MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

MATRIX-MFO Tandem Workshop/Small Collaboration: Rough Wave Equations (hybrid meeting)

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ABSTRACT. The consideration of wave propagation in inhomogeneous media or the modelling of nonlinear waves often requires the study of wave equations with low regularity data and/or coefficients. Several Australian-European collaborations have recently led to deeper analytical understanding of rough wave equations. This tandem workshop provided a platform for such collaborations and brought together early career researchers and leading experts in harmonic analysis, microlocal analysis and spectral theory. The workshop focused on collaboration and technical knowledge exchange on topics such as local smoothing, spectral multipliers, restriction estimates, Hardy spaces for Fourier integral operators, and nonlinear partial differential equations.

Introduction by the Organizers

Interactions between harmonic analysis, microlocal analysis and spectral theory have led to exciting advances on topics such as rough wave equations, oscillatory integral estimates, local smoothing, and spectral multipliers. Here the additional flexibility and insight that comes with taking a microlocal perspective on classical problems allows one to incorporate powerful tools from phase space analysis.

In particular, such tools include invariant spaces for wave and Schrödinger propagators, Hamiltonian flows on phase space and Egorov theorems, (multilinear)

Fourier integral operators, semiclassical analysis, and refined conditions on oscillatory integrals that allow for optimal restriction and extension estimates. On the other hand, classical tools from probability theory have led to a deeper understanding of the quantum separation effect, and adapted Hardy and bmo spaces continue to be very useful for harmonic analysis on non-doubling manifolds with ends, and Heisenberg groups. Moreover, results in harmonic and microlocal analysis have been used to obtain new spectral multiplier theorems, both in concrete and in abstract settings, and they have found applications to various classes of nonlinear partial differential equations. Most relevant for this workshop are the nonlinear wave, Schrödinger, Dirac and Maxwell equations.

The workshop brought together 23 early career researchers and leading experts working on these topics, 8 of which were physically present at MFO and 15 of which participated virtually from Australia and New Zealand. This hybrid format led to new interactions between different areas, both geographically and scientifically. In particular, due to inherent obstacles coming from the distance between Europe and Oceania, as well as due to border closures, the MATRIX-MFO collaboration made for a workshop that would not have been conceivable in another manner. It also allowed for existing Australian-European collaborations to continue in a 'research in pairs' format.

On an organizational level, the time difference between the continents led to a novel approach, whereby the participants were asked to pre-record their talks and make them available online. A number of talks were then discussed every day during the European morning and the late afternoon and evening in Oceania, around which thematic sessions were organized that involved lively discussions. Apart from this, time was reserved for collaboration in smaller groups, both in person and virtually.

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Abstracts

Intersections between microlocal, semiclassical and harmonic analysis
Melissa Tacy

In this talk I will briefly survey the intersection between microlocal, semiclassical and harmonic analysis and discuss some of the types of problems that benefit from being viewed as sitting in this intersection.

Microlocal analysis broadly speaking is concerned with the analysis of the local structure of singularities. In particular microlocal analysis interests itself not only in the local properties of singularities (such as where their singular support sits) but also in the directions in which the singularity is 'felt'. Adding the directional information makes microlocal analysis the perfect setting to discuss solutions to evolution equations and associated PDE.

Semiclassical analysis offers a refinement of microlocal analysis by also addressing scale. In semiclassical analysis one typically considers scale dependent functions (rather than singularities). However as the semiclassical scale $h \to 0$ these functions become singular. Therefore semiclassical analysis is an excellent setting to study singularity formation but it also provides the correct setting to study fine scale properties of solutions to PDE.

For example consider studying the L^p norms of solutions u to a PDE (or pseudo-differential equation). Different values of p 'see' different features. A common feature is a local point-like concentration where $u(x)$ is particularly large at one value of x. Such a feature may be invisible to L^2 theory. For instance if $|u(x)| \approx h^{-\frac{n-1}{2}}$ on a h^n region (this feature is for example found on zonal spherical harmonics). The contribution of this feature to the L^p norm of u is $h^{-\frac{n-1}{2}} h^{\frac{n}{p}}$ so in the $h \to 0$ limit this feature is invisible unless $p \geq \frac{2n}{n-1}$. Since semiclassical analysis naturally formulates everything in terms of the semiclassical parameter it provides a suitable framework for studying these type of problems.

The key ingredients to placing a problem in the semiclassical setting are the semiclassical Fourier transform $\mathcal{F}_h$ and semiclassical pseudodifferential operators $p(x, hD)$. The semiclassical Fourier transform is given by

$$\mathcal{F}_h[u] = \frac{1}{(2\pi h)^{n/2}} \int e^{-\frac{i}{h}\langle x, \xi \rangle} u(x) dx$$

with inverse

$$\mathcal{F}_h^{-1}[u] = \frac{1}{(2\pi h)^{n/2}} \int e^{\frac{i}{h}\langle x, \xi \rangle} u(\xi) d\xi.$$

The choice of $(2\pi h)^{-n/2}$ as a pre-factor ensures that the semiclassical Fourier transform is an isometry on $L^2(\mathbb{R}^n)$. Like the standard Fourier transform differentiation of u corresponds to multiplication of $\mathcal{F}_h[u]$, however the scale factor h is also built it to this relationship. We have that

$$\mathcal{F}_h\left[hD_{x_j}u\right] = \xi_j \mathcal{F}_h[u].$$

The definition of a semiclassical pseudodifferential operator is also similar to that of a standard pseudodifferential operator. The operator $p(x, hD)$ associated with $p(x, \xi)$ by the left quantisation is

$$p(x, hD)u = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h} \langle x-y, \xi \rangle} p(x, \xi) u(y) d\xi dy.$$

The symbol $p(x, \xi)$ must obey the same type of 'polynomial like' bounds as for a standard pseudodifferential operator but in practice the symbol $p(x, \xi)$ can often be taken to be compactly supported. For full details on the semiclassical calculus see for example [16]. Typically we study u that are approximate solutions, quasimodes, to a pseudodifferential equation $p(x, hD)u = 0$. In particular we are often interested in order h quasimodes. These functions have the property that

$$\|p(x, hD)u\|_{L^2} \lesssim h \|u\|_{L^2}.$$

Order h quasimodes are particularly convenient for local analysis as localising near a point x (or localising their semiclassical Fourier transform near a point ξ) preserves the order h quasimode property.

The ability to study fine scale properties via semiclassical analysis is exactly what makes this theory so useful in problems arising form harmonic analysis. Many problems in harmonic analysis are concerned with L^p (or other function space norms) of solutions to PDE or spectral clusters associated with differential operators, those associated with the Laplacian often forming the canonical examples. Questions about spectral clusters can easily be converted to questions about quasimodes. Here we give an example of the conversion when the underlying operator is the Laplacian on a compact manifold. We denote the eigenfunctions by ϕ_j with eigenvalue λ_j^2. Then a spectral cluster of window width W is given by

$$u_\lambda = \sum_{\lambda_j \in [\lambda, \lambda+W]} c_j \phi_j$$

and the associated cluster projection operator by

$$P_\lambda = \sum_{\lambda_j \in [\lambda, \lambda+W]} P_{\lambda_j}$$

where P_{λ_j} is the projection onto the λ_j eigenspace. Now let $h = \lambda^{-1}$ then

$$\|(h^2 \Delta - 1)u_\lambda\|_{L^2} = \left\| \sum_{\lambda_j \in [\lambda, \lambda+W]} (h\lambda_j + 1)(h\lambda_j - 1)\phi_j \right\|_{L^2}$$
$$\lesssim hW \|u_\lambda\|_{L^2}.$$

So for instance width one window spectral clusters are in fact order h quasimodes. A similar calculation show that for any v, $P_\lambda v$ is a quasimode with error hW. This conversion allows us to see harmonic analysis problems about operators as semiclassical problems about quasimodes. We can then make use of the considerable technical machinery of semiclassical analysis to examine these functions

and therefore make conclusion about the mapping properties of the associated operators.

Below is an (incomplete) list of harmonic analysis type problems successfully treated through semiclassical techniques.

- Linear or bilinear L^p estimates of order h quasimode (or joint quasimode) of semiclassical pseudodifferential operators $p(x, hD)$; [15] [9],[6],[12],[13].
- Estimates on the L^p norm of an order h quasimode of a semiclassical pseudodifferential operator $p(x, hD)$ restricted to lower dimensional submanifold; [10],[8].
- Control of the L^2 norm of weighted Cauchy or Neumann data on interior hypersurfaces or on the boundary; [14], [3], [2], [1], [11].
- Control of the mapping norms of single and double layer operators; [7],[5], [4].

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Spectral multipliers and wave equation for sub-Laplacians
Alessio Martini

Let $\mathcal{L}$ be a sub-Laplacian on a sub-Riemannian manifold of dimension n. It has been known for a long time that, under fairly general assumptions on the sub-Laplacian and the underlying sub-Riemannian structure, an operator of the form $F(\mathcal{L})$ is bounded on L^p $(1 < p < \infty)$ whenever the multiplier F satisfies a scale-invariant smoothness condition of sufficiently larger order. For example, if $\mathcal{L}$ satisfies Gaussian-type heat kernel bounds and the underlying metric-measure space is doubling, then a Mihlin–Hörmander type multiplier theorem holds for $\mathcal{L}$, where the required order of differentiability depends on the "homogeneous dimension" of the underlying geometry, which may be larger than the topological dimension n. The problem of determining the minimal smoothness assumptions, however, remains widely open in general, and is intimately connected with that of proving sharp L^p estimates for the corresponding wave equation. The talk focuses on two instances of recent progress on this problem.

In joint work with Detlef Müller and Sebastiano Nicolussi Golo [2], we prove universal necessary conditions for spectral multiplier and wave equation estimates for sub-Laplacians, only depending on the topological dimension of the underlying manifold. Namely, we show that the ranges of validity of spectral multiplier estimates of Mihlin–Hörmander type and wave propagator estimates of Miyachi–Peral type for $\mathcal{L}$ cannot be wider than the corresponding ranges for the Laplace operator on $\mathbb{R}^n$ — despite the lack of ellipticity of $\mathcal{L}$. A similar result was previously known for 2-step structures [1], but the method of proof that we develop in the new work is substantially different and allows us to tackle the case of arbitrary step. Specifically, the proof hinges on the construction of a suitable Fourier integral representation for the wave propagator associated with $\mathcal{L}$, which allows us to exploit known nondegeneracy properties of the sub-Riemannian geodesic flow.

