Mathematisches Forschungsinstitut Oberwolfach

Report No. 46/2021

DOI: 10.4171/OWR/2021/46

Homotopical Algebra and Higher Structures (hybrid meeting)

Organized by
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19 September – 25 September 2021

Abstract. Homotopical algebra and higher category theory play an increasingly important role in pure mathematics, and higher methods have seen tremendous development in the last couple of decades. The talks delivered at the workshop described some of the latest progress in this area and applications to various problems of algebra, geometry, and combinatorics.

Mathematics Subject Classification (2010): 18N65, 18N50, 18M70, 18M30, 18M85, 18E30, 18D50.

Introduction by the Organizers

The workshop *Homotopical algebra and higher structures*, organized by Michael Batanin (Prague), Andrey Lazarev (Lancaster), Muriel Livernet (Paris) and Martin Markl (Prague) was a hybrid workshop with 23 participants attending in person, and a further 27 joining online. It represented a geographically broad selection of pure mathematicians based in Europe, Asia, North and South America, and Australia. Particular care was taken to promote an appropriate gender balance among participants and speakers of the workshop.

There were five talks per day; three in the morning and two in the afternoon, with the exception of Wednesday when, according to a well-established tradition, participants went on a hike in Black Forest in the afternoon. For many participants, this was the first live event after a period of one and a half years of lockdown and online-only events, and there can be no doubt that they felt very happy to be part of a traditional conference, particularly one held in such a special place as Oberwolfach.

One novel feature of the workshop was having a survey talk at the end of each afternoon session. These talks were technically of unlimited duration (although the dinner held at 18:30 provided a de facto deadline), and they were solicited by the organizers from a selection of high-profile experts in the field. Three of these talks were delivered online and one – in person. The latter was especially interesting to watch as there were significantly more questions than is usually the case with a conference talk, and the talk almost morphed into a lively discussion towards the end. Judging by the reaction from participants, this format was popular, and if we were to organize another workshop in Oberwolfach (hopefully in a more traditional format), we would like to repeat this feature.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows".

Worskhop (hybrid meeting): Homotopical Algebra and Higher Structures

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Abstracts

Derived modular operads and metric ribbon graphs

Clemens Berger
(joint work with Ralph M. Kaufmann)

In this partly expository talk we revisit the known combinatorial models of the moduli space $M_{g,n}$ of Riemann surfaces of genus g with n punctures [3, 13] from a modular operad perspective. We relate the space $MRG_{g,n}$ of metric ribbon graphs of type (g, n) – known to be equivalent to $M_{g,n}$ – to some kind of derived modular operad associated with the cyclic associative operad.

This derived modular operad is the modular envelope of the W-construction applied to the cyclic associative operad. The framework [4] of Feynman categories allows us to define in a very explicit manner a W-construction and a modular (resp. surface-modular) envelope for any cyclic (resp. planar-cyclic) operad. These newly defined planar-cyclic and surface-modular operads (called non-Σ-cyclic and non-Σ-modular by Markl [5]) arise by means of a Grothendieck construction for symmetric monoidal set-valued functors out of Feynman categories, cf. [1].

We also relate our model of $MRG_{g,n}$ to Igusa's model [2] (the nerve of a suitably defined ribbon category of type (g, n)) and mention related combinatorial models for Riemann surfaces with boundary components marked by finite sets of points. The latter is done by allowing metric ribbon graphs to have outer flags.

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Calculus of fractions

Denis-Charles Cisinski

The purpose of abstract homotopy theory is to provide category theoretic constructions which are compatible with suitable notions of weak homotopy equivalences. One can revisit the concepts that have lead to Quillen's notion of model category as devices to compute mapping spaces of localizations in terms of Kan extensions, which, in turns provide tools to compute (co)limits in localized infinity-categories as homotopy (co)limits. From there, one can produce a perfect dictionary between (co)complete infinity-categories and their models together with a good theory of derived functors as Kan extensions, revisiting the work of Szumilo, Kapulkin and Mazel-Gee. Reformulating homotopy theory properly as suggested above, using mainly the language of Kan extensions is not only a pleasant way to revisit classical constructions (although that would be good enough), but also a way to internalize homotopy theory in any higher topos (in fact in any directed type theory). This will have applications, for instance, to formulate and prove the universal property of Morel and Voevodsky's motivic homotopy theory (possibly formulated within derived geometry), as well as to study condensed/pyknotic mathematics (e.g. one can see the pro-etale topos of a scheme as a condensed/pyknotic presheaf topos on the the associated Galois category constructed by Barwick, Glasman and Haine).

Deformation cohomology of tensor categories

Alexei Davydov

Possible monoidal structures on a given tensor category naturally form an object of an algebro-geometric nature (the moduli space). The tangent space to the moduli space of tensor structures is computed by the third cohomology of a certain complex, the deformation complex of the tensor category [4, 7]. The tangent cones to this moduli space are controlled by a degree 2-bracket on the deformation cohomology. This graded Lie bracket is compatible with the $\cup$-product on the deformation cohomology making it a 3-algebra.

This is in parallel with the deformation theory of associative algebras, where the role of the tangent cohomology is played by Hochschild cohomology. The tangent cones to the moduli space of associative algebras are given by a degree 2-bracket, the Gerstenhaber bracket [5]. The Gerstenhaber bracket is a Lie bracket and is compatible with the $\cup$-product on the Hochschild cohomology making it a 2-algebra. According to the celebrated Deligne's conjecture this 2-algebra structure lifts to an action of the E_2-operad on the Hochschild complex.

The analogous statement is true for the deformation cohomology of a tensor category - the 3-algebra structure lifts to an action of the E_3-operad on the deformation complex [3]. The proof is in some sense easier than existing proofs of the Delinge's conjecture and is based on the internal structure of the deformation complex of a tensor category. As well as Hochschild complex the deformation complex is a cosimplicial complex. The special feature of the deformation complex is that it is a cosimplicial complex of algebras. This together with a lattice paths model of the E_3-operad provides an explicit E_3-algebra structure on the deformation complex [3, Theorem 2.56].

It follows from a very general prediction of M. Kontsevich on deformations of identity morphisms that the 2-bracket is trivial in characteristic zero [6]. In finite characteristic the 2-bracket is non-trivial [3, Example 4.15]. The case of finite characteristic will be studied systematically in a future work.

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Fibrations of $(\infty, 2)$-categories

Andrea Gagna
(joint work with Yonatan Harpaz, Edoardo Lanari)

The classical work of Grothendieck introduced a fibrational framework to study stacks and more generally pseudo-functors valued in small categories. This amounts to an equivalence of 2-categories

$$\mathbf{PsFun}(A^{\mathrm{op}}, \mathbf{Cat}) \simeq \mathbf{Cart}(A),$$

between **Cat**-valued pseudo-functors with source the opposite of a small category A and Grothendieck (or cartesian) fibrations over A. Lurie [1] generalized this equivalence to the realm of ∞-categories, by showing that (co)cartesian fibrations of simplicial sets over a base ∞-category $\mathcal{A}$ are in complete correspondence with functors $\mathcal{A} \to \mathbf{Cat}_\infty$, from $\mathcal{A}$ to the ∞-category of small ∞-categories, in the cocartesian case, and contravariant such functors in the cartesian case. That is to say, he proves an equivalence

$$\mathbf{Fun}_\infty(\mathcal{A}, \mathbf{Cat}_\infty) \simeq \mathbf{CoCart}_\infty(\mathcal{A}).$$

This Grothendieck–Lurie correspondence plays a key role in higher category theory, as it allows one to handle highly coherent pieces of structure, such as functors, in a relatively accessible manner. They are also crucial to define and develop the formalism of ∞-operads.

Recent progress in derived algebraic geometry and topological quantum field theory requires a solid formalism of fibrations in the $(\infty, 2)$-categorical context in order to perform important constructions. For instance, Gaitsgory and Rozenblyum [2] need such a tool to:

- formally extend the quasi-coherent sheaves ∞-functor from the ∞-category of derived schemes to that of derived prestacks;
- define symmetric monoidal $(\infty, 2)$-categories and prove a Serre duality for Ind-coherent sheaves of derived schemes of finite type;
- prove the existence and the coherences of a six-functor formalism for $\mathcal{D}$-modules.

In a couple of recent articles [3, 4], together with my collaborators we have introduce a plethora of fibrations of $(\infty, 2)$-categories, modelled by maps of scaled simplicial sets satisfying suitable lifting properties. Scaled simplicial sets were introduced by Lurie and serve as a model for $(\infty, 2)$-categories. The advantage of using this model is that the combinatorics needed to deal with most proofs and constructions can sometimes be adapted from the current literature. Having to deal with the intricated combinatorics of simplices instead of that of globes is the main disadvantage, which also prevents us to have a complete theory of dualities/variances on-the-nose.

Fibrations can be thought of as a collection of $(\infty, 2)$-categories indexed by the base $(\infty, 2)$-category, which vary functorially. With this picture in mind, the most general case requires us to deal with fibrations with $(\infty, 2)$-categorical fibres and four possible variances must be encoded: we can reverse all the 1-cells, all the 2-cells, none or both. These correspond to the functorial dependence, *i.e.* covariant/contravariant, of the 1-cells as well as the functorial dependence of the 2-cells (modelled by triangles). By doing so, we also recover in particular the case of $(\infty, 2)$-categories fibred in ∞-categories. These are important because one of the most natural example, namely the fibration $\mathcal{A}_{/x} \to \mathcal{A}$ for x an object of $\mathcal{A}$, is of this kind and corresponds to the ∞-functor $\mathcal{A}^{\mathrm{op}} \to \underline{\mathbf{Cat}}_\infty$ represented by x, where $\underline{\mathbf{Cat}}_\infty$ here is the $(\infty, 2)$-category of small ∞-categories. These fibrations enjoy a list of nice properties and they are stable under composition and base change. More importantly they are equivalent to their enriched counterparts, that is with a natural notion of fibrations for categories strictly enriched in ∞-categories, which has the advantage of having all the variances already built-in.

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Homology of strict ω-categories and the bubble-free conjecture

Leonard Guetta

The object of this talk is to present some aspects of the homotopy theory of *strict ω-categories*. This theory started with the introduction by Street [Str87] of a nerve functor

$$N \colon \mathbf{Str\omega Cat} \to \widehat{\Delta},$$

which generalizes the usual nerve for (small) categories. Then, we can define the homology of a strict ω-category C as the homology of its nerve $N(C)$. In fact, as proved by Gagna [Gag18], if we transfer the (Kan–Quillen) weak equivalences of simplicial sets to strict ω-categories via this nerve functor, then the homotopy theory obtained is equivalent to the usual homotopy theory of spaces. Hence, the homology of ω-categories as defined above is just another way of looking at the usual homology of spaces.

On the other hand, there is a well-studied class of strict ω-categories, called *polygraphs* or *computads*, for which we can define what is usually known as *polygraphic homology*. Intuitively, polygraphs are the analogues of CW-complexes and polygraphic homology is the analogue of cellular homology. Then, in analogy with the topological case, a natural question to ask is whether the polygraphic homology of a polygraph coincide with the homology of its nerve. As it happens, the general answer to this question is *no*. However, there are many examples of "well-behaved" polygraphs for which both homologies do coincide, but trying to figure out exactly what is a well-behaved polygraph seems not an easy task and has been studied in [Gue21].

In the case of 2-dimensional polygraphs, there is a particular class of polygraphs, the *bubble-free polygraphs*, which seems to stand out. These are the 2-dimensional polygraphs that contain no non-trivial 2-cells of the form

$$A \underset{1_A}{\overset{1_A}{\Downarrow}} A,$$

that is to say 2-cells whose source and target are units on a object. (Note that this is required for *all* 2-cells of the polygraph and not only the generating ones). The ultimate goal of this talk is to present the "bubble-free conjecture" which simply asserts that the polygraphic homology of a 2-dimensional polygraph coincide with the homology of its nerve if and only if it is bubble-free.

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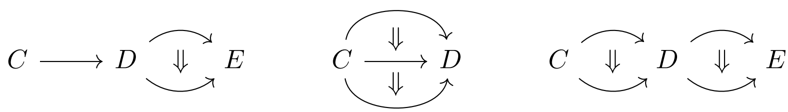
On pasting in $(\infty, 2)$-categories

Philip Hackney

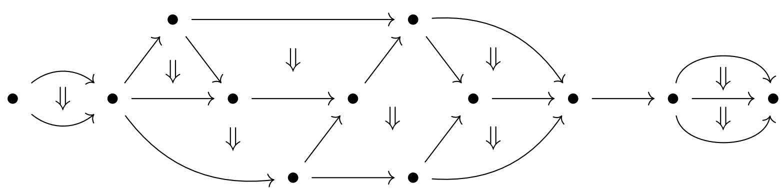
(joint work with Viktoriya Ozornova, Emily Riehl, Martina Rovelli)

Given a category C and a sequence of morphisms $f_1, f_2, \ldots, f_m$ with the target object of f_i equal to the source object of f_{i+1}, there is a unique composite morphism $f_m \circ f_{m-1} \circ \cdots \circ f_1$. If instead C is an $(\infty, 1)$-category, there need not be a unique composite in the usual sense. Instead, there is a *space* parameterizing both the possible compositions $f_m \circ f_{m-1} \circ \cdots \circ f_1$ of the given morphisms, and the relationships among all of these possibilities. Composition in an $(\infty, 1)$-category is *homotopically* unique in the sense that this space is always contractible.

In a 2-category, there are more complicated sorts of compositions possible among collections of 1- and 2-morphisms, which should be familiar to anyone who has worked with natural transformations of functors between categories. For instance, natural transformations between functors can be whiskered along other functors, or composed vertically or horizontally.



One can of course put these basic pictures together into more complicated ones, but the same picture can arise in multiple ways. Power proved in [4] that general pasting in a 2-category is well-defined. *Pasting schemes* are planar directed graphs satisfying certain conditions, such as the following.



A labeling of a pasting scheme by a 2-category X is given by assigning an object to each vertex, a 1-morphism to each directed edge (so that the source and target of the 1-morphism match the labelings of the end vertices), and a 2-morphism to each interior face (whose source and target match the composites of the 1-morphism labelings on the boundaries). Power showed that given such a labeling of a pasting scheme by a 2-category, there is a unique composite 2-morphism in the 2-category.

When working on some problems in $(\infty, 2)$-category theory, we found ourselves reasoning via pasting diagrams, but were surprised to discover that this fundamental result had not yet been established in this context. We sought to rectify this, and proved in [3] that a labeling of a pasting scheme by an $(\infty, 2)$-category X has a homotopically unique composite.

Theorem. *Given a pasting scheme and a labeling of the pasting scheme by an $(\infty, 2)$-category X, the associated space of composites is contractible.*

The labeling in this context is defined in terms of homotopy colimits of the constituent cells, which handles, for instance, the possible ambiguity about 1-morphism composition in the underlying $(\infty,1)$-category of X. The idea of the proof is to show that the free $(\infty,2)$-category on a pasting scheme and the free 2-category on the pasting scheme are equivalent as $(\infty,2)$-categories, and then apply Power's theorem. Establishing this equivalence is not formal, as the functor from 2-categories to $(\infty,2)$-categories is not cocontinuous. Indeed, our proof of the equivalence relies heavily on delicate calculations in a particular model for $(\infty,2)$-categories. We work in the simplicial categories model of Lurie, which has the benefit that many of the horizontal compositions come 'for free' and do not need to be added by hand. The actual computations utilize a new sharpening of a result of Thomason about how the nerve functor interacts with pushouts of Dwyer maps between small categories [5, 4.3]. There is a different, earlier proof of this same equivalence (in the same model) in the unpublished PhD thesis [1] of Tobias Columbus, who contacted us after our initial posting; a revised version of this thesis is now readily available [2].

One of course hopes for a similar theorem for pasting in (∞,n)-categories when $n>2$, though this is a more difficult problem. Indeed, the combinatorial aspects of n-categorical pasting are more subtle, and there are not simple extensions of our methods to higher dimensions.

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Categorical Koszul Duality

Julian Holstein

(joint work with Andrey Lazarev)

The algebraic analogue of the loop space construction of topological spaces is Adams' cobar construction. Together with the bar construction it induces a Koszul duality between algebras and coalgebras, providing an equivalence of suitable homotopy theories of augmented differential graded (dg) algebras and dg conilpotent coalgebras. Interesting things happen as one generalises this result, in particular dropping the augmentation on the dg algebra side corresponds to introducing a curvature term on the coalgebra side, this has been investigated by Positselski.

