivalent to the following spin ELSV-type formula: for $r = 2s$ and for $\mu = (\mu_1, \ldots, \mu_n)$ odd*
(2)
$$h^{r,\theta}_{g;\mu} = r^{\frac{(r+1)(2g-2+n)+d}{r}} \left(\prod_{i=1}^{n} \frac{\left(\frac{\mu_i}{r}\right)^{[\mu_i]}}{[\mu_i]!} \right) \int_{\overline{\mathcal{M}}_{g,n}} \frac{(\epsilon^2_1)_* \left(c^{(2)}_{W,g,n} \cdot (\epsilon^r_2)_* \Omega^{[1]}_{g,n}(r,1;\langle\vec{\mu}\rangle) \right)}{\prod_{i=1}^{n} (1 - \frac{\mu_i}{r}\psi_i)}$$
where $\mu_i = r[\mu_i] + r - (2\langle\mu_i\rangle + 1)$, with $0 \leq \langle\mu_i\rangle \leq s - 1$, and $\epsilon^{a\cdot b}_b$ is the forgetful map from the moduli space of $a \cdot b$ spin structures to the moduli space of b spin structures, and $c^{(2)}_W$ is Witten's 2-spin class.

Example 1 ($s = 1$)**.** In particular, the ELSV for $\mu_i = 2b_i - 1$ reads

$$h^{2,\theta}_{g;\mu} = 2^{4g-4+2n} \left(\prod_{i=1}^{n} \frac{\mu_i^{b_i-1}}{(b_i-1)!} \right) \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda(1)\Lambda(-\frac{1}{2})}{\prod_{i=1}^{n} \left(1 - \frac{\mu_i}{2}\psi_i\right)}, \quad (3)$$

expressing spin single Hurwitz numbers with 3-completed cycles in terms of double Hodge integrals.

As open directions of research, we mention the understanding of the Witten spin parameter t (which is currently $t = 2$) in the ELSV formula, and the understanding of the spin enriched version of the GW/H correspondence, which we plan to investigate in future works.

References

[1] A. Alexandrov and S. Shadrin, *Elements of spin Hurwitz theory: closed algebraic formulas, blobbed topological recursion, and a proof of the Giacchetto-Kramer-Lewański conjecture*, arXiv:2105.12493.
[2] A. Giacchetto, R. Kramer, and D. Lewański, *A new spin on Hurwitz theory and ELSV via theta characteristics*, arXiv:2104.05697.

The incidence variety compactifications of strata of k-differentials in genus 0

Duc-Manh Nguyen
(joint work with Vincent Koziarz)

Given $k \in \mathbb{Z}_{\geq 2}$ and $n \in \mathbb{Z}_{\geq 3}$, let $\mathcal{H}^{(k)}_{0,n}$ denote the space of tuples $(X, x_1, \ldots, x_n, q)$ where $(X, x_1, \ldots, x_n)$ is a stable curve of genus 0, and q a holomorphic section of the line bundle $k\omega_X + \sum_{i=1}^{n}(k-1)x_i$ on X (ω_X is the dualizing sheaf of X). The space $\mathcal{H}^{(k)}_{0,n}$ is a vector bundle of rank $k(n-2)+1-n$ over $\overline{\mathcal{M}}_{0,n}$. Denote by $\mathbb{P}\mathcal{H}^{(k)}_{0,n}$ the associated projective bundle.

Let $\underline{k} := (k_1, \ldots, k_n)$ be a family of n integers such that $k_i \geq 1 - k$, and $k_1 + \cdots + k_n = -2k$. Denote by $\Omega^k\mathcal{M}_{0,n}(\underline{k})$ the stratum of k-differentials whose zeros and poles have orders prescribed by $(k_1, \ldots, k_n)$. Elements of $\Omega^k\mathcal{M}_{0,n}(\underline{k})$ are tuples $(\mathbb{P}^1_{\mathbb{C}}, x_1, \ldots, x_n, q)$, where $(\mathbb{P}^1_{\mathbb{C}}, x_1, \ldots, x_n) \in \mathcal{M}_{0,n}$ and q is a (meromorphic) k-differential on $\mathbb{P}^1_{\mathbb{C}}$ whose zeros and poles are contained in the set $\{x_1, \ldots, x_n\}$

with order at x_i being k_i. Denote the by $\mathbb{P}\Omega^k\mathcal{M}_{0,n}(\underline{k})$ the projectivization of $\Omega^k\mathcal{M}_{0,n}(\underline{k})$.

The stratum $\Omega^k\mathcal{M}_{0,n}(\underline{k})$ and its projectivization $\mathbb{P}\Omega^k\mathcal{M}_{0,n}(\underline{k})$ are naturally subspaces of $\mathcal{H}^{(k)}_{0,n}$ and of $\mathbb{P}\mathcal{H}^{(k)}_{0,n}$ respectively. By definition, the *Incidence Variety Compactification* of $\mathbb{P}\Omega^k\mathcal{M}_{0,n}(\underline{k})$, which will be denoted by $\mathbb{P}\Omega^k\overline{\mathcal{M}}_{0,n}(\underline{k})$, is its closure in $\mathbb{P}\mathcal{H}^{(k)}_{0,n}$ (see [5, 1, 2]).

The boundary $\partial\overline{\mathcal{M}}_{0,n} := \overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n}$ is a simple normal crossing divisor. The irreducible components of $\partial\overline{\mathcal{M}}_{0,n}$ are in bijection with partitions of $\{1,\dots,n\}$ into two subsets $\{I_0, I_1\}$ such that $\min\{|I_0|, |I_1| \geq 2\}$. Denote by $\mathcal{P}$ the set of such partitions. For any $\mathcal{S} = \{I_0, I_1\} \in \mathcal{P}$, we will always choose the numbering of the subsets I_0, I_1 such that

$$\sum_{i\in I_0} k_i \leq -k \leq \sum_{i\in I_1} k_i \tag{1}$$

(recall that $k_1 + \cdots + k_n = -2k$). The component of $\partial\overline{\mathcal{M}}_{0,n}$ associated to $\mathcal{S}$ will be denoted by $D_{\mathcal{S}}$.

Consider now a point $\mathbf{x}$ in $\overline{\mathcal{M}}_{0,n}$. Assume that the curve $C_{\mathbf{x}}$ parametrized by $\mathbf{x}$ has $(r+1)$ irreducible components, which will be denoted by $C^0_{\mathbf{x}},\dots,C^r_{\mathbf{x}}$. Let $\mathcal{N}(\mathbf{x})$ be the set of nodes of $C_{\mathbf{x}}$. Note that $|\mathcal{N}(\mathbf{x})| = r$. For any $\alpha \in \mathcal{N}(\mathbf{x})$, let $\mathcal{S}_\alpha := \{I_{0,\alpha}, I_{1,\alpha}\}$ be the associated partition of $\{1,\dots,n\}$. Let $\mathcal{U}_{\mathbf{x}}$ be a neighborhood of $\mathbf{x}$ that satisfies the following

(i) $\mathcal{U}_{\mathbf{x}}$ does not intersect any boundary divisor $D_{\mathcal{S}}$ such that $\mathcal{S} \notin \{\mathcal{S}_\alpha,\ \alpha \in \mathcal{N}(\mathbf{x})\}$,
(ii) $\mathcal{U}_{\mathbf{x}}$ can be identified with an open subset of $\mathbb{C}^{n-3}$ such that for each $\alpha \in \mathcal{N}(\mathbf{x})$, there is a coordinate function t_α such that $D_{\mathcal{S}_\alpha} \cap \mathcal{U}_{\mathbf{x}}$ is defined by the equation $t_\alpha = 0$.

As $\overline{\mathcal{M}}_{0,n}$ is a projective variety, we can actually choose $\mathcal{U}_{\mathbf{x}}$ to be an open affine of $\overline{\mathcal{M}}_{0,n}$ such that t_α is an element of the coordinate ring of $\mathcal{U}_{\mathbf{x}}$.

Splitting a node $\alpha \in \mathcal{N}(\mathbf{x})$ into two points, we obtain two subcurves $\hat{C}^0_{\mathbf{x},\alpha}\hat{C}^1_{\mathbf{x},\alpha}$ of $C_{\mathbf{x}}$, where $\hat{C}^k_{\mathbf{x},\alpha}$ contains the i-marked points with $i \in I_{k,\alpha}$. For $j \in \{0,\dots,r\},\ \alpha \in \mathcal{N}(\mathbf{x})$, define

$$\beta_{j,\alpha} = \begin{cases} k + \sum_{i\in I_{1,\alpha}} k_i & \text{if } C^j_{\mathbf{x}} \subset \hat{C}^1_{\mathbf{x},\alpha}, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

$$\beta_j = (\beta_{j,\alpha})_{\alpha\in\mathcal{N}(\mathbf{x})}, \quad \text{and} \quad t^{\beta_j} = \prod_{\alpha\in\mathcal{N}(\mathbf{x})} t_\alpha^{\beta_{j,\alpha}}. \tag{3}$$

Let $\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}$ be the structure sheaf of $\overline{\mathcal{M}}_{0,n}$, and $\mathcal{I}_{\mathcal{U}_{\mathbf{x}}}$ be the ideal sheaf of $\mathcal{O}_{\overline{\mathcal{M}}_{0,n}|\mathcal{U}_{\mathbf{x}}}$ generated by $\{t^{\beta_0},\dots,t^{\beta_r}\}$. We will prove

Theorem 1.

(i) *The family* $\{\mathcal{I}_{\mathcal{U}_{\mathbf{x}}},\, \mathbf{x} \in \overline{\mathcal{M}}_{0,n}\}$ *defines a sheaf of ideals* $\mathcal{I}$ *of* $\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}$.

(ii) *The incidence variety compactification* $\mathbb{P}\Omega^k\overline{\mathcal{M}}_{0,n}(\underline{k})$ *of* $\mathbb{P}\Omega^k\mathcal{M}_{0,n}(\underline{k})$ *is isomorphic to the blow-up* $\widehat{\mathcal{M}}_{0,n}(\underline{k})$ *of* $\overline{\mathcal{M}}_{0,n}$ *along* $\mathcal{I}$.

(iii) *Let* $\hat{p} : \widehat{\mathcal{M}}_{0,n}(\underline{k}) \to \overline{\mathcal{M}}_{0,n}$ *be the blow-up projection. For every* $\mathbf{x} \in \overline{\mathcal{M}}_{0,n}$ *the fiber* $\hat{p}^{-1}(\{\mathbf{x}\})$ *is always a projective space, whose dimension is determined by* $\underline{k}$ *and the stratum of* $\overline{\mathcal{M}}_{0,n}$ *to which* $\mathbf{x}$ *belongs.*

Remark 1.

- The blow-up $\widehat{\mathcal{M}}_{0,n}(\underline{k})$ can be equal to $\overline{\mathcal{M}}_{0,n}$ for instance in the case $k_i < 0$ for all $i = 1, \dots, n$.
- When $n = 4$, we always have $\widehat{\mathcal{M}}_{0,4}(\underline{k}) \simeq \overline{\mathcal{M}}_{0,4}$. When $n = 5$, it can be shown that $\widehat{\mathcal{M}}_{0,5}(\underline{k})$ is always an orbifold. However, for $n > 5$, the space $\widehat{\mathcal{M}}_{0,n}(\underline{k})$ is highly singular in general.

Our second main result shows that for all strata $\Omega^k\mathcal{M}_{0,n}(\underline{k})$ such that k does not divide any of the k_i, $i \in \{1, \dots, n\}$, $\mathrm{vol}_1(\mathbb{P}\Omega^k\mathcal{M}_{0,n}(\underline{k}))$ can be computed by the self-intersection number of an explicit divisor on $\widehat{\mathcal{M}}_{0,n}(\underline{k})$. Specifically, define

$$\mu_i := -\frac{k_i}{d},\ i = 1, \dots, n, \quad \text{and} \quad \mu := (\mu_1, \dots, \mu_n). \tag{4}$$

We will call μ_i the weight of the i-th marked point on the pointed curves parametrized by $\overline{\mathcal{M}}_{0,n}$. We associate to each component $D_{\mathcal{S}}$ of $\partial\overline{\mathcal{M}}_{0,n}$, where $\mathcal{S} = \{I_0, I_1\}$, a weight $\mu_{\mathcal{S}}$ given by

$$\mu_{\mathcal{S}} := \frac{1}{2} \cdot \left(\sum_{i \in I_0} \mu_i - \sum_{i \in I_1} \mu_i \right) = 1 - \sum_{i \in I_1} \mu_i. \tag{5}$$

Note that $\mu_{\mathcal{S}}$ is always non-negative. Recall that ψ_i, $i = 1, \dots, n$, is a divisor in $\overline{\mathcal{M}}_{0,n}$ which represents the line bundle $\sigma_i^* K_{\overline{\mathcal{C}}_{0,n}/\overline{\mathcal{M}}_{0,n}}$. Define

$$\mathcal{D}_\mu := \frac{d}{2} \left(\sum_{i=1}^{n} -\mu_i \psi_i + \sum_{\mathcal{S} \in \mathcal{P}} (1 - \mu_{\mathcal{S}}) D_{\mathcal{S}} \right). \tag{6}$$

By construction $\hat{p}^{-1}\mathcal{I} \cdot \mathcal{O}_{\widehat{\mathcal{M}}_{0,n}(\underline{k})}$ is isomorphic to the line bundle associated to an explicit Cartier divisor $\mathcal{E}$ on $\widehat{\mathcal{M}}_{0,n}(\underline{k})$.

Theorem 2. *Define* $\hat{\mathcal{D}}_\mu := \hat{p}^*\mathcal{D}_\mu + \mathcal{E}$. *Then the restriction of the tautological line bundle* $\mathcal{O}(-1)_{\mathbb{P}\mathcal{H}_{0,n}^{(d)}}$ *to* $\widehat{\mathcal{M}}_{0,n}(\underline{k})$ *is isomorphic to the one associated to* $\hat{\mathcal{D}}_\mu$. *Moreover, if* k *does not divide* k_i, *for all* $i = 1, \dots, n$, *then*

$$\mathrm{vol}_1(\mathbb{P}\Omega^k\mathcal{M}_{0,n}(\underline{k})) = \frac{(-1)^{n-3}}{d^{n-2}} \cdot \frac{(2\pi)^{n-2}}{2^{n-2}(n-2)!} \cdot \hat{\mathcal{D}}_\mu^{n-3} \tag{7}$$

where $\hat{\mathcal{D}}_\mu^{n-3}$ *means the self-intersection number* $\underbrace{\hat{\mathcal{D}}_\mu \cdots \hat{\mathcal{D}}_\mu}_{n-3}$ *of* $\hat{\mathcal{D}}_\mu$ *in* $\widehat{\mathcal{M}}_{0,n}(\underline{k})$.

Remark 2.

- The $\mathbb{Q}$-divisor $\mathcal{D}_\mu$ actually represents a line bundle $\bar{\mathcal{L}}_\mu$ over $\overline{\mathcal{M}}_{0,n}$.
- That $\hat{\mathcal{D}}_\mu^{n-3}$ computes the volume of $\mathbb{P}\Omega^k\mathcal{M}_{0,n}(\underline{k})$ follows from the results of [4].
- In the cases where $k < k_i < 0$, for all $i = 1, \ldots, n$, we have $\widehat{\mathcal{M}}_{0,n}(\underline{k}) \simeq \overline{\mathcal{M}}_{0,n}$, $\hat{\mathcal{D}}_\mu \simeq \mathcal{D}_\mu$, and (7) is the content of [6, Th. 1.1].

References

[1] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller, *Compactification of strata of Abelian differentials*, Duke Math. Journ. **167** (2018), no. 12, 2347–2416.
[2] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller, *Strata of k-differentials*, Algebr. Geom. **6** (2019), no. 2, 196–233.
[3] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller, *The moduli spaces of multi-scale differentials*, preprint, arXiv:1910.13492.
[4] M. Costantini, M. Möller, and J. Zachhuber, *The area is a good enough metric*, preprint, arXiv:1910.14151.
[5] Q. Gendron, *The Deligne-Mumford and the Incidence Variety Compactifications of the strata of the moduli space of Abelian differentials*, Annales de l'Institut Fourier, **68**, no. 3 (2018), pp. 1169–1240.
[6] V. Koziarz and D.-M. Nguyen, *Complex hyperbolic volume and intersection of boundary divisors in moduli spaces of pointed genus zero curves*, Ann. Sci. de l'É.N.S. (4) **51** (2018), no. 6, 1549–1597.

Quantum Hall effect via the GRR formula

Dimitri Zvonkine
(joint work with Semyon Klevtsov)

1. The Hall effect.

The classical Hall effect is the deviation of the electric current in a thin conducting plate traversed by a perpendicular magnetic field. At low temperature surprising quantum effects kick in: for instance, for a range of values of the magnetic field, the current in the plate flows in the direction perpendicular to the difference of potentials.

The suggested explanation is that the ground state of the Hamiltonian of a charged particle on the conducting surface is highly degenerate (that is, the eigenspace with lowest eigenvalue has a high dimension), and the quantum Hall effect occurs when it is completely filled with particles.

In this work we study the quantum Hall effect on a closed surface of genus g.

2. Gauge theory.

Let C be a conducting surface with a Riemannian metric that we represent as a pair ω, I, where ω is a symplectic form and I a complex structure. In particular, C acquires the structure of a Riemann surface.

A magnetic field B over C is described by a principal $U(1)$-bundle $P \to C$ endowed with a connection ∇, up to a connection-preserving isomorphism. The intensity of the magnetic field is the curvature of ∇. The degree d of P is the total magnetic flux through C. From now on we will work under the *constant magnetic field assumption*, in other words, we assume that $B = \beta\omega$, where β is the intensity of the magnetic field independent of time and of the point on C.