On the side of sufficient conditions, in joint work with Gian Maria Dall'Ara [3], we consider the case of Grushin operators $\mathcal{L} = -\partial_x^2 - V(x)\partial_y^2$ on the plane $\mathbb{R}^2 = \mathbb{R}_x \times \mathbb{R}_y$. Under the sole assumptions that $V(-x) \simeq V(x) \simeq xV'(x)$ and $|x^2 V''(x)| \lesssim V(x)$ (the assumption on the second derivative V'' can actually be weakened to a Hölder-type condition on V'), we prove a spectral multiplier theorem of Mihlin–Hörmander type for $\mathcal{L}$, whose smoothness requirement is optimal and independent of V, and coincides with that for the classical Laplacian on $\mathbb{R}^2$. The independence from V of the smoothness requirement in our multiplier theorem is particularly striking, when compared with the classical result obtained via heat kernel bounds, as the homogeneous dimension associated to $\mathcal{L}$ depends on the degree of polynomial growth of V and can be arbitrarily large. The proof is fundamentally based on the spectral analysis of one-dimensional Schrödinger operators with single-well potentials, including universal estimates of eigenvalue gaps and matrix coefficients of the potential.

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Boundedness of multilinear oscillatory integral operators in local Hardy spaces

DAVID RULE

(joint work with Aksel Bergfeldt, Salvador Rodriguez-Lopez, Wolfgang Staubach)

In this talk we discuss the global boundedness of multilinear oscillatory integral operators that are connected to nonlinear dispersive equations. These are operators that take the form

$$
(1) \qquad T_\sigma^\Phi(f_1,\dots,f_N)(x) = \int_{\mathbf{R}^{nN}} \sigma(x,\Xi) \prod_{j=1}^N \left(\widehat{f_j}(\xi_j) e^{ix\cdot\xi_j} \right) e^{i\Phi(\Xi)}\, d\Xi,
$$

where σ is a Hörmander class multilinear amplitude, satisfying the bounds

$$
|\partial_\Xi^\alpha \partial_x^\beta \sigma(x,\Xi)| \le C_{\alpha,\beta} \langle\Xi\rangle^{m-|\alpha|}
$$

for all multi-indices α and β, and

$$
\Phi(\Xi) = \varphi_0(\xi_1 + \dots + \xi_N) + \sum_{j=1}^N \varphi_j(\xi_j) \quad (\Xi = (\xi_1,\dots,\xi_N))
$$

is a combination of *phase functions* φ_j $(j = 0, 1, \dots, N)$. We study Φ which take this form as it is this structure that appears in applications, as described in [1]. The assumptions we make on each φ_j depend on the PDE from which the operator is derived, and is quantified by its *order* s: For a given $s \in (0,\infty)$ we assume each $\varphi_j\colon \mathbf{R}^n \to \mathbf{R}$ belongs to $\mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$ and satisfies

$$
(2) \qquad |\partial^\alpha \varphi_j(\xi)| \le c_\alpha |\xi|^{s-|\alpha|} \quad \text{for } \xi \ne 0 \text{ and } |\alpha| \ge 1.
$$

Alternatively, instead of (2), we assume each ϕ_j is positively homogeneous of degree one. This alternative is an stronger assumption corresponding to the case $s = 1$, in which case (1) is usually called a Fourier integral operator.

We have proved Hölder-type boundedness results of the operators in (1) in the local Hardy spaces h^p $(0 < p < \infty)$ and related spaces. We define the functions spaces X^p as

$$
X^p := \begin{cases} h^p & \text{if } p \le 1 \\ L^p & \text{if } 1 < p < \infty \\ \mathrm{bmo} & \text{if } p = \infty, \end{cases}
$$

where L^p is the usual Lebesgue space, h^p is the local Hardy space, and bmo is the dual space of h^1. The following preliminary results are partially published in [2] and are partially a work in progress we hope to publish soon. The first preliminary result concerns Fourier integral operators.

Theorem 1. *For integers $n, N \geq 2$, let the exponents $p_j \in (\frac{n}{n+1}, \infty]$ ($j = 0, \ldots, N$) satisfy*

$$(3) \qquad \frac{1}{p_0} = \sum_{j=1}^{N} \frac{1}{p_j}.$$

Moreover let

$$m \leq -(n-1) \left(\sum_{j=1}^{N} \left| \frac{1}{p_j} - \frac{1}{2} \right| + \left| \frac{1}{p_0} - \frac{1}{2} \right| \right)$$

and let each phase φ_j being smooth outside the origin and positively homogeneous of degree one. Then the multilinear operator T_σ^Φ extends to a bounded multilinear operator from $X^{p_1} \times \ldots \times X^{p_N}$ to X^{p_0}.

The second preliminary result concerns oscillatory integral operators with phases satisfying (2).

Theorem 2. *For integers $N, n \geq 1$, and real number $s \in (0, \infty)$, assume that the exponents $p_j \in (n/(n + \min(1, s)), \infty]$ ($j = 0, \ldots, N$) satisfy (3) and*

$$m \leq -sn \left(\sum_{j=1}^{N} \left| \frac{1}{p_j} - \frac{1}{2} \right| + \left| \frac{1}{p_0} - \frac{1}{2} \right| \right).$$

Then the multilinear operator T_σ^Φ extends to a bounded multilinear operator from $X^{p_1} \times \ldots \times X^{p_N}$ to X^{p_0}. Moreover, if the functions φ_j are all in $\mathcal{C}^\infty(\mathbb{R}^n)$ (the Schrödinger-case is an example of such a case), then the ranges of p_j's in the theorem could be extended to $\in (0, \infty]$.

In the talk we discuss only the bilinear case, as the methods are the same as in the general multilinear case, and we look at a couple of details in the proof (in particular, Lemma 8.1 in [2]). Finally, we discuss how to show that the results are sharp, at least for some exponents of X^p spaces.

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Nonlinear Dirac Equations

SEBASTIAN HERR

(joint work with Ioan Bejenaru and Timothy Candy)

The talk started with an introduction to the Dirac Equation

$$-i\gamma^\mu \partial_\mu \psi + M\psi = 0,$$

for spinors $\psi : \mathbb{R}^{1+3} \to \mathbb{C}^4$ and (rescaled) mass parameter $M \in \mathbb{R}$, with Dirac matrices $\gamma^\mu \in \mathbb{C}^{4\times 4}$. The dispersive features of its solutions have been analyzed and the connection to classical topics in harmonic analysis – in particular related to the Fourier restriction theory – have been described.

Then, nonlinear Dirac Equations have been considered, starting with the Soler model

(cD) $$-i\gamma^\mu \partial_\mu \psi + M\psi = (\psi^\dagger \gamma^0 \psi)\psi$$

and the corresponding initial value problem (IVP). After a discussion of its basic features, the results from [3, 2, 4] on global well-posedness and scattering have been summarized as follows:

Theorem 1. *Let $(d, D) = (2, 2)$, or $(d, D) = (3, 4)$. For small initial data $\psi(0) \in H^{\frac{d-1}{2}}(\mathbb{R}^d; \mathbb{C}^D)$ the IVP (cD) is globally well-posed. Given the solution $\psi : \mathbb{R}^{1+d} \to \mathbb{C}^D$ of (cD), there is a solution $\psi^\pm$ of the linear equation $-i\gamma^\mu \partial_\mu \psi^\pm + M\psi^\pm = 0$ satisfying*

$$\lim_{t\to\pm\infty} \|\psi^\pm(t) - \psi(t)\|_{H^{\frac{d-1}{2}}(\mathbb{R}^d; \mathbb{C}^D)} = 0.$$

The key ideas of the proof are a construction of microlocal endpoint Strichartz- and Energy-estimates in adapted coordinate frames, similar to work of Daniel Tataru on wave maps. These can be used if waves are transversal, otherwise the null-structure is effective.

Then, the Dirac-Klein-Gordon system

(DKG) $$-i\gamma^\mu \partial_\mu \psi + M\psi = \phi\psi$$
$$\partial_t^2 \phi - \Delta\phi + m^2\phi = \psi^\dagger \gamma^0 \psi$$

for a spinor $\psi : \mathbb{R}^{1+3} \to \mathbb{C}^4$ and scalar field $\phi : \mathbb{R}^{1+3} \to \mathbb{R}$ has been considered.

After a short introduction, the results from [1, 7] on global well-posedness and scattering have been summarized as follows:

Theorem 2. *Let $M, m > 0$. For small initial data*

$$(\psi(0), \phi(0), \partial_t\phi(0)) \in H^\varepsilon(\mathbb{R}^3; \mathbb{C}^4) \times H^{\frac{1}{2}+\varepsilon}(\mathbb{R}^3; \mathbb{R}) \times H^{-\frac{1}{2}+\varepsilon}(\mathbb{R}^3; \mathbb{R}),$$

either for $\varepsilon > 0$ and $2M > m$, or for $\varepsilon = 0$ and additional spherical regularity, the IVP (DKG) is globally well-posed. For solution (ψ, ϕ) we have

$$\lim_{t\to\pm\infty} (\psi^\pm - \psi, \phi^\pm - \phi, \partial_t(\phi^\pm - \phi))(t) = 0$$

for some solution $(\psi^\pm, \phi^\pm)$ of the linear problem.

In the proof, a key idea is to use the bilinear Fourier restriction theory in the case of transversal wave interactions. More precisely, a version of the bilinear Fourier restriction estimate for the hyperboloid and transference to function spaces is required. Some conditional results [6, 5] for large data and open questions have been briefly discussed, too.

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Possible Exact Egorov theorem on hyperbolic surfaces

ANTOINE GANSEMER

(joint work with Andrew Hassell)

Egorov's theorem is a well known result in semiclassical analysis. Given a Riemannian manifold (X, g), it is a relation between the quantum evolution of a system, given by the Schrödinger propagator $e^{itH/h}$, where $0 < h \ll 1$ is a small semiclassical parameter which one takes to 0 in the semiclassical limit and H is a quantisation of the classical Hamiltonian $H(x, \xi) \in C^\infty(T^*X)$, and the classical evolution of a system, namely the Hamiltonian flow, the solutions to Hamilton's equations,

$$\dot{x} = \frac{\partial H}{\partial \xi} \quad \dot{\xi} = -\frac{\partial H}{\partial x}$$

for a given Hamiltonian $H(x, \xi)$. The free Hamiltonian $H(x, \xi) = |\xi|_g^2$ is often of interest, it represents the dynamics of a particle on a space with no potential. The quantisation of $|\xi|_g^2$ is often chosen as $h^2 \Delta_g$. A classical observable is given by a smooth function on phase space, $a(x, \xi) \in C^\infty(T^*X)$. A (dense) subspace of classical observables, called symbols, is associated to a quantum observable, $\mathrm{Op}_h(a)$, in a process called quantisation. $\mathrm{Op}_h(a)$ is an operator on $C^\infty(X)$ and is the "quantum observable" of $a(x, \xi)$. It allows a way to measure the "position" and "momentum" of a wavefunction, generally a wavelike object. Egorov's theorem states that the Schrödinger propagator of a quantum observable is approximately

equal to the quantised observable given by the pullback by classical flow, φ_t. In other words,

$$e^{-ith\Delta}\mathrm{Op}_h(a)e^{ith\Delta} \approx \mathrm{Op}_h(a \circ \varphi_t)$$

with an error which grows in time but small in semiclassical parameter, usually it is $O(h)$ in some sense. In fact, it holds for much more general Hamiltonians than just the free Hamiltonian. On flat $\mathbb{R}^n$, or spaces whose universal cover is $\mathbb{R}^n$, there exists the so-called Weyl quantisation which satisfies an exact version of Egorov's theorem:

$$e^{-ith\Delta}\mathrm{Op}_h^{weyl}(a)e^{ith\Delta} = \mathrm{Op}_h^{weyl}(a \circ \varphi_t)$$

where $\varphi_t : (x,\xi) \mapsto (x + 2t\xi, \xi)$ is the Hamiltonian flow associated to the free Hamiltonian on $\mathbb{R}^n$. We explore the existence of such a quantisation on hyperbolic surfaces, which have rather different dynamics. We follow the work of Anantharaman and Zelditch, [1], [2], who explore two families of distributions on T^*X, where X is a compact hyperbolic surface. One family of distributions is the Wigner distributions, they are matrix elements of the Zelditch left quantisation (see [3]), with respect to Laplacian eigenfunctions. $W_{\lambda_j,\lambda_j}(a) := \langle \mathrm{Op}(a)\varphi_j, \varphi_k \rangle_{L^2(X)}$, where $\Delta\varphi_j = -(1/4 + \lambda_j^2)\varphi_j$ and $\Delta\varphi_k = -(1/4 + \lambda_k^2)\varphi_k$. The other distribution is the family of Patterson-Sullivan distributions, PS_{λ_j,λ_k}, which involve the dynamical resonances ϵ_{λ_j}, that are eigendistributions of the geodesic flow on the hyperbolic surface and have certain anisotropic regularity properties. Anantharaman and Zelditch construct an explicit intertwining operator $\mathcal{L}$ between these two distributions such that $PS_{\lambda_j,\lambda_k}(\mathcal{L}a) = W_{\lambda_j,\lambda_k}(a)$. We show that a slightly modified normalisation of these two distributions implies that $\mathcal{L}$ is a unitary operator on $L^2(T^*X)$ with respect to a certain Haar density, (i.e. the density is invariant under the natural isometry group on X). It is our belief that the intertwining operator and the family of Patterson-Sullivan distributions may come in useful for defining a quantisation that satisfies the exact Egorov theorem on hyperbolic surfaces.

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Hardy spaces adapted to Fourier Integral Operators

Andrew Hassell

(joint work with Jan Rozendaal, Pierre Portal)

In this first short talk, I discuss the Hardy spaces $H_{FIO}^p(\mathbb{R}^n)$ adapted to Fourier Integral Operators introduced first by Hart Smith (but only for $p = 1$) and then in full generality by Portal, Rozendaal and myself in 2020.

The background for this work is the celebrated work of Seeger, Sogge and Stein [1] on the optimal mapping properties for FIOs of order zero (associated to a canonical transformation) on standard Sobolev spaces. They showed that such operators map between L^p-based Sobolev spaces with a loss of $(n-1)|1/2 - 1/p|$ derivatives, and this loss is sharp. To do this, they employed a second dyadic decomposition of frequency space, in which dyadic annuli of radius $\sim |\xi|$ are further decomposed into angular 'petals' of angular range $\sim |\xi|^{-1/2}$.

Smith [2] then found a function space contained in $L^1(\mathbb{R}^n)$ invariant under the action of FIOs of order zero. He did this by building the second dyadic decomposition into the function space and using the fact that, intuitively, FIOs essentially 'permute' such regions in a measure-preserving way.

Portal, Rozendaal and myself then developed this idea from a slightly different point of view [3]. We consider a wave packet transform W associated to the second dyadic decomposition, mapping functions on $\mathbb{R}^n$ to $\mathbb{R}^{2n}$, and view functions on $\mathbb{R}^{2n}$ as (potentially) lying in tent spaces $T^p(S^*\mathbb{R}^n)$ over the cosphere bundle $S^*\mathbb{R}^n = \mathbb{R}^n \times S^{n-1}$. H^p_{FIO} is then the space of functions that get mapped by W into the tent space $T^p(S^*\mathbb{R}^n)$. In this way the generalization to all $p \in [1, \infty]$ is natural, and we are able to exploit standard tent space properties such as interpolation and atomic decompositions.

Our main results are that

- FIOs of order zero are bounded on the spaces H^p_{FIO}, and
- There are sharp embeddings between standard Sobolev spaces and H^p_{FIO}, of the form

$$W^{s(p),p}(\mathbb{R}^n) \hookrightarrow H^p_{FIO}(\mathbb{R}^n) \hookrightarrow W^{-s(p),p}(\mathbb{R}^n), \quad s(p) = \frac{n-1}{2}\left|\frac{1}{2} - \frac{1}{p}\right|.$$

Together, these results allow one to deduce the Seeger-Sogge-Stein result but is in a sense much more precise.

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Rough wave equations and regularity in L^p and H^p_{FIO} spaces

ANDREW HASSELL

(joint work with Jan Rozendaal)

In this second talk, I discuss a recent result [1] proved with Jan Rozendaal on well-posedness for the rough wave equation in Sobolev spaces $H^{s,p}_{FIO}$ over the H^p_{FIO} spaces introduced in the first talk, where this space consists of functions

with s derivatives in H_{FIO}^p. We typically consider the wave equation with $C^{1,1}$ coefficients. More precisely, we consider the equation

$$(1) \qquad Lu(t,x) = F(t,x), \quad L = D_t^2 + \sum_{i,j=1}^n D_i a_{ij}(x) D_j, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n,$$

with initial conditions

$$(2) \qquad u(0,\cdot) = f, \quad D_t u(0,\cdot) = g.$$

We assume that $a_{ij} \in C^{1,1}$ are uniformly elliptic, and that

$$(3) \qquad f \in H_{FIO}^{s,p}(\mathbb{R}^n), \quad g \in H_{FIO}^{s-1,p}(\mathbb{R}^n), \quad F \in L^1(\mathbb{R}; H_{FIO}^{s-1,p}(\mathbb{R}^n)).$$

We say that this equation is well-posed if there exists a function

$$(4) \qquad u(t,x) \in C(\mathbb{R}; H_{FIO}^{s,p}(\mathbb{R}^n)) \cap C^1(\mathbb{R}; H_{FIO}^{s-1,p}(\mathbb{R}^n)) \cap W^{2,1}(\mathbb{R}; H_{FIO}^{s-2,p}(\mathbb{R}^n))$$

satisfying initial conditions (2) and satisfying (1) as an identity in $H_{FIO}^{s-2,p}(\mathbb{R}^n)$ for a.e. time t.

For context, we note that a well-posedness statement of this type is not possible in standard Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ for $p \neq 2$, even for wave equations with smooth coefficients, due to the loss of derivatives of FIOs of order zero acting on such spaces, as discussed in the previous talk. However, there is no loss of derivatives for FIOs of order zero acting on the H_{FIO}^p scale of spaces, and that suggests that well-posedness results of this type may be valid on Sobolev spaces over these spaces.

Our main result is that well-posedness holds in the above sense for suitable p and s depending on dimension n. More precisely we have well-posedness provided that

$$(5) \qquad s(p) := \frac{n-1}{2}\left|\frac{1}{2} - \frac{1}{p}\right| < 1/2, \text{ and}$$
$$-1 + s(p) < s < 2 - s(p).$$

In particular, for $n = 2$ or 3, this permits a full range of exponents $p \in (1,\infty)$.

The proof proceeds by writing $L = L_1 + L_2$, where L_1 is a pseudodifferential operator derived from L by smoothing the coefficients so that only frequencies less than $\sim |\xi|^{1/2}$ are included in the symbol $\sigma(L_1)(x,\xi)$ — a process familiar from paradifferential theory. We then take an approximate square root $b(x,D)$ of L_1 and construct an accurate parametrix for the first order evolution equation $(D_t + b(x,D))u(t,x) = 0$, using the wave packet transform W described in the previous talk to move to phase space and following the bicharacteristic flow of the symbol of b on phase space. As for the operator L_2, this is in effect a first, not second, order operator when the coefficients are $C^{1,1}$. This follows from estimates due to Rozendaal [2] of rough pseudodifferential operators acting on $H_{FIO}^{s,p}$ spaces. Given this, L_2 can be treated similarly to the inhomogeneous term F. We then employ a standard iterative construction to obtain the exact solution. For this to work, it is absolutely crucial that there is no loss of derivatives for the solution operator on $H_{FIO}^{s,p}$ spaces, as any such loss would accumulate infinitely many times.

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L^p estimates for certain wave equations with Lipschitz coefficients
Pierre Portal
(joint work with Dorothee Frey)

In 1980, Peral and Miyachi proved that the operator $(I - \Delta)^{-\frac{\alpha}{2}} \exp(i\sqrt{-\Delta})$ is bounded on $L^p(\mathbb{R}^d)$ if and only if $\alpha \geq s_p := (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|$. Their result was then extended to general Fourier integral operators (FIO) in a celebrated theorem of Seeger, Sogge, and Stein [4], leading, in particular, to $L^p(\mathbb{R}^d)$ well-posedness results for wave equations with smooth variable coefficients on $\mathbb{R}^d$ or driven by the Laplace-Beltrami operator on a compact manifold. A new approach to these results has recently been designed in [1], building on groundbreaking work of Smith [5]. It introduces a scale of Hardy spaces H^p_{FIO} that are invariant under the action of FIO, just like standard Hardy spaces $H^p(\mathbb{R}^d)$ are invariant under the action of pseudo-differential operators. These spaces can then be compared to standard Sobolev spaces through the embedding

$$W^{\frac{s_p}{2},p} \subset H^p_{FIO} \subset W^{-\frac{s_p}{2},p},$$

to recover Peral/Miyachi or Seeger-Sogge-Stein results. This approach has the advantage that one only needs to loose derivatives once through these embeddings. Computing with FIO on H^p_{FIO} itself can be done without loss of derivatives. In 2020, this has been used by Hassell and Rozendaal [2] to prove well posedness of wave equations with $C^{1,1}$ coefficients. In this talk, we discussed complementary joint work with Dorothee Frey [3] focusing on wave equations with $C^{0,1}$ coefficients having a specific algebraic structure. It should be noted that, for many reasons, $C^{1,1}$ is a natural level of regularity for wave equations, and that Strichartz estimates are known to fail, in general, below this level of regularity.

We consider wave equations $\partial_t^2 u = Lu$ for $L = \sum_{j=1}^{d} a_{j+d}\partial_j a_j \partial_j$, where the coefficients $a_j \in C^{0,1}$ and $a_{j+d} \in C^{0,1}$ only depend on the j-th variable. Assuming that these coefficients are bounded above and below and have bounded derivatives, we construct a scale of $H^p_{FIO,a}$ spaces that is invariant under the action of $\exp(i\sqrt{-L})$, and still satisfy the Sobolev embedding properties

$$W^{\frac{s_p}{2},p} \subset H^p_{FIO,a} \subset W^{-\frac{s_p}{2},p}.$$

This thus proves Peral/Miyachi estimates: $(I - L)^{-\frac{\alpha}{2}} \exp(i\sqrt{-L}) \in B(L^p)$ for $\alpha \geq s_p$.

The central idea in [1] is to implement the refined Littlewood-Paley decomposition of [4] through a wave packet transform. Considering two cut-off functions $\Psi, \psi \in C_c^\infty$ with $supp\Psi \subset [\frac{1}{4}, 2]$, one defines

$$W_\sigma(f)(x, \omega) := \psi_{\omega,\sigma}(\nabla)f(x) := \Psi(\sigma\omega.\nabla)\psi(\sigma^{\frac{1}{2}}\omega_1.\nabla)...\psi(\sigma^{\frac{1}{2}}\omega_{d-1}.\nabla)f(x),$$

where $(\omega, \omega_1, ..., \omega_{d-1})$ is an orthonormal basis. This gives an orthogonal decomposition of L^2

$$\|f\|_{L^2(\mathbb{R}^d)}^2 \sim \int\limits_0^1 \|W_\sigma f\|_{L^2(\mathbb{R}^d \times S^{d-1})}^2 \frac{d\sigma}{\sigma} + \int\limits_1^\infty \|\Psi(\sigma^2\Delta)f\|_{L^2(\mathbb{R}^d)}^2 \frac{d\sigma}{\sigma},$$

which decomposes a function f into pieces that are "asymptotically one dimensional" in the sense that, when their Fourier transform is located within a dyadic Littlewood-Paley annulus far from 0 (i.e. σ is close to 0), then it is also located in a narrow slice of the annulus, where the angular part of the momentum is within $\sqrt{\sigma}$ of a fixed direction ω. One then defines H_{FIO}^p by forcing this decomposition to be unconditional in L^p, using a norm such as

$$\|\omega \mapsto [(\sigma, x) \mapsto 1_{(1,\infty)}(\sigma)\Psi(\sigma^2\Delta)f(x) + 1_{[0,1]}(\sigma)W_\sigma(f)(x,\omega)]\|_{L^p(S^{d-1};T^{p,2}(\mathbb{R}^d))}.$$

To adapt these spaces to the coefficients $(a_j)_{j=1,...,2d}$, we replace the partial derivatives ∂_j in the definition of the wave packet transform by Dirac operators

$$e_j.D_a := \begin{pmatrix} 0 & -a_{j+d}\partial_j \\ a_j\partial_j & 0 \end{pmatrix}.$$

The algebraic condition on the coefficients guarantees that the square of these operators (i.e. the generators of one dimensional half-wave groups adapted to the coefficients in each direction) do commute. This allows us to use Coifman-Weiss's transference principle to prove that operators such as

$$\left(\exp(i\sqrt{-L}) - \exp(i\sum_{j=1}^d \omega_j \sqrt{(e_j.D_a)^2}) \right) \int_0^1 \psi_{\omega,\sigma}(D_a)\frac{d\sigma}{\sigma}$$

are bounded on L^p (because they are transferred Fourier multipliers with appropriate symbols), and consequently that $\exp(i\sqrt{-L}) \in B(H_{FIO,a}^p)$, since the boundedness of $\exp(i\sum_{j=1}^d \omega_j \sqrt{(e_j.D_a)^2})$ on L^p follows from standard one dimensional results via a bilipschitz change of variables.

To prove the Sobolev embedding properties, one exploits dispersive properties, expressed as off-diagonal decay with respect to a family of anisotropic distances associated with the Seeger-Sogge-Stein refinement of the Littlewood-Paley decomposition. These are estimates of the form: for every $M \in \mathbb{N}$, there exists $C_M > 0$ such that for all $E, F \subset \mathbb{R}^d$ Borel sets, $\sigma \in (0,1)$ and $\omega \in S^{d-1}$, we have

$$\|1_E\psi_{\omega,\sigma}(D_a)(1_F f)\|_{L^2(\mathbb{R}^d)} \le C_M\sigma^{-\frac{d}{2}}(1 + \frac{d_\omega(E,F)}{\sigma})^{-M}\|1_F f\|_{L^1(\mathbb{R}^d)}$$

for all $f \in L^1(\mathbb{R}^d)$, where

$$d_\omega(x, y) := |\langle \omega, x - y \rangle| + \sum_{j=1}^{d-1} \langle \omega_j, x - y \rangle^2 \quad \forall x, y \in \mathbb{R}^d.$$

Such estimates are precisely what allows us to use techniques designed for parabolic and elliptic problems (and classical Hardy spaces) in the context of wave equations.

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Spectral multiplier estimates for abstract differential operators
HIMANI SHARMA

For an operator generating a group on L^p spaces transference results give bounds on the Phillips functional calculus also known as spectral multiplier estimates. In the talk, we consider specific group generators which are abstraction of first order differential operators and show similar spectral multiplier estimates assuming only that the group is bounded on L^2 rather than L^p. The motivation behind not asking for any L^p information on the group is the fact that for a simple differential operator on L^2 it is generally difficult to decide whether or not it generates a group on L^p. For instance, the first derivative on $L^p(\mathbb{R})$ generates the group of translations but i times the second derivative generates a group only if $p = 2$.

We thus start with a self adjoint operator D on $L^2(\mathbb{R}^d; \mathbb{C}^n)$ such that the C_0-group e^{itD} generated by iD has finite propagation speed $\kappa_D \leq \kappa$, which happens if

$$\text{supp}(e^{itD}u) \subset K_{\kappa|t|} := \{x \in \mathbb{R}^d; \ \text{dist}(x, K) \leq \kappa|t|\} \quad \forall t \in \mathbb{R}$$

whenever $\text{supp}(u) \subset K \subset \mathbb{R}^d$, for some compact set K. For one of our results, we also assume some Sobolev embedding property, that is, $(I + D^2)^{-\frac{1}{2}} \in B(L^{p_*}, L^p)$, where $p_* = \frac{dp}{d+p}$. Under these assumptions we obtain L^p boundedness of operators $a(D)$ for $a \in C_c^\infty$. The bound, however, is not optimal but has important applications in dispersive PDE, where one needs to project precisely on part of the spectrum, i.e. use a compactly supported a. The key idea is that one can use off-diagonal arguments that are abstractions of Calderon-Zygmund theoretic arguments directly on the representation of $a(D)$ as an integral of e^{itD}.

An even better result on the Hardy space H_D^p associated with D is also obtained by eliminating the Sobolev embedding assumption. The use of such an adapted

space is not a problem, as it can be combined with results that identify the range of values of p for which $H_D^p = L^p$. One such operator D is the perturbed Hodge-Dirac operator $\Pi_B = \begin{pmatrix} 0 & -divA \\ \nabla & 0 \end{pmatrix}$ (defined in the sense of Frey-McIntosh-Portal [1]) acting on $L^2(\mathbb{R}^d; \mathbb{C}^{1+d})$, where A is a real, smooth $d \times d$ matrix. In this case we get $\mathcal{N}_p(\Pi_B) \oplus H_{\Pi_B}^p = L^p$ on the range of p for which Π_B Hodge decomposes $L^p(\mathbb{R}^d; \mathbb{C}^{1+d})$ and hence the norm of the operators $a(D)$ on $H_{\Pi_B}^p$ and L^p are equivalent.

Under the same assumption that iD generates a group with finite propagation speed we also improve an R-bounded Hörmander calculus result of Kriegler and Weis—where they only assume that D^2 generates an analytic semigroup—by weakening their technical R-boundedness assumption into a $L^{p_*} - L^p$ boundedness assumption. This proves, in particular, that the square of the above defined Hodge-Dirac operator has a bounded Hörmander calculus (recovering, in particular, the result for uniformly elliptic divergence form operators with L^∞ coefficients).

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Quantum separation effect for Feynman-Kac semigroup

Adam Sikora

(joint work with Jacek Zienkiewicz)

The quantum tunnelling phenomenon allows a microscopic particle in Schrödinger mechanics to tunnel through a barrier that it classically could not overcome. It even allows the particle to pass through the potential barrier, even if its height is infinite. The tunnelling phenomenon is easily predicted and explained by Schrödinger mechanics - the eigenstates of Hamiltonian of the system cannot be localised. In our project we consider the quantum well and investigate the possibility that the particle trapped in the well cannot escape, that is the possibility that the barrier separates two regions. To be more precise, we consider the domain $D \subset \mathbb{R}^d$ and its boundary $K = \partial D$ that separates D and its complement D^c. Next, we define the distance from K by the formula $d_K(x) = \inf\{d(x,y): y \in K\}$ and set

$$V_\beta = C d_K^{-\beta}$$

for some $\beta > 0$. We investigate the Hamiltonian of the system, that is the operator

$$H_V = \Delta - V_\beta$$

initially defined for function belonging to $C_c^\infty(\mathbb{R}^d \setminus K)$. Here Δ is the positive standard Laplace operator. Then we consider Feynman-Kac semigroup $\exp(-tH_V)$

generated by the Hamiltonian H_{V_β} and we denote by $p_t^V(x,y)$ the corresponding heat kernel. We address the question for which values of β, $\exp(-tH_{V_\beta})$ separates D and D^c, that is when $p_t^V(x,y) = 0$ for $x \in D$ and $y \in D^c$. If the domain D has a smooth boundary, the problem considered by us has been satisfactory resolved by Wu in [7]. In our study we generalise the results described in [7]. First, the domains we consider are irregular fractals of some special but still general type including van Koch snowflake domain. Second, we study the quantitative estimates of the tunnelling effect in our setting. We consider cut-off potential and we estimate the rate it suppresses the semigroup kernel $p_t(x,y)$ when x, y are separated by the boundary $K = \partial D$. The essential difference compared to [7] is that we do not require smoothness of the considered domain D. Instead we consider only fractal type regularity conditions for K. In order to deal with irregular domains, we develop a new approach different than in Wu. For the case of separation problem, it is still an elementary and simple probabilistic argument based on the Paley-Zygmund inequality and Blumenthal's zero-one law. It becomes more involved Brownian paths analysis for the quantitative description of the tunnelling.

The questions concerning separation can be posed for any semigroup of operators, even without direct relations to Schrödinger mechanics. We mention here [2] where the authors study similar phenomena for certain types of divergence form degenerate elliptic operators. The separation phenomenon for semigroups is also related to the regularity theory for the solutions of Partial Differential Equations corresponding to their generators which was investigated in [3] and [5]. Essentially one can say that separation and regularity are mutually excluding properties. In our study we also discuss a regularity result for integrable potentials, which shows that the range of β for which we verify separation is optimal.

The estimates we discuss in our note are strictly connected to the boundary behaviour of the Brownian motion. Motivation for the techniques we use partially comes from the analysis in [1] and [6]. The detailed discussion of the described results can be found in [4].

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Sharp L^p estimates for oscillatory integral operators of arbitrary signature

Marina Iliopoulou

(joint work with Jonathan Hickman)

The restriction problem in harmonic analysis asks for L^p bounds on the Fourier transform of functions defined on curved surfaces. In this talk, we present improved restriction estimates for hyperbolic paraboloids, that depend on the signature of the paraboloids. These estimates still hold, and are sharp, in the variable coefficient regime. This is joint work with Jonathan Hickman.

In particular, we denote by $B^d(0,\lambda)$ the ball with centre 0 and radius λ in $\mathbb{R}^d$. For any hypersurface $\Sigma := \{(\omega, \Sigma(\omega)) : \omega \in B^{n-1}(0,1)\}$ in $\mathbb{R}^n$, with non-vanishing Gaussian curvature, the extension operator associated to Σ is defined by

$$Ef(x) = \int e^{2\pi i \langle x, (\omega, \Sigma(\omega)) \rangle} f(\omega) d\omega, \text{ for all } f : B^{n-1}(0,1) \to \mathbb{C} \text{ and } x \in \mathbb{R}^n.$$

Let σ be the signature of Σ. We prove that

$$\|Ef\|_{L^p(B^n(0,\lambda))} \leq c_{\epsilon,\Sigma} \lambda^\epsilon \|f\|_{L^p(B^{n-1}(0,1))} \text{ for all } \lambda \geq 1 \text{ and } \epsilon > 0,$$

whenever p satisfies

$$p \geq \begin{cases} 2 \cdot \frac{\sigma + 2(n+1)}{\sigma + 2(n-1)}, & \text{if } n \text{ is odd} \\ 2 \cdot \frac{\sigma + 2n + 3}{\sigma + 2n - 1}, & \text{if } n \text{ is even} \end{cases} .$$

We further prove the same estimates for Hörmander-type operators (with general phase functions) - in this generality, the above estimates are sharp.

In the talk we discuss the differences between the extention operator and general Hörmander-type operators, and explain the role of signature in the problem.

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A Transference Principle for Bilinear Restriction Estimates

Timothy Candy

(joint work with Sebastian Herr and Kenji Nakanishi)

This talk describes two recent results. The first is joint work with S. Herr and K. Nakanishi [3], and gives a fairly general method to show that bilinear restriction estimates on $\mathbb{R}^4$ for free (or homogeneous) Schrödinger waves can be extended to all elements $u \in Z$, where Z is an (endpoint) inhomogeneous Strichartz space defined via the norm

$$\|u\|_Z = \|u\|_{L_t^\infty L_x^2(\mathbb{R}^{1+4})} + \|(i\partial_t + \Delta)u\|_{L_t^2 L_x^4(\mathbb{R}^{1+4})}.$$

In other words, the Banach space Z satisfies a *transference principle*. The transference argument described below also works in much greater generality (other dispersive PDE, general dimensions, multilinear estimates...) but here we simply illustrate the argument in the bilinear setting for the Schrödinger equation in 4 dimensions. The second result is a bilinear restriction estimate for wave-Schrödinger interactions [2]. This bilinear estimate is sharp under a transversality type assumption. However once this transversality assumption is dropped it is unclear what the sharp estimates are. This open problem is of particular interest due to connections with the regularity theory for the Zakharov equation.

In the context of dispersive PDE, the *transference principle* refers to an extremely useful property in dispersive PDE which allows us to transfer estimates from free waves (i.e. homogeneous solutions) to all elements of suitable Banach space. This Banach space typically contains solutions to a corresponding *nonlinear* PDE, and thus the transference principle can be thought of as a way to transfer estimates for free waves, to estimates for nonlinear waves. The transference principle was first observed to hold for $X^{s,b}$ spaces (or Bourgain spaces/wave-Sobolev spaces) [4, 7], and under certain conditions also holds in the U^p/V^p framework [5]. To illustrate the transference principle in the bilinear setting, consider the following bilinear restriction estimate.

Theorem 1 (Bilinear Restriction for Paraboloid [9, 8, 1])**.** *Let $r > \frac{5}{3}$ and $\mu \in 2^{\mathbb{Z}}$. If $u = e^{it\Delta}f$ and $v = e^{it\Delta}g$ are free solutions to the Schrödinger equation, then*[1]

$$\|P_\mu(\overline{u}v)\|_{L_t^1 L_x^r(\mathbb{R}^{1+4})} \lesssim \mu^{2-\frac{4}{r}} \|f\|_{L^2(\mathbb{R}^4)} \|g\|_{L^2(\mathbb{R}^4)}.$$

Theorem 1 gives a substantial improvement over the endpoint Strichartz estimate $\|e^{it\Delta}f\|_{L_t^2 L_x^4} \lesssim \|f\|_{L^2}$, which essentially corresponds to the case $r = 2$. The fact that Theorem 1 allows $r < 2$ is extremely useful, as it shows that the product $\overline{u}v$ decays faster than expected. This additional decay played a key role in proving global existence for the Zakharov equation [3], a nonlinear coupled wave-Schrödinger system. However the arguments in [3] required a version of Theorem 1 which applied to *nonlinear* solutions. More precisely, the following bilinear estimate was needed.

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[1]Here P_μ restricts Fourier support to the set $|\xi| \approx \mu$.

Theorem 2 (Inhomogeneous Bilinear Restriction [3]). *Let* $r > \frac{5}{3}$. *Then for any* $u, v \in Z$ *we have*

$$\|P_\mu(\overline{u}v)\|_{L_t^1 L_x^r(\mathbb{R}^{1+4})} \lesssim \mu^{2-\frac{4}{r}} \|u\|_Z \|v\|_Z.$$

The standard strategy to obtain Theorem 2 from Theorem 1 would be to apply a transference principle. This is fairly straightforward in the $X^{s,b}$ setting, as elements $u \in X^{s,b}$ can be written as L^1 averages of free waves. In other words, provided $b > \frac{1}{2}$, we can write

$$u(t,x) = \int_{\mathbb{R}} e^{it\tau} (e^{it\Delta} f_\tau)(x) \, d\tau, \qquad \text{with} \qquad \int_{\mathbb{R}} \|f_\tau\|_{L_x^2} d\tau \lesssim \|u\|_{X^{0,b}}$$

where $\|u\|_{X^{s,b}} = \|\langle \xi \rangle^s \langle i\partial_t + \Delta \rangle^b u\|_{L_{t,x}^2}$. It is then straightforward to conclude the $X^{0,b}$ version of Theorem 2 from Theorem 1. However this argument fails in the case of the inhomogeneous Strichartz space Z, as it is in general not possible to write elements $u \in Z$ as L^1 averages of free waves. Instead, the proof of Theorem 2 proceeds by first extending Theorem 1 to *vector valued* free Schrödinger waves, namely

$$(1) \qquad \left\| \left(\sum_{j,k} |\overline{e^{it\Delta} f_j} e^{it\Delta} g_k|^2 \right)^{\frac{1}{2}} \right\|_{L_t^1 L_x^r} \lesssim \left(\sum_j \|f_j\|_{L^2}^2 \right)^{\frac{1}{2}} \left(\sum_k \|g_k\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

This improvement follows from a standard randomisation argument, see for instance the discussion in [2]. The equation (1) is the key bilinear restriction input in the proof of Theorem 2, and the remainder of the proof essentially follows via the Duhamel formula, the decomposition

$$\int_0^t e^{i(t-s)\Delta} F(s) ds = \sum_{\lambda \in 2^{\mathbb{Z}}} \sum_{\substack{I,J \subset \mathbb{R} \text{ intervals} \\ |I| \approx |J| \approx \text{dist}(I,J) \approx \lambda}} \chi_I(t) \int_J e^{i(t-s)\Delta} F(s) ds$$

and an inhomogeneous Strichartz estimate [6]. The vector valued bilinear restriction estimate (1) controls the sum over the intervals I, J, while the inhomogeneous Strichartz estimate makes it possible to sum over j, see [3, Section 4] for the details.

The second main result we present is the following bilinear restriction estimate for wave-Schrödinger interactions.

Theorem 3 (Bilinear restriction for wave-Schrödinger interactions [1, 2]). *Let* $d \geqslant 2$, $1 \leqslant q, r \leqslant 2$ *with* $\frac{1}{q} + \frac{d+1}{r} < \frac{d+1}{2}$. *Let* $|\xi_0| \approx |\eta_0| \approx 1$ *and define* $\alpha = |\frac{\eta_0}{|\eta_0|} + 2\xi_0|$. *Assume that*

$$(2) \qquad \left| \left(\frac{\eta_0}{|\eta_0|} + 2\xi_0 \right) \cdot \frac{\eta_0}{|\eta_0|} \right| \gtrsim \alpha.$$

If $f, g \in L^2(\mathbb{R}^d)$ *with supp* $\widehat{f} \subset \{|\xi - \xi_0| \ll \alpha\}$ *and supp* $\widehat{g} \subset \{|\xi| \approx 1, \angle(\xi, \eta_0) \ll \alpha\}$, *then we have*

$$\|e^{it\Delta} f e^{it|\nabla|} g\|_{L_t^q L_x^r(\mathbb{R}^{1+d})} \lesssim \alpha^{d - \frac{d}{r} - \frac{2}{q}} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}.$$

This result is sharp in the sense that the range q, r, and the dependence on α cannot be improved. However it is interesting to understand what estimates of this form are possible when the transversality assumption (2) is dropped. This is of particular interest in view of the fact that dealing with a wave-Schrödinger product is one of the main difficulties in studying the Zakharov system. Following the discussion in [2], we give counterexamples that may provide some guide as to the possible range of the exponents q, r when the assumption (2) fails.

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BMO spaces associated to operators with generalised Poisson bounds on non-doubling manifolds with ends

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(joint work with Peng Chen, Ji Li, Liang Song, Lixin Yan)

The space BMO of functions of bounded mean oscillation on $\mathbb{R}^n$, which was originally introduced by John and Nirenberg [5] has been very useful in the study of partial differential equations. The BMO space was identified as the dual space of the classical Hardy space H^1 in the celebrated work by Fefferman and Stein [4]. Since then the BMO function space and its predual H^1 are considered as the natural substitutions for the Lebesgue spaces L^∞ and L^1 respectively in the study of singular integrals. The BMO space and Hardy space have been extended from the space $\mathbb{R}^n$ to the case of spaces of homogeneous type (X, d, μ), i.e. the underlying measure μ satisfies the doubling (volume) property. For a singular integral operator T which is bounded on $L^2(X)$ with a doubling space X, a sufficient condition for T to be bounded from $H^1(X)$ to $L^1(X)$, and from $L^\infty(X)$ into $BMO(X)$ is that the associated kernel of T satisfies the (integral) Hörmander condition with respect to variable x and y, respectively.

If the underlying space X does not satisfy the doubling condition but its volume growth is at most polynomial, then many classical results of harmonic analysis on doubling spaces are still true by the works of Nazarov, Treil, Volberg, Tolsa and others [6], [7]. For these non-homogenous spaces, regularized BMO spaces were introduced by Tolsa [7]; however, for an L^2 bounded singular operator T to be bounded from L^∞ to the regularized BMO space, strong conditions on the associated kernel of T are required, for example the Hörmander condition is not sufficient for that purpose.

We aim to study singular integrals with rough kernels i.e. the associated kernels do not satisfy the Hörmander condition, acting on non-homogeneous spaces. Our model of the underlying space is a non-doubling manifold with ends $M = \mathfrak{R}^n \sharp \mathbb{R}^m$ where $\mathfrak{R}^n = \mathbb{R}^n \times \mathbb{S}^{m-n}$ for $m > n \geq 3$. We say that an operator L has a generalised Poisson kernel if $\sqrt{L}$ generates a semigroup $e^{-t\sqrt{L}}$ whose kernel $p_t(x,y)$ has an upper bound similar to the kernel of $e^{-t\sqrt{\Delta}}$ where Δ is the Laplace-Beltrami operator on M. An example for operators with generalised Gaussian bounds is the Schrödinger operator $L = \Delta + V$ where V is an arbitrary non-negative locally integrable potential. We note that without further condition on the potential V, then the kernel $p_t(x,y)$ can be discontinuous, hence the associated kernels of certain standard singular integrals like $L^{is}, s \in \mathbb{R}$ are rough and do not satisfy the Hörmander condition, see for example [1].

In this talk, our aim is to introduce the BMO space $BMO_L(M)$ associated to operators with generalised Poisson bounds (see [2, 3] for the setting of doubling spaces) which serves as an appropriate setting for certain singular integrals with rough kernels to be bounded from $L^\infty(M)$ into this new $BMO_L(M)$. On our $BMO_L(M)$ spaces, we show that the John-Nirenberg inequality holds and we show an interpolation theorem for a holomorphic family of operators which interpolates between $L^q(M)$ and $BMO_L(M)$. As an application, we show that the holomorphic functional calculus $m(\sqrt{L})$ is bounded from $L^\infty(M)$ into $BMO_L(M)$, and bounded on $L^p(M)$ for $1 < p < \infty$.

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Local smoothing and Hardy spaces for Fourier integral operators
JAN ROZENDAAL

This presentation concerns the local smoothing conjecture for the Euclidean wave equation on $\mathbb{R}^n$. The phenomenon of local smoothing revolves around determining, for each $1 < p < \infty$, the minimal $s \in \mathbb{R}$ for which there exists a $C \geq 0$ such that

$$(1) \qquad \left(\int_0^1 \|e^{it\sqrt{-\Delta}} f\|_{L^p(\mathbb{R}^n)}^p \mathrm{d}t \right)^{1/p} \leq C\|f\|_{W^{s,p}(\mathbb{R}^n)}$$

for all $f \in W^{s,p}(\mathbb{R}^n)$. The optimal fixed-time estimates for the half-wave propagator $e^{it\sqrt{-\Delta}}$ imply that one may let $s = 2s(p)$, where

$$s(p) = \frac{n-1}{2}\left|\frac{1}{2} - \frac{1}{p}\right|.$$

This estimate is sharp for $1 < p \leq 2$, but it can be improved for $p > 2$. In fact, the local smoothing conjecture, as originally formulated by Sogge in [6], stipulates that (1) should hold with $s = \sigma(p) + \varepsilon$ for each $\varepsilon > 0$, where $\sigma(p) = 0$ for $2 < p \leq 2n/(n-1)$, and $\sigma(p) = 2s(p) - 1/p$ for $p > 2n/(n-1)$.

Although the local smoothing conjecture is still open in full generality, there are many partial results available, and the conjecture was proved by Guth, Wang and Zhang [2] for $n = 2$. For the purposes of the present article, however, it is relevant to highlight the article [1] by Bourgain and Demeter. Building on work by Wolff [7], they showed that (1) holds with $s = \sigma(p) + \varepsilon$ for $p \geq 2(n+1)/(n-1)$, by proving the ℓ^2 decoupling conjecture. More precisely, one has

$$(2) \qquad \left(\int_0^1 \|e^{it\sqrt{-\Delta}} f\|_{L^p(\mathbb{R}^n)}^p \mathrm{d}t \right)^{1/p} \leq C\|f\|_{W^{d(p)+\varepsilon,p}(\mathbb{R}^n)}$$

for $s = d(p) + \varepsilon$, where

$$d(p) := \begin{cases} 2s(p) - \frac{1}{p} & \text{if } p \geq \frac{2(n+1)}{n-1}, \\ s(p) & \text{if } 2 \leq p < \frac{2(n+1)}{n-1}. \end{cases}$$

Main results. The local smoothing conjecture is sharp, in the sense that (1) does not hold for $s < \sigma(p)$. Nonetheless, in [4] we have obtained improved local smoothing estimates for $p \geq 2(n+1)/(n-1)$, by working with a different space of initial data in (2). Our main result, formulated in terms of the Sobolev spaces $\mathcal{H}_{FIO}^{s,p}(\mathbb{R}^n) = \langle D \rangle^{-s} \mathcal{H}_{FIO}^p(\mathbb{R}^n)$ over the Hardy spaces for Fourier integral operators $\mathcal{H}_{FIO}^p(\mathbb{R}^n)$, is as follows.

Theorem 1. *Let $p \in (2, \infty)$ and $\varepsilon > 0$. Then there exists a $C > 0$ such that*

$$(3) \qquad \left(\int_0^1 \|e^{it\sqrt{-\Delta}} f\|_{L^p(\mathbb{R}^n)}^p \mathrm{d}t \right)^{1/p} \leq C\|f\|_{\mathcal{H}_{FIO}^{d(p)-s(p)+\varepsilon,p}(\mathbb{R}^n)}$$

for all $f \in \mathcal{H}_{FIO}^{d(p)-s(p)+\varepsilon,p}(\mathbb{R}^n)$.

The Hardy space $\mathcal{H}_{FIO}^1(\mathbb{R}^n)$ for Fourier integral operators (FIOs) was introduced by Smith in [5], and his construction was extended by Hassell, Portal and the author [3] to a scale $(\mathcal{H}_{FIO}^p(\mathbb{R}^n))_{1 \leq p \leq \infty}$ of invariant spaces for Fourier integral operators. More precisely, $\mathcal{H}_{FIO}^p(\mathbb{R}^n)$ is invariant under FIOs of order zero which have a compactly supported Schwartz kernel and are associated with a local canonical graph, and one has

$$(4) \qquad W^{s(p),p}(\mathbb{R}^n) \subseteq \mathcal{H}_{FIO}^p(\mathbb{R}^n) \subseteq W^{-s(p),p}(\mathbb{R}^n)$$

for $1 < p < \infty$, with the natural modifications involving the local Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ for $p = 1$, and $\mathrm{bmo}(\mathbb{R}^n)$ for $p = \infty$. In particular,

$$(5) \qquad W^{d(p)+\varepsilon,p}(\mathbb{R}^n) \subseteq \mathcal{H}_{FIO}^{d(p)-s(p)+\varepsilon,p}(\mathbb{R}^n) \subseteq W^{d(p)-2s(p)+\varepsilon,p}(\mathbb{R}^n)$$

for $2 < p < \infty$, and (3) recovers (2). However, since the exponents in (4) and (5) are sharp, (3) is in fact a strict improvement of (2). And since $d(p) = \sigma(p)$ for $p \geq 2(n+1)/(n-1)$, Theorem 1 improves upon the local smoothing conjecture for such p. Also note that

$$d(p) - s(p) = 0 = \sigma(p) \quad \text{for} \quad 2 < p \leq 2\frac{n}{n-1},$$

and

$$d(p) - s(p) = 0 < 2s(p) - \frac{1}{p} = \sigma(p) < d(p) \quad \text{for} \quad 2\frac{n}{n-1} < p < 2\frac{n+1}{n-1}.$$

Hence the sharpness of the embeddings in (5) implies that, for $2 < p < 2(n+1)/(n-1)$, (3) neither follows from the local smoothing conjecture nor implies it.

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Improved discrete restriction for the parabola

Po-Lam Yung

(joint work with Shaoming Guo, Zane Kun Li)

It was noted that Fourier decoupling inequalities played a key role in the recent advances regarding the local smoothing conjecture for the wave equation (c.f. Bourgain and Demeter [2]). In fact, examples that almost extremizes Fourier decoupling inequalities for the sphere provide initial data that motivates the numerology in the local smoothing conjecture, although it was also explained that Kakeya type phenomenon from incidence geometry produces examples for the local smoothing conjecture that are worse by a logarithm.

The talk then turned to another recent progress about Fourier decoupling, this time for the parabola in the plane. Guth, Maldague and Wang [4] proved an improved decoupling inequality for the parabola in $\mathbb{R}^2$: they showed that there exists some finite constant A, so that if $R \gg 1$ and $\{\theta\}$ is a family of finitely overlapping rectangles of dimensions $R^{-1/2} \times R^{-1}$ that cover an R^{-1} neighborhood of the unit parabola $\{(\xi, \xi^2) \colon |\xi| \le 1\}$, then whenever $\{f_\theta\}$ is a family of Schwartz functions indexed by $\{\theta\}$ so that $\widehat{f_\theta}$ is supported in θ for every θ, and $f = \sum_\theta f_\theta$, one has

$$(1) \qquad \int_{\mathbb{R}^2} |f|^6 \lesssim (\log R)^{6A} \Big(\sum_\theta \|f_\theta\|_{L^\infty(\mathbb{R}^2)}^2 \Big)^2 \int_{\mathbb{R}^2} |f|^2.$$

This in turn allowed them to conclude that if $u(x,t)$ is the solution to the periodic Schrödinger equation on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with initial data h, i.e.

$$u(x,t) = \sum_{n \in \mathbb{Z}} \widehat{h}(n) e^{2\pi i(nx + n^2 t)},$$

then there exists a finite constant A so that whenever $\widehat{h}(n) = 0$ for all $|n| > M$, one has

$$(2) \qquad \|u\|_{L^6(\mathbb{T}^2)} \lesssim (\log M)^A \|h\|_{L^2(\mathbb{T})}.$$

The idea of Guth, Maldague and Wang is that to estimate $\int_{\mathbb{R}^2} |f|^6$ in (1) one should exploit efficiently what one knows at L^4. For instance, a classical estimate of Fefferman and Cordoba says that

$$(3) \qquad \int_{\mathbb{R}^2} |f|^4 \lesssim \int_{\mathbb{R}^2} |g|^2, \qquad g := \sum_\theta |f_\theta|^2.$$

One may pigeonhole so that all non-zero $\|f_\theta\|_{L^\infty(\mathbb{R}^2)}$ are comparable to each other. If it holds true that

$$(4) \qquad \int_{\mathbb{R}^2} g^2 \lessapprox \int_{\mathbb{R}^2} \sum_\theta |f_\theta|^4,$$

then we obtain the desired bound by noting that

$$\int_{\mathbb{R}^2} |f|^6 \leq \|f\|_{L^\infty(\mathbb{R}^2)}^2 \int_{\mathbb{R}^2} |f|^4 \overset{(3)}{\lesssim} \|f\|_{L^\infty(\mathbb{R}^2)}^2 \int_{\mathbb{R}^2} g^2 \overset{(4)}{\lesssim} \|f\|_{L^\infty(\mathbb{R}^2)}^2 \int_{\mathbb{R}^2} \sum_\theta |f_\theta|^4$$

$$\lesssim \left(\sum_\theta \|f_\theta\|_{L^\infty(\mathbb{R}^2)} \right)^2 \sup_\theta \|f_\theta\|_{L^\infty(\mathbb{R}^2)}^2 \int_{\mathbb{R}^2} \sum_\theta |f_\theta|^2$$

$$\lesssim \left(\sum_\theta \|f_\theta\|_{L^\infty(\mathbb{R}^2)}^2 \right)^2 \int_{\mathbb{R}^2} |f|^2.$$

The problem is that (4) is not always true. Roughly speaking, (4) holds only when $g = \sum_\theta |f_\theta|^2$ is a sum of functions that are (almost) orthogonal to each other. At this stage it is advantageous to introduce an intermediate scale R_*, satisfying $R^{1/2} < R_* < R$. One covers an R_*^{-1} neighborhood of the unit parabola by finitely overlapping rectangles $\{\tau\}$ of dimensions $R_*^{-1/2} \times R_*^{-1}$, and write

$$g_* := \sum_\tau |f_\tau|^2, \qquad f_\tau := \sum_{\theta \subset \tau} f_\theta.$$

On one hand g is essentially the low frequency part of g_*: essentially

$$\widehat{g}(\xi) = \mathbb{1}_{|\xi| \leq R^{-1/2}} \widehat{g_*}(\xi).$$

On the other hand, the high frequency part of g_* is a sum of functions that are suitably orthogonal to each other, and hence a version of (4) holds: if g_*^{high} is the inverse Fourier transform of $\mathbb{1}_{|\xi| > R^{-1/2}} \widehat{g_*}(\xi)$, then

$$(5) \qquad \int_{\mathbb{R}^2} |g_*^{\text{high}}|^2 \lesssim (R/R_*)^{1/2} \int_{\mathbb{R}^2} \sum_\tau |f_\tau|^4.$$

One wants to keep R/R_* small in the last estimate. Thus one is led to introducing many more intermediate scales and appealing to induction on scales. This allows one to systematically exploit gains in (5), and eventually proves (1).

Chasing through the proof of Guth, Maldague and Wang, one would get a constant A in (2) that is at least 30. In joint work with Shaoming Guo and Zane Kun Li [3], we sharpened this bound for A, and showed that one can indeed take A to be $2 + \varepsilon$ for any $\varepsilon > 0$. We did that by using Fourier decoupling for the parabola in $\mathbb{Q}_p^2$ (where $\mathbb{Q}_p$ is the p-adic field) instead of over $\mathbb{R}^2$. This is possible because the Lebesgue exponent 6 is an even integer, and advantageous because the uncertainty principle for the Fourier transform over the p-adic fields is much cleaner than the one over $\mathbb{R}$. We also additionally used a Whitney decomposition to bilinearize, that saved us some powers of log. A counter-example of Bourgain [1] showed that A has to be at least $1/6$. The value $A = 1/6$ seems out of reach of our current methods.

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Pointwise decay in time of solutions to energy critical nonlinear Schrödinger and wave equations

ZIHUA GUO

(joint work with Chunyan Huang, Liang Song)

Consider the nonlinear Schrödinger equation

$$(1) \qquad \begin{cases} i\partial_t u + \Delta u = \lambda |u|^4 u, & (x,t) \in \mathbb{R}^3, \\ u(x,0) = u_0(x). \end{cases}$$

It is well-known that the linear solutions of Schrödinger equation satisfy the following dispersive estimates

$$(2) \qquad \|e^{it\Delta} u_0\|_{L_x^\infty} \le C|t|^{-3/2} \|u_0\|_{L_x^1}.$$

It is natural to ask whether one can obtain global solutions to (1) with the same time decay

$$(3) \qquad \|u(t)\|_{L_x^\infty} \le C|t|^{-3/2}.$$

It is well-known that (1) is energy critical. Global well-posedness and scattering theory were extensively studied in last two decades (see [2]). A global solution of (1) scatters in the energy space means there exists $\phi_\pm \in H^1$ such that

$$(4) \qquad \lim_{t \to \pm\infty} \|u(t) - e^{it\Delta}\phi_\pm\|_{H^1} = 0.$$

Even if we have scattering, to get pointwise-in-time decay (3) is not trivial. On one hand, pointwise-in-time decay is not expected if assuming data only in $H^1(\mathbb{R}^3)$. On the other hand, assuming $u_0 \in L^1$ one cannot ensure the scattering state $\phi_\pm \in L^1$. Thus it requires extra effort. Recently, by contradiction argument Fan and Zhao [1] proved that scattering solution of (1) satisfies (3) assuming $u_0 \in L^1 \cap H^k$ for some k. We give a direct and simpler proof of their result. Our main ingredient is the boundedness for Schrödinger propagator in Hardy space. Our argument gives a quantitative bound and also works for energy-critical wave equation. More precisely, we prove:

Assume $f \in H^3 \cap L^1(\mathbb{R}^3)$. Then

$$\|u(t,x)\|_{L_x^\infty} \le C|t|^{-3/2}.$$

Moreover, $\phi_+ \in H^3 \cap L^1$ such that

$$\lim_{t \to \infty} \|u(t) - e^{it\Delta}\phi_+\|_{H^3} = 0, \quad \|u(t) - e^{it\Delta}\phi_+\|_{L^\infty} \le C|t|^{-3}.$$

Conversely, for any $\phi_+ \in H^3 \cap L^1$, there exists $u \in C_t H^3$ such that

$$\lim_{t \to \infty} \|u(t) - e^{it\Delta}\phi_+\|_{H^3} = 0, \quad \|u(t) - e^{it\Delta}\phi_+\|_{L^\infty} \leq C|t|^{-3}.$$

The main ingredient in the proof is the following result due to Miyachi [3]:

$$\|e^{it\Delta}f\|_{h^1} \leq C(1 + |t|)^{n/2}\|(1 - \Delta)^{n/2}f\|_{h^1},$$

where h^1 is the local Hardy space.

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Quasilinear Maxwell equations

ROBERT SCHIPPA

(joint work with Roland Schnaubelt)

The system of Maxwell equations in media in the absence of currents are given by

$$(1) \quad \begin{cases} \partial_t D & = \nabla \times H, \quad \nabla \cdot D = \rho_e \quad x = (t, x') \in \mathbb{R} \times \mathbb{R}^3, \\ \partial_t B & = -\nabla \times E, \quad \nabla \cdot B = 0, \\ E(0, \cdot) & = E_0, \quad B(0, \cdot) = B_0, \end{cases}$$

where $(E, D) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ are referred to as electric and displacement field and $(B, H) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ as magnetic and magnetizing field.

These equations have to be supplemented with material laws linking, e.g., E with D and H with B. Here we consider the pointwise constitutive relations

$$D(x) = \varepsilon(x)E(x), \qquad \varepsilon : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^{3 \times 3},$$
$$B(x) = \mu(x)H(x), \qquad \mu : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^{3 \times 3}.$$

ε and μ are required to be symmetric and uniformly positive-definite. As an intermediate step to consider interface problems, we consider $\varepsilon \in C^s(\mathbb{R} \times \mathbb{R}^3)$, $\mu \in C^s(\mathbb{R} \times \mathbb{R}^3)$ with $0 < s \leq 2$. If $\varepsilon_{j3} = \varepsilon_{3j} = 0$ for $j \in \{1, 2\}$, if E_0, $B_0 = H_0$, and ρ_e in (1) only depend on $(x, y) \in \mathbb{R}^2$, and if the components E_{03}, H_{01} and H_{02} vanish, then the solutions (E, H) to (1) have the same properties. Hence, the resulting Maxwell system in two spatial dimensions is given by

$$(2) \quad \begin{cases} \partial_t D = \nabla_\perp H, \quad \nabla \cdot D = \rho_e, \\ \partial_t H = -\nabla \times E, \\ D(0, \cdot) = D_0, \quad H(0, \cdot) = H_0. \end{cases}$$

In the above display we have $D, E : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ and $H, \rho_e : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$. In this case $\varepsilon : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^{2\times2}$, $\mu : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$. The relation with wave equations with rough coefficients becomes evident after considering $\partial_t^2 H$:

$$\partial_t^2 H = (\partial_1(\varepsilon_{22}\partial_1) - \partial_2(\varepsilon_{12}\partial_1) - \partial_1(\varepsilon_{21}\partial_2) + \partial_2(\varepsilon_{11}\partial_2))H.$$

(ε_{ij}) denotes the components of ε^{-1}. Tataru proved sharp Strichartz estimates for wave equations with C^s-coefficients in a series of papers (see [3] and references therein) by using the FBI transform, which is defined by

$$T_\lambda f(z) = C_d \lambda^{\frac{3d}{4}} \int_{\mathbb{R}^d} e^{-\frac{\lambda}{2}(y-z)^2} f(y)dy \quad z = x - i\xi \in T^*\mathbb{R}^d.$$

The idea is to conjugate a pseudo-differential operator with rough x-dependence to a multiplication operator in phase space with an error estimate in L_Φ^2. Tataru proved for $a \in C_x^1 C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with $a_\lambda(x,\xi) = a(x,\xi/\lambda)$ precisely

$$\|T_\lambda(a_\lambda(x,D)) - a(x,\xi)T_\lambda\|_{L_\Phi^2} \le C\lambda^{-\frac{1}{2}}.$$

We denote the Maxwell operator in two dimensions by

$$P(x,D) = \begin{pmatrix} \partial_t & 0 & -\partial_2 \\ 0 & \partial_t & \partial_1 \\ -\partial_2(\varepsilon_{11}\cdot) + \partial_1(\varepsilon_{12}\cdot) & \partial_1(\varepsilon_{22}\cdot) - \partial_2(\varepsilon_{21}\cdot) & \partial_t \end{pmatrix}.$$

The principal symbol is given by

$$p(x,\xi) = \begin{pmatrix} i\xi_0 & 0 & -i\xi_2 \\ 0 & i\xi_0 & i\xi_1 \\ -i\xi_2\varepsilon_{11}(x) + i\xi_1\varepsilon_{12}(x) & i\xi_1\varepsilon_{22}(x) - i\xi_2\varepsilon_{12}(x) & i\xi_0 \end{pmatrix}.$$

Diagonalizing the principal symbol yields

$$p(x,\xi) = m(x,\xi)d(x,\xi)m^{-1}(x,\xi)$$

with $d(x,\xi) = i\mathrm{diag}(\xi_0, \xi_0 - \|\xi\|_{\varepsilon'}, \xi_0 + \|\xi\|_{\varepsilon'})$, which means Maxwell equations in two dimensions can be diagonalized to two non-degenerate half-wave equations and one degenerate wave equation. The degeneracy comes from the possibility of stationary solutions due to large charges. The degenerate component is ameliorated in case of small charges as follows from the form of m^{-1}. In [1] we recover the dispersive properties of wave equations with rough coefficients for solutions to Maxwell equations with small charges. We supplement the established Strichartz estimates with examples, which show the derivative loss to be sharp. However, diagonalizing the principal symbol is not enough: We diagonalize the equation with pseudo-differential operators

$$P = \mathcal{M}\mathcal{D}\mathcal{N} + E$$

with $\|E\|_{L^2 \to L^2} \le C$ and $\mathcal{D} = \mathrm{diag}(\partial_t, \partial_t - iD_{\varepsilon'}, \partial_t + iD_{\varepsilon'})$ to prove sharp Strichartz estimates for C^2-coefficients. From the analysis follows well-posedness for quasi-linear Maxwell equations with Kerr nonlinearity

$$\varepsilon = (1 + |E|^2)$$

in $H^s(\mathbb{R}^2)$ for $s > 11/6$. This improves on the energy method, which yields local well-posedness for $s > 2$. In [2] the results are partially extended to the three-dimensional case: For $\varepsilon(x) = \operatorname{diag}(\varepsilon_1(x), \varepsilon_1(x), \varepsilon_2(x))$ and $\mu(x) \equiv 1$ the Strichartz estimates for wave equations with rough coefficients are recovered for solutions to Maxwell equations with small charges. We apply the Strichartz estimates to show local well-posedness for Maxwell equations in three dimensions with Kerr nonlinearity with initial data in $H^s(\mathbb{R}^3)$, $s > 13/6$. This improves on the previous regularity threshold $s > 5/2$ from the energy method.

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Characterising Hardy Spaces

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(joint work with Peng Chen, Ming-Yi Lee, Ji Li, Alessandro Ottazzi)

In the early 1970s, C. Fefferman and Stein wrote a fundamental paper on Hardy spaces, in which they characterised Hardy spaces in different ways. Shortly after, Coifman and Weiss wrote a long paper on the atomic theory of Hardy spaces, including Hardy spaces on "spaces of homogeneous type".

A Hardy space $H^1(\mathbb{H}^n)$ on the Heisenberg group $\mathbb{H}^n$ was developed by Christ and Geller in the 1980s, following earlier work of Folland and Stein. This space may be described in many ways, much like the Hardy space on $\mathbb{R}^n$. The polydisc $\{z \in \mathbb{C}^n : |z_j| < 1\}$ was studied by various authors, including Chang, R. Fefferman, Gundy and Stein. Again, in this context, there is a theory of Hardy spaces. See [4] for many references on the development of Hardy spaces.

Nagel, Stein and others (Ricci, Wainger, . . .) considered boundary behaviour in domains in $\mathbb{C}^n$. The case of $u \in H(D)$, where D is smooth and strictly pseudoconvex is well understood. Such domains are modelled on the Heisenberg group, and "flag geometry" appears. These authors did not develop an appropriate Hardy space theory, but Stein made several conjectures about what this theory might look like. Recently, Han, Lu and Sawyer [2] defined a flag Hardy space, in terms of square functions, and established its interpolation properties.

We have recently developed a complete Hardy space theory on the Heisenberg group. This has several novel features, as we now explain.

Suppose that we wish to use Hardy spaces in a different context: on the Heisenberg group, for instance, or associated to a rough Laplacian. In general contexts, we have fairly good estimates for heat kernels, hence also for Poisson kernels by

subordination. For instance, on the Heisenberg group,

$$
(1) \qquad |D^\alpha p_1(g)| \lesssim \frac{1}{(1+|g|^2)^{(Q+1+|\alpha|)/2}},
$$

where D^α is a left-invariant horizontal differential operator of order $|\alpha|$, and Q is the homogeneous dimension. But we do not have an exact formula in terms of elementary functions. We say that a function that satisfies (1) is Poisson bounded, and we need a theory that works for general Poisson bounded functions. In the classical case, this is more general than the original Fefferman–Stein theory. For maximal functions, Uchiyama [5] has filled the gap, but his arguments do not work in contexts such as square functions. We need to extend both classical and more recent arguments.

The basic geometrical objects associated to the Folland–Stein–Christ–Geller Hardy space are the Korányi balls

$$
B^{(1)}(0,r) := \left\{ (z,t) \in \mathbb{C}^n \times \mathbb{R} : (|z|^4 + t^2)^{1/4} < r \right\},
$$

and their translates. The basic objects in flag geometry are "tubes", which are products of these balls with balls

$$
B^{(2)}(0,s) := \{ t \in \mathbb{R} : |t| < s \}
$$

in the last variable (and their translates).

In classical Hardy space theory, cubes and dyadic decompositions play an important role. In the Heisenberg group, there is a similar theory, due to Strichartz and Tyson, where the basic objects are "tiles", of the form

$$
\{ (z,t) \in \mathbb{C}^n \times \mathbb{R} : z \in Q(w,r), f(z) \le t < f(z) + s \},
$$

where $Q(w,r)$ is a dyadic cube in $\mathbb{C}^n$ of centre w and side r, and f is a continuous *fractal* function. These tiles look like broken wooden sticks. We also stack tiles on top of each other. These tiles and stacks of tiles are used by Han, Lu and Sawyer. We use these objects too, to make it easier to compare our work with theirs, but it would also be possible to use the theory of almost dyadic decompositions due to Christ [1] and to Hytönen and Kairema [3].

Here is a short version of our main theorem.

Theorem 1. *There is a space $H^1_F(\mathbb{H}^n)$ that may be characterised by:*

- *an atomic decomposition;*
- *boundedness of certain singular integrals (Riesz transforms);*
- *boundedness of certain maximal functions;*
- *boundedness of certain square functions;*
- *boundedness of certain Lusin area functions.*

We say that $f \in H^1(\mathbb{H}^n)$ if and only if we may write

$$
(2) \qquad f = \sum_{j \in \mathbb{N}} \lambda_j a_j,
$$

where all a_j are atoms and $\sum_{j \in \mathbb{N}} |\lambda_j| < \infty$. The atomic norm of f is the infimum of all sums $\sum_{j \in \mathbb{N}} |\lambda_j|$ over representations (2) of f. In turn, each atom a is an L^2

sum of *fundamental particles* a_T, of the form $\Delta_{(1)}^M \Delta_{(2)}^N b_T$, where $\Delta_{(1)}$ and $\Delta_{(2)}$ are sub-Laplacians on $\mathbb{H}^n$ and on $\mathbb{R}$, the b_T are in the L^2 domain of these operators, and supported in tubes T, and, for all ± 1-valued functions σ,

$$\left| \bigcup_{T \in \mathfrak{T}} T \right|^{1/2} \left\| \sum_{T \in \mathfrak{T}} \sigma_T a_T \right\|_{L^2(\mathbb{H}^n)} \le 1.$$

Christ and Geller used Riesz transformations R_j on $\mathbb{H}^n$ to characterise their Hardy space. The singular integral version of flag Hardy space is defined by requiring that all transforms $R_j f$ and $R_j H f$ lie in $L^1(\mathbb{H}^n)$, where H is the Hilbert transformation H on $\mathbb{R}$.

Given Poisson bounded functions $\phi^{(1)}$ on $\mathbb{H}^n$ and $\phi^{(2)}$ on $\mathbb{R}$, we define

$$\phi_{r,s} = \phi_r^{(1)} *_{\mathbb{R}} \phi_s^{(2)},$$

$\phi_r^{(1)}$ and $\phi_s^{(2)}$ are normalised dilates of $\phi^{(1)}$ and $\phi^{(2)}$ (here $r, s \in \mathbb{R}^+$). We then consider two-parameter extensions of functions f on $\mathbb{H}^n$ of the form $f * \phi_{r,s}$. We use these functions to define the other possible Hardy spaces, by estimating them over "cones", that is, sets of the form

$$\Gamma(z, t, r, s) := (z, t) \cdot \bar{B}^{(1)}(o, \alpha r) \cdot \bar{B}^{(2)}(0, \beta s).$$

When $\phi^{(1)}$ and $\phi^{(2)}$ have integral 1, we take maximal functions over cones, while when $\phi^{(1)}$ and $\phi^{(2)}$ have integral 0, we define area functions using integrals over cones. There are also square functions, and discrete versions of square functions and area functions. We require that the relevant maximal function, or area function, or square function, is in $L^1(\mathbb{H}^n)$.

As suggested above, all these Hardy spaces coincide. They almost coincide with the space defined by Han, Lu and Sawyer, who required that a discrete square function is in $L^1(\mathbb{H}^n)$; however, these authors considered convolutions of the form $\phi_{r,s} * f$ rather than $f * \phi_{r,s}$, and these convolutions do not fit the Heisenberg group geometry quite as well. But reflection of functions is isomorphism of our space with theirs that allows us to conclude from their work that the complex interpolation space $[H^1, L^2]_\theta$ is an L^p space.

Some parts of the proof of our theorem are routine. For example, to show that if $S(f) \in L^1(\mathbb{H}^n)$, where $S(f)$ is a discrete square function associated to f, then f has an atomic decomposition, a tent space argument works. One just has to be a little careful.

Other parts of the generalisation are tricky. To connect maximal functions and area functions, C. Fefferman and Stein used the Cauchy–Riemann equations in $\mathbb{R}^n \times \mathbb{R}^+$. For product Hardy spaces, Merryfield's lemma is used. Neither of these is are available in our context, and we need new techniques.

The square and area function characterisations are also complicated: some nondegeneracy of ϕ is needed. It would suffice, for instance, to suppose that $\phi = \Delta \psi$, where ψ has integral 1, but this does not describe all ϕ that characterise $H^1(\mathbb{R}^n)$. It is still not entirely clear to us what the right description is in a noncommutative setting, but it seems to be tied up with the Calderón reproducing formula.

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