Mathematisches Forschungsinstitut Oberwolfach

Report No. 47/2021

DOI: 10.4171/OWR/2021/47

Singularities (hybrid meeting)

Organized by
Javier Fernandez de Bobadilla, Bilbao
Francois Loeser, Paris
András Némethi, Budapest
Duco van Straten, Mainz

26 September – 2 October 2021

Abstract. Singularity theory concerns local and global structure of singularities of (algebraic) varieties and maps. As such, it combines tools from algebraic geometry, complex analysis, topology, algebra and combinatorics.

Mathematics Subject Classification (2010): 14Bxx, 14Exx, 32Sxx, 58Kxx.

Introduction by the Organizers

The workshop *Singularities* took place in September, 2021. It was organized by Javier Fernandez de Bobadilla (Bilbao), Francois Loeser (Paris), András Némethi (Budapest) and Duco van Straten (Mainz). It continued a long tradition of Oberwolfach meetings on this subject, aiming at gathering experts studying similar objects – singularities of varieties and maps – from various points of view.

The meeting followed a standard schedule of three talks in the morning and two in the afternoon, with a longer break in between to give an opportunity for longer discussion and collaboration in smaller groups. Traditionally, Wednesday afternoon was kept free for a hike.

Because of the pandemics, the meeting was organized in a hybrid format: almost 30 participants in Oberwolfach were joined by around 20 online participants, both for the talks and for the discussions afterwards. Out of 21 talks, 8 were given by online participants: this did not, however, in any way affect the quality of the talks, or the stimulating atmosphere the Oberwolfach workshops are known for.

We will now give a short overview of multiple themes covered during the week, and refer to the abstracts below for a detailed explanation of each.

The meeting begun with a talk of Gert-Martin Greuel, focused on semicontinuity results for singularity invariants: one of these results, concerning completed fiber dimension over an arbitrary noetherian ring, was presented here for the first time. Such an algebraic description of deformations was also one of the themes of Jan Steven's talk, who presented an unprojection method (curiously dubbed "Tom & Jerry" by Miles Reid) to construct codimension 4 Gorenstein rings.

The algebraic approach was also represented in the talk of Eleonore Faber, who used cluster algebras to classify specific maximal Cohen-Maculay models. Interestingly, the same objects where the topic of Augustín Romanos's talk, who approached them from a geometric perspective. Links between singularity theory, algebra and combinatorics were also a recurring theme in the talk of Baldur Sigurðson, who exhibited subtle applications of computation sequences to Weil divisors in toric varieties, which, for example, allow in some cases to compute p_g combinatorially. Similar formulas holds for *generic* surface singularities, which were explored in the talk of János Nagy: in particular, he showed a combinatorial formula for their multiplicities.

Beautiful interplay between geometry and combinatorics was explored during three talks concerning characteristic classes, by Irma Pallarés, Richárd Rimányi and Andrzej Weber. The first one gave a general solution to an important conjecture of Brasselet, Shürmann and Yokura, relating Goresky–MacPherson L-class with a Hirzebruch Todd class in Hodge theory, just like Hodge index theroem computes the signature in terms of Hodge numbers. The latter talks were more specialized. Richárd Rimányi focused on quiver and bow varieties, where 3d mirror symmetry can be expressed in a simple combinatorial way. In his talk, he outlined a proof of a conjecture relating characteristic classes of such variety and its mirror, which up to now was known only in certain special cases. In turn, Andrzej Weber introduced methods to study stable envelopes for cotangent bundles, under certain assumptions which are satisfied whenever the BB-decomposition looks like a Whitney stratification.

In general, constructing nice stratifications is an important technique to understand singularities. Immanuel Halupczok in his talk presented a new way strategy, based on reinterpreting the notion of d-triviality from Lipschitz geometry in the non-archimedean setting. Non-archimedean geometry was also a theme of the talk of Raf Cluckers, focused on C^r-parametrizations with a view towards Wilkie conjecture and its analogues.

A recent breakthrough in Lipschitz geometry was presented by María Pe, who introduced *moderately discontinuous* (MD) homology and homotopy theory, which see the bilipschitz information: more precisely, viewing a subanalytic germ as a cone over its link, MD theory captures different speeds in which the topology of the link collapses towards the origin. This opens a new way, for example, to test whether some analytic invariants are in fact bilipschitz. We note that Lipschitz geometry occurs naturally in the theory of real singularities: for example, the Fukui-Kudryka-Paunescu conjecture, which is a real version of the Zariski multiplicity conjecture, asks if the multiplicity is invariant under a subanaylitic, arc-analytic bilipschitz homeomorphism. The proof of this conjecture was presented in the talk of Edson Sampaio. Moreover, Lipschitz geometry distinguishes broad class of *Lipschitz normally embedded* (LNE) singularities, which played an important role in the talk of Anne Pichon. She showed that the topology of an LNE surface singularity determines the geometry of its polar curve: in general, given a link, this curve can take finitely many shapes. These results fit into a program of "polar exploration" whose goal is to find a generic polar curve for a surface singularity, and as a result, relate two ways of resolving it: by blowups and by Nash transforms.

Pursuit for more and more subtle topological obstructions still occupies an important place in singularity theory. One such obstruction was presented by Maciej Borodzik: it uses Ozsváth–Szabó d-invariant, coming from Heegaard Floer homology, to give strong restrictions on the semigroups of plane curve singularities. Even more refined obstructions can be obtained in the symplectic category, for example, by studying the symplectic representative of the monodromy, as explained by Norbert A'Campo. In turn, Mark McLean in his talk showed how to apply fixed-point Floer cohomology of this symplectomorphism to show that the multiplicity is in fact an embedded contact invariant of the link. He also announced a proof of the conjecture of Budur, de Bobadilla, Lê and Nguyen, which asserts that the above Floer cohomology computes the cohomology of contact loci.

The latter are expected to play a role in the monodromy conjecture, which relates the poles of certain zeta function associated with a singularity with the geometry of its resolution. Such zeta functions, and related exponential sums, were a subject of the talk by Kien Huu Nguyen. One of the main tools used in his talk was the minimal model program (MMP): as it turns out, the geometric information captured by the zeta functions can be seen on the minimal model. This powerful method, coming from birational geometry, was also extensively used in the talk of Chengyang Xu, concerning stability of klt singularities.

Explicit description of the minimizing process can be achieved in dimension two via *almost* MMP, explained in a detailed, three-lecture course by Karol Palka. Applied to a log smooth surface (X, D), or, more generally, to (X, rD) for $r \in [0, 1]$, it gives a way to control the MMP by introducing singularities as late as possible. In case $r = \frac{1}{2}$, this method has already been used to classify planar rational cuspidal curves, and smooth $\mathbb{Q}$-acyclic surfaces, assuming the Negativity Conjecture $\kappa(K_X + \frac{1}{2}D) = -\infty$. Here, (X, D) is a log smooth completion of the surface or of the complement of the curve. In his series of talks, Karol Palka presented a new application of this method to classify normal del Pezzo surfaces of Picard rank 1.

A similar approach – namely, considering log surfaces with various coefficients in the boundary – was used in the talk by Jonathan Wahl to extend the theory of splice singularities to universal abelian (orbifold) covers. Just as in the classical case, given a (now - extended) splice diagram, one gets explicit equations for such a cover, as well as a description of orbifold homology.

As can be seen form the broad range of topics outlined above – and presented in detail in the following abstracts – the meeting indeed gathered experts from various fields, and therefore created a great opportunity to exchange experiences and establish new collaborations. We are extremely grateful to the MFO staff for making this possible by keeping us safe – first and foremost, from the pandemics; and second, from all sorts of distractions which might elsewhere obstruct efficient research, but are known to vanish at this particular Institute.

We hope that new techniques, intuitions and conjectures, conveyed both during the talks and between them, will stimulate further research resulting in interesting discoveries to be presented during the next Oberwolfach *Singularities* meeting.

Workshop (hybrid meeting): Singularities

Table of Contents

Abstracts

Symplectic Monodromy

Norbert A'Campo

The computation of the geometric monodromy, i.e, the monodromy diffeomorphism up to isotopy, of an isolated complex hypersurface singularity is in general a difficult task. For instance, it seems that no algorithm for the computation of the action of the geometric monodromy on the integral homology of the Milnor fiber is known.

A way to progress is to make the problem more difficult and to solve it in a special case. The more difficult problem is to compute the geometric monodromy group of the universal unfolding of an isolated complex hypersurface singularity. This problem is solved for plane curve singularities using a real morsification [1, 2, 4, 3, 5, 8], which gives a finite system of generators for the monodromy group. In [9, 11, 10], Michael Lönne studies the relations of such systems of generators. In the work of Pablo Portilla Cuadrado and Nick Salter [13] the geometric monodromy group is computed by providing a geometric property that characterizes the elements of the mapping class group belonging to the monodromy group. Here we see two approaches, so to say from the inside and from the outside.

In this talk we propose a third approach that up to now only works fine for the A_μ curve singularities [6].

Let the polynomial mapping $f : \mathbb{C}^{n+1} \to \mathbb{C}$ define a hypersurface $X_0 = \{f = 0\}$ having an isolated singularity at the origin. Let $(x^\alpha), \alpha \in A$, be a system of μ monomials that span linearly the quotient ring $\mathbb{C}[[x_0, x_1, \cdots, x_n]]/(\partial_x f)$ by the Jacobian ideal $(\partial_x f)$. The family of polynomial mappings $f_\lambda = f + \sum_\alpha \lambda_\alpha x^\alpha :$ $\mathbb{C}^{n+1} \to \mathbb{C}$ gives by $X_\lambda = \{f_\lambda = 0\}$ the universal unfolding of the singularity at the origin of X_0.

For $\epsilon > 0$ small enough define the local singular fibre to be $F_0 = \{p \in \mathbb{C}^{n+1} \mid f(p) = 0, N_f(p) \leq \epsilon\}_0$, where $N_f(p) = *(df_p \wedge * d\bar{f}_p)$ is the square of the norm of the differential df_p. The suffix $\}_0$ means here that we take the connected component containing the origin. By the curve selection lemma [12] of Henri Cartan and François Bruhat the local singular fibre F_0 is well defined and independent of $\epsilon > 0$ if chosen small enough. For $\delta > 0$ small enough let $\Lambda = B_\delta \subset \mathbb{C}^A$ be the ball centered at the origin. Define $F_\lambda = \{p \in \mathbb{C}^{n+1} \mid f_\lambda(p) = 0, N_{f_\lambda}(p) \leq \epsilon\}_0$, $\lambda \in \Lambda$. The family $(X_\lambda), \lambda \in \Lambda$, is a representative of the universal unfolding of the isolated singularity of F_0. Fibres $F_\lambda, \lambda \neq 0$, are called nearby fibres. Define $\Lambda^* \subset \Lambda$ such that the fibre F_λ is a smooth manifold with boundary if $\lambda \in \Lambda^*$. The family $(F_\lambda)_{\lambda \in \Lambda`*}$ defines a smooth locally trivial fibre bundle $\pi : E_\Lambda \to \Lambda^*$ of smooth manifolds with boundary. Its geometric monodromy is a representation of the fundamental group $\pi_1(\Lambda^*, \lambda_0)$ into the relative mapping class group of $(F_{\lambda_0}, \partial F_{\lambda_0})$ and its image is called the geometric monodromy group of the singularity.

The fibres $F_\lambda, \lambda \in \Lambda^*$, carry a symplectic structure ω_λ and a complex structure J_λ both induced from the ambient space. Moreover, on each fibre is defined the function $p \in F_\lambda \mapsto N_{f_\lambda}(p) \in \mathbb{R}_{\geq 0}$.

Vladimir Arnol'd made the important observation that the symplectic structure ω_λ is locally constant up to stretching. More precisely, first make the above ϵ depending on λ. Put $\epsilon_0 = \epsilon$ and define ϵ_λ implicitly by requiring that the function $\lambda \in \Lambda \mapsto \mathrm{Vol}_\omega(\{p \in \mathbb{C}^{n+1} \mid f_\lambda(p) = 0, N_{f_\lambda}(p) \leq \epsilon_\lambda\}_0,\ \lambda \in \Lambda$, is constant.

The speed vector field $\dot\gamma$ of a smooth path $\gamma : [0.1] \to \Lambda^*$ lifts in a unique way to a vector field $\dot\Gamma$ above the path γ such that $(D\pi)_p(\dot\Gamma_p) = \dot{\gamma(t)}$ and $\dot\Gamma_p \perp_\omega F_{\pi(p)}, \pi(p) = \gamma(t), t \in [0,1]$, hold. From the Cartan homotopy formula follows that the flow of the field $\dot\Gamma$ generates an injective smooth symplectomorphism $\phi_\gamma : (F_{\gamma(0)}, \omega_{\gamma(0)}) \to (X_{\gamma(1)}, \omega_{\gamma(1)})$. The Rigidity Theorem of Moser allows to post compose ϕ_γ by an isotopy yielding a symplectic diffeomorphism $\psi_\gamma : (F_{\gamma(0)}, \omega_{\gamma(0)}) \to (F_{\gamma(1)}, \omega_{\gamma(1)})$ and confirming the observation of Arnol'd. By choosing $\delta \ll \epsilon$ we may choose this isotopy to be small and such that for a closed path the diffeomorphism ψ_γ is the identity on $\partial F_{\gamma(0)}$. Now, the system ψ_γ where γ runs over finitely many generators of $\pi_1(\Lambda^*, \lambda_0)$ will give a monodromy representation into the relative symplectic mapping class group of $(F_{\lambda_0}, \partial F_{\lambda_0})$.

The function $N_\lambda = N_{f_\lambda}$ on F_λ has a symplectic gradient Y_λ defined by $i_{Y_\lambda}\omega_\lambda = dN_\lambda$. The vector field $X_\lambda = JY_\lambda$ is the metrical gradient of the function $-N_\lambda$. The flow lines of X_0 contract the singular fibre F_0 to 0, as the flow lines on F_λ, $\lambda \in \Lambda$, contract the smooth fibre to a skeleton. Define $\Lambda^{**} \subset \Lambda^*$ to be the subset for which the function N_λ is a Morse function. Define the beginning of a stratification on Λ^* by declaring the connected components of Λ^{**} to be its top dimensional strata. Now the speculation is to compute the above symplectic monodromy representation by a wall-crossing method.

For the moment, the above is a speculation, but we will explain how it already works for A_μ curve singularities.

Let $f = y^2 - x^{\mu+1}$ define by $f = 0$ the hypersurface singularity A_μ at $0 \in \mathbb{C}^2$. Its universal unfolding $f_\lambda = y^2 - P_\lambda(x)$ is parametrized by the vector space $\Lambda = \mathbb{C}^n$ of balanced univariate polynomials $P_\lambda(x) = x^{\mu+1} + \lambda_{\mu-1}x^{\mu-1} + \cdots + \lambda_1 x + \lambda_0$ of degree $\mu + 1$. Only monomials x^k appear, which allows us to define N_λ as the norm square of the partial differential $\partial_y f_\lambda = \frac{1}{2}y$. The unfolding is weighted homogeneous, which allows us to define $F_\lambda = X_\lambda$, i.e. think of $\epsilon = \delta = \infty$. The function N_λ on X_λ is in this case $N_\lambda = \frac{1}{4}|P_\lambda(x)|^2 : X_\lambda \to \mathbb{R}$.

As before, Λ^* is the set of λ's for which X_λ is smooth and Λ^{**} the set for which N_λ^2 is a Morse function, i.e. the set such that the polynomial P_λ has $\mu + 1$ distinct roots and that the polynomial P'_λ has μ distinct roots. We need an extra condition: define Λ^{***} to be the set of $\lambda \in \Lambda^{**}$ for which the gradient lines of $-|P_\lambda|$ that start at any $a \in \mathbb{R} \subset \mathbb{C}, a \ggg 1$, have a root of P_λ in their closure. Call $f_\lambda, \lambda \in \Lambda^{***}$ very generic.

Top dimensional strata in Λ^* will be the connected components of Λ^{***}. Let $U \subset \Lambda^{***}$ be a top dimensional stratum with $\lambda \in U$. First we introduce combinatorial data that are attached to the Morse functions $N_\lambda : X_\lambda \to \mathbb{R}$ and $|P_\lambda(x)|^2 : \mathbb{C} \to \mathbb{R}$.

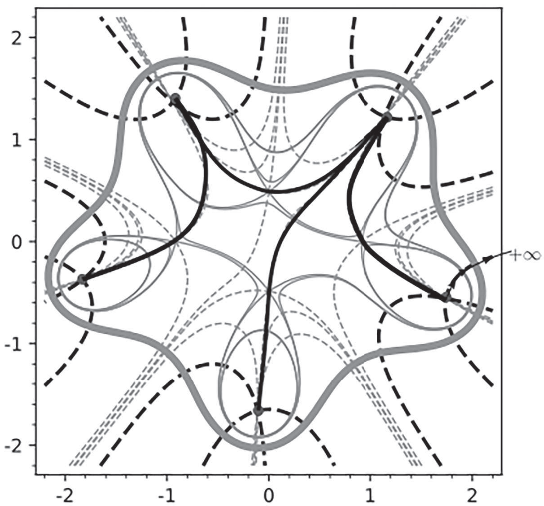


FIGURE 1. Rooted tree Γ_{P_λ} for $P_\lambda(x) = x^5 - x^3 - x + 15i + 5$. The subtree E_{P_λ} having as vertices the roots of P_λ is the Dynkin diagram D_5.

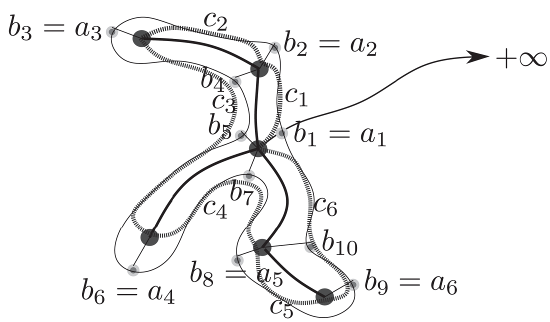


FIGURE 2. The tree Γ_P in black, 6 vertices corresponding to roots of $P(x)$, $\mu = 5$, images of cycles $c_1, c_2, \cdots, c_5, c_6$ in gray, cycles $c_1, c_2, \cdots, c_5$ form the Dynkin diagram in the fibre $\{(x, y) \in \mathbb{C}^2 \mid y^2 + P(x) = 0\}$.

See Fig. 1, an example for the very generic polynomial $P_\lambda(x) = x^5 - x^3 - x + 15i + 5$. The thin gray lines are critical levels of $|P_\lambda(x)|^2$, the thick gray line is the level $|P_\lambda(x)| = 25$, the thin dashed gray lines are flow lines of $-\text{grad}(|P_\lambda|^2)$. Moreover, the thick dashed gray ones are the inverse image by P_λ of the real and imaginary axis. The black ones are saddle connections of roots of P_λ and the (thin) flow line starting at $+\infty$.

The black part is a planar tree Γ_{P_λ}, rooted at $+\infty$ with degree(P_λ) vertices in the plane. As first result we state that the top dimensional strata of Λ^* correspond bijectively to isotopy classes of at $+\infty$ rooted planar trees with $\mu+1 = \text{degree}(P_\lambda)$ vertices. So the number of top dimensional strata is the Catalan number $\text{Cat}(\mu) = \frac{1}{\mu+1}\binom{2\mu}{\mu}$.

There are two types of codimension one wall strata. The first type, a real codimension one stratum consists of a connected component of those λ's for which the gradient line of $-\text{grad}(|P_\lambda|)$ that starts at $+\infty$ has a zero of P'_λ in its closure and for which the function $|P_\lambda|^2$ is a Morse function. The second consists of a connected component of those λ's for which P_λ is very generic, except that P'_λ has one root of multiplicity 2.

The inverse image of Γ_{P_λ} by the map $(x, y) \in X_\lambda \mapsto x \in \mathbb{C}$ is a system V_λ of vanishing cycles together with a relative cycle. There exists a prefered orientation for these cycles. The Dynkin diagram is far from being of type A_μ in general.

In order to obtain a Dynkin diagram of type A_μ one draws gray curves as in Fig. 2. Start at b_1, the root in the closure of the flow line from $+\infty$, go counter clock wise around the subtree E_{P_λ} and visit each root of P_λ once. The inverse image of the gray cycle in X_λ is a configuration of vanishing cycles that realizes the extended Dynkin diagram $\tilde{A}_\mu$.

Wall crossings lead to relabelings and Whitehead moves. For instance crossing walls of the first type changes the label of the root b_1 of P_λ, which has many relabelings as a consequence. Crossing a wall of second type corresponds to a Whitehead move of the planar tree E_{P_λ}. The systems $V_{\lambda'}, V_{\lambda''}$ on both sides of a wall are related by a well defined wall crossing element $\phi_{\lambda',\lambda''}$ in the mapping class group of F_λ, if one crosses at λ a codimension one wall. Composing the wall crossing elements along a loop computes the geometric monodromy.

For more details we refer to our unstable preprint [6], that is available on demand. The above approach was motivated by my try to understand cluster monodromy, as explained to me, CH, by Roger Casals, CA, in his zoom talk at Sabir Gusein-Zade seminar, September 2020, in Moscow, see [7].

References

[1] N. A'Campo, *Le groupe de monodromie du déploiement des singularités isolées de courbes planes. I*, Math. Ann. **213** (1975), 1–32.

[2] ———, *Le groupe de monodromie du déploiement des singularités isolées de courbes planes. II*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, 1975, pp. 395–404.

[3] ———, *Generic immersions of curves, knots, monodromy and Gordian number*, Inst. Hautes Études Sci. Publ. Math. (1998), no. 88, 151–169 (1999).

[4] ———, *Real deformations and complex topology of plane curve singularities*, Ann. Fac. Sci. Toulouse Math. (6) **8** (1999), no. 1, 5–23.

[5] ———, *Quadratic vanishing cycles, reduction curves and reduction of the monodromy group of plane curve singularities*, Tohoku Math. J. (2) **53** (2001), no. 4, 533–552.

[6] ———, *Flowbox decomposition for gradients of univariate polynomials. billiards. Tree like configurations of vanishing cycles for A_n curve singularities. Geometric cluster monodromy group*, preprint, 2021.

[7] R. Cassals, *Lagrangian skeleta and plane curve singularities*, arXiv:2009.06737, 2020.

[8] S. M. Guseĭn-Zade, *Intersection matrices for certain singularities of functions of two variables*, Funkcional. Anal. i Priložen. **8** (1974), no. 1, 11–15.
[9] M. Lönne, *Fundamental group of discriminant complements of Brieskorn-Pham polynomials*, C. R. Math. Acad. Sci. Paris **345** (2007), no. 2, 93–96.
[10] ———, *Fundamental groups of moduli stacks of smooth Weierstrass fibrations*, arXiv:0602371, 2007.
[11] ———, *Braid monodromy of some Brieskorn-Pham singularities*, Internat. J. Math. **21** (2010), no. 8, 1047–1070.
[12] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, vol. 61, Princeton University Press, 1968.
[13] P. Portilla Cuadrado and N. Salter, *Vanishing cycles, plane curve singularities, and framed mapping class groups*, arXiv:2004.01208, 2021.