3. One charged particle.

We define a complex hermitian line bundle $L \to C$ by $L = \mathbb{C} \times_{U(1)} P$. The hermitian structure on L (inherited from $\mathbb{C}$) and the complex structure on C automatically endow L with the structure of a holomorphic line bundle via the operator $\overline{\partial} = \nabla^{0,1}$.

The wave function of a charged particle on C is a section ψ of L. The Hamiltonian operator is

$$H\psi = (\nabla^*\nabla + \nabla\nabla^*)\psi = [(\overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*) + \beta]\psi,$$

where the second equality is the Weitzenböck identity between laplacians.

Thus ground states of the Hamiltonian, i.e., eigenfunctions of H with lowest eigenspaces, are holomorphic sections of L.

4. Many charged particles.

It is not possible to solve exactly the system of N interacting particles. Instead, physicists introduced several phenomenological models to describe their behavior. The simplest is the Laughlin model. Denote by $\mathbb{L} \to C^N$ the line bundle

$$\mathbb{L} = \pi_1^* L \otimes \pi_2^* L \otimes \cdots \otimes \pi_N^* L.$$

Further, let $\Delta = \bigcup_{i<j}\{z_i = z_j\}$ be the union of diagonals in C^N. For a positive integer b, a *Laughlin state* of N particles on C is a holomorphic S_N-invariant section of $\mathbb{L}(-b\Delta)$, where S_N permutes the factors of C^N and, for odd b, multiplies ψ by the sign of the permutation.

5. Question 1:

What is the dimension of the space of Laughlin states?

6. Working in families.

Up to now we have considered one line bundle L over one Riemann surface C. It turns out that switching on the electric field corresponds to changing the line bundle L. Indeed, the electric field changes the connection ∇ with time while keeping its curvature (corresponding to the magnetic field) invariant. This changed the operator $\overline{\partial}$ and thus the holomorphic structure of L. In other words, the electric field makes us travel over $\mathrm{Pic}^d(C)$. Laughlin states form a vector bundle over $\mathrm{Pic}^d(C)$. It turns out that the Hall conductivity (electric current divided by the electric field) equals the slope (1st Chern class divided by rank) of this vector bundle.

7. Question 2:

Find the characteristic classes of the vector bundle of Laughlin states over $\mathrm{Pic}^d(C)$.

An even larger family can be obtained by also varying the surface C itself, but we will not touch this here.

8. Method.

The line bundle $\mathbb{L}(-b\Delta)$ descends to the symmetric power $S^N C$ to the curve. By abuse of notation, we will still denote the obtained line bundle by $\mathbb{L}(-b\Delta)$. Answers to Questions 1 and 2 are obtained by applying the Grothendieck-Riemann-Roch theorem to this line bundle. By a theorem of Mattuck, the natural map $S^N C \to \mathrm{Pic}^N(C)$ is a the projectivization of a vector bundle. It has two natural 2-cohomology classes: Θ (the pull-back of the theta-class from $\mathrm{Pic}^N(C)$) and $\xi = c_1(\mathcal{O}(1))$. We have

$$c_1(\mathbb{L}(-b\Delta)) = b\Theta + p\xi,$$

where $p = d - b(N + g - 1)$ measures how many more particles could have been added to the ground state. For $p \geq 0$ the Kodaira vanishing applies to $\mathbb{L}(-b\Delta)$; thus the GRR formula gives the Chern character of the vector bundle of sections of $\mathbb{L}(-b\Delta)$ on the symmetric power $S^N C$ of the curve, which are precisely the Laughlin states.

Applying the GRR formula leads to nontrivial computations in the cohomology ring of $S^N C$, solved using the Lagrange inversion theorem.

9. Results.

The Chern character of the vector bundle V of Laughlin states over $\mathrm{Pic}^d(C)$ equals

$$\mathrm{ch}_m(V) = \sum_{k=m}^{g} \binom{g-m}{k-m} \binom{N-g+p}{k-g+p} b^{k-m} \frac{\Theta^m}{m!}.$$

In particular, its rank equals

$$\sum_{k=0}^{g} \binom{g}{k} \binom{N-g+p}{k-g+p} b^k.$$

The simplest particular case is $p = 0$. In this case one gets

$$\mathrm{ch}(V) = b^m \exp(\Theta/b).$$

Wall-crossing for abelian differentials

Maxim Kontsevich
(joint work with Yan Soibelman)

For an abelian differential on a complex curve one can count saddle connections in all possible relative homology classes. These numbers jump when one crosses a wall in the moduli space of abelian differentials. I will show that the jumping formula is a particular case of the general wall-crossing formalism of Y. Soibelman and myself. The corresponding graded Lie algebra is the algebra of matrices over the ring of Laurent polynomials in several variables. The wall-crossing structure is explicitly calculable, and is determined by a finite collection of invertible matrices over the field of rational functions. The whole story generalizes from curves to higher-dimensional complex algebraic varieties.

Curves on the Hilbert scheme of a K3 surface

Georg Oberdieck

1. An enumerative problem

Let S be a smooth complex projective surface which we assume here for simplicity to be Fano (in particular, $p_g = q = 0$). Let L be a line bundle with no higher cohomology. We are interested in counting curves in the linear system $|L|$ of given geometric genus and *gonality*.

Definition 1. A smooth proper connected curve C is n-gonal if there exists a morphism $C \to \mathbb{P}^1$ of degree n.

Definition 2. Let $N_{g,n}(L)$ be the number of irreducible curves $C \in |L|$ such that:

(i) the normalization $\widetilde{C}$ is n-gonal of genus g
(ii) C passes through $\ell(g,n)$ generic points.

Here $\ell(g,n)$ is the number of points which makes the problem of expected dimension 0. To find it recall first that because the Brill-Noether number reads $\rho(g,a,d) = g - (a+1)(g-d+a)$, in a given family of genus g curves the loci of n-gonal curves has expected codimension $-\rho(g,1,n) = g+2-2n$. Second, the locus of geometric genus g curves in a family arithmetic genus p_a curves is of expected codimension $p_a - g$. Let $p_a(L)$ be the arithmetic genus of a curve in $|L|$. Hence

$$\begin{aligned}\ell = \ell(g,n) &= \dim|L| - (p_a(L) - g) + \rho(g,1,n)\\ &= \frac{1}{2}L\cdot(L-K) - \left(\frac{1}{2}L\cdot(K+L) + 1 - g\right) - (g+2-2n)\\ &= c_1(S)\cdot L - 1 + 2n - 2.\end{aligned}$$

2. Hilbert schemes

By a classical idea of Graber, the Hilbert scheme of n points $S^{[n]}$ can be used to approach the count $N_{g,n}(L)$. By definition a morphism $T \to S^{[n]}$ from a Noetherian scheme T corresponds to a closed subscheme $C \subset T \times S$ flat over T of degree n. Hence we find the natural bijection:

$$\left\{ \text{maps } f : \mathbb{P}^1 \to S^{[n]} \right\} \xrightarrow{\cong} \left\{ \begin{matrix} \text{subcurves } C \subset \mathbb{P}^1 \times S \\ \text{flat over } \mathbb{P}^1 \text{ of degree } n \end{matrix} \right\}.$$

Moreover, as explained in [3, Sec. 1] the map f has class $\beta + kA$ under the natural isomorphism $H_2(S^{[n]}, \mathbb{Z}) \cong H_2(S, \mathbb{Z}) \oplus \mathbb{Z}A$ if and only if we have $[C] = \beta + n[\mathbb{P}^1] \in H_2(S \times \mathbb{P}^1, \mathbb{Z})$ and $\chi(\mathcal{O}_C) = k + n$. Similarly, the projection of C to S is incident to a point $P \in S$ if and only if $f(\mathbb{P}^1)$ is incident to the cycle $I(P) = \{\xi \in S^{[n]} | P \in \xi\}$.

Define the genus g Gromov-Witten invariant of the Hilbert scheme:

$$\big\langle \alpha; \gamma_1, \ldots, \gamma_N \big\rangle^{S^{[n]}}_{g,\beta+kA} := \int_{[\overline{M}_{g,N}(S^{[n]},\beta+kA)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\gamma_1) \cdots \mathrm{ev}_N^*(\gamma_N) \tau^*(\alpha)$$

where α is a tautological class on $\overline{M}_{g,N}$, which is the target of the forgetful morphism τ. A virtual count $H_{g,n}(\beta)$ of n-gonal genus g curves on S in class β passing through ℓ points is then defined by

$$\sum_{k \in \mathbb{Z}} \big\langle I(P)^\ell \big\rangle^{S^{[n]}}_{0,\beta+kA} p^k = \sum_g H_{g,n}(\beta) (p^{-1/2} + p^{1/2})^{2n+2g-2}.$$

The justification for this is that for an isolated genus g curves $C \subset S \times \mathbb{P}^1$, the corresponding map $f : \mathbb{P}^1 \to S^{[n]}$ meets the diagonal $\Delta_{S^{[n]}}$ in $2n + 2g - 2$ points, and by Graber each of these intersection points should contribute $p^{-1/2} + p^{1/2}$ to the left hand side. In particular Graber proves:

Theorem 1 ([1])**.** *For $S = \mathbb{P}^2$ the count $H_{g,2}(\beta)$ is enumerative, or in other words equal to $N_{g,2}(\beta)$. For an explicit recursion see* [1].

3. K3 surfaces

The above discussion motivates the study of the Gromov-Witten theory of the Hilbert scheme of points of a K3 surface. We state a triality of conjectures which governs the structure of the theory. Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with section B and fiber class F. We define potential of reduced Gromov-Witten invariants:

$$F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_N) = \sum_{d=-m}^{\infty} \sum_{r \in \mathbb{Z}} \langle \alpha; \gamma_1, \ldots, \gamma_N \rangle^{S^{[n]}}_{g,m(B+F)+dF+kA} q^d (-p)^k.$$

By deformation invariance these series determine all Gromov-Witten invariants of hyper-Kähler varieties of $K3^{[n]}$-type [4]. By convention, we assume $k = 0$ for $n = 1$. Recall the algebra QJac of quasi-Jacobi forms [3].

Conjecture 2. *$F_{g,m}(\alpha;\gamma_1,\dots,\gamma_N)$ is a quasi-Jacobi form of index $n-1$ and weight $n(2g-2)+\sum_i \underline{\deg}(\gamma_i)-10$ of the form*

$$F_{g,m}(\alpha;\gamma_1,\dots,\gamma_N)\in \frac{1}{\Delta(q)}\mathsf{QJac}.$$

Here, if $\gamma \in H^*(S^{[n]})$ is written in terms of the action of Nakajima operators

$$\gamma = \prod_i \mathfrak{q}_{a_i}(\delta_i)1, \quad 1\in H^*(S^{[0]})$$

where δ_i are elements of a fixed basis $\{W := B+F, F, \mathsf{p}, 1, e_3, \dots, e_{22}\}$ with $e_i \in H^2(S)$ orthogonal to W, F, then the modified degree function $\underline{\deg}$ is defined by

$$\underline{\deg}(\gamma) = \deg(\gamma) + w(\gamma) - f(\gamma)$$

where $w(\gamma)$ and $f(\gamma)$ are the number of classes δ_i equal to W and F respectively.

Conjecture 3. *We have the multiple cover conjecture:*

$$F_{g,m}(\alpha;\gamma_1,\dots,\gamma_N) = m^{\sum_i \deg(\gamma_i)-\underline{\deg}(\gamma_i)}\cdot T_{m,\ell}F_{g,1}(\alpha;\gamma_1,\dots,\gamma_N)$$

where $\ell = n(2g-2)+\sum_i \underline{\deg}(\gamma_i)$ and $T_{m,\ell}$ is the formal Hecke operator on Jacobi forms, see [4, 2.6].

Conjecture 3 implies that every $F_{g,m}$ is a quasi-Jacobi form (with poles at $q=0$) of index $m(n-1)$ for the congruence subgroup $\Gamma_0(n)\rtimes \mathbb{Z}^2$. The weight is as before.

Conjecture 4. *We have the holomorphic anomaly equation:*

$$\begin{aligned}\frac{d}{dG_2}F_{g,m}(\alpha;\gamma_1,\dots,\gamma_N) =& F_{g,m}(\alpha;\gamma_1,\dots,\gamma_N,U)\\ &+2\sum_{\substack{g=g_1+g_2\\ \{1,\dots,N\}=A\sqcup B}} F_{g_1,m}(\alpha_1;\gamma_A,U_1)F^{vir}_{g_2}(\alpha_2;\gamma_B,U_2)\\ &-2\sum_{i=1}^N F_{g,m}(\alpha\cdot q^*(\psi_i);\gamma_1,\dots,\gamma_{i-1},U\gamma_i,\gamma_{i+1},\dots,\gamma_N)\\ &-\frac{1}{m}\sum_{a,b}(G^{-1})_{ab}T_{e_a}T_{e_b}F_{g,m}(\alpha;\gamma_1,\dots,\gamma_N)\end{aligned}$$

with the following notations:

- *by convention the last term vanishes in case $m=0$,*
- *the intersection matrix G of the e_a is defined by $G_{ab}=\langle e_a,e_b\rangle$,*
- *we let $\rho : \wedge^2 H^2(X)\cong \mathfrak{so}\left(H^2(X)\right)\to \mathrm{End}H^*(X)$ be the Looijenga-Lunts-Verbitsky algebra action for $X=S^{[n]}$ with the conventions of* [2],
- *$U=\tilde{f}_F=\rho(-f\wedge F)$,*
- *$T_\lambda F_{g,m}(\alpha;\gamma_1,\dots,\gamma_N)=\sum_{i=1}^N F_{g,m}(\alpha;\gamma_1,\dots,\gamma_{i-1},\rho(\lambda\wedge F)\gamma_i,\gamma_{i+1},\dots,\gamma_N)$,*

- *$q : \overline{M}_{g,N}(S^{[n]}, \beta) \to \overline{M}_{g,N}(\mathbb{P}^n, \pi_*\beta)$ is induced by the Lagrangian fibration $\pi : S^{[n]} \to \mathbb{P}^n$ associated to $S \to \mathbb{P}^1$,*
- *F_g^{vir} is the potential of ordinary (non-reduced) Gromov-Witten invariants.*

The first two conjectures can be found in [3] and [4]. The last one generalizes the K3 surface case [5]. Example calculations will be discussed elsewhere.

References

[1] T. Graber, *Enumerative geometry of hyperelliptic plane curves*, J. Algebraic Geom. **10** (2001), no. 4, 725–755.
[2] A. Negut, G. Oberdieck, and Q. Yin, *Motivic decompositions for the Hilbert scheme of points of a K3 surface*, J. Reine Angew. Math., to appear, arXiv:1908.08830.
[3] G. Oberdieck, *Gromov-Witten invariants of the Hilbert scheme of points of a K3 surface*, Geom. Topol. **22** (2018), no. 1, 323–437.
[4] G. Oberdieck, *Gromov-Witten theory and Noether-Lefschetz theory for holomorphic-symplectic varieties*, arXiv:2102.11622.
[5] G. Oberdieck and A. Pixton, *Holomorphic anomaly equations and the Igusa cusp form conjecture*, Invent. Math. **213** (2018), no. 2, 507–587.

Integrals of ψ-classes on spaces of differentials and twisted double ramification cycles

Johannes Schmitt
(joint work with Matteo Costantini, Adrien Sauvaget)

Intersection numbers of strata of k-differentials

Let $g, n, k \geq 0$ be integers such that $2g - 2 + n > 0$ and let $a \in \mathbb{Z}^n$ such that $\sum_i a_i = k(2g - 2 + n)$. Then we define the closed subset

$$\mathcal{M}_g(a) = \left\{ (C, p_1, \ldots, p_n) : \omega_{\log}^{\otimes k} \cong \mathcal{O}_C \left(\sum_{i=1}^{n} a_i p_i \right) \right\} \subseteq \mathcal{M}_{g,n} \tag{1}$$

inside the moduli space of smooth curves. Note that the condition on the line bundles in (1) is equivalent to the existence of a meromorphic k-differential on C with zeros and poles of orders $a_i - k$ at the marked points p_i. Thus we call the closure $\overline{\mathcal{M}}_{g,n}(a) \subseteq \overline{\mathcal{M}}_{g,n}$ inside the moduli space of stable curves a *stratum of k-differentials.* The first part of the talk studies the following concrete problem:

Question 1. *Can we give a formula in terms of g, a for the intersection numbers*

$$\mathcal{B}_g(a) = \int_{[\overline{\mathcal{M}}_g(a)]} \psi_1^{2g-3+n} \tag{2}$$

of the fundamental class of $\overline{\mathcal{M}}_g(a)$ with a power of the first ψ-class?

Here, the exponent $2g-3+n$ of ψ_1 is chosen since for $a \notin k \cdot \mathbb{Z}^n_{>0}$ this is the dimension of $\overline{\mathcal{M}}_g(a)$, see e.g. [7, 13, 2, 3]. Computing such numbers is natural from the point of view of the enumerative geometry of the strata of k-differentials, but they also appear in recursive formulas for volumes of spaces of flat surfaces [12] and Euler characteristics of minimal strata of differentials (see below).