In this talk I discuss joint work with Andrey Lazarev [1], in which we generalise this situation to a categorical Koszul duality and find a category of coalgebras

Quillen equivalent to differential graded categories (with their usual Dwyer-Kan-Tabuada model structure). To do this we reinterpret categories as monoids in bicomodules over a coalgebra of objects. Then the bar construction associates to any small dg category a pointed curved coalgebra, and dually there is a cobar construction from pointed curved coalgebras to small dg categories.

To work more easily with curved coalgebras we introduce an uncurving functor adjoint to the natural inclusion from dg coalgebras to curved coalgebras.

We also show that there is an equivalence of comodules over a pointed coalgebra and modules over its cobar construction.

The most important example of a pointed curved coalgebra is (a twisted version of) the chain coalgebra of a simplicial set, where we may view the simplicial set either as a space or an $(\infty, 1)$-category.

We finally show that the bar/cobar construction is closely related to the coherent nerve construction from simplicial categories to quasicategories.

The additional structures on chain coalgebras and their comodules provide promising directions for further research.

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Higher mathematics from a topos theoretic perspective

André Joyal

(joint work with Mathieu Anel, Georg Biedermann, Eric Finster)

I am very excited by the beautiful ideas that are coming out of this conference. There are plenty of connections between the ideas and it is tempting to look for some kind of big picture. Higher topos theory has proved to be a unifying force for homotopy theory, higher category theory, higher algebra and the foundation of mathematics. I will sketch it in my talk, although the picture is incomplete and far from definitive.

The fact that category theory can be wholly extended to quasicategories [24, 29, 15], hence also to all $(\infty, 1)$-categories [11, 35], is a kind of mathematical miracle. The two theories are very similar, despite some important differences in higher topos theory and stable homotopy theory. The similarity is very helpful for understanding and discussing the theory of $(\infty, 1)$-categories in general. It is no secret that in private discussion, mathematicians are often dropping the prefix ∞ when refering to ∞-categories and ∞-topoi. An ordinary category can be said to be a 1-*category* if the occasion arise. We may also omit the word "homotopy" when refering to homotopy limits and colimits. All pullback squares can be supposed to be homotopy pullback and all pushouts to be homotopy pushout. An ∞-groupoid can be called a *space* and the $(\infty, 1)$-category of ∞-groupoids called *the category of spaces* $\mathcal{S}$. We may also say that a homotopy equivalence is an *isomorphism*. Any ambiguity can be resolved by reverting temporarily to the traditional terminology.

We are using these conventions in our papers [3, 4]. We are influenced in this by the philosophy of homotopy type theory, according to which the building blocks of mathematics are (homotopy) types [17, 34, 20, 6].

We will argue that the theory of higher topoi

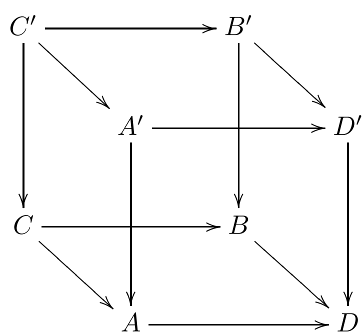
(1) is formally simpler than the theory of 1-topoi [29];
(2) is analogous to the theory of commutative rings [29, 7];
(3) includes stable and unstable homotopy theory [29, 30, 22];
(4) is a natural context for Goodwillie calculus [1];
(5) is foundational for mathematics [42, 9, 39].

I will present a few arguments in support of each of the statements above.

(1) *The theory of higher topoi is formally simpler than the theory of 1-topoi* [29]. A (higher) topos can be defined to be a presentable category satisfying Rezk's descent principle. If $\mathcal{E}$ is a presentable category, consider the contravariant functor $Slice : \mathcal{E} \to \mathbf{CAT}$ defined by putting $Slice(A) := \mathcal{E}/A$ for every object $A \in \mathcal{E}$ and putting $Slice(u) := u^\star : \mathcal{E}/B \to \mathcal{E}/A$ for every map $u : A \to B$. The *descent principle* [37, 29] states that a presentable category $\mathcal{E}$ is a topos if and only if the contravariant functor *Slice* takes colimits to limits. It follows that colimits in $\mathcal{E}$ are *universal* and *effective*. The universality condition means that the base change functor $u^\star : \mathcal{E}/B \to \mathcal{E}/A$ preserves colimits for every map $u : A \to B$ in $\mathcal{E}$. The effectiveness condition means that if $\alpha : D' \to D$ is a cartesian natural transformation between two diagrams $D, D' : I \to \mathcal{E}$, then the following square is cartesian for every $i \in I$.

$$\begin{array}{ccc} D'(i) & \xrightarrow{\mathsf{in}'(i)} & colim(D') \\ {\scriptstyle \alpha(i)}\downarrow & & \downarrow{\scriptstyle colim(\alpha)} \\ D(i) & \xrightarrow{\mathsf{in}(i)} & colim(D) \end{array}$$

In particular, if the top and bottom faces of the following cube in $\mathcal{E}$ are cocartesian, and the vertical faces that contain the arrow $C' \to C$ are cartesian, then the vertical faces that contain the arrow $D' \to D$ are cartesian.



This statement is false in the 1-topos of sets, as we can easily see in the case where $A = B = \{1\}$ and $C = \{1, 2\}$.

If $\mathcal{E}$ is a topos, then so is the category $\mathcal{E}/A$ for every object $A \in \mathcal{E}$. If $u : A \to B$ is a map in $\mathcal{E}$, then the base change functor $u^\star : \mathcal{E}/B \to \mathcal{E}/A$ has both a left adjoint $u_!$ and a right adjoint $u_\star$.

(2) *The theory of higher topoi is analogous to the theory of commutative rings* [29, 7]. Recall that a continuous maps between two topological spaces $f : X \to Y$ induces a pair of adjoint functors $f^\star : Sh(Y) \leftrightarrow Sh(X) : f_\star$ between the category of set-valued sheaves. The inverse image functor $f^\star$ always preserves finite limits. Recall that a *topos morphism* $(\phi^\star, \phi_\star) : \mathcal{F} \to \mathcal{E}$ is defined to be a pair of adjoint functors $\phi^\star : \mathcal{E} \leftrightarrow \mathcal{F} : \phi_\star$, in which the functor $\phi^\star$ preserves finite limits. Notice that the functor $\phi^\star : \mathcal{E} \to \mathcal{F}$ preserves colimits, since it is a left adjoint. An *algebraic morphism of topoi* $\phi^\star : \mathcal{E} \to \mathcal{F}$ is by definition a functor which preserves colimits and finite limits. Every algebraic morphism $\phi^\star : \mathcal{E} \to \mathcal{F}$ has a right adjoint $\phi_\star : \mathcal{F} \to \mathcal{E}$ since every cocontinuous functor between presentable categories has a right adjoint. Hence the notions of topos morphism and of algebraic morphism of topoi are essentially equivalent, *except* that they are running in opposite directions. We shall denote by Topos the category of topoi and topos morphisms, and by Topos^{op} the category of topoi and algebraic morphisms. An object of the category Topos^{op} is called a *logos* in [7]. The categories of topoi Topos and of logoi $\mathsf{Logos} := \mathsf{Topos}^{op}$ are mutually opposite. The duality between topoi and logoi is similar to the duality between affine schemes and commutative rings. A logos $\mathcal{E}$ is the same thing as a topos, except that the category $\mathcal{E}$ should be viewed as a ring-like structure not as a space (a topos). Taking a colimit in $\mathcal{E}$ is a form of addition and taking a finite limit is a form of multiplication. A morphism of logoi $\phi^\star : \mathcal{E} \to \mathcal{F}$ is by definition a functor preserving colimits and finite limits, like a ring homomorphism $f : A \to B$ is a map preserving sums and products. The category of logoi Logos has many properties in common with the category of commutative rings.

algebra	**geometry**
logoi	topoi
frames	locales
commutative rings	affine schemes
Boolean algebras	Stone spaces
commutative $\mathbb{C}^\star$-algebras	compact spaces

Commutative rings	**Logoi**
the ring of integers: $\mathbb{Z}$	the logos of spaces: $\mathcal{S}$
coproduct of rings: $A \otimes B$	coproduct of logoi: $\mathcal{E} \otimes \mathcal{F}$
polynomial ring: $\mathbb{Z}[x]$	polynomial logos: $\mathcal{S}[X]$
ideal: $J \subseteq A$	logos congruence: $W \subseteq \mathcal{E}$
quotient ring $A \to A/J$	lex localization $\mathcal{E} \to Loc(\mathcal{E}, W)$
sum of ideals: $\sum_i J_i$	supremum of congruences: $\bigvee_i W_i$
product of ideals: $J_1 \cdot J_2$	product of congruences: $W_1 \cdot W_2$
division of ideals: $J_1 \backslash J_2$	division of congruences: $W_1 \backslash W_2$

The coproduct of two logoi $\mathcal{E}$ and $\mathcal{F}$ is their tensor product $\mathcal{E}\otimes\mathcal{F}$ in the category of presentable categories, like the coproduct of two commutative rings A and B is their tensor product $A\otimes B$ in the category of abelian groups. The logos $\mathcal{S}[X]$ freely generated by an object X is the category $Fun(\mathsf{Fin}, \mathcal{S})$ of functor $\mathsf{Fin} \to \mathcal{S}$, where Fin is the category of finite spaces and X is the canonical functor $\mathsf{Fin} \to \mathcal{S}$. If $\mathcal{E}$ is a logos, then so is the arrow category $\mathcal{E}^{[1]} = Fun([1], \mathcal{E})$. We shall say that a class of maps $W \subseteq \mathcal{E}$ is a *congruence* if it contains the isomorphisms, if it is closed under composition and the full subcategory of $\mathcal{E}^{[1]}$ spanned by the arrows in W a sub-logos (= it is closed under colimits and finite limits). For example, if $\phi : \mathcal{E} \to \mathcal{F}$ is a morphism of logoi, then the class of maps $W_\phi = \{f \in \mathcal{E} \mid \phi(f) \text{ is invertible}\}$ is a congruence in $\mathcal{E}$. Every congruence W has the 3-for-2 property. Every class of maps $\Sigma \subseteq \mathcal{E}$ generates a congruence $\Sigma^c \subseteq \mathcal{E}$. If $\Sigma \subseteq \mathcal{E}$ is a set of maps, then the full subcategory of Σ^c-local objects in $\mathcal{E}$ is reflective, the reflector $\rho : \mathcal{E} \to Loc(\mathcal{E}, \Sigma^c)$ is a morphism of logoi, $W_\rho = \Sigma^c$ and every morphism of logoi $\phi : \mathcal{E} \to \mathcal{F}$ which inverts the maps in Σ factors uniquely through ρ. The pushout product $f \square g$ of maps in a logos $\mathcal{E}$ is the tensor product of a symmetric monoidal closed structure on $\mathcal{E}^{[1]}$. The *dot product* of two congruences W_1 and W_2 on $\mathcal{E}$ is defined by putting $W_1 \cdot W_2 := (W_1 \square W_2)^c$. If W_1 and W_2 are two congruences, then so is $W_1 \backslash W_2 := \{f \in \mathcal{E} \mid f \square W_1 \subseteq W_2\}$.

(3) *The theory of higher topoi includes stable and unstable homotopy theory* [29, 30, 2, 22]. Classical homotopy theory and stable homotopy theory can be developed in any logos. We shall say that a map $u : A \to B$ is *(-1)-truncated* if it is a monomorphism. A map $u : A \to B$ is (-1)-truncated if and only if the diagonal map $A \to A \times_B A$ is invertible. If $n \geq 0$, a map $u : A \to B$ is said to be n-*truncated* if the diagonal map $A \to A \times_B A$ is $(n-1)$-truncated. An object A is said to be n-*truncated* if the map $A \to 1$ is n-truncated. The full subcategory of n-truncated objects in $\mathcal{E}$ is a $(n+1)$-*logos* denoted $\mathcal{E}^{\leq n}$. A 0-logos is just a frame in the usual sense of the word (dual to a locale) and a 1-logos is just a 1-topos. The inclusion functor $\mathcal{E}^{\leq n} \subseteq \mathcal{E}$ has a left adjoint $\tau_n : \mathcal{E} \to \mathcal{E}^{\leq n}$ which takes an object A to its n-*truncation* $\tau_n(A)$. The sequence

$$\cdots \to \tau_2(A) \to \tau_1(A) \to \tau_0(A) \to \tau_{-1}(A)$$

is the *Postnikov tower* of the object A. An object A is said to be n-*connected* if $\tau_n(A) = 1$. A map $u : A \to B$ is said to be n-*connected* if the object (A, u) of $\mathcal{E}/B$ is n-connected. A (-1)-connected map is a surjection. Every map $u : A \to B$ is the composite of a n-connected map $p : A \to E$ followed by a n-truncated map $v : E \to B$ and this decomposition is unique. An object A is said to be ∞-*connected* if it is n-connected for every $n \geq 0$. Not every ∞-connected objects in a logos is contractible (we shall see examples later). A map $u : A \to B$ is said to be ∞-*connected* if it is n-connected for every $n \geq 0$. A logos $\mathcal{E}$ is said to be *hyper-complete*, or *reduced*, if every ∞-connected map is invertible. The class W_∞ of ∞-connected maps in a logos $\mathcal{E}$ is a congruence and the logos $Loc(\mathcal{E}, W_\infty)$ is reduced.

An object 0 in a category $\mathcal{E}$ is said to be *nul* if it is both initial and terminal. A category with a nul object is said to be *pointed*. The *suspension* ΣX of an object X in a pointed category $\mathcal{E}$ is defined by a pushout square, while its *loop space* ΩX is defined by a pullback square:

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array} \qquad\qquad \begin{array}{ccc} \Omega X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X \end{array}$$

The functor $\Sigma : \mathcal{E} \to \mathcal{E}$ is left adjoint to the functor $\Omega : \mathcal{E} \to \mathcal{E}$. A pointed category $\mathcal{E}$ with finite limits and finite colimits is said to be *stable* [30] if cartesian and cocartesian squares in $\mathcal{E}$ coincide. In which case the functors $\Omega, \Sigma : \mathcal{E} \to \mathcal{E}$ are mutually inverse. The homotopy category $ho(\mathcal{E})$ of a stable category $\mathcal{E}$ is triangulated. We shall denote the category of stable presentable categories and cocontinuous functors by Stable. The category of spectra $\mathcal{S}p$ is stable, presentable and freely generated by one object $\mathbb{S} \in \mathcal{S}p$, the *sphere spectrum*.

If $\mathcal{E}$ is category and $A \in \mathcal{S}$, then a functor $A \to \mathcal{E}$ is a *family* of objects of $\mathcal{E}$ indexed by A. Consider the contravariant functor

$$Fun(-, \mathcal{E}) : \mathcal{S} \to \mathbf{CAT}$$

which takes a space $A \in \mathcal{S}$ to the category $Fun(A, \mathcal{E}) = \mathcal{E}^A$. The *category of families* $\mathsf{Fam}(\mathcal{E})$ is obtained by applying the Grothendieck construction to the functor $Fun(-, \mathcal{E})$

$$\mathsf{Fam}(\mathcal{E}) := \int^{A \in \mathcal{S}} \mathcal{E}^A$$

If $\mathcal{S}p$ is the category of spectra, then $\mathsf{Fam}(\mathcal{S}p)$ is the category of *parametrised spectra.* If the category $\mathcal{E}$ is stable and presentable, then the category $\mathsf{Fam}(\mathcal{E})$ is a logos [22]. In particular, the category of parametrised spectra $\mathsf{Fam}(\mathcal{S}p)$ is a logos (Biedermann, Rezk). A parametrised spectrum $A \to \mathcal{S}p$ is ∞-connected in the logos $\mathsf{Fam}(\mathcal{S}p)$ if and only if $A = 1$. The canonical functor $p : \mathsf{Fam}(\mathcal{E}) \to \mathcal{S}$ is a left exact localisation and $W_p = W_\infty$. Moreover, the functor

$$\mathsf{Fam} : \mathsf{Stable} \to \mathsf{Logos}/\mathcal{S}$$

is fully faithful.