Heegaard Floer homology and plane curves

Maciej Borodzik
(joint work with Beibei Liu, Ian Zemke)

The aim of the project is to study which configurations of singularities can occur on complex curves in $\mathbb{C}P^2$ of given degree. We present some obstructions for realization of given configurations, stemming from Heegaard Floer theory.

Let $C \subset \mathbb{C}P^2$ be a reduced complex curve of degree d. Let N be its tubular neighborhood, intepreted as the set of points in $\mathbb{C}P^2$ at distance less than $\varepsilon \ll 1$ from N. The boundary $Y = \partial N(C)$ is a smooth, three-dimensional manifold. On the other hand, $Y = \partial X$, where $X = \mathbb{C}P^2 \setminus N(C)$. If C is rational cuspidal, X is a rational homology ball. In that case, if $\mathfrak{s}$ is a spin-c structure on Y that extends over X, by [6] we have $d(Y, \mathfrak{s}) = 0$. Here $d(Y, \mathfrak{s})$ is the Ozsváth–Szabó d-invariant from Heegaard Floer homology.

On the other hand, as Y can be shown to be a surgery on the (connected sum of) links of singularities, its d-invariants can be computed from invariants of the underlying singularity. To give a brief overview, let $z \in C$ be a cuspidal singular point and S_z be its semigroup (refer to [8] for precise definition and properties of S_z). The *semigroup counting function* is

$$R_z(j) := \#\{x \in S_z \colon x < j\}.$$

In [4], based on calculations of Ozsváth and Szabó [6], the invariants $d(Y, \mathfrak{s})$ were expressed in terms of functions R_z. The condition $d(Y, \mathfrak{s}) = 0$, translates into restrictions on the values of the semigroup counting function for singularities on rational cuspidal curves; see [4] for more detail. These restrictions resemble the conjecture [5].

If C has only cuspidal singularities, but positive geometric genus, Bodnár, Celoria, and Golla, and — independently — Borodzik, Hedden, and Livingston [1, 2] proved analogous restrictions. The method uses d_{top} and d_{bot} invariants associated with three-manifolds having $b_1 > 0$, instead of ordinary d-invariants. The key difficulty is to express these invariants of $Y = \partial N(C)$ in terms of the semigroup counting function. The formula is more complex than in the $b_1 = 0$ case, because Y is a surgery on a connected sum of links of singularities and several Borromean rings. To compute $d(Y,\mathfrak{s})$ one needs to understand the knot Floer chain complex of Borromean rings, but this was already done in [6].

Suppose C has non-cuspidal singularities. In [3], we show that the three-manifold $Y=\partial N(C)$ can be expressed as a surgery on connected sum of links of cuspidal singularities connected with several copies of Borromean rings *and* with knotification of links of non-cuspidal singularities. Here, a *knotification* assigns to an n-component link $L\subset S^3$ a knot $\widehat{L}\subset \#_{n-1}S^2\times S^1$ in a canonical way. In [7] there is a comparison of link Floer chain complexes of L and $\widehat{L}$, but only for the hat version of Heegaard Floer theory. This is usually not sufficient to compute d-invariants of surgeries.

In [3], based on previous results of the third author (in a series of papers starting from [9]), we compare the Floer chain complex for $\mathrm{CFK}^\infty(L)$ and $\mathrm{CFK}^\infty(\widehat{L})$. We are also able to recover the action of $\Lambda^*H_1(Y;\mathbb{Z})/\operatorname{Tors}$ on $\widehat{L}$. These results are of interest on their own. As an application, we compute Floer chain complex of knotification of the $T(2,2n)$ torus link, including Hopf links. Repeating the procedure started in [4], we obtain various restrictions for possible configurations of singular points on curves in $\mathbb{C}P^2$. One of the results that we prove is the following (item (a) was proved in [1, 2] and is given here for comparison).

Proposition 1. Let C be a degree d curve with one cuspidal singular point, whose semigroup counting function is denoted by R. Assume C has at most one ordinary double point and no other singularities. For all $k=1,\dots,d-2$ set $K_k=\frac{1}{2}(j+1)(j+2)$.

(a) If $g=1$ and C has no double points, then $R(kd-1)\in\{K_k-1,K_k\}$, $R(kd+1)\in\{K_k,K_k+1\}$ for all k;

(b) If $g=0$ and C has a single positive double point, then $R(kd-1)\in\{K_k-1,K_k\}$, $R(kd+1)\in\{K_k,K_k+1\}$ for all k, but also
$$R(kd)\leq K_k.$$

(c) If $g=0$ and C has a single negative double point, then $R(kd-1)\in\{K_k-1,K_k\}$, $R(kd+1)\in\{K_k,K_k+1\}$ for all k, but also
$$R(kd)\geq K_k.$$

We remark that a complex curve C cannot have negative double point. In that case, the correct assumptions on C might be that C is a smooth curve except at singular points; near singular points C is diffeomorphic to the cone on the link of singularity. See [3] for more detail.

Proposition 1 allows us to obstruct deforming curves of genus 1 to a curve of genus 0 at the expense of creating an extra double point.

References

[1] J. Bodnár, D. Celoria, and M. Golla, *Cuspidal curves and Heegaard Floer homology*, Proc. Lond. Math. Soc. **112** (2016), no. 3, 512–548.

[2] M. Borodzik, M. Hedden, and C. Livingston, *Plane algebraic curves of arbitrary genus via Heegaard Floer homology*, Comment. Math. Helv. **92** (2017), no. 2, 215–256.

[3] M. Borodzik, B. Liu, and I. Zemke, *Heegaard Floer homology and plane curves with non-cuspidal singularities*, preprint, arxiv: 2104.13709.
[4] M. Borodzik, and C. Livingston, *Heegaard Floer homology and rational cuspidal curves*, Forum Math. Sigma **2** (2014), Paper No. e28, 23 pp.
[5] J. Fernández de Bobadilla, I. Luengo, A. Melle-Hernández and A. Némethi, *On rational cuspidal projective plane curves*, Proc. of London Math. Soc., **92** (2006), 99–138.
[6] P. Ozsváth and Z. Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math., **173** (2003), 179–261.
[7] ———, *Holomorphic disks, link invariants and the multi-variable Alexander polynomial*, Algebr. Geom. Topol. **8** (2008), no. 2, 615–692.
[8] C. T. C. Wall, *Singular Points of Plane Curves* London Mathematical Society Student Texts, 63. Cambridge University Press, Cambridge, 2004.
[9] I. Zemke, *A graph TQFT for hat Heegaard Floer homology*, arXiv:1503.05846, 2020.

On C^r-parametrizations in and outside o-minimality, and point counting

Raf Cluckers

C^r parameterizations for real semi-algebraic sets were discovered by Yomdin and Gromov [10, 9, 7] in the late eighties (with several alternative and more detailed proofs to follow, e.g. by Burguet, Pila-Wilkie, Kocel-Pawłucki-Valette, Binyamini-Novikov). Such parameterizations were initially used in the study of entropy by Yomdin and Gromov. More recently, their generalizations to the o-minimal setting by Pila and Wilkie [8] have given striking diophantine applications. Since then, many variants of the C^r parameterizations and their applications have emerged, both in o-minimal and non-archimedean settings.

I will discuss some of such recent variants. One such variant is an improved result on parameterizations for (real) globally subanalytic sets, by Pila, Wilkie and myself [6]. This came in fact in analogy to results on the subanalytic sets in the p-adics, by Comte, Forey, Loeser and myself [2, 3], leading to improved point counting results on subanalytic sets (both real and p-adic), in a direction towards Wilkie's Conjecture (namely with polylogarithmic instead of subexponential upper bounds).

In fact, bounds for rational points on definable sets in non-archimedean settings have many faces. On subsets of $\mathbb{Q}_p^n$, one can count actual rational points of bounded height, where on subsets of $\mathbb{C}((t))^n$ one rather "counts" the polynomials in t of bounded degree. But, the latter can be infinite! For $\mathbb{C}((t))$, we discuss:

1) the setting of subanalytic sets, where we show finiteness of the rational points of bounded height (on the transcendental part of any subanalytic set) but where growth can be aribitrarily fast with the degree in t (playing the role of the height);

2) the setting of Pfaffian sets, which is new in the non-archimedean world and for which we show an analogue of Wilkie's Conjecture in all dimensions;

3) the (axiomatic) Hensel minimal setting, which is most general and where finiteness starts to fail, even for definable transcendental curves. In this infinite case, one bounds the dimension rather than the (infinite) cardinality.

Items 1)–3) represent joint work with Binyamini, Novikov [1], Halupczok, Rideau, Vermeulen [5], and separate work by Cantoral-Farfan, Nguyen, Vermeulen [4].

References

[1] G. Binyamini, R. Cluckers, and D. Novikov, *Point counting and Wilkie's conjecture for non-archimedean Pfaffian and Noetherian functions*, arXiv:2009.05480, to appear in Duke Math. J., 2021.

[2] R. Cluckers, G. Comte, and F. Loeser, *Non-Archimedean Yomdin-Gromov parametrizations and points of bounded height*, Forum Math. Pi **3** (2015), e5, 60.

[3] R. Cluckers, A. Forey, and F. Loeser, *Uniform Yomdin-Gromov parametrizations and points of bounded height in valued fields*, Algebra Number Theory **14** (2020), no. 6, 1423–1456.

[4] V. Cantoral-Farfán, K. H. Nguyen, and F. Vermeulen, *A Pila-Wilkie theorem for Hensel minimal curves*, arXiv:2107.03643, 2021.

[5] R. Cluckers, I. Halupczok, S. Rideau, and F. Vermeulen, *Hensel minimality. II: Mixed characteristic and a diophantine application*, arXiv:2104.09475, 2021.

[6] R. Cluckers, J. Pila, and A. Wilkie, *Uniform parameterization of subanalytic sets and diophantine applications*, Ann. Sci. Éc. Norm. Supér. (4) **53** (2020), no. 1, 1–42.

[7] M. Gromov, *Entropy, homology and semialgebraic geometry*, no. 145-146, 1987, Séminaire Bourbaki, Vol. 1985/86, pp. 5, 225–240.

[8] J. Pila and A. J. Wilkie, *The rational points of a definable set*, Duke Math. J. **133** (2006), no. 3, 591–616.

[9] Y. Yomdin, *Volume growth and entropy*, Israel J. Math. **57** (1987), no. 3, 285–300.

[10] ———, *C^k-resolution of semialgebraic mappings. Addendum to: "Volume growth and entropy"*, Israel J. Math. **57** (1987), no. 3, 301–317.

From line singularities to Grassmannian categories of infinite rank

Eleonore Faber

(joint work with Jenny August, Man-Wai Cheung, Sira Gratz and Sibylle Schroll)

This talk is about a categorification of the coordinate rings of Grassmannians of infinite rank in terms of graded maximal Cohen-Macaulay modules over the coordinate ring of line singularities $\operatorname{Spec}(\mathbb{C}[x,y]/(x^k))$, for $k \geq 2$. This yields an infinite rank analogue of the Grassmannian cluster categories introduced by Jensen–King–Su. In the case $k = 2$, the line singularity is of countable Cohen-Macaulay type, and our category has a combinatorial model by an "∞-gon".

Our work brings together concepts from singularity theory (Cohen–Macaulay type) and combinatorial representation theory (cluster algebras and categories, see e.g. [Kel08] for an introduction). Cluster algebras were introduced by Fomin–Zelevinsky in the early 2000s to study questions related to Lusztig's dual canonical basis and total positivity. Our motivation is from categorification: one tries to interpret cluster algebras as combinatorial invariants associated to categories of representations. From the rich structure of these so-called cluster categories one then hopes to prove results on cluster algebras which seem beyond the scope of purely combinatorial methods. In the other direction, one tries to find combinatorial models for certain triangulated categories, which give a simple way to understand the structure of these categories.

One of the most striking examples of a cluster algebra is the homogeneous coordinate ring of the Grassmannian $\mathbb{C}[\mathrm{Gr}(k,n)]$, which carries a natural cluster structure. Jensen–King–Su [JKS16] introduced an additive categorification of $\mathbb{C}[\mathrm{Gr}(k,n)]$ of finite rank (i.e., $n<\infty$) via G-equivariant maximal Cohen–Macaulay modules over the plane curve singularities $R_{(k,n)} = \mathbb{C}[x,y]/(x^k - y^{n-k})$, where G is the cyclic group of order n acting on $R_{(k,n)}$ in a natural way. They show that rank 1 modules in $\mathrm{MCM}_G R_{(k,n)}$ are in bijection with Plücker coordinates in $\mathbb{C}[\mathrm{Gr}(k,n)]$, and that this bijection is structure preserving: rigidity of subcategories of rank 1 modules is translated to compatibility of the corresponding Plücker coordinates (i.e. pairwise noncrossing of k-subsets). This connects two examples of a priori unrelated ADE classifications: Grassmannian cluster algebras $\mathbb{C}[\mathrm{Gr}(k,n)]$ of finite type, and simple plane curve singularities $R_{(k,n)}$, which occur in the cases $k=2$ and $n\geq 4$ (type A_{n-3}), $k=3$ and $n=6$ (type D_4), $k=3$ and $n=7$ (type E_6), and $k=3$ and $n=8$ (type E_8). Here, the type indicates both cluster algebra type and singularity type respectively.

Now fix $k\geq 2$ and let n go to infinity: then a natural object to consider on the cluster algebra side is the ring

$$\mathcal{A}_k = \frac{\mathbb{C}[p_I \mid I\subseteq \mathbb{Z}, |I|=k]}{\langle \text{Plücker relations}\rangle}.$$

One can show that this ring can be equipped with the structure of a cluster algebra of infinite rank in the sense of [GG14] and further be interpreted as the homogeneous coordinate ring of an infinite Grassmannian under a generalized Plücker embedding, see the appendix by Groechenig [GG14].

We construct an analogue of the Jensen, King, and Su Grassmannian cluster categories in this infinite setting: For a fixed $k\geq 2$, the Grassmannian category of infinite rank $\mathcal{C}_k$ is defined to be the category of finitely generated $\mathbb{Z}$-graded maximal Cohen–Macaulay modules over the graded ring $R_k = \mathbb{C}[x,y]/(x^k)$, where $\deg(x)=1$, and $\deg(y)=-1$. This is a natural construction: the singularity $\mathrm{Spec}(\mathbb{C}[x,y]/(x^k))$ is the limit of $\mathrm{Spec}(\mathbb{C}[x,y]/(x^k-y^{n-k}))$ as $n\to\infty$, and the cyclic group actions yield a circle action in the limit, giving rise to the $\mathbb{Z}$-grading.

Thus we find a categorical incarnation of the combinatorics of the infinite Grassmannians. As one of our main results, we rediscover the combinatorial description of Plücker coordinates through certain indecomposable objects in $\mathcal{C}_k$:

Theorem 1 (Thm. 3.9, 4.1 in [ACF$^+$20])**.** *(1) There is a bijection between Plücker coordinates in $\mathcal{A}_k$ and generically free rank 1-modules in $\mathcal{C}_k = \mathrm{MCM}_{\mathbb{Z}} R_k$.*

(2) Let I and J be generically free modules of rank 1 in $\mathcal{C}_k$ corresponding, under the bijection from (1), to Plücker coordinates p_I and p_J respectively. Then $\mathrm{Ext}^1_{\mathcal{C}}(I,J)=0$ if and only if p_I and p_J are compatible.

The correspondence from this theorem connects the concept of rigidity in $\mathcal{C}_k$ with the concept of compatibility of Plücker coordinates. Theorem 1 is a direct consequence of a general formula for the dimension of the Ext^1-space between any two given generically free modules of rank 1. As a corollary, we obtain that Ext^1

on generically free modules of rank 1 is symmetric in its argument, and further that the full subcategory of generically free MCM modules is stably 2-Calabi-Yau.

Let us now consider the case $k = 2$. In this case, we obtain a combinatorial model for our category: The ring $R_2 = \mathbb{C}[x, y]/(x^2)$ can be seen as coordinate ring of a singularity of type A_∞ (these have been studied as isolated line singularities by Siersma [Sie83]). The ring R_2 has countable Cohen–Macaulay type, and indecomposable objects in the category $\mathcal{C}_2$ can be classified via matrix factorizations:

Theorem 2 (Buchweitz–Greuel–Schreyer [BGS87])**.** *Let M be an indecomposable graded* MCM*-module over R_2. Then M is determined by a graded shift of one of the following matrix factorizations of rank $m = 1, 2$ of x^2:*
(1) $M = \operatorname{Coker}(x^2) = R_2$ is defined by the graded matrix factorization of rank 1

$$\mathbb{C}[x, y](-2) \xrightarrow{1} \mathbb{C}[x, y](-2) \xrightarrow{x^2} \mathbb{C}[x, y] \ .$$

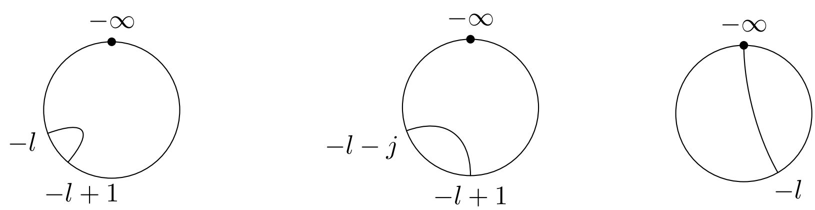
(2) $M = \operatorname{Coker}(x) \cong \mathbb{C}[y]$ is defined by the graded matrix factorization of rank 1

$$\mathbb{C}[x, y](-2) \xrightarrow{x} \mathbb{C}[x, y](-1) \xrightarrow{x} \mathbb{C}[x, y] \ .$$

(3) For $k = 1, 2, \ldots,$ and $A = B = \begin{pmatrix} x & y^j \\ 0 & -x \end{pmatrix}$, the module $M = \operatorname{Coker}(A) \cong (x, y^j)$ is defined by the graded matrix factorizations of rank 2

$$\mathbb{C}[x, y](-2) \oplus \mathbb{C}[x, y](j-1) \xrightarrow{B} \mathbb{C}[x, y](-1) \oplus \mathbb{C}[x, y](j) \xrightarrow{A} \mathbb{C}[x, y] \oplus \mathbb{C}[x, y](j+1) \ .$$

These MCM-modules can be classified, using a geometric Dynkin type A_∞ model, by arcs in an ∞-gon with one marked accumulation point, see Fig. 1.



(1) Arcs corresponding to $R_2(l)$.

(2) Arcs corresponding to $(x, y^j)(l)$.

(4) Arcs corresponding to $\mathbb{C}[y](l)$.

FIGURE 1. Types of arcs in the completed infinity-gon.

Furthermore, $\mathcal{C}_2$ has cluster tilting subcategories, which we determine in [ACF$^+$21], recovering the one-accumulation point case by Paquette–Yıldırım [PY21] from a different perspective. Also see [HJ12, IT15] for related constructions.

The $k \geq 3$ case, is of a different nature: already in the finite rank setting (bar a handful of exceptions), we are in wild Cohen–Macaulay type. This wildness persists in $\mathcal{C}_k$ and a classification of indecomposable objects in $\mathcal{C}_k$ is out of reach. However, via Theorem 1 we can still classify the generically free rank 1 MCM modules, and it would be interesting to study the subcategory of generically free MCM modules from this combinatorial point of view.

References

[ACF+20] J. August, M.-W. Cheung, E. Faber, S. Gratz, and S. Schroll. *Grassmannian cluster categories of infinite rank.* arXiv:2007.14224, 2020.

[ACF+21] ———, *Grassmannian cluster categories of type A_∞.* 2021. in preparation.

[BGS87] R.-O. Buchweitz, G.-M. Greuel, and F.-O. Schreyer. *Cohen-Macaulay modules on hypersurface singularities. II.* Invent. Math., **88**, no. 1, 165–182, 1987.

[GG14] J. E. Grabowski and S. Gratz. *Cluster algebras of infinite rank.* J. Lond. Math. Soc. (2) **89**, no. 2, 337–363, 2014. With an appendix by Michael Groechenig.

[HJ12] T. Holm and P. Jørgensen. *On a cluster category of infinite Dynkin type, and the relation to triangulations of the infinity-gon.* Math. Z., **270**, no. 1-2, 277–295, 2012.

[IT15] K. Igusa and G. Todorov. *Cluster categories coming from cyclic posets.* Comm. Algebra, **43**, no. 10, 4367–4402, 2015.

[JKS16] B. T. Jensen, A. D. King, and X. Su. *A categorification of Grassmannian cluster algebras.* Proc. Lond. Math. Soc. (3), **113**, no. 2, 185–212, 2016.

[Kel08] B. Keller. *Categorification of acyclic cluster algebras: an introduction,* Higher structures in geometry and physics, Progr. Math. vol. 287, Birkhäuser/Springer, New York 2011, pp. 227–241.

[PY21] C. Paquette and E. Yildirim. *Completions of discrete cluster categories of type $\mathbb{A}$.* Trans. Lond. Math. Soc. **8**, no.1, 35–64, 2021.

[Sie83] D. Siersma. *Isolated line singularities.* Singularities, Part 2 (Arcata, Calif., 1981, Proc. Sympos. Pure Math. vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 485–496.

Semicontinuity for families of power series and applications to singularity invariants and computational algorithms

Gert-Martin Greuel

(joint work with Gerhard Pfister and Hans Schönemann)

We report on recent results about the semicontinuity of singularity invariants in families of formal power series. The problem came up in connection with the classification of singularities in positive characteristic (cf. [1]). Then it is important that certain invariants like the determinacy can be bounded simultaneously in families of formal power series parametrized by some algebraic variety and not just by the spectrum of a complete local ring. In the case of complex analytic families such a bound is well known due to the finite coherence theorem, saying that a quasi-finite morphism is finite in a small neighbourhood. However, in our families the fibers are complete local rings and the base rings are affine rings. Then the problem is rather subtle, since the modules defining the invariants are quasi-finite but not finite over the base space. In fact, in general the fibre dimension is not semicontinuous and the quasi-finite locus is not open. However, if we pass to the completed fibers in a family of rings or modules we can prove that their fibre dimension is semicontinuous, even if the base space is the spectrum of an arbitrary Noetherian ring.

The semicontinuity theorem has several applications in singularity theory and we report on two of them in the second part. There we consider first families of parameterizations of reduced curve singularities over a Noetherian base scheme and prove that the delta invariant is semicontinuous. Secondly we provide an algorithm for standard basis computations of a 0-dimensional ideal in a power series ring or in the localization of a polynomial ring in finitely many variables over a field. The algorithm provides a remarkable speed up if the field is the quotient field of a Noetherian integral domain, like the integers or a polynomial ring, when coefficient swell occurs during the computation.

Most of the results appeared in the papers [4], [5], [6]. However, the result that the (semicontinuity) Theorem 1 holds for an arbitrary Noetherian ring is new and was presented in the talk at Oberwolfach for the first time.