A relation to the double ramification cycles

To approach this problem, we use a relation between the strata of k-differentials and the so-called *(twisted) double ramification cycles* $\mathrm{DR}_g(a)$. This was proposed as Conjecture A in [7, 13] and recently proven in [9, 1]. Stated informally, it says that for $a \notin k \cdot \mathbb{Z}^n_{>0}$ we have

$$[\overline{\mathcal{M}}_g(a)] + (\text{boundary terms}) = \mathrm{DR}_g(a) \in H^{2g}(\overline{\mathcal{M}}_{g,n})\,. \tag{3}$$

Here the class $\mathrm{DR}_g(a)$ has an explicit formula [10] in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ proposed by Pixton and is a cycle-valued polynomial in a by [11]. Using the concrete form of the boundary terms above (which are described as products of further strata of k- and 1-differentials), it is easy to see that for vectors a with $a_1 \notin [1, k(2g-1)] \cap k \cdot \mathbb{Z}$, the class ψ_1^{2g-3+n} vanishes on these boundary terms, so that the function $\mathcal{B}_g(a)$ agrees with the intersection number

$$\mathcal{A}_g(a) = \int_{\mathrm{DR}_g(a)} \psi_1^{2g-3+n}\,. \tag{4}$$

Our first main result is a concrete formula for $\mathcal{A}_g$, generalizing a similar formula from [5] for vectors a with $\sum_i a_i = 0$.

Theorem 2. *Given $g, n \geq 0$, $a \in \mathbb{Z}^n$ and setting $k = (\sum_i a_i)/(2g-2+n)$, we have*

$$\mathcal{A}_g(a) = [z^{2g}] \exp\left(\frac{a_1 z \cdot \mathcal{S}'(z)}{\mathcal{S}(kz)}\right) \cdot \frac{\prod_{i=2}^n \mathcal{S}(a_i z)}{\mathcal{S}(z)\mathcal{S}(kz)^{2g-1+n}}, \text{ for } \mathcal{S}(z) = \frac{\sinh(z/2)}{z/2}\,, \tag{5}$$

where $[z^{2g}]$ denotes the operation of taking the coefficient of z^{2g} in the subsequent power series.

To prove this result, we show a series of properties and recursive formulas for the function $\mathcal{A}_g$ (in particular including the fact that the function is polynomial in a, as follows from [11]). Then we give a combinatorial argument that these properties (together with the initial data of $\mathcal{A}_0, \mathcal{A}_1$) uniquely determine all functions $\mathcal{A}_g$. The proof of Theorem 2 is then finished by showing that the formula from the theorem satisfies these universal properties.

Finally, as we saw above, this allows to compute $\mathcal{B}_g(a)$ on many input vectors a. Then, with some more work, one can in fact compute all intersection numbers

$$\int_{[\overline{\mathcal{M}}_g(a)]} \psi_1^u \text{ for } a \text{ with } k = k(a) > 0, \text{ for all } u.$$

Euler characteristics of minimal strata of differentials

For $k = 1$, the paper [6] gives a formula for the *orbifold Euler-characteristic* $\chi(\mathcal{M}_g(a))$ of the strata of 1-differentials in terms of intersection numbers on the spaces of *multi-scale differentials*, a compactification of $\mathcal{M}_g(a)$ constructed in [4]. For the case $n = 1$, i.e. the *minimal* strata of holomorphic differentials $\mathcal{M}_g(2g-1)$, we are able to use divisorial relations in the spaces of multi-scale differentials to bring this formula into a shape where all terms can be computed explicitly. Due to space constraints, we only state the following informal version here.

Theorem 3 (in progress)**.** *For any $g \geq 1$, the Euler characteristic $\chi(\mathcal{M}_g(2g-1))$ is given by an explicit formula involving numbers $a_{g'}, b_{g'}$, defined using power series identities involving $\overline{\mathcal{S}}(z) = \sin(z/2)/(z/2)$, and the function $\mathcal{A}_{g'}$ above, for $g' \leq g$.*

Spin refinements

For $k, a_1, \ldots, a_n$ odd and $(C, p_1, \ldots, p_n) \in \mathcal{M}_g(a)$ we have

$$\omega_C^{\otimes k} \cong \mathcal{O}_C\big(\sum_{i=1}^n (a_i - k)p_i\big) \iff \omega_C \cong \underbrace{\big(\omega_C^{-(k-1)/2}\big(\sum_{i=1}^n \frac{a_i - k}{2} p_i\big)\big)}_{=:\mathcal{L}}{}^{\otimes 2},$$

so that the line bundle $\mathcal{L}$ above defines a *spin structure* on C, i.e. a root of ω_C. For such bundles $\mathcal{L}$ the *parity* $p_{\mathcal{L}} = (h^0(\mathcal{L}) \bmod 2)$ is a deformation invariant, so that $\overline{\mathcal{M}}_g(a)$ decomposes into odd and even components. We define the *spin class*

$$[\overline{\mathcal{M}}_g(a)]^{\text{spin}} = [\overline{\mathcal{M}}_g(a)]^{\text{even}} - [\overline{\mathcal{M}}_g(a)]^{\text{odd}}$$

and the variant

$$\mathcal{B}_g^{\text{spin}}(a) = \int_{[\overline{\mathcal{M}}_g(a)]^{\text{spin}}} \psi_1^{2g-3+n}$$

of the function $\mathcal{B}_g$ above. Then we make the following predictions concerning intersection numbers of these classes and functions:

Conjecture 4. *There exists a tautological class $\mathrm{DR}_g^{\text{spin}}(a) \in H^{2g}(\overline{\mathcal{M}}_{g,n})$, given by a cycle-valued polynomial of degree $2g$ in a, equivariant with respect to permutations of markings, such that for vectors a with $a_1 \notin [1, k(2g-1)] \cap k \cdot \mathbb{Z}$, we have $\mathcal{B}_g^{\text{spin}}(a) = \mathcal{A}_g^{\text{spin}}(a)$ for*

$$\mathcal{A}_g^{\text{spin}}(a) = \int_{\mathrm{DR}_g^{\text{spin}}(a)} \psi_1^{2g-3+n}.$$

Conjecture 5. *We have*

$$\mathcal{A}_g^{\text{spin}}(a) = 2^{-g}[z^{2g}] \exp\left(\frac{a_1 z \cdot \mathcal{S}'(z)}{\mathcal{S}(kz)}\right) \cdot \frac{\cosh(z/2)}{\mathcal{S}(z)} \frac{\prod_{i=2}^n \mathcal{S}(a_i z)}{\mathcal{S}(kz)^{2g-1+n}}.$$

For now, we have a sketch of proof that Conjecture 4 implies Conjecture 5. Furthermore, for the cycle $\mathrm{DR}_g^{\text{spin}}(a)$ we have an explicit proposed formula, given by a variation of Pixton's formula for $\mathrm{DR}_g(a)$. This proposal is inspired by formulas in the recent paper [8] studying a spin-variant of the r-spin Hurwitz numbers.

References

[1] Y. Bae, D. Holmes, R. Pandharipande, J. Schmitt, and R. Schwarz, *Pixton's formula and Abel-Jacobi theory on the Picard stack*, arXiv:2004.08676.

[2] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller, *Compactification of strata of abelian differentials*, Duke Mathematical Journal **167.12** (2018), 2347–2416.

[3] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller, *Strata of k-differentials*, Algebraic Geometry **6.2** (2019), 196–233.

[4] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller, *The moduli space of multi-scale differentials*, arXiv:1910.13492.

[5] A. Buryak, S. Shadrin, L. Spitz, and D. Zvonkine, *The moduli space of multi-scale differentials*, American Journal of Mathematics **137** (2015), 699–737.

[6] M. Costantini, M. Möller, and J. Zachhuber, *The Chern classes and the Euler characteristic of the moduli spaces of abelian differentials*, arXiv:2006.12803.

[7] G. Farkas and R. Pandharipande, *The moduli space of twisted canonical divisors*, Journal of the Institute of Mathematics of Jussieu **17** (2018), 615–672.

[8] A. Giacchetto, R. Kramer, and D. Lewański, *A new spin on Hurwitz theory and ELSV via theta characteristics*, arXiv:2104.05697.

[9] D. Holmes and J. Schmitt, *Infinitesimal structure of the pluricanonical double ramification locus*, Compositio Mathematica, to appear.

[10] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine, *Double ramification cycles on the moduli spaces of curves*, Publications Mathématiques. Institut de Hautes Études Scientifiques **125** (2017), 221–266.

[11] A. Pixton and D. Zagier, *On combinatorial properties of the explicit expression for double ramification cycles*, in preparation.

[12] A. Sauvaget, *Volumes of moduli spaces of flat surfaces*, arXiv:2004.03198.

[13] J. Schmitt, *Dimension theory of the moduli space of twisted k-differentials*, Documenta Mathematica **23** (2018), 871–894.

An analogue of measured foliations on K3 surfaces

Simion Filip

Measured foliations and laminations on surfaces are fundamental tools in the study of the topology and dynamics of surfaces. The space of all measured foliations is used to compactify the Teichmüller space of all complex structures on a given compact surface. Masur's criterion [2] relates the recurrence in moduli space with unique ergodicity properties of measured foliations.

The talk presented an analogue of some of the above concepts on K3 surfaces, developed in joint work with Valentino Tosatti [1]. Recall that a *K3 surface* is a compact complex 2-dimensional manifold which is simply-connected and has trivial canonical bundle (i.e. a nowhere vanishing holomorphic 2-form). K3 surfaces carry canonical Ricci-flat metrics, which exist and are unique in any cohomology class which admits *some* Kähler metric, by a theorem of Yau. From a different point of view, some of the results described in the talk can be related to the degeneration of the Ricci-flat Kähler metrics as the cohomology class approaches the boundary of the ample cone of the K3 surface.

Recall that the ample cone $Amp(X)$ consists of all cohomology classes which are represented by ample $\mathbb{R}$-divisors, and its boundary is denoted $\partial Amp(X)$. The setting of [1] is assuming that the group of holomorphic automorphisms $Aut(X)$

of the K3 surface X is a lattice in the corresponding group $SO_{1,\rho-1}(\mathbb{R})$, where ρ is the rank of the Neron-Severi group. We have

Theorem 1. *There exists a unique map*

$$\eta\colon \partial_c Amp(X) \to \mathcal{Z}^{1,1}_{pos}(X)$$

where $\mathcal{Z}^{1,1}_{pos}(X)$ denotes closed positive currents, which is characterized by the properties:

(1) *The map η is $Aut(X)$-equivariant.*
(2) *The map η is continuous, for the weak topology of convergence on currents.*

Additionally:

- *The currents in the image of η have continuous potentials.*
- *(also established by Verbitsky-Sibony) For irrational cohomology classes in $\partial_c Amp(X)$, the closed positive representatives are unique.*

Note that the boundary $\partial_c Amp(X)$ that appears in the statement of the theorem is a slightly modified version of the usual boundary, taking into account the behavior of hyperbolic geodesics which enter certain cusps of hyperbolic manifolds.

An analogous theorem is proved to show the existence of height functions, when the K3 surface X is defined over a number field. In that case, similar characterizations of the heights are available.

References

[1] Simion Filip and Valentino Tosatti, *Canonical currents and heights for K3 surfaces*, arXiv:2103.02095.

[2] Howard Masur, *Interval exchange transformations and measured foliations*, Ann. of Math. (2), **115** (1982), no. 1, 169–200.

Hyperbolic cone surfaces and billiards

Viveka Erlandsson

(joint work with Christopher J. Leininger and Chandrika Sadanand)

Let S be a closed surface of genus at least 2 and $C \subset S$ a finite set of marked points. A hyperbolic cone metric on S is a metric φ which is hyperbolic on $S \setminus C$ and has cone singularities at the points of C with cone angles greater than 2π. We call a geodesic η in the universal cover $p : (\tilde{S}, \tilde{\varphi}) \to (S, \varphi)$ non-singular if it does not pass through any of the cone points in $p^{-1}(C)$ and we call it a *basic geodesic* if it is either non-singular or a limit of non-singular geodesics. We are interested in the set

$$\mathcal{G}_{\tilde{\varphi}} \subset \partial\tilde{S} \times \partial\tilde{S} \setminus \Delta/(x, y) \sim (y, x)$$

of pairs of endpoints of basic $\tilde{\varphi}$-geodesics and *to what extent this set determines the metric* φ. (Here $\partial\tilde{S}$ denotes the Gromov boundary and Δ the diagonal.) More precisely, we say that a hyperbolic cone surface (S, φ) is *rigid* if whenever $\mathcal{G}_{\tilde{\varphi}} = \mathcal{G}_{\tilde{\varphi}'}$ for some hyperbolic cone metric φ', we must have that φ and φ' is the same metric (up to isotopy).

The motivation is marked length spectrum rigidity. Recall that we say a family of metrics $\mathcal{M}$ on S is marked length spectrum rigid if the mapping

$$\mathcal{M} \to \mathbb{R}^{\mathcal{S}}, \quad \varphi \mapsto (\ell_\varphi(\eta))_{\eta \in \mathcal{S}}$$

is injective, where $\mathcal{S}$ is the set of homotopy classes of closed curves on S and $\ell_\varphi(\eta)$ is the φ-length of a geodesic representative. Otal [8] proved that the family of negatively curved Riemannian metrics is marked length spectrum rigid by showing that a metric's *Liouville current* determines the metric. This result has been generalized in various directions (see [3, 4, 7]), including to non-positively curved cone metrics on S through work by Duchin-Leininger-Rafi [5], Bankovic-Leininger [1], and Constantine [2], showing that such metrics are also determined by their Liouville current and hence are marked length spectrum rigid. However, while the Liouville current has full support in the Riemannian case, it does not for cone metrics. In fact, the Liouville current for a hyperbolic cone metric φ is exactly the set $\mathcal{G}_{\tilde{\varphi}}$ introduced above. Hence a hyperbolic cone metric is rigid if it is determined by only the *support* of its Liouville current.

In [6] we show that hyperbolic cone metrics are indeed generically rigid and characterize exactly the flexible case. If (S, φ) admits a locally isometric branched covering of a hyperbolic orbifold $\mathcal{O}$ and every cone point maps to an even order orbifold point, then any non-trivial orbifold deformation of $\mathcal{O}$ lifts to a non-trivial deformation φ' of φ with $\mathcal{G}_{\tilde{\varphi}} = \mathcal{G}_{\tilde{\varphi}'}$. However, this is the only way (S, φ) can fail to be rigid:

Theorem 1. *Suppose (S, φ) and (S, φ') are hyperbolic cone surfaces with $\mathcal{G}_{\tilde{\varphi}} = \mathcal{G}_{\tilde{\varphi}'}$. Then either $\varphi = \varphi'$ or (S, φ) branch cover an orbifold and the two metrics are related by an orbifold deformation.*

See [6] for the precise statement.

As an application we parameterize the space of hyperbolic polygons with the same symbolic coding for their billiard dynamics. Given a compact, simply connected hyperbolic polygon P with cyclically labeled sides, any billiard trajectory on P results in a bi-infinite sequence in the labels. The bounce spectrum $\mathcal{B}(P)$ of P is the collection of all such sequences and we say that P is *billiard rigid* if it is determined by its bounce spectrum. By "unfolding" P into a hyperbolic cone surface we use rigidity of such surfaces to show that generically $\mathcal{B}(P) = \mathcal{B}(P')$ if and only if P and P' are isometric. For instance, in the cases where P has an interior angle which is an irrational multiple of π or when it has no angle of the form $\pi/2k$ for some $k \in \mathbb{N}$, we have that P is billiard rigid.

References

[1] A. Bankovic and C.J. Leininger, *Marked-length-spectral rigidity for flat metrics*, Trans. Amer. Math. Soc., **370** (2018), no 3, 1867–1884.

[2] D. Constantine, *Marked length spectrum rigidity in non-positive curvature with singularities*, Indiana Univ. Math. J., **67** (2018), no. 6, 2337–2361.

[3] C. Croke, *Rigidity for surfaces of nonpositive curvature*, Comment. Math. Helv., **65** (1990), no. 1, 150–169.

[4] C. Croke, A. Fathi, and J. Feldman, *The marked length-spectrum of a surface of nonpositive curvature*, Topology, **31** (1992), no. 4, 847–855.

[5] M. Duchin, C. J. Leininger, and K. Rafi, *Length spectra and degeneration of flat metrics*, Invent. Math., **182** (2010), no. 2, 231–277.

[6] V. Erlandsson, C.J. Leininger, and C. Sadanand, *Hyperbolic cone metrics and billiards*, arXiv:2104.12176.

[7] J. Frazier, *Length spectral rigidity of non-positively curved surfaces*, ProQuest LLC, Ann Arbor, MI (2012), PhD Thesis, University of Maryland, College Park.

[8] J-P. Otal, *Le spectre marqué des longueurs des surfaces à courbure négative*, Ann. of Math. (2), **131** (1990), no. 1, 151–162.

Large Genus Asymptotics for Intersection Numbers

Amol Aggarwal

1. Results

Fix integers $g, n \geq 0$ with $2g+n \geq 3$. Let $\overline{\mathcal{M}}_{g,n}$ denote the moduli space of genus g stable curves with n marked points. Letting $\psi_1, \psi_2, \ldots, \psi_n$ denote the tautological ψ-classes on $\overline{\mathcal{M}}_{g,n}$, denote the intersection number, or *correlator*, by

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \psi_2^{d_2} \cdots \psi_n^{d_n}, \tag{1}$$

for any nonnegative integer sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$; it is nonzero only if $|\mathbf{d}| = \sum_{i=1}^{n} d_i = 3g + n - 3$. We will be interested in the asymptotics of this correlator as g tends to ∞. The below theorem states that

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_{g,n} \xrightarrow{g \to \infty} \frac{(6g+2n-5)!!}{24^g g! \prod_{i=1}^{n} (2d_i+1)!!} \big(1 + o(1)\big), \tag{2}$$

uniformly in $\mathbf{d}$, as long as $n = o(g^{1/2})$. To make this more precise, define a normalization of the correlator (1) given by

$$\langle \mathbf{d} \rangle = \langle \mathbf{d} \rangle_{g,n} = \frac{24^g g! \prod_{i=1}^{n} (2d_i+1)!!}{(6g+2n-5)!!} \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_{g,n};$$

then, (2) suggests it should tend to 1. For any $\varepsilon > 0$, further define the set

$$\Delta(g; \varepsilon) = \big\{ \mathbf{d} = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}_{\geq 0}^n : |\mathbf{d}| = 3g + n - 3, n < \varepsilon g^{1/2} \big\},$$

which will be used to make sense of "uniformity in $\mathbf{d}$, as long as $n = o(g^{1/2})$," when g and n are simultaneously tending to ∞.