Commutative rings	**Logoi**
ideal: $J \subseteq A$	logos congruence: $W \subseteq \mathcal{E}$
J-adic filration: $A \supseteq J \supseteq J^2 \supseteq J^3 \supseteq \cdots$	W-adic filration: $\mathcal{E} \supseteq W \supseteq W^{\cdot 2} \supseteq W^{\cdot 3} \supseteq \cdots$
J-adic tower: $\ldots \to A/J^3 \to A/J^2 \to A/J$	Goodwillie's tower: $\ldots \to \mathcal{E}/W^{\cdot 3} \to \mathcal{E}/W^{\cdot 2} \to \mathcal{E}/W$
Ring of dual numbers: $\mathbb{Z}[\epsilon]/(\epsilon^2)$	Logos of parametrised spectra: $\mathsf{Fam}(\mathcal{S}p)$
A-modules	$\mathcal{E}$-stacks of stable categories

(4) *The theory of higher topoi is a natural context for Goodwillie functor calculus* [1]. The n-fold *dot power* $W^{\cdot n}$ of a congruence W in a logos $\mathcal{E}$ is defined by induction on $n \geq 1$:

$$W^{\cdot 1} = W \quad \text{and} \quad W^{\cdot(n+1)} = W^{\cdot n} \cdot W$$

This defines a decreasing chain of congruences

$$W \supseteq W^{\cdot 2} \supseteq W^{\cdot 3} \supseteq \cdots$$

and hence a sequence of left exact localizations

$$P_n : \mathcal{E} \to Loc(\mathcal{E}, W^{\cdot(n+1)})$$

for $n \geq 0$. The object $P_n F$ is the n-th *excisive approximation* of F in the sense of Goodwillie. The (generalised) *Goodwillie tower* of an object $F \in \mathcal{E}$ with respect to the congruence W is the sequence

$$\cdots \to P_2 F \to P_1 F \to P_0 F$$

The classical Goodwillie tower was initially defined for finitary functors $\mathcal{S}' \to \mathcal{S}$, where $\mathcal{S}' = 1\backslash\mathcal{S}$ is the category of pointed spaces [12, 8]. Recall that a functor $\mathcal{S}' \to \mathcal{S}$ is *finitary* if it preserves directed colimits. A finitary functor $\mathcal{S}' \to \mathcal{S}$ is the left Kan extension of its restriction to the full category of finite pointed spaces $\mathsf{Fin}' \subset \mathcal{S}'$. Hence the restriction functor $Fun^{\mathsf{fin}}(\mathcal{S}', \mathcal{S}) \to Fun(\mathsf{Fin}', \mathcal{S})$ is an equivalence of categories. But the category $Fun(\mathsf{Fin}', \mathcal{S})$ is the logos $\mathcal{S}[X']$ freely generated by a *pointed* object X' (the object X' is the canonical functor $\mathsf{Fin}' \to \mathcal{S}$). There is a unique morphism of logoi $\epsilon : \mathcal{S}[X'] \to \mathcal{S}$ such that $\epsilon(X') = 1$. We have $\epsilon(F) = F(\star)$ for every $F : \mathsf{Fin}' \to \mathcal{S}$. The classical Goodwillie tower of the functor F is the generalised tower of F with respect to the congruence $W = W_\epsilon$. In this case, P_0F is the constant functor with value $F(\star)$. When $F(\star) = \star$ (the reduced case) the functor $P_1(F)$ can be constructed as the colimit

$$P_1(F)(A) = colim_{n\to\infty}\Omega^n F(\Sigma^n(A))$$

The resulting functor $P_1(F)$ is 1-excisive, which means that it takes every pushout square to a pullback square. It follows that we have

$$P_1(F)(A) = \Omega^\infty(D^1(F) \wedge A)$$

for every pointed space A, for a spectrum $D^1(F)$ called the *first derivative* of the functor F. In the general case (the non-reduced case) the first derivative $D^1(F)$ of the functor $F : \mathsf{Fin}' \to \mathcal{S}$ is a parametrised spectrum, parametrised by the space $F(\star)$. It can be computed as follows. Observe that the sphere spectrum $\mathbb{S}$ is an object of the logos of parametrised spectra $\mathsf{Fam}(\mathcal{S}p)$, since $\mathbb{S} \in \mathcal{S}p \subset \mathsf{Fam}(\mathcal{S}p)$; the object $\mathbb{S}$ is naturally pointed; hence there exists a unique morphism of logoi $D^1 : \mathcal{S}[X'] \to \mathsf{Fam}(\mathcal{S}p)$ such that $D^1(X') = \mathbb{S}$, since the logos $\mathcal{S}[X']$ is freely generated by the pointed object X'. The image of $F \in \mathcal{S}[X']$ by the morphism of logoi $D^1 : \mathcal{S}[X'] \to \mathsf{Fam}(\mathcal{S}p)$ is a parametrised spectrum $D^1(F)$. For every pointed space A we have

$$P_1(F)(A) = \bigsqcup_{x\in F(\star)} \Omega^\infty(D^1(F)(x) \wedge A)$$

where the coproduct is a symbolic notation for the total space of a family of spaces parametrised by the space $F(\star)$. It turns out that

$$Loc(\mathcal{S}[X'], W^{\cdot 2}) = \mathsf{Fam}(\mathcal{S}p).$$

(5) *The theory of higher topoi is foundational for mathematics* [42, 9, 39]. Colimits and finite limits in topoi are by definition preserved by algebraic morphisms of topoi. But there are many other interesting operations in topoi that are not preserved by algebraic morphisms. For example, every topos is cartesian closed but the internal hom $Hom(X, Y) := Y^X$ is generally not preserved by algebraic morphisms. If $\mathcal{E}$ is a topos, then the contravariant functor $Sub : \mathcal{E}^{op} \to Set$ which takes an object X to the set of sub-objects of X is representable by an object $\Omega \in \mathcal{E}$ equipped with a monomorphism $t : 1 \to \Omega$. The object Ω is 0-truncated, since the space $Map(X, \Omega) = Sub(X)$ is a set for every object $X \in \mathcal{E}$. Thus Ω belongs to the 1-topos $\mathcal{E}^{\leq 0}$. It follows that the map $\Omega^{\tau_0 X} \to \Omega^X$ induced by the canonical map $X \to \tau_0 X$ is invertible for every object $X \in \mathcal{E}$. In particular, if

X is connected then $\Omega^X \simeq \Omega$ and Ω^X contains no information about X in this case. If κ is a regular cardinal, let us say that a map $f : X \to B$ is *κ-small* if the object (X, f) of the category $\mathcal{E}/B$ is κ-small. Then the contravariant functor $Slice_\kappa^{core} : \mathcal{E}^{op} \to \mathcal{S}$ which takes an object B to the core of the full subcategory of $\mathcal{E}/B$ spanned by the κ-small objects is representable by an object $\Omega_\kappa \in \mathcal{E}$ equipped with a κ-small map $p_\kappa : \Omega'_\kappa \to \Omega_\kappa$ [Lur1]. The κ-small map p_κ is universal in the following sense: for every κ-small map $f : X \to B$ there exists a unique cartesian square

$$\begin{array}{ccc} X & \longrightarrow & \Omega'_\kappa \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p_\kappa} \\ B & \longrightarrow & \Omega_\kappa \end{array}$$

When the cardinal κ is inaccessible, the pair $(\Omega_\kappa, p_\kappa)$ is a *universe* in Voevodski's sense; the topos $\mathcal{E}$ becomes a model of Voevodsky's type theory [42, 9, 39, 16]. Martin-Lof type theory was initially developped as a constructive foundation of mathematics [17]. A type theory of parametrised spectra was developed in [36].

I am ending my talk with the following questions:

- Q1 Is there a natural notion of (∞, n)-topos for $1 < n \leq \infty$?
- Q2 Is there a Goodwillie calculus for (∞, n)-functors ?
- Q3 What should be the role of higher operads ?
- Q4 What should be the role of factorisation algebra and homology?
- Q5 What should be the role of cobordism?

Comment Q1: See [40, 41] for the notion of cosmoi, see [34, 35] for the notion of ∞-cosmoi and see [43] for the notion of 2-toposes. See [32] for the notion of presentable $(\infty, 2)$-categories and [21] for a higher Grothendieck construction. A type theory for (weak) ω-categories was introduced in [18, 14].

Comment Q2: That is a wild question.

Comment Q3: Higher operads have an important role in higher mathematics since the foundational work of Batanin [10], Baez and Dolan [13], Leinster [26, 27] and Maltsiniotis [33]. Ordinary operads are playing a role in the chain rule for derivatives in Goodwillie's calculus [8].

Comment Q4: See [19] for factorisation algebra.

Comment Q5: See [31] for cobordism.

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Derived Decorated Feynman categories

Ralph M. Kaufmann

(joint work with Clemens Berger)

The naive inclusion of trees into all graphs gives rise to a natural construction of moduli spaces of curves via a formalization using Feynman categories [10]. There are three basic ingredients: the first is the identification of a categories encoding these graphs. The second is a is the theory of decorations, defined in joint work with Jason Lucas [9], which are a version of the Grothendieck construction later refined in [2]. The last ingredient is a W-construction from [10]. Applying this machine results in cubical complexes which we prove can be identified with moduli spaces, Outer space and further cubical complexes arising from mathematical physics. This gives an *ab initio* construction of these intricate geometric objects from combinatorial data. These results have been announced in [8].

1. Feynman categories corresponding to trees in graphs

The categories in question are graphical Feynman categories based on trees and graphs. A *Feynman category* $\mathfrak{F}$ is a special type of monoidal category $\mathcal{F}$ with a choice of basic objects, that is a groupoid $\mathcal{V}$ and a functor $\imath : \mathcal{V} \to \mathcal{F}$, which satisfy three axioms. The first says that the objects and their isomorphisms are freely monoidally generated by the objects of $\mathcal{V}$, the second says that the morphisms (and their isomorphisms in the category of arrows) are freely generated by basic morphisms, where is morphism is basic if its target lies in $\imath(\mathcal{V})$. The third axiom is a technical size condition that guarantees that certain colimits exist. see [10] for further details.

The strong symmetric monoidal functors from the monoidal category to a symmetric monoidal category $\mathcal{C}$ are called $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}} = [\mathcal{F}, \mathcal{C}]_{\otimes}$. Other names would be algebras or representations. This is avoided, because for certain Feynman categories the $\mathcal{F}$-$\mathcal{O}ps_{\mathcal{C}}$ are algebras and representations in $\mathcal{C}$ [8]. The $\mathcal{O}ps$ for graphical Feynman categories are versions and generalizations of operads or PROPs. All other graphical Feynman categories are obtained by restriction or decoration, [10, 9].

There is a *basic graphical Feynman category* $\mathfrak{F}^{\mathfrak{G}}$ whose underlying monoidal category is a full subcategory of the Borisov–Manin [5] category of graphs, whose objects are aggregates of corollas, that is graphs without any edges—tails are allowed. The morphisms in this category are intricate, but have an underlying graph called the ghost graph of the morphism

The Feynman categories of particular interest are subcategories of $\mathfrak{F}^{\mathfrak{G}}$: the wide subcatgory $\mathfrak{F}^{\mathfrak{G}^{ctd}}$ whose basic morphisms have underlying graphs that are connected, and the wide subcategory $\mathfrak{F}^{cyc}$ whose basic morphisms have underlying graphs that are trees. The $\mathfrak{F}^{cyc}$-$\mathcal{O}ps_{\mathcal{C}}$ are cyclic operads in $\mathcal{C}$ whence the name. The $\mathfrak{F}^{\mathfrak{G}^{ctd}}$-$\mathcal{O}ps$ are non–genus graded modular operads introduced in [11].

Morphisms of Feynman categories are pairs of compatible functors (v, f) on the groupoids and monoidal categories. It is proven in [10] that these morphisms induce an adjunction on $\mathcal{O}ps$ via $f_! = Lan_f$ is the pointwise left Kan extension.

$$f_! : [\mathcal{F}, \mathcal{C}]_{\otimes} \leftrightarrows [\mathcal{F}', \mathcal{C}]_{\otimes} : f^*$$

The trivial functor $\mathcal{T}_{\mathcal{F}} \in [\mathcal{F}, \mathcal{C}]_{\otimes}$ is defined by sending all objects to the monoidal unit $\mathcal{T}(X) = 1_{\mathcal{C}}$ and all morphisms to the identity of the monoidal unit. A morphisms of Feynman categories is called *connected* if $f_!(\mathcal{T}_{\mathcal{F}}) = \mathcal{T}_{\mathcal{F}}$.

2. Decorations

Decorations are a Grothendieck construction. The data is a Feynman category $\mathfrak{F}$ and $\mathcal{O} \in [\mathcal{F}, \mathcal{C}]_\otimes$. This yields a new Feynman category $\mathfrak{F}_{\mathrm{dec}\,\mathcal{O}}$ whose underlying category is $el(\mathcal{O})$. The objects are pairs (X, a_x) of an object and a "decoration" $a_x \in \mathcal{O}(X)$. Forgetting the decoration is morphism of Feynman categories $\mathfrak{F}_{\mathrm{dec}\,\mathcal{O}} \to \mathfrak{F}$ called a *covering.*

The relevant functor will be the set valued cyclic operad $CycAss : \mathfrak{F}^{cyc} \to \mathcal{S}et$. The value of $CycAss$ on a corolla with tails S is the set of cyclic orders of S. The morphisms of $\mathfrak{F}^{cyc}_{\mathrm{dec}\,CycAss}$ have underlying graphs of basic morphisms are trees with a cyclic order at each vertex, that is planar trees. The $\mathcal{O}ps$ are precisely planar cyclic operads: $\mathfrak{F}^{cyc}_{\mathrm{dec}\,CycAss} = \mathfrak{F}^{\neg\Sigma cyc}$. Decorations are functorial and behave well with respect to morphisms of Feynman categories and the associated adjoint functors for $\mathcal{O}ps$.

In [2] it is proven that for set-valued $\mathcal{O}ps$ the connected morphisms and coverings form a factorization system. That is every morphisms of Feynman categories factors uniquely (up to isomorphism) extending the factorization

$$f : \mathcal{F} \xrightarrow{i} \mathcal{F}'_{\mathrm{dec}\,f_!(\mathcal{T}))} \xrightarrow{\pi} \mathcal{F}'$$

3. W-Construction

The W–construction of [10] is defined for so–called *cubical* Feynman categories. This is a special condition which basically means that there is a $\mathbb{N}_0$ valued degree function deg on the morphisms of $\mathcal{F}$ that is additive under $\circ$ and $\otimes$. Furthermore the degree 0 morphisms are precisely the isomorphism and that there is a free and transitive Σ_n action on each class of composable sequence of n degree 1 morphisms. Here the classes are taken with respect to isomorphisms on the sequence. This allows to construct a cubical complex by assigning transitions times in $[0, 1]$ to each degree 1 morphism of a chain

$$X \xrightarrow[f_1]{t_1} X_1 \xrightarrow[f_2]{t_2} X_2 \to \cdots \to X_{n-1} \xrightarrow[f_n]{t_n} Y$$

There is a realization functor by taking a colimit which assembles a cubical complex: Let $\mathcal{O} \in [\mathcal{F}, \mathcal{T}op]_\otimes$. For $Y \in ob(\mathcal{F})$ we define $W(\mathcal{O})(Y) := colim_{w(\mathfrak{F},Y)}\mathcal{O} \circ s(-)$ where $w(\mathfrak{F}, Y)$ is a slice type indexing category whose objects are of sequences as above with source X, see [10] for details. This construction generalizes that of Boardman–Vogt [4] and [1].

4. Results (w/ C. Berger)

On the combinatorial level applying the above to the inclusion $i : \mathfrak{F}^{cyc} \to \mathfrak{F}^{\mathfrak{G}^{ctd}}$ one obtains a square with the horizontal arrows being connected and the vertical arrows being covers:

$$\begin{array}{ccc}
\mathfrak{F}^{cyc}_{dec\,CycAss} = \mathfrak{F}^{\neg cyc} & \xrightarrow{i^{CycAss}} & \mathfrak{F}^{mod}_{dec\,i_!(CycAss)} = \mathfrak{F}^{\neg\Sigma\,mod} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\mathfrak{F}^{cyc} & \xrightarrow{i} & \mathfrak{F}^{\mathfrak{G}^{ctd}}_{j_!(\mathcal{T})} = \mathfrak{F}^{mod} \\
& \searrow{\scriptstyle j} & \downarrow{\scriptstyle \pi} \\
& & \mathfrak{F}^{\mathfrak{G}^{ctd}}
\end{array}$$

Where we proved the indicated identifications. The names indicate that the $\mathcal{O}ps$ for $\mathfrak{F}^{\neg\Sigma cyc}, \mathcal{F}^{mod}, \mathcal{F}^{\neg\Sigma mod}$ are planar cyclic operads, modular operads and, respectively, non-Sigma modular operads, as introduced by Markl in [12]. The basic objects of $\mathfrak{F}^{\neq\Sigma mod}$ are decorated corollas $*_{g,s,S_1 \amalg \cdots \amalg S_b}$ where the index is a surface type: genus g, b boundaries marked respectively by $S_1, \dots S_b$ points and s internal punctures.

On the topological level we prove the following results for the moduli space $M_{g,s,S_1 \amalg \cdots \amalg S_b}$ of Riemann surfaces of the given type.