1. Semicontinuity for families of power series

Let $\mathbb{k}$ denote an arbitrary field, A a Noetherian ring, $R := A[[x]], x = (x_1, \cdots, x_n)$, the formal power series ring over A and M a finitely generated R-module. Set $k(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = \mathrm{Quot}(A/\mathfrak{p})$, the residue field of A at $\mathfrak{p} \in \mathrm{Spec}\, A$,

$$M(\mathfrak{p}) := M \otimes_A k(\mathfrak{p}) \text{ the fiber of } M \text{ over } \mathfrak{p},$$

and $d_\mathfrak{p}(M) := \dim_{k(\mathfrak{p})} M(\mathfrak{p})$ the *fiber dimension* of M over $\mathfrak{p}$. Semicontinuity of the fiber dimension at $\mathfrak{p}$ would mean that $d_\mathfrak{q}(M) \leq d_\mathfrak{p}(M)$ for all $\mathfrak{q}$ in some neighbourhood of $\mathfrak{p}$.

The example $A = \mathbb{k}[t], M = A[[x]]/\langle x - t\rangle$ shows, however, that $d_\mathfrak{p}(M)$ is not semicontinuous at $\langle 0\rangle$. Namely, $\dim_{k(\mathfrak{p})} M(\mathfrak{p}) = 1$ for $\mathfrak{p} = \langle t\rangle$ and 0 for $\mathfrak{p} = \langle t - c\rangle, c \in \mathbb{k} \setminus 0$, while $\dim_{k(\mathfrak{p})} M(\mathfrak{p}) = \infty$ for $\mathfrak{p} = \langle 0\rangle$ (the generic point $\langle 0\rangle$ is in the neighbourhood of any other point).

On the other hand, the fiber $M(\mathfrak{p})$ is not right object in the applications. For example, the ring $A[[x]](\langle 0\rangle)$ is not local, while we are interested in the local ring $A(\langle 0\rangle)[[x]] = \mathbb{k}(t)[[x]]$, which is the $\langle x\rangle$-adic completion of $A[[x]](\langle 0\rangle)$.

To put the individual $\langle x\rangle$-adic completions of $M(\mathfrak{p})$ into a family over $\mathrm{Spec}\, A$, we introduce the completed tensor product. The *completed tensor product* of R and an A-algebra B is the ring

$$R\hat{\otimes}_A B := \varprojlim \big((R/\langle x\rangle^m) \otimes_A B\big),$$

while the *completed tensor product of M and N over A*, for N an A-module, is

$$M\hat{\otimes}_A N := \varprojlim \big((M/\langle x\rangle^m M) \otimes_A N\big).$$

We define for $\mathfrak{p} \in \mathrm{Spec}\, A$

$\hat{M}(\mathfrak{p}) := M\hat{\otimes}_A k(\mathfrak{p})$ the *completed fiber* of M over $\mathfrak{p}$

$\hat{d}_\mathfrak{p}(M) := \dim_{k(\mathfrak{p})} \hat{M}(\mathfrak{p})$ the *completed fiber dimension* of M over $\mathfrak{p}$.

We have $\hat{M}(\mathfrak{p}) = M(\mathfrak{p})^\wedge$, the $\langle x\rangle$-adic completion of $M(\mathfrak{p})$, and $\hat{R}(\mathfrak{p}) = k(\mathfrak{p})[[x]]$. If $\mathfrak{p}$ is a closed point (a maximal ideal), then $\hat{M}(\mathfrak{p}) = M(\mathfrak{p})$.

The following semicontinuity theorem is new in this generality.

Theorem 1. *Let $\mathfrak{p} \in \mathrm{Spec}\, A$ such that $\hat{d}_\mathfrak{p}(M) < \infty$. Then there is an open neighbourhood $U \subset \mathrm{Spec}\, A$ of $\mathfrak{p}$ such that*

$$\hat{d}_\mathfrak{q}(M) \leq \hat{d}_\mathfrak{p}(M) \textit{ for all } \mathfrak{q} \in U.$$

Since the inequality holds also if $\hat{d}_{\mathfrak{p}}(M) = \infty$, the function $\mathfrak{p} \mapsto \hat{d}_{\mathfrak{p}}(M)$ is upper semicontinuous on $\operatorname{Spec} A$ and for each $N = 0, 1, 2, ..., \infty$ the set

$$\{\mathfrak{p} \in \operatorname{Spec} A | \; \hat{d}_{\mathfrak{p}}(M) \leq N\}$$

is open in $\operatorname{Spec} A$.

For the proof we use that a presentation matrix of $\hat{M}(\mathfrak{p})$ is finitely determined (cf. [3, Theorem 3.2]) and hence can be replaced by a polynomial matrix and then apply [4, Theorem 42].

Corollary 2. *The Tjurina number $\tau(\hat{M}(\mathfrak{p}))$ is semicontinuous on* $\operatorname{Spec} A$*.*

The *Tjurina number* of $\hat{M}(\mathfrak{p}))$ is defined as the $k(\mathfrak{p})$-codimension of the extended tangent image (cf. [3]) of the presentation matrix of $\hat{M}(\mathfrak{p})$ in the space of all matrices (of the corresponding size) with entries in $k(\mathfrak{p})[[x]]$. It coincides with the usual Tjurina number if $\hat{M}(\mathfrak{p}) = \hat{R}(\mathfrak{p})/I$, where I defines a complete intersection singularity, e.g. a hypersurface singularity. The same semicontinuity holds for the Milnor number of a hypersurface singularity.

Let us look at an example with $A = \mathbb{Z}$, $F(x) \in \mathbb{Z}[[x]] \subset \mathbb{Q}[[x]]$, $p \in \mathbb{Z}$ a prime number, $F_p(x)$ the image of F in $\mathbb{F}_p[[x]]$. Then the Milnor number satisfies

$$\mu(F) \leq \mu(F_p) \;\text{ for all prime numbers } p.$$

We mention that such a result cannot be proved just by Gröbner basis methods (one can prove the inequality for almost all prime numbers p). The proof uses essentially the semicontinuity for the completed fiber dimension over non-closed points and will be important for the algorithm in the last section.

2. Semicontinuity of the delta invariant in a family of parameterizations.

The delta invariant is the most important numerical invariant of a reduced curve singularity. We call a morphism of A-algebras,

$$\varphi_A : P_A := A[[x]] = A[[x_1, ..., x_n]] \to \tilde{R}_A := A[[t]],$$

a *family of parameterizations of reduced curve singularities over* A (for simplicity we consider here only one branch). For a prime ideal $\mathfrak{p}$ of A the map

$$\varphi_{\mathfrak{p}} : P_{\mathfrak{p}} := k(\mathfrak{p})[[x_1, ..., x_n]] \to \tilde{R}_{\mathfrak{p}} := k(\mathfrak{p})[[t]]$$

is called a *parameterization of the reduced curve singularity*

$$R_{\mathfrak{p}} := P_{\mathfrak{p}}/Ker(\varphi_{\mathfrak{p}}) \cong \varphi_{\mathfrak{p}}(P_{\mathfrak{p}}) \subset \tilde{R}_{\mathfrak{p}}$$

if $\varphi_{\mathfrak{p}} \neq 0$ ($R_{\mathfrak{p}}$ is then a 1-dimensional reduced ring). We have $P_{\mathfrak{p}} = P_A \hat{\otimes}_A k(\mathfrak{p})$ and $\tilde{R}_{\mathfrak{p}} = \tilde{R}_A \hat{\otimes}_A k(\mathfrak{p})$ (but $R_{\mathfrak{p}} \neq R_A \hat{\otimes}_A k(\mathfrak{p})$ in general, even if $\mathfrak{p}$ is maximal).

We define the *delta-invariant* of $R_{\mathfrak{p}}$ ($\bar{R}_{\mathfrak{p}}$ the normalization of $R_{\mathfrak{p}}$) as

$$\delta(R_{\mathfrak{p}}) := \dim_{k(\mathfrak{p})} \bar{R}_{\mathfrak{p}}/R_{\mathfrak{p}}$$

Theorem 3. *Let $\varphi_A : P_A \to \tilde{R}_A$ be a family of parameterizations, $\mathfrak{p} \in \operatorname{Spec} A$. If $\varphi_\mathfrak{p} : P_\mathfrak{p} \to \tilde{R}_\mathfrak{p}$ is a (primitive) parameterization of the reduced curve singularity $R_\mathfrak{p}$, then there exists an open neighbourhood $U \subset \operatorname{Spec} A$ of $\mathfrak{p}$ such that*

(1) $R_\mathfrak{q} \hookrightarrow \tilde{R}_\mathfrak{q}$ is the normalization of $R_\mathfrak{q}$ for each $\mathfrak{q} \in U$ and
(2) $\mathfrak{q} \mapsto \delta(R_\mathfrak{q})$ is upper semicontinuous on U.

Not all singularity invariants are semicontinuous, e.g. the dimension of the conductor, $\dim_{k(\mathfrak{p})} \bar{R}_\mathfrak{p} / \operatorname{Ann}_{R_\mathfrak{p}}(\bar{R}_\mathfrak{p}/R_\mathfrak{p})$ is in general neither upper nor lower semicontinuous.

The proof (cf. [5, Theorem 18]) is similar to the proof of Theorem 1 but has to be modified since here we are not dealing with fibers of a morphism but with families of subalgebras. We need the following determinacy theorem ([5, Proposition 11]) for parameterizations.

Proposition 4. The parameterization $\varphi_\mathfrak{p} : P_\mathfrak{p} \to \tilde{R}_\mathfrak{p}$ is $4\delta(R_\mathfrak{p})-2$ left-determined.

3. Semicontinuity and standard basis computations.

Let A be a domain[1], $K := \operatorname{Quot}(A) = k(\langle 0 \rangle)$ and $>$ a local monomial ordering on $A[x] = A[x_1, ..., x_n]$. Let I be an ideal in $A[x]$. We want to compute

$$\dim_{k(\mathfrak{p})} k(\mathfrak{p})[x]_{\langle x \rangle}/I(\mathfrak{p}) = \dim_{k(\mathfrak{p})} k(\mathfrak{p})[[x]]/\hat{I}(\mathfrak{p}),$$

$\mathfrak{p} \in \operatorname{Spec} A$, for $\mathfrak{p} = \langle 0 \rangle$ by using the computation for $\mathfrak{p} \neq \langle 0 \rangle$. Here $I(\mathfrak{p})$ resp. $\hat{I}(\mathfrak{p})$ is the ideal generated by I in $k(\mathfrak{p})[x]_{\langle x \rangle}$ resp. in $k(\mathfrak{p})[[x]]$. For the results of this section see [6].

A monomial m is the *highest corner* of an ideal $I \subset A[x]$, denoted by HC_I, if m is the smallest (w.r.t. $>$) monomial not in the leading ideal of I. Using Theorem 1, we prove that the highest corner behaves semicontinuos, in particular that the degree of the highest corner of I over the generic point $\langle 0 \rangle$ is smaller or equal than over any other point $\mathfrak{p} \in \operatorname{Spec} A$, provided the codimension of $I(\langle 0 \rangle) \subset K[x]_{\langle x \rangle}$ and $I(\mathfrak{p}) \subset k(\mathfrak{p})[x]_{\langle x \rangle}$ are the same. This is used in the following algorithm ([6, Algorithm 3.2]).

INPUT: $S \subset A[x]$ a finite set of polynomials. $I(\langle 0 \rangle) := S \cdot K[x]_{\langle x \rangle}$.
Assume that $d(0) := \dim_K K[x]_{\langle x \rangle}/I(\langle 0 \rangle) < \infty$.
OUTPUT: $G \subset K[x]$ a standard basis w.r.t. local degree ordering $>$ for the ideal $I(\langle 0 \rangle)$.

(1) Choose a prime ideal $\mathfrak{p} \neq \langle 0 \rangle$ in A and compute a standard basis $G(\mathfrak{p})$ w.r.t. $>$ of the ideal $I(\mathfrak{p})$ generated by S in $k(\mathfrak{p})[x]_{\langle x \rangle}$.
(2) Use $G(\mathfrak{p})$ to compute $d(\mathfrak{p}) := \dim_{k(\mathfrak{p})} k(\mathfrak{p})[x]_{\langle x \rangle}/I(\mathfrak{p})$.
(3) If $d(\mathfrak{p}) = \infty$ choose another $\mathfrak{p} \neq \langle 0 \rangle$ and continue from the beginning.
(4) Assume $d(\mathfrak{p}) < \infty$ and compute $HC_{I(\mathfrak{p})}$.

[1] The results apply analogously to every reduced Noetherian ring A and $K = \operatorname{Quot}(A/\mathfrak{P})$ $\mathfrak{P}$ a generic point of $\operatorname{Spec} A$.

(5) Compute a standard basis G (using Algorithm 1.7.1 and 1.7.6 from [2]), starting with $S \subset K[x]$, and omit any non-vanishing term of degree $>$ $\deg HC_{I(\mathfrak{p})} + 1$ during the computation.
G is a standard basis of an ideal $I'(\langle 0\rangle) \supset I(\langle 0\rangle)$ in $K[x]_{\langle x\rangle}$.
(6) Compute $d'(0) := \dim_K K[x]_{\langle x\rangle}/I'(\langle 0\rangle)$.
(7) If $d'(0) = d(\mathfrak{p})$ return G, else
choose another $\mathfrak{p} \neq \langle 0\rangle$ and continue from the beginning.

Theorem 5.

(1) The algorithm is correct.
(2) If A contains infinitely many prime ideals, the algorithm terminates for a random choice of $\mathfrak{p}$ in steps 3. and 7. of the algorithm.
(3) If A is a principal ideal domain with infinitely many prime ideals (e.g. $A = \mathbb{Z}$ or $A = \Bbbk[t]$), then the algorithm terminates after finitely many steps for any (not necessarily random) choice of $\mathfrak{p}$ in steps 3. and 7. of the algorithm.

The speed up of the new algorithm can be dramatic, as is illustrated in the following table, where we computed $\dim_K K[x]_{\langle x\rangle}/I(\langle 0\rangle)$ for eight examples, the first four example with $A = \mathbb{Z}$, and the others with $A = \mathbb{Z}[t]$. The examples are listed in [6] (Milnor or Tjurina number in 1,2,3,5,6,7 and a random ideal in 4,8) as well as the Singular code. The examples were computed on a Linux machine with i7-6700 CPU @ 3.40GHz. Times are in sec, RAM in MB. `modStd` is the modular standard basis algorithm in Singular.

Ex.	new Alg.	`std` without HC	RAM	`modStd`
1	0.03	2115.29	7571	2.60
2	0.16	210.69	1213	5.95
3	3.36	$> 2h$	$> 64GB$	$> 2h$
4	124.94	2457.66	7938	141.74
5	0.01	$> 2h$	$> 120GB$	$--$
6	0.16	$> 2h$	$> 80GB$	$--$
7	13.59	$> 2h$	$> 80GB$	$--$
8	50.32	1881.670	10003	$--$

References

[1] G.-M. Greuel and H. D. Nguyen. *Right simple singularities in positive characteristic.* J. Reine Angew. Math. **712**, 81–106, 2016.
[2] G.-M. Greuel and G. Pfister. *A Singular introduction to commutative algebra*, 2nd extended edition, with contributions by O. Bachmann, C. Lossen and H. Schönemann, Springer, Berlin, 2008.
[3] G.-M. Greuel and T. H. Pham. *On finite determinacy for matrices of power series.* Math. Z., **290** no. 3-4, 759–774, 2018.
[4] G.-M. Greuel and G. Pfister. *Semicontinuity of singularity invariants in families of formal power series.* Singularities and their interaction with geometry and low dimensional topology, Trends Math., Birkhäuser/Springer Cham, 2021, pp. 207–245.

[5] ______, *On Delta for parameterized curve singularities.* arXiv:2101.01784, 2021.
[6] G.-M. Greuel, G. Pfister, H. Schönemann, *Using semicontinuity for standard bases computations.* arXiv:2108.09735, 2021.

Canonical stratifcations using non-archimedean methods

Immanuel Halupczok
(joint work with David Bradley-Williams)

One approch to understand the singularities of a (say algebraic) set $X \subset \mathbb{C}^n$ consists in finding a stratification of that set. If a point $x \in X$ lies in some stratum S_i, then on a neighbourhood B of x, X is "trivial along S_i". In particular, if $\dim S_i = d$, then there exists a map $\phi\colon B \to B' \subset \mathbb{C}^n$ such that $\phi(X \cap B)$ is of the form $B'' \times F$, for some ball $B'' \subset \mathbb{C}^d$ and some $F \subset \mathbb{C}^{n-d}$. (I will call X "d-trivial on B" if such ϕ, B', B'', F exist.) Different types of stratifications yield different kinds of triviality. For example, Whitney stratifications yield topological triviality, meaning that ϕ is a homeomorphism, whereas Mostowski's Lipschitz stratification yield billipschitz triviality, meaning that ϕ is a bilipschitz map.

It might be tempting to use the notion of d-triviality as definition of a stratification, i.e., let the stratification be the coarsest one such that for each stratum S_i, each $x \in S_i$ has a neighbourhood on which X is ($\dim S_i$)-trivial. However, this does not work well, since the existence of homeomorphisms or of bilipschitz maps is a very non-algebraic condition. Instead, in this talk, I presented a different notion of triviality for which this naive approach works.

To this end, one has to work in a spherically complete valued field K of characteristic $(0, 0)$. If one is interested in an algebraic set $X(\mathbb{C}) \subset \mathbb{C}^n$, then one can e.g. take K to be the field of Hahn series $K = \mathbb{C}((t^{\mathbb{Q}}))$ (and consider $X(K) \subset K^n$). Intuitively, one should think of t as an infinitesimal element. Our notion of d-triviality (of $X(K)$ on a valuative ball $B \subset K^n$) intuitively imposes that $X(K) \cap B$ is "translation invariant up to infinitesimal error".[1] The main result[2] is that for each d, the set $T_{d,X}$ of all valuative balls $B \subset K^n$ on which $X(K)$ is d-trivial in our sense is first-order definable. Using this, one can then construct a canonical stratification of $X(\mathbb{C})$ consisting of algebraic subsets of $\mathbb{C}$ via the above "naive approach".

The stratification obtained in this way is at least a Whitney stratification, and it is probably much stronger, though we do not yet know what properties exactly it has. However, instead of only considering the stratification, one should probably rather consider the sets of balls $T_{d,X}$, which contain interesting additional information about the singularities. As an example, note that from those $T_{d,X}$, one can obtain a set of candidate poles for the Poincaré series of X. Since everything is canonical, we hope that this set of candidate poles is closer to the acual set of poles than the set of candidate poles obtained using a resolution of singularities.

---

[1] The map $\phi\colon B \to B'$ is required to satisfy $\mathrm{rv}(x_1 - x_2) = \mathrm{rv}(\phi(x_1) - \phi(x_2))$ for all $x_1, x_2 \in B$, where $\mathrm{rv}(a) = \mathrm{rv}(a')$ holds iff the difference betwwen a and a' is infinitesimal compared to a, i.e., if $v(a - a') > v(a)$.

[2] work in progress, hopefully to be finished in 2021

Floer Cohomology and Arc Spaces

Mark McLean

1. Introduction

Seidel observed in [DL02, Remark 2.7] that there is a similarly between Floer cohomology and arc spaces. *Floer cohomology* [Flo88] is the cohomology of a chain complex which is built using an infinite dimensional Morse function. The *arc space* of a singularity is a space of holomorphic maps from the unit disk passing through this singularity. A general goal for us is to understand the relationship between Floer cohomology and arc spaces better. We explore this in the case of isolated hypersurface singularities.

2. Floer cohomology of the Milnor monodromy map

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial with an isolated singularity at 0 and let $S_\epsilon \subset \mathbb{C}^n$ be the sphere of some small radius $\epsilon > 0$. The *link* of f is the submanifold $L_f = f^{-1}(0) \cap S_\epsilon$. The *Milnor map* is the fibration:

$$\frac{f}{|f|} : S_\epsilon - L_f \longrightarrow S^1. \tag{1}$$

The *Milnor fiber* M_f is a fiber of this map and the *Milnor monodromy map* ϕ_f is the monodromy around the base S^1 of this fibration after choosing a connection. The standard symplectic form on $\mathbb{C}^n$ gives the Milnor fiber the structure of a symplectic manifold so that ϕ_f is a symplectomorphism for an appropriate connection [CNPP06, Proposition 3.9].

Floer cohomology $HF^*(\phi)$ of a symplectomorphism $\phi : M \to M$ is the homology of a chain complex freely generated by fixed points of ϕ. The differential is a matrix with respect to this basis of fixed points whose entries are counts of holomorphic sections of a certain mapping cylinder of ϕ, see [DS94].

The group $HF^*(\phi)$ has the property that its Euler characteristic is the Lefschetz number of ϕ. Therefore, the groups $HF^*(\phi_f^d)$, $d \in \mathbb{N}$ can be thought of as giving more information than the monodromy zeta function.

Theorem 1 ([McL19, Section 3])**.** *Suppose $\check{f}$ is another polynomial with an isolated singularity at 0 so that there is a contactomorphism of S_ϵ, with its standard contact structure $TS_\epsilon \cap iTS_\epsilon$, sending the link of $\check{f}$ to the link of f. Then $HF^*(\phi_f^d) = HF^*(\phi_{\check{f}}^d)$ for all $d \in \mathbb{N}$.*

3. Contact loci

Contact loci should be thought of as jets of arcs. The *d-th contact locus* $\chi_d(f)$ of our polynomial f is the subspace:

$$\{u(t) \in \mathrm{Jet}^d(\mathbb{C}^n)|_0 \ : \ f(u(t)) = t^d \in \mathrm{Jet}^d(\mathbb{C})|_0\} \tag{2}$$

where $\mathrm{Jet}^d(\mathbb{C}^n)|_0$ and $\mathrm{Jet}^d(\mathbb{C})|_0$ is the space of d-jets of arcs in $\mathbb{C}^n$ and $\mathbb{C}$ respectively mapping 0 to 0. In other words, the d-th contact locus is the subspace of

d-jets that map via f to the d-jet of the map $z \mapsto z^d$. More concretely, it is the space:

$$\chi_d(f) := \left\{ u(t) = \sum_{i=1}^{d} a_i t^i \in \mathbb{C}^n[t] \ : \ f(u(t)) = t^d \bmod t^{d+1} \right\}. \tag{3}$$

4. Main result

We have the following conjecture:

Conjecture 2 ([BdBLN20]). *There is an isomorphism*

$$HF^*(\phi_f^d) \cong H_c^{*+2nd+n-1}(\chi_d(f))$$

where H_c^* means compactly supported cohomology.

Such a conjecture was verified by the same authors when d is equal to the multiplicity of f at 0.

