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MFO-RIMS Tandem Workshop: Symmetries on Polynomial Ideals and Varieties (hybrid meeting)

Organized by
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Abstract. The study of symmetry as a structural property of algebraic objects is one of the fundamental pillows of the developments of modern mathematics, most prominently beginning with the work of Abel and Galois. The focus of the workshop was on permutation actions of the symmetric group on polynomial rings and algebraic and semi-algebraic sets. More concretely, it was centered around recent developments in the asymptotic setup of symmetric ideals in the polynomial ring in infinitely many variables.

Mathematics Subject Classification (2010): 13A99, 13F20, 13P10, 20B30.

Introduction by the Organizers

The MFO-RIMS Tandem Workshop *Symmetries on Polynomial Ideals and Varieties* organised by Gunnar Fløystad (Bergen), Satoshi Murai (Tokyo), Cordian Riener (Tromsø), and Kohji Yanagawa (Osaka) took place between September 5–11 2021 as a joint hybrid workshop hosted at the Mathematical Research Institute Oberwolfach and the Research Institute for Mathematical Sciences in Kyoto. Unfortunately, the ongoing Covid-19 pandemic made it impossible to have a physical meeting in Kyoto and the workshop was therefore attended by 14 participants physically at MFO and another 24 participants who joint in virtually.

The main focus of the workshop was on the study of properties of ideals in the infinite polynomial ring which are invariant with respect to actions of the symmetric group. As has been observed by various authors, many properties of classical commutative algebra, which fail for the infinite polynomial ring, can be restored when considered in an equivariant setup, up to symmetry. Most notably, it had been already observed by Cohn in the 1980s that although the infinite polynomial ring is not Noetherian symmetric, ideals can always be generated by finitely many orbits of polynomials. In recent years renewed motivation to study such ideals stemmed from the observation that one can view such ideals as limits of sequences of algebraic objects which arise in the area of algebraic statistics and algebraic chemistry.

One main goal of the joint workshop between the two research institutes was to bring together different mathematicians working on various aspects of this thematic area and, in particular, build new research connections between Europe and Asia around this topic. The technical facilities in the lecture halls allowed recording of the talks and interactions between remote participants. We would say this worked well, with some minor technical problems. On the Japanese side they had the disadvantage that each participant was in their home or office, so all interaction was digital. Nevertheless, questions after talks, and problem discussions worked well interactively. Also, the time difference between the two groups of participants made it necessary to distinguish between talks, which could followed by all participants live and talks which were recorded because their time slot was falling into very early hours in Europe or very late hours at night in Japan. But all in all, the hybrid setup allowed for a rich program of diverse talks, joint problem sessions as well as discussions.

The workshop had talks on the following topics:

- Stability of inc- and sym- chains of ideals.
- Hilbert functions and free resolutions of chains of ideals.
- SAGBI basis approach to invariant rings
- Polynomial representations.
- Specht ideals.
- Equivariant modules and sheaves.
- Cohomology of symmetric semi-algebraic sets.

The presentations featured both recent developments on these topics as well introductory expositions. On Monday we had a shorter problem session with spontaneous volunteers. For Wednesday we took a more active approach. On Tuesday we requested and approached participants to prepare and inform on problems. The whole Wednesday morning we then had a planned session of problem presentations. First, two hours from the Oberwolfach side, and then one hour from the RIMS side. This worked very well. Both more systematic programs of problems were presented, for instance on asymptotic sequences of symmetric ideals, as well as more concrete specific problems. The workshop was on a timely topic with recent surging interest. We believe it was very inspiring and informative to both the Oberwolfach and RIMS participants. From the Oberwolfach side it was especially useful to be informed on the diversity of activity in Japan, and similarly the talks in Oberwolfach were a strong inspiration for participants in Japan.

We would like to especially thank the staff of MFO for their excellent additional efforts which ensured to provide perfect working conditions also under the special circumstances of the Covid-19 pandemic and allowed for collaboration between on site and remote participants of the workshop.

MFO-RIMS Tandem Workshop (hybrid meeting): Symmetries on Polynomial Ideals and Varieties

Table of Contents

Abstracts

Stabilization in Sequences of Symmetric Ideals

Uwe Nagel

Sequences of ideals arising in various contexts often have a rich algebra structure, as indicated, for example, by a large automorphism group. It is natural to expect that properties of related ideals remain somewhat invariant or become predictable eventually even though they involve more and more variables. We discussed a framework for studying and justifying this belief, namely the category of FI-modules and its ordered analog, the category of OI-modules, over a noetherian polynomial FI- or OI-algebra.

In the 1890ties, Hilbert established for a polynomial algebra P in finitely many variables over a field three fundamental results: (1) Every ideal of P is finitely generated. (2) Any finitely generated graded P-module has a rational Hilbert series. (3) Any finitely generated P-module admits a finite free resolution. By work in [2, 3] (see also [1]) analogous results are true for any finitely generated OI-module $\mathbf{M}$ over a noetherian polynomial OI-algebra $\mathbf{P}$. Such a module may be thought of as a sequence $(\mathbf{M}_n)_{n\in\mathbb{N}}$ of $\mathbf{P}_n$-modules $\mathbf{M}_n$, where every $\mathbf{P}_n$ is a polynomial ring in finitely many variables. The numbers of variables grows linearly in n. The finiteness results on $\mathbf{M}$ imply that invariants of $\mathbf{M}_n$ such as the Krull dimension and the degree grow eventually linearly and exponentially in n, respectively. Homological invariants such as the projective dimension and the Castelnuovo-Mumford regularity are conjectured to also grow linearly in n eventually. This is true in various cases, but remains open in general.

Similar stabilization results for OI-modules over non-noetherian polynomial OI-algebras have been established in some interesting cases, but the boundaries for such results are not clear yet, and there are many open questions.

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Projective dimension and regularity of equivariant filtrations of ideals

Hop D. Nguyen
(joint work with Dinh Van Le)

Ideals with large symmetry groups have been investigated by many authors in recent times. For many reasons, we are interested in ideals fixed the monoid $\mathrm{Inc}(\mathbb{N})$ of increasing functions $\mathbb{N} \to \mathbb{N}$, or the group of permutations $\mathrm{Sym}(\mathbb{N}) = \bigcup_{n=1}^{\infty} \mathrm{Sym}(n)$. In this talk, we discuss two conjectures in [3, 4] on the eventual behavior of projective dimension and Castelnuovo–Mumford regularity of Inc-invariant chains of homogeneous ideals.

Let k be a field, and $R_n = k[x_1, \dots, x_n]$ a polynomial ring over k for $n = 1, 2, \dots$. Denote $R = \bigcup_{n=1}^{\infty} R_n = k[x_1, x_2, \dots]$. Then $\mathrm{Inc}(\mathbb{N})$ acts on R by endomorphisms in such a way that $\pi(x_i) = x_{\pi(i)}$ for any $\pi \in \mathrm{Inc}(\mathbb{N})$. An Inc-invariant filtration of ideals is a collection of ideals $\mathcal{I} = (I_n)_{n\geq 1}$, where $I_n \subseteq R_n$, such that for any $n \geq m \geq 1$ and any increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $f(m) \leq n$, we have $f(I_m) \subseteq I_n$. Similarly we can define Sym-invariant filtrations of ideals. By work of Nagel-Römer, Kahle-Le-Römer [2, 6], any such Inc-invariant chain of ideals eventually stabilizes, in the sense that there exists an integer $r \geq 1$ such that I_n is generated by the Inc-orbits of elements in I_r for all $n \geq r$. The smallest such r is called the *stability index* of the chain $(I_n)_{n\geq 1}$. We call the chain $\mathcal{I} = (I_n)_{n\geq 1}$ *saturated* if for all $n \geq 1$, it holds that $I_n = I_{n+1} \cap R_n$. Saturated filtrations are usually rather well-behaved, but little is known about non-saturated filtrations.