Theorem 1 ([1, Theorem 1.5]). *We have* $\lim_{\varepsilon \to 0} \Big(\lim_{g \to \infty} \Big(\max_{\mathbf{d} \in \Delta(g;\varepsilon)} \big| \langle \mathbf{d} \rangle - 1 \big| \Big) \Big) = 0.$

Remark 3. It can be shown for large g that $\langle 3g-2, 1^{n-1} \rangle_{g,n} \approx e^C$, if $n \approx \sqrt{12Cg}$. Thus, the constraint $n = o(\sqrt{g})$ cannot be directly removed in the asymptotic (2).

For $n = 1$, the approximation provided by (2) is exact, a fact which follows from the Kontsevich-Witten theorem [5, 6]. For $n = 2$, it was proven in [3], based on a simple explicit formula for the two-point correlator derived in [8]. In general, the proof of the theorem appearing [1] is based on a comparison between the recursive relations (Virasoro constraints) that uniquely determine the correlators (1) with the jump probabilities of a certain asymmetric simple random walk.

2. Applications

In this section we focus on an application of Theorem 1 to flat surfaces of large genus. Let $\mathcal{Q}_g$ denote the moduli space of pairs (X, q), where X is a Riemann surface of genus g, and q is a quadratic differential on X with n simple poles; q induces a flat metric on X, and so (X, q) can be viewed as a flat surface. This moduli space of quadratic differentials $\mathcal{Q}_g$ admits a volume form Vol, known as the Masur-Veech measure, given by the pullback of the Lebesgue measure under the period map. It was shown by Delecroix-Goujard-Zograf-Zorich [3] that $\mathrm{Vol}\mathcal{Q}_g$ can be expressed in terms of the correlators (1). In particular, they wrote

$$\mathrm{Vol}\mathcal{Q}_g = \sum_{\Gamma} \mathrm{Vol}(\Gamma), \tag{3}$$

where Γ is summed over all stable graphs of genus g, and $\mathrm{Vol}(\Gamma)$ is an explicit polynomial in the correlators $\langle \mathbf{d} \rangle$. The quantity $\mathrm{Vol}(\Gamma)$ admits a geometric interpretation. It can be viewed as the contribution to $\mathrm{Vol}\mathcal{Q}_g$ coming from those surfaces whose horizontal cylinder decomposition has "backbone" given by Γ; for example, each edge of Γ corresponds to a horizontal cylinder of the surface.

Using (3), Theorem 1, and the explicit expression for $\mathrm{Vol}(\Gamma)$ from [3], each quantity $\mathrm{Vol}(\Gamma)$ can be analyzed explicitly in the large genus limit. One can then sum over all stable graphs Γ as in (3) (which requires some effort, since this sum involves exponentially in g many terms). This leads to the following asymptotics for the volume $\mathrm{Vol}\mathcal{Q}_g$ of the moduli space of quadratic differentials.

Theorem 2 ([1, Theorem 1.7])**.** *As g tends to ∞,* $\mathrm{Vol}\mathcal{Q}_g = \frac{1}{\pi}\left(\frac{8}{3}\right)^{4g-4}\big(1 + o(1)\big)$.

More precise asymptotics for $\mathrm{Vol}\mathcal{Q}_g$, namely, an all-order expansion of the form

$$\mathrm{Vol}\mathcal{Q}_g = \frac{1}{\pi}\left(\frac{8}{3}\right)^{4g-4}\left(1 + \sum_{j=1}^{K} c_j g^{-j} + \mathcal{O}(g^{-K-1})\right),$$

for constants $c_1, c_2, \ldots$, have been predicted by Yang-Zagier-Zhang in [7]. Additionally, predictions on the large genus asymptotics for other (non-principal) strata volumes of quadratic differentials can be found in [2].

The geometric interpretation described above for the quantities $\mathrm{Vol}(\Gamma)$ from (3) implies that $(\mathrm{Vol}\mathcal{Q}_g)^{-1}\mathrm{Vol}(\Gamma)$ can be viewed as the probability of a random surface admitting backbone Γ under its horizontal cylinder decomposition; observe that Theorem 2 enables an approximation for the prefactor $(\mathrm{Vol}\mathcal{Q}_g)^{-1}$. Using this, together with more intricate geometric and combinatorial considerations, the recent work [4] provides a very precise description for the geometry of random square-tiled surfaces. For example, they show that the number of cylinders under its horizontal cylinder decomposition (equivalently, the number of edges in its backbone Γ) converges in law, as g tends to ∞, to a Poisson random variable of mean $\frac{1}{2}\big(\log(24g-24)+\gamma+o(1)\big)$, where γ is the Euler-Mascheroni constant.

References

[1] A. Aggarwal, *Large Genus Asymptotics for Intersection Numbers and Principal Strata Volumes of Quadratic Differentials*, arXiv:2004.05042.
[2] A. Aggarwal, V. Delecroix, É. Goujard, P. G. Zograf, and A. Zorich, *Conjectural Large Genus Asymptotics of Masur-Veech Volumes and of Area Siegel-Veech Constants of Strata of Quadratic Differentials*, Arnold Math. J. **6** (2020), 149–161.
[3] V. Delecroix, É. Goujard, P. G. Zograf, and A. Zorich, *Masur-Veech Volumes, Frequencies of Simple Closed Geodesics, and Intersection Numbers on Moduli Spaces of Curves*, To appear in Duke Math. J., arXiv:1908.08611.
[4] V. Delecroix, É. Goujard, P. G. Zograf, and A. Zorich, *Large Genus Asymptotic Geometry of Random Square-Tiled Surfaces and of Random Multicurves*, arXiv:2007.04740.
[5] M. Kontsevich, *Intersection Theory on the Moduli Space of Curves and the Matrix Airy Function*, Commun. Math. Phys. **147** (1992), 1–23.
[6] E. Witten, *Two-Dimensional Gravity and Intersection Theory on Moduli Space*, Surveys in Differential Geometry **1** (1992), 243–310.
[7] D. Yang, D. Zagier, and Y. Zhang, *Masur-Veech Volumes of Quadratic Differentials and Their Asymptotics*, J. Geom. Phys. **158** (2020), 103870.
[8] P. G. Zograf, *An Explicit Formula for Witten's 2-Correlators*, J. Math. Sci. **240** (2019), 535–538.

Extremal hyperbolic surfaces and the Selberg trace formula

Bram Petri
(joint work with Maxime Fortier Bourque)

1. The invariants

Our work is about extremal problems in hyperbolic geometry. These extremal problems are the natural hyperbolic analogues of classical problems in the theory of Euclidean sphere packings.

Suppose we are given an oriented connected closed hyperbolic manifold M – that is, a Riemannian manifold with a metric of constant sectional curvature $\equiv -1$. In what follows, we will focus on two geometric and two spectral invariants of M.

On the geometric side, we will consider the *systole* $\mathrm{sys}(M)$ of M – the length of the shortest closed geodesic on M – and the *kissing number* $\mathrm{kiss}(M)$ of M – the number of oriented closed geodesics realizing its systole[1].

To define the spectral invariants, recall that the *Laplacian* on M is the differential operator $\Delta : C^\infty(M) \to C^\infty(M)$ given by $\Delta f = -\mathrm{div} \circ \mathrm{grad}\, f$, where grad

[1]Note that with this convention the kissing number of a hyperbolic manifold is always an even number. The reason for choosing this convention is that it mimics the kissing number of a Euclidean lattice, which also counts oriented geodesics on the torus corresponding to the lattice.

denotes the gradient of a function and div denotes the divergence of a vector field. The spectral theorem for Δ states that the solutions to the eigenvalue problem $\Delta\varphi = \lambda \cdot \varphi$ consist of a discrete set of real eigenvalues

$$0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \dots$$

corresponding to eigenfunctions $(\varphi_n)_n$ that – when normalized – form an orthonormal basis of $L^2(M)$. We will concentrate on the smallest non-zero eigenvalue $\lambda_1(M)$ and its multiplicity $m_1(M)$.

2. The questions

The basic question we ask is: suppose we fix the volume of M – in dimension two, this is the same as fixing its genus – how large can sys(M), kiss(M), $\lambda_1(M)$ and $m_1(M)$ be? That is, determine the following functions:

$$\mathrm{S}(n,v) = \max_{M\in\mathcal{H}(n,v)}\{\mathrm{sys}(M)\}, \quad \mathrm{K}(n,v) = \max_{\mathcal{H}(n,v)}\{\mathrm{kiss}(M)\},$$

$$\Lambda(n,v) = \max_{\mathcal{H}(n,v)}\{\lambda_1(M)\}, \quad M(n,v) = \max_{\mathcal{H}(n,v)}\{m_1(M)\},$$

where $\mathcal{H}(n,v)$ denotes the set of connected oriented closed hyperbolic n-manifolds of volume $\leq v$. Of course, stated like this, this is a very ambitious question. But even the question of determining the maximizers for low volumes or determining the asymptotic behavior of these functions as $v \to \infty$ is open in most cases.

Let us focus on the two-dimensional case and discuss some of the things we do know. Jenni [9] proved that the Bolza surface maximizes the systole among closed hyperbolic surfaces of genus 2 and it follows from a topological argument that it also maximizes the kissing number in genus 2 (see eg. [11]). The same surface is also expected to maximize λ_1 and m_1 among hyperbolic surfaces of genus 2, but this currently remains open (for this and several other conjectures on which surfaces should maximize what invariant, see [15]). One of our results (in progress) below determines the m_1-maximizer in genus 3: the Klein quartic. For as far as we're aware, the above is a complete list of known maximizers.

Concerning the large genus behavior of the functions defined above also plenty of open questions remain. It's known that S grows logarithmically as a function of genus. The upper bound comes from a simple argument based on area growth and the lower bound from an explicit arithmetic construction due to Buser-Sarnak [1]. An asymptotic equivalent for S however remains elusive, in particular, the multiplicative constant in front of the logarithm is not known. Schmutz [16], again using arithmetic surfaces, proved that K grows faster than $g^{4/3-\varepsilon}$ for every $\varepsilon > 0$ and Parlier [13] proved that it grows slower than $g^2/\log(g)$. Our methods also give an alternative proof of Parlier's result. On the spectral side, the asymptotes are also still to be determined. Cheng [2] proved that $\limsup_{v\to\infty} \Lambda(2,v) \leq \frac{1}{4}$ and it is believed that this limit exists and equals $\frac{1}{4}$. This would for instance follow from the Selberg conjecture. The best known results towards the Selberg conjecture, due to Kim-Sarnak [10], prove that the $\liminf_{v\to\infty} \Lambda(2,v) \geq \frac{975}{4096}$. Finally, Colin de

Verdière [4] proved that the maximal multiplicity M grows faster than $\sqrt{g}$ and Sévennec [17] proved that it grows at most linearly as a function of the genus g.

Of course, the set of closed hyperbolic surfaces of genus g forms a moduli space $\mathcal{M}_g$, so we can also ask for local maximizers for any of our invariants. This has been mostly studied for the systole (see [16, 8, 7]).

3. Our results

Our program with Maxime Fortier Bourque is to attack the problems above using the Selberg trace formula – a formula that links the Laplacian spectrum of a hyperbolic manifold to the lengths of closed geodesics on the same manifold. Our method is the hyperbolic analogue of a method developed by Cohn-Elkies [3] in the setting of Euclidean sphere packings.

Using these methods, we first of all recover Parlier's two-dimensional kissing bound and are able to generalize it to higher dimensions [5]:

Theorem 1 (Parlier, Fortier Bourque-P.)**.** *For every $n \geq 2$, there exists a constant $C_n > 0$ such that:*

$$\mathrm{K}(n,v) \leq C_n \cdot \frac{v^2}{\log(v)}$$

for all $v > 0$.

More recently, we've started applying our methods to the low genus case in dimension two [6]. One of our results is:

Theorem 2 (Fortier Bourque-P. in progress)**.** *We have*

$$\max\{m_1(M);\ M \in \mathcal{M}_3\} = 8$$

and this maximum is realized by the Klein quartic.

The proof of this result relies on a patchwork of arguments. We combine our methods based on the Selberg trace formula with bounds on λ_1 due to Ros [14], a bound on the number of small eigenvalues due to Otal-Rosas [12] and topological ideas due to Sévennec [17] to prove that $m_1(M) \leq 8$ for hyperbolic surfaces M of genus 3. To prove that the Klein quartic realizes the upper bound, we again use the Selberg trace formula, but now combined with information on the length spectrum the automorphism group of the Klein quartic. Some of our work is based on computer searches for optimal test functions for the Selberg trace formula. We are currently in the process of performing the final rigorous verifications.

References

[1] P. Buser and P. Sarnak, *On the period matrix of a Riemann surface of large genus*, Invent. Math. **117** (1994), 27–56. With an appendix by J. H. Conway and N. J. A. Sloane.

[2] S. Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*, Math. Z. **143** (1975), no. 3, 289–297.

[3] H. Cohn and N. Elkies, *New upper bounds on sphere packings. I*, Ann. of Math. (2) **157** (2003), no. 2, 689–714.

[4] Y. Colin de Verdière, *Spectres de variétés riemanniennes et spectres de graphes*, Proceedings of the International Congress of Mathematicians **1**, 2 (Berkeley, Calif., 1986), 522–530. Amer. Math. Soc., Providence, RI, 1987.
[5] M. Fortier Bourque and B. Petri, *Kissing numbers of closed hyperbolic manifolds*, Amer. J. Math., to appear, (2021+).
[6] M. Fortier Bourque and B. Petri, *Multiplicity in genus* 3, in preparation, (2021+).
[7] M. Fortier Bourque and K. Rafi, *Local maxima of the systole function*, J. Eur. Math. Soc. (JEMS), to appear (2021+).
[8] U. Hamenstädt, *New examples of maximal surfaces*, Enseign. Math. (2) **47** (2001), no. 1-2, 65–101.
[9] F. Jenni, *Über den ersten Eigenwert des Laplace-Operators auf ausgewählten Beispielen kompakter Riemannscher Flächen*, Comment. Math. Helv. **59** (1984), no. 2, 193–203.
[10] H.H. Kim. *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183, with appendix 1 by D. Ramakrishnan and appendix 2 by Kim and P. Sarnak.
[11] J. Malestein, I. Rivin, and L. Theran, *Topological designs*, Geom. Dedicata **168** (2014), 221–233.
[12] J.-P. Otal and E. Rosas, *Pour toute surface hyperbolique de genre g*, $\lambda_{2g-2} > 1/4$, Duke Math. J. **150** (2009), no. 1, 101–115.
[13] H. Parlier, *Kissing numbers for surfaces*, J. Topol. **6** (2013), no. 3, 777–791.
[14] A. Ros, *On the first eigenvalue of the laplacian on compact surfaces of genus three*, Journal of the Mathematical Society of Japan, to appear, 2021+.
[15] P. Schmutz Schaller, *Systoles on Riemann surfaces*, Manuscripta Math. **85** (1994), no. 3-4,429–447.
[16] P. Schmutz Schaller, *Geometry of Riemann surfaces based on closed geodesics*, Bull. Amer. Math. Soc. (N.S.) **35** (1998), no. 3, 193–214.
[17] B. Sévennec, *Multiplicity of the second Schrödinger eigenvalue on closed surfaces*, Math. Ann. **324** (2002), no. 1, 195–211.

Effective counting estimates for filling closed geodesics on hyperbolic surfaces

Francisco Arana-Herrera

Counting problems for closed geodesics on hyperbolic surfaces have been extensively studied since the 1950s. Huber's prime geodesic theorem [10] can be considered as one of the first major breakthroughs in this subject. According to this theorem, the number $p(X,L)$ of primitive closed geodesics of length $\leq L$ on a closed, orientable hyperbolic surface X admits the following estimate,

$$p(X,L) = \mathrm{Li}\left(e^L\right) + O_X\left(e^{(1-\kappa)L}\right), \tag{1}$$

where the gap $\kappa = \kappa(X) > 0$ depends only on the smallest non-zero eigenvalue of the Laplacian of X, and where $\mathrm{Li}\colon [2,+\infty) \to \mathbf{R}$ is the Eulerian logarithmic integral

$$\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t}.$$

Let us highlight the fact that $\mathrm{Li}(x)$ is asymptotic to $x/\log(x)$ as $x \to +\infty$. The backbone of Huber's proof, and of many of the subsequent improvements and generalizations due to several authors [9, 19, 22], is Selberg's famous trace formula [23]. A proof of an estimate analogous to (1) for arbitrary compact, negatively curved

Riemannian manifolds was given by Margulis in his thesis [11] using dynamical and geometric arguments.