Theorem 3 (M., in progress). *The conjecture above is true in general.*

In order to prove this theorem, one needs to first construct a natural morphism (called a PSS map):

$$\mathrm{ev} : HF^*(\phi_f^d) \longrightarrow H_c^{*+2nd+n-1}(\chi_d(f)). \tag{4}$$

On the chain level, this will be a constructed from the moduli space of holomorphic disks whose boundary limits to an orbit of the mapping torus of ϕ_f^d and so that the d-jet of these disks sweep out a cycle in a thickening of $\chi_d(f)$ inside $\mathrm{Jet}^d(f)|_0$. This map is similar in spirit to the log PSS map defined by Ganatra and Pomerleano [GP20].

One then needs to construct filtrations F and F' on the chain complexes computing $HF^*(\phi_f^d)$ and $H_c^{*+2nd+n-1}(\chi_d(f))$ and show that our PSS map ev

(1) respects these filtrations and
(2) induces an isomorphism on the associated graded filtration.

These filtrations come from spectral sequences in [McL19] and [BdBLN20] respectively.

5. Further Directions

The bi-graded group $\oplus_d HF^*(\phi_f^d)$ also has a product, called the pair of pants product, making it into a graded algebra [Sei08, Section 2]. What does this product correspond to in $\oplus_d H_c^*(\chi_d(f))$? It might be related to an arc space version of the string product [CS04], in which the boundaries of the arcs are considered as loops. In fact, there might be the structure of a framed little disks operad, see [Vor05], and other interesting structures as well, see [Sei19]. These algebras should also be functorial under adjacency. In other words, if f_0 and f_1 are polynomials with isolated singularities at zero and f_1 is a very small perturbation of f_0 then there should be a corresponding morphism of algebras:

$$\oplus_d H_c^*(\chi_d(f_0)) \to \oplus_d H_c^*(\chi_d(f_1)). \tag{5}$$

It might also be interesting to see whether these techniques can be used to compute other Floer groups. Consider a variety $A \subset \mathbb{C}^N$ of dimension n with an isolated singularity at 0. The *link* of A is the contact manifold $L_A := A \cap S_\epsilon$ with contact structure $\xi_A := TL_A \cap iTL_A$. Associated to such a contact manifold, we have a Floer homology algebra called *full contact homology* $CH_*(L_A, \xi_A)$ [EGH00]. There should be a spectral sequence computing this algebra, which is similar to the one in [McL19]. Can one compute such an algebra from the space of short arcs on A [KN14]? Is there an arc space version of this spectral sequence? Serious technical difficulties appear when one tries to construct a PSS map here. There might be some methods that one can use for complete intersection singularities. If the singularity A is of dimension 2, then can one describe *embedded contact homology* [Hut14] of its link in terms of arcs?

References

[BdBLN20] N. Budur, J. Fernández de Bobadilla, Q. T. Lê, and H. D. Nguyen. *Cohomology of contact loci.* to appear in J. Differential Geom., arXiv:1911.08213, 2020.

[CNPP06] C. Caubel, A. Némethi, and P. Popescu-Pampu. *Milnor open books and Milnor fillable contact 3-manifolds.* Topology, **45**, no. 3, 673–689, 2006.

[CS04] M. Chas and D. Sullivan. *Closed string operators in topology leading to Lie bialgebras and higher string algebra.* The legacy of Niels Henrik Abel, Springer, Berlin, 2004, pp. 771–784.

[DL02] J. Denef and F. Loeser. *Lefschetz numbers of iterates of the monodromy and truncated arcs.* Topology, **41**, no. 5, 1031–1040, 2002.

[DS94] S. Dostoglou and D. Salamon. *Self-dual instantons and holomorphic curves.* Ann. of Math. (2), **139**, no. 3, 581–640, 1994.

[EGH00] Y. Eliashberg, A. Givental, and H. Hofer. *Introduction to symplectic field theory.* GAFA 2000 (Tel Aviv, 1999). Geom. Funct. Anal. Special Volume, Part II, 560–673, 2000.

[Flo88] A. Floer. *Morse theory for lagrangian intersections.* J. Differential Geom., **28** 335–356, 1988.

[GP20] S. Ganatra and D. Pomerleano. *Symplectic cohomology rings of affine varieties in the topological limit.* Geom. Funct. Anal., **30**, no. 2, 334–456, 2020.

[Hut14] Michael Hutchings. *Lecture notes on embedded contact homology.* Contact and symplectic topology, Bolyai Soc. Math. Stud. vol. 26, János Bolyai Math. Soc., Budapest, 2014, pp. 389–484.

[KN14] J. Kollár and A. Némethi. *Holomorphic arcs on singularities.* Invent. Math. **200**, no. 1, 97–127, 2014.

[McL19] M. McLean. *Floer cohomology, multiplicity and the log canonical threshold.* Geom. Topol., **23**, no. 2. 957–1056, 2019.

[Sei08] P. Seidel, *Lectures on four-dimensional Dehn twists.* Symplectic 4-manifolds and algebraic surfaces, Lecture Notes in Math. vol. 1938, Springer, Berlin, 2008, pp. 231–267.

[Sei19] ——— *Fukaya A_∞-structures associated to Lefschetz fibrations. IV.* Breadth in contemporary topology, Proc. Sympos. Pure Math. vol. 102, Amer. Math. Soc., Providence, RI, 2019, pp. 195–276.

[Vor05] A. Voronov, *Notes on universal algebra.* Graphs and patterns in mathematics and theoretical physics, Proc. Sympos. Pure Math. vol. 73. Amer. Math.Soc., Providence, RI, 2005. pp. 81–103

Invariants of generic analytic structures of normal surface singularities

János Nagy
(joint work with András Némethi)

For a given negative definite resolution graph $\mathcal{T}$, usually there are several analytically different surface singularities with resolution graph $\mathcal{T}$. Several analytic invariants like the multiplicity, geometric genus or cohomology numbers of line bundles can change if we change the analytic structure.

In [NN19] the authors investigated Abel maps on normal surface singularities with rational homology sphere resolution graphs.

It turned out that while many analytic invariants depend on the analytic type of the resolution, surprisingly the cohomology numbers of a generic line bundle with given Chern class depends just on the resolution graph $\mathcal{T}$ and can be computed combinatorially.

In [NN20a], the authors following the local deformation theory of resolutions of normal surface singularities developed by Laufer [Lau73], computed several invariants of generic normal surface singularities with fixed rational homology sphere resolution graph, like the geometric genus or the analytic Poincaré series.

It turns out that the cohomology numbers of some special line bundles, called natural line bundles on generic surface singularities coincide in many cases with the cohomology numbers of generic line bundles with the same Chern class.

In the most special case, if we have a generic normal surface singularity (X, o) and one of its generic good resolutions $\widetilde{X}$ with rational homology sphere resolution graph $\mathcal{T}$ then we have the combinatorial formula $p_g = 1 - \min_{0<l\in L} \chi(l)$ for the geometric genus (here $\chi(l)$ denotes the Riemann Roch function of the structure sheaf $\mathcal{O}_l$ corresponding to the cycle l, which is topological).

Next we wish to determine the multiplicity of generic normal surface singularities:

In [Wag70] Wagreich proved that in the presence of a resolution $\widetilde{X} \to X$, if Z_{max} is the exceptional part of the divisor of the lift to $\widetilde{X}$ of the maximal ideal of X, and $\mathcal{O}_{\widetilde{X}}(-Z_{max})$ has no base points, then $\text{mult}(X, o) = -Z_{max}^2$.

In general there are two difficulties: to determine Z_{max}, and to characterize the base points of $\mathcal{O}_{\widetilde{X}}(-Z_{max})$.

In [NN20a] the cycle Z_{max} was determined for the generic analytic structure together with the 'analytic semigroup' of divisors of analytic functions of (X, o).

For $\widetilde{X}$ generic, and (X, o) non–rational, Z_{max} is determined as follows. Set $\mathcal{M} = \{Z : \chi(Z) = \min_{l\in L} \chi(l)\}$. Then the unique maximal element of $\mathcal{M}$ is the maximal ideal cycle of $\widetilde{X}$.

The next theorem provides the structure of base points, together with a formula for the multiplicitiy from [NN20b]:

Theorem ([NN20b]). *Consider a resolution $\widetilde{X} \to X$ with generic analytic structure and assume that the link of the normal surface singularity (X, o) is a rational* *homology sphere. Let E be the exceptional curve $\cup_{v\in\mathcal{V}}E_v$. We say that the irreducible component E_v satisfies the property $(*_v)$ if*

$$(*_v) \qquad \min_{l\geq E_v}\{\chi(Z_{max}+l)\} = \chi(Z_{max})+1.$$

Then the following facts hold.

- *If p is a base point of $\mathcal{L} := \mathcal{O}_{\widetilde{X}}(-Z_{max})$ then p is a regular point of E.*
- *If $p \in E_v$ is a base point of $\mathcal{L}$ then E_v satisfies $(Z_{max}, E_v) < 0$ and $(*_v)$.*
- *If $(Z_{max}, E_v) < 0$ and E_v satisfies $(*_v)$ then $\mathcal{L}$ has exactly $-(Z_{max}, E_v)$ base points on E_v.*

If the property $(*_v)$ holds for a vertex v, then let us define $t(v) := m_v^+ - m_v$, where m_v is the E_v–coefficient of Z_{max} and

$$m_v^+ = \max\{\, E_v\text{-coefficient of } Z_{max}+l \,:\, l\geq E_v,\ l\in L,\ \chi(Z_{max}+l) = \chi(Z_{max})+1\}.$$

We get the following formula for the multiplicity of a generic normal surface singularity:

$$\mathrm{mult}(X,o) = -Z_{max}^2 - \sum_v t(v)\cdot(Z_{max},E_v),$$

where the sum is over all $v\in\mathcal{V}$ with $(Z_{max},E_v)<0$ and satisfying $(*_v)$.

References

[Lau73] H. B. Laufer, *Deformations of resolutions of two-dimensional singularities*, Rice Univ. Stud. **59** (1973), no. 1, 53–96.

[NN19] J. Nagy and A. Némethi, *The Abel map for surface singularities I: generalities and examples*, Math. Ann. **375** (2019), no. 3-4, 1427–1487.

[NN20a] ———, *The Abel map for surface singularities II. Generic analytic structure*, Adv. Math. **371** (2020), 107268, 38.

[NN20b] ———, *The multiplicity of generic normal surface singularities*, arXiv:2005.10867, 2020.

[Wag70] P. Wagreich, *Elliptic singularities of surfaces*, Amer. J. Math. **92** (1970), 419–454.

Igusa's conjecture for exponential sums: optimal estimates for non-rational singularities

Kien Huu Nguyen

(joint work with Raf Cluckers, Mircea Mustaţă and Wim Veys)

Let $f\in\mathbb{Z}[x_1,...,x_n]$ be a non-constant polynomial. Let p be a prime number and m be a positive integer. The exponential sum associated with f and p,m is given by

$$E_f(p,m) := \frac{1}{p^{mn}}\sum_{x\in(\mathbb{Z}/p^m\mathbb{Z})^n}\exp(2\pi i f(x)/p^m).$$

Let σ be a positive real number. Suppose that for each prime number p, there is a positive constant c_p such that

$$|E_f(p,m)| \leq c_p p^{-m\sigma}$$

for all $m \geq 2$. Igusa's conjecture on exponential sums predicts that in the above inequality one can take c_p independent of p. This conjecture relates to the validity of a certain adèlic Poisson summation formula and the estimation of the major arcs in the Hardy-Littlewood circle method towards the Hasse principle of f.

In a recent work with Cluckers and Mustaţă, we proved this conjecture for any $\sigma < \sigma(f) = \min_{a \in \mathbb{C}^n} \mathrm{lct}_a(f - f(a))$, where $\mathrm{lct}_x(g)$ is the log-canonical threshold of g at x. In particular, if $f - b$ has non-rational singularities for some critical value b of f, we can claim that the exponent $\sigma(f)$ is optimal in the sense that if $\sigma > \sigma_0$ then for infinitely many prime numbers p one cannot find a constant c_p as above. The proof of this result follows by a relation between exponential sums and Igusa's local zeta functions. Using Denef's formula to compute Igusa's local zeta functions will help us to reduce Igusa's conjecture on exponential sums for $\sigma < \sigma(f)$ to study a special phenomenon related to log-resolution of f and its numerical data. The work of Birkar, Cascini, Hacon and McKernan on Minimal Model Program provides a tool to explain this phenomenon and thus prove the conjecture.

To go further on applications of Igusa's conjecture on exponential sums, we need to work with the case of rational singularities. Recently, in a joint work with Veys, we proved this conjecture when f is in at most three variables. The proof follows by the classification for rational singularities of polynomials in at most three variables.

On the other hand, let f_d be the homogeneous part of highest degree of f and s be the dimension of the critical locus of f_d. In a recent joint work with Cluckers, we focused Igusa's conjecture on exponential sums for the exponent $\sigma < (n-s)/d$ as follows:

Conjecture. *Let $f \in \mathbb{Z}[x_1, ..., x_n]$ be a non-constant polynomial of degree $d > 1$. Let f_d be the homogeneous part of highest degree of f and s be the dimension of the critical locus of f_d. For each $\epsilon > 0$, each prime number p, there is a constant $C_{p,\epsilon} > 0$ such that*

$$|E_f(p,m)| \leq C_{\epsilon,p} p^{m(-\frac{n-s}{d}+\epsilon)},$$

for all $m \geq 1$. Moreover, one can take $C_{\epsilon,p} = 1$ if p is large enough.

This conjecture may be viewed as an analogue of the theorem of Deligne for exponential sums over finite fields in the finite ring setting. Together with Cluckers, we checked this conjecture when f is in at most four variables or f satisfies some non-degeneracy condition.

References

[1] R. Cluckers, M. Mustaţă, and K. H. Nguyen, *Igusa's conjecture for exponential sums: optimal estimates for nonrational singularities*, Forum Math. Pi **7** (2019), e3, 28.

[2] R. Cluckers and K. H. Nguyen, *Combining Igusa's conjectures on exponential sums and monodormy with semi-continuity of the minimal exponent*, arXiv:2005.04197, 2021.

[3] K. H. Nguyen and W. Veys, *On the motivic oscillation index and bound of exponential sums modulo p^m via analytic isomorphisms*, arXiv:2008.11637, to appear in Journal de Mathématiques Pures et Appliqueés, 2021.

Avoiding singularities in log MMP - almost minimal models and applications

Karol Palka
(joint work with Mariusz Koras and Tomasz Pełka)

Assume we want to study a smooth quasi-projective surface S, for instance we want to study a curve embedded into a smooth projective surface and we take S to be the complement. We begin by taking a log smooth completion, that is, a smooth projective surface X and a reduced simple normal crossing divisor D on X (a boundary) such that $X \setminus D \cong S$. Let K_X denote the canonical divisor of X. A strong tool in the analysis is to run the logarithmic version of the minimal model program (MMP), which finds a birational model of (X, D) for which the log canonical divisor, the direct image of $K_X + D$, has very special numerical properties. Thus one studies a minimal model of (X, D), hoping that by tracking back the minimization process one can understand better (X, D), and hence S. For surfaces the MMP is a sequence of contractions of *log exceptional* curves L, for which by definition

$$L^2 < 0 \quad \text{and} \quad L \cdot (K_X + D) < 0. \tag{1}$$

A log surface (X, D) is *minimal* if it has no log exceptional curve. If $D = 0$ then $L \cdot K_X < 0$, so L is a (-1)-curve and the image of X after the contraction is smooth. On the contrary, a minimal model of a log smooth surface with nonzero boundary may very well be singular. Indeed, assume that L_1 is a component of D which is smooth, rational, has a negative self-intersection number $L_1^2 \leq -2$, and for which $\beta_D(L_1) := L_1 \cdot (D - L_1) \leq 1$ (a rational tip of D). By adjunction $L_1 \cdot (K_X + D) = -2 + L_1 \cdot (D - L_1) < 0$, so L_1 is log exceptional and after its contraction the surface becomes singular. This makes further analysis more difficult.

For surfaces one can modify the MMP run by reordering contractions, which leads to the notion of an *almost minimal model* and allows to delay the appearance of singularities. Let $\operatorname{ctr}(L_1)\colon X \to \overline{X}$ be the contraction of L_1 and let $\overline{D}$ be the direct image of D. A maximal sequence of contractions of log exceptional curves contained in D and its images is called a *peeling of* D. If the image of a peeling is a minimal log surface then (X, D) is *almost minimal*. Suppose now that $\operatorname{ctr}(L_1)$ is the peeling morphism but $(\overline{X}, \overline{D})$ is not almost minimal. That is, on $(\overline{X}, \overline{D})$ there is a log exceptional curve L_2 not contained in $\operatorname{Supp} \overline{D}$. Let $(\overline{X}', \overline{D}')$ be the image of $(\overline{X}, \overline{D})$ after its contraction. Then the proper transform $L_2' := \operatorname{ctr}(L_1)_*^{-1} L_2$ on X, called *an almost log exceptional curve*, is a (-1)-curve, usually not log exceptional for (X, D). Still, it is worth contracting it first, before L_1. Since X' is normal, we get a commutative diagram, where the right vertical arrow, here a contraction of the image of L_1, turns out to be a partial run of an MMP:

$$\begin{tikzcd}
(X,D) \arrow[r, "\mathrm{ctr}(L_2')"] \arrow[d, "\mathrm{ctr}(L_1)"'] & (X',D') \arrow[d] & \text{log smooth} \arrow[d, "peeling"] \\
(\overline{X},\overline{D}) \arrow[r, "\mathrm{ctr}(L_2)"] & (\overline{X}',\overline{D}') & \text{log terminal}
\end{tikzcd} \tag{2}$$

By repeating this reordering of contractions we can delay the moment surfaces singularities appear in the process of minimalization to the moment when an almost minimal model is reached. The price one has to pay is that (X', D') may have in principle worse log singularities than (X, D), which requires some additional care.

Making the above modification of the usual MMP run turns out to be very beneficial for the analysis of the initial surface S or the log surface (X, D). The theory was developed by M. Miyanishi and collaborators in case (X, D) is log smooth and D is reduced, see the 'theory of peeling' [Miy01, §2.3]. Recently we generalized it to general log surfaces, see [Pal19]. When applied to (X, rD), where $r \in [0, 1]$ and D is reduced, it produces interesting (almost log exceptional) curves lying outside of D and meeting D in a controlled manner. For instance, for we have the following results. A (-2)-*twig* of D is a chain of (-2)-curves in D such that for each component $\beta_D \leq 2$ and for one of them $\beta_D \leq 1$.

Lemma. Let S be a smooth affine surface and (X, D) its log smooth completion. If L is almost minimal on (X, D) then $L \cdot D \leq 1$. If L is almost minimal on $(X, \frac{1}{2}D)$ then $L \cdot D \leq 2$ and if the equality holds then L meets D in two distinct points, exactly one of which belongs to a (-2)-twig of D.

Note that in particular $L \cap S$ is isomorphic to $\mathbb{A}^1$ or $\mathbb{A}^1_* := \mathbb{A}^1 \setminus \{0\}$, so the Euler characteristic of $L \cap S$ is non-negative. Mastering the process of creation of an almost minimal model of (X, D) led for instance to a considerable progress in understanding the class of complex rational planar curves with unibranched singularities, so-called *rational cuspidal curves.* In particular, we proved the Coolidge-Nagata conjecture asserting that they are Cremona-equivalent to a line [Pal14], [KP17] and we showed that except a unique 4-cuspidal quintic all of them have at most three cusps [KP20]. Later we completely classified all planar rational cuspidal curves up to a projective equivalence under the assumption that the Negativity Conjecture holds for their complements [PP17], [PP19].

Conjecture (Negativity Conjecture, [Pal19, 4.7])**.** *Let S be a complex $\mathbb{Q}$-acyclic surface and let (X, D) be a log smooth completion. Then $\kappa(K_X + \frac{1}{2}D) = -\infty$.*

Finding almost log exceptional curves allowed to group rational cuspidal curves into series in which one curve is obtained inductively from another in a simple manner. Classification results on rational cuspidal curves were extended in the Thesis by T. Pełka, who completed the classification of smooth $\mathbb{Q}$-acyclic surfaces for which the Negativity Conjecture holds. Such surfaces have been studied for a long time. Again, almost log exceptional curves were used to group them into series. As a corollary from the classification Pełka showed that several other conjectures related to $\mathbb{Q}$-acyclic surfaces formulated by tom-Dieck-Petrie, Flenner-Zaidenberg and Koras follow from the Negativity Conjecture. So far the latter conjecture remains open. It would be desirable to formulate it for a broader class of surfaces and for $\mathbb{Q}$-acyclic affine higher-dimensional varieties, too.

A new application of the almost minimal model techniques has led to a progress in a different area. A normal projective log surface $(\overline{X}, \overline{D})$ is called a *log del Pezzo of rank* 1 if $\overline{X}$ has Picard rank 1, that is, all divisors are numerically proportional, and $-(K_{\overline{X}} + \overline{D})$ is ample. Log del Pezzo surfaces with $D = 0$ are of great interest because they appear as minimal models of projective surfaces of negative Kodaira dimension. Numerous authors contributed to their partial classification using a variety of techniques, with strongest results by Keel-McKernan [KM99] and Lacini [Lac21]. In 2019 during the Affine Algebraic Geometry Meeting in Osaka, (March 7-10) we discussed the following result and we sketched our approach to prove it. The proof will appear in a forthcoming paper.

Theorem. *Let $\overline{X}$ be a normal del Pezzo surface of rank* 1 *defined over an algebraically closed field $\Bbbk$. Assume that* $\operatorname{char}\Bbbk \neq 2$. *Then $\overline{X} \cong \mathbb{P}^2$ or the minimal log resolution (X, D) of $\overline{X}$ has a $\mathbb{P}^1$-fibration such that $f \cdot D \leq 4$ for a general fiber f.*

In case $\operatorname{char}\Bbbk = 2$ the exceptions with $f \cdot D > 4$ can be described precisely. The bound on the intersection of f with D is low enough that a complete classification of log del Pezzo surfaces of rank 1 will follow using standard techniques related to the analysis of degenerate fibers of $\mathbb{P}^1$-fibrations. The main ingredient of the proof of the theorem is to shift the attention from analyzing arbitrarily singular del Pezzos with no boundary to analyzing $\frac{1}{2}$-log terminal log del Pezzos with boundary. This is where the technique of almost minimal models plays a crucial role.

References

[KM99] Seán Keel and James McKernan, *Rational curves on quasi-projective surfaces*, Mem. Amer. Math. Soc. **140** (1999), no. 669, viii+153.

[KP20] M. Koras and K. Palka, *Complex planar curves homeomorphic to a line have at most four singular points*, arXiv:1905.11376, to appear in J. Math. Pures Appl., 2020

[KP17] ———, *The Coolidge-Nagata conjecture*, Duke Math. J. **166** (2017), no. 16, 3085–3145.

[Lac21] J. Lacini, *On rank one log del Pezzo surfaces in characteristic different from two and three*, arXiv:2005.14544, 2021.

[Miy01] M. Miyanishi, *Open algebraic surfaces*, CRM Monograph Series, vol. 12, American Mathematical Society, Providence, RI, 2001.

[Pal14] K. Palka, *The Coolidge–Nagata conjecture, part I*, Adv. Math. **267** (2014), 1–43.