Example 1. The ideals $I_n = (x_ix_j : 1 \leq i, j \leq n, j - i \geq 3, (i, j) \neq (1, n))$ form an Inc-invariant chain of squarefree monomial ideals. We have $I_n = 0$ for $n \leq 4$, $I_5 = (x_1x_4, x_2x_5)$, $I_6 = (x_1x_4, x_1x_5, x_2x_5, x_2x_6, x_3x_6)$, and so on. The stability index is 5. This is a non-saturated filtration, as $x_1x_n \in (I_{n+1} \cap R_n) \setminus I_n$ for all $n \geq 5$. Computations with Macaulay2 [1] show that the regularity and the projective dimension of I_n are $\operatorname{reg} I_n = 3$ and $\operatorname{pd} I_n = n - 3$ for all $n \geq 6$.

Denote by $G(J)$ the set of minimal monomial generators of a monomial ideal J. For a monomial $m = x^{\boldsymbol{a}} \in k[x_1, \dots, x_n]$, denote $w(m) = \max\{a_1, \dots, a_n\}$. For a monomial ideal J, denote $w(J) = \min\{w(m) : m \in G(J)\}$. Hence, for example, $w((x^5y, x^2y^2)) = 2$ and $w((x^4, y^5, xyz)) = 1$. It is interesting to study finiteness/stability properties of invariant chains of ideals. In this direction, the following two conjectures on homological invariants of such chains were proposed.

Conjecture 2 ([4, Conjectures 1.1 and 4.12])**.** Let $\mathcal{I} = (I_n)_{n\geq 1}$ be an $\mathrm{Inc}(\mathbb{N})$-invariant filtration of proper homogeneous ideals. Denote $r = \mathrm{ind}(\mathcal{I})$ the stability index of $\mathcal{I}$. Then the sequence $(\mathrm{reg}(I_n))_{n\geq 1}$ is eventually linear.

Moreover, if each I_n is a monomial ideal, then there exists an integer D such that $\mathrm{reg}(I_n) = (w(I_r) - 1)n + D$ for all $n \gg 0$.

This conjecture holds true if $\mathcal{I}$ is moreover saturated and Sym-invariant, as showed by Murai [5] and Raicu [7]. It is also known to be true for I_n defining artinian rings [4]. But it remains open even for non-saturated filtrations of squarefree monomial ideals.

Conjecture 3 ([3, Conjecture 1.3])**.** Let $\mathcal{I} = (I_n)_{n\geq 1}$ be an Inc($\mathbb{N}$)-invariant filtration of non-zero proper homogeneous ideals. Then the sequence $(\mathrm{pd}(R_n/I_n))_{n\geq 1}$ is eventually linear.

For Conjecture 2, we prove an asymptotic version for all (saturated or not) chains of monomial ideals. For Conjecture 3, we prove the exact statement for all *saturated* chains of monomial ideals.

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Symmetric spectrahedra with respect to the permutation and orthogonal group

Mario Kummer

Consider a set $M \subset \mathbb{R}^n$ which is invariant under the permutation group $\mathfrak{S}_n$. We let $S^2(n)$ be the vector space of real symmetric $n \times n$ matrices. For $A \in S^2(n)$ we denote by $\lambda(A)$ the unordered tuple of eigenvalues of A. Then the set $\Lambda(M) = \{A \in S^2(n) : \lambda(A) \in M\}$ is invariant under the orthogonal group $\mathrm{O}(n)$ acting on $S^2(n)$ via conjugation and inherits a lot of desirable properties like convexity and (semi-)algebraicity from M. Recall that a *spectrahedron* is a convex semi-algebraic set that can be described via a linear matrix inequality $x_1A_1 + \ldots + x_nA_n \succeq 0$ where the A_i are real symmetric matrices and $A \succeq 0$ means that the matrix A is positive semidefinite. We conjecture that M is a spectrahedron if and only if $\Lambda(M)$ is a spectrahedron. This would be true if the *generalized Lax conjecture* [2] was true. We present a result from [1] which gives some representation theoretic sufficient conditions for $\Lambda(M)$ being a spectrahedron. This result implies that the $\mathrm{O}(n)$-invariant set of all symmetric matrices for which the kth derivative of their characteristic polynomial has only nonnegative zeros is a spectrahedron for all k.

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Stabilization of chains of cones and monoids

Thomas Kahle
(joint work with Dinh Van Le, Tim Römer)

In equivariant algebra and geometry there are many results about Noetherianity, defined as a finite generation property that holds for all objects in some class. For example, any ideal in a Noetherian ring is finitely generated. In this talk I discussed non-Noetherian situations concerning chains $(C_n)_n$ of convex cones $C_n \subset \mathbb{R}^n_{\geq 0}$ or, similarly, chains of monoids $M_n \subset \mathbb{N}^n$. As there are cones and monoids that are not finitely generated, there is no Noetherianity. In joint work with Dinh van Le and Tim Römer we constructed a framework which allows one to understand conditions for the equivalence of finite generation of a limit object $\bigcup_n C_n$ (the *global* situation) and stabilization of the chain $(C_n)_n$ (the *local* situation).

To see a specific example, we define a Sym-*invariant chain* of cones C_n to be one that satisfies

$$\operatorname{cone}(\operatorname{Sym}(n)(C_m)) \subseteq C_n, \qquad \text{for all } m \leq n.$$

Here $\operatorname{Sym}(n)$ is the ordinary symmetric group, acting on the m-th cone embedded in $\mathbb{R}^n$, by permuting the coordinates. The chain *stabilizes*, if for sufficiently large m, n all containments above are equalities. A specific example of the above mentioned equivalence is the following theorem, quoted from [1, Corollary 5.4].

Theorem 1. Let $\mathcal{C} = (C_n)_{n\geq 1}$ be a Sym-invariant chain of convex cones $C_n \subseteq \mathbb{R}^n_{\geq 0}$ with limit cone $C_\infty = \bigcup_n C_n$. Then the following statements are equivalent:

(1) $\mathcal{C}$ stabilizes and is eventually finitely generated;
(2) There exists an $r \in \mathbb{N}$ such that for all $n \geq r$ the following hold:
  (a) $C_\infty \cap \mathbb{R}^n_{\geq 0} = C_n$,
  (b) C_n is finitely generated by elements of support size at most r;
(3) C_∞ is generated by finitely many Sym-orbits.

The paper [1] contains many examples which illustrate that the theorem cannot be significantly strengthened. The proof of the theorem (which appears as a corollary in the paper) is via a generalization to a much more general framework in which invariance and stabilization can be formulated. In this framework the operation cone can be replaced by an arbitrary closure operation (like monoid or ideal closure). The action of the symmetric group $\operatorname{Sym}(n)$ on the m-th cone C_m can be replaced by a suitable family of embeddings of C_m into the n-th position of the chain, for example by increasing maps on the coordinate indices (i.e. Inc).