Neither Huber's nor Margulis's methods can be used to prove an estimate for the number $s(X, L)$ of simple closed geodesics of length $\leq L$ on a closed, orientable hyperbolic surface X. This problem, which garnered great interest towards the end of the last century [5, 13, 14, 21], would witness major developments through the work of Mirzakhani. In her thesis [15, 17], Mirzakhani showed that the following asymptotic estimate holds as $L \to +\infty$,

$$s(X, L) \sim n(X) \cdot L^{6g-6}, \tag{2}$$

where $n(X) > 0$ is a constant depending on the geometry of X, and where $g \geq 2$ is the genus of X. An important driving force behind Mirzakhani's proof is the ergodicity of the action of the mapping class group on the space of singular measured foliations, a result proved by Masur in [12].

The methods introduced in Mirzakhani's thesis did not provide an error term. It would actually take more than 15 years and several developments in Teichmüller dynamics for Eskin, Mirzakhani, and Mohammadi [6] to show that

$$s(X, L) = n(X) \cdot L^{6g-6} + O_X\left(L^{6g-6-\kappa}\right), \tag{3}$$

where the gap $\kappa = \kappa(g) > 0$ depends only on the genus of X. The main source of effective estimates in the proof of (3) is the exponential mixing property of the Teichmüller geodesic flow, a result proved by Avila and Resende [4] building on previous work of Avila, Gouëzel, and Yoccoz [3].

Both the methods introduced in Mirzakhani's thesis and those introduced in her later work with Eskin and Mohammadi can be used to study more refined counting problems of closed geodesics on hyperbolic surfaces. Two closed curves on homeomorphic surfaces are said to have the same topological type if there exists a homeomorphism between the surfaces identifying their free homotopy classes. Given a closed, orientable hyperbolic surface X of genus $g \geq 2$, a closed curve β on X, and $L > 0$, denote by $c(X, \beta, L)$ the number of closed geodesic on X of the same topological type as β and length $\leq L$. In [6], Eskin, Mirzakhani, and Mohammadi showed that if β is simple then

$$c(X, \beta, L) = n(X, \beta) \cdot L^{6g-6} + O_{X,\beta}\left(L^{6g-6-\kappa}\right), \tag{4}$$

where $n(X, \beta) > 0$ is a constant depending on the geometry of X and the topological type of β, and where the gap $\kappa = \kappa(g) > 0$ depends only on the genus of X. Let us highlight the fact that the proof of this estimate makes crucial use of the assumption that the closed curve β is simple.

As the estimates in (1), (2), and (3) show, simple closed geodesics account for just a tiny fraction of all primitive closed geodesics of a closed, orientable hyperbolic surface. Furthermore, primitive closed geodesics are generically filling, i.e., all but a quantitatively small number of primitive closed geodesics of a closed, orientable hyperbolic surface cut the surface into discs.

Counting problems for closed geodesics of non-simple topological types had been previously studied by Mirzakhani. A closed curve on a closed, orientable surface is said to be filling if it intersects every homotopically non-trivial closed curve. In [18], Mirzakhani showed that if X is a closed, orientable hyperbolic surface of genus $g \geq 2$ and β is a filling closed curve on X, then, asymptotically as $L \to +\infty$,

$$c(X, \beta, L) \sim n(X, \beta) \cdot L^{6g-6}, \tag{5}$$

where $n(X, \beta) > 0$ is a constant depending on the geometry of X and the topological type of β. As in her thesis, the proof of this estimate is also based on dynamical arguments, but the key player in this case is the ergodicity of the earthquake flow, a result proved earlier by herself in [16]. An asymptotic estimate analogous to (5) for closed geodesics of any topological type was later proved by Erlandsson and Souto [8] using original arguments introduced in their earlier work [7]. Neither Mirzhakani's methods nor the methods of Erlandsson and Souto provide an error term.

The problem of proving a quantitative estimate with a power saving error term for the counting function $c(X, \beta, L)$ in the case where β is non-simple has since remained open. This problem was alluded to in work of Mirzakhani [18, §1.6.9] and was recently advertised by Wright [24, Problem 18.2]. In this talk we discuss a recent solution to this problem in the generic case where β is filling.

We address this problem using a novel method we refer to as the tracking method. This method relies on recent progress made in the prequels [1] and [2] on the study of the effective dynamics of the mapping class group on Teichmüller space and the space of closed curves of a closed, orientable surface. These recent developments in turn rely on the exponential mixing rate, the hyperbolicity, and the renormalization dynamics of the Teichmüller geodesic flow as their main driving forces.

References

[1] F. Arana-Herrera, *Effective mapping class group dynamics I: Counting lattice points in Teichmüller space*, arXiv:2010.03123.

[2] F. Arana-Herrera, *Effective mapping class group dynamics II: Geometric intersection numbers*, arXiv:2104.01694.

[3] A. Avila, S. Gouëzel, and J.-C. Yoccoz, *Exponential mixing for the Teichmüller flow*, Publ. Math. Inst. Hautes Études Sci. (2006), no. 104, 143–211.

[4] A. Avila and M.J. Resende, *Exponential mixing for the Teichmüller flow in the space of quadratic differentials*, Comment. Math. Helv. **87** (2012), no. 3, 589–638.

[5] J.S. Birman and C. Series, *Geodesics with bounded intersection number on surfaces are sparsely distributed*, Topology **24** (1985), no. 2, 217–225.

[6] A. Eskin, M. Mirzakhani, and A. Mohammadi, *Effective counting of simple closed geodesics on hyperbolic surfaces*, arXiv:1905.04435.

[7] V. Erlandsson and J. Souto, *Counting curves in hyperbolic surfaces*, Geom. Funct. Anal. **26** (2016), no. 3, 729–777.

[8] V. Erlandsson and J. Souto, *Mirzakhani's Curve Counting*, arXiv:1904.05091.

[9] D.A. Hejhal, *The Selberg trace formula for* $\mathrm{PSL}(2, \mathbf{R})$. *Vol. I*, Lecture Notes in Mathematics, **548** (1976), Springer-Verlag, Berlin-New York.

[10] H. Huber, *Über eine neue Klasse automorpher Funktionen und ein Gitterpunktproblem in der hyperbolischen Ebene. I*, Comment. Math. Helv. **30** (1956), 20–62.

[11] G.A. Margulis, *On some aspects of the theory of Anosov systems*, PhD Thesis, 1970, Springer-Verlag, Berlin, 2003.

[12] H. Masur, *Ergodic actions of the mapping class group*, Proc. Amer. Math. Soc. **94** (1985), no. 3, 455–459.
[13] G. McShane and I. Rivin, *A norm on homology of surfaces and counting simple geodesics*, Internat. Math. Res. Notices (1995), no. 2, 61–69.
[14] G. McShane and I. Rivin, *Simple curves on hyperbolic tori*, C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), no. 12, 1523–1528.
[15] M. Mirzakhani, *Simple geodesics on hyperbolic surfaces and the volume of the moduli space of curves*, ProQuest LLC, Ann Arbor, MI, 2004, PhD Thesis, Harvard University.
[16] M. Mirzakhani, *Ergodic theory of the earthquake flow*, Int. Math. Res. Not. IMRN (2008), no. 3, Art. ID rnm116, 39.
[17] M. Mirzakhani, *Growth of the number of simple closed geodesics on hyperbolic surfaces*, Ann. of Math. (2) **168** (2008), no. 1, 97–125.
[18] M. Mirzakhani, *Counting Mapping Class group orbits on hyperbolic surfaces*, arXiv:1601.03342.
[19] B. Randol, *The Riemann hypothesis for Selberg's zeta-function and the asymptotic behavior of eigenvalues of the Laplace operator*, Trans. Amer. Math. Soc. **236** (1978), 209–223.
[20] K. Rafi and J. Souto, *Geodesic currents and counting problems*, Geom. Funct. Anal. **29** (2019), no. 3, 871–889.
[21] I. Rivin, *Simple curves on surfaces*, Geom. Dedicata **87** (2001), no. 1-3, 345–360.
[22] P.C. Sarnak, *PRIME GEODESIC THEOREMS*, ProQuest LLC, Ann Arbor, MI, 1980, PhD Thesis, Stanford University.
[23] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. (N.S.) **20** (1956), 47–87.
[24] A. Wright, *A tour through Mirzakhani's work on moduli spaces of Riemann surfaces*, Bull. Amer. Math. Soc. (N.S.) **57** (2020), no. 3, 359–408.

Towards optimal spectral gap in large genus

Alex Wright

The first non-zero Laplace eigenvalue λ_1 of a hyperbolic surface controls the error term in the Geometric Prime Number Theorem, the speed of mixing of geodesic flow, and measures the extent to which the surface is an expander.

Let Λ_g denote the maximum value of λ_1 over all genus g hyperbolic surfaces. Proving that $\limsup_{g\to\infty} \Lambda_g \leq \frac{1}{4}$ is not hard. The question of whether $\Lambda = \frac{1}{4}$ goes back at least to 1987 [11], and may go as far back as the early 1970s [6]. More concretely:

Question 1. *Is there a sequence of hyperbolic surfaces S_n with genus going to infinity and $\lambda_1(S_n) \to \frac{1}{4}$?*

The best lower known bound for Λ is $\Lambda \geq \frac{1}{4} - (\frac{7}{64})^2$, which can be derived from the appendix of [2] (written by Kim and Sarnak). A positive answer would follow from Selberg's famous $\frac{1}{4}$ conjecture.

Despite the difficulty of this open problem, it is natural to conjecture that λ_1 of a typical genus g surface is close to $\frac{1}{4}$ for large g; see, for example, [12].

Conjecture 2. *For all $\epsilon > 0$, the Weil-Petersson probability that a surface of genus g has $\lambda_1 < \frac{1}{4} - \epsilon$ goes to zero as $g \to \infty$.*

In [3], we obtained the following weaker result:

Theorem 3 (Lipnowski-W, Wu-Xue). *For all $\epsilon > 0$, the Weil-Petersson probability that a surface of genus g has $\lambda_1 < \frac{3}{16} - \epsilon$ goes to zero as $g \to \infty$.*

The same result was obtained independently by Wu and Xue in [13]. Related results, again with $\frac{3}{16}$ appearing, for random covers of a fixed surface were obtained previously in [5, 4].

Mirzakhani pioneered the study of Weil-Petersson random surfaces [8], and devoted her ICM address to this topic [7]. She proved in particular a version of Theorem 3 with $\frac{3}{16}$ replaced with $\frac{1}{4}\left(\frac{\log(2)}{2\pi+\log(2)}\right)^2 \approx 0.002$ [8].

Our proof of Theorem 3 involved computing the average number of closed geodesics of length $C\log(g)$ on for surfaces of genus g, up to a $1/g$ multiplicative error. Our work is inspired by and builds on recent work of Mirzakhani and Petri [9], and hinges on the idea that at length scales growing slowly with genus, most geodesics are simple and non-separating.

On the topic of Conjecture 2, see also the work of Anantharaman, Monk, and Thomas [1, 10].

References

[1] N. Anantharaman and L. Monk, *A high-genus asymptotic expansion of Weil-Petersson volume polynomials*, arXiv:2011.14889.

[2] H.H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183, With appendix 1 by D. Ramakrishnan and appendix 2 by Kim and P. Sarnak.

[3] M. Lipnowski and A. Wright, *Towards optimal spectral gaps in large genus*, arXiv:2103.07496.

[4] M. Magee and F. Naud, *Explicit spectral gaps for random covers of Riemann surfaces*, Publ. Math. Inst. Hautes Études Sci. **132** (2020), 137–179.

[5] M. Magee, F. Naud, and D. Puder, *A random cover of a compact hyperbolic surface has relative spectral gap* $\frac{3}{16} - \epsilon$, arXiv:2003.10911.

[6] H.P. McKean, *Correction to: "Selberg's trace formula as applied to a compact Riemann surface" (Comm. Pure Appl. Math.* **25** *(1972), 225–246)*, Comm. Pure Appl. Math. **27** (1974), 134.

[7] M. Mirzakhani, *On Weil-Petersson volumes and geometry of random hyperbolic surfaces*, Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi (2010), pp. 1126–1145.

[8] M. Mirzakhani, *Growth of Weil-Petersson volumes and random hyperbolic surfaces of large genus*, J. Differential Geom. **94** (2013), no. 2, 267–300.

[9] M. Mirzakhani and B. Petri, *Lengths of closed geodesics on random surfaces of large genus*, Comment. Math. Helv. **94** (2019), no. 4, 869–889.

[10] L. Monk and J. Thomas, *The tangle-free hypothesis on random hyperbolic surfaces*, arXiv:2008.09363.

[11] P. Sarnak, *Review of the article "Riemann surfaces of large genus and large* λ_1*" by P. Buser, M. Burger, and J. Dodziuk*, Mathematical Reviews (1987).

[12] A. Wright, *A tour through Mirzakhani's work on moduli spaces of Riemann surfaces*, Bull. Amer. Math. Soc. (N.S.) **57** (2020), no. 3, 359–408.

[13] Y. Wu and Y. Xue, *Random hyperbolic surfaces of large genus have first eigenvalues greater than* $\frac{3}{16} - \epsilon$, arXiv:2102.05581.

Volumes of odd strata of quadratic differentials

Élise Goujard
(joint work with Eduard Duryev)

In this work in progress we express the Masur-Veech volumes of "completed" odd strata of quadratic differentials as a sum over stable graphs. This formula is a direct generalization of the formula of Delecroix-G-Zograf-Zorich in the case of principal strata. The coefficients of the formula are in this case intersection number of psi classes with the Witten-Kontsevich combinatorial classes; they naturally appear in the count of integer metrics on ribbon graphs with prescribed odd valencies. The main issue here is that this formula computes the Masur-Veech volumes of odd strata together with the weighted volumes of some degenerated strata. In this talk I describe our attempts to understand the weights appearing in the formula and I list all the cases where we are able to explain them (work in progress with E. Duryev).

The moduli space $\mathcal{Q}_{g,n}$ of integrable meromorphic quadratic differentials with n simple poles on genus g Riemann surfaces identifies with the cotangent bundle to the moduli space of Riemann surfaces of genus g with n marked points, by classical Teichmüller theory. It is stratified with respect to the orders of the zeros of the differentials, so we have $\mathcal{Q}_{g,n} = \bigsqcup_{\underline{k}\vdash 4g-4+n} \mathcal{Q}(\underline{k}, -1^n)$ where $\underline{k}$ is a partition of $4g-4+n$. Each stratum is a complex orbifold of dimension $d = 2g-2+\ell(\underline{k})+n$, which is locally modelled on $\mathbb{C}^d$ via the period map. The Lebesgue measure on local coordinates is preserved by change of coordinates and gives rise to a well defined measure μ on each stratum. The Masur-Veech measure defined by disintegration of μ on the level sets of the area function is finite, as proven by Masur and Veech independently. The Masur-Veech volume of some stratum is then defined as

$$\mathrm{Vol}(\mathcal{Q}(\underline{k})) = c \cdot \mu\Big(\Big\{ (x,q) \in \mathcal{Q}(\underline{k}),\ \int_X |q| \leq 1 \Big\} \Big)$$

where c depends on the normalization choice, and it can be evaluated by counting integer points in the moduli space, namely square-tiled surfaces:

$$\mathrm{Vol}(\mathcal{Q}(\underline{k})) = c \lim_{N\to\infty} \frac{\mathrm{Card}\{(X,q) \in \mathcal{Q}^{\mathbb{Z}}(\underline{k}),\ \int_X |q| \leq N\}}{N^d}.$$

The formula that we present here is a natural generalization of the formula for the volumes of principal strata given in [3].

For $m_* = (m_0, m_1, \dots)$ a sequence of non-negative integers being almost all zero, denote by $W_{m_*,n}$ the Witten-Kontsevich combinatorial cycle defined in [10] from the space of equivalence classes of metric ribbon graphs with n boundary components and m_i vertices of valency $2i-1$ (no vertices of even valency) in the combinatorial moduli space. These cycles and their relation to the algebraic geometry of moduli space were studied in particular in [1], [11], [6], [7], [8]. The intersection numbers of the ψ_i classes along the $W_{m*,n}$ are denoted by

$$\langle \tau_{\underline{d}} \rangle_{m_*} = \int_{W_{m_*,n}} \psi_1^{d_1} \dots \psi_n^{d_n}.$$

The exponential generating series for these intersection numbers is an asymptotic expansion of a matrix integral, as proven in [10]. The results of [4] allow to compute these numbers from the usual intersection numbers $\langle\tau_{\underline{d}}\rangle$. The homogeneous polynomial

$$N_{g,n}^{m*}(b_1,\dots,b_n)=\frac{1}{2^{5g-6+2n-2M}}\sum_{\underline{d}\dashv 3g-3+n-M}\frac{\langle\tau_{\underline{d}}\rangle_{m_*}}{d_1!\dots d_n!}b_1^{2d_1}\dots b_n^{2d_n},$$

where $M=\sum m_i(i-1)$, is equal, up to lower order terms and outside a finite number of hyperplanes, for all integer b_i such that $\sum b_i$ is even, to the weighted count of integer metrics on ribbon graphs of genus g of type m_* with n boundaries of length b_i, by a result of [10].

Theorem 1. *The completed Masur-Veech volumes of odd strata of quadratic differentials are given by*

$$\overline{Vol}(\mathcal{Q}(\underline{k}))=\frac{2^d\cdot 2d}{d!}\prod_i m_i!\sum_{\Gamma\in G_{g,n}^{m_*}}\frac{1}{2^{|V(\Gamma)|-1}}\frac{1}{\mathrm{Aut}(\Gamma)}\mathcal{Z}\left(\prod_{e\in E(\Gamma)}b_e\cdot\prod_{v\in V(\Gamma)}N_{g_v,n_v}^{m_{*,v}}(\underline{b}_v)\right)$$

where $G_{g,n}^{m_*}$ is the set of decorated stable graphs of genus g with n leaves and decoration m_*, the partition $\underline{k}=((2i-1)^{m_i})$, and $\mathcal{Z}$ is a linear operator defined on monomials by $\mathcal{Z}(\prod_i b_i^{l_i})=\prod_i l_i!\cdot\zeta(l_i+1)$.