[Pal19] ———, *Cuspidal curves, minimal models and Zaidenberg's finiteness conjecture*, J. Reine Angew. Math. **747** (2019), 147–174.

[PP17] K. Palka and T. Pełka, *Classification of planar rational cuspidal curves I. $\mathbb{C}^{**}$-fibrations*, Proc. Lond. Math. Soc. **115** (2017), no. 3, 638–692.

[PP19] ———, *Classification of planar rational cuspidal curves II. Log del Pezzo models*, Proc. Lond. Math. Soc. **120** (2019), no. 5, 642–703.

The Brasselet-Schürmann-Yokura conjecture for rational homology manifolds

Irma Pallarés

(joint work with Javier Fernández de Bobadilla and Morihiko Saito)

In [11], Hirzebruch introduced the χ_y-characteristic $\chi_y(X)$ of a compact complex manifold X. For $y = -1, 0, 1$, it specializes to the Euler characteristic, the arithmetic genus, and the signature of X, respectively. The generalized Hirzebruch-Riemann-Roch Theorem states that

$$\chi_y(X) = \int_X T_y^*(TX) \cap [X]$$

where $T_y^*(TX) \in H^*(X;\mathbb{Q})[y]$ is the Hirzebruch characteristic class of the tangent bundle TX of X. For $y = -1, 0, 1$, this class specializes to the total Chern class, the total Todd class, and the total Thom-Hirzebruch L-class, respectively.

In [4], this theory was generalized for singular complex algebraic varieties. Let $K_0(var/X)$ be the relative Grothendieck group of complex algebraic varieties over a variety X. There is a natural transformation

$$T_{y,*}\colon K_0(var/-) \to H_{2*}^{BM}(-;\mathbb{Q})[y]$$

defined by using Hodge theory, such that, $T_{y,*}([X \to X]) = T_y^*(TX) \cap [X]$ if X is non-singular.

The class $T_{y,*}([X \to X])$ specializes, for $y = -1$, to the rationalized Schwartz-MacPherson Chern class defined by MacPherson in [12] by a natural transformation which coincides with the Schwartz class [17] via Alexander duality [3]. For $y = 0$, it specializes to the Baum-Fulton-MacPherson Todd class [2] if X has only du Bois singularities. For $y = 1$, Brasselet, Schürmann and Yokura in [4] conjectured the following equality of characteristic classes:

Theorem 1 ([9]). *Let X be a compact complex algebraic variety that is a rational homology manifold, then*

$$T_{1,*}([X \to X]) = L_*(X)$$

where $L_*(X)$ is the Goresky-MacPherson L-class of X [10].

This conjecture is the characteristic class version for rational homology manifolds of the Hodge Index Theorem that expresses the signature of a compact complex algebraic manifold in terms of the Hodge numbers.

Theorem 1 was proved in special cases; in [5] for hypersurfaces with isolated singularities, in [6] for quotient singularities, in [13] for some toric varieties, in [1] for certain projective threefolds. The projective case of this conjecture is shown in [8] using the theory of cubical hyperresolutions, the Decomposition Theorem and classical Hodge theory.

In [9], we prove Theorem 1 using the theory of mixed Hodge modules. To show this conjecture we consider a generalization of the Goresky-MacPherson L-class given by Cappell and Shaneson in [7]. Let $\Omega_{\mathbb{K}}(X)$ be the cobordism group of self-dual $\mathbb{K}$-complexes of sheaves on X, where $\mathbb{K}$ is a subfield of $\mathbb{R}$, see [7], [18], [4], [9].

The elements in $\Omega_{\mathbb{K}}(X)$ are equivalence classes of pairs $(\mathcal{F}, S)$ where $\mathcal{F} \in D_c^b(X)$ and $S\colon \mathcal{F} \otimes \mathcal{F} \to \mathbb{D}_X$ is a perfect pairing, see [9].

There is a natural transformation

$$L_*\colon \Omega_{\mathbb{K}}(-) \to H_{2*}(-;\mathbb{Q})$$

such that, the class $L_*([IC_X, S])$, where IC_X is the intersection cohomology sheaf complex on X, coincides with the Goresky-MacPherson L-class $L_*(X)$ of X.

Theorem 2 ([4]). *There is a unique natural transformation*

$$sd_{\mathbb{K}}\colon K_0(var/-) \to \Omega_{\mathbb{K}}(-)$$

such that, for X non-singular, $sd_{\mathbb{K}}([X \to X]) = [\mathbb{K}_X[\dim_{\mathbb{C}} X]]$, where $\mathbb{K}_X[\dim_{\mathbb{C}} X]$ is the shifted constant sheaf. Moreover, the following diagram commutes

$$\begin{tikzcd} K_0(var/-) \arrow[rr, "sd_{\mathbb{K}}"] \arrow[dr, "T_{1,*}"'] & & \Omega_{\mathbb{K}}(-) \arrow[dl, "L_*"] \\ & H_{2*}(-;\mathbb{Q}) & \end{tikzcd}$$

In [4], the authors formulated the following stronger statement in $\Omega_{\mathbb{R}}(X)$:

Theorem 3 ([9]). *If X is a compact complex algebraic variety that is a rational homology manifold, then*

$$sd_{\mathbb{R}}([X \to X]) = [IC_X] \in \Omega_{\mathbb{R}}(X).$$

The proof of Theorem 3 follows from the results introduced below, also a counter-example of this theorem with $\mathbb{Q}$-coefficients is given in [9].

Let $\mathcal{M}$ be a pure $\mathbb{K}$-Hodge module of weight ω on X. Let $K_{\mathbb{R}}$ be its underlying $\mathbb{R}$-complex and let $S_{\mathbb{R}}\colon K_{\mathbb{R}} \otimes K_{\mathbb{R}} \to \mathbb{D}_X(-\omega)$ be the scalar extension of the polarization S of the $\mathbb{K}$-Hodge module $\mathcal{M}$, see [14], [15], [16], [9].

Proposition 4 ([9], [8]). The cobordism class $[K_{\mathbb{R}}, S_{\mathbb{R}}] \in \Omega_{\mathbb{R}}(X)$ is independent of the choice of the polarization S.

Let $K_0(MHM(X))$ be the Grothendieck group of mixed $\mathbb{K}$-Hodge modules on X, by Proposition 4 the following theorem holds:

Theorem 5 ([9]). *There is a natural transformation*

$$\mathrm{Pol}\colon K_0(MHM(X)) \to \Omega_{\mathbb{R}}(X)$$

defined by

$$\mathrm{Pol}([\mathcal{M}]) = [K_{\mathbb{R}}, (-1)^{\omega(\omega+1)/2} S_{\mathbb{R}}],$$

such that, $\mathrm{Pol} \circ f_* = f_* \circ \mathrm{Pol}$, for $f\colon X \to Y$ a proper morphism.

Corollary 6 ([9]). *The following diagram commutes*

$$\begin{tikzcd} K_0(var/X) \arrow[rr, "\mathrm{Hdg}"] \arrow[dr, "sd_{\mathbb{R}}"'] & & K_0(MHM(X)) \arrow[dl, "\mathrm{Pol}"] \\ & \Omega_{\mathbb{R}}(X) & \end{tikzcd}$$

where the morphism Hdg *is defined by* $[f\colon Y \to X] \mapsto \sum_j (-1)^j [H^j f_! \mathbb{K}_Y^H]$, *see* [4].

Theorem 3 follows after applying Corollary 6 to the identity class $[X \to X]$ in $K_0(var/X)$, noticing that $IC_X^H = \mathbb{R}_X^H[\dim_{\mathbb{C}} X]$ when X is a rational homology manifold, and taking into account the sign relation between polarizations of Hodge modules and polarizations of variations of Hodge structures. Theorem 1 holds after applying the L-transformation to the equality in Theorem 3.

A slight extension of the Brasselet-Schürmann-Yokura conjecture holds when $X \setminus \Sigma$ is a rational homology manifold with Σ a finite set of points, satisfying certain vanishing condition at Σ, see [9].

References

[1] M. Banagl, *Topological and Hodge L-classes of singular covering spaces and varieties with trivial canonical class*, Geom. Dedicata **199** (2019), 189–224.

[2] P. Baum, W. Fulton, and R. MacPherson, *Riemann-Roch for singular varieties*, Inst. Hautes Études Sci. Publ. Math. (1975), no. 45, 101–145.

[3] J.-P. Brasselet and M.-H. Schwartz, *Sur les classes de Chern d'un ensemble analytique complexe*, The Euler-Poincaré characteristic (French), Astérisque, vol. 82, Soc. Math. France, Paris, 1981, pp. 93–147.

[4] J.-P. Brasselet, J. Schürmann, and S. Yokura, *Hirzebruch classes and motivic Chern classes for singular spaces*, J. Topol. Anal. **2** (2010), no. 1, 1–55.

[5] S. E. Cappell, L. G. Maxim, J. Schürmann, and J. L. Shaneson, *Characteristic classes of complex hypersurfaces*, Adv. Math. **225** (2010), no. 5, 2616–2647.

[6] ———, *Equivariant characteristic classes of singular complex algebraic varieties*, Comm. Pure Appl. Math. **65** (2012), no. 12, 1722–1769.

[7] S. E. Cappell and J. L. Shaneson, *Stratifiable maps and topological invariants*, J. Amer. Math. Soc. **4** (1991), no. 3, 521–551.

[8] J. Fernández de Bobadilla and I. Pallarés, *The Brasselet-Schürmann-Yokura conjecture on L-classes of projective varieties*, arXiv:2007.11537, 2020.

[9] J. Fernández de Bobadilla, I. Pallarés, and M. Saito, *Hodge modules and cobordism classes*, arXiv:2103.04836, 2021.

[10] M. Goresky and R. MacPherson, *Intersection homology theory*, Topology **19** (1980), no. 2, 135–162.

[11] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer-Verlag New York, Inc., New York, 1966,

[12] R. D. MacPherson, *Chern classes for singular algebraic varieties*, Ann. of Math. (2) **100** (1974), 423–432.

[13] L. G. Maxim and J. Schürmann, *Characteristic classes of singular toric varieties*, Comm. Pure Appl. Math. **68** (2015), no. 12, 2177–2236.

[14] M. Saito, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849–995 (1989).

[15] ———, *Introduction to mixed Hodge modules*, no. 179-180, 1989, Actes du Colloque de Théorie de Hodge (Luminy, 1987), pp. 10, 145–162.

[16] ———, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), no. 2, 221–333.

[17] M. H. Schwartz, *Classes caractéristiques définies par une stratification d'une variété analytique complexe*, C. R. Acad. Sci. Paris **260** (1965).

[18] B. Youssin, *Witt groups of derived categories*, K-Theory **11** (1997), no. 4, 373–395.

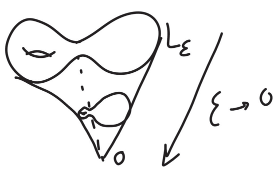
Moderately Discontinuous Algebraic Topology

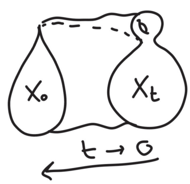
María Pe Pereira
(joint work with J. Fernández de Bobadilla, S. Heinze and E. Sampaio)

In the works [4] and [3] we develope a new metric algebraic topology, called the Moderately Discontinuous (MD) homology and homotopy in the context of subanalytic germs in $\mathbb{R}^n$ with a supplementary metric structure. This theory satisfies the analogues of usual theorems in algebraic topology: long exact sequences for the relative case and for coverings of certain type as in Mayer Vietoris; a Seifert van Kampen theorem for special coverings; a Hurewicz morphism relating MD-homology and MD-homotopy; etc.

The theory captures bilipschitz information, or in other words, quasi isometric invariants. The typical examples to which it applies are subanalytic germs with the inner metric d_{inn} (the length metric induced by the euclidean metric) or with the outer metric d_{out} (the restriction of the euclidean metric).

A subanalytic germ (X, O) is topologically a cone over its link and the MD theory captures the different speeds, with respect to the distance to the origin, in which the topology of the link collapses towards the origin (see picture on the left). Moreover, in the work in progress [5] we generalise this theory to subanalytic one parameter families in order to capture the different collapsing rates with respect to the family parameter (see picture on the right).





The MD homology and homotopy is functorial for lipschitz maps $f\colon (X, O, d_X) \to (Y, O, d_Y)$ for which there exists $K > 0$ such that

$$(*) \qquad \frac{1}{K}||x|| \leq ||f(x)|| \leq K||x|| \text{ for every } x \in X.$$

In particular, the MD theory gives subanalytic bilipschitz invariants for germs with the inner or the outer metric. Moreover the theory is functorial for *b-maps* which are defined in [4]. A *b-map* is a collection of mappings satisfying $(*)$ whose domains give a subanalytic covering of X and such that in the intersections of two of these domains, the difference (measured in the metric of the target space) is of order less than t^b where t is the distance to the origin. See Definition 5.3 in [4] for an accurate definition.

Given $b \in (0, +\infty)$, two subanalytic arcs $\mathfrak{p} : (0, \epsilon) \to X$ and $\mathfrak{q} : (0, \epsilon) \to X$ satisfying $(*)$ are called *b-equivalent points* (they will be *points* in our category of subanalytic sets), and we write $\mathfrak{p} \sim_b \mathfrak{q}$, if

$$\lim_{t \to 0} \frac{d_X(\mathfrak{p}(t), \mathfrak{q}(t))}{t^b} = 0.$$

We call *b-points* the equivalence classes of arcs/points with respect to $\sim_b$. Then, the MD homology/homotopy theory for parameter $b \in (0, +\infty]$ can be thought, very roughly speaking, as the homology/homotopy theory of the space of b-points. Also, we can say that a b-map, that is only a piecewise map between germs, is an actual map between the spaces of b-points of the corresponding germs.

Given an abelian coefficient group A, the MD homology is a diagram of graded abelian groups indexed by $b \in (0, +\infty]$, where each $MDH^b_\bullet(X, O, d; A)$ is a graded abelian group, called the b MD homology group, and for any $b \geq b'$ there is a homomorphism

$$h_{b,b'} : MDH^b_\bullet(X, O, d; A) \to MDH^{b'}_\bullet(X, O, d; A)$$

of graded abelian groups. These homomorphisms are a very important part of the invariant, and sometimes contain the most interesting information (see the curve case below). In particular, we call b_0 a *jump* if $h_{b_0-\epsilon,b_0+\epsilon}$ is never an isomorphism for every ϵ. Given a b-point $\mathfrak{p} : (0, \epsilon) \to (X, O)$ in a metric germ (X, O), the MD homotopy based in $\mathfrak{p}$ is a diagram of graded groups indexed by $b \in (0, +\infty]$ where $MD\pi^b_\bullet(X, O, d; \mathfrak{p})$ is a graded group and for any $b \geq b'$ we have connecting homomorphisms similarly to $h_{b,b'}$.

Our original motivation was the understanding of complex singularities; I summarize here some of the main results in this direction in the following sections.

1. MD groups are finitely generated. Finiteness and rationality of jumps.

For germs $(X, O) \subset (\mathbb{R}^n, O)$ with the outer metric we have the following equality for any $b' > b$ small enough

$$MDH^b_n(X, d_{out}; A) = H_n(\mathcal{H}_{b,\eta}(X) \setminus \{O\}; A),$$

where $\mathcal{H}_{b,\eta}(X) := \bigcup_{x \in X} B(x, \eta\|x\|^b)$ is a thickening of $X \setminus \{O\}$ in $\mathbb{R}^n$. This equality was conjecture by L. Birbrair and we proved it in [4] (see Section 11 in [4]), Corollary 11.12). A similar result is true for the MD Homotopy groups (see Theorem 5.2.7 in [3]).

As a consequence, since b' can be chosen rational in the above equality, we get that the MD homology groups are finitely generated for a germ X with the outer metric. Moreover, the jumps (both for the MD homology and MD homotopy) are finite and rational (see [4, Theorems 11.14-11.15] and [3, Corollary 5.3.1]).

Knowing by [1] that, up to subanalytic reembedding, any inner metric is an outer metric, we get the same result about finite generacy and the nature of jumps of the MD homology/homotopy for germs with the inner metric.

2. Plane branches

Let $(C, 0, d_{out})$ be a complex plane branch with the outer metric. In Corollary 15.5 in [4] we have proved that

- For any $b \in [1, \infty]$

$$MDH^b_0(C; \mathbb{Z}) \cong MDH^b_1(C; \mathbb{Z}) \cong \mathbb{Z} \quad \text{and} \quad MDH^b_n(C; \mathbb{Z}) \cong \{0\} \text{ if } n > 1.$$

- For any $b_1, b_2 \in [1, \infty]$, if $(b_2, b_1]$ contains exactly one Puiseux exponent with corresponding Puiseux pair (m, k) for some $m \in \mathbb{N}$, we have
$$h_1^{b_1,b_2}(x) = kx.$$
If $(b_2, b_1]$ contains more than one Puiseux exponent, $h_1^{b_1,b_2}$ can be determined by concatenation. Otherwise, $h_*^{b_1,b_2} = id_{\mathbb{Z}}$

Recall the well known fact that the Puiseaux exponents give the outer metric type up to a bilipschitz homeomorphism (also up to an abstract homeomorphism, see [6]). So, in this case the MD homology completely codifies the outer metric type up to bilipschitz homeomorphism.

With the same kind of arguments we get, as stated in [3, Theorem 6.0.11], that the MD-fundamental group of an irreducible plane curve singularity coincides with its first MD-homology group.

3. Normal Surfaces

Let $(X, 0, d_{inn})$ be a complex normal surface singularity with its inner metric. Its Lipschitz geometry is completely described in the thick-thin decomposition of [2]. We recall that this decomposition induces in the link $X_\epsilon := X \cap \mathbb{S}_\epsilon$ a non-minimal JSJ-decomposition. We denote by Y_1 the thick part of X, that is, a piece that is bilipschitz subanalytic homeomorphic to the 1-cone over its link with the inner metric. Then, it is shown in [2] that there is a subanalytic map $\xi : \overline{X \setminus Y_1} \to D_\epsilon^*$ which is a locally trivial fibration, where D_ϵ^* denotes the punctured disc of a certain radius ϵ. The map ξ restricts to a locally trivial fibration over each piece of the decomposition of $X \setminus Y_1$.

Now, in order to compute the MD algebraic topology of X we introduced in Section 6 in [3] what we call the $(b,1)$-*homotopy model* of X and that we denote by X_ϵ^b. The space X_ϵ^b can be understood as the result of fibrewise identifying to a point the connected components of the part of the fibres of ξ restricted to the link that collapse to a rate higher than b. It has the homotopy type of a plumbed 3-manifold in which several circles are identified.

Observe that if $b \geq b'$ we have a natural continuous map
$$\alpha_{b,b'} : X_\epsilon^b \to X_\epsilon^{b'}.$$

Then, as stated in [3], for every $n \in \mathbb{N}$ we have isomorphisms from
$$... \to H_n(X_\epsilon^b) \to H_1(X_\epsilon^{b'}) \to ...$$
whose mappings are induced by the $\alpha_{b,b'}$, to
$$... \to MDH_n^b(X) \to MDH_1^{b'}(X) \to ....$$
induced by the $h_{b,b'}$, and moreover, given a point p in the link X_ϵ and any b-point $\mathfrak{p} : (0, \epsilon) \to X$ in X there is an isomorphism from
$$... \to \pi_1(X_\epsilon^b, p) \to \pi_1(X_\epsilon^{b'}, p) \to ...$$
to
$$... \to MD\pi_1^b(X, \mathfrak{p}) \to MD\pi_1^{b'}(X, \mathfrak{p}) \to ....$$

These computations use a version of Mayer Vietoris Theorem for the MD homology, see [4, Theorem 8.11], and an analogoue of the Seifert van Kampen Theorem for moderately discontinuous groupoids, see [3, Theorem 4.2.5].

4. Smooth point

Among complex germs, the smooth point is characterised by its integral MD homology, see [4, Theorem 13.7]. More precisely, if (X, x_0, d_{out}) is a complex analytic germ endowed with the outer metric such that $MDH_*^\bullet(X, x_0, d_{out}; \mathbb{Z})$ coincides with the moderately discontinuous homology of a smooth germ, then (X, x_0) is a smooth germ.

5. Further questions

We belive that the MD algebraic topology can be a good framework to pursue the understanding not only of bilipschitz geometry of a germ itself, but other questions about singularities where bilipschitz geometry may play a role. Some questions that we have in mind:

- Does constancy of MD homology in a family imply constancy of the lipschitz type?.
- Is it enough to have the equality for the MD theory of a germ with respect to the inner and the outer metric to say that the inner and the outer metric are equivalent?
- Is MD theory useful to study the bilipschitz nature of certain analytical invariants?
- Does the monodromy of a hypersurface singularity admit a lipschitz representative with fixed b-points? In this direction we are working in [5] on a Fixed Point Theorem analogous to the Lefschetz fixed point theorem which may have structure to this study.

Furthermore, the theory is susceptible to be generalized in many different directions.

References

[1] L. Birbrair and T. Mostowski, *Normal embeddings of semialgebraic sets*, Michigan Math. J. **47** (2000), no. 1, 125–132.
[2] L. Birbrair, W. D. Neumann, and A. Pichon, *The thick-thin decomposition and the bilipschitz classification of normal surface singularities*, Acta Math. **212** (2014), no. 2, 199–256.
[3] J. Fernández de Bobadilla, S. Heinze, and M. Pe Pereira, *Moderately discontinuous homotopy*, arXiv:2007.01538, to appear in IMRN., 2020.
[4] J. Fernández de Bobadilla, S. Heinze, M. Pe Pereira, and E. Sampaio, *Moderately discontinuous homology*, arXiv:1910.12552, to appear in Comm. Pure App. Math., 2020.
[5] J. Fernández de Bobadilla and M. Pe Pereira, *A fixed point theorem in the md algebraic topology*, work in progress.
[6] W. D. Neumann and A. Pichon, *Lipschitz geometry of complex curves*, J. Singul. **10** (2014), 225–234.

Polar exploration of complex surface germs

Anne Pichon
(joint work with André Belotto da Silva, Lorenzo Fantini and Andràs Némethi)

"Polar exploration" is the quest to determine the generic polar curve, i.e., the apparent shape, of a singular surface germ.

A normal complex surface singularity $(X,0)$ can be resolved either by a sequence of normalized point blowups, following seminal work of Zariski [14] from the late 1930s, or by a sequence of normalized Nash transforms, as was done half a century later by Spivakovsky [13]. One of the main motivation of polar exploration is to shed some light on the relationship between these two resolution algorithms, which despite their importance and their centrality in modern mathematics is still quite mysterious, providing some evidence of a duality between the two which was initially observed by Lê [8, §4.3].

While the blowup $\mathrm{Bl}_0 X$ of the maximal ideal of $(X,0)$ is the minimal transformation which resolves the family of generic hyperplane sections of $(X,0)$, the Nash transform ν of $(X,0)$ is the minimal transformation that resolves the family of the polar curves associated with the generic plane projections of $(X,0)$. Therefore, the study of the duality of resolution algorithms translates into the study of the relative positions on $(X,0)$ of those two families of curves. This is the viewpoint we adopted in our recent works [2] and [1], which answer respectively the two following natural questions:

Question 1. Does the topological type of $(X,0)$ "restrict" the possibilities of relative positions between these families of curves?