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Strong Lefschetz property and prehomogeneous vector space

Takahiro Nagaoka
(joint work with Akihito Wachi)

Let S be a polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ and F be a homogeneous polynomial of degree r in S. Define a homogeneous ideal $\mathrm{Ann}F$ of S as

$$\mathrm{Ann}F := \left\{ p \in S \;\middle|\; p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) F = 0 \right\}.$$

Then one can consider the quotient algebra $R_F := S/\mathrm{Ann}F$, and it is known that $R_F = R_F^0 \oplus \cdots \oplus R_F^r$ is a standard graded Artinian Gorenstein algebra. For a standard graded Artinian Gorenstein algebra $R = \bigoplus_{i=0}^r R^i$, we can consider the strong Lefschetz property (SLP). We say R satisfies the strong Lefschetz property for a degree one element $L \in R^1$ if for any $0 \le k \le r$, the multiplication map

$$\times L^{r-2k} : R^k \to R^{r-k}$$

is an isomorphism. Historically, SLP was proved for cohomology rings of Kähler manifold (M, ω) and a Kähler class $L = [\omega]$. Recently, many mathematicians studied SLP for many other Artinian Goremstein algebras associated with some combinatorial objects and gave many applications to combinatorics.

Inspired by [1] and [2], we consider SLP for the Artinian Gorenstein algebra associated with the Kirchhoff polynomial $F_{K_{r+1}}$ of the complete graph K_{r+1}, which corresponds to Type (C_r, r) in the following Theorem 1. The main idea for the proof is that we identify the Kirchhoff polynomial $F_{K_{r+1}}$ with the determinant polynomial of symmetric matrices and we take advantage of the theory of prehomogeneous vector spaces.

Theorem 1. For (S, F) in the following table and any $t \in \mathbb{Z}_{>0}$, the Artinian Gorenstein algebra $R_{F^t} := S/\mathrm{Ann}F^t$ satisfies the SLP.

Type	S	F
(C_r, r)	$\mathbb{C}[\mathrm{Sym}_{r\times r}] = \mathbb{C}[x_{ij}]_{1\le i,j\le r}/\langle x_{ij} - x_{ji}\rangle$	$\det(x_{ij})$
(A_{2r-1}, r)	$\mathbb{C}[\mathrm{Mat}_{r\times r}] = \mathbb{C}[x_{ij}]_{1\le i,j\le r}$	$\det(x_{ij})$
$(\mathrm{D}_{2m}, 2m)$	$\mathbb{C}[\mathrm{Alt}_{2m\times 2m}] = \mathbb{C}[x_{ij}]_{1\le i,j\le 2m}/\langle x_{ij} + x_{ji}\rangle$	$\mathrm{Pf}(x_{ij})$
$(\mathrm{B}_r, 1)$	$\mathbb{C}[x_1, \dots, x_r]$	$x_1^2 + \cdots + x_r^2$
$(\mathrm{E}_7, 7)$	$\mathbb{C}$[27 variables]	a polynomial of deg 3

,

where $\mathrm{Mat}_{r\times r} := \{\, r \times r\text{-matrices} \,\}$
$\mathrm{Sym}_{r\times r} := \{\, r \times r\text{-symmetric matrices} \,\}$
$\mathrm{Alt}_{2m\times 2m} := \{\, 2m \times 2m\text{-antisymmetric matrices} \,\}$

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On Artinian Gorenstein algebras associated to the face posets of regular polyhedra

Akiko Yazawa

For a polyhedra $\mathcal{P}$, let $V(\mathcal{P})$ and $F(\mathcal{P})$ be the collection of 0-dimensional faces and facets, respectively. For a regular polyhedra $\mathcal{P}$, we define a homogeneous polynomial by

$$F_{\mathcal{P}} = \sum_{F \in F(\mathcal{P})} \prod_{v \in F} x_v.$$

Then we consider the Artinian Gorenstein algebra

$$A_{\mathcal{P}} = \mathbb{R}[\partial_v | v \in V(\mathcal{P})] / \operatorname{Ann}(F_{\mathcal{P}}),$$

where $\partial_v = \frac{\partial}{\partial x_v}$ is the partial derivative operator of x_v. We discuss the strong Lefschetz property of $A_{\mathcal{P}}$ and Hodge–Riemann relation with respect to the Poincaré duality

$$P^k_{F_{\mathcal{P}}} : A_k \times A_{s-k} \to \mathbb{R}, \qquad (f, g) \mapsto fgF_{\mathcal{P}}.$$

Theorem 1. The following hold for the Platonic solids:

- The Artinian Gorenstein algebras have the strong Lefschetz property.
- Except for the dodecahedron, the sum of all variables is a strong Lefschetz element of the Artinian Gorenstein algebras. For the dodecahedron, the sum of all variables is not a strong Lefschetz element, but the sum of all variables except for one is a strong Lefschetz element.
- For the regular tetrahedron and octahedron, the algebra satisfy the Hodge–Riemann relation on the positive orthant. For the others, the algebras do not satisfy the Hodge–Riemann relation with respect to those strong Lefschetz elements.
- The Hilbert series are as follows:
  - Tetrahedron: $(1, 4, 4, 1)$.
  - Hexahedron: $(1, 8, 18, 8, 1)$.
  - Octahedron: $(1, 3, 3, 1)$.
  - Dodecahedron: $(1, 20, 90, 90, 20, 1)$.
  - Icosahedron: $(1, 12, 12, 1)$.

This article is an extended abstract of the paper [1]. We omit proofs and details. See [1] for details.

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Elementary construction of minimal free resolutions of the Specht ideals of $(n-d,d)$

Kosuke Shibata
(joint work with Kohji Yanagawa)

In this talk, we present the results of a generalization of [2]. The paper on this result is under construction.

Throughout this paper, we assume that K is a field with $\mathrm{char}(K)=0$. For a positive integer n, a *partition* of n is a sequence $\lambda=(\lambda_1,\dots,\lambda_l)$ of integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 1$ and $\sum_{i=1}^{l}\lambda_i = n$. A partition λ is frequently represented by its Young diagram. The *Young tableau* of shape λ is a bijective filling of the Young diagram of λ by the integers in $[n] := \{1,2,\dots,n\}$. Let $\mathrm{Tab}(\lambda)$ be the set of Young tableaux of shape λ.

Let $R = K[x_1,\dots,x_n]$ be a polynomial ring over K. For a tableau $T \in \mathrm{Tab}(\lambda)$ of λ, the *Specht polynomial* of T is denoted by f_T. The symmetric group $\mathfrak{S}_n$ acts on the vector space spanned by $\{\, f_T \mid T \in \mathrm{Tab}(\lambda)\}$. As an $\mathfrak{S}_n$-module, this vector space is isomorphic to the *Specht module* V_λ, which is very important in the representation theory of symmetric groups. Here we study the Specht *ideal*

$$I_\lambda^{\mathrm{Sp}} := (\, f_T \mid T \in \mathrm{Tab}(\lambda))$$

of R. In [3], Yanagawa proved that $R/I^{\mathrm{Sp}}_{(n-d,d)}$ and $R/I^{\mathrm{Sp}}_{(d,d,1)}$ are Cohen–Macaulay, and $R/I^{\mathrm{Sp}}_{(n-2,2)}$ is Gorenstein.