(1) There is an identification

$$W(i_!^{CycAss}(\mathcal{T}))(*_{g,s,S_1 \amalg \cdots \amalg S_b}) = Cone(\bar{M}^{comb}_{g,s,S_1 \amalg \cdots \amalg S_b})$$

The space is the space of metric almost ribbon graphs with the cone point being a one vertex graph without edges corresponding to all edge length being 0 and $M^{comb}_{g,s,S_1 \amalg \cdots \amalg S_b} \subset \bar{M}^{comb}_{g,s,S_1 \amalg \cdots \amalg S_b}$ is the combinatorial compactification of Penner. In the case of emtpy boundary this is also homotopic to the Kontsevich compatification,

(2) There is a strong deformation retraction

$$i_!^{cycAss}(W(\mathcal{T}))(*_{g,s,S_1 \amalg \cdots \amalg S_b}) \simeq M_{g,s,S_1 \amalg \cdots \amalg S_b}$$

For as given topological surface type $(g, s, S_1 \amalg \cdots \amalg S_b)$ is a surface type: genus g. This is a generalization of Igusa's result identifying $M_{g,n}$ with the nerve a category of forest contractions [7].

There is a similar results for the non–decorated case relating to Outer space of [6] and, via restriction, to cubical complexes appearing in physics [3].

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B_∞-structures, monoidal categories and singularity categories

Bernhard Keller

This is a report on the B_∞-structures present on the Hochschild cochain complexes of algebras, differential graded categories and in particular the differential graded enhancements of singularity categories. For simplicity, we work over a field k throughout.

We start by recalling the history of the investigation of the structure of Hochschild cohomology starting from Gerhard Hochschild [14] with subsequent contributions by Cartan–Eilenberg [3], Gerstenhaber [10], Baues [1], Kadeishvili [16] and Getzler–Jones [11], who defined a B_∞-structure as a dg (=differential graded) bialgebra structure on the free cocomplete coalgebra

$$B^+(V) = T^c(\Sigma V)$$

whose augmented graded coalgebra structure is the canonical one, where V is a graded vector space and ΣV its suspension: $(\Sigma V)^p = V^{p+1}$. Such structures are instrumental in (almost?) all positive answers to Deligne's question on the action of the little squares operad on the Hochschild cochain complex. We observe that B_∞-structures are often concomitant with the existence of monoidal structures on suitable triangulated categories. For example, the derived category of A-bimodules becomes monoidal for the derived tensor product over A, concomitant with Getzler–Jones B_∞-structure on the Hochschild cochain complex; the derived category of sheaves of abelian groups on a topological space becomes monoidal for the derived tensor product of sheaves, a structure concomitant with

Baues' B_∞-structure on the cochain complex of a simplicial set X with punctual X_0 and whose geometric realization has trivial 1-skeleton.

We observe that if B is a B_∞-algebra with homologically unital underlying A_∞-algebra, then its derived category $\mathcal{D}B$ of homologically unital A_∞-modules carries a structure of monoidal triangulated category whose unit is the free B-module of rank one. Thus, the perfect derived category of B (the subcategory of compact objects in $\mathcal{D}B$) becomes a *unitally generated* monoidal triangulated category (i.e. a triangulated category with a monoidal structure whose unit is a classical generator of the triangulated category). More precisely, it is a k-linear, monoidal, stable, small ∞-category generated (as a triangulated category) by its unit. We would expect every such category to be of this form. Some evidence for this is provided by Proposition 7.1.2.6 of Lurie's [25], which provides an E_2-structure on the endomorphism spectrum of the unit in the non k-linear setting. Interestingly, giving an E_2-structure (up to homotopy) is equivalent to giving a B_∞-structure (up to homotopy) such that the brace operations $m_{k,l}$ vanish for $k > 1$. Indeed, by definition, the B_∞-algebras satisfying this property are precisely the algebras over the braces operad. Now, as shown in section 3.5 of [35], the braces operated is isomorphic to Kontsevich–Soibelman's [21] 'minimal' operad M and they show in [loc. cit.] that M is quasi-isomorphic to E_2. The idea that (algebraic) monoidal triangulated categories should have B_∞-structures on the derived endomorphism algebras of their unit objects is further supported by the following theorem.

Theorem (Lowen–Van den Bergh [24]). *Let $(\mathcal{A}, \otimes, I)$ be a monoidal k-category such that*

a) *$\mathcal{A}$ is abelian (but $\otimes$ is* not *supposed to be biexact!) and*
b) *$\mathcal{A}$ has enough projectives and $? \otimes P$ is exact for each projective P.*

Then the derived endomorphism algebra $V = \mathrm{REnd}(I)$ carries a B_∞-structure such that the canonical functor from the perfect derived category of V to the thick subcategory of the derived category of $\mathcal{A}$ generated by I becomes monoidal.

Notice that this result is inspired by Reiner Hermann's Ph. D. thesis [13] under the supervision of R.-O. Buchweitz and H. Krause. It applies for example to the category $\mathcal{A}$ of bimodules over an algebra A endowed with the tensor product $\otimes_A$ and the unit object A. In this case, the derived endomorphism algebra is represented by the Hochschild cochain complex with the cup product and Lowen–Van den Bergh show that their construction yields the classical B_∞-structure (up to quasi-isomorphism).

On the other hand, the theorem does not apply to the category $\mathcal{A}$ of sheaves of abelian groups on a topological space X with the canonical tensor product and the constant sheaf $I = \underline{k}_X$ as its unit. Indeed, this category does not have enough projectives in general. The derived endomorphism algebra is represented by the singular cochains on X and we would expect to obtain Baues' B_∞-structure.

The construction of the Hochschild cochain complex with its B_∞-structure generalizes from k-algebras to k-categories (which we view as 'k-algebras with several objects' as in Mitchell's [28])) and further to dg (=differential graded) categories. This extension is of importance because of applications in the deformation theory of abelian and of triangulated categories. It is a classical fact that the center, i.e. the zeroth Hochschild cohomology, of an algebra coincides with the center of its module category, i.e. the endomorphism algebra of its identity functor. The 'derived version' of this fact is due, independently, to Lowen–Van den Bergh [26] and to Toën [31]. It states that there is a canonical algebra isomorphism between the Hochschild cohomology of an algebra and the Hochschild cohomology of its (unbounded) derived category *endowed with its canonical dg enhancement*, i.e. the *dg-derived category* $\mathcal{D}_{dg}A$. Moreover, by [17], this isomorphism lifts to an isomorphism in the homotopy category of B_∞-algebras between the corresponding Hochschild cochain complexes. We see in particular that the center of A identifies with the center of the dg category $\mathcal{D}_{dg}A$. This is a desirable property and one for which it is crucial to take into account the dg structure. Indeed, the computations by Krause–Ye [23] show that the center of the underlying category $\mathcal{D}A$ is a pathological object.

Let now A be a right noetherian algebra. Let $\mathrm{mod}A$ denote the abelian category of finitely generated right A-modules and $\mathcal{D}^b(\mathrm{mod}A)$ its bounded derived category. The *perfect derived category* $\mathrm{per}(A)$ is the thick subcategory generated by the free A-module A_A. The *singularity category of* A is by definition the Verdier quotient

$$\mathrm{sg}(A) = \mathcal{D}^b(\mathrm{mod}A)/\mathrm{per}(A).$$

It first appears in Buchweitz' unpublished manuscript [2] in this algebraic setting and was rediscovered by Orlov [29] in a geometric setting motivated by mirror symmetry. Notice that it vanishes if A is 'smooth', i.e. has finite global dimension. Now assume that $A^e = A \otimes A^{op}$ is also noetherian. Then one defines the *singular Hochschild cohomology* or *Tate Hochschild cohomology* of A to be the Yoneda algebra of the identity bimodule in the singularity category of bimodules:

$$HH^*_{sg}(A) = \mathrm{Ext}^*_{\mathrm{sg}(A^e)}(A, A).$$

It is not hard to show directly that this is a graded commutative algebra although the quotient $\mathrm{sg}(A^e)$ does not carry any obvious monoidal structure. It is a natural question to ask whether nevertheless, singular Hochschild cohomology carries a Gerstenhaber bracket and whether it is the homology of a canonical B_∞-algebra. Both questions are hard but were answered in the affirmative in the recent work of Zhengfang Wang [34]. Thus, we see that there is a complete structural analogy between Tate–Hochschild cohomology and classical Hochschild cohomology. It is therefore natural to ask whether Tate–Hochschild cohomology is not an instance of classical Hochschild cohomology, i.e. whether the Tate–Hochschild cohomology of A is classical Hochschild cohomology of some more complicated object associated with A. Recall that a dg category $\mathcal{A}$ is *smooth* if the identity bimodule

$$I_{\mathcal{A}} : (X, Y) \mapsto \mathcal{A}(X, Y)$$

is perfect in the derived category $\mathcal{D}(\mathcal{A}^e)$ of bimodules. Define the *dg singularity category of A* as the dg quotient

$$\mathrm{sg}_{dg}(A) = \mathcal{D}^b_{dg}(\mathrm{mod}A)/\mathrm{per}_{dg}(A).$$

Theorem ([18]). *There is a canonical morphism of graded algebras*

$$HH^*_{sg}(A) \to HH^*(\mathrm{sg}_{dg}(A)).$$

It is an isomorphism if the dg category $\mathcal{D}^b_{dg}(\mathrm{mod}A)$ is smooth.

According to Theorem A of Elagin–Lunts–Schnürer's [9], the dg-derived category $\mathcal{D}^b_{dg}(\mathrm{mod}A)$ is smooth if A is a finite-dimensional algebra such that $A/\mathrm{rad}(A)$ is separable over k (which is automatic if k is perfect). By Theorem B of [loc. cit.], it also holds if the algebra A is right noetherian and finitely generated over its center and the center is a finitely generated algebra over k.

Conjecture. *The morphism of the theorem lifts to a morphism in the homotopy category of B_∞-algebras.*

Note that this morphism will be an isomorphism if the bounded dg derived category $\mathcal{D}^b_{dg}(\mathrm{mod}A)$ is smooth. In particular, this should hold for each finite-dimensional algebra defined by a quiver with an admissible ideal of relations. The following theorem confirms the conjecture for radical square 0 algebras.

Theorem (Chen–Li–Wang [4]). *The conjecture holds if $A = kQ/(Q_1)^2$, where Q is a finite quiver without sinks or sources and $(Q_1)^2$ the square of the ideal of the path algebra kQ generated by the arrows.*

Although Theorem for the moment only yields a graded algebra isomorphism, it is sufficient to prove useful reconstruction theorems for isolated hypersurface singularities. We refer to [15] for the precise statements and links to the theory of cluster categories.

I thank the organizers for the invitation to this inspiring conference and Vladimir Hinich for interesting discussions and for the reference to Proposition 7.1.2.6 in [25]. I am grateful to Bruno Vallette for the reference to section 3.5 of [35].

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Operadic categories and simplicial 'sets'

Joachim Kock

(joint work with Michael Batanin, Mark Weber)

The talk started with some motivation for the notion of operadic category: just as operads have algebras, operadic categories have operads. It was explained how this works for the operadic category Δ_+ whose operads are nonsymmetric operads (cf. Day and Street), and there was mention also of $\mathbb{F}$, a skeleton of the category of finite sets, whose operads are symmetric operads. Then came an outline of the definition of operadic category by listing the data (but not all the axioms): An operadic category is a small category $\mathcal{C}$ with a chosen terminal object in each connected component, equipped with a cardinality functor $|{-}| : \mathcal{C} \to \mathbb{F}$, and with a fibre structure: for each morphism $f : T \to S$ in $\mathcal{C}$ and for each element $i \in |S|$, there is given an object denoted $f^{-1}(i)$ (not necessarily given by preimage). Some of the axioms and their subtleties were described, pointing out in particular that the notion is not invariant under equivalence of categories.

After brief mention of the alternative formulation of the axioms by Lack in terms of skew monoidal structures, the next topic was the alternative formulation by Garner, Kock, and Weber [3]. One main ingredient here is the observation that the chosen-local-terminals structure amounts precisely to saying that $\mathcal{C}$ is a coalgebra for the (upper) decalage comonad D on ***Cat***. This was explained in some detail since it plays an important role. The comonad D induces a monad $\widetilde{\mathsf{D}}$ on D-***Coalg***, whose algebras are unary operadic categories. To get the multi-aspect, [3] introduced a certain modification of D.

The rest of the talk explained the new material. It is a new interpretation of operadic categories, in which all the axioms end up as simplicial identities. Again the starting point is the D-coalgebra viewpoint, but now the multi-aspect is encoded in a different way, in terms of the symmetric-monoidal-category monad S. After some discussion of the compatibilities between D and S, the first theorem was stated: an operadic category is the same thing as a $\widetilde{\mathsf{D}}$-pseudo-algebra in the Kleisli category for S. Some of the arguments in the proof were explained along the way.

The second main theorem is the simplicial interpretation. Following Garner–Kock–Weber we observe that the unary-operadic-category axioms amount to an *undecking* of the nerve of $\mathcal{C}$. This means a simplicial set X such that $\mathsf{D}X = N\mathcal{C}$. To capture general operadic categories, it is necessary to undeck in the Kleisli category for S. The final theorem states that an operadic category is the same thing as a pair $(\mathcal{C}, X)$ where $\mathcal{C}$ is a small category and X is a pseudo-simplicial groupoid such that $\mathsf{D}X = \mathsf{S}N\mathcal{C}$. In other words, the category of operadic categories ***OpCat*** can be characterised as the pullback

$$\begin{array}{ccc} \boldsymbol{OpCat} & \longrightarrow & \boldsymbol{sGrpd}^{\mathrm{ps}} \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathsf{D}} \\ \boldsymbol{Cat} & \xrightarrow[\mathsf{S}\circ N]{} & \boldsymbol{sGrpd}. \end{array}$$

The operadic-category axioms concerning the local terminal objects are now simplicial identities of the additional top degeneracy maps in the undecking. The axioms concerning the fibre functor are simplicial identities of the additional top face maps in the undecking. The structure and axioms relating to the cardinality functor are not immediately visible, but they are encoded by S. A subtlety regarding the pseudo-ness of the simplicial groupoid X was pointed out: it is concentrated in the top face maps, and it is not just something one can suppress. Actually in the proof of the theorem it is revealed that the coherence isomorphisms of the pseudo-simplicial identities correspond precisely to one of the axioms of the cardinality functor of an operadic category. The coherence isomorphisms are thus an essential part of the structure.

To finish, two benefits of the new characterisation were explained, in addition to the general pleasure of expressing things simplicially: one is that one can now obtain an equivalence-invariant notion of operadic category simply by replacing the equation $\mathsf{D}X = \mathsf{S}N\mathcal{C}$ by an equivalence. One important example of this situation is the two-sided bar construction of a symmetric operad is an operadic category. Second, the pullback-diagram characterisation can be copied to the realm of infinity-categories in the sense of Segal spaces to give a definition of infinity operadic category. (Here simplicial spaces stand in for both simplicial sets and simplicial groupoids, and there is no need to say ‘pseudo’.) The development of such a theory has only just begun. A motivating example is that the category of configuration spaces is (strongly suspected to be) an infinity operadic category.

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Higher algebra of A_∞-algebras and the n-multiplihedra

Thibaut Mazuir

The structure of strong homotopy associative algebra, or equivalently A_∞ -algebra, was introduced in the seminal paper of Stasheff [Sta63]. It provides an operadic model for the notion of differential graded algebra whose product is associative up to homotopy. It is defined as the datum of a set of operations $\{m_m : A^{\otimes m} \to$

$A\}_{m\geq 2}$ of degree $2-m$ on a dg-$\mathbb{Z}$ -module (A,∂), which satisfy the sequence of equations

$$[\partial, m_m] = \sum_{\substack{i_1+i_2+i_3=m\\ 2\leq i_2\leq m-1}} \pm m_{i_1+1+i_3}(\mathrm{id}\, e^{\otimes i_1}\otimes m_{i_2}\otimes \mathrm{id}\, e^{\otimes i_3}).$$

Similarly, the notion of A_∞ -morphism between two A_∞ -algebras A and B offers an operadic model for the notion of morphism of strong homotopy associative algebras which preserves the product up to homotopy. It is defined as the datum of a set of operations $\{f_m : A^{\otimes m}\to B\}_{m\geq 1}$ of degree $1-m$ which satisfy the sequence of equations

$$[\partial, f_m] = \sum_{\substack{i_1+i_2+i_3=m\\ i_2\geq 2}} \pm f_{i_1+1+i_3}(\mathrm{id}\, e^{\otimes i_1}\otimes m_{i_2}\otimes \mathrm{id}\, e^{\otimes i_3}) + \sum_{\substack{i_1+\cdots+i_s=m\\ s\geq 2}} \pm m_s(f_{i_1}\otimes\cdots\otimes f_{i_s})\ .$$

A_∞ -algebras and A_∞ -morphisms between them provide a satisfactory framework for homotopy theory. The most famous instance of this statement is the homotopy transfer theorem. Let (A,∂_A) and (H,∂_H) be two cochain complexes together with a diagram

$$h \circlearrowright (A, d_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)\ ,$$

where i and p are dg-morphisms of degree 0 and $[\partial,h]=\mathrm{id}\, e - ip$. If (A,∂_A) is endowed with an A_∞ -algebra structure, then H can be made into an A_∞ -algebra such that i and p extend to A_∞ -morphisms and h extends to an A_∞ -homotopy (see next paragraph). The first (weaker) version of this theorem dates back to [Kad80] and the previous formulation is due to [Mar06]. See also [Val20] and [LH02] for an extensive study on the homotopy theory of A_∞ -algebras.

An A_∞ -algebra structure on a dg-module A is equivalent to a coderivation D_A on its suspended bar construction $\overline{T}(sA)$ such that $D_A^2=0$. An A_∞ -morphism between two A_∞ -algebras A and B is then equivalent to a morphism of dg-coalgebras $F : (\overline{T}(sA), D_A)\to(\overline{T}(sB), D_B)$. Following [LH02], one can then define an A_∞ -homotopy between two A_∞ -morphisms F and G to be a morphism of dg-coalgebras $H : \mathbf{\Delta}^1\otimes\overline{T}(sA)\longrightarrow\overline{T}(sB)$ where $\mathbf{\Delta}^1$ is a dg-coalgebra model for the interval and H maps the points 0 and 1 to F and G respectively. This defines a satisfactory notion of homotopy, as the relation *being A_∞-homotopic* is an equivalence relation and is stable under composition. This definition can moreover be rephrased in terms of operations, using the universal property of the bar construction. An A_∞ -homotopy between two A_∞ -morphisms $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ of A_∞-algebras A and B corresponds to a collection of maps

$$\begin{aligned}[\partial, h_n] =& g_n - f_n + \sum_{\substack{i_1+i_2+i_3=m\\ i_2\geq 2}} \pm h_{i_1+1+i_3}(\mathrm{id}\, e^{\otimes i_1}\otimes m_{i_2}\otimes \mathrm{id}\, e^{\otimes i_3})\\ &+ \sum_{\substack{i_1+\cdots+i_s+l\\ +j_1+\cdots+j_t=n\\ s+1+t\geq 2}} \pm m_{s+1+t}(f_{i_1}\otimes\cdots\otimes f_{i_s}\otimes h_l\otimes g_{j_1}\otimes\cdots\otimes g_{j_t})\end{aligned}$$

A_∞ -homotopies between A_∞ -algebras being defined, one can now ask what a good notion of a homotopy between homotopies is. And of a homotopy between two homotopies between homotopies. And so on. This problem is solved in [Maz21b]. After introducing the cosimplicial dg-coalgebra $\mathbf{\Delta}^n$ together with the language of overlapping partitions of [MS03], we define a n-morphism between two A_∞ -algebras A and B to be a morphism of dg-coalgebras $F : \mathbf{\Delta}^n \otimes \overline{T}(sA) \longrightarrow \overline{T}(sB)$. These higher morphisms are such that 0-morphisms correspond to A_∞ -morphisms and 1-morphisms correspond to A_∞ -homotopies between A_∞ -morphisms. They have an equivalent definition in terms of operations. A n-morphism from A to B corresponds to a collection of maps $f_I^{(m)} : A^{\otimes m} \longrightarrow B$ of degree $1 - m - \dim(I)$ for $I \subset \Delta^n$ and $m \geq 1$, that satisfy

$$\left[\partial, f_I^{(m)}\right] = \sum_{j=0}^{\dim(I)} (-1)^j f_{\partial_j I}^{(m)} + (-1)^{|I|} \sum_{\substack{i_1+i_2+i_3=m \\ i_2 \geq 2}} \pm f_I^{(i_1+1+i_3)}(\mathrm{id}\, e^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}\, e^{\otimes i_3})$$
$$+ \sum_{\substack{i_1+\cdots+i_s=m \\ I_1\cup\cdots\cup I_s = I s\geq 2}} \pm m_s(f_{I_1}^{(i_1)} \otimes \cdots \otimes f_{I_s}^{(i_s)}).$$

where the last sums runs over all overlapping s-partitions $I_1 \cup \cdots \cup I_s = I$ of the face $I \subset \Delta^n$. The set of higher morphisms between two A_∞ -algebras then defines a simplicial set which has the property of being a Kan complex. This Kan complex is in fact in particular an algebraic ∞-category, which means that all fillers for an inner horn inclusion can be explicitly described. The simplicial homotopy groups of this Kan complex can moreover be conveniently computed. The HOM-simplicial sets $\mathrm{HOM}_{\mathsf{A}_\infty-\mathsf{Alg}}(A, B)_\bullet$ fall however short of defining a natural simplicial enrichment of the category $\mathsf{A}_\infty - \mathsf{Alg}$: the composition of A_∞ -morphisms cannot be naturally lifted to define a composition between $n - A_\infty$-morphisms.

Denote A_∞ for the operad encoding A_∞ -algebra structures and $A_\infty - \mathrm{Morph}$ for the operadic bimodule encoding A_∞-morphisms between A_∞ -algebras. They each can be realized as operadic objects in polytopes : the operad A_∞ stems from the associahedra K_m and the operadic bimodule $A_\infty - \mathrm{Morph}$ stems from the multiplihedra J_m. See [MTTV19] and [MMLA] for instance. In [Maz21b], we construct a family of polytopes encoding the A_∞ -equations for n-morphisms : *the n-multiplihedra $n - J_m$*. They are defined as the polytopes $\Delta^n \times J_m$ endowed with a thinner polytopal subdivision that is obtained by lifting the Alexander-Whitney coproduct AW to level of the polytopes Δ^n. Let M be an oriented closed Riemannian manifold endowed with a Morse function f together with a Morse-Smale metric. Following Abouzaid in [Abo11] who drew from earlier works by Fukaya ([Fuk97] for instance), the Morse cochains $C^*(f)$ can be endowed with a geometric A_∞ -algebra stucture by counting moduli spaces of perturbed Morse gradient ribbon trees. See also [Mes18] and [AL18]. In [Maz21a] we prove that this A_∞ -algebra structure actually stems from an ΩB As -algebra structure. We prove in [Maz21a] and [Maz21b] that given two Morse functions f and g, one can in fact construct n-morphisms between their Morse cochain complexes $C^*(f)$ and $C^*(g)$

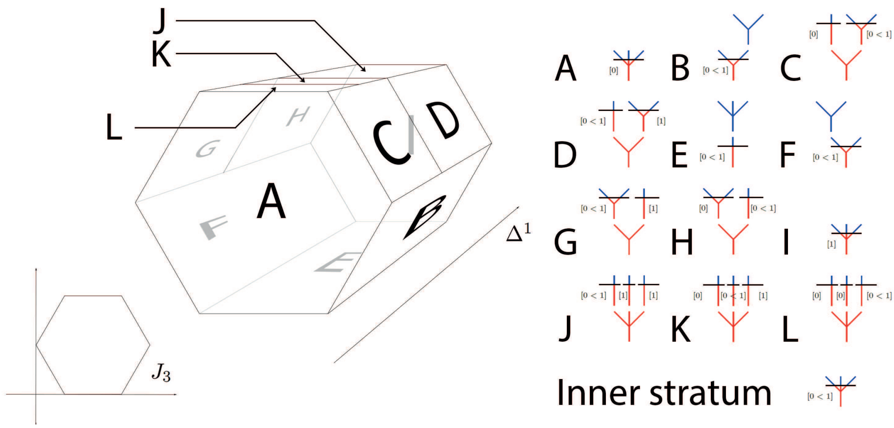

FIGURE 1. *The* 1*-multiplihedron* $\Delta^1 \times J_3$ ...

through a count of geometric moduli spaces of perturbed Morse gradient trees. This gives a realization of this higher algebra of A_∞ -algebras in Morse theory. These constructions stem from the fact that the associahedra can be realized as the compactified moduli spaces of stable metric ribbon trees and the multiplihedra can be realized as the compactified moduli spaces of stable two-colored metric ribbon trees. It is also quite clear that given two compact symplectic manifolds M and N, one should be able to construct n-morphisms between their Fukaya categories $\mathrm{Fuk}(M)$ and $\mathrm{Fuk}(N)$ through counts of moduli spaces of quilted disks (under the correct technical assumptions). In collaboration with Nate Bottman, we may also inspect in the near future possible links between the n-multiplihedra and the 2-associahedra (see [Bot19a] and [Bot19b] for instance), that Bottman introduced in order to define the notion of an $(A_\infty, 2)$-category.

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Gravity properad and moduli spaces

Sergei Merkulov

Let $M_{g,m+n}$ be the moduli space of algebraic curves of genus g with $m+n$ marked points decomposed into the disjoint union of two sets of cardinalities m and n, and $H_c^\bullet(M_{m+n})$ its compactly supported cohomology group. We prove that the collection of $S_m^{op} \times S_n$-modules

$$\left\{H_c^{\bullet-m}(M_{g,m+n})\right\}_{m\geq 1, n\geq 0, 2g+(m+n)\geq 3} =: GRav,$$

has the structure of a properad (called the gravity properad) such that it contains the (degree shifted) E. Getzler's gravity operad [2] as the sub-collection $\{H_c^{\bullet-1}(M_{0,1+n})\}_{n\geq 2}$. The properadic structure in $GRav$ is non-trivial and generates higher genus cohomology classes from lower (even genus zero) ones; we found infinitely many non-trivial examples. Moreover, we prove that the generators of the 1-dimensional cohomology groups $H_c^{\bullet-1}(M_{0,1+2})$, $H_c^{\bullet-2}(M_{0,2+1})$ and $H_c^{\bullet-3}(M_{0,3+0})$ satisfy with respect to this properadic structure the relations of the (degree shifted) quasi-Lie bialgebra, a fact making the totality of cohomology groups

$$\prod_{\substack{g,n\geq 0, m\geq 1\\ p2g+n+m\geq 3}} H_c^\bullet(M_{g,m+n}) \otimes_{S_m^{op}\times S_n} (sgn_m \otimes id_n)$$

into a complex with the differential fully determined by the just mentioned three cohomology classes (which should be understood in this context as a hyperbolic sphere S^2 with (i) one geodesic boundary and two cusps, (ii) two geodesic boundaries and one cusp, and (iii) three geodesic boundaries, respectively). It is proven that this complex contains infinitely many cohomology classes, all coming from a morphism from M. Kontsevich's odd graph complex GC_{-1}.

The prop structure in *GRav* is established with the help of T. Willwacher's [4] twisting endofunctor *tw* (in the category of properads under the operad of Lie algebras) applied to the properad of ribbon graphs $RGra_d$ introduced earlier by T. Willwacher and the author in [3]. As a family of complexes, the twisted properad $twRGra_d = \{twRGra_d(m,n)\}_{m\geq 1, n\geq 0}$ can in turn be identified with the cell complexes of K. Costello's family of moduli spaces [1], $\{D_{g,m,0,n}\}_{m\geq 1, n\geq 0}$ of nodal disks with m marked boundaries and n internal marked points (such that each disk contains at most one internal marked point) making the latter S-bimodule into a dg properad called the chain gravity properad. According to K. Costello [1], $D_{g,m,0,n}$ is homotopy equivalent to $M_{g,m+n}$ so that the properadic structure on the above collection *GRav* of cohomology groups follows immediately from the explicit purely combinatorial properadic structure in $twRGra_d$ and K. Costello's homotopy equivalence.

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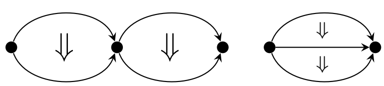
Learning about complicial sets

Viktoriya Ozornova

Many applications in modern algebraic topology and algebraic geometry reveal a need for a higher-categorical framework. Strict higher categories are easy to define inductively: an n-category is defined as a category enriched in $(n-1)$-categories. A prototypical example of such an object is the category of small categories: between any two small categories, it is customary to consider not only a *set* of functors between them, but rather a *category* of functors and natural transformations between those functors. In particular, we observe that natural transformations can be composed in two different 'basic' ways, which can be schematically pictured as follows:



However, the examples of strict n-categories are rare in nature, mostly given by categorical examples as above or archetypical examples encoding certain features as e.g. compositions, which can be intuitively seen as defined by the pictures above. Typical topological applications, as for example cobordism categories, require a weak version of higher categories. A particular choice of such weakening is known as '(∞, n)-categories'. However, this is not a unique mathematical notion; there is rather a variety of ways to make this concept precise, and we refer to these different ways as *models* of (∞, n)-categories. One consequence of weakening the axioms of higher categories is the fact that it turns out to be appropriate to consider homotopy theories of (∞, n)-categories as opposed to mere categories of (∞, n)-categories.

One family of models, often called *cellular*, is based on the idea of encoding sets (or spaces) of compositions for every 'basic' way of composition of higher morphisms. Since the composition is not strictly associative any more, we need to remember also compositions of $3, 4, \ldots$ higher morphisms composed at once. A precise way of encoding these compositions is using an indexing category Θ_n of such compositions, based on pioneering work of Joyal, further studied by Berger and Makkai–Zawadowski, with homotopy theories for cellular models constructed by Rezk [10] and Ara [1].

Cellular models have an advantage of being very intuitive; moreover, it is easy to retrieve compositions of higher morphisms. However, a major disadvantage of cellular models is their growth with n; the combinatorics of the problem get out of hand very quickly. Complicial sets, due to Verity [12], also developed further in [11] and [9], are a member of the family of *simplicial* models. They are based on an easy modification of simplicial sets, and the underlying category does not change with n - the varying value of n is encoded in the model structure. However, it is not immediate whether this approach is equivalent to the cellular one in a precise sense, and the status depends on n. For $n = 1$, the comparison was established by Joyal–Tierney [7] and Verity [12]. For $n = 2$, the missing piece for the comparison was recently established by Gagna–Harpaz–Lanari [6], and the indirect equivalence is also based on work of Bergner–Rezk [4, 5] and Lurie [8]. In a joint work with Bergner and Rovelli, we were able to derive a direct comparison between these models, based on the just cited work as well as results by Barwick–Schommer-Pries [2] :

Theorem ([3]). *There is a direct Quillen equivalence between a model structure on Θ_2-spaces and the model structure for 2-complicial sets.*

In joint work in progress with Rovelli, we made progress towards the comparison for $n > 2$, which is completely open as of now.

90%-Theorem. *Under mild hypothesis, there is a weak equivalence between the nerve of the suspension of an n-category and the suspension of its nerve in the model structure for n-complicial sets.*

The main consequence of this slightly technical result would be as follows.

90%-Corollary. *The same formula as for $n = 2$ defines a left Quillen functor between a model structure on Θ_n-spaces and the model structure for n-complicial sets.*

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Weakly globular double categories and weak units

Simona Paoli

Higher category theory is a rapidly developing field with applications to disparate areas, from homotopy theory, mathematical physics, algebraic geometry to, more recently, logic and computer science.

Higher categories comprise not only objects and morphisms (like in a category) but also higher morphims, which compose and have identities. A key point in higher category theory is the behaviour of these compositions. In a category, composition of morphisms is associative and unital. Higher categories in which these rules for compositions hold for morphisms in all dimensions are called strict higher categories: they are not difficult to formalize, but they are of limited use in applications. A striking example is the case of strict n-groupoids, which are strict n-categories with invertible higher morphisms. These are algebraic models for the building blocks of topological spaces (the n-types) only when $n = 0, 1, 2$, see [6] for a counterexample.

To model n-types for all n (that is, to satisfy the 'homotopy hypothesis'), a more complex class of higher structures is needed, the weak n-categories. In a weak n-category, compositions are associative and unital only up to an invertible cell in the next dimension, in a coherent way.

There are several different models of weak n-categories: a survey was given in [3], and several new approaches appeared later on. Of particular relevance for us are the Segal-type models [4], based on multi-simplicial structures. These comprise the Tamsamani model $\mathsf{Ta^n}$, originally introduced by Tamsamani [8] and further studied by Simpson [7] as well as two new models I introduced in [4]: the weakly globular Tamsamani n-categories $\mathsf{Ta^n_{wg}}$ and the weakly globular n-fold categories $\mathsf{Cat^n_{wg}}$. These models use a new paradigm to encode weakness in a higher category, the notion of weak globularity. The sets of higher morphisms in dimensions $0, \cdots, n$ in $\mathsf{Ta^n}$ are replaced in $\mathsf{Ta^n_{wg}}$ and $\mathsf{Cat^n_{wg}}$ by homotopically discrete structures which are only equivalent of sets. This allows to obtain the model $\mathsf{Cat^n_{wg}}$ of weak n-categories based on the simple structure of n-fold categories. The three Segal-type models $\mathsf{Ta^n}, \mathsf{Ta^n_{wg}}, \mathsf{Cat^n_{wg}}$ are proved in [4] to be equivalent up to homotopy .

A model of higher categories with associative compositions and weak units was proposed by Joachim Kock [2] and called fair n-categories $\mathsf{Fair^n}$. This model is similar in spirit to $\mathsf{Ta^n}$, but with the simplicial category Δ replaced by the 'fat delta' category $\underline{\Delta}$ of coloured finite non-empty semi-ordinals. To date it is not yet known if this model satisfies the homotopy hypothesis, except for the special case of 1-connected 3-types [1]. It was conjectured earlier on by Simpson [6] that there should exist a model of higher categories with associative compositions and weak units that satisfies the homotopy hypothesis and is suitably equivalent to the fully weak models.

Here we concentrate on the case $n = 2$. We construct a pair of functors between $\mathsf{Cat^2_{wg}}$ and $\mathsf{Fair^2}$ and show they induce an equivalence of categories after localization with respect to the 2-equivalences. This equivalence is not surprising, since both models are known to be equivalent to bicategories [5], [2]. The significance and novelty of our result lies in the method of proof: we establish a direct comparison between $\mathsf{Cat^2_{wg}}$ and $\mathsf{Fair^2}$, which does not use their equivalence to bicategories. This direct comparison is very non-trivial, and makes use of several novel ideas and constructions, which we believe will lead to higher dimensional generalizations.

The passage from weakly globular double categories to fair 2-categories uses a property of $\mathsf{Cat^2_{wg}}$ that was not observed so far, namely that it is possible to extract from it a strictly associative (though not strictly unital) composition. It also gives a new meaning to the weak globularity condition of weakly globular double categories as encoding the category of weak units.

The functor in the other direction, from fair 2-categories to weakly globular double categories, factors thorough the category of Segalic pseudo-functors $\mathsf{SegPs}[\Delta^{\mathsf{op}}, \mathsf{Cat}]$ from Δ^{op} to Cat, already introduced in [4]; it also uses novel properties of $\mathsf{Fair^2}$ and of the 'fat delta' category $\underline{\Delta}$.

Finally, the categories $\mathsf{Fair^2}$, $\mathsf{Cat^2_{wg}}$ and $\mathsf{SegPs}[\Delta^{\mathsf{op}}, \mathsf{Cat}]$ are not sufficient to prove the final comparison result. To establish the zig-zags of 2-equivalences giving rise to the equivalence of categories after localization between $\mathsf{Cat^2_{wg}}$ and $\mathsf{Fair^2}$, we need to enlarge the context by introducing two new players: the category of Segalic pseudo-functors $\mathsf{SegPs}[\underline{\Delta}^{\mathsf{op}}, \mathsf{Cat}]$ from from the opposite of the 'fat delta' category to Cat and the category $\mathsf{Fair}^2_{\mathsf{wg}}$ of weakly globular fair 2-categories.

We envisage that the new ideas and techniques of this work will provide a basis for higher dimensional generalizations.

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Polyhedra for $\mathcal{V}_\infty$-algebras, string topology, and moduli spaces

Kate Poirier

(joint work with Thomas Tradler)

Where associahedra are polyhedra that organize operations and relations in an A_∞-algebra, assocoipahedra are polyhedra that organize operations and relations in a $\mathcal{V}_\infty$-algebra, a homotopy version of an associative algebra that has a compatible co-inner product. Assocoipahedra appear in the study of spaces of string topology operations—both on the chains or homology of the loop space of a closed, oriented manifold (the topological side) and on the Hochschild cochains or cohomology of a $\mathcal{V}_\infty$-algebra (the algebraic side). Here, we present five roles that assocoipahedra play on both sides, including a conjecture relating these spaces of operations to the moduli space of Riemann surfaces.

Combinatorics of directed planar trees. An *α-tree* is a directed planar tree such that every interior vertex has at least one outgoing edge and no bivalent vertices have exactly one incoming and one outgoing edge. A familiar example is a planar rooted tree, with edges directed toward the root. An *edge expansion* of α-tree T is an α-tree from which T is obtained by contracting interior edges. Given an α-tree T, its *space of edge expansions* is a convex polyhedron in Euclidean space whose faces correspond to edge expansions of T. When T is a corolla, its space of edge expansions is called an *assocoipahedron.* When T is a corolla with exactly one outgoing edge, the corresponding assocoipahedron is the usual associahedron.

Theorem (P.–Tradler [3])**.** *The space of edge expansions of an α-tree is a decomposition of $K \times \Delta$ where K is an associahedron and Δ is a simplex.*

Koszuality of the $\mathcal{V}^{(d)}$-dioperad. A $\mathcal{V}^{(d)}$*-algebra* is an algebra with an associative product and a symmetric, invariant co-inner product. An example of such an algebra is the singular cohomology of an even-dimensional manifold M; the product is the cup product and the co-inner product is the Thom class of the diagonal. The combinatorics of assocoipahedra are used to prove the following theorem:

Theorem (P.–Tradler [4])**.** *The dioperad governing* $\mathcal{V}^{(d)}$*-algebras is Koszul.*

Further, assocoipahedra describe the operations and relations in the $\mathcal{V}_\infty^{(d)}$-dioperads in the same way that associahedra describe the operations and relations in the A_∞-operad.

The space of directed graphs. A *directed* graph is a directed fatgraph with no bivalent vertices with one incoming and one outgoing edge. An *edge expansion* of a directed graph G is a directed graph from which G is obtained by contracting interior edges without changing the topological type. Tradler–Zeinalian show that a chain complex $\mathcal{DG}_*$—given by directed graphs and their edge expansions—acts on the Hochschild cochains of a $\mathcal{V}_\infty^{(d)}$-algebra (the algebraic side of string topology) [5].

In work in progress with T. Tradler, we use assocoipahedra to build a cell complex $\mathcal{DG}$ whose complex of cellular chains is exactly the chain complex $\mathcal{DG}_*$. The subcomplex of directed graphs with at least one "input" vertex and no directed cycles is denoted $\mathcal{NDG}$. The directed graphs in $\mathcal{NDG}$ may be assembled from disjoint "input" vertices and collections of α-trees attached inductively along their leaves.

Directed graphs and string diagrams. A *short-branched tree* is a metric fatgraph tree subject to a particular length condition. A *string diagram* is a metric fatgraph assembled from disjoint "input" circles and collections of short-branched trees attached inductively along their leaves, in much the same way as graphs in $\mathcal{NDG}$ are constructed. String diagrams form a cell complex $\mathcal{SD}$. In joint work with G.C. Drummond-Cole and N. Rounds, we show that $\mathcal{SD}$ parametrizes operations on the singular chains of the loop space of a closed, oriented manifold and that composition is respected on homology (the topological side of string topology) [2].

In work in progress with T. Tradler, we use assocoipahedra to construct a homotopy equivalence $\mathcal{NDG} \to \mathcal{SD}$.

Assocoipahedra and the moduli space of Riemann surfaces. The metric condition defining short-branched trees is related to an equivalence relation appearing in Bodigheimer's work on the moduli space of Riemann surfaces [1]; the short-branched trees in some sense "interpolate" between equivalent cells in his harmonic compactification of moduli space. In compelling examples, identifications on faces of assocoipahedra create small circles in moduli space near its harmonic boundary, providing evidence for the following conjecture:

Conjecture. *The space of string diagrams* $\mathcal{SD}$ *is homotopy equivalent to the moduli space of Riemann surfaces with boundary.*

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Canonical E_∞-operads involved in homotopy colimits of $\mathcal{I}$-chain complexes

Birgit Richter

Work starting in the late 1960s of Quillen, Sullivan, Bousfield-Gugenheim, Neisendorfer and others enables us to study rational nilpotent spaces of finite type via algebraic models. For instance Quillen developed models in the categories of differential graded cocommutative coalgebras and Lie algebras and Sullivan's differential graded commutative model of the rational cochains on a space allowed for the important concept of minimal models in rational homotopy theory.

Mandell proved in 2006 that two finite type nilpotent spaces are weakly equivalent if and only if their integral singular cochains are quasi-isomorphic as E_∞-algebras. Thus, if you don't want to restrict to *rational* homotopy theory, then you need the full information of the E_∞-structure on the cochains and this is quite an intricate structure.

One can ask whether one can replace the E_∞-algebra of cochains $C^*(X;k)$ on a space X by a strictly commutative model, if k is any commutative ring. Of course this cannot be done in the context of differential graded commutative algebras, because the Steenrod operations for $k = \mathbb{F}_p$ witness that this isn't possible. The existence as a commutative $\mathcal{I}$-chain algebra is guaranteed by [3]. Here, $\mathcal{I}$ is the skeleton of the category of finite sets and injections. In [2] we develop an explicit model $A_*^{\mathcal{I}}(X;k)$ that generalizes Sullivan's model to arbitrary commutative rings k and that detects the homotopy type of nilpotent spaces of finite type.

Can we use $\mathcal{I}$-chains to obtain models of spaces in the setting of differential graded cocommutative coalgebras and Lie-algebras? An obstacle is that the homotopy colimit, that allows us to pass from $\mathcal{I}$-chain complexes to ordinary chain complexes is only lax monoidal, but *not* lax symmetric monoidal or lax symmetric comonoidal. In fact we prove in [2] (modifying a construction from [4, Proposition 6.5] for spaces) that the homotopy colimit sends commutative $\mathcal{I}$-chain algebras to algebras over the Barratt-Eccles operad. So this is one canonical E_∞-operad occurring in this setting.

There is an inclusion of categories $i\colon \Sigma \subset \mathcal{I}$, where Σ is the skeleton of the category of finite sets and bijections. This inclusion and the left Kan extension of symmetric sequences along i already features prominently in the work of Church-Ellenberg-Farb [1].

If Z_* is a symmetric sequence in chain complexes, then the left Kan extension can be explicitly described as

$$i_!(Z_*)(\mathbf{m}) = \mathrm{colim}_{i(\mathbf{n})\downarrow \mathbf{m}} Z_*(\mathbf{n}) \cong \bigoplus_{n\geq 0} k\{\mathcal{I}(\mathbf{n},\mathbf{m})\} \otimes_{k[\Sigma_n]} Z_*(\mathbf{n}).$$

We use a canonical operad in the category of small category as the means to describe the homotopy colimits of $\mathcal{I}$-chains of the form $i_!(Z_*)$. The mth arity of the operad is the category of objects under $\mathbf{m}$, $C(m) := \mathbf{m} \downarrow \mathcal{I}$, with $\mathbf{m} = \{1,\dots,m\}$. We can use the nerve functor, the free module functor and the associated chain complex functor to produce an operad O in chain complexes with $O(m) = C_*(k\{N(\mathbf{m}\downarrow\mathcal{I})\})$. This operad is an E_∞-operad in the category of chain complexes. Note that if one restricts to bijections, then this corresponds to the Barratt-Eccles operad.

We show that for any symmetric sequence in chain complexes Z_* one has

$$\mathsf{hocolim}_{\mathcal{I}} i_! Z_* \cong \bigoplus_{m\geq 0} O(m) \otimes_{\Sigma_m} Z_*(\mathbf{m}).$$

This yields the main result:

Theorem. *For all chain complexes C_* and all operads $(P(m))_{m\geq 0}$ in the category of modules $\mathsf{hocolim}_{\mathcal{I}} i_!(P(F_1^\Sigma(C_*))$ is the free $O\otimes P$-algebra generated by C_*.*

Here, $F_1^\Sigma(C_*)$ denotes the free symmetric sequence on C_* at the object $\mathbf{1} = \{1\}$. In particular, for $P = \mathrm{Lie}$ we get that $\mathsf{hocolim}_{\mathcal{I}} i_!(\mathrm{Lie}(F_1^\Sigma(C_*))$ is a free $O \otimes \mathrm{Lie}$-algebra generated by C_*.

For cocommutative comonoids we obtain:

Theorem. *If Z_* is a cocommutative comonoid in symmetric sequences of chain complexes, then $i_!(Z_*)$ is a cocommutative monoid in $\mathcal{I}$-chain complexes and $\mathsf{hocolim}_{\mathcal{I}} i_!(Z_*)$ is an E_∞ differential graded coalgebra.*

For this structure we use a deconcatenation product on the operad O and this in turn relies on the Alexander-Whitney map.

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A modular operad of seemed surfaces and subgroups of $\widehat{\mathsf{GT}}$

Marcy Robertson
(joint work with Luciana Basualdo Bonatto)

The absolute Galois group of $\mathbb{Q}$, denoted throughout by $Gal(\mathbb{Q})$, is the (topological) group of automorphisms of the separable closure $\bar{\mathbb{Q}}$ over $\mathbb{Q}$ which fix $\mathbb{Q}$. The group $Gal(\mathbb{Q})$ is an example of a *profinite group* which means that it is defined as the inverse limit $\widehat{G} = \lim G/N$ of all of its finite quotients. This means, if we wished to describe an element $g \in Gal(\mathbb{Q})$ we would require a description of the image of g in each of the finite quotients. But we don't know the finite quotients of $Gal(\mathbb{Q})$!

This motivates the study of "Grothendieck-Teichmuller" or "Lego-Teichmuller" theory, laid out in *Esquisse d'un Programme* [Gro97], is to identify each $g \in Gal(\mathbb{Q})$ with a pair

$$(\chi(g), f_g) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2.$$

Here $\chi(g)$ is the *cyclotomic character*, $\chi : Gal(\mathbb{Q}) \to \widehat{\mathbb{Z}}^*$, which gives the action of $Gal(\mathbb{Q})$ on roots of unity and is well-understood. The more difficult part is to find necessary and sufficient conditions for an element of f of the free group $\widehat{F}_2$ to come from a $g \in Gal(\mathbb{Q})$. Since $\widehat{F}_2 = \pi_1(\mathcal{M}_{0,4})$ it is reasonable to conjecture that by studying the geometric actions of $Gal(\mathbb{Q})$ on the fundamental groups of all moduli spaces $\pi_1(\mathcal{M}_{g,n})$ one could gain some insight into which elements $f \in \widehat{F}_2$ come from $g \in Gal(\mathbb{Q})$. This approach has yielded some necessary conditions, but sufficient conditions are still a mystery. The goal of the work described is to introduce a modular operad whose automorphisms correspond to pairs $(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2$ and show that some of these automorphisms correspond to elements $g \in Gal(\mathbb{Q})$.

A modular operad of surfaces. Let $\Sigma_{g,n}$ be a Riemann surface with n-boundaries. We want to assume that boundary has a collar and that the boundaries have a labelling. If we choose a surface $\Sigma_{g,n}$ as a basepoint, loops (up to homotopy) in $\mathcal{M}_{g,n}$ are exactly diffeomorphism classes of $\Sigma_{g,n}$ (up to those homotopic to the identity). That means we can identify the *profinite mapping class group*, denoted $\widehat{\Gamma}_{g,n}$, with the fundamental group of $\pi_1(\mathcal{M}_{g,n})$.

In joint work with Luciana Basualdo Bonatto we are constructing a modular operad of seamed surfaces. We define a family of groupoids $\mathcal{S}_{g,n}$ whose objects will be surface of genus g with n boundaries together with a chosen "atomic" quilt decomposition. A *pants decomposition* of a surface $\Sigma_{g,n}$ is a collection of simple closed curves which cuts the surface $\Sigma_{g,n}$ into surfaces of type $\Sigma_{0,3}$ (pairs of pants). *Quilted pants decompositions*, introduced in [NS00] require that every circle in the pants decomposition of $\Sigma_{g,n}$ has two marked points, called *vertices* and three disjoint lines between the vertices. These "quilts" cut each pair of pants in the decomposition into two hexagonal patches.

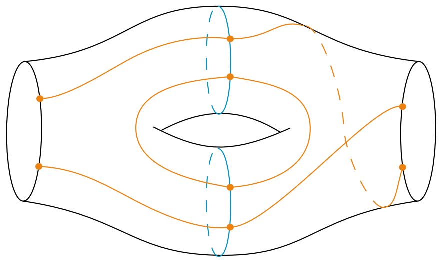

FIGURE 1. An example of a quilted pants decomposition of $\Sigma_{1,2}$

Definition. *Let $\Sigma_{g,n}$ be a fixed surface. We define a groupoid $\mathcal{S}_{g,n}$ whose objects are a chosen quilt decomposition Q/P. Morphisms are homotopy classes of orientation preserving diffeomorphisms preserving the collars and labels of the boundaries (up to isotopy).*

We define composition

$$\mathcal{S}_{g,n} \times_{ij} \mathcal{S}_{h,k} \to \mathcal{S}_{g+h,n+k-2}$$

and contraction operations

$$\mathcal{S}_{g,n} \to \mathcal{S}_{g+1,n-2}$$

on objects by gluing of surfaces and on morphisms as the "combination" of the maps on the subsurfaces.

Theorem. *The collection of groupoids $\mathcal{S} = \{\mathcal{S}_{g,n}\}$ assemble into a modular operad. What's more, if we take the classifying space functor $B\mathcal{S}_{g,n} \simeq B\Gamma_{g,n}$.*

$\widehat{GT}$**.** The profinite Grothendieck-Teichmuller group $\widehat{\mathsf{GT}}$ is an intermediate object between the absolute Galois group and the fundamental groups of the moduli spaces. Even though it is a profinite group, it is easier to understand and has a, by now, well-understood description in terms of automorphisms of operads [Hor17, BdBHR19]. In [NS00] they define a subgroup of the Grothendieck-Teichmuller group which contains the absolute absolute Galois group. Before we can define this group we must introduce some notation. For any homomorphism of profinite groups

$$\widehat{F}_2 \longrightarrow G$$

$$(x, y) \longmapsto (a, b)$$

we write $f(a,b)$ for the image of any $f \in \widehat{F}_2$. For example, given the map $\widehat{F}_2 \to \widehat{F}_2$ which swaps generators x and y we write $f \mapsto f(y,x)$.

Definition. *The group $\Lambda \subseteq \widehat{\mathsf{GT}}$ is the group of pairs*

$$(\lambda, f) \in \widehat{\mathbb{Z}}^* \times \widehat{F}_2'$$

which satisfy the property that

$$x \mapsto x^{\lambda} \quad \textit{and} \quad y \mapsto f^{-1} y^{\lambda} f$$

induce an automorphism of $\widehat{F}_2$ *and :*

(I) $f(x,y)f(y,x) = 1$,

(II) $f(x,y)x^m f(z,x)z^m f(y,z)y^m = 1$ *where* $xyz = 1$ *and* $m = (\lambda - 1)/2$,

(III) $f(x_{34}, x_{45})f(x_{51}, x_{12})f(x_{23}, x_{34})f(x_{45}, x_{51})f(x_{12}, x_{23}) = 1$ *in* $\widehat{\Gamma}_{0,5}$ *where* x_{ij} *is a Dehn twist along a loop surrounding boundaries* i *and* j*.*

(iv) $f(e_1, a_1)a_3^{-8\rho_2} f(a_2^2, a_3^2)(a_3a_2a_3)^{2m} f(e_2, e_1)e_2^{2m} f(e_3, e_2)a_2^{-2m}(a_1a_2a_1)^{2m}$ $f(a_1^2, a_2^2)a_1^{8\rho_2} f(a_3, e_3) = 1$ *where* a_1, a_2, a_3, e_1, e_2 *are the generating Dehn twists in* $\widehat{\Gamma}_{1,2}$*.*

Let $\widehat{\mathcal{S}}$ denote the profinite completion of the modular operad $\mathcal{S}$ as defined in [Hor17]. Let $\mathrm{End}_0(\widehat{\mathcal{S}})$ denote the endomorphisms of the modular operad $\mathcal{S}$ which fix objects. Our main theorem is:

Theorem. *There is an isomorphism of profinite groups:*

$$\mathrm{End}_0(\widehat{\mathcal{S}}) \cong \Lambda.$$

It remains to show that the Galois group $Gal(\mathbb{Q})$ acts non-trivially on the operad $\widehat{\mathcal{S}}$ but we expect this will follow from a similar result in [BdBHR19].

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Deformations, cohomology and homotopy theory of relative Rota-Baxter Lie algebras

Yunhe Sheng

(joint work with Chengming Bai, Li Guo, Andrey Lazarev, Rong Tang)

The concept of Rota-Baxter operators on associative algebras was introduced by G. Baxter [6] in his study of fluctuation theory in probability. A linear map $T : A \to A$ on an associative algebra A is a Rota-Baxter operator if it satisfies

$$T(x)T(y) = T(T(x)y + xT(y))$$

for all $x, y \in A$.

Recently it has found many applications, including Connes-Kreimer's [11] algebraic approach to the renormalization in perturbative quantum field theory. Rota-Baxter operators lead to the splitting of operads [3, 36], and are closely related to noncommutative symmetric functions and Hopf algebras [13, 22, 43]. Recently the relationship between Rota-Baxter operators and double Poisson algebras were studied in [19]. For further details on Rota-Baxter operators, see [20, 21].

In the Lie algebra context, a Rota-Baxter operator was introduced independently in the 1980s as the operator form of the classical Yang-Baxter equation. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. A linear operator $T : \mathfrak{g} \longrightarrow \mathfrak{g}$ is called a **Rota-Baxter operator** if

$$[T(x), T(y)]_{\mathfrak{g}} = T\big([T(x), y]_{\mathfrak{g}} + [x, T(y)]_{\mathfrak{g}}\big)$$

for all $x, y \in \mathfrak{g}$.

Rota-Baxter operators on Lie algebras play important roles in many subfields of mathematics and mathematical physics such as integrable systems [37].

To better understand the classical Yang-Baxter equation and related integrable systems, the more general notion of an $\mathcal{O}$-operator (later also called a relative Rota-Baxter operator or a generalized Rota-Baxter operator) on a Lie algebra was introduced by Kupershmidt [27]; this notion can be traced back to Bordemann [7]. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra, $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$ on a vector space V. A linear map $T : V \to \mathfrak{g}$ is called a **relative Rota-Baxter operator** if

$$[Tu, Tv]_{\mathfrak{g}} = T(\rho(Tu)(v) - \rho(Tv)(u))$$

for all $u, v \in V$.

Relative Rota-Baxter operators provide solutions of the classical Yang-Baxter equation in the semidirect product Lie algebra and give rise to pre-Lie algebras [2].

Deformations. The concept of a formal deformation of an algebraic structure began with the seminal work of Gerstenhaber [17, 18] for associative algebras. Nijenhuis and Richardson extended this study to Lie algebras [34, 35]. More generally, deformation theory for algebras over quadratic operads was developed by Balavoine [4]. There is a well known slogan, often attributed to Deligne, Drinfeld and Kontsevich: *every reasonable deformation theory is controlled by a differential graded (dg) Lie algebra, determined up to quasi-isomorphism.* It is also meaningful to deform *maps* compatible with given algebraic structures. Recently, the deformation theory of morphisms was developed in [8, 15, 16] and the deformation theory of diagrams of algebras was studied in [5, 14] using the minimal model of operads and the method of derived brackets [26, 30, 42]. Sometimes a dg Lie algebra up to quasi-isomorphism controlling a deformation theory manifests itself naturally as an L_∞*-algebra.* This often happens when one tries to deform several algebraic structures as well as a compatibility relation between them, such as diagrams of algebras mentioned above. We will see that this also happens in the study of deformations of a relative Rota-Baxter Lie algebra, which consists of a Lie algebra, its representation and a relative Rota-Baxter operator.

Cohomology theories. A classical approach for studying a mathematical structure is associating invariants to it. Prominent among these are cohomological invariants, or simply cohomology, of various types of algebras. Cohomology controls deformations and extension problems of the corresponding algebraic structures. Cohomology theories of various kinds of algebras have been developed and studied in [10, 17, 24, 25]. More recently these classical constructions have been extended to strong homotopy (or infinity) versions of the algebras, cf. for example [23].

Homotopy invariant construction of Rota-Baxter Lie algebras. Homotopy invariant algebraic structures play a prominent role in modern mathematical physics [40]. Historically, the first such structure was that of an A_∞-algebra introduced by Stasheff in his study of based loop spaces [38]. Relevant later developments include the work of Lada and Stasheff [28, 39] about L_∞-algebras in mathematical physics and the work of Chapoton and Livernet [9] about pre-Lie_∞-algebras. Strong homotopy (or infinity-) versions of a large class of algebraic structures were studied in the context of operads in [29, 32].

Our results. We apply Voronov's higher derived brackets construction [42] to construct the L_∞-algebra that characterizes relative Rota-Baxter Lie algebras as Maurer-Cartan (MC) elements in it. This leads, by a well-known procedure of twisting, to an L_∞-algebra controlling deformations of relative Rota-Baxter Lie algebras. Moreover, we show that this L_∞-algebra is an extension of the dg Lie algebra that controls deformations of LieRep pairs (a LieRep pair consists of a Lie algebra and a representation) given in [1] by the dg Lie algebra that controls deformations of relative Rota-Baxter operators given in [41].

We study the cohomology theory for relative Rota-Baxter Lie algebras. A relative Rota-Baxter Lie algebra consists of a Lie algebra, its representation and an operator on it together with appropriate compatibility conditions. Constructing the corresponding cohomology theory is not straightforward due to the complexity of these data. We solve this problem by constructing a deformation complex for a *relative* Rota-Baxter Lie algebra and endowing it with an L_∞*-structure.* Infinitesimal deformations of relative Rota-Baxter Lie algebras are classified by the second cohomology group. Moreover, we show that there is a long exact sequence of cohomology groups linking the cohomology of LieRep pairs introduced in [1], the cohomology of $\mathcal{O}$-operators introduced in [41] and the cohomology of relative Rota-Baxter Lie algebras. The above general framework has two important special cases: Rota-Baxter Lie algebras and triangular Lie bialgebras and we introduce the corresponding cohomology theories for these objects. We also show that infinitesimal deformations of Rota-Baxter Lie algebras and triangular Lie bialgebras are classified by the corresponding second cohomology groups.

Dotsenko and Khoroshkin studied the homotopy of Rota-Baxter operators on associative algebras in [12], and noted that "in general compact formulas are yet to be found". For Rota-Baxter Lie algebras, one encounters a similarly challenging situation. We use the approach of L_∞-algebras and their MC elements to formulate the notion of a (strong) homotopy version of a relative Rota-Baxter Lie algebra, which consists of an L_∞-algebra, its representation and a homotopy relative Rota-Baxter operator. We show that strict homotopy relative Rota-Baxter operators give rise to pre-Lie$_\infty$-algebras, and conversely the identity map is a strict homotopy relative Rota-Baxter operator on the subadjacent L_∞-algebra of a pre-Lie$_\infty$-algebra. An r_∞-matrix gives rise to a homotopy relative Rota-Baxter operator.

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Higher Lie theory

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(joint work with Daniel Robert-Nicoud)

State of the art. At the center of classical Lie theory lies Lie's third theorem, which tells us how to integrate a finite dimensional real Lie algebra in order to obtain a simply connected real Lie group, providing an equivalence between the respective categories. The Baker–Campbell-Hausdorff (BCH) formula plays a fundamental role, inducing the group structure. We developed the *higher Lie theory*, i.e. the extension of this theory to the derived world of differential graded Lie algebras and — further still — to homotopy Lie algebras.

Replacing classical Lie algebras by their derived versions is mandatory in deformation theory, whose fundamental theorem, due to Pridham [Pri10] and Lurie [Lur11], shows that every deformation problem is controlled by a differential graded Lie algebra in characteristic zero, formalizing a heuristic principle that dates back to Deligne and others, see [Toe17] for a detailed account. In this domain, given an underlying "space" together with a type of structure, one would like to classify all the possible structures present on that space up to some equivalence relations. For example, one can study the classification of associative algebras structures on a chain complex up to isomorphisms [Ger64], the classification of complex structures on Riemannian manifolds up to diffeomorphisms [KS58], or the classification of Poisson structures on manifolds up to diffeomorphisms [Kon03]. In each of these cases, there is a differential graded Lie algebra whose Maurer–Cartan elements are in one-to-one correspondence with structures and whose action of the gauge group, obtained via the BCH formula, models the equivalence relation of interest.

This deformation theoretical information is contained in the *Deligne groupoid*, whose points are the Maurer–Cartan elements of the differential graded Lie algebra and whose morphisms are the gauge equivalences. However, in the world of homotopy theory we are interested in comparing these equivalences, and then the equivalences between equivalences, and so on. This raises the question of finding a Deligne "∞-groupoid" which would faithfully encode this higher data. The picture that comes to mind is that of a topological space with points related by paths subject to homotopies and then homotopies of homotopies, etc. Indeed, according to Grothendieck's homotopy hypothesis, this is exactly the type of object that one should be looking for in order to get a suitable notion of an ∞-groupoid.

The study of the rational homotopy theory of spaces is yet another domain where differential graded Lie algebras play a key role: as shown by Quillen [Qui69] with homotopical methods, they faithfully model the rational homotopy type of connected and simply-connected spaces. Using geometrical methods, Sullivan [Sul77] settled a Eckmann–Hilton or Koszul dual version of rational homotopy theory with quasi-free differential graded commutative algebras as models. Although the notion was not yet defined at the time, such a Sullivan model is equivalent to the structure of a *homotopy Lie algebra* on the dual of its generators. This latter notion, which generalizes that of a differential graded Lie algebra by relaxing relations up to homotopy, has come to play a ubiquitous role in recent years.

In the world of homotopy theory, where one considers objects up to quasi-isomorphisms, the notions of a differential graded Lie algebra and of a homotopy Lie algebra are equivalent. For instance, the Lurie–Pridham theorem can equally be stated using homotopy Lie algebras as it is formulated in terms of an equivalence of ∞-categories. But in the algebraic world, where one works up to isomorphisms, this does not hold true and the notion of a homotopy Lie algebra is mandatory to encode some deformation problems in the way explained above. Homotopy Lie algebras also give rise to a natural notion of "morphisms up to homotopy" — commonly called ∞-morphisms — that also play a very important role in deformation theory, see the deformation quantization of Poisson manifolds by Kontsevich [Kon03]. If one wants to do deformation theory using homotopy Lie algebras, the main issue is to coin the right generalization of the gauge group: what is the nature of the object that integrates homotopy Lie algebras?

Higher Lie theory. Our work lies at the cornerstone of Lie theory, deformation theory, and rational homotopy theory. Hinich [Hin97] and then Getzler [Get09] introduced two deformation ∞-groupoids naturally associated to homotopy Lie algebras, the latter one being denoted by $\gamma_\bullet(\mathfrak{g})$. Since its introduction, Getzler's ∞-groupoid was rarely studied or used directly, contrarily to Hinich's version. The reason for this lies in the complicated form of its intrinsic definition. This is unfortunate since, in the authors' opinion, this object should be central in both deformation theory and rational homotopy theory. Driven by this, the starting goal of our project was to present an alternative definition of Getzler's ∞-groupoid with which it is easier to work and for which we can give explicit formulas. We construct an explicit cosimplicial (complete shifted) homotopy Lie algebra

$$\mathfrak{mc}^\bullet := \widehat{\mathfrak{s}\mathscr{L}_\infty}\,(\mathrm{C}^\bullet)$$

which is freely generated by the normalized chain complex of the geometric simplices. Then an idea going back to Kan [Kan58] tells us how to use it to obtain a couple of adjoint functors

$$\mathfrak{L} \;:\; \mathsf{sSet} \underset{\perp}{\rightleftarrows} \mathfrak{s}\mathscr{L}_\infty\text{-}\mathsf{alg} \;:\; \mathrm{R}\ .$$

Theorem. *Getzler's functor is naturally isomorphic to the above integration functor:*

$$\mathrm{R}(\mathfrak{g}) := \mathrm{Hom}_{\mathfrak{s}\mathscr{L}_\infty\text{-}\mathsf{alg}}(\mathfrak{mc}^\bullet, \mathfrak{g}) \cong \gamma_\bullet(\mathfrak{g})\ .$$

This new description has many advantages. For example, it is known that Hinich's space $\mathrm{MC}_\bullet(\mathfrak{g})$ is functorial with respect to ∞-morphisms, while Getzler's space was known to be functorial with respect to strict morphisms only. With the present new construction, we are able to prove functoriality of R with respect of a refined version of ∞-morphisms. Moreover, this approach automatically gives us a left adjoint functor $\mathfrak{L}$ which provides us with explicit homotopy Lie algebra models for non-necessarily connected nor simply-connected spaces.

Higher Baker–Campbell–Hausdorff products. The Kan property for $\mathrm{R}(\mathfrak{g})$ can be established by providing canonical horn fillers, so that we actually get a *canonical algebraic* ∞*-groupoid*. This latter notion [Nik11] is defined by a structure (the canonical horn fillers) and not by a property (existence); as such, it behaves much better than generic Kan complexes with respect to algebraic properties. This represents a *change of paradigm* which makes Getzler's construction leave homotopy theory and enter the world of algebra. These horns fillers define a whole hierarchy of *higher Baker–Campbell–Hausdorff products* for homotopy Lie algebras.

Rational homotopy theory. Finally, the higher Lie theory admits a salient application in rational homotopy theory: the adjoint functors $\mathfrak{L}$ and R directly relate homotopy Lie algebras and simplicial sets, i.e. spaces, faithfully preserving their rational homotopy type. First, these two functors induce bijections between path-connected points and gauge equivalent Maurer–Cartan elements. Then, they are shown to preserve the composition into connected components. So it remains to study their behaviour on any connected component alone. We introduce a pointed version $\widetilde{\mathfrak{L}} \dashv \widetilde{\mathrm{R}}$ of the aforementioned adjunction.

Theorem. *For $X_\bullet$ a pointed connected finite type simplicial set, the unit*

$$X_\bullet \longrightarrow \widetilde{\mathrm{R}}\widetilde{\mathfrak{L}}(X_\bullet)$$

of the $\widetilde{\mathfrak{L}} \dashv \widetilde{\mathrm{R}}$-adjunction is homotopically equivalent to the $\mathbb{Q}$-completion of Bousfield–Kan. In particular, it a rationalization when $X_\bullet$ is nilpotent.

This simplifies drastically the Lie side of rational homotopy theory: recall that Quillen's original construction, associating a Lie algebra to a space, is done via the composite of many consecutive adjunctions. It also has the great advantage of extending it outside the connected and the simply-connected cases. And it settles a faithful Eckmann–Hilton or Koszul dual to Sullivan's approach: recall that the linear dual of the generating space of a Sullivan model form a homotopy Lie algebra. A similar approach — which was a source of inspiration of our work — was developed by Buijs–Felix–Tanre–Murillo, see e.g. [BFMT20], but with a theory working only for differential graded Lie algebras and without the explicit formulas provided by the operadic calculus.

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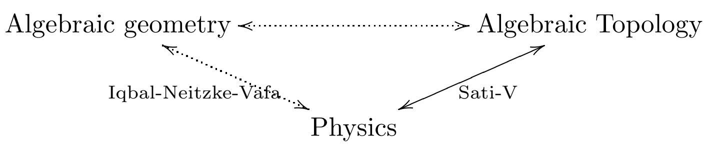
Rational homotopy theory and mysterious duality

Alexander A. Voronov

(joint work with Hisham Sati)

Mysterious duality was discovered by Iqbal, Neitzke, and Vafa [3] in 2001 as a convincing, yet mysterious correspondence between certain symmetry patterns in toroidal compactifications of M-theory and del Pezzo surfaces [2, 4], both governed by the root system series E_k. It turns out that the sequence of del Pezzo surfaces is not the only sequence of objects in mathematics which gives rise to the same E_k symmetry pattern. I will present a sequence of topological spaces, starting with the four-sphere S^4, and then forming its iterated cyclic loop spaces $\mathcal{L}_c^k S^4$, whose rational homotopy theory is subject to the same E_k symmetry properties. For this sequence of spaces, the correspondence between its E_k symmetry pattern and that of toroidal compactifications of M-theory is no longer a mystery, as each space $\mathcal{L}_c^k S^4$ is naturally related to the compactification of M-theory on the k-torus via identification of the equations of motion of $(11-k)$-dimensional supergravity as the defining equations of the Sullivan minimal model of $\mathcal{L}_c^k S^4$, cf. [1]. This gives an explicit duality between rational homotopy theory and physics and uncovers the mystery of Mysterious Duality between mathematics and physics:



Conjecture. *There must be an explicit relation between the series of del Pezzo surfaces* $\mathbb{B}_k$*,* $0 \leq k \leq 8$*, and the series of iterated cyclic loop spaces* $L_c^k S^4$*,* $0 \leq k \leq 8$*. This relation should match the* E_k *symmetry patterns occurring in both series, as well as relate other geometric data, such as the volumes of curves on del*

Pezzo surfaces, with some geometric data, such as the radii of S^4 and S^1s, for the iterated cyclic loop spaces $L_c^k S^4$.

Thus, we shift the mystery of Mysterious Duality into a duality between algebraic geometry and algebraic topology:

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What is a 2-dimensional field theory and how might one construct one?

Nathalie Wahl

Let Cob_2 denote the surface cobordism category: the objects of Cob_2 are the natural numbers, where we think of n as representing a disjoint union of n circles, and morphisms all the topological types of cobordisms between circles. By a $2d$–TFT (2*–dimensional topological field theory*), we mean here a symmetric monoidal functor

$$F\colon Cob_2 \longrightarrow \text{Vect}$$

from Cob_2 to the category of vector spaces.

Theorem (Folklore theorem)**.** *There is an equivalence of categories between the category of* $2d$*-TFT and the category of commutative Frobenius algebras.*

In the statement, a *commutative Frobenius algebra* means a vector space V equipped with a commutative multiplication $\mu : V \otimes V \to V$ and a non-degenerate trace $\tau : V \to k$, where k denotes the ground field. In the stated equivalence, $V = F(1)$ is the value of the $2d$–TFT on the object 1, μ the value of F on the pair of pants, and τ the value of F on a disc. (A proof of this theorem can be found in Kock's book [4].)

Now we can replace Cob_2 by a category $Cob_2^{C_*}$ with the same objects but with morphisms from n to m the chain complex

$$\bigoplus_{\Sigma \in Cob_2(n,m)} C_*(\mathcal{M}(\Sigma))$$

where Σ runs over the morphisms in Cob_2, i.e., the topological types of cobordisms from n to m circles, and $\mathcal{M}(\Sigma)$ denotes the moduli space of Riemann structures on Σ, and where the chains are some appropriate cellular chains on the moduli space. Likewise, one can consider the category $Cob_2^{H_*}$ with the same objects and morphisms now the graded vector spaces

$$\bigoplus_{\Sigma \in Cob_2(n,m)} H_*(\mathcal{M}(\Sigma)).$$

We will call symmetric monoidal functors

$$F\colon Cob_2^{C_*} \longrightarrow \text{Chain}$$

from $Cob_2^{C_*}$ to the dg-category of chain complexes a 2d–TCFT (2–*dimensional topological conformal field theory*), and symmetric monoidal functors

$$F\colon Cob_2^{H_*} \longrightarrow \text{gVect}$$

from $Cob_2^{C_*}$ to the graded linear category of graded vector spaces a 2d–HCFT (2–*dimensional homological conformal field theory*).

In this talk, we considered the following questions:

(1) What is a 2d–TCFT or 2d–HCFT as an algebraic structure? i.e., what could be an analogue of the above theorem in those cases?
(2) Are there non-trivial examples?
(3) Are such field theories useful for anything?

We only gave very partial answers, surveying some of the results and ideas in the papers [1, 2, 3].

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Substitudes, Bousfield localization, higher braided operads, and Baez-Dolan stabilization

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(joint work with Michael Batanin)

In 1995, Baez and Dolan introduced the *stabilization hypothesis*, which loosely states that k-tuply monoidal weak n-categories are the same as $(k+1)$-tuply monoidal weak n-categories as long as $k \geq n+2$ [BD95]. Here *k-tuply monoidal* signifies the additional structure you get on a weak n-category from reindexing from an $(n+k)$-category with one cell in each dimension $< k$. For example, if $\mathcal{C}$ is a 2-category with one object and one morphism, and we reindex two levels, then we obtain a 0-category (i.e., a set) with two commuting operations, corresponding to horizontal and vertical composition in the 2-cells of $\mathcal{C}$. By the Eckmann-Hilton argument, this yields the structure of a commutative monoid. Reindexing three levels, from a 3-category with only one cell in dimensions 0, 1, and 2, does not yield any additional structure on the resulting 0-category.

In [BW15], we sketched a proof of the stabilization hypothesis depending on the homotopy theory of k-operads (which encode k-tuply monoidal structure). We made good on this promise by proving the [BW20b, Theorem 14.2.1]:

Theorem A (Baez-Dolan Stabilization)**.** *Let $0 \leq n$ and $\mathcal{M}$ an n-truncated monoidal combinatorial model category with cofibrant unit. Then $i_! : B_k(\mathcal{M}) \to B_{k+1}(\mathcal{M})$ and $(j_k)_! : B_k(\mathcal{M}) \to E_\infty(\mathcal{M})$ are left Quillen equivalences for $k \geq n+2$.*

Here $\mathcal{M}$ should be thought of as a model category of weak n-categories (e.g., Rezk's model via Θ_n-spaces), $B_k(\mathcal{M})$ is the category of algebras over a k-operad G_k (the cofibrant replacement of the terminal k-operad) encoding k-tuply monoidal weak n-categories, and i and j are comparison functors (based on suspension and symmetrization) between k-operads, $(k+1)$-operads, and symmetric operads, previously constructed by Batanin [Bat10]. To say $\mathcal{M}$ is *n-truncated* means its simplicial mapping spaces are W_n-local (defined below).

Such a result (but requiring a standard system of simplices on $\mathcal{M}$) had previously been proven by Batanin, but we deduce Theorem A from a much stronger result [BW20b, Theorem 14.1.2], where SO is the category of symmetric operads:

Theorem B. *Let $\mathcal{M}$ a combinatorial monoidal model category with cofibrant unit. For $k \geq 3$ and $2 \leq n+1 \leq k$, the symmetrization functor $sym_k : Op_k^{W_n}(\mathcal{M}) \to SO(\mathcal{M})$ and the suspension functor $\Sigma_! : Op_k^{W_n}(\mathcal{M}) \to Op_m^{W_n}(\mathcal{M})$ (for $k < m \leq \infty$) are left Quillen equivalences. Moreover, for $1 \leq n \leq \infty$, the braided symmetrization functor $bsym_2 : Op_2^{W_n}(\mathcal{M}) \to BO(\mathcal{M})$ is a left Quillen equivalence with the category of braided operads.*

Here $Op_k^{W_n}(\mathcal{M})$ denotes the category of *locally constant k-operads*, relative to the localizer W_n that encodes n-types. As developed by Cisinski, a *fundamental localizer* [BW20b, Definition 9.1.1] is a class of functors between small categories that contains all identity functors, satisfies the two out of three property, is closed under retracts, contains functors $A \to 1$ where 1 is the terminal category and A is a category with terminal objects, and such that, if $u/c : A/c \to B/c$ is in W for each object $c \in C$ (where u is a morphism in Cat/C) then u is in W.

The localizer W_n is the smallest localizer containing the unique functor from the $(n+1)$-sphere (viewed as a category) to the terminal category. That minimal fundamental localizers such as W_n exist is a theorem of Cisinski. We recall that a category A is said to be *W-aspherical* if the unique functor from A to 1 is in W.

To study the homotopy theory of $Op_k^{W_n}(\mathcal{M})$, we encode categories of k-operads as algebras over substitudes. A substitude [BW20b, Definition 5.1.1] is equivalent to the data of a colored operad P with a category A of unary operations. We use techniques from [BB17] and [WY18] to transfer model structures from presheaf categories $[A, \mathcal{M}]$ to categories algebras over what we call Σ-free tame unary substitudes with faithful unit, a class that includes categories of k-operads.

We generalize work of Cisinski to prove the existence of left Bousfield localizations $[A, \mathcal{M}]^W$ for any proper fundamental localizer W, with respect to the projective, injective, or Reedy model structure on presheaves [BW20b, Theorem 9.3.5]. In these local model structures, local objects $F : A \to \mathcal{M}$ are *W-locally constant presheaves*, i.e., for any W-aspherical category A', and any functor $u : A' \to A$, the induced functor $u^*(F) : A' \to \mathcal{M}$ is isomorphic to a constant presheaf in $Ho[A', \mathcal{M}]$. The local equivalences are morphisms $u : A \to B$ inducing right Quillen equivalences on categories of locally constant presheaves.

W_∞ is the minimal fundamental localizer making categories with terminal objects W-aspherical. Equivalently, W_∞ is the class of functors whose nerve is a weak equivalence. W_∞-locally constant functors F are those taking all morphisms f in A to weak equivalences. This is analogous to [CW18] where the local objects are the homotopy functors (i.e., those preserving weak equivalences). If $\mathcal{M}$ is n-truncated then $[A, \mathcal{M}]^{W_r} \to [A, \mathcal{M}]^{W_\infty}$ is a Quillen equivalence for all $r \geq n+1$.

To get from $Op_k(\mathcal{M})$ to $Op_k^{W_n}(\mathcal{M})$, we must left Bousfield localize. Unfortunately, categories of algebras over substitudes are often not left proper. To remedy this, we develop a theory of left Bousfield localization that does not require left properness, and results in a semi-model structure. A semi-model category [BW20b, Definition 2.1.1] has three classes of morphisms that satisfy all of the model category axioms except that we only know that trivial cofibrations *with cofibrant domain* lift against fibrations, and that morphisms *with cofibrant domain* admit factorizations into trivial cofibrations followed by fibrations. Because semi-model categories admit cofibrant replacement, and because the subcategory of cofibrant objects behaves exactly like a model category, every result about model categories has a semi-model categorical analogue, and semi-model categories are equally useful in practice. We state our localization theorem [BW20a, Theorem A]:

Theorem C (Bousfield localization without left properness)**.** *Suppose that $\mathcal{M}$ is a combinatorial semi-model category whose generating cofibrations have cofibrant domain, and $\mathcal{C}$ is a set of morphisms of $\mathcal{M}$. Then there is a semi-model structure $L_\mathcal{C}(\mathcal{M})$ on $\mathcal{M}$, whose weak equivalences are the $\mathcal{C}$-local equivalences, whose cofibrations are the same as $\mathcal{M}$, and whose fibrant objects are the $\mathcal{C}$-local objects. Furthermore, $L_\mathcal{C}(\mathcal{M})$ satisfies the universal property that, for any any left Quillen functor of semi-model categories $F : \mathcal{M} \to \mathcal{N}$ taking $\mathcal{C}$ into the weak equivalences of $\mathcal{N}$, then F is a left Quillen functor when viewed as $F : L_\mathcal{C}(\mathcal{M}) \to \mathcal{N}$.*

This theorem is of independent interest for a host of applications, detailed in [BW20a], as lack of left properness has bedeviled researchers seeking to left Bousfield localize for years. Examples in [BW20a], show that sometimes the classes of morphisms above do not satisfy the model category axioms, so only a semi-model structure is possible.

There are two ways to get from $[A, \mathcal{M}]$ to $Op_k^{W_n}(\mathcal{M})$. One can either localize first, then lift the resulting model structure (as in [Whi17, Whi21]), or one can lift first (using the transfer theorem) and then attempt to localize. As proven in [BW21, Theorem 5.6], these two approaches are equivalent (when both work).

In addition to these localization results, to prove Theorem B we develop a theory of homotopical Beck-Chevalley squares [BW20b, Theorem 4.2.2] to lift Quillen equivalences of presheaf categories to Quillen equivalences of algebras over substitudes. This vastly generalizing previous work on such problems (e.g., [WY19]).

Locally constant k-operads are a model for higher braided operads [Bat10], and the category of unary operations Q_k^{op} has $Q_k \cong \coprod Q_k(m)$ such that the nerve of $Q_k(m)$ is homotopy equivalent to the unordered configuration space of points in $\mathbb{R}^k$. An analysis of this homotopy type [BW20b, Theorem 11.1.7] is the last ingredient in the proof of Theorems A and B, and the reason for the inequalities featuring k and n. We also lift various equivalences of homotopy categories in this setting (known since [Bat10]) to Quillen equivalences [BW20b, Proposition 12.2.1]. Another consequence of Theorem B is:

Theorem D (Stabilization for Higher Braided Operads)**.** *If $\mathcal{M}$ is a n-truncated, combinatorial, monoidal model category with cofibrant unit, and $n \geq 0$ and $3 \leq n+2 \leq k \leq \infty$, then the symmetrization functor $sym_k : Op_k^{W_\infty}(\mathcal{M}) \to SO(\mathcal{M})$ and the suspension functor $\Sigma_! : Op_k^{W_\infty}(\mathcal{M}) \to Op_m^{W_\infty}(\mathcal{M})$ (for $k < m \leq \infty$) are left Quillen equivalences. Moreover, for $1 \leq n \leq \infty$, $bsym_2 : Op_2^{W_\infty}(\mathcal{M}) \to BO(\mathcal{M})$ is a left Quillen equivalence.*

Finally, we obtain a stabilization result for $(n+m, n)$-categories, rather than just weak n-categories, stated below for Rezk's model of $(n+m, n)$-categories (where Sp_m models m-types), as a consequence of the stronger results listed above:

Theorem E. *The suspension functor induces a left Quillen equivalence*

$$i_! : B_k(\Theta_n Sp_m) \to B_{k+1}(\Theta_n Sp_m)$$

for $k \geq m + n + 2$ and, hence, an equivalence between homotopy categories of Rezk's k-tuply monoidal $(n + m, n)$-categories and Rezk's $(k + 1)$-tuply monoidal $(n + m, n)$-categories.

Our methods are general enough to apply to other definitions of higher categories, including Tamsamani, Segal, and n-quasi-categories.

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