Question 2. Can we find special classes of surfaces $(X,0)$ for which the relative positions between these families of curves are determined by the topological type?

In order to give precise statements of our results we need to introduce some additional notation. Let $\pi\colon X_\pi \to X$ be a good resolution of $(X,0)$, by which we mean a proper bimeromorphic morphism from a smooth surface X_π to X which is an isomorphism outside of a simple normal crossing divisor $E=\pi^{-1}(0)$, and denote by $V(\Gamma_\pi)$ the set of vertices of the dual graph Γ_π of π, so that every element v of $V(\Gamma_\pi)$ corresponds to an irreducible component E_v of E. We weight Γ_π by attaching to each vertex v the genus $g(v)\geq 0$ of the complex curve E_v and the self-intersection $e(v)<0$ of E_v.

For each v in $V(\Gamma_\pi)$, we denote by l_v the intersection multiplicity of the zero locus $h^{-1}(0)$ of a generic hyperplane section $h\colon (X,0)\to(\mathbb{C},0)$ with E_v, and we call *$\mathcal{L}$-vector* of $(X,0)$ the vector $L_\pi=(l_v)_{v\in V(\Gamma_\pi)}\in\mathbb{Z}_{\geq 0}^{V(\Gamma_\pi)}$. Whenever $\pi\colon X_\pi\to X$ factors through $\mathrm{Bl}_0 X$, the strict transform of such a generic hyperplane section via π consists of a disjoint union of smooth curves that intersect transversely E at smooth points of E, and l_v is the number of such curves passing through the component E_v. Similarly, we denote by p_v the intersection multiplicity of the strict transform of the polar curve of a generic plane projection $\ell\colon (X,0)\to(\mathbb{C}^2,0)$ with E_v and we call *$\mathcal{P}$-vector* of $(X,0)$ the vector $P_\pi=(p_v)_{v\in V(\Gamma_\pi)}\in\mathbb{Z}_{\geq 0}^{V(\Gamma_\pi)}$.

Whenever $\pi\colon X_\pi \to X$ factors through ν then such a strict transform consists of smooth curves intersecting E transversely at smooth points, and p_v equals the number of such curves through E_v.

The following result gives a complete positive answer to Question 1:

Theorem 1 ([2, Theorem A])**.** *Let M be a real 3-manifold. There exists finitely many triplets (Γ, L, P), where Γ is a weighted graph and L and P are vectors in $(\mathbb{Z}_{\geq 0})^{V(\Gamma)}$, such that there exists a normal surface singularity $(X, 0)$ satisfying the following conditions:*

(1) The link of $(X, 0)$ is homeomorphic to M.
(2) $(\Gamma, L, P) = (\Gamma_\pi, L_\pi, P_\pi)$, where $\pi\colon X_\pi \to X$ is the minimal good resolution of $(X, 0)$ which factors through the blowup of the maximal ideal and the Nash transform of $(X, 0)$.

Sketch of the proof. The first step of the proof is the fact that the topological type M of a normal surface singularity gives an effective bound n_M of the multiplicity of the germs realizing it [2, Proposition C]. The proof of this is based on a construction of Clément Caubel, Andràs Némethi and Patrick Popescu-Pampu in [5].

Given a weighted graph Γ, the bound n_M would then be sufficient to prove the finiteness of the set of the $\mathcal{L}$-vectors L such that the pair (Γ, L) can be realized by a surface singularity $(X, 0)$. By a procedure that we call *gardening*, we then obtain the finiteness of pairs (Γ, L) such that Γ is the graph of the minimal good resolution factoring through the blowup of the maximal ideal.

In order to obtain the finiteness of the P-vector, we then use the well-known Lê–Greuel–Teissier formula [9] to deduce from n_M a bound on the multiplicity of the polar curve of $(X, 0)$ in terms of n_M and of the Euler characteristic of the Milnor–Lê fiber of a generic linear form on $(X, 0)$, which can be computed in terms of the graph Γ.

We now need to prove that the topological type of $(X, 0)$ provide a bound of the number of point blowups necessary to go from any good resolution of $(X, 0)$ to one factoring through the Nash transform of $(X, 0)$. We do this by considering a set of invariants, the so-called *Mather discrepancies* introduced by de Fernex, Ein, and Ishii [6], and proving that they are bounded from above by another invariant ν_v which only depends on the topological type of $(X, 0)$. We conclude by showing that the Mather discrepancies grow faster than the ν_v do when we perform any blowup necessary to achieve factorization through the Nash transform, which permits us to set up an inductive argument. The key technical result allowing us to do this is [2, Theorem 5.1], which proves the existence of a suitable sheaf of Kähler 2-forms that only depends on the topological type of $(X, 0)$, leading to the definition of the invariants ν_v. □

A germ of a real or complex analytic space $(X, 0)$ embedded in $(\mathbb{R}^n, 0)$ or in $(\mathbb{C}^n, 0)$ is equipped with two natural metrics: its *outer metric*, induced by the standard metric of the ambient space, and its *inner metric*, which is the associated arc-length metric on the germ. The germ $(X, 0)$ is said to be *Lipschitz Normally Embedded* (*LNE* for short) if the identity map of $(X, 0)$ is a bilipschitz homeomorphism between the inner and the outer metrics. This property only depends on the analytic type of $(X,0)$, and not on the choice of an embedding in some smooth ambient space $(\mathbb{R}^n,0)$ or $(\mathbb{C}^n,0)$ [12, Proposition 7.2.13]. Lipschitz Normally Embedded germs are fairly common among surface singularities, including in particular all minimal surface singularities [11] and also the superisolated surface singularities with LNE tangent cone [10]. The following result gives a positive answer to Question 2 (Part (1)) extended to the question of the determination of the discriminant curve (Part (2)). Part (1) is a generalization of a result of Spivakovsky on minimal surface singularities [13, Theorem 5.4] while Part (2) generalizes a result proved later by Bondil [3, Theorem 4.1], [4, Proposition 5.4].

Theorem 2. *Let $(X,0)$ be a LNE surface singularity. Then the topological type of $(X,0)$ determines*

(1) the triple $(\Gamma_\pi, L_\pi, P_\pi)$;
(2) the topological type of the discriminant curve of a generic projection.

References

[1] A. Belotto da Silva, L. Fantini and A. Pichon, *On Lipschitz Normally Embedded complex surface germs*, arXiv:2006.01773, to appear in Compositio Math., 2020.
[2] A. Belotto da Silva, L. Fantini, A. Pichon and A. Némethi, *Polar exploration of complex surface germs*, arXiv:2103.15444, 2021.
[3] R. Bondil, *Discriminant of a generic projection of a minimal normal surface singularity*, C. R. Math. Acad. Sci. Paris **337** (3) (2003), 195–200.
[4] ———, *Fine polar invariants of minimal singularities of surfaces*, J. Singul. **14** (2016), 91–112.
[5] C. Caubel, A. Némethi and P. Popescu-Pampu, *Milnor open books and Milnor fillable contact* 3*-manifolds*, Topology **45** (3) (2006), 673–689.
[6] T. de Fernex, L. Ein and S. Ishii, *Divisorial valuations via arcs*, Publ. Res. Inst. Math. Sci. **44**(2) (2008), 425–448.
[7] J. Kollár, *Toward moduli of singular varieties*, Compositio Math. **56**(3) (1985), 369–398.
[8] D. T. Lê, *Geometry of complex surface singularities*, in Singularities - Sapporo 1998, volume 29 of Adv. Stud. Pure Math., pages 163–180. Kinokuniya, Tokyo, 2000.
[9] D. T. Lê and B. Teissier, *Variétés polaires locales et classes de Chern des variétés singulières*, Ann. of Math. (2), **114**(3) (1981), 457–491.
[10] F. Misev and A. Pichon, *Lipschitz normal embedding among superisolated singularities*, Int. Math. Res. Not. **17** (2021), 13546–13569.
[11] W. D. Neumann, H. Møller Pedersen and A. Pichon, *Minimal surface singularities are Lipschitz normally embedded*, J. Lond. Math. Soc. (2) **101** (2020), no. 2, 641–658.
[12] A. Pichon, *An introduction to Lipschitz geometry of complex singularities*, Introduction to Lipschitz geometry of singularities, Lecture Notes in Math. vol. 2280, Springer, Cham, 2020, pp. 167–216.
[13] M. Spivakovsky, *Sandwiched singularities and desingularization of surfaces by normalized Nash transformations*, Ann. of Math. (2), **131**(3) (1990), 411–491.
[14] O. Zariski, *The reduction of the singularities of an algebraic surface*, Ann. of Math. (2) **40** (1939), 639–689.

3D Mirror Symmetry for characteristic classes of singularities

Richárd Rimányi

Assigning characteristic classes to singular subvarieties of a smooth ambient space is an important tool in enumerative geometry. This technique is proved to be effective in at least two settings of singularity theory:

(1) The ambient space is the vector space of map germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^m, 0)$, and the singular subvariety Σ is the collection of germs with a prescribed multiplicity algebra. In this setting the equivariant charcterisitic class of Σ is called a Thom polynomial.

(2) The ambient space is a homogeneous space, like a Grassmannian, and the singular subvariety is a Schubert variety. In this setting the associated characteristic class is called a Schubert class.

Works of Okounkov and his co-authors ([MO, O, AO]) triggered new developments in the theory of characteristic classes of singularities. In particular, now we think that characteristic classes should come with the following improvements.

- There is a 1-parameter deformation of the notion of fundamental class. Due to some relations with physics we denote the parameter by $\hbar$. In cohomology the 1-parameter deformation—in certain situations—coincides with the concept of *Chern-Schwartz-MacPherson class* [RV, FR, AMSS1]. In K theory the 1-parameter deformation—in certain situations—coincides with the concept of *motivic Chern class* [FRW, AMSS2].
- While traditionally characteristic classes are studied in cohomology and K theory, the cohomology theory corresponding to the third one dimensional algebraic group, the elliptic curve, is more general. The $\hbar$-deformed characteristic class notion exists in elliptic cohomology in many interesting cases. These cases include important singularities living in Nakajima quiver varieties (works of Aganagic-Okounkov [AO]), and Schubert varieties in homogeneous spaces ([RW1, KRW, R]).

In this talk we will review a further development: the pool of ambient spaces will be enlarged from type-A Nakajima quiver varieties [N1] to type-A Cherkis bow varieties [Ch1, Ch2, Ch3, NT, N2, BFN, RS]. Bow varieties are associated with NS5-D5-brane diagrams [HW]. They are smooth, holomorphic symplectic varieties with a torus action. We will show how the the finitely many fixed points of the action are in bijection with 0-1 matrices with prescribed row and column sums ("binary contingency tables").

If we chose a 1-parameter subgroup of the torus acting on the bow varieties, then every fixed point p defines its attractive (stable) manifold. Let $\mathrm{Stab}^{ell}(p)$ be the $\hbar$-deformed elliptic characteristic class of this attracting manifold. A remarkable new feature of these elliptic classes is that—besides $\hbar$—they also depend on two sets of variables: the equivariant variables a_i, and the Kähler variables z_i.

A combinatorially very simple involution, namely switching NS5 branes and D5 branes, defines an involution on the pool of bow varieties. In fact, this is one of the main advantages of bow varieties over quiver varieties. Following $N = 4$, $d = 3$ supersymmetric gauge theory we will call this involution the 3d mirror symmetry.

In the lecture we describe a conjecture relating characteristic classes of singularities in 3d mirror dual bow varieties. Namely let X and $X^!$ be a pair of mirror pairs. We will show a natural bijection between the torus fixed points of X and $X^!$, defined via the combinatorics of binary contingency tables. Corresponding fixed points will be denoted by $p \leftrightarrow p^!$. The 3d mirror symmetry conjecture for characteristic classes of singularities is:

$$\mathrm{Stab}^{ell}(p)|_q(a_i, z_i, \hbar) = \pm\,\mathrm{Stab}^{ell}(q^!)|_{p^!}(z_i, a_i, \hbar^{-1}),$$

where $x|_q$ means the restriction of the (elliptic) cohomology class to the fixed point q. The switching of equivariant parameters with Kähler parameters is predicted by superstring theory, where they play the role of *mass-* and *Fayet-Iliopoulos-*parameters of the Higgs branch of the two mirror dual theories.

The conjecture is proved for three infinite series of pairs of spaces:

(i) $X = T^*$Grassmannian, in which case the dual variety is a quiver variety [RSVZ1];

(ii) $X = T^*G/B$, in which case the dual variety is the Langlands dual full flag variety [RSVZ2, RW2];

(iii) for pairs of hypertoric varities [SZ]; as well as for finitely many sporadic cases.

Our proposed proof is via an elliptic cohomological Hall algebra associated to brane diagrams. Namely, we expect that for bow varieties the shuffle product of the Hall algebra is consistent with the characteristic classes. That way the $\mathrm{Stab}^{ell}(p)$ classes, and their conjectured 3d mirror symmetry property, can be studied inductively.

References

[AO] M. Aganagic, A. Okounkov. *Elliptic stable envelopes.* arXiv:1604.00423 (2020).

[AMSS1] P. Aluffi, L. C. Mihalcea, J. Schürmann, C. Su. *Shadows of characteristic cycles, Verma modules, and positivity of Chern-Schwartz-MacPherson classes of Schubert cells.* arXiv:1709.08697, 2017.

[AMSS2] ———, *Motivic Cherm classes of Schubert cells, Hecke algebras, and applications to Casselman's problem.* arXiv:1902.10101, 2019.

[BFN] A. Braverman, M. Finkelberg, H. Nakajima. *Coulomb branches of 3d $N = 4$ quiver gauge theories and slices in the affine Grassmannian.* With two appendices by Braverman, Finkelberg, Kamnitzer, Kodera, Nakajima, Webster and Weekes. Adv. Theor. Math. Phys., 23(1), 75–166, 2019.

[Ch1] S. A. Cherkis. *Moduli spaces of instantons on the Taub-NUT space*, Comm. Math. Phys. 290 (2009), no. 2, 719–736.

[Ch2] ———, *Instantons on the Taub-NUT space,* Adv. Theor. Math. Phys. 14 (2010), no. 2, 609–641.

[Ch3] ——— *Instantons on gravitons*, Comm. Math. Phys. 306 (2011), no. 2, 449–483.

[FR] L. Fehér, R. Rimányi. *Chern-Schwartz-MacPherson classes of degeneracy loci.* Geom. Topol. 22 (2018) 3575–3622.

[FRW] L. M. Fehér, R. Rimányi, A. Weber. *Motivic Chern classes and K-theoretic stable envelopes.* Proc. London Math. Soc., Vol. 122, Issue 1, January 2021, 153–189.

[HW] A. Hanany, E. Witten. *Type IIB superstrings, BPS monopoles, and threedimensional gauge dynamics.* Nucl. Phys., B492:152–190, 1997.

[KRW] S. Kumar, R. Rimányi, A. Weber. *Elliptic classes of Schubert varieties.* Math. Ann., 378(1), 703-728, 2020.

[MO] D. Maulik, A. Okounkov. *Quantum Groups and Quantum Cohomology.* Astérisque 408, Société Mathématique de France, 2019.

[N1] H. Nakajima. *Lectures on Hilbert Schemes of Points on Surfaces* University Lecture Series 18; AMS 1999.

[N2] ———, *Towards geometric Satake correspondence for Kac-Moody algebras—Cherkis bow varieties and affine Lie algebras of type A.* arXiv:1810.04293, 2021.

[NT] H. Nakajima, Y. Takayama. *Cherkis bow varieties and Coulomb branches of quiver gauge theories of affine type A*, Selecta Mathematica 23 (2017), no. 4, 2553–2633.

[O] A. Okounkov. *Lectures on K-theoretic computations in enumerative geometry.* In: *Geometry of Moduli Spaces and Representation Theory*, IAS/Park City Math. Ser., 24, AMS, Providence, RI, (2017), pp. 251–380.

[R] R. Rimányi. *$\hbar$-deformed Schubert calculus in equivariant cohomology, K theory, and elliptic cohomology.* In *Singularities and their interaction with geometry and low dimensional topology*, Trends Math., pages 73–96. Birkhäuser/Springer, Cham, 2021.

[RS] R. Rimányi, Y. Shou. *Bow varieties—geometry, combinatorics, characteristic classes.* arXiv:2012.07814, 2020.

[RSVZ1] R. Rimányi, A. Smirnov, A. Varchenko, Z. Zhou. *3d mirror symmetry and elliptic stable envelopes.* arXiv:1902.03677, 2020.

[RSVZ2] ———, *Three dimensional mirror self-symmetry of the cotangent bundle of the full flag variety.* SIGMA 15 (2019), 093, 22 pages.

[RV] R. Rimányi, A. Varchenko. *Equivariant Chern-Schwartz-MacPherson classes in partial flag varieties: interpolation and formulae.* In *Schubert Varieties, Equivariant Cohomology and Characteristic Classes*, IMPANGA2015 (eds. J. Buczynski, M. Michalek, E. Postingel), EMS 2018, pp. 225–235.

[RW1] R. Rimányi, A. Weber. *Elliptic classes of Schubert varieties via Bott-Samelson resolution.* J. of Topology, Vol. 13, Issue 3, September 2020, 1139–1182.

[RW2] ———, *Elliptic classes on Langlands dual flag varieties.* arXiv:2007.08976, To appear in Comm. in Contemp. Math. 2020.

[SZ] A. Smirnov, Z. Zhou. *3d Mirror Symmetry and Quantum K-theory of Hypertoric Varieties.* arXiv:2006.00118, 2020.

New developments in the study of Maximal Cohen-Macaulay modules over normal surface singularities

Faustino Agustín Romano-Velázquez

(joint work with J. Fernández de Bobadilla)

Our objective is to fully understand the maximal Cohen-Macaulay (MCM) modules over a normal surface singularity. In particular, we are interested in: their classification, deformation theory and moduli space. Some part is done by a joint work with J. F. de Bobadilla.

In the case of a rational surface singularity, the classification of MCM modules is well understood (under the hypothesis of specialty on the modules), also its Cohen-Macaulay representation type is known. Some references are: Artin and Verdier [1], Esnault [3], Wunram [12], Drozd, Greuel and Kashuba [2] and Riemenschneider [10]. In the case of a non-rational surface singularity there are some results in concrete families of singularities: a complete classification and description of the Cohen-Macaulay representation type in the case of minimally elliptic singularities by Kahn [5] and Drozd, Greuel and Kashuba [2], some results on matrix factorizations by Laza, Pfister and Popescu in [7] and Laza, O'Carroll and Popescu [6].

In a collaboration with Bobadilla [4] we obtain almost complete answers for the case of normal Gorenstein singularities. We begin by establishing some notation.

By (X, x) we denote a complex analytic normal Gorenstein surface germ. As usual $\mathcal{O}_X$ denotes the structure sheaf. We denote by $\pi\colon \tilde{X} \to X$ a resolution of singularities. The exceptional set is denoted by E, and its decomposition into irreducible components is $E = \cup_i E_i$.

Since (X, x) is Gorenstein, there is a Gorenstein 2-form $\Omega_{\tilde{X}}$ which is meromorphic in $\tilde{X}$, and has neither zeros nor poles in $\tilde{X} \setminus E$; it is called the *Gorenstein form.* Let $div(\Omega_{\tilde{X}}) = \sum q_i E_i$ be the divisor associated with the Gorenstein form. The *canonical cycle* is defined by $Z_k := \sum_i -q_i E_i$, where the q_i are the coefficients of the Gorenstein form. We say that $\tilde{X}$ is *small with respect to the Gorenstein form* if Z_k is greater than or equal to 0. The *geometric genus* denoted by p_g of X is defined to be the dimension as a $\mathbb{C}$-vector space of $R^1\pi_*\mathcal{O}_{\tilde{X}}$ for any resolution.

Let (X, x) be a normal surface singularity and $\pi : \tilde{X} \to X$ be a resolution. Let M be a reflexive $\mathcal{O}_X$-module (equivalently a MCM module). Its associated *full* $\mathcal{O}_{\tilde{X}}$-module is $\mathcal{M} := (\pi^*M)^{\vee\vee}$ (where $(\quad)^\vee$ denotes the dual with respect to the structure sheaf). A crucial theorem in our classification of MCM modules is the computation of the first cohomology of full sheaves.

Theorem 1 (J. F. de Bobadilla, —). *Let (X, x) be the germ of a normal Gorenstein surface singularity. Let M be a reflexive $\mathcal{O}_X$-module of rank r. Let $\pi : \tilde{X} \to X$ be a small resolution with respect to the Gorenstein form, let Z_k be the canonical cycle at $\tilde{X}$. Let $\mathcal{M}$ be the full $\mathcal{O}_{\tilde{X}}$-module associated to M. Then*

$$\dim_{\mathbb{C}}(R^1\pi_*\mathcal{M}) = \dim_{\mathbb{C}}(R^1\pi_*\mathcal{M}^\vee) - [c_1(\mathcal{M})] \cdot [Z_k].$$

As a corollary of this theorem, we construct the *minimal adapted resolution* of M. Such resolution is characterized as the minimal resolution $\pi : \tilde{X} \to X$ such that the full $\mathcal{O}_{\tilde{X}}$-module $\mathcal{M}$ associated with M is generated by global sections. The minimal adapted resolution motives the following definition: we say that M is a *special reflexive* module if the associated full sheaf $\mathcal{M}$ at the minimal adapted resolution satisfies $\dim_{\mathbb{C}}(R^1\pi_*(\mathcal{M}^\vee)) = rp_g$, where r is the rank of M. One of the main results of our work is a determination of special reflexive modules in terms of a first Chern class.

Theorem 2 (J. F. de Bobadilla, —). *Let (X, x) be a normal Gorenstein surface singularity. Let M be a special $\mathcal{O}_X$-module without free factors. Let $\pi : \tilde{X} \to X$ be the minimal resolution adapted to M, and $\mathcal{M}$ the full $\mathcal{O}_{\tilde{X}}$-module associated to M. The module $\mathcal{M}$ (and equivalently M) is determined by its first Chern class in $Pic(\tilde{X})$.*

Using the previous theorem we classify the special MCM modules over normal Gorenstein surface singularities as follows.

Theorem 3 (J. F. de Bobadilla, —)**.** *Let (X, x) be a normal Gorenstein surface singularity. Then, there exists a bijection between the following sets:*

(1) The set of special indecomposable reflexive $\mathcal{O}_X$-modules up to isomorphism.
(2) The set of irreducible divisors E over x, such at any resolution of X where E appears, the Gorenstein form has not either zeros or poles along E.

We also study the deformation theory and moduli questions of MCM modules. Our first application of the study of the deformations of MCM modules is an affirmative answer of a conjecture of Drozd, Greuel and Kashuba [2] and, together with previous work in [1] and [2] completes the classification of normal Gorenstein surface singularities in Cohen-Macaulay representation types.