In this talk, we construct the minimal free resolution of $I^{\mathrm{Sp}}_{(n-d,d)}$ for $1 \leq d \leq n/2$. Minimal free resolutions of these ideals had been studied by Berkesch Zamaere, Griffeth, and Sam [1]. More precisely, [1] determined the $\mathfrak{S}_n$-module structure of $\mathrm{Tor}_i^R(K, R/I^{\mathrm{Sp}}_{(n-d,d)})$ (They called $I^{\mathrm{Sp}}_{(n-d,d)}$ the "$(d+1)$-equal ideal"). However the paper [1] dose not give the differential maps of their resolutions, and uses highly advanced tools of the representation theory (rational Cherednik algebras, Jack polynomials, etc). By contrast, we describe the differential maps explicitly, and uses only the basic theory of Specht modules. Here we do not use results of [1].

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The geometry of polynomial representations

Jan Draisma, Andrew Snowden
(joint work with A. Bik, A. Danelon, R.H. Eggermont)

These talks concerned the geometry of GL-*varieties*: infinite-dimensional affine varieties X equipped with an action of the infinite general linear group $\mathrm{GL} = \bigcup_n \mathrm{GL}_n$, in such a manner that the coordinate ring $K[X]$ is a polynomial GL-representation and generated as an algebra by finitely many GL-orbits of elements. The role of *affine spaces* in this theory is played by the spaces $\mathbf{A}^{\underline{\lambda}} = \mathbf{A}^{\lambda_1} \times \cdots \times \mathbf{A}^{\lambda_k}$ for a k-tuple $\underline{\lambda} = [\lambda_1, \ldots, \lambda_k]$ of partitions representing Schur functors. For instance, $\mathbf{A}^{[(2),(1,1)]}$ is the space of infinite-by-infinite matrices, acted upon by GL via $(g, A) \mapsto gAg^T$, and $\mathbf{A}^{(d)}$ is the space of homogeneous degree d polynomials in infinitely many variables. A typical, and interesting, example of a GL-variety is the closure of the strength $\leq s$ locus in $\mathbf{A}^{(d)}$.

An important technique on GL-varieties is *shifting*, which is replacing the action of GL by the action of its subgroup $G(n)$, isomorphic to GL, consisting of block diagonal matrices $\mathrm{diag}(I_n, g)$ for $g \in \mathrm{GL}$ and I_n the $n \times n$ identity matrix. The variety X equipped with the action of GL via the isomorphism $\mathrm{GL} \to G(n)$, $g \mapsto \mathrm{diag}(I_n, g)$ is denoted $\mathrm{Sh}_n(X)$.

Theorem 1 (Noetherianity, [1])**.** Closed GL-subvarieties of a GL-variety satisfy the descending chain condition.

Theorem 2 (Shift theorem, [2])**.** For any nonempty GL-variety X there exists a nonnegative integer n, a nonzero $G(n)$-fixed function $h \in K[X]$, a finite-dimensional variety B, and a tuple $\underline{\lambda}$ such that $(\mathrm{Sh}_n X)[1/h] \cong B \times \mathbf{A}^{\underline{\lambda}}$.

Theorem 3 (Unirationality theorem, [2])**.** Every irreducible GL-variety X admits a dominant GL-equivariant morphism $B \times \mathbf{A}^{\underline{\lambda}} \to X$, where B is irreducible. The inclusion-wise minimal $\underline{\lambda}$ with this property is unique, and so is the dimension of B; it equals the transcendence degree of the field of invariants $K(X)^{\mathrm{GL}}$.

An *elementary* GL-variety is one of the form $B \times \mathbf{A}^{\underline{\lambda}}$, as appearing in the above theorems. A map of elementary GL-varieties $\phi\colon B \times \mathbf{A}^{\underline{\lambda}} \to C \times \mathbf{A}^{\underline{\mu}}$ is *elementary* if the tuple $\underline{\lambda}$ contains the tuple $\underline{\mu}$, and ϕ has the form $\psi \times \pi$ where $\psi\colon B \to C$ is a surjective map of varieties and $\pi\colon \mathbf{A}^{\underline{\lambda}} \to \mathbf{A}^{\underline{\mu}}$ is the projection. A *locally elementary* GL-variety X is one admitting a $G(n)$-stable open subvariety U such that U is elementary (after identifying $G(n)$ with GL) and $X = \bigcup_{g \in \mathrm{GL}} gU$. An example of a locally elementary GL-variety is $\mathbf{A}^{\underline{\lambda}} \setminus \{0\}$. Locally elementary morphisms are defined similarly.

Theorem 4 (Decomposition theorem, [2])**.** Let $\phi\colon X \to Y$ be a morphism of GL-varieties. Then there are finite decompositions $X = \bigsqcup_{i=1}^n X_i$ and $Y = \bigsqcup_{j=1}^m Y_j$ where X_i and Y_j are locally closed GL-subvarieties that are locally elementary, and for each i, ϕ induces a locally elementary map from X_i to some Y_j.

The following is an important consequence of the decomposition theorem:

Theorem 5 (Chevalley's theorem [2])**.** Let $\phi\colon X \to Y$ be a morphism of GL-varieties and let E be a GL-constructible subset of X (i.e., E is a finite union of locally closed GL-stable sets). Then $\phi(E)$ is GL-constructible.

This theorem implies, for instance, that the strength exactly s locus in $\mathbf{A}^{(d)}$ is GL-constructible. It turns out that on GL-varieties acts the monoid $E \supseteq \mathrm{GL}$ of matrices with finitely many nonzero elements in each row.

Theorem 6 (Maximal and minimal dense orbits [3])**.** Among the K-points in $\mathbf{A}^{\underline{\lambda}}$ with dense GL-orbits, there is one whose E-orbit contains all other elements in $\mathbf{A}^{\underline{\lambda}}$. If $\underline{\lambda}$ contains only partitions of the same number, then there is also a "minimal" element with dense orbit, namely, one which is contained in the E-orbit of any other element with a dense orbit.

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Equivariant Hilbert series of hierarchical models

Aida Maraj

(joint work with Uwe Nagel)

Hierarchical models are discrete statistical models that record the dependency relations among discrete random variables in given data. They are fully determined by a simplicial complex that present these relations, and a vector of positive integers where one records the number of states each variable has. Hierarchical models are described explicitly by certain toric ideals in polynomial rings in variables indexed by the state space [1]. Intuitively, if one increases the number of states in a model one expects essential properties to be preserved. In order to make this precise, one takes advantage of symmetry and studies asymptotic behaviours of related models. Algebraically, increasing the number of states leads to ideals with more generators in polynomial rings in increasingly many variables; see [2]. There are two approaches we can take; we can work with sequences of related ideals in larger and larger polynomial rings, or we can work with a single ideal in a polynomial ring with infinitely many variables.

In some cases symmetry can be used to capture qualitative information by showing the existence of a finite set of "master generators" from which the rest of generators can be obtained [2]. Quantitative information is classically captured via a generating function that uses formal power series in one variable, called the Hilbert series. In order to analyze infinitely many models simultaneously we introduce in [4] multivariate equivariant Hilbert series with the action of a product of symmetric groups. We give a hypothesis which ensures that these are rational functions with rational coefficients, which allow us to derive quantitative information about each individual ideal. In the language of algebraic geometry, these are filtrations of some generalized Segre embeddings of infinite dimensional planes. Inspired by [3], we involve the theory of regular languages and finite automata, and provide explicit formulas. The method provided in this work allows for applications to other filtrations of algebraic objects [5]. A complete classification of hierarchical models with rational Hilbert series is yet to be found.

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Specht Ideals, their varieties and connections to symmetric ideals

Cordian Riener

(joint work with Arne Lien, Philippe Moustrou, Hugues Verdure)

The construction of the irreducible representations of the symmetric group S_n was given by Wilhelm Specht [3] in terms of polynomials. To each partition λ of n one can associate a Young tableau, filled with the numbers $1, \ldots, n$. The Specht polynomial associated to the tableau then is obtained by multiplying the difference products $\prod_{i<j}(X_i - X_ik$, where i, j range over the columns of the tableau. Specht showed that the S_n module generated by a Specht polynomial corresponding to a partition of n is an irreducible S_n module and that by considering this construction for all partitions of n one obtains a complete list of all irreducible S_n representations. Given a partition λ we consider the ideal I_λ^{sp} generated by all Specht polynomials associated to λ and V_λ the corresponding variety. In [2] it is shown that the inclusions of the Specht ideals rand their varieties are related to the dominance order " $\trianglelefteq$ " on partitions of n. More concretely:

Theorem 1. Let λ and μ be two partitions of n. Then the following assertions are equivalent:

i) The partition μ dominates λ, *i.e.* $\lambda \trianglelefteq \mu$,
ii) The ideal I_μ^{sp} contains I_λ^{sp}, *i.e.* $I_\lambda^{\text{sp}} \subset I_\mu^{\text{sp}}$,
iii) The variety V_λ contains V_μ, *i.e.* $V_\mu \subset V_\lambda$.

It was conjectured by Yanagawa [4] that the Specht ideals all are radical. Using the above Theorem one can obtain the primary decomposition of each $\sqrt{I_\lambda^{\text{sp}}}$ quite explicitly and hopefully opening a way to verify Yanagawa's conjecture. A second result links Specht polynomials to general symmetric ideals, i.e. S_n-invariant ideals, via monomials. Given a monomial m of weight l and degree d and assuming that $l+d \leq n$, we can define a partition $\mu(m)$ of n by setting

$$\mu(m) = (\lambda_1+1, \lambda_2+1, \dots, \lambda_l+1, \underbrace{1, \dots, 1}_{n-d-l}).$$

With this notion the following holds.

Theorem 2. Let $I \subset \mathbb{K}[X_1, \dots, X_n]$ be a symmetric ideal. Assume that there exists $P \in I$ of degree d, such that the degree d homogeneous parts P_d of P does not contain the variables $X_n, \dots, X_{n-d}$. Then, for every monomial $m \in Mon(P_d)$ we have $I_\lambda^{\mathrm{sp}} \subset I$ for every $\lambda \trianglelefteq \mu(m)^\perp$, where $\mu(m)^\perp$ denotes the partition dual to $\mu(m)$.

The above theorems have several interesting applications which underline that the Specht ideals provide an interesting class of ideals which link the combinatorial theory of the symmetric group to symmetric ideals. In particular the following conjecture provides a further interesting connection.

Conjecture 1. Let $\lambda \vdash n$, then the Specht polynomials of shape $\mu \vdash n$, where $\lambda \trianglerighteq \mu$, form a Gröbner basis of I_λ.

The above conjecture is indeed true for the particular case of hock-shaped partitions, i.e., partitions of the form $\lambda_{n,k} = (n-k, 1^k)$ as was shown by Arne Lien in his master thesis [1].

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Finitely generated polynomial subalgebras without finite SAGBI bases

Shigeru Kuroda

Let $k[\boldsymbol{x}] = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field k, and $\preceq$ a monomial order on $k[\boldsymbol{x}]$. For each $f \in k[\boldsymbol{x}]$, we denote by $\mathrm{in}_\preceq(f)$ the initial monomial of f with respect to $\preceq$. Then, for any k-subalgebra A of $k[\boldsymbol{x}]$, the k-vector subspace $\mathrm{in}_\preceq(A)$ of $k[\boldsymbol{x}]$ generated by $\mathrm{in}_\preceq(f)$ for $0 \neq f \in A$ is a k-subalgebra of $k[\boldsymbol{x}]$, which we call the *initial algebra* of A. A subset S of A is called a *SAGBI* (*Subalgebra Analogue to Gröbner Bases for Ideals*) *basis* of A if $\{\mathrm{in}_\preceq(f) \mid f \in S\}$ generates the k-algebra $\mathrm{in}_\preceq(A)$ (cf. [6]). By definition, A has a finite SAGBI basis if and only if the k-algebra $\mathrm{in}_\preceq(A)$ is finitely generated.

The notion of initial algebra is of great importance in the study of deep problems in Affine Algebraic Geometry, such as the problems of $\mathrm{Aut}_k(k[\boldsymbol{x}])$ (cf. [3]). However, for a given k-subalgebra A, it is quite difficult to determine the structure of $\mathrm{in}_{\preceq}(A)$ in general. Even if A is finitely generated, $\mathrm{in}_{\preceq}(A)$ is not necessarily finitely generated, and no general criterion for finite generation of $\mathrm{in}_{\preceq}(A)$ is known.

The best known result about finite generation of initial algebra is the following: For a subgroup G of the symmetric group S_n, we consider the invariant ring

$$k[\boldsymbol{x}]^G = \{f \in k[\boldsymbol{x}] \mid f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_1, \dots, x_n) \quad (\forall \sigma \in G)\}.$$

Then, $\mathrm{in}_{\preceq}(k[\boldsymbol{x}]^G)$ is finitely generated if and only if the group G is generated by transpositions (see [1] when $\preceq$ is the lexicographic order, and [2], [5] and [7] for the general case). There are several generalizations of this result.

In this talk, we present a new construction of finitely generated subalgebras whose initial algebras are not finitely generated. Our construction is very general, so that we obtain a large class of subalgebras of this property. Except for a class of invariant rings mentioned above, no such class of subalgebras are previously found.

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The Lefschetz property and the Schur-Weyl duality

Akihito Wachi

The Aim of this talk is to introduce part of results of "Zero-dimensional Gorenstein Algebras with the Action of the Symmetric Group" [1] by Morita, Wachi and J. Watanabe. This paper contains results related to Specht polynomials.

Let K be a field of characteristic zero. $GL_n(K)$ acts on K^n, and the symmetric group S_k acts on the k-fold tensor product $(K^n)^{\otimes k}$. The celebrated Schur-Weyl duality describes the $(GL_n(K) \times S_k)$-irreducible decomposition of $(K^n)^{\otimes k}$:

$$(K^n)^{\otimes k} \cong \bigoplus_{\lambda=(\lambda_1,\dots,\lambda_n)\vdash k} V_\lambda \otimes_K S_\lambda \qquad (\textbf{Schur-Weyl duality}),$$

where $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, V_λ is the irreducible representation of $GL_n(K)$ corresponding to λ, and S_λ is the irreducible representation of S_k corresponding to λ. In view of S_k-modules, the Schur-Weyl duality gives multiplicities of the Specht modules S_λ, which is equal to $\dim_K V_\lambda$.

We consider a modification of the Schur-Weyl duality. By changing the n-dimensional K-vector space K^n to the quotient ring $K[x]/(x^n)$ (standard basis $e_i \in K^n \leftrightarrow x^{i-1} \in K[x]/(x^n)$), we have the following:

$$K[x_1, \ldots, x_k]/(x_1^n, \ldots, x_k^n) \cong \bigoplus_{\lambda=(\lambda_1,\ldots,\lambda_n)\vdash k} V_\lambda \otimes_K S_\lambda$$

(**'Modified' Schur-Weyl duality**).

Because S_k-action preserves degrees of polynomial, $\dim_K V_\lambda$ copies of the Specht module S_λ in $K[x_1, \ldots, x_k]/(x_1^n, \ldots, x_k^n)$ can be taken in homogeneous components. Thus we are interested in the 'Hilbert series' of multiplicity of S_λ in $K[x_1, \ldots, x_k]/(x_1^n, \ldots, x_k^n)$ as an S_k-module. It turns out to be equal to the q-dimension of V_λ, which is defined as

$$\dim_q V_\lambda = q^{n(\lambda)} \frac{\Delta_q(\lambda + (n-1, n-2, \ldots, 0))}{\Delta_q(n-1, n-2, \ldots, 0)} (\text{'Hilbert series' of multiplicity}),$$

where $n(\lambda) = \lambda_2 + 2\lambda_3 + \cdots + (n-1)\lambda_n$, and $\Delta_q(x_1, \ldots, x_n) = \prod_{i<j}[x_i - x_j]$ ($[a] = 1 + q + \cdots + q^{a-1}$).

It is known that $K[x_1, \ldots, x_k]/(x_1^n, \ldots, x_k^n)$ has the strong Lefschetz property (see Yazawa's or Nagaoka's abstract for the definition). In particular, the symmetric polynomial $L = x_1 + \cdots + x_n$ is a Lefschetz element, and $\times L$ maps a simple S_k-submodule equivalent to S_λ onto another one (or zero). In this talk, for small n or small k, using Specht polynomials and higher Specht polynomials, we explicitly give a basis of $K[x_1, \ldots, x_k]/(x_1^n, \ldots, x_k^n)$ which is a union of bases of simple S_k-submodules, and is compatible with $\times L$.

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Symmetries and local-global principles

Tim Römer

Let $(A_n)_{n\geq 1}$ be an ascending chain of objects of interest in algebra or geometry. Examples we have in mind are families of ideals in polynomial rings, affine monoids, or polyhedral cones. Assume that a limit object A_∞ exists which is often just the union of the elements of the given chain. One is interested in the following fundamental questions:

(1) Determine the asymptotic behaviour of properties of interest of the A_n;
(2) Study corresponding properties of A_∞;
(3) Show a connection between the properties of A_n and the ones of A_∞.

Here (1) can be considered as the *local* and (2) the *global* information. (3) is then a *local-global principle* which one would like to understand. Many times in (2) an additional obstacle is that nice properties (e.g., being noetherian) are not given and standard tools from algebra or geometry can not be used.

In this generality, the three problems from above can not studies without further assumptions. A very successful approach from the recent years (see, e.g., [1, 2, 4, 5, 11, 12]) is that suitable groups act on A_n as well as A_∞ and one can use symmetries to obtain answers. A typical example is that the n-th symmetric group $\mathrm{Sym}(n)$ acts on A_n and $\mathrm{Sym}(\infty) = \bigcup_{n\geq 1} \mathrm{Sym}(n)$ acts on A_∞. It is necessary that all group actions are compatible in a natural way. Note that a categorial way to study such a situation is the consideration of FI-objects and their limits (see, e.g., [3, 10]).

In this talk we report on recent results in [6, 7, 8, 9, 10] related to local-global principles with the emphasis on the situation of chain of ideals as they occur in commutative algebra. Here considered properties include stabilisation of the chains as well as being finitely generated up to symmetry of limit objects.

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Introduction to equivariant modules and sheaves

Mitsuyasu Hashimoto

(joint work with Mitsuhiro Miyazaki, Masahiro Ohtani)

This is a survey on equivariant modules and sheaves under actions of affine algebraic group schemes over a field. The speaker is mainly interested in the applications to the ring-theoretic aspects of invariant theory.

The talk consists of three parts.

The first part is on the basic constructions containing (the five of) the 'six operations' of equivariant sheaves, including the equivariant version of the twisted inverse [1]. In particular, the correspondence between the sheaves and the modules is important.

The second part is on so-called Matijevic–Roberts type theorem. The following is one form of the theorem proved in [4]. Let $R = \bigoplus_{\lambda \in \mathbb{Z}^n} R_\lambda$ be a $\mathbb{Z}^n$-graded Noetherian commutative ring, and P its prime ideal. If R_{P_*} is Cohen–Macaulay, then R_P is so, where P_* is the prime ideal generated by all the homogeneous elements of P. Viewing this theorem as a theorem on torus actions, Miyazaki and the speaker [3] proved a theorem on group actions, generalizing this theorem.

The third part is on Grothendieck's theorem on the equivalence between the category of quasi-coherent equivariant sheaves and the category of quasi-coherent sheaves on the base of the principal bundle. Although this theorem best describes the motivation of considering equivariant sheaves on varieties with group actions, principal bundles rarely appear in invariant theory. We consider a weaker version of it, almost principal bundles [2]. We give some examples, and give a theorem on the (quasi-)Gorenstein property of the invariant subring, generalizing Watanabe's theorem on the action of finite subgroup of $\mathrm{GL}(V)$ on V without pseudo-reflections.

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Cohomology of symmetric semi-algebraic sets and a stability conjecture

Saugata Basu

(joint work with Daniel Perrucci, Cordian Riener)

The decomposition of the cohomology modules of a closed semi-algebraic set $S \subset \mathrm{R}^k$ defined by symmetric polynomials having degrees at most d into isotypic components was studied in [2], as well as in [3] where several results were proved. The first important result was a severe restriction on the partitions that are allowed to appear in the isotypic decomposition of the cohomology – which cuts down the possibilities for the allowed partitions *from exponential to polynomial* (for fixed d).

The second key result obtained in [2] is a *polynomial bound* (again for fixed d) on the multiplicities of the various Specht modules occurring in the isotypic decomposition of $\mathrm{H}^*(S, \mathbb{Q})$.

Independent of the above results, the phenomenon of representational and homological stability (see for example [4]) is an active topic of research in algebraic topology. The basic phenomenon of (homological) stability that motivates this study is the fact that for any fixed p, and any manifold X, the rank of the p-th homology group, $\mathrm{H}_p(C_n(X))$ of $C_n(X)$ (the n-th configuration space of X), is eventually given by a polynomial in n.

Inspired by the above example, (with C. Riener in [2]) we made a conjecture about the growth-rate of the cohomology groups of certain natural sequences of symmetric semi-algebraic sets.

More precisely, let R be a real closed field, and $\Lambda_n = \mathrm{R}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$ be the graded ring of invariant polynomials, with natural graded homomorphisms $\Lambda_{n+1} \to \Lambda_n$ obtained by setting X_{n+1} to 0. Denote by $\Lambda = \operatorname{proj\,lim} \Lambda_n$ (where the limit is taken in the category of graded rings), and denote by $\phi_n : \Lambda \to \Lambda_n$ the graded homomorphisms induced by the limit.

Suppose $I = (f_1, \ldots, f_k)$ is a finitely generated ideal of Λ. Then, I defines in a natural way symmetric real subvarieties $V_n = \mathrm{Zer}(\phi_n(f_1), \ldots, \phi_n(f_k)) \subset \mathrm{R}^n, n > 0$. For any fixed partition $\lambda \vdash d$, denote for $n > d$, by $\{\lambda\}_n$ the partition of n obtained by increasing the first row of λ by $n - d$.

The following conjecture was made in [2].

Conjecture 1. [2] For any fixed $p \geq 0$, the multiplicity of the Specht module corresponding to $\{\lambda\}_n$ in $\mathrm{H}^p(V_n, \mathbb{Q})$ is eventually given by a polynomial in n.

We study a special case of this conjecture (joint work with D. Perrucci). We have proved the following theorem.

Theorem 1. [1] Let $f = \sum_{i=0}^{d} a_i \sigma_i \in \Lambda$ be a linear combination of the elementary symmetric functions $\sigma_i \in \Lambda, 0 \leq i \geq d$, and let $I = (f)$. Then, for any partition λ and n large enough, the multiplicity of the Specht module corresponding to $\{\lambda\}_n \in \mathrm{H}^0(V_n, \mathbb{Q})$ equals 0 if $\mathrm{length}(\lambda) > 1$, and stablizes to a constant if $\mathrm{length}(\lambda) = 1$.

Theorem 1 verifies Conjecture 1 for ideals generated by one linear combination of elementary symmetric functions, with $p = 0$.

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How to count irreducible components up to symmetry

Rob Eggermont
(joint work with Jan Draisma, Azhar Farooq)

In recent work with Jan Draisma and Azhar Farooq, we showed that if X_n is a variety of $c \times n$-matrices that is stable under the group $\mathrm{Sym}([n])$ of column permutations and if forgetting the last column maps X_n into X_{n-1}, then the number of $\mathrm{Sym}([n])$-orbits on irreducible components of X_n is a quasipolynomial in n for all sufficiently large n. In this talk, I want to show a few examples to highlight some of the difficulties one may encounter when trying to count components up to symmetry. If there is time, I hope to present a bit of insight in the main ideas behind our proof.

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Bi-graded Koszul modules, K3 carpets, and Green's conjecture

Steven V Sam
(joint work with Claudiu Raicu)

Let C be a smooth curve of genus g over an algebraically closed field $\mathbf{k}$ of characteristic 0 and define its canonical ring via

$$\Gamma_C = \bigoplus_{d \geq 0} \mathrm{H}^0(C; \omega_C^{\otimes d}).$$

This is a finitely generated module over $A = \mathrm{Sym}(\mathrm{H}^0(C; \omega_C)) \cong \mathbf{k}[x_1, \ldots, x_g]$. We are concerned with the vanishing of the graded Betti numbers

$$\beta_{i,j}(C) = \dim_{\mathbf{k}} \mathrm{Tor}_i^A(\Gamma_C, \mathbf{k})_j.$$

Green's conjecture states that $\beta_{i,i+2}(C) = 0$ for $i < \mathrm{Cliff}(C)$, the Clifford index of C. This governs for how many steps the equations of C have only linear syzygies. Rather than define the Clifford index, we just remark that for most curves (in a sense which can be made precise using the moduli space of curves), the Clifford index of C is $\mathrm{gon}(C) - 2$ where $\mathrm{gon}(C)$ (the gonality of C) is the minimum degree of a non-constant map from C to the projective line.

Voisin [4, 5] showed that Green's conjecture holds generically. That is, there is a nonempty Zariski dense set in the moduli space of curves such that Green's conjecture holds for the curves in this set. The Clifford index of a curve is bounded from above by $(g-1)/2$ and a finer version of the result shows that this set contains curves of every gonality. Various strengthenings or refinements were introduced since then, but of interest to us is a recent proof by Aprodu, Farkas, Papadima, Raicu, and Weyman [1] which reproves Voisin's result using ideas from representation theory. The method of proof is in some ways simpler and extends the result to fields of characteristic $p \geq (g+2)/2$. (Of relevance is work of Schreyer [3] which shows that Green's conjecture does fail in small characteristic.)

The idea behind [1] is to consider rational cuspidal curves. These are smoothable, and due to the upper semicontinuity of Betti numbers in flat families, it suffices to prove that a version of Green's conjecture holds for them. Rational cuspidal curves (in their canonical embedding) with g cusps can be realized as hyperplane sections of the tangential surface T_g of the degree g rational normal curve, and hence the rational cuspidal curve and T_g have the same graded Betti numbers. There is a short exact sequence of graded modules

$$0 \to \mathbf{k}[T_g] \to \widetilde{\mathbf{k}[T_g]} \to M(-1) \to 0$$

where the first term is the homogeneous coordinate ring of T_g, the middle term is its normalization, and M is the canonical module of the homogeneous coordinate ring of the degree g rational normal curve (the (-1) denotes a grading shift). The Tor groups of M can be computed explicitly using the Eagon–Northcott complex, and the Tor groups of the middle term can be computing using the method of Kempf collapsing of vector bundles. Hence one is left to analyze the corresponding long exact sequence on Tor, and the goal is to prove that the following map is surjective for $i \geq (g-1)/2$:

$$\mathrm{Tor}_i^A(\widetilde{\mathbf{k}[T_g]}, \mathbf{k})_{i+1} \to \mathrm{Tor}_i^A(M, \mathbf{k})_i.$$

Using careful analysis of the representation theory of $\mathbf{SL}_2(\mathbf{k})$, the authors show that the cokernel of this map is identified with the middle homology of the following complex

$$\mathrm{Sym}^{g-2-i}(\mathrm{D}^{i+1}\mathbf{k}^2) \otimes \mathrm{D}^{2i}(\mathbf{k}^2) \to \mathrm{Sym}^{g-1-i}(\mathrm{D}^{i+1}\mathbf{k}^2) \otimes \mathrm{D}^{i+1}(\mathbf{k}^2) \to \mathrm{Sym}^{g-i}(\mathrm{D}^{i+1}\mathbf{k}^2)$$

where D denotes the divided power functor. The upshot is that by summing over all g, this gives a Koszul module, in the following sense. Given a vector space V and a linear subspace $K \subset \bigwedge^2 V$, the Koszul module $W(V, K)$ is the middle homology of the modified Koszul complex

$$\mathrm{Sym}(V) \otimes K \to \mathrm{Sym}(V) \otimes V(1) \to \mathrm{Sym}(V)(2).$$

In the above setting, $V = \mathrm{D}^{i+1}(\mathbf{k}^2)$ and $K = \mathrm{D}^{2i}(\mathbf{k}^2)$. One of the main results of [1] is that the following are equivalent:

- $K^{\perp} \subset \bigwedge^2(V^*)$ contains no nonzero rank 2 matrix,
- $W(V, K)$ is finite-dimensional,
- $W(V, K)_d = 0$ for all $d \geq \dim V - 3$.

This translates to the desired vanishing result for the rational cuspidal curve.