Our main result is that the completed volume $\overline{\mathrm{Vol}}(\mathcal{Q}(\underline{k}))$ is a sum of the actual Masur-Veech volume $\mathrm{Vol}(\mathcal{Q}(\underline{k}))$ with additional weighted contributions of certain adjacent strata. The weights appearing in this sum are still conjectural, we give a list of those conjectural weights in some of the first cases.

Conjecture 2. *For any $\nu=(1^k,-1^n)$ such that the following strata exist, we have:*

$$\begin{aligned}
\overline{\mathrm{Vol}}\mathcal{Q}(3,\nu)&=\mathrm{Vol}\,\mathcal{Q}(3,\nu)+\mathrm{Vol}\left(\mathcal{H}(0)\times\mathcal{Q}(-1,\nu)\right)\\
\overline{\mathrm{Vol}}\mathcal{Q}(5,\nu)&=\mathrm{Vol}\,\mathcal{Q}(5,\nu)+3\cdot\mathrm{Vol}\left(\mathcal{H}(0)\times\mathcal{Q}(1,\nu)\right)\\
\overline{\mathrm{Vol}}\mathcal{Q}(7,\nu)&=\mathrm{Vol}\,\mathcal{Q}(7,\nu)+5\cdot\mathrm{Vol}\left(\mathcal{H}(0)\times\mathcal{Q}(3,\nu)\right)+3\cdot\mathrm{Vol}\left(\mathcal{H}(2)\times\mathcal{Q}(-1,\nu)\right)\\
&\quad+\frac{7}{2}\cdot\mathrm{Vol}\left(\mathcal{H}(0)^2\times\mathcal{Q}(-1,\nu)\right)\\
\overline{\mathrm{Vol}}\mathcal{Q}(9,\nu)&=\mathrm{Vol}\,\mathcal{Q}(9,\nu)+7\cdot\mathrm{Vol}\left(\mathcal{H}(0)\times\mathcal{Q}(5,\nu)\right)+9\cdot\mathrm{Vol}\left(\mathcal{H}(2)\times\mathcal{Q}(1,\nu)\right)\\
&\quad+\frac{27}{2}\cdot\mathrm{Vol}\left(\mathcal{H}(0)^2\times\mathcal{Q}(1,\nu)\right)\\
\overline{\mathrm{Vol}}\mathcal{Q}(3,3,\nu)&=\mathrm{Vol}\,\mathcal{Q}(3,3,\nu)+2\cdot\mathrm{Vol}\left(\mathcal{H}(0)\times\mathcal{Q}(-1,3,\nu)\right)\\
&\quad+\mathrm{Vol}\left(\mathcal{H}(0)^2\times\mathcal{Q}(-1,-1,\nu)\right)\\
\overline{\mathrm{Vol}}\mathcal{Q}(5,3,\nu)&=\mathrm{Vol}\,\mathcal{Q}(5,3,\nu)\\
&\quad+3\cdot\mathrm{Vol}\left(\mathcal{H}(0)\times[\mathcal{Q}(1,3,\nu)\right)+\mathrm{Vol}\left(\mathcal{H}(0)\times\mathcal{Q}(-1,5,\nu)\right)\\
&\quad+3\cdot\mathrm{Vol}\left(\mathcal{H}(0)^2\times\mathcal{Q}(-1,1,\nu)\right)\\
\overline{\mathrm{Vol}}\mathcal{Q}(11,\nu)&=\mathrm{Vol}\,\mathcal{Q}(11,\nu))+9\cdot\mathrm{Vol}\left(\mathcal{H}(0)\times\mathcal{Q}(7,\nu)\right)
\end{aligned}$$

$$+ 15 \cdot \mathrm{Vol}\big(\mathcal{H}(2) \times \mathcal{Q}(3,\nu)\big) + 5 \cdot \mathrm{Vol}\big(\mathcal{H}(4) \times \mathcal{Q}(-1,\nu)\big)$$
$$+ \frac{55}{2} \cdot \mathrm{Vol}\big(\mathcal{H}(0)^2 \times \mathcal{Q}(3,\nu)\big) + 33 \cdot \mathrm{Vol}\big(\mathcal{H}(0) \times \mathcal{H}(2) \times \mathcal{Q}(-1,\nu)\big)$$
$$+ \frac{33}{2} \cdot \mathrm{Vol}\big(\mathcal{H}(0)^3 \times \mathcal{Q}(-1,\nu)\big)$$

The formula for the completed Masur-Veech volumes together with an analysis of the contributions of the adjacent strata should allow to study some large genus asymptotics questions, as the contribution of the one cylinder surfaces, the distribution of the numbers of cylinders, etc, similarly to the principal case studied in [3].

We already wish to thank Adrien Sauvaget for finding a nice general formula for all these completion coefficients after the talk, and more generally, we sincerely thank the workshop participants for interesting discussions that will certainly help us to finish this project. We also thank the MFO for making this progress possible and for providing such a nice research environment.

References

[1] E. Arbarello and M. Cornalba, *Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves*. J. Algebraic Geom. **5** (1996), no. 4, 705–749.

[2] G. Bini, *A combinatorial algorithm related to the geometry of the moduli space of pointed curves*. (English summary) J. Algebraic Combin. **15** (2002), no. 3, 211–221.

[3] V. Delecroix, E. Goujard, P. Zograf, and A. Zorich, *Masur-Veech volumes, frequencies of simple closed geodesics and intersection numbers of moduli spaces of curves* arXiv:2011.05306.

[4] P. Di Francesco, C. Itzykson, and J.-B. Zuber, *Polynomial averages in the Kontsevich model.* Comm. Math. Phys. **151** (1993), no. 1, 193–219.

[5] A. Goupil and G. Schaeffer, *Factoring n-cycles and counting maps of given genus*. Eur. J. Comb. **19** (1998), no. 7, 819–834.

[6] K. Igusa, *Combinatorial Miller-Morita-Mumford classes and Witten cycles*. Algebr. Geom. Topol. **4** (2004), 473–520.

[7] K. Igusa, *Graph cohomology and Kontsevich cycles*. Topology **43** (2004), no. 6, 1469–1510.

[8] K. Igusa and M. Kleber, *Increasing trees and Kontsevich cycles*. Geom. Topol. **8** (2004), 969–1012.

[9] D.M. Jackson and T.I. Visentin, *An atlas of the smaller maps in orientable and nonorientable surfaces*. CRC Press Series on Discrete Mathematics and its Applications. Chapman Hall/CRC, Boca Raton, FL, 2001.

[10] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*. Comm. Math. Phys. **147** (1992), no. 1, 1–23.

[11] G. Mondello, *Combinatorial classes on $\overline{\mathcal{M}_{g,n}}$ are tautological*. Int. Math. Res. Not. **2004**, no. 44, 2329–2390.

Counts of bitangents of tropical plane quartics

Hannah Markwig
(joint work with María Angélica Cueto, Yoav Len, Sam Payne, Kristin Shaw)

A plane quartic curve defined over an algebraically closed field has exactly 28 bitangent lines. This beautiful geometric count is known since a long time, going back to Plücker in 1834 [10]. If we consider a quartic over the reals, then the number of bitangents depends on the topology of the curve. There can be 4, 8, 16 or 28 real bitangents.

The aim of this talk was to use this old counting problem to showcase how tropical geometry can be used as a tool to study geometric counting problems simultaneously over different fields.

Tropical geometry can be viewed as a degeneration of algebraic geometry, where algebraic curves are degenerated to certain piecewise linear graphs in the plane called tropical curves [4, 8]. As many important properties of the underlying algebraic curves can be read off the tropical curves, tropical geometry allows to infuse combinatorial methods into algebraic geometry.

Tropical geometry has proved to be a successful tool for the study of enumerative questions. This was pioneered by Mikhalkin [9] who proved that you can determine both Gromov-Witten invariants and Welschinger invariants of the plane using the same count of tropical curves, only with different multiplicity. The Gromov-Witten invariants count complex curves of a certain degree and genus satisfying point conditions. The Welschinger invariant is the analogous signed count of real curves. The fact that the tropical count can be used simultaneously for the real and the complex invariants also demonstrates the usefulness of this tool for counts over various fields.

For tropical plane quartics, there can be infinitely many bitangents. But we can define an equivalence relation declaring two bitangents equivalent if we can transform one into the other while maintaining bitangency. Then, it has been shown that there are 7 equivalence classes of tropical bitangents [1, 6]. Figure 1 shows an example of a tropical quartic and its 7 bitangent classes. For a given equivalence class, we can ask which members lift to a bitangent of an algebraic plane curve tropicalizing to our given tropical curve. These are 4 when we work over an algebraically closed field [2, 7], yielding $7 \cdot 4 = 28$ complex bitangent lines, as expected. If we study lifts over the real numbers, it turns out that a tropical bitangent class can have 0 or 4 lifts [3]. The question remains how to recover the 4 different possibilities for real counts mentioned above. This is being studied by my PhD student Geiger together with Panizzut and they will publish a preprint about it soon.

In the last part of the talk, we give an outlook on how to use tropical bitangents for arithmetic counts. Arithmetic counts have been pioneered by Kass-Wickelgren and Levine and aim at a universal theory of geometric counting, valid over any field. Counting should not be taken should not be taken too literally in this setting. We do not count geometric objects one by one, but we associate an element in the

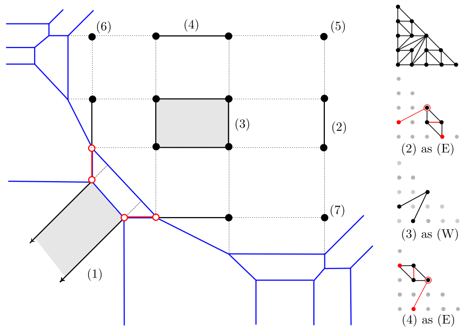

Figure 1. A tropical plane quartic with its seven bitangent classes. On the right, the dual Newton subdivision and the relevant parts of it for the bitangents. (Picture taken from [3].)

Grothendieck-Witt ring to a geometric object, and then add the elements in the Grothendieck-Witt ring for all our objects. The Grothendieck-Witt ring contains all formal sums of isomorphism classes of quadratic forms $q : V \times V \to K$, where V is a K-vector space. We can think of such an element as a matrix A, since after choosing a basis for V, a quadratic form is given by $(x, y) \mapsto x^T \cdot A \cdot y$. As an example, the elements in the Grothendieck-Witt ring over $\mathbb{C}$ defined by the two matrices below are equivalent:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ since } \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Viewed as elements in the Grothendieck-Witt ring over $\mathbb{R}$ however, they are not equivalent.

Larson and Vogt studied an arithmetic count of bitangents of a plane quartic recently [5]. We end the talk by showing how this arithmetic count can be simplified by means of tropical geometry.

References

[1] M. Baker, Y. Len, R. Morrison, N. Pflueger, and Q. Ren. *Bitangents of tropical plane quartic curves*, Mathematische Zeitschrift **282** (2015), 1017–1031.

[2] M. Chan and P. Jiradilok, *Theta characteristics of tropical k_4-curves*, arXiv:1503.05776.

[3] M.A. Cueto and H. Markwig, *The combinatorics and real lifting of tropical bitangents to quartics*, arXiv:2004.10891.

[4] I. Itenberg, G. Mikhalkin, and E. Shustin, *Tropical algebraic geometry*, Oberwolfach Seminars **35**, Birkhäuser Verlag, Basel, second edition, 2009.

[5] H. Larson and I. Vogt, *An enriched count of the bitangents to a smooth plane quartic curve*, arXiv:1909.05945.

[6] H. Lee and Y. Len, *Bitangents of non-smooth tropical quartics*, arXiv:1710.10126.

[7] Y. Len and H. Markwig, *Lifting tropical bitangents*, J. Symbolic Comput., **96** (2020), 122–152.

[8] D. Maclagan and B. Sturmfels, *Introduction to tropical geometry*, Graduate Studies in Mathematics **161** (2015), American Mathematical Society, Providence, RI.

[9] G. Mikhalkin, *Enumerative tropical geometry in* $\mathbb{R}^2$, J. Amer. Math. Soc. **18** (2005), 313–377.
[10] J. Plücker, *Solution d'une question fondamentale concernant la théorie générale des courbes*, J. Reine Angew. Math. **12** (1834), 105–108.

Generating function of monodromy symplectomorphism, isomonodromic tau-function and its WKB expansion

Dmitry Korotkin
(joint work with M. Bertola)

Consider the Fuchsian linear system

$$\frac{\partial \Psi}{\partial z} = \sum_{i=1}^{N} \frac{A_i}{z - t_i}\Psi \ , \qquad \Psi(z=\infty) = I; . \tag{1}$$

We assume that all residues $A_j \in SL(n)$ are diagonalizable, $A_j = G_j L_j G_j^{-1}$, where L_j are diagonal and such that the eigenvalues of each A_j do not differ by an integer. Moreover, we assume that $\sum_{j=1}^{N} A_j = 0$. For fixed values of poles t_j the monodromy map sends the set of coefficients A_j (moduli simultaneous adjoint action by an arbitrary $SL(n)$ matrix) to a point of $SL(n)$ character variety of the N-punctured sphere. Let us order the monodromies M_j around t_j such that $M_1 \dots M_N = I$.

The space of coefficients A_j is equipped with the Kirillov-Kostant Poisson bracket; this bracket is degenerate with Casimirs given by matrices L_j. There exists an extension of the Kirillov-Kostant bracket to the space of matrices (G_j, C_j) given by a quadratic Poisson bracket defined by the dynamical r-matrix [1], which is non-degenerate. The corresponding symplectic form is given by

$$\Omega_A = \sum_{j=1}^{N} \mathrm{tr}(\mathrm{d}L_j \wedge G_j^{-1}\mathrm{d}G_j) - \mathrm{tr}(L_j G_j^{-1}\mathrm{d}G_j \wedge G_j^{-1}\mathrm{d}G_j) \tag{2}$$

which can also be written as $\Omega_A = \mathrm{d}\theta_A$ with $\theta_A = \sum_{j=1}^{N} \mathrm{tr}(L_j \wedge G_j^{-1}\mathrm{d}G_j)$.

The monodromy map can be extended to the map $\mathcal{F}$ from the set of matrices (L_j, G_j) to the set of matrices (Λ_j, C_j) where $\Lambda_j = \exp 2\pi i L_j$ and $M_j = C_j^{-1}\Lambda_j C_j$. The natural symplectic form Ω_M on the (Λ_j, C_j) - space is given by the inverse of an extension of the Goldman bracket from the character variety to the (Λ_j, C_j) - space. The form Ω_M can be expressed in terms of Fock-Goncharov coordinates and a set of additional coordinates ρ_j (logarithms of normalizing factors of eigenvectors of M_j), which are canonically conjugate to the (logarithms of) Casimirs of the Goldman bracket [1]. The general structure of Ω_M looks as follows:

$$\Omega_M = \sum_{j<k} n_{jk}\mathrm{d}\zeta_j \wedge \mathrm{d}\zeta_k + n\sum_{k=1}^{N}\sum_{j=1}^{n-1} \mathrm{d}\log m_{k;j} \wedge \mathrm{d}\rho_{k;j} \tag{3}$$

where σ_j are Fock-Goncharov coordinates corresponding to some triangulation Σ of the Riemann sphere with N vertices; $\rho_{k;j}$, $j = 1, \dots, N$, $k = 1, \dots, n-1$ are the "toric variables" corresponding to the Casimirs $m_{k;j}$. The integer coefficients n_{jk} were computed in [1]. The symplectic potential such that $\mathrm{d}\theta_M = \Omega_M$ can then be chosen as follows:

$$\theta_M = \frac{1}{2}\sum_{j<k} n_{jk}(\zeta_j \mathrm{d}\zeta_k - \zeta_k \mathrm{d}\zeta_j) + n\sum_{k=1}^{N}\sum_{j=1}^{n-1} \log m_{k;j} \wedge \mathrm{d}\rho_{k;j}$$

It can be shown [5, 2] that the monodromy map $\mathcal{F}$ is a symplectomorphism, i.e. $\mathcal{F}^*\Omega_M = \Omega_A$ for fixed values of t_j. If d contains also the derivatives in t_j and the forms Ω_A and Ω_M are extended to the spaces including the positions t_j of the singularities, one gets the relation $\mathcal{F}^*\Omega_M = \Omega_A - \sum_{j=1}^N \mathrm{d}H_j \wedge \mathrm{d}t_j$ [2], where $H_j = \sum_{k\neq j} \frac{\mathrm{tr} A_j A_k}{t_j - t_k}$ are the standard hamiltonians of the Schlesinger system. This relation implies the existence of (local) function τ depending both on t_j's and monodromy data such that

$$\mathrm{d}\log\tau = \theta_A - \mathcal{F}^*\theta_M - \sum_{j=1}^N H_j \mathrm{d}t_j \tag{4}$$

where in the definitions of the 1-forms θ_A (2) and θ_M (3) the "d"-operator is assumed to contain also t_j-part. The definition (4) implies in particular that the t_j-dependence of τ is given by the Jimbo-Miwa equations $\partial_{t_j}\log\tau = H_j$. In addition, the equations (4) define the dependence of τ on the monodromy data, including Casimirs.

The monodromy dependence of τ becomes especially simple in $SL(2)$ case when the Fock-Goncharov coordinates coincide with (complexified) Thurston's shear coordinates; to each edge e of Σ one associates the coordinate ζ_e. Then the equations (4) give rise to

$$\frac{\partial}{\partial\zeta_e}\log\tau = \sum_{j=1}^N \mathrm{trace}\left(L_j G_j^{-1}\frac{\partial G_j}{\partial\zeta_e}\right) - \frac{1}{4\pi i}\left(\sum_{\substack{e'\perp v_1\\ e\prec e'}}\zeta_{e'} - \sum_{\substack{e'\perp v_1\\ e'\prec e}}\zeta_{e'} + \sum_{\substack{e'\perp v_2\\ e\prec e'}}\zeta_{e'} - \sum_{\substack{e'\perp v_2\\ e'\prec e}}\zeta_{e'}\right) \tag{5}$$

where v_1 and v_2 denote vertices of Σ connected by the edge e $\prec$ denotes the counterclockwise ordering of edges at each vertex of Σ. The change of τ under a flip of an edge e of Σ looks as follows:

$$\frac{\tilde{\tau}}{\tau} = \exp\left[-\frac{1}{2\pi i}L\left(\frac{\mathrm{e}^{2\zeta_e}}{\mathrm{e}^{2\zeta_e}+1}\right)\right] \tag{6}$$

where L is the Roger's dilogarithm; the relation (6) means that (an appropriate power of) the tau-function defined via (4) can be interpreted as a section of the natural "dilogarithm line bundle" on the character variety.

This formalism turns out to be convenient in analyzing the expansions of WKB type [3, 4]. For example, let us introduce the small parameter $\hbar$ in the system (1):

$$\frac{\mathrm{d}\Psi}{\mathrm{d}z} = \frac{1}{\hbar}\sum_{j=1}^N \frac{A_j}{z-t_j}\Psi\ ; \tag{7}$$

from now on we are not going to assume that the sum of A_j vanishes i.e. we allow Fuchsian singularity at $z = \infty$. For, say, the 11 component of the matrix Ψ the system (7) implies the scalar equation of second order

$$\varphi_{zz} - \left(\frac{Q_0}{\hbar^2} + \frac{Q_1}{\hbar} + Q_2\right)\varphi = 0 \tag{8}$$

where meromorphic functions Q_j are expressed in terms of coefficients of (7); for example, $Q_0 = -\det\left(\sum_{j=1}^N \frac{A_j}{z-t_j}\right)$. The graph Σ can then be naturally constructed from critical horizontal trajectories of the meromorphic quadratic differential $Q_0(z)(\mathrm{d}z)^2$. Another key ingredient of the WKB analysis is the hyperelliptic curve C of genus $N-2$ defined by $\mu^2 = Q_0$. The WKB ansatz then looks as follows: $\varphi = \mu^{-1/2}\exp\int_{z_0}^z(\hbar^{-1}v_{-1} + v_0 + \hbar v_1 + \dots)$ and the WKB differentials on C are found recursively from (8); in particular $v_{-1} = \mu\mathrm{d}z$, $v_0 = \frac{Q_1(\mathrm{d}z)^2}{2v_{-1}}$ etc. The periods of v_j on C are called the Voros symbols. To each edge e of Σ on can the associate the 1-cycle l_e on C; then one can prove the following asymptotic expansion of ζ_e in terms of Voros symbols: $\zeta_e = \sum_{k=-1}^\infty \int_{l_e} v_k$. This allows to compute the $\hbar$-expansion of the generating function of the monodromy map, at least in the first several orders [4] Notice that the $\hbar$-contribution to the generating function is a special case of the Joyce function of Bridgeland [6].

Similar strategy can be applied to the WKB analysis of the second order equation on an arbitrary Riemann surface; in this case the leading coefficient is given by the Bergman tau-function (a version of $\det\bar\partial$-operator) [3].

References

[1] M. Bertola and D. Korotkin, *Extended Goldman symplectic form in Fock-Goncharov coordinates*, arXiv:1910.06744, to appear in J.Diff.Geom.

[2] M. Bertola and D. Korotkin, *Tau-function and monodromy symplectomorphism*, arXiv:1903.09197

[3] M. Bertola and D. Korotkin, *WKB expansion of Yang-Yang generating function and Bergman tau-function*, arXiv:1910.07140, to appear in Theor.Math.Phys.

[4] M. Bertola, D. Korotkin, and F. Del Monte, *Generating function of monodromy symplectomorphism for* 2×2 *Fuchsian systems and its WKB expansion*, to appear.

[5] P. Boalch, *Quasi-Hamiltonian geometry of meromorphic connections*, Duke Math. J. **139** (2007), Issue 2, 369–405

[6] T. Bridgeland, *Geometry from Donaldson-Thomas invariants*, arXiv:1912.06504.

Tropical ψ-classes

Renzo Cavalieri

(joint work with Andreas Gross, Hannah Markwig)

The main goal of this talk is to introduce ψ *classes* for moduli spaces of tropical curves of arbitrary genus, as in [1]. The notion of ψ classes on moduli spaces of rational tropical curves is introduced by Mikhalkin in [5] and later investigated by Kerber and Markwig [4], who prove a correspondence theorem stating the equality of intersection numbers of algebraic and tropical ψ classes. Katz [2] gives a non-computational proof of such equality by combining the connection of tropical and toric intersection theory with the fact that $\overline{M}_{0,n}$ gives a tropical compactification.

In this talk we introduce a construction of ψ classes which lies entirely in the tropical world. The main technical issue to overcome is how to make the integral lattices of adjacent cones of the extended cone complex $\overline{M}_{g,n}$ *communicate* with each other. The solution we propose is that families of tropical curves must be endowed with such information, in the form of a sheaf of functions to be considered *affine*. Germs of functions at points of faces then provide the appropriate transition data among the integral lattices of adjacent cones. Once one knows how to transition affine functions across faces, it is possible to define a notion of balancing, which is the key tool for tropical intersection theory. One may also define sheaves $A_{\mathcal{C}}(ks)$ of affine functions with prescribed order along a linear section, which are torsors over affine functions. This allows to sidestep the technical issue of making sense of the notions of tangent bundles or relative dualizing sheaves in the category of tropical spaces: the *i-th cotangent line bundle* of a family of tropical curves $\mathcal{C}$ is defined to be $A_{\mathcal{C}}(-s_i)$, drawing from the algebraic identification of the i-th cotangent line bundle with the conormal bundle to the i-th section. The tropical ψ class then admits the natural definition

$$\psi_i := c_1(A_{\mathcal{C}}(-s_i)),$$

where c_1 denotes the first Chern class of a line bundle on a stack, i.e. a coherent assignment of a degree one cycle on the base of any family of tropical curves.

Given the absence of any algebraic input, it is not surprising that one obtains a combinatorial theory which is broader and not as well behaved as the algebraic theory. Tropical ψ classes in general do not enjoy the positivity properties of their algebraic counterparts. It appears however that when a family of tropical curves arises from an algebraic one, then the combinatorial theory agrees with the algebraic one. As evidence of the compatibility of our constructions with the algebraic theory, the computation of the degree of ψ on $\overline{M}_{1,1}$ is very much parallel to its classical counterpart [8, Section 3.13]: a pencil of plane cubics has nine base points, and hence it provides a family of genus one curves with nine sections, with total space $Bl_{p_1,\dots,p_9}\mathbb{P}^2$. The ψ classes on this family are dual to the self-intersections of the exceptional divisors, and hence have degree one. Since the family has twelve rational fibers, the pencil gives a degree-twelve covering of $\overline{M}^{alg}_{1,1}$. We consider tropical stable maps instead of cubic curves to obtain a covering of $\overline{M}_{1,1}$, as it would be impossible for curves with very large j-invariant to satisfy the point constraints without contracting any edge. Well-spacedness, which is also a realizability condition [7, 6], ensures that the family is pure-dimensional. After that, the proof is parallel to the classical one: the degree of the covering is twelve, as a consequence of the count of rational curves in the family, or from [3]. The class ψ_i is supported on the unique curve where the i-th leg is incident to a four-valent vertex, and an explicit computation shows that the multiplicity is one.

The aim of this talk is to present the main ideas and intuition behind the technical constructions in [1]. Our goal is to provide motivation for the concepts through the illustration of several simple examples for the various constructions.

References

[1] R. Cavalieri, A. Gross, and H. Markwig, *Tropical ψ classes*, arXiv:2009.00586.

[2] E. Katz, *Tropical intersection theory from toric varieties*, Collect. Math. **63** (2012), no. 1, 29–44.

[3] M. Kerber and H. Markwig, *Counting tropical elliptic plane curves with fixed j-invariant*, Com-ment. Math. Helv. **84** (2009), no. 2, 387–427.

[4] M. Kerber and H. Markwig, *Intersecting Psi-classes on tropical $M_{0,n}$*, Int. Math. Res. Not. IMRN (2009), no. 2, 221–240.

[5] G. Mikhalkin, *Moduli spaces of rational tropical curves*, Proceedings of Gokova Geometry-Topology Conference 2006, Gokova Geometry/Topology Conference (GGT), Gokova, (2007), 39–51.

[6] D. Ranganathan, K. Santos-Parker, and J. Wise, *Moduli of stable maps in genus one and logarithmic geometry I*, arXiv:1708.02359.

[7] D. E. Speyer, *Parameterizing tropical curves. I: Curves of genus zero and one.*, Algebra Number Theory **8** (2014), no. 4, 963–998.

[8] R. Vakil, *The moduli space of curves and Gromov-Witten theory*, Enumerative invariants in algebraic geometry and string theory, Lecture Notes in Math. **1947**, Springer, Berlin, 2008, 143–198.

Volume of the moduli space of super Riemann surfaces

Paul Norbury

Stanford and Witten defined the volume of the moduli space of super hyperbolic surfaces via a natural measure on the character variety of a surface into a supergroup [5]. Alternatively, the volume of a symplectic supermanifold can be quite generally expressed in terms of the Euler class of a bundle over an underlying symplectic manifold together with its symplectic form. These two viewpoints lead to a conjectural relationship between a collection of integrals of known cohomology classes over the moduli space of stable Riemann surfaces and the volume of the moduli space of super Riemann surfaces.

Define the moduli space of super hyperbolic surfaces via the character variety

$$\widehat{\mathcal{M}}_{g,n} := \{\rho : \pi_1(\Sigma) \to OSp(1|2) \stackrel{\text{hyp}}{\to} SL(2,\mathbb{R})\}/\sim$$

where $\Sigma = \overline{\Sigma} - \{p_1, ..., p_n\}$ for $\overline{\Sigma}$ a smooth compact genus g surface. Each representation into the supergroup $OSp(1|2)$ is required to live over a hyperbolic (Fuchsian) representation $\pi_1(\Sigma) \stackrel{\text{hyp}}{\to} SL(2,\mathbb{R})$ which defines a spin structure over a hyperbolic surface and gives a map to the moduli space of spin hyperbolic surfaces

$$\widehat{\mathcal{M}}_{g,n} \to \mathcal{M}_{g,n}^{spin}$$

known as its reduced space. This map possess a natural section $\mathcal{M}_{g,n}^{spin} \to \widehat{\mathcal{M}}_{g,n}$ and we denote the normal bundle of its image by $\nu_{g,n} \to \mathcal{M}_{g,n}^{spin}$. The tangent space of the character variety at a super hyperbolic surface $\widehat{\Sigma}$, which lives above the spin hyperbolic surface Σ, is naturally identified with the first cohomology group of the sheaf of locally constant sections of the tangent space to $\widehat{\Sigma}$

$$H^1(T_{\widehat{\Sigma}}) \cong H^1(T_\Sigma) \oplus H^1(T_\Sigma^{\frac{1}{2}}).$$

Here $T_\Sigma^{\frac{1}{2}}$ is the flat rank two bundle over Σ associated to the natural action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$. The $H^1(T_\Sigma)$ summand describes the classical, bosonic directions $T_\Sigma \mathcal{M}_{g,n}^{spin}$ while the $H^1(T_\Sigma^{\frac{1}{2}})$ summand describes the fermionic directions which represent the normal bundle $H^1(T_\Sigma^{\frac{1}{2}}) \hookrightarrow \nu_{g,n} \to \mathcal{M}_{g,n}^{\text{spin}}$. This leads to the dimension of the moduli space given by $\dim \widehat{\mathcal{M}}_{g,n} = (6g-6+2n \mid 4g-4+2n)$.

Torsion of the cochain complex produces a measure μ on $H^1(T_{\widehat{\Sigma}})$ which is used in [5] to define the volume of the moduli space of super hyperbolic surfaces

$$\text{Vol}(\widehat{\mathcal{M}}_{g,n}) = \int_{\widehat{\mathcal{M}}_{g,n}} \mu = \int_{\mathcal{M}_{g,n}^{\text{spin}}} e(\nu_{g,n}) \exp(\omega^{WP}).$$

The second equality, valid for super symplectic manifolds, expresses the volume using the Euler class and Weil-Petersson symplectic form on the reduced space.

The moduli space of stable spin curves is defined by

$$\overline{\mathcal{M}}_{g,n}^{\text{spin}} = \{(\mathcal{C}, L, p_1, ..., p_n, \phi) \mid \phi : L^2 \xrightarrow{\cong} \omega_{\mathcal{C}}^{\log}\}$$

where $\mathcal{C}$ is a stable twisted curve, or stack, with group $\mathbb{Z}_2$ such that generic points have trivial isotropy group and non-trivial orbifold points have isotropy group $\mathbb{Z}_2$. Denote by $\mathcal{E}$ the universal spin structure over $\overline{\mathcal{M}}_{g,n}^{\text{spin}}$ and define the bundle $E_{g,n} = -R\pi_*\mathcal{E}^\vee$ over $\overline{\mathcal{M}}_{g,n}^{\text{spin}}$ with fibre $H^1(L^\vee) \hookrightarrow E_{g,n} \to \overline{\mathcal{M}}_{g,n}^{\text{spin}}$.

Theorem 1 ([3])**.** *There is a natural extension of $\nu_{g,n}$ to the compactification:*

$$\begin{array}{ccc} \nu_{g,n} & \to & E_{g,n} \\ \downarrow & & \downarrow \\ \mathcal{M}_{g,n}^{\text{spin}} & \to & \overline{\mathcal{M}}_{g,n}^{\text{spin}}. \end{array}$$

defined by a canonical isomorphism of the sheaf of locally constant sections and the sheaf of locally holomorphic sections:

$$H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}}) \cong H^1(\overline{\Sigma}, L^\vee).$$

The theorem gives a cohomological expression for a measure $e(\nu_{g,n})$, although it remains to determine if $E_{g,n}$ is the correct extension. The Euler class of $E_{g,n}$ can be evaluated via push-forward [4] using the forgetful map $p : \overline{\mathcal{M}}_{g,n}^{\text{spin}} \to \overline{\mathcal{M}}_{g,n}$

$$\Theta_{g,n} := (-1)^n 2^{g-1+n} p_* c_{2g-2+n}(E_{g,n}) \in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n}).$$

A deformation of the super symplectic form gives

$$\widehat{V}_{g,n}(L_1, ..., L_n) = \int_{\widehat{\mathcal{M}}_{g,n}(\vec{L})} \mu.$$

Analogous to Wolpert's formula for Weil-Petersson volumes, define

$$V^{\Theta}_{g,n}(L_1,...,L_n) := \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \exp\left\{2\pi^2\kappa_1 + \frac{1}{2}\sum_{i=1}^{n} L_i^2\psi_i\right\}.$$

The naturality of the extension of $\nu_{g,n}$ to $E_{g,n}$ leads to the following conjecture.

Conjecture 2.

$$\widehat{V}_{g,n}(L_1,...,L_n) = V^{\Theta}_{g,n}(L_1,...,L_n).$$

There exists a natural Hermitian metric and connection on $E_{g,n}$ related to the construction of the Weil-Petersson metric. This produces a differential form representing $e(E_{g,n})$ which may coincide with $e(\nu_{g,n})$ to prove the conjecture.

Define kernels $\widehat{H}(x,y) = \frac{1}{2\pi}\left(\frac{e^{\frac{-x+y}{4}}}{1+e^{\frac{-x+y}{2}}} - \frac{e^{\frac{x+y}{4}}}{1+e^{\frac{x+y}{2}}}\right)$

$$\widehat{D}(x,y,z) = \widehat{H}(x,y+z), \quad \widehat{R}(x,y,z) = \frac{1}{2}\widehat{H}(x+y,z) + \frac{1}{2}\widehat{H}(x-y,z)$$

and $P^{\Theta}_{g,n}(x,y,L_K) = V^{\Theta}_{g-1,n+1}(x,y,L_K) + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=K}} V^{\Theta}_{g_1,|I|+1}(x,L_I)V^{\Theta}_{g_2,|J|+1}(y,L_J)$

for $K = \{2,..,n\}$

Theorem 3 ([3])**.** *The following two statements are equivalent.*

(1) $$L_1V^{\Theta}_{g,n}(L_1,...,L_n) = \frac{1}{2}\int_0^\infty\int_0^\infty xy\widehat{D}(L_1,x,y)P^{\Theta}_{g,n}(x,y,L_2,...,L_n)dxdy + \frac{1}{2}\sum_{j=2}^{n}\int_0^\infty x\widehat{R}(L_1,L_j,x)V^{\Theta}_{g,n-1}(x,L_2,...,\hat{L}_j,...,L_n)dx$$

(2) $$Z^{BGW} = \exp\sum\frac{1}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\Theta_{g,n}\cdot\prod_{i=1}^{n}\psi_j^{k_i}t_{k_i} \text{ is a KdV tau function.}$$

The recursion (1) matches Mirzakhani's recursion for Weil-Petersson volumes [2] except for the definition of the kernels. The KdV tau function in (2) is the Brézin-Gross-Witten tau function and (2) has been verified up to genus six. The proof of Theorem 3 is analogous to Mirzakhani's proof of the Kontsevich-Witten KdV tau function [1, 6] from intersection numbers of tautological classes on $\overline{\mathcal{M}}_{g,n}$.