Theorem 4 (J. F. de Bobadilla, —)**.** *A Gorenstein surface singularity is of finite Cohen-Macaulay representation type if and only if it is a rational double point. Rational surface singularities of tame Cohen-Macaulay representation type are precisely the log-canonical ones. The remaining Gorenstein surface singularities are of wild Cohen-Macaulay representation type.*

The second application is the construction of fine moduli spaces of special modules without free factors of prescribed graph and rank on normal Gorenstein surface singularities. The interested reader may consult [4] for details.

In [11], we use the theory of blowing-up an algebraic variety X along a coherent sheaf M by Raynaud and Gruson [8, 9] to continue our study of MCM modules. Using the Raynaud-Gruson blow-up we give a different construction of the minimal adapted resolution, the interested reader may consult [11] for details. Using the results given in [4] we know how to recover the normalization of the blow-up at MCM modules using the minimal adapted resolution.

Theorem 5 (—)**.** *Let (X, x) be the germ of a normal surface singularity. Let M be a reflexive $\mathcal{O}_X$-module of rank r. Let $\pi \colon \tilde{X} \to X$ be the minimal adapted resolution associated to M. Let $\mathcal{M} := (\pi^* M)^{\vee\vee}$ be the full sheaf associated to M. Then, the normalization of the Raynaud-Gruson blow-up of X at M is obtained by contracting the irreducible components E_i of E such that $c_1(\mathcal{M}) \cdot E_j = 0$.*

In general, the blow-up of X at M a MCM module may be very difficult to describe. For example, it is difficult to guarantee the normality of such blow-up. Our last result solves the problem of the normality of the Raynaud-Gruson blow-up as follows.

Theorem 6 (—)**.** *Let (X, x) be the germ of a normal Gorenstein surface singularity and M be a special module. Then, the Raynaud-Gruson blow-up of X at M is normal.*

References

[1] M. Artin and J.-L. Verdier. *Reflexive modules over rational double points.* Math. Ann., 270:79–82, 1985.
[2] Y. Drozd, G. M. Greuel, and I. Kashuba. *On Cohen-Macaulay modules on surface singularities.* Moscow Mathematical Journal, v.3, 397-418, 742 (2003), 3, 01 2003.
[3] H. Esnault. *Reflexive modules on quotient surface singularities.* J. Reine Angew. Math., 362:63–71, 1985.
[4] J. Fernández de Bobadilla and A. Romano-Velázquez. *Reflexive modules on normal gorenstein stein surfaces, their deformations and moduli.* arXiv:1812.06543, 2018.
[5] C. P. Kahn. *Reflexive modules on minimally elliptic singularities.* Math. Ann., 285(1):141–160, 1989.
[6] R. Laza, L. O'Carroll, and D. Popescu. *Maximal Cohen-Macaulay modules over* $Y_1^3+\cdots+Y_n^3$ *with few generators.* Math. Rep., Buchar., 3(2):177–185, 2001.
[7] R. Laza, G. Pfister, and D. Popescu. *Maximal Cohen-Macaulay modules over the cone of an elliptic curve.* J. Algebra, 253(2):209–236, 2002.
[8] M. Raynaud. *Flat modules in algebraic geometry.* Compos. Math., 24:11–31, 1972.
[9] M. Raynaud and L. Gruson. *Critères de platitude et de projectivité. Techniques de "platification" d'un module.* Invent. Math., 13:1–89, 1971.
[10] O. Riemenschneider. *Special representations and the two-dimensional McKay correspondence.* Hokkaido Math. J., 32(2):317–333, 2003.
[11] A. Romano-Velázquez. *On the blow-up of a normal singularity at maximal Cohen-Macaulay modules.* arXiv:2004.05441, 2020.
[12] J. Wunram. *Reflexive modules on quotient surface singularities.* Math. Ann., 279(4):583–598, 1988.

On the Fukui-Kurdyka-Paunescu Conjecture

José Edson Sampaio
(joint work with Alexandre Fernandes and Zbigniew Jelonek)

This is an extended abstract of the very recent joint work [1] with Fernandes and Jelonek.

Zariski's famous Multiplicity Conjecture, stated by Zariski in 1971 (see [8]), is formulated as follows:

Conjecture 1 (Zariski's Multiplicity Conjecture)**.** *Let $f,g\colon(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be two reduced complex analytic functions. If there is a homeomorphism $\varphi\colon(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ such that $\varphi(V(f))=V(g)$, then $m(V(f),0)=m(V(g),0)$.*

This is still an open problem, and in the real case it does not hold in the same form as in the complex case. However, we have the following conjecture, stated by Fukui, Kurdyka and Paunescu [2, Conjecture 3.3]:

Conjecture 2 (Fukui-Kurdyka-Paunescu's Conjecture)**.** *Let $X,Y\subset\mathbb{R}^n$ be two germs at the origin of irreducible real analytic subsets. If $h\colon(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ is the germ of a subanalytic, arc-analytic and bi-Lipschitz homeomorphism such that $h(X)=Y$, then $m(X,0)\equiv m(Y,0) \bmod 2$.*

Several authors approached this conjecture, for instance:

(1) Risler in [3] showed that Conjecture 2 holds true for $n=2$;

(2) Fukui, Kurdyka and Paunescu in [2] showed that Conjecture 2 holds true for $n = 2$ even for equality without mod 2;
(3) Valette in [7] showed that Conjecture 2 holds true for hypersurfaces, i.e., when $\dim X = \dim Y = n - 1$ (see also [5]);
(4) Sampaio in [4] proved the real version of Gau-Lipman's theorem, i.e., multiplicity $\bmod 2$ of real analytic sets is invariant under homeomorphisms $\varphi\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that φ and φ^{-1} have a derivative at the origin (see also [6]).

In order to understand better the statements here, let us recall some definitions.

Definition 3. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be two sets and let $h\colon X \to Y$.

- We say that h is **Lipschitz** if there exists a positive constant C such that
$$\|h(x) - h(y)\| \leq C\|x - y\|, \quad \forall x, y \in X.$$
- We say that h is **semi-Lipschitz at** $x_0 \in X$ if there exist a positive constant C such that
$$\|h(x) - h(x_0)\| \leq C\|x - x_0\|, \quad \forall x \in X.$$
- We say that h is **semi-bi-Lipschitz** if h is a homeomorphism, it is semi-Lipschitz at x_0 and its inverse is also semi-Lipschitz at $h(x_0)$.

Definition 4. Let M and N be analytic manifolds. Let $X \subset M$ and $Y \subset N$ be analytic subsets. We say that a mapping $f\colon X \to Y$ is **arc-analytic** if for any analytic arc $\gamma\colon (-1, 1) \to X$, the mapping $f \circ \gamma$ is an analytic arc as well.

In the paper [1], we present the following example:

Example. Consider $X = \{(x, y, z) \in \mathbb{R}^3; z(x^2 + y^2) = y^3\}$ and $Y = \{(x, y, z) \in \mathbb{R}^3 : z(x^4 + y^4) = y^5\}$. Let $h\colon (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be the mapping given by
$$h(x, y, z) = \begin{cases} \left(x, y, z - \frac{y^3}{x^2+y^2} + \frac{y^5}{x^4+y^4}\right) & \text{if } x^2 + y^2 \neq 0 \\ (0, 0, z) & \text{if } x^2 + y^2 = 0. \end{cases}$$
Then X and Y are irreducible real analytic sets such that $m(X, 0) = 3$ and $m(Y, 0) = 5$. Moreover, h is a semialgebraic arc-analytic bi-Lipschitz homeomorphism such that $h(X) = Y$.

Thus, we cannot expect invariance of multiplicity without mod 2 in Fukui-Kurdyka-Paunescu's Conjecture. Still in the same paper [1], we give a positive answer to Fukui-Kurdyka-Paunescu's Conjecture. More precisely, we prove the following:

Theorem 5 ([1, Theorem 4.2])**.** *Let $(X, 0) \subset (\mathbb{R}^n, 0), (Y, 0) \subset (\mathbb{R}^m, 0)$ be germs of real analytic sets and let $h\colon (X, 0) \to (Y, 0)$ be a subanalytic arc-analytic bi-Lipschitz homeomorphism. Then $m(X, 0) \equiv m(Y, 0) \bmod 2$.*

Remark. Theorem 5 proves even more than stated in Fukui-Kurdyka-Paunescu's Conjecture, since we do not require in Theorem 5 that the sets X and Y have to be irreducible or that h has to be defined on a neighbourhood of $0 \in \mathbb{R}^n$.

Moreover, with the following definition, we also prove a global version of Theorem 5.

Definition 6. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be analytic subsets and let $\overline{X}$ be the closure of X in $\mathbb{P}^n(\mathbb{R})$. We say that a mapping $f\colon X \to Y$ is **arc-analytic at** $\overline{X}$ if for any analytic arc $\gamma\colon (-1,1) \to \mathbb{P}^n(\mathbb{R})$ such that $\gamma((-1,0)\cup(0,1)) \subset X$, we have $f \circ \gamma|_{(-1,0)\cup(0,1)}$ extends to an analytic arc $\tilde{\gamma}\colon (-1,1) \to \mathbb{P}^m(\mathbb{R})$.

Thus, the global version of Theorem 5 is the following:

Theorem 7 ([1, Corollary 4.7]). *Let $A \subset \mathbb{R}^n, B \subset \mathbb{R}^m$ be real algebraic sets, let $\overline{A}$ be the closure of A in $\mathbb{P}^n(\mathbb{R})$ and let $h\colon A \to B$ be a semialgebraic and semi-bi-Lipschitz homeomorphism. Assume that h is arc-analytic at $\overline{A}$. Then* $\deg(A) \equiv \deg(B) \bmod 2$.

References

[1] A. Fernandes, Z. Jelonek and J. E. Sampaio *On the Fukui-Kurdyka-Paunescu Conjecture.* arXiv:2108.01179 (2021)

[2] T. Fukui, K. Kurdyka and L. Paunescu *An inverse mapping theorem for arc-analytic homeomorphisms.* Geometric singularity theory, pp. 49–56, Banach Center Publ., 65, Polish Acad. Sci. Inst. Math., Warsaw, 2004.

[3] J.-J. Risler *Invariant Curves and Topological Invariants for Real Plane Analytic Vector Fields.* J. Differential Equations, vol. 172 (2001), 212–226.

[4] J. E. Sampaio *Differential invariance of the multiplicity of real and complex analytic sets.* to appear in Publ. Mat., arXiv:2009.13643 (2020)

[5] ———, *Multiplicity, regularity and Lipschitz geometry of real analytic hypersurfaces.* to appear in Israel J. of Math. `https://www.researchgate.net/publication/336369451`.

[6] ———, *Multiplicity, regularity and blow-spherical equivalence of real analytic sets.* to appear in Math. Z., arXiv:2105.09769 (2021)

[7] G. Valette *Multiplicity mod 2 as a metric invariant.* Discrete Comput. Geom., vol. 43 (2010), 663-679.

[8] O. Zariski *Some open questions in the theory of singularities.* Bull. Amer. Math. Soc., **77** (1971), no. 4, 481-491.

Local Newton nondegenerate Weil divisors in toric varieties

Baldur Sigurðsson

(joint work with András Némethi)

Let $N \cong \mathbb{Z}^r$ be a free abelian group of rank $r > 1$, and $M = N^\vee$ its dual. As a base element, we choose a finitely generated, rational and strictly convex cone $\Sigma \subset N_\mathbb{R}$ of dimension r. Correspondingly, we have an affine toric variety $Y = U_\Sigma$. To any face $\sigma \subset \Sigma$ of dimension 1, there corresponds a Weil divisor $D_\sigma \subset Y$. A sum of such divisors is called *invariant.*

Let $f \in \mathcal{O}_{Y,0}$ be a germ of a holomorphic function at the origin $0 \in Y$, that is, the unique fixed point by the toric action. Denote by $(X',0)$ the Cartier divisor defined by f, and let $(X,0)$ be the union of the components of $(X',0)$ which are not invariant. Any effective local analytic Weil divisor $(X,0) \subset (Y,0)$, none of whose components are invariant, can be obtained by this construction.

Given an $f \in \mathcal{O}_{Y,0}$, write

$$f = \sum_{p \in S_\Sigma} a_p x^p,$$

where $S_\Sigma = \Sigma^\vee \cap M$ is the *semigroup* associated with S. The *support* of f is the set of $p \in S_\Sigma$ for which $a_p \neq 0$, and the *Newton polyhedron* $\Gamma_+(f)$ associated with f is the convex closure of the union of $p + \Sigma^\vee$ for $p \in \operatorname{supp}(f)$. The *Newton diagram* $\Gamma(f)$ is the union of compact faces $F \subset \Gamma_+(f)$. For such a face F, set $f = \sum_{p \in F \cap M} a_p x^p$. Since F is compact, this is a Laurent polynomial. We say that f (or $(X,0)$) is *Newton nondegenerate* if for all such compact faces f, the hypersurface in $(\mathbb{C}^*)$ defined by f_F is a smooth variety. In this case, a resolution of singularities of $\tilde{X} \to X$ can be obtained by a toric morphism $\pi : \tilde{Y} \to Y$, induced by a regular subdivision $\tilde{\triangle}_f$ of the dual fan $\triangle_f$ to the Newton polyhedron $\Gamma_+(f)$.

For each facet $F \subset \Gamma_+(f)$, that is, face of dimension $n-1$, there exists a unique primitive element $\ell_F \in N$ so that F is the minimal set of the restriction of ℓ_F to $\Gamma_+(f)$. The Newton polyhedron $\Gamma_+(f)$ can there be defined by inequalities $\ell_F \geq m_F$, where $m_F \in \mathbb{Z}$. Define the larger polyhedron

$$\Gamma_+^*(f) = \{p \in M_\mathbb{R} | \forall F \in \mathcal{F}_{\text{nc}} : \ell_F \geq m_F\}$$

where $\mathcal{F}_{\text{nc}}$ is the set of noncompact facets of $\Gamma_+(f)$.

Theorem 1. *Let $(X,0) \subset (Y,0)$ be a Newton nondegenerate Weil divisor of dimension $d = r-1$.*

(1) We have the following canonical identifications

$$\overline{\mathcal{O}}_{X,0}/\mathcal{O}_{X,0} \cong \bigoplus_{p \in M} \tilde{H}^0(\Gamma_+(x^p f) \setminus \Sigma^\vee, \mathbb{C}),$$

$$H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong \bigoplus_{p \in M} \tilde{H}^i(\Gamma_+(x^p f) \setminus \Sigma^\vee, \mathbb{C}), \quad i > 0.$$

In particular, if these vector spaces have finite dimension, then

$$\delta(X,0) = \sum_{p \in M} \tilde{h}^0(\Gamma_+(x^p f) \setminus \Sigma^\vee, \mathbb{C}),$$

$$p_g(X,0) = (-1)^{d-1} \sum_{p \in M} \tilde{\chi}(\Gamma_+(x^p f) \setminus \Sigma^\vee, \mathbb{C}),$$

where $\tilde{\chi}$ denotes the reduced Euler characteristic, *that is, the alternating sum of ranks of reduced singular cohomology groups.*

(2) We have

$$\tilde{h}^{d-1}(\Gamma_+(x^p f) \setminus \Sigma^\vee, \mathbb{C}) = \begin{cases} 1 & \textit{if} \quad 0 \in \Gamma_+^*(x^p f)^\circ \setminus \Gamma_+(x^p f)^\circ, \\ 0 & \textit{else}. \end{cases}$$

In particular, $h^{d-1}(\tilde{X}, \mathcal{O}_{\tilde{X}}) = |M \cap \Gamma_+^*(f)^\circ \setminus \Gamma_+(f)^\circ|$.

Theorem 2. *Assume that $(X,0) \subset (Y,0)$ is a Newton nondegenerate Weil divisor of dimension 2, and is normal. Then $(X,0)$ is Gorenstein if and only if there exits an integral point $p \in M$ so that if $F \in \mathcal{F}_{\mathrm{nc}}$ is so that $F \cap \Gamma(f)$ has dimension 1, then $\ell_F(p) = m_F + 1$.*

From now on, we assume that $(X,0)$ is a normal Gorenstein surface singularity, and that $p \in M$ is as in the above theorem. For each compact facet $F \subset \Gamma_+(f)$, let C_F be the convex hull of $F \cup \{p\}$. Combining the two theorems, we find that $p_g(X,0)$ is the number of integral points in the union of these C_F. We fix a refinement of the dual fan which induces a proper morphism $\tilde{Y} \to Y$, restricting to a resolution of singularities $\tilde{X} \to X$.

A *computation sequence* is a finite sequence of *cycles*, i.e. divisors in $\tilde{X}$, supported in the exceptional divisor $E = \cup_{v\in\mathcal{V}} E_v$, of the form $0 = Z_0 < \cdots < Z_k$, where for each i, there is a $v(i) \in \mathcal{V}$ so that $Z_i = Z_{v(i)}$, and $Z_k = Z_K$ is the anticanonical cycle which satisfies the adjunction formula $(E_v, Z_K) = 2 - \delta_v + E_v^2$, where δ_v is the degree of v in the resolution graph. If $Z = \sum_{v\in\mathcal{V}} a_v E_v$ is a divisor supported on the exceptional divisor, write $m_v(Z) = a_v$. There is a bijection between the compact facets of $\Gamma_+(f)$ and a subset $\mathcal{N} \subset \mathcal{V}$, the *nodes* of the resolution graph. The *diagonal computation sequence* is defined as follows. Start by setting $\bar{Z}_0 = 0$. Assuming that $\bar{Z}_i$ has been defined, choose $\bar{v}(i) \in \mathcal{N}$ to minimize the fraction

$$\frac{m_{\bar{v}(i)}(\bar{Z}_i)}{m_{\bar{v}(i)}(\bar{Z}_K - E)}.$$

The cycle $\bar{Z}_{i+1}$ is then obtained by applying a *generalized Laufer algorithm* to $\bar{Z}_i + E_{\bar{v}(i)}$, that is, iteratively add a component E_w to the cycle as long as it has positive intersection with E_w.

It follows from general properties that given a computation sequence $(Z_i)_i$, we have an inequality

$$p_g \leq \sum_{i=0}^{k-1} \max\{0, (-Z_i, E_{v(i)}) + 1\}. \tag{1}$$

Denote by $S(Z_i)$ the right hand side above. The *minimal path lattice cohomology* associated with $(X,0)$ is the smallest value $S(Z_i)$ obtained by any computation sequence. This is a topological invariant, that is, it depends only on the oriented differomorphism type of the link of $(X,0)$. Note that the ith term in (1) vanishes if $(Z_i, E_{v(i)}) > 0$,

If there exists a computation sequence for which equality holds in (1), then p_g equals the minimal path lattice cohomology. Furthermore, in this case, the geometric genus is maximal among all singularities sharing the given topological type.

Theorem 3. *Assume that $(X,0)$ is a normal and Gorenstein Newton nondegenerate local Weil divisor whose link is a rational homology three-sphere, and let $(\bar{Z}_i)$ be the diagonal computation sequence on the resolution given by Oka's algorithm.*

Then

$$p_g = \sum_{i=0}^{\bar{k}-1} \max\{0, (-\bar{Z}_i, E_{\bar{v}(i)}) + 1\}.$$

In particular, the geometric genus is a topological invariant.

References

[1] A. Némethi and B. Sigurðsson. *Local Newton nondegenerate Weil divisors in toric varieties* arXiv:2102.02948 (2021).

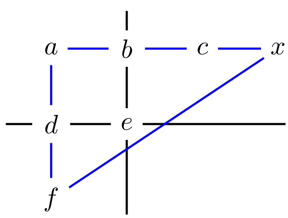
Tom and Jerry

Jan Stevens

(joint work with Gavin Brown and Miles Reid)

It remains difficult to construct Gorenstein rings in codimension 4. One approach is to use unprojection, that is projection undone.

We show how this works for a specific example, the anticanonical cone over $\mathbb{P}(1,2,3)$. This Gorenstein singularity can also be described torically by a lattice polytope with vertices $(1,0)$, $(-2,3)$ and $(-1,0)$ with dual polygon as in the picture on the left with the names of the variables we use shown, or as the cone over a singular del Pezzo surface given by the linear system of cubics in the plane with a fixed flectional tangent in a given point, with a base of the linear system shown on the right.



$$\begin{array}{cccc} U^3 & U^2V & UV^2 & V^3 \\ U^2W & UVW & & \\ UW^2 & & & \end{array}$$

Each of these descriptions provides generators of the ring, and from there equations are easily obtained. Projecting gives a way to organise the equations: one variable occurs only linearly, in four of the 9 equations, and can be elimnated. The remaining equations are the 4×4 Pfaffians of a skew-symmetric 5×5 matrix

$$\begin{pmatrix} 0 & c & b & d \\ & x & c & e \\ & & e & f \\ & & & 0 \end{pmatrix}$$

Here we omit the zeros on the diagonal and the $m_{ji} = -m_{ij}$ below the diagonal. The projection leads to an exceptional divisor, which is a linear codimension 4 complete intersection $(c = e = f = x = 0)$ in the codimension 3 Pfaffian variety.

This exemplifies the general set-up of unprojection, of a Gorenstein codimension one divisor $D \subset X$ in a Gorenstein variety X. The adjunction sequence

$$0 \to \omega_X \to \mathrm{Hom}(\mathcal{I}_D, \omega_X) \to \omega_D \to 0$$

provides a generator s of $\mathrm{Hom}(\mathcal{I}_D, \omega_X)$ projecting on a generator of ω_D. This in turn leads to a rational function and the closure of its graph defines the unprojected variety.

In the codimension 4 case often Pfaffians, as in our example, occur. For skew 5×5 matrices there are two ways to ensure that the Pfaffians lie in the ideal $I = (c, e, f, x)$ of the divisor D:

$$\mathrm{Tom}_1\colon \quad \left(\begin{array}{|cccc} * & * & * & * \\ \hline & x & c & e \\ & & m_1 & f \\ & & & m_2 \end{array}\right) \qquad \mathrm{Jerry}_{4,5}\colon \quad \left(\begin{array}{cc|cc} * & * & m_2 & m_3 \\ & * & e & c \\ & & f & x \\ \hline & & & m_1 \end{array}\right)$$

where $m_1, m_2 \in I$, respectively $m_1, m_2, m_3 \in I$. These neutral names were given by Miles Reid and are not associated to any mathematical meaning.

Extending the matrix of our example as a Tom matrix in a maximal way defines a deformation whose general fibre can be given by the 2×2 minors of a 3×3 matrix. The general variety of this type is the total space of one deformation component of the cone over a del Pezzo surface of degree 6. The other component, which also exists for our singular del Pezzo, is connected to the Jerry format.