In [2] we follow a similar strategy using ribbons, a different degeneration of canonical curves. These are non-reduced double structures on the projective line, which can be realized as hyperplane sections of non-reduced double structures on rational normal scrolls known as K3 carpets. For the scroll, we choose two parameters a and g and consider the join of the degree a rational normal curve with the degree $g-1-a$ rational normal curve (we assume that $a \leq g-1-a$). The corresponding ribbon is a degeneration of a genus g curve with Clifford index a and gonality $a+2$. In this case, we get an extension

$$0 \to \omega_B \to B' \to B \to 0$$

where B is the homogeneous coordinate ring of the rational normal scroll, ω_B is its canonical module, and B' is the homogeneous coordinate ring of the K3 carpet. As before, the Eagon–Northcott complex can be used to explicitly compute the Tor groups of B and ω_B, so one needs to analyze the corresponding long exact sequence and show that the following map is surjective for $i < a$:

$$\mathrm{Tor}^A_{i+1}(B, \mathbf{k})_{i+2} \to \mathrm{Tor}^A_i(\omega_B, \mathbf{k})_{i+2}.$$

Everything in sight is bi-graded, and so can be further refined. We omit this detail but it is possible to identify the cokernel with bi-graded components of a bi-graded analogue of a Koszul module, in the following sense. Let V_1, V_2 be vector spaces and let K be a linear subspace of $V_1 \otimes V_2 \subset \bigwedge^2(V_1 \oplus V_2)$. Then the bi-graded Koszul module $W(V, K)$ is the middle homology of the modified Koszul complex (which is now bi-graded):

$$\mathrm{Sym}(V_1 \oplus V_2) \otimes K \to \begin{matrix} \mathrm{Sym}(V_1 \oplus V_2) \otimes V_1(0,1) \\ \mathrm{Sym}(V_1 \oplus V_2) \otimes V_2(1,0) \end{matrix} \to \mathrm{Sym}(V_1 \oplus V_2)(1,1)$$

We prove in [2] that the following are equivalent for bi-graded Koszul modules:

- $K^\perp \subset V_1^* \otimes V_2^*$ contains no nonzero rank ≤ 2 matrix,
- $W(V, K)_{d,e} = 0$ for $d, e \gg 0$,
- $W(V, K)_{d,e} = 0$ for $d \geq \dim V_2 - 2$ and $e \geq \dim V_1 - 2$.

Modulo the missing explanation above, this proves that an analogue of Green's conjecture holds for ribbons of Clifford index a and genus g. In fact, this result holds as long as the characteristic of $\mathbf{k}$ is at least a, so that we get a bound for where the refined generic Green conjecture holds in positive characteristic. Furthermore, this also improves the bound in [1] since the Clifford index of a genus g curve is at most $(g-1)/2$.

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Quadratic Gröbner bases of block matching field ideals

Hidefumi Ohsugi
(joint work with Akihiro Higashitani)

This talk is based on the paper [2]. In the present talk, we show that toric ideals of certain s-block diagonal matching fields have quadratic Gröbner bases. Thus, in particular, those are quadratically generated. By using this result, we provide a new family of toric degenerations of Grassmannians. These results are a generalization of the results for toric ideals of 2-block diagonal matching fields [1, 3].

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The radicalness of Specht ideals and the principal radical system

Junzo Watanabe
(joint work with Chris McDaniel, Endicott College)

Let $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$ be a partition of an integer n. Let $I(\lambda)$ be the ideal in the polynomial ring $K[x_1, \dots, x_n]$ generated by the Specht polynomials defined by λ. We prove that R/I is reduced if λ is almost a rectangle, i.e., $\lambda = (h, \dots, h, r)$, $n = qh + r$ with $0 \leq r < h$. Our method uses a simplified version of the principal radical system, which was originally introduced by Hochster and Eagon to prove that the determinantal ideals are radical in 1971.

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Betti tables of $\mathfrak{S}_n$-invariant monomial ideals

Claudiu Raicu
(joint work with Satoshi Murai)

Consider the polynomial ring $R = \mathbb{C}[x_1, \cdots, x_n]$, together with the action of the symmetric group $\mathfrak{S}_n$ by coordinate permutations. For a monomial ideal I in R which is invariant under the $\mathfrak{S}_n$-action, it is natural to consider associated homological invariants, and to study how the symmetry is reflected in the structure of the invariants. Some of the most fundamental invariants associated to an ideal in a polynomial ring are the Betti numbers,

$$\beta_{i,j}(I) = \dim_{\mathbb{C}} \operatorname{Tor}_i^R(I, \mathbb{C})_j,$$

which measure the complexity of the syzygies between the generators of I. The vanishing and non-vanishing behavior of Betti numbers leads to further invariants such as the Castelnuovo-Mumford regularity

$$\operatorname{reg}(I) = \max\{j - i : \beta_{i,j}(I) \neq 0\},$$

or the projective dimension

$$\operatorname{pdim}(I) = \max\{i : \beta_{i,j}(I) \neq 0 \text{ for some } i\},$$

both of which may be viewed as approximations of the size of the Betti table $(\beta_{i,j}(I))_{i,j}$. Concrete results about regularity and projective dimension in the case of $\mathfrak{S}_n$-invariant monomial ideals I are presented in [2] and [3], leaving open the question of how to understand the individual Betti numbers $\beta_{i,j}(I)$.

In my talk I present some of the main ideas in [1], whose goal is to analyze the $\mathfrak{S}_n$-module structure of the groups $\operatorname{Tor}_i^R(I, \mathbb{C})$, which in turn can be viewed as a way to produce a recipe for computing $\beta_{i,j}(I)$. The main result of [1] shows how the groups $\operatorname{Tor}_i^R(I, \mathbb{C})$ can be assembled together from basic building blocks constructed as induced representations on tensor products of Specht modules for smaller symmetric groups, leading to an equivariant version of the classical Hochster's formula for computing multigraded Betti numbers of monomial ideals. One application of the structure that we discover identifies the $\mathfrak{S}_n$-invariant part of the Tor groups in terms of Betti numbers for the "unsymmetrization" of I.

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Matroid stratifications of determinantal hypergraph varieties and their realization spaces

Fatemeh Mohammadi

(joint work with Oliver Clarke, Kevin Grace, and Harshit Motwani)

Motivated by the realizability and causality problems in statistics, we study associated varieties of hypergraphs from the point of view of projective geometry and matroid theory. In [1], we describe the decompositions of these varieties into matroid varieties, which may be reducible and can have arbitrary singularities by the Mnëv-Sturmfels universality theorem. In particular, we focus on various families of hypergraph varieties for which we explicitly compute an irredundant irreducible decomposition. Our main findings in this direction are threefold: (1) we describe minimal matroids of such hypergraphs; (2) we prove that the varieties of these matroids are irreducible and their union is the hypergraph variety; and (3) we show that every such matroid is realizable over real numbers. As corollaries, we give conceptual decompositions of various, previously-studied, varieties in algebraic statistics (see, e.g., [CMR20, CMM21, EHHM13]). In particular, our decomposition strategy gives immediate matroid interpretations of the irreducible components of multiple families of varieties associated to conditional independence (CI) models in statistical theory and unravels their symmetric structures which hugely simplifies the computations.

Using ideas from projective geometry we show that the variety associated to each hypergraph Δ is the union of certain matroid varieties. The union is taken over all realizable matroids M whose dependent sets contain the edges of Δ. Then, we study the realization spaces of matroids of the form point and line configurations, and we show that the realization spaces of configurations with at most 6 lines are irreducible. Applying tools from matroid theory we prove that for every matroid M, there exists a so-called 'grid' hypergraph $\Delta^{s,t}$ and a dependent matroid M' for $\Delta^{s,t}$ such that a *restriction* of M' is isomorphic to M. Hence, our goal is to understand the geometric properties of the matroids arising from grids. For example, we provide certain sufficient conditions on the parameters s and t of the grids to guarantee that it has a unique minimal matroid.

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