Stanford and Witten [5] constructed the kernels $\widehat{D}(x,y,z)$ and $\widehat{R}(x,y,z)$ by analysing super hyperbolic pairs of pants in a similar way to Mirzakhani's derivation of $D(x,y,z)$ and $R(x,y,z)$ via hyperbolic pairs of pants. They used this to prove that $\widehat{V}_{g,n}(L_1,...,L_n)$ satisfies the recursive formulae (1).

References

[1] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function.* Comm. Math. Phys. **147**, (1992), 1-23.

[2] M. Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces.* Invent. Math. **167** (2007), 179-222.

[3] P. Norbury, *Enumerative geometry via the moduli space of super Riemann surfaces.* arXiv:2005.04378
[4] P. Norbury, *A new cohomology class on the moduli space of curves*, arXiv:1712.03662
[5] D. Stanford and E. Witten, *JT Gravity and the Ensembles of Random Matrix Theory*, arXiv:1907.03363
[6] E. Witten *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry (Cambridge, MA, 1990), 243–310, Lehigh Univ., Bethlehem, PA, 1991.

Geometric Recursion

Jørgen Ellegaard Andersen
(joint work with Gaetan Borot and Nicolas Orantin)

In low dimensional topology, the following setup is very common. One has a functor, say E from the category of compact oriented surfaces with boundary to the category of vector spaces and for each such surface Σ, one has an assignment of a vector

$$\Sigma \longmapsto \Omega_\Sigma \in E(\Sigma)$$

which is invariant under the natural action of the mapping class group of Σ acting on $E(\Sigma)$.

To mention a few examples, one has various kinds of assignments of functions on Teichmüller spaces $\mathcal{T}$, such as the trivial constant function 1, sums over multicurves of functions applied to the hyperbolic length of the multi-curves, functions applied to the spectrum of the spectrum of the Laplace operator acting on functions on each hyperbolic surface, in particular spectral determinants of relevance in string theory. If E is the vector space of smooth forms on $\mathcal{T}$, then the Weil-Petersson symplectic form on $\mathcal{T}$ is another example or its associated volume form. When E is the vector space of smooth forms on the irreducible part of the moduli space of flat G-connections M_G, G is a semi-simple Lie group, then the Goldman Symplectic form is another example. Further one can consider the vector space of smooth functions on Teichmüller space times M_G and as Ω_Σ consider the Ricci potential for the family of complex structor induced on M_G parameterised by $\mathcal{T}$. In fact this family of complex structure is another examples in it self. If one consider the vector space of smooth tensors on the higher Teichmüller spaces, then the Pressure metric is yet another good example.

Indeed, *Geometric Recursion* is also about constructing such $\Omega_\Sigma \in E(\Sigma)$ for each compact oriented surface with boundary Σ, but the big difference to the usual constructions mentioned above, is the recursive scheme we propose. In fact, in our Geometric Recursion the construction proceeds by successive excisions of homotopy classes of embedded pairs of pants, and thus by induction on the Euler characteristic. We provide sufficient conditions to guarantee the infinite sums appearing in this construction converge in a very general setting. In particular, we can generate mapping class group invariant vectors $\Omega_\Sigma \in E(\Sigma)$. The initial data for the recursion encode the cases when Σ is a pair of pants or a torus with one boundary, as well as the "recursion kernels" used for glueing.

As a first application, we demonstrate that our formalism produce a large class of measurable functions on the moduli space of bordered Riemann surfaces. Under certain conditions, the functions produced by the geometric recursion can be integrated with respect to the Weil-Petersson measure on moduli spaces with fixed boundary lengths, and we show that the integrals satisfy a topological recursion generalizing the one of Eynard and Orantin. We establish a generalization of Mirzakhani-McShane identities, namely that multiplicative statistics of hyperbolic lengths of multicurves can be computed by the geometric recursion, and thus their integrals satisfy the topological recursion. As a corollary, we find an interpretation of the intersection indices of the Chern character of bundles of conformal blocks in terms of the aforementioned statistics. Please see [1] for more details.

The theory has however a wider scope than functions on Teichmüller space, which will be explored in future work; one expects that many functorial objects in low-dimensional geometry, including at least the above mentioned, could be constructed by variants of our geometric recursion.

This work grew out of the search for an intrinsic geometric meaning of the topological recursion of [2], inspired by the famous but isolated example of Mirzakhani-McShane identities and Mirzakhani's recursion for the Weil-Petersson volume of the moduli spaces of bordered Riemann surfaces [3].

References

[1] J.E. Andersen, G. Borot, and N. Orantin, *Geometric Recursion*, arXiv:1711.04729.

[2] B. Eynard and N. Orantin, *Topological recursion in enumerative geometry and random matrices*, J. Phys. A **42** (2009), no. 29, 293001, 117 pp.

[3] M. Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, Invent. Math. **167** (2007), 179–222.

Hurwitz degrees and counting linear systems with prescribed incidences

Gavril Farkas

(joint work with Carl Lian)

The results in this talk summarize our recent preprint [3]. Having fixed positive integers r and s and setting $g = rs + s$ and $d = rs + r$, in a celebrated paper [1], Castelnuovo computed the number of linear series of type g^r_d on a general curve C of genus g. By degeneration to a g-nodal rational curve, he argued that this number equals the degree of the Grassmannian $G(r+1, d+1)$ in its Plücker embedding, that is,

$$g! \cdot \frac{1! \cdot 2! \cdot \cdots \cdot r!}{s! \cdot (s+1)! \cdot \cdots \cdot (s+r)!}.$$

Motivated by two very recent papers of Tevelev [4] and Cela-Pandharipande-Schmitt [2] we consider a variant of this problem, where we impose incidence conditions on the corresponding maps to projective spaces. Let $[C, x_1, \ldots, x_n] \in \mathcal{M}_{g,n}$ be a general n-pointed complex curve of genus g. We denote by $G^r_d(C)$ the variety of linear systems $\ell = (L, V)$ of type g^r_d on C. A general $\ell \in G^r_d(C)$ corresponds to a regular map $\phi_\ell \colon C \to \mathbb{P}^r$. Evaluation at the points $x_1, \ldots, x_n$ induces a rational map

$$\text{(1)} \qquad \mathrm{ev}_{(x_1,\ldots,x_n)} \colon G^r_d(C) \dashrightarrow \left(\mathbb{P}^r\right)^n / PGL(r+1) =: P^n_r,$$

to the moduli spaces of n points in $\mathbb{P}^r$. We study the degree $L_{g,r,d}$ of the map $\mathrm{ev}_{(x_1,\ldots,x_n)}$ in the case when this map is generically finite and both spaces have non-negative dimension. Since $G^r_d(C)$ is a smooth variety of dimension $\rho(g, r, d) = g - (r+1)(g - d + r)$, whereas $\dim(P^n_r) = rn - r^2 - 2r$ as long as $n \geq r + 2$, one expects $\mathrm{ev}_{(x_1,\ldots,x_n)}$ to be generically finite precisely when

$$\text{(2)} \qquad n = \frac{dr + d + r - rg}{r}.$$

If $y_1, \ldots, y_n \in \mathbb{P}^r$ are general points, $L_{g,r,d}$ counts the number of morphisms $f \colon C \to \mathbb{P}^r$ of degree d satisfying $f(x_i) = y_i$ for $i = 1, \ldots, n$. Since the points y_i are consider up to projective equivalence, these incidence conditions are intrinsic to ℓ. For large d, it turns out there is a very simple formula for this degree:

Theorem 1. *Suppose $d \geq rg + r$, or equivalently $n \geq d + 2$. Then*

$$L_{g,r,d} = (r+1)^g.$$

We remark that the hypothesis $n \geq d + 2$ is automatically satisfied whenever $g \leq 1$. Indeed, if instead $n \leq d+1$ and $g \leq 1$, then $d+1 \geq n = d+1+\frac{d}{r}-g \geq d+\frac{d}{r}$, hence $n \leq d + 1 \leq r + 1$, a contradiction.

When $r = 1$, the special case $d = g + 1$ was studied under the guise of *scattering amplitudes* by Tevelev [4], who found the strikingly simple formula $L_{g,1,g+1} = 2^g$. This raised the possibility, confirmed by Theorem 1, that in the range when d is relatively large, the degree $L_{g,r,d}$ has a simple expression. Using Hurwitz space techniques, Cela-Pandharipande-Schmitt [2] obtained general formulas for $L_{g,1,d}$, which they called *Tevelev degrees*; in particular, when $d \geq g + 1$, they found again $L_{g,1,d} = 2^g$.

When either $r = 1$ or $n = r + 2$, or under the hypotheses of Theorem 1, we obtain a more general formula in terms of Schubert calculus. For a positive integer a, we recall the notation σ_a for the class of the special Schubert cycle of codimension a consisting of those $(r+1)$-planes $V \in G(r+1, d+1)$ meeting a fixed subspace $W \subseteq \mathbb{C}^{d+1}$ of dimension $d - a$. We also recall that σ_{1^r} denotes the class of the special Schubert cycle of codimension r consisting of those $(r+1)$-planes $V \in G(r+1, d+1)$ whose intersection with a fixed codimension 2 linear subspace $U \subseteq \mathbb{C}^{d+1}$ has dimension at least r. Our main result is as follows:

Theorem 2. *Suppose that either:*

- *$d \geq rg + r$, (i.e., the same hypothesis as in Theorem 1),*
- *$d = r + \frac{rg}{r+1}$ (in which case $n = r + 2$), or*
- *$r = 1$.*

In each of these cases,

$$L_{g,r,d} = \int_{G(r+1,d+1)} \sigma_{1^r}^g \cdot \left[\sum_{\alpha_0+\cdots+\alpha_r=(r+1)(d-r)-rg} \left(\prod_{i=0}^r \sigma_{\alpha_i} \right) \right].$$

References

[1] G. Castelnuovo, *Numero delle involuzioni razionali giacenti sopra una curva di dato genere*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **5** (1889), 130–133.

[2] A. Cela, R. Pandharipande, and J. Schmitt, *Tevelev degrees and Hurwitz moduli spaces*, arXiv:2103.14055.

[3] G. Farkas and C. Lian, *Linear series on general curves with prescribed incidence conditions*, arXiv:2105.09340

[4] J. Tevelev, *Scattering amplitudes of stable curves*, arXiv:2007.03831.

Reporter: Reinier Kramer

Participants

Prof. Dr. Jørgen E. Andersen
Centre for Quantum Mathematics
University of Southern Denmark
Campusvej 55
5230 Odense M
DENMARK

Francisco Arana-Herrera
Department of Mathematics
Stanford University
450 Jane Stanford Way
Stanford CA 94305-2125
UNITED STATES

Prof. Dr. Jayadev S. Athreya
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle, WA 98195-4350
UNITED STATES

Dr. David Aulicino
Department of Mathematics
CUNY - Brooklyn College
2900 Bedford Avenue
Brooklyn NY 11210-2889
UNITED STATES

Prof. Dr. Matthew Bainbridge
Department of Mathematics
Indiana University
Bloomington IN 47405-4301
UNITED STATES

Frederik Benirschke
Department of Mathematics
Stony Brook University
Math. Tower
Stony Brook, NY 11794-3651
UNITED STATES

Prof. Dr. Gaëtan Borot
Institut für Mathematik
Humboldt-Universität Berlin
Rudower Chaussee 25
12489 Berlin
GERMANY

Dr. Renzo Cavalieri
Department of Mathematics
Colorado State University
Weber Building
Fort Collins, CO 80523-1874
UNITED STATES

Prof. Dr. Dawei Chen
Department of Mathematics
Boston College
Chestnut Hill MA 02467-3806
UNITED STATES

Matteo Costantini
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Vincent Delecroix
LaBRI
Université de Bordeaux
office 314
351, cours de la Liberation
33405 Talence Cedex
FRANCE

Dr. Norman Do
School of Mathematics
Monash University
Monash University VIC 3800
AUSTRALIA

Prof. Dr. Benjamin Dozier
Department of Mathematics
Stony Brook University
Math. Tower
Stony Brook, NY 11794-3651
UNITED STATES

Prof. Dr. Philip M. Engel
Department of Mathematics
University of Georgia
Athens GA 30602
UNITED STATES

Dr. Viveka Erlandsson
School of Mathematics
University of Bristol
Fry Building
Woodland Road
Bristol BS81UG
UNITED KINGDOM

Dr. Federica Fanoni
CNRS and Laboratoire d'analyse et de mathématiques appliquées, Université Paris-Est, Creteil
61, Avenue du General de Gaulle
94010 Créteil Cedex
FRANCE

Prof. Dr. Gavril Farkas
Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
10117 Berlin
GERMANY

Prof. Dr. Simion Filip
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago IL 60637-1514
UNITED STATES

Dr. Quentin Gendron
Centro de Investigación en Matemáticas
Jalisco S/N
Guanajuato Col. Valenciana CP: 36023
MEXICO

Alessandro Giacchetto
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Dr. Elise Goujard
Département de Mathématiques
Université Bordeaux
351, cours de la Liberation
33405 Talence Cedex
FRANCE

Prof. Dr. Samuel Grushevsky
Department of Mathematics
Stony Brook University
Stony Brook NY 11794-3651
UNITED STATES

Prof. Dr. Ursula Hamenstädt
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Felix Janda
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556-5683
UNITED STATES

Prof. Dr. Bruno Klingler
Fachbereich Mathematik
Humboldt Universität Berlin
Unter den Linden 6
10099 Berlin
GERMANY

Prof. Dr. Dmitry Korotkin
Department of Mathematics & Statistics
Concordia University
1400 De Maisonneuve Blvd. West
Montréal QC H3G 1M8
CANADA

Dr. Reinier Kramer
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Erwan Lanneau
Institut Fourier
UMR 5582, CNRS
Université Grenoble-Alpes
BP 74
38402 Saint-Martin-d'Hères
FRANCE

Dr. Danilo Lewanski
IHES and Institut de Physique
Théorique Paris
91191 Gif-sur-Yvette Cedex
FRANCE

Prof. Dr. Hannah Markwig
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Howard Masur
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Prof. Dr. Curtis T. McMullen
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Prof. Dr. Martin Möller
FB 12 - Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Straße 6-8
60325 Frankfurt am Main
GERMANY

Prof. Dr. Gabriele Mondello
Dipartimento di Matematica "Guido
Castelnuovo"
Sapienza Università di Roma
Piazzale Aldo Moro, 5
00185 Roma
ITALY

Dr. Scott Mullane
Institut für Mathematik
Goethe-Universität Frankfurt
Postfach 111932
60054 Frankfurt am Main
GERMANY

Prof. Dr. Duc-Manh Nguyen
IMB Bordeaux
Université de Bordeaux
Batiment A33
351, cours de la Liberation
33405 Talence Cedex
FRANCE

Prof. Dr. Paul Norbury
School of Mathematics and Statistics
University of Melbourne
Melbourne VIC 3010
AUSTRALIA

Dr. Chaya Norton
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Georg Oberdieck
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Hugo Parlier
Unite de Recherche en Mathématiques
Université du Luxembourg
Campus Belval, Maison du Nombre
6, Avenue de la Fonte
4364 Esch-sur-Alzette
LUXEMBOURG

Dr. Bram Petri
Institut de Mathématiques de Jussieu
Paris Rive Gauche (IMJ-PRG)
4 place Jussieu
P.O. Box 247
75252 Paris Cedex 05
FRANCE

Aaron Pixton
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Prof. Dr. Kasra Rafi
Department of Mathematics
University of Toronto
40 St George Street
Toronto ON M5S 2E4
CANADA

Dr. Anja Randecker
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Dr. Jenya Sapir
Department of Mathematical Sciences
Binghamton University
State University of New York
P.O. Box 6000
Binghamton, NY 13902-6000
UNITED STATES

Dr. Adrien Sauvaget
Département de Mathématiques
Université de Cergy-Pontoise
Site Saint-Martin, BP 222
2, avenue Adolphe Chauvin
95302 Cergy-Pontoise Cedex
FRANCE

Johannes Schmitt
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Johannes Schwab
Institut für Mathematik, FB 12
Goethe-Universität Frankfurt
Robert-Mayer-Straße 6-8
60325 Frankfurt am Main
GERMANY

Prof. Dr. Ravi Vakil
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
UNITED STATES

Campbell Wheeler
Max Planck Institute for Mathematics
Bonn International Graduate School
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Scott A. Wolpert
Department of Mathematics
4176 Campus Drive
University of Maryland
College Park, MD 20742-4015
UNITED STATES

Dr. Alex M. Wright
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Prof. Dr. Don B. Zagier
Max Planck Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Peter Zograf
Steklov Mathematical Institute
PDMI
Fontanka 27
St. Petersburg 191 023
RUSSIAN FEDERATION

Prof. Dr. Anton Zorich
Institut de Mathématiques de Jussieu
Université de Paris VII
Paris Rive Gauche (IMJ-PRG)
8 Place Aurélie Nemours
75013 Paris Cedex
FRANCE

Prof. Dr. Dimitri Zvonkine
Département de Mathématiques
Université de Versailles Saint-Quentin
45, Avenue des Etats-Unis
78035 Versailles Cedex
FRANCE