These two components can also be retrieved by computing the unprojection of the versal deformation $i_S\colon D_S \hookrightarrow X_S$ of the map $i\colon D \hookrightarrow X$, leading to a deformation Y_S of the unprojection Y of $D \subset X$. In our example the computation becomes very easy, if one does not determine the remaining four equations (this is better done with a computer algebra system). This procedure works more generally, for example for a general hyperplane section, a simple elliptic singularity of multiplicity 6. By a judicious choice of section the computation fits in the same pattern, and one quickly finds the base space of the deformation, which is irreducible: up to a smooth factor it is the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$. The smooth factor comes from the modulus of the elliptic curve, and a second modulus, related to the point used to project. The resulting matrix is neither a Tom nor a Jerry matrix.

References

[1] G. Brown, M. Reid and J. Stevens, *Tutorial on Tom and Jerry: the two smoothings of the anticanonical cone over* $\mathbb{P}(1, 2, 3)$. EMS Surv. Math. Sci. **8** (2021), 25–38.

Orbifold splice quotients and log covers of surface pairs

Jonathan Wahl

(joint work with Walter D. Neumann)

Our goal is to extend to surface pairs (X, C) the notions associated with splice quotient singularities $(X, 0)$; the work appears in [7].

In a series of papers [5], the authors introduced the class of complex normal surface singularities called *splice quotient singularities*. These are germs, whose link Σ is a rational homology sphere ($\mathbb{Q}$HS), for which the defining equations can be written down explicitly from the graph Γ of the minimal good resolution $(\tilde{X}, E) \to (X, 0)$. This class includes many important examples: rational singularities [8]; minimally elliptic singularities with $\mathbb{Q}$HS link; weighted homogeneous singularities with rational central curve. These singularities have also been studied by other authors (e.g. [2] and [3]).

The key point is that when Σ is a $\mathbb{Q}$HS, i.e., Γ is a tree, then one has the finite universal abelian covering $\Sigma' \to \Sigma$, and this extends to a finite map $(X', 0) \to (X, 0)$ (also called the UAC). The covering group is $H_1(\Sigma; \mathbb{Z})$, which can be computed from Γ as the *discriminant group* $D(\Gamma)$. The authors realized that in many cases the UAC $(X', 0)$ is in fact a complete intersection, with defining equations of $(X', 0)$ of very special type (e.g., a Brieskorn complete intersection in the weighted homogeneous case). Here is a relaxed version of one of the results:

Theorem 1 ([5])**.** *Under some mild conditions ("semigroup and congruence conditions") on a minimal good resolution graph Γ with t ends, one can construct*

(1) explicit classes of complete intersection singularities $(X', 0) \subset (\mathbb{C}^t, 0)$
(2) a diagonal representation $D(\Gamma) \subset (\mathbb{C}^*)^t$ acting freely on each $X' - \{0\}$
(3) $(X', 0) \to (X', 0)/D(\Gamma) \equiv (X, 0)$, the UAC of a singularity with graph Γ.

Such X, called "splice quotient singularities," are thus explicit examples having a resolution with the given graph Γ. The name arises because one first passes from Γ to an auxiliary graph Δ called a *splice diagram*, by collapsing all vertices of valence 2, and assigning a positive integer weight to every node and emanating edge (cf. [1]).

The "end-curve theorem" of [6] characterizes those singularities $(X, 0)$ which are splice quotients. The t ends of the graph Γ of the minimal good resolution correspond to t isotopy classes of knots in the link Σ. Splice quotient singularities are exactly those for which each such class is represented (up to a multiple) by the zero-set of a function on $(X, 0)$.

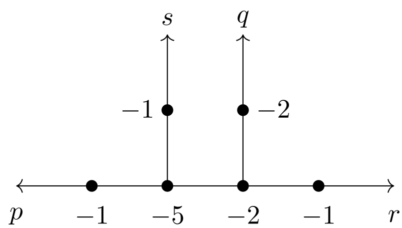
It is natural to consider a "log situation" (X, C), or surface pair, where $(X, 0)$ is a normal surface singularity and $C = \sum_{i=1}^{r} n_i C_i$ is a positive integral linear combination of Weil divisors ("irreducible curves") on X. On the boundary, this provides an *orbifold*; that is, a three-manifold Σ with a class of r knots γ_i and multiplicities n_i. (This definition is compatible with more standard ones [4].) There is already a well-defined notion of *orbifold cover*, which considers orbifolds mapping to (Σ, γ_i, n_i) with bounded ramification. When further Σ is a $\mathbb{Q}$HS, there is a finite *universal abelian orbifold cover* (or UAOC) mapping finitely onto (Σ, γ_i, n_i), with covering group called the *orbifold homology* $H_1^{orb}(\Sigma)$. The question becomes: what are the analogues in the algebraic setting, for a surface pair (X, C)? One defines *log covers*, which are analogues of orbifold covers, leading to

Theorem 2. *Suppose $(X, C = \sum_{i=1}^r n_i C_i)$ is a surface pair for which the link Σ is a $\mathbb{Q}HS$. Then there exists a unique* universal abelian log cover *(UALC) $(X'', C'') \to (X, C)$, inducing the UAOC on the boundary. The covering group $G = H_1^{orb}(\Sigma)$ sits in a short exact sequence*

$$0 \to \mathbb{Z}/(n_1) \oplus \cdots \oplus \mathbb{Z}/(n_r) \to G \to H_1(\Sigma; \mathbb{Z}) \to 0\,.$$

To understand the covering group and construct the UALC of (X, C), one considers the *minimal orbifold log resolution* of (X, C). This is a resolution $(\tilde{X}, E) \to (X, 0)$ for which the inverse image of C has strong normal crossings. To imitate the UAC situation, one blows up enough so that the proper transforms of the C_i intersect an end of E, and each end intersects at most one C_i. Such ends are called *special.* The appropriate graph of this picture is a decorated version Γ^* of Γ, which identifies those ends intersecting a C_i, plus the multiplicities.

Example. If $X = \mathbb{C}^2$, $C = \{x^p y^q (x-y)^r (y^2 - x^3)^s = 0\}$, the diagram for Γ^* is



Note that Γ is usually *not* the graph of the minimal good resolution of $(X, 0)$. From Γ^*, one can compute the orbifold homology.

Theorem 3. *Consider the minimal orbifold log resolution $(\tilde{X}, E = \sum_{j=1}^m E_j) \to (X, C)$. To every exceptional E_j which is not a special end, assign weight $n_j = 1$; the special ends already have weights n_i. Then the orbifold homology $H_1^{orb}(\Sigma)$ is generated by $e_1, \cdots, e_m$, modulo the relations*

$$\sum_{j=1}^m n_i (E_i \cdot E_j) e_j = 0, \quad i = 1, 2, \cdots, m.$$

One now knows the ingredients for a UALC.

Theorem 4. *Suppose Γ and Γ^* satisfy the semigroup and congruence conditions. Then one can construct, analogously to the UAC case, explicit equations and group action for the UALC of pairs (X, C) whose minimal orbifold resolution gives Γ and Γ^*.*

A singular pair (X, C) which can be constructed as in the Theorem is called an *orbifold splice quotient.* Such a singularity $(X, 0)$ is automatically a splice quotient, but the converse is not true; one requires the semigroup and congruence conditions not on the minimal good resolution, but on the minimal orbifold log resolution. Further, the C_i in an orbifold splice quotient must be $\mathbb{Q}$-Cartier. However, one does have the

Theorem 5. *A singular pair (X, C) for which $(X, 0)$ has a rational surface singularity is automatically an orbifold splice quotient.*

References

[1] D. Eisenbud and W. D. Neumann, *Three-dimensional link theory and invariants of plane curve singularities.* Ann. Math. Stud. **110**, Princeton. Princeton Univ. Press (1985).
[2] A. Némethi and T. Okuma, *On the Casson invariant conjecture of Neumann-Wahl*, J. Algebraic Geometry **18** (2009), no. 1, 135–149.
[3] ———, *The Seiberg-Witten invariant conjecture for splice-quotients*, J. London Math. Soc.(2) **78** (2008), 143–154.
[4] W.D. Neumann, *Notes on Geometry and 3-Manifolds, Low Dimensional Topology*, Böröczky, Neumann, Stipsicz, eds., Bolyai Society Mathematical Studies **8** (1999), 191–267.
[5] W. D. Neumann and J. Wahl, *Complete intersection singularities of splice type as universal abelian covers*, Geom. Topol. **9** (2005), 699–755.
[6] ———, *The End Curve Theorem for normal complex surface singularities*, J. Eur. Math. Soc. **12** (2010), 471-503.
[7] ———, *Orbifold splice quotients and log covers of surface pairs*, Journal of Singularities **23** (2021), 151–169.
[8] T. Okuma, *Universal abelian covers of certain surface singularities*, Math. Ann. **334** (2006), 753–773.

From elliptic characteristic classes to twisted motivic

Andrzej Weber
(joint work with Jakub Koncki)

Recent development in representation theory and enumerative geometry led A. Okounkov and his coauthors to the definition of a stable envelope, which is a cohomological object and can be interpreted as a characteristic class associated to a torus acting on a complex symplectic algebraic manifold. If the symplectic manifold is the cotangent bundle of a base manifold, then the stable envelopes can be expressed by invariants of Białynicki-Birula cells: Chern-Schwartz-MacPherson classes in cohomology, motivic Chern classes in K-theory or elliptic classes of Borisov-Libgober in elliptic cohomology, depending which cohomology theory we consider. We briefly review the notion of stable envelopes and report on recent developments obtained in [3, 7, 4, 5].

We consider smooth complex algebraic variety M with an action of $\mathbb{C}^*$. For each fixed point component $F \subset M^{\mathbb{C}^*}$ we define the set of points which converge to that component as $t \to 0$:

$$M_F^+ = \{x \in M \ : \ \lim_{t\to 0} t \cdot x \in F\}\,.$$

If M is complete then the sets M_F^+ cover whole M. In beginning of 70s Białynicki-Birula proved basic properties of this decomposition, which from that time is often called BB-decomposition. One should note that in general $\{M_F^+\}$ is not a stratification. If the variety M is not complete, then the BB-cells not necessarily cover whole M. Nevertheless the BB-decomposition is a powerful tool to study topology and geometry of algebraic varieties. An important class of examples consists of homogeneous varieties G/P, where G is a semisimple algebraic group and P is a parabolic subgroups. With properly chosen $\mathbb{C}^*$ in the maximal torus the Białynicki-Birula cells coincides with Schubert cells.

In the last decade a significant activity of research was initiated by A. Okounkov, who defined characteristic classes associated to the $\mathbb{C}^*$-action on symplectic manifolds. These classes are called stable envelopes. In the case of isolated fixed points the stable envelope is a collection of cohomology classes associated to the fixed points. Here three types of cohomologies $\mathcal{H}$ are considered: the classical cohomology, K-theory and elliptic theory. We use the equivariant theory, for a bigger torus $\mathbb{T}$ which acts on the manifold M and contains $\mathbb{C}^*$ defining decomposition. The restriction of cohomology classes to the fixed points is meaningful. With a mild assumption about M the restriction map $\mathcal{H}_\mathbb{T}(M) \to \mathcal{H}_\mathbb{T}(M^\mathbb{T})$ is an isomorphism up to a $\mathcal{H}_\mathbb{T}(pt)$-torsion. If $M^\mathbb{T}$ is finite then to describe a class in $\mathcal{H}_\mathbb{T}(M)$ it is enough to give a list of elements of $\mathcal{H}_\mathbb{T}(pt)$ indexed by fixed points. The equivariant cohomology of a point is an object with which it is much easier to work.

(1) If $\mathcal{H}$ is the Borel equivariant cohomology, then $\mathcal{H}_\mathbb{T}(pt) = H^*_\mathbb{T}(pt;\mathbb{C}) = \mathbb{C}[\mathfrak{t}]$ is the polynomial ring,
(2) If $\mathcal{H}$ is K-theory, then $\mathcal{H}_\mathbb{T}(pt) = K_\mathbb{T}(pt)_\mathbb{C} = \mathbb{C}[\mathbb{T}]$ is the Laurent polynomial ring,
(3) If $\mathcal{H}$ is elliptic cohomology, then the cohomology of a point can be identified with sections of some bundles over a product of elliptic curves. In down-to-earth terms one works with double quasi-periodic functions on $\mathfrak{t}$.

The stable envelopes are subjects of axioms, and existence of stable envelopes is not clear a priori. We discuss one of the axioms, called *smallness*. We consider the case of the cotangent bundle $M = T^*N$, where N is a variety with an action of a torus $\mathbb{A}$ and $\mathbb{T} = \mathbb{A} \times \mathbb{C}^*$, with the second factor acting on the fibers of T^*N as scalar multiplication. The Okounkov axioms can be rephrased in terms of data associated to the fixed points in N. The smallness axiom is the following. Suppose $p, q \in N^\mathbb{A}$, $p \neq q$. Then the components of the stable envelope associated to these points satisfy: $Stab(p)_{|q}$ is *small* comparing with $Stab(q)_{|q}$. The notion of smallness depends on the cohomology theory we use. For the classical theory it is just the comparison of the degrees. For K-theory we compare the elements of the representation ring of the torus, which is the Laurent polynomial ring. One polynomial is smaller than other if we have an inclusion of the associated Newton polytopes. For elliptic cohomology the elements we would have to compare sections of different line bundles. Such comparison is irrelevant and the smallness axiom is empty.

The case of the generalized full flag variety $N = G/B$ was extensively studied. In this case the stable envelopes up to a normalization coincide with the following invariants of singular spaces

(1) in cohomology: Chern-Schwartz-MacPherson classes [6, 1],
(2) in K-theory: motivic Chern classes of Brasselet-Schürmann-Yokura [2, 3],
(3) in elliptic theory: elliptic classes of Borisov-Libgober [7].

For K-theory only particular stable envelopes are under control - those associated to the trivial slope. Let us explain this dependence in detail. The Okounkov smallness axiom is expressed in terms of a *slope*, i.e. a chosen ,,fractional" bundle $s \in Pic(M) \otimes \mathbb{Q}$. For $p \in M^{\mathbb{A}}$ the restriction $s_{|p}$ determines a $\mathbb{Q}$-weight $\mathbf{w}_p(s)$ of the torus $\mathbb{A}$. Smallness axiom for K-theory takes the form:

$$\text{If } p, q \in M^{\mathbb{A}},\ p \neq q \quad \text{then} \quad \mathcal{N}(Stab^s(p)_{|q}) - \mathbf{w}_p(s) \ \subset\ \mathcal{N}(Stab^s(q)_{|q}) - \mathbf{w}_q(s),$$

i. e. the inclusion of the Newton polytope $\mathcal{N}(Stab^s(p)_{|q})$ into $\mathcal{N}(Stab^s(q)_{|q})$ is up to a shift prescribed by the slope. Here we assume that the slope s is generic. The current work [5] shows that using a normalized limit of elliptic classes one can define twisted motivic Chern classes which form a stable envelope, provided that BB-decomposition is regular enough. The construction works for cotangent bundle, provided that the fixed point set is finite. Let us present some details.

A necessary ingredient for construction of elliptic classes is the Jacobi Theta Function $\vartheta_q(x)$. We write it in the multiplicative notation. It depends on $q = e^{2\pi i\tau}$ belonging to the unit disk in $\mathbb{C}$, or treated as a formal parameter. Let

$$\delta_q(x,y) = \frac{\vartheta_q'(1)\vartheta_q(xy)}{\vartheta_q(x)\vartheta_q(y)} = \frac{1-(xy)^{-1}}{(1-x^{-1})(1-y^{-1})} + q(x^{-1}y^{-1} - xy) + q^2 \ldots$$

The elliptic characteristic class of a variety X together with a $\mathbb{Q}$-divisor D is defined via a resolution of singularities $f : \widetilde{X} \to X$. We assume that $K_X + D$ is $\mathbb{Q}$-Cartier divisor and

$$f^*(K_X + D) = K_{\widetilde{X}} + \widetilde{D},$$

where $supp(\widetilde{D})$ together with the exceptional divisor has simple normal crossings. Then

$$\mathcal{E}\ell\ell(X; D) = f_*\mathcal{E}\ell\ell(\widetilde{X}; \widetilde{D})\,.$$

For the resolution $\widetilde{X}$ the elliptic class is defined directly using certain combination of Chern classes and components of $\widetilde{D}$. For simplicity assume that the torus $\mathbb{A}$ is acting on $\widetilde{X}$ with finitely many fixed points. We give a formula for the elliptic class via Localization Theorem. Suppose $\tilde{p} \in \widetilde{X}^{\mathbb{A}}$ and $z_1, z_2, \ldots z_n$ are $\mathbb{A}$ invariant coordinates at $\tilde{p}$. Assume $\widetilde{D}_{|\tilde{p}} = \sum_{i=1}^n a_i H_i$, where $H_i = \{z_i = 0\}$. Then

$$\mathcal{E}\ell\ell(\widetilde{X}, \widetilde{D})_{|\tilde{p}} = e(T_{\tilde{p}}\widetilde{X}) \prod_{i=1}^n \delta_q(x_i, h^{1-a_i})\,.$$

Here x_i is the character of z_i and $e(T_{\tilde{p}}\widetilde{X})$ denotes K-theoretic Euler class. In [7] we have shown that for a fixed point σ in the generalized flag variety

$$Stab^s(\sigma) = \mathcal{E}\ell\ell(X_\sigma, \partial X_\sigma + s)\,,$$

where X_σ is the corresponding Schubert variety. This result cannot be easily extended. The singularities of the Schubert varieties in G/B have very special form: the divisor $K_{X_\sigma} + \partial X_\sigma$ is $\mathbb{Q}$-Cartier and it admits a resolution (by Bott-Samelson construction) $f : \widetilde{X}_\sigma \to X_\sigma$, such that

$$f^*(K_{X_\sigma} + \partial X_\sigma) = K_{\widetilde{X}_\sigma} + \partial \widetilde{X}_\sigma\,,$$

where $\partial \widetilde{X}_\sigma$ is taken with multiplicities one. Taking the limit as the modular parameter $q \to 0$, we obtain

$$\lim_{q\to 0} \mathcal{E\ell\ell}(X_\sigma, \partial X_\sigma + s) = f_*\Big(\mathcal{O}(\lceil f^*(s)\rceil)\cdot mC(\widetilde{X}_\sigma \setminus f^{-1}(\partial X_\sigma) \hookrightarrow \widetilde{X}_\sigma)\Big)\,.$$

Here $mC(\widetilde{X} \setminus f^{-1}(\partial X_\sigma) \hookrightarrow \widetilde{X})$ is the motivic Chern class of the resolution. Regardless that the elliptic classes are not always defined the limit classes make sense in much wider context.

Let X be a (possibly singular) variety, D a $\mathbb{Q}$-Cartier divisor. Suppose $X \setminus |D|$ is smooth. Let $f : \widetilde{X} \to X$ be a resolution of the pair (X, D) with $E = f^{-1}(|D|)$ a normal crossing divisor. Define the twisted motivic Chern class by the formula

$$mC(X; D) := f_*\Big(\mathcal{O}(\lceil f^*(s)\rceil)\cdot mC(\widetilde{X} \setminus E \hookrightarrow \widetilde{X})\Big)\,.$$

Theorem 1 ([5])**.** *The twisted motivic Chern class $mC(X; D)$ does not depend on the resolution.*

Our proof is based on the equivariant version of the Weak Factorization Theorem. Moreover

Theorem 2 ([5])**.** *The twisted motivic Chern classes satisfy the smallness axiom of K-theoretic stable envelopes*

$$\mathcal{N}(mC(\overline{M}^+_p; s)_{|q}) - \mathbf{w}_p(s) \;\subset\; \mathcal{N}(mC(\overline{M}^+_q; s)_{|q}) - \mathbf{w}_q(s)\,.$$

The support axiom of stable envelopes for cotangent bundles forces the BB-decomposition to be a Whitney stratification (at least the condition A is necessary). Otherwise the stable envelopes are not defined. With a little bit stronger condition expressed in terms of a resolution we can show that the twisted motivic Chern classes satisfy all axioms of stable envelopes. Nevertheless we think that the Whitney condition should be enough.

References

[1] P. Aluffi, L. C. Mihalcea, J. Schürmann, and C. Su. *Shadows of characteristic cycles, Verma modules, and positivity of Chern-Schwartz-MacPherson classes of Schubert cells*, arXiv:1709.08697 (2017)

[2] ———, *Motivic Chern classes of Schubert cells, Hecke algebras, and applications to Casselman's problem*, arXiv:1902.10101 (2019)

[3] L. M. Fehér, R. Rimányi, and Andrzej Weber, *Motivic Chern classes and K-theoretic stable envelopes* Proc. Lond. Math. Soc. (3), 122(1):153–189, 2021.

[4] J. Koncki, *Comparison of motivic Chern classes and stable envelopes for cotangent bundles,* arXiv:2006.02403 (2020).

[5] J. Koncki and A. Weber, *Twisted motivic Chern class and stable envelopes*, arXiv:2101.12515 (2021).
[6] R. Rimányi and A. Varchenko, *Equivariant Chern-Schwartz-MacPherson classes in partial flag varieties: interpolation and formulae.* In *Schubert varieties, equivariant cohomology and characteristic classes—IMPANGA 15*, EMS Ser. Congr. Rep., pages 225–235. Eur. Math. Soc., Zürich, 2018.
[7] R. Rimányi and A. Weber, *Elliptic classes of Schubert varieties via Bott-Samelson resolution.* J. Topol., 13(3):1139–1182, 2020.

Stability of singularities

Chenyang, Xu

In higher dimensional geometry, there is a local-to-global principle, which makes the analogue between Fano varieties and Kawamata log terminal (klt) singularities. One topic of Fano varieties, which has been rapidly moving forward in recent years, is its K-stability theory. One might naturally ask whether there exists a stability theory for klt singularities. Recall that K-stability was introduced by differential geometers to characterise the existence of Kähler-Einstein metric on a Fano variety. There is a local differential geometry theory which is similar to the global study of Kähler-Einstein metrics, namely the theory of Sasaki-Einstein metrics. However, it only considers klt singularities $x \in X$ with a torus group $\mathbb{T}$-action, such that x is the unique fixed point contained in the closure of any orbit.

A general klt singularity will not admit a torus $\mathbb{T}$-action, therefore the setting of the Sasaki-Einstein theory is rather special from a birational geometry viewpoint. In [2], Donaldson-Sun study the singularity $(x \in X)$ where X is a singular Fano variety appearing as the Gromov-Hausdorff limit of a sequence of Kähler-Einstein metric. Then they show that one can use the (singular) Kähler-Einstein metric to construct a metric cone, which degenerates $(x \in X)$ to a singularity $(o \in C)$ with a Sasaki-Einstein metric. While the assumption that the singularity has to been on a Fano variety with a metric condition is rather restrictive for establishing a purely local singularity theory, Donaldson-Sun's theory suggests a picture of canonically degenerating an arbitrary singularity to a singularity with a torus action.

Inspired by earlier work of Martelli-Sparks-Yau [7], in [3] Li made the remarkable observation that Donaldson-Sun's degeneration should be given by a purely algebraic process, using a valuation which minimizes the normalized volume function invented by him. This conjectural picture has the spectacular feature that it should hold for *any* klt singularity, without any extra assumption.

More precisely, for any klt singularities $(x \in X = \mathrm{Spec}(R))$, denote the non-archