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Numerical Methods for Fully Nonlinear and Related PDEs (hybrid meeting)

Organized by
Sören Bartels, Freiburg
Susanne C. Brenner, Baton Rouge
Xiaobing Feng, Knoxville
Michael Neilan, Pittsburgh

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Abstract. The aim of this workshop was to discuss the challenges, latest trends and advancements on numerical methods for fully nonlinear PDEs. The construction of numerical schemes and their convergence analysis is still an emerging field in computational mathematics with several fundamental open problems. Nonetheless, significant breakthroughs have recently appeared, including the design of accurate finite element schemes for non-variational problems, a priori error estimates for monotone schemes, and the construction of high-order and adaptive methods.

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Introduction by the Organizers

Fully nonlinear PDEs arise in several applications including economics, geometric optics, meteorology, cosmology, and optimal transport. Despite significant progress in PDE analysis for this class of problems, the construction of numerical schemes and their convergence analysis is still an emerging field in computational mathematics with several fundamental open problems. Nonetheless, significant breakthroughs have recently appeared in the field including the design of high-order finite element schemes for non-variational problems, a priori error estimates for monotone schemes, and the construction of high-order and adaptive methods.

This Oberwolfach workshop brought together 27 leading experts to discuss the latest trends and advancements on numerical methods for fully nonlinear PDEs. Broadly speaking, the talks of the workshop fell into two intersecting categories: (i)

Numerical methods for elliptic PDEs in non-divergence form and Hamilton-Jacobi-Bellman problems and (ii) Computational methods for Monge-Ampère-type equations. Specific topics in the first category include finite element discretizations for linear and nonlinear PDEs in non-divergence form with non-smooth coefficients, the design of high-order finite element methods, a priori and a posteriori error estimates, and adaptive schemes. Themes of the second category include enforcement of convexity at the discrete level, discretization of optimal transport, and error estimates for monotone discretizations.

The workshop was well attended with broad geographic representation from Europe, the USA, and China. A total of 17 research talks were given by a blend of senior and early-career researchers. Due to the Covid-19 pandemic, the workshop took place in a hybrid format, with 6 participants attending in-person, while 21 participants attended virtually.

Workshop (hybrid meeting): Numerical Methods for Fully Nonlinear and Related PDEs

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Abstracts

Convergence of adaptive discontinuous Galerkin and C^0-interior penalty finite element methods for Hamilton–Jacobi–Bellman and Isaacs equations

Iain Smears
(joint work with Ellya L. Kawecki)

Hamilton–Jacobi–Bellman (HJB) and Isaacs equations arise in stochastic optimal control problems and two-player stochastic games, and find diverse applications in engineering, industry, economics, and finance. In many cases, these equations are fully nonlinear second-order PDE, i.e. the nonlinearity includes the second derivatives. As a model problem, we consider the Isaacs equation

$$\begin{aligned} \inf_{\alpha\in\mathscr{A}}\sup_{\beta\in\mathscr{B}}\left[L^{\alpha\beta}u - f^{\alpha\beta}\right] &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a nonempty bounded convex polytopal open set in $\mathbb{R}^d$, $d \in \{2,3\}$, where $\mathscr{A}$ and $\mathscr{B}$ are nonempty compact metric spaces, and where the second-order nondivergence form elliptic operators $L^{\alpha\beta}$, $\alpha\in\mathscr{A}, \beta\in\mathscr{B}$, are defined by

$$L^{\alpha\beta}v := a^{\alpha\beta} : \nabla^2 v \quad \forall v \in H^2(\Omega). \tag{2}$$

It is also possible to consider problems with lower-order terms in the operators $L^{\alpha\beta}$. The infimum and supremum in (1) are understood in the pointwise sense over Ω, and it is also possible to consider (1) with the infimum and supremum in reverse order without affecting our results. If either $\mathscr{A}$ or $\mathscr{B}$ is a singleton set, then the Isaacs equation (1) reduces to a HJB equation for the value function of the associated stochastic optimal control problem. Many other nonlinear PDE can also be reformulated as HJB and Isaacs equations, a notable example being the (simple) Monge–Ampère equation, which admits a reformulation as an HJB equation. The importance of these equations therefore stems from both their applications and their connections to other PDE.

We are interested here in *adaptive* methods for Isaacs and HJB equations with Cordes coefficients, based on successive mesh refinements driven by computable error estimators. Building on our recent analysis of *a posteriori* error estimators in [3], we prove in [2] the convergence of a broad family of adaptive discontinuous Galerkin and C^0-interior penalty methods using adaptively refined conforming simplicial meshes in two and three space dimensions, with fixed but arbitrary polynomial degrees greater than or equal to two. Convergence is shown for all choices of penalty parameters that are sufficient for stability of the discrete problems.

Our analysis rests upon a detailed theory for the limit spaces, which are nonstandard function spaces that describe the limiting structure of the finite element spaces under adaptive mesh refinement. A key ingredient of our approach is a novel intrinsic characterization of the limit space in terms of characterizations of the distributional derivatives, which enables us to identify the weak limits of bounded sequences of finite element functions. We provide a detailed theory for the limit spaces, and also some original auxiliary function spaces, that resolves some foundational challenges and that is of independent interest to adaptive nonconforming methods for more general problems. These include Poincaré and trace inequalities, a proof of the density of functions with nonvanishing jumps on only finitely many faces of the limit skeleton, symmetry of the Hessians, approximation results by finite element functions and weak convergence results.

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Auxiliary space preconditioners for C^0 finite element approximation of Hamilton–Jacobi–Bellman equations with Cordes coefficients

Shuonan Wu

In this talk, we first propose the C^0 finite element approximation of the Hamilton–Jacobi–Bellman (HJB) equations with Cordes coefficients. Motivated by the Miranda–Talenti estimate, a discrete analog is proved once the finite element space is C^0 on the $(n-1)$-dimensional subsimplex (face) and C^1 on $(n-2)$-dimensional subsimplex. The main novelty of the non-standard finite element methods is to introduce an interior penalty term to argument the PDE-induced variational form of the linear elliptic equations in the HJB equations. As a consequence, the monotonicity constant the HJB equations at discrete level is exactly the same as that from PDE theory.

In the semi-smooth steps, the linearised systems have large condition numbers, which depend not only on the mesh size, but also on the parameters in the Cordes condition. We then design and analyze the auxiliary space preconditioners for the linearised systems. Based on the stable decomposition on the auxiliary spaces, we propose both the additive and multiplicative preconditoners which converge uniformly in the sense that the resulting condition number is independent of both the number of degrees of freedom and the parameter λ in Cordes condition. Numerical experiments are provided to validate the convergence theory and to illustrate the efficiency of the proposed preconditioners.

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Adaptive discretization of HJB equations with Cordes coefficients

Dietmar Gallistl
(joint work with Endre Süli)

The contribution discusses mixed finite element methods of elliptic problems in nondivergence form with coefficients satisfying the Cordes condition. In case of a second-order linear differential operator $\sum_{j,k=1}^{d} A_{jk}\partial_{jk}^2$ without lower-order terms (for simplicity), the Cordes condition on A reads

$$\sum_{j,k=1}^{d} A_{jk}^2 \Big/ (\operatorname{tr} A)^2 \leq \frac{1}{d-1+\varepsilon}$$

for some $\varepsilon \in (0,1]$. Here, A is a uniformly elliptic L^∞ tensor field over the convex domain $\Omega \subseteq \mathbb{R}^d$. The Cordes condition guarantees that solutions to $\sum_{j,k=1}^{d} A_{jk}\partial_{jk}^2 = f$ with homogeneous Dirichlet boundary conditions and $f \in L^2(\Omega)$ possess a square-integrable Hessian. In this case, mixed discretizations methods [1, 2] provide a simple means of approximationg the solution to the Dirichlet problem with finite elements. The main focus of this contributinon is on partial differential equations with Hamilton–Jacobi–Bellman (HJB) structure: Given a compact metric space Λ, seek $u \in H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$\sup_{\alpha\in\Lambda}(L_\alpha u - f_\alpha) = 0 \quad \text{a.e. in } \Omega.$$

Here, the usual notation on L^2-based Sobolev spaces is employed. The linear differential operator (including lower-order terms) inside the supremum is defined by

$$L_\alpha(v) := a_\alpha : D^2 v + b_\alpha \cdot \nabla v - c_\alpha v.$$

It is shown that, under a Cordes condition, the approximation error in the $H^2(\Omega)$ norm by a mixed finite element method is proportional to (strong) residuals and that an adaptive mesh-refining algorithm steered by these residuals is convergent. Details on these results can be found in [3].

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Semi-Lagrangian schemes for nonlocal HJB-equations and Mean Field Games

Espen R. Jakobsen

(joint work with Fabio Camilli, Indranil Chowhury, Olav Ersland)

Mean Field Games (MFGs) [1] is currently a very active area of research. These games involve a large/infinite number of agents and can be described by a coupled system of nonlinear PDEs: (i) a HJB equation from optimal control describing the decisions of a generic agent, and (ii) a Fokker-Planck equation for the distribution of agents. In the presence of noise, both equations include diffusion terms, and in the presence of long-distance interactions/fat tails, this diffusion is nonlocal and can often be modelled by a Levy jump process. Nonlocal diffusions are sometimes called anomalous diffusions, and are common in e.g. Physics and Finance.

In this talk we consider (i) Semi-Lagrangian (SL) approximations for HJB equations, (ii) how to use these to systematically build (dual) SL-schemes for Fokker-Planck equations, and (iii) SL-schemes for the full MFGs. SL schemes are pointwise finite difference-interpolation schemes that can be defined on general triangulations [7]. For HJB equations they have very good consistency, monotonicity, stability, and convergence properties [5], including degenerate problems with no restriction on the diffusion matrix (like diagonal-dominance). For local MFGs there are SL schemes and results by Carlini and Silva [2]. In the non-local case there are extra challenges due to infinite activity of small jumps/hypersingular integrals. Our analytical results include a number of stability and convergences results for degenerate and nondegenerate problems. We prove subsequence convergence to viscosity/very weak solutions, explain how to obtain full convergenec and convergence to classical solutions, and prove such results under additional assumptions.

The the numerical schemes and analysis are taken form a recent preprint with Chowdhury and Ersland [4] (MFGs) and a paper joint with Camilli [3] (HJB equations). Well-posedness results for nonlocal MFGs from [6] are also used.

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Numerical methods for second-order linear elliptic PDEs in non-divergence form with continuous coefficients

Stefan Schnake

(joint work with Xiaobing Feng, Tom Lewis, Michael Neilan)

This talk focuses on two numerical methods for second-order linear elliptic PDEs in non-divergence form whose coefficients are merely continuous. These PDEs present themselves in the nonlinear Hamilton-Jacobi-Bellman equations, which have applications in stochastic optimal control and mathematical finance, as well as the linearization of the Monge-Ampère equations. Galerkin-type numerical methods for these equations have only been recently developed due to the non-variational structure of the PDE. The first method presented is an interior-penalty discontinuous Galerkin (IP-DG) finite element method. The method is closely related to IP-DG methods for advection-diffusion equations and is easily implementable on existing finite element software. We study the convergence of the method in a discrete $W^{2,p}$-norm; showing the stability of the method using a discrete version of the freezing coefficient technique. Additionally the similarities in this scheme and the standard IP-DG methods for diffusion equations are shown. The second method presented is the vanishing moment method applied to our second order PDE. The vanishing moment method seeks to approximate the second order PDE by a family of fourth order PDEs created by the addition of a small bi-Laplacian term. Since the highest-order term is in divergence form, a trivial C^1 conforming finite element method can been applied. We show the stability and convergence of the method in the H^2-norm. Numerical tests for each method are shown as well.

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Optimal transport and mesh generation on the plane and the sphere

Chris Budd
(joint work with Andrew McRae, Colin Cotter, Tristan Pryer, Simone Appella)

Adaptive mesh generation for the computational solution of PDEs can be done effectively by using optimal transport (OT) methods. This proves to be a robust and reliable way of generating meshes with provable regularity and skewness measures. Finding the best mesh on which to solve a PDE then becomes a problem of solving a fully nonlinear time-evolving system on an appropriate manifold with appropriate boundary conditions. The regularity of the mesh then follows from estimates of the solution of the fully nonlinear problem.

In this talk I will describe the ideas behind using OT methods for mesh generation. These involve solving Monge-Ampére equations (or Monge-Ampére like equations in the case of the sphere) of the general form

$$m(\nabla\phi)H(\phi) = \theta$$

where $m(\nabla\phi)$ is a 'monitor function' related to the solution of the underlying PDE which is constructed to be a measure of the solution error.

I will then show how these OT mesh generation methods are implemented on both the plane and on the sphere by solving the Monge-Ampére or Monge-Ampére like equations by using fast quasi Newton methods. I will give some examples of using these methods to generate meshes to solve the shallow water equations on the sphere and also Poisson's equation in a non convex domain. In the latter case the meshes generated lead to optimal error estimates despite the singularities in the solution of the PDE.

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Finite Element Methods for Bellman operators with nonlinear mixed boundary conditions

Max Jensen

(joint work with Bartosz Jaroszkowski)

The value function of an optimal control problem is the solution of a Hamilton-Jacobi-Bellman (HJB) equation, provided basic conditions are satisfied. In this work we consider the numerical solution of HJB equations with mixed boundary conditions of the form:

$$-\partial_t v + \sup_{\alpha\in A}(L^\alpha \; v - f^\alpha) = 0 \qquad \text{in } [0,T)\times\Omega, \tag{1}$$

$$-\partial_t v + \sup_{\alpha\in A}\big(L^\alpha_{\partial\Omega} v - g^\alpha\big) = 0 \qquad \text{on } [0,T)\times\partial\Omega_t, \tag{2}$$

$$\sup_{\alpha\in A}\big(L^\alpha_{\partial\Omega} v - g^\alpha\big) = 0 \qquad \text{on } [0,T)\times\partial\Omega_R, \tag{3}$$

$$v - g \;= 0 \qquad \text{on } [0,T)\times\partial\Omega_D, \tag{4}$$

$$v - v_T \;= 0 \qquad \text{on } \{T\}\times\overline{\Omega}. \tag{5}$$

Here L^α and $L^\alpha_{\partial\Omega}$ denote operators on the domain Ω and its boundary, respectively. The sets $\partial\Omega_t$, $\partial\Omega_R$ and $\partial\Omega_D$ form a decomposition of $\partial\Omega$. While leaving the precise details of the notation to [2], it is already apparent how the basic fully nonlinear structure of the PDE operator, meaning the left-hand side of (1), is mirrored in the Robin-type boundary conditions (2) and (3). But there is a crucial, additional complication of the boundary operators $L^\alpha_{\partial\Omega}$: They will in general depend on the full gradient ∇v and not just on the tangential gradient $\nabla_{\partial\Omega} v$, meaning that $L^\alpha_{\partial\Omega} v$ cannot be evaluated with knowledge of $v|_{\partial\Omega}$ only.

For the authors the problem of primary interest is the Heston model of financial interest rates with an uncertain market price of volatility risk [3]. The Heston equation is most naturally posed on an unbounded domain, where already with a certain market price of volatility risk it appears with mixed boundary terms corresponding to (2), (3) as well as (4). All those types of boundary conditions remain when introducing uncertainty and when truncating the domain for the purposes of numerical approximation.

The aim of this work is to introduce a finite element method capable of computing approximations to viscosity solutions for the aforementioned problems. This paper extends results of [4] with the inclusion of mixed, fully nonlinear boundary conditions. The presented method permits degenerate diffusions. Boundary operators may exhibit discontinuities across face boundaries and where the type of boundary condition changes.

A challenge for problems of this type is the discretisation of the first-order directional derivatives in (2) and (3) which needs to be simultaneously consistent and monotone. On the one hand establishing monotonicity with an artificial diffusion approximating the Laplace-Beltrami operator of $\partial\Omega$ would not be sufficient because of the normal component in the directional derivatives of (2) and (3).

On the other hand an artificial diffusion approximating the Laplace operator of Ω would not vanish under refinement due to different scaling of boundary and domain terms, thus leading to an inconsistent method. Our formulation is based on the observation that lower Dini directional derivatives exist for all functions in the P1 approximation space whenever the direction in question is in the tangent cone of Ω at the position of interest.

Ultimately, we show uniform convergence of monotone P1 finite element methods to the viscosity solution of isotropic parabolic Hamilton-Jacobi-Bellman equations with mixed boundary conditions on unstructured meshes and for possibly degenerate diffusions. In time the Bellman equation is approximated through IMEX schemes. Existence and uniqueness of numerical solutions follows through Howard's algorithm. We also discuss extension to Isaacs equations, in particular Dirichlet conditions, which face additional difficulties in the game-theoretical setting [1].

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The second boundary value problem for a discrete Monge-Ampère equation

Gerard Awanou

In this work we propose a natural discretization of the second boundary condition for the Monge-Ampère equation of geometric optics and optimal transport. It is the natural generalization of the popular Oliker-Prussner method proposed in 1988. For the discretization of the differential operator, we use a discrete analogue of the subdifferential. Existence, unicity and stability of the solutions to the discrete problem are established. Convergence results to the continuous problem are given.

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A convexity enforcing finite element method for the Monge-Ampère equation in two dimensions

Li-yeng Sung
(joint work with Susanne C. Brenner, Zhiyu Tan, Hongchao Zhang)

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain. We show that the Dirichlet boundary value problem

$$\det D^2 u = \psi \quad \text{in } \Omega \quad \text{and} \quad u = \phi \quad \text{on } \partial\Omega$$

for the Monge-Ampère equation can be solved by a nonlinear least-squares method with box constraints. It is based on an enhanced cubic Lagrange element that can enforce the elementwise convexity of the approximate solutions.

The enhanced cubic Lagrange element is affine equivalent. On the reference simplex $\hat{T}$, the space of shape functions is the direct sum of $P_3(\hat{T})$ and the three dimensional bubble function space $[\hat{x}_1\hat{x}_2(1-\hat{x}_1-\hat{x}_2)]^3 P_1(\hat{T})$. The dofs of a shape function v consist of the 10 dofs from the cubic Lagrange element and Δv at $(1,1)$, $(-1,1)$ and $(1,-1)$. The idea is that v is strictly convex if and only if $\det D^2 v > 0$ and $\Delta v \geq 0$. Since the positivity of $D^2 v$ can be enforced through the Monge-Ampère equation, it suffices to enforce the nonnegativity of Δv, and the three additional dofs can be used for this purpose.

Under the assumptions that (i) $\psi \in H^2(\Omega)$ is strictly positive on $\bar{\Omega}$, (ii) $\phi \in H^4(\Omega)$ and (iii) $u \in H^4(\Omega)$ is strictly convex, we prove that an approximate solution u_h obtained by the least-squares method satisfies the following error estimate:

$$\Big(\|D_h^2(u-u_h)\|_{L_2(\Omega)}^2 + \sum_{e\in\mathcal{E}_h^i} |e|^{-1}\|[\![\partial(u-u_h)/\partial n]\!]\|_{L_2(e)}^2\Big)^{\frac{1}{2}} \leq Ch^2$$

where D_h^2 is the piecewise Hessian, $\mathcal{E}_h^i$ is the set of the edges of the simplicial triangulation $\mathcal{T}_h$, $|e|$ is the length of the edge e, and $[\![\cdot]\!]$ is the jump across the interior edges.

We also show that

$$\eta_h(u_h) = \|\det D_h^2 u_h - \psi\|_{L_2(\Omega)} + \Big(\sum_{e\in\mathcal{E}_h^i} |e|^{-1}\|[\![\partial u_h/\partial n]\!]\|_{L_2(e)}^2\Big)^{\frac{1}{2}}$$

is a reliable and locally efficient error estimator.

The key ingredients of the error analysis are (i) the *a priori* bounds that follow from the least-squares formulation, (ii) the elementwise convexity of u_h, and (iii) a stability result for discontinuous Galerkin finite element methods for elliptic problems in nondivergence form (cf. [3, 2]).

Details can be found in [1].

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Numerical Methods with Convergence Rates for the Monge-Ampère Equation

Ricardo H. Nochetto

(joint work with W. Li, D. Ntogkas, W. Zhang)

We analyze the Oliker-Prussner method [2] and a two-scale method [3, 4, 5] for the Monge-Ampère equation with Dirichlet boundary condition, and explore connections with a Bellman formulation [5]. We derive pointwise error estimates that rely on the discrete Alexandroff maximum principle [1] and the geometric structure of the Monge-Ampère equation for both classical and non-classical solutions. The two-scale method and analysis extend to the approximation of convex envelopes [6]. We conclude with a two-scale method, via an integro-differential formulation, for linear elliptic PDEs in non-divergence form with continuous coefficients [1].

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Asymptotic analysis of domain decomposition for optimal transport

Bernhard Schmitzer
(joint work with Mauro Bonafini, Ismael Medina)

Let $X = [0,1]^d$, $Y \subset \mathbb{R}^d$ compact, $c \in C^1(X \times Y)$ a differentiable cost function, μ and ν probability measures on X and Y and let $\varepsilon \geq 0$ be an entropic regularization parameter. We are concerned with solving the (optionally) entropy regularized optimal transport problem

$$\inf \left\{ \int_{X\times Y} c \, \mathrm{d}\pi + \varepsilon \cdot \mathrm{KL}(\pi|\mu \otimes \nu) \middle| \pi \in \Pi(\mu,\nu) \right\}$$

where $\Pi(\mu,\nu)$ denotes set of transport plans and KL is the Kullback–Leibler divergence.

Benamou proposed [1] a domain decomposition algorithm which is amenable for large-scale parallelization. The idea is to split the domain X into two sets of 'staggered' partitions J_A and J_B. Starting from an initial feasible coupling $\pi^{(0)} \in \Pi(\mu,\nu)$, for a cell $X_J \in J_A$ one optimizes over $\pi^{(0)}$ on $X_J \times Y$ while keeping it fixed on $(X \setminus X_J) \times Y$. This is done for every cell of J_A, and in fact this can be done in parallel for all cells. Then one repeats the procedure on the cells of J_B and iterates this until convergence. Benamou showed that it solves the problem for $c(x,y) = \|x-y\|^2$, $\varepsilon = 0$ under suitable assumptions on J_A, J_B and μ. More recently the feasibility of the method was shown [2] for the case $\varepsilon > 0$ for arbitrary bounded costs c. Empirically, on problems with sufficient geometric structure (such as the Wasserstein-2 problem on convex domains), fast convergence is observed. To obtain a theoretical understanding in [3] the dynamics of the algorithm in the limit of increasingly fine cells are studied. The main result can roughly be summarized as follows.

At discretization level n, let J_A and J_B be given by two staggered Cartesian grids with edge length $2/n$ and one iteration corresponds to a time step of $\Delta t = 1/n$. The Γ-limit of the local π-updates at time t and location $x \in X$ becomes

$$\inf \left\{ \int_{Z\times Y} \langle \nabla_X c(x,y), z \rangle \, \mathrm{d}\lambda(z,y) + \eta \cdot \mathrm{KL}(\lambda|\sigma \otimes \pi_{t,x}) \middle| \lambda \in \Pi(\sigma, \pi_{t,x}) \right\}$$

where $Z = [-1,1]^d$ is a rescaled asymptotic copy of the individual partition cells, σ is a locally rescaled version of μ, $\pi_{t,x}$ is the (t,x)-disintegration of the limit of algorithm trajectories and η is a suitable limit of the regularization parameter ε (which will generally depend on n). Note that in this limit the cost function c is locally replaced by a linear expansion in X. From the solutions λ of the above problem a velocity field $v \in L^1([0,T] \times X \times Y, X, \pi)$ can be extracted which describes the evolution of the trajectory π via

$$\partial_t \pi_t + \mathrm{div}_X(\pi_t \cdot v_t) = 0.$$

In a nutshell this means that the limit of the trajectories of the domain decomposition algorithm is described by a limit version of the algorithm via a continuity equation. The required number of iterations will therefore (under suitable conditions) be proportional to n. Heuristically this also explains the good performance of the algorithm in combination with coarse-to-fine schemes as observed in [2].

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Narrow-stencil finite difference methods for approximating fully nonlinear elliptic PDEs

Thomas Lewis

A new narrow-stencil finite difference method for approximating viscosity solutions of fully nonlinear elliptic partial differential equations will be presented. The finite difference method naturally extends the Lax-Friedrichs method for first order problems to second order problems by introducing a key stabilization term called a numerical moment that is based on the difference of two discrete Hessian operators. We will discuss novel techniques for proving the admissibility, stability, and convergence of the proposed narrow-stencil method that were developed to overcome the challenges associated with analyzing a non-monotone method. We will also discuss a related convergent method for approximating stationary Hamilton-Jacobi equations that utilizes a numerical moment to formally achieve a second-order local truncation error. This talk is based on the joint work with Xiaobing Feng in [1].

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First-Order System Least-Squares Finite Element Methods for Second Order Elliptic Equations in Non-Divergence Form

Weifeng Qiu
(joint work with Shun Zhang)

We studies adaptive first-order system least-squares finite element methods for second-order elliptic partial differential equations in non-divergence form. The first-order least-squares formulations naturally have stable weak forms without using integration by parts, allow simple finite element approximation spaces, and have build-in a posteriori error estimators for adaptive mesh refinements.

The non-divergence equation is first written as a system of first-order equations by introducing the gradient as a new variable. Then least-squares finite element methods using simple C^0 finite elements are developed in the paper. Under a very mild assumption that the PDE has a unique solution, optimal a priori and a posteriori error estimates are proved. With an extra assumption on the operator regularity which is weaker than traditionally assumed, convergences in standard norms are also discussed. L^2-error estimates are derived.

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Monotone finite difference discretization of the Monge-Ampère equation of optimal transport

Guillaume Bonnet
(joint work with J. Frédéric Bonnans, Jean-Marie Mirebeau)

We introduce [2] a monotone finite difference scheme for the second boundary value problem for the Monge-Ampère equation:

$$\begin{cases} \det D^2u(x) = f(x)/g(Du(x)) & \text{in } X, \\ D^2u(x) \succeq 0 & \text{in } X, \\ \overline{Du(X)} = \overline{Y}. \end{cases}$$

One difficulty when discretizing this system is that it is not well-posed in the usual sense since its set of solutions is stable by addition of a constant; this causes some discretizations to admit no solutions. The proposed scheme, for which both the existence of solutions and the convergence of those solutions is proved, is a discretization of a reformulation of the above system in the form

$$\max\{F(x, Du(x), D^2u(x)) + \alpha,\, H(Du(x))\} = 0 \quad \text{in } X,$$

where $F(x,p,M) := \max_{\mathcal{D}\succeq 0,\, \mathrm{Tr}(\mathcal{D})=1}(2(f(x)/g(p)\det \mathcal{D})^{1/2} - \mathrm{Tr}(\mathcal{D}M))$ is a reformulation of the Monge-Ampère operator previously used in [3], and H is a defining function of the set Y. Taking the maximum between the reformulation of the Monge-Ampère operator and the reformulation of the boundary condition is justified in [4]; the real number α is an additional unknown that we add to the equation in order to guarantee existence of solutions to its finite difference discretization, following the numerical experiments in [1].

We need to discretize in a monotone way some second-order terms of the form $\mathrm{Tr}(\mathcal{D}D^2u(x))$. To this end, we use Selling's formula, a tool originating from the geometry of low-dimensional lattices. We prove that, up to an approximation of the parameter set, the maximum in the discrete counterpart to the operator F admits a closed-form formula, improving the efficiency of the resulting finite difference scheme.

The same method is used to discretize the more general Monge-Ampère equation

$$\det(D^2u(x) - A(x, Du(x))) = B(x, Du(x)) \quad \text{in } X.$$

In this setting, we prove the existence of solutions to the numerical scheme, but the convergence of those solutions remains an open problem. We present a numerical application to the far field refractor problem in nonimaging optics.

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Accelerating solvers for nonlinear PDEs

Sara Pollock

Anderson acceleration (AA) is an extrapolation technique used to accelerate the convergence of fixed-point iterations. It requires the storage of a (usually) small number of solution and update vectors, and the solution of an optimization problem that is generally posed as least-squares and solved efficiently by a thin QR decomposition. First developed in 1965 [1] in the context of integral equations, it has recently been increasing in popularity as a Jacobian-free approach to converging to discrete nonlinear PDE solutions, not to mention applications in optimization and machine learning. The convergence behavior of AA depends on the selection of parameters including the algorithmic depth and the damping (or mixing) factor.

Here we review recent theory from [4], which shows a one-step residual bound. This bound holds under the conditions that the operator is sufficiently smooth and is either contractive, as in [3, 5], or that it satisfies a Newton-like nondegeneracy condition. We then explore tuning the algorithm through a process of dynamic parameter updates and a novel filtering strategy suggested by the new theory.

To give some specifics, suppose the underlying fixed-point iteration $x_{k+1} = \phi(x_k)$ is written in the form $x_{k+1} = x_k + w_{k+1}$, where w_{k+1} is the update step or nonlinear residual. The depth-m Anderson-accelerated update has the form $x_{k+1} = x_k + \beta_k w_{k+1} - (E_k + \beta_k F_k)\gamma_k$, where β_k is a relaxation factor and γ_k solves the optimization problem $\min_{\gamma \in \mathbb{R}^m} \|w_{k+1} - F_k\gamma_k\|$. The least-squares solution can be written in terms of the factors $F_k = QR$. The filtering strategy works by computing the m constants $c_i = |r_{ii}|/\|f_i\|$, with r_{ii} a diagonal value of R and f_i the corresponding column of matrix F_k. The columns of both E_k and F_k for which c_i is small are filtered out at each iteration. This strategy can been viewed as pragmatically controlling the magnitude of the optimization coefficents.

Numerical examples include monotone iterations for quasilinear PDE from [2], as well as the p-Laplace equation in both degenerate and singular regimes. In agreement with th theory, the filtering and dynamic depth strategies improve the robustness and the efficiency of the iteration.

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Modeling and Simulation of Thin Sheet Folding

Philipp Tscherner

(joint work with Sören Bartels, Peter Hornung, Andrea Bonito)

We derive a model for thin sheet folding along possibly curved arcs $\mathcal{C}$ by incorporating a material weakening mechanism into general 3D bending theory. To describe our approach let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $\Omega_\gamma = \Omega \times (-\gamma/2, \gamma/2)$ denote a sheet of thickness γ. For a deformation $y_\gamma : \Omega_\gamma \to \mathbb{R}^3$ we consider the elastic energy

$$E_\gamma[y_\gamma] = \int_{\Omega_\gamma} f_{\varepsilon,r}(x)\, W(\nabla y_\gamma)\, \mathrm{dx}\,,$$

where W is some stored energy functional and $f_{\varepsilon,r}$ coincides with ε in the folding area $B_r(\mathcal{C})$ and is 1 otherwise. The results of Duncan & Duncan [5] motivate the existence of non-trivial solutions. By properly relating the sheet thickness γ, the fold with r and the material weakening parameter ε, an adaptation of the well-known Γ-convergence results of Friesecke, James & Müller [1] allows for a justification of a 2D model as the thickness of the sheet goes to zero, see also [7, 8]. The limit model is subject to weak continuity $y \in H^1(\Omega;\mathbb{R}^3) \cap H^2(\Omega \setminus \mathcal{C};\mathbb{R}^3)$ across the folding curve and the isometry constraint $(\nabla y)^\top \nabla y = I_{2\times 2}$ in $\Omega \setminus \mathcal{C}$. We refer to [3] for existing results on the numerical approximation of the isometry condition. The resulting problem can be discretized using the discontinuous Galerkin method [6]. By dropping gradient jumps of the deformation along the folding curve the dG formulation naturally allows for configurations with folds.

In the linear case, where only small deflections in vertical direction are considered, the isometry constraint can be dropped. Assuming that the folding curve is accurately resolved by an isoparametric mesh, an error estimate of order $k-1$ can be shown for piecewise polynomial approximations of order $k \geq 2$.

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Pointwise error estimates of two-scale methods for the convex envelope problem and normalized infinity Laplace equation

Wenbo Li

(joint work with Ricardo H. Nochetto, Abner J. Salgado)

We consider two different nonlinear problems in this talk: the convex envelope problem and the normalized infinity Laplace equation.

The convex envelope u of $f \in C(\overline{\Omega})$ is the viscosity solution of the fully nonlinear elliptic PDE $\min\{f - u, \lambda_1[D^2u]\} = 0$ where $\lambda_1[D^2u]$ denotes the smallest eigenvalue of D^2u. Motivated by the wide stencil method by Oberman, we construct a monotone scheme with two different scales δ, h [1]. The main difficulty in deriving pointwise error estimates is that the solution u in general does not have regularity higher than $C^{1,1}(\overline{\Omega})$, which usually implies an $O(1)$ consistency error for second-order operators. In [1], we overcome this difficulty through utilizing geometric structures of the problem and prove pointwise error estimate for discrete solution $u_{\delta,h}$:

$$\|u - u_{\delta,h}\|_{L^\infty(\Omega)} \lesssim \left(\frac{h^2}{\delta^2} + \delta^2\right) |u|_{C^{1,1}(\overline{\Omega})} + \delta^2 |f|_{C^{1,1}(\overline{\Omega})}.$$

Choosing $\delta = O(h^{1/2})$ leads to a linear decay rate $O(h)$.

Given $f \in C(\overline{\Omega})$, the normalized infinity Laplace equation $-\frac{(Du)^T}{|Du|} D^2 u \frac{Du}{|Du|} = f$ in Ω with Dirichlet boundary condition $u = g$ on $\partial\Omega$ admits a unique solution u under the assumption $f \equiv 0$ or $\min f > 0$ or $\max f < 0$. Inspired by the tug-of-war game closely related to the problem and the wide stencil method by Oberman, we consider the following monotone two-scale method

$$\frac{1}{\delta^2}\left(2u_{\delta,h}(x_i) - \sup_{\mathcal{N}_h \cap \overline{B}(x_i,\delta)} u_{\delta,h} - \inf_{\mathcal{N}_h \cap \overline{B}(x_i,\delta)} u_{\delta,h}\right) = f(x_i),\ x_i \in \mathcal{N}_h^o \quad \text{(interior nodes)}.$$

The lower regularity of the continuous solution u again brings difficulties in the pointwise error estimates. By using the structures of the problems, we derive a better control of the consistency error and use discrete comparison principle and barrier functions to prove convergence rates $O(h^{1/3})$ if $\min f > 0$ or $\max f < 0$, and $O(h^{1/4})$ if $f \equiv 0$.

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Reporter: Philipp Tscherner

Participants

Prof. Dr. Gerard Awanou
Department of Mathematics
University of Illinois at Chicago
Chicago, IL 60607
UNITED STATES

Prof. Dr. Sören Bartels
Abteilung für Angewandte Mathematik
Universität Freiburg
Hermann-Herder-Straße 10
79104 Freiburg i. Br.
GERMANY

Guillaume Bonnet
Département de Mathematiques
Faculté des Sciences d'Orsay
Université Paris-Saclay
Batiment 307
91405 Orsay Cedex
FRANCE

Prof. Dr. Susanne C. Brenner
Department of Mathematics
Louisiana State University
303 Lockett Hall
Baton Rouge LA 70803-4918
UNITED STATES

Prof. Dr. Chris J. Budd
Dept. of Mathematical Sciences
University of Bath
Claverton Down
Bath BA2 7AY
UNITED KINGDOM

Prof. Dr. Xiaobing H. Feng
Department of Mathematics
University of Tennessee
218 Ayres Hall
Knoxville TN 37996-1300
UNITED STATES

Prof. Dr. Dietmar Gallistl
Institut für Mathematik
Universität Jena
Ernst-Abbe-Platz 2
07743 Jena
GERMANY

Prof. Dr. Espen Jakobsen
Department of Mathematical Sciences
NTNU
7491 Trondheim
NORWAY

Dr. Max Jensen
Department of Mathematical Sciences
University of Sussex
Falmer
Brighton BN1 9QH
UNITED KINGDOM

Dr. Tom Lewis
Department of Mathematics & Statistics
University of North Carolina at
Greensboro
116 Petty Building
Greensboro, NC 27402-6170
UNITED STATES

Dr. Wenbo Li
Department of Mathematics
University of Tennessee
Knoxville TN 37996
UNITED STATES

Dr. Jean-Marie Mirebeau
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 425
91405 Orsay Cedex
FRANCE

Prof. Dr. Michael J. Neilan
Department of Mathematics
University of Pittsburgh
139 University Place
Pittsburgh PA 15260
UNITED STATES

Prof. Dr. Ricardo H. Nochetto
Department of Mathematics
Institute for Physical Science and Technology
University of Maryland
College Park MD 20742-2431
UNITED STATES

Prof. Dr. Sara Pollock
Department of Mathematics
University of Florida
358 Little Hall
P.O. Box P.O.Box 118105
Gainesville, FL 32611-8105
UNITED STATES

Dr. Weifeng Qiu
Department of Mathematics
City University of Hong Kong
Office: Y6627 Academic 1
83 Tat Chee Avenue, Kowloon
999077 Hong Kong
CHINA

Prof. Dr. Abner J. Salgado
Department of Mathematics
University of Tennessee
Knoxville 37996
UNITED STATES

Prof. Dr. Bernhard Schmitzer
Institut für Informatik
Universität Göttingen
Goldschmidtstraße 7
37077 Göttingen
GERMANY

Dr. Stefan Schnake
Oak Ridge National Laboratory
Building 920 A
P. O. Box Y
Oak Ridge, TN 37831
UNITED STATES

Dr. Iain Smears
Department of Mathematics
University College London
Gower Street
London WC1H 0AY
UNITED KINGDOM

Prof. Dr. Endre Süli
Mathematical Institute
University of Oxford
Radcliffe Observatory Quarter
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Prof. Dr. Li-yeng Sung
Department of Mathematics
Louisiana State University
208 Lockett Hall
Baton Rouge LA 70803-4918
UNITED STATES

Dr. Zhiyu Tan
Center for Computation and Technology
Louisiana State University
Baton Rouge LA 70803-4918
UNITED STATES

Philipp Tscherner
Mathematisches Institut
Universität Freiburg
Ernst-Zermelo-Straße 1
79104 Freiburg i. Br.
GERMANY

Dr. Shuonan Wu
Room 1494E, Sciences Building No. 1
Department of Mathematics
Peking University
No.5 Yiheyuan Road, Haidian District
Beijing 100 871
CHINA

Dr. Yunan Yang
Courant Institute of Mathematical Sciences
New York University
251, Mercer Street
New York NY 10012-1110
UNITED STATES

Dr. Wujun Zhang
Department of Mathematics
Rutgers University
Busch Campus, Hill Center
New Brunswick, NJ 08854-8019
UNITED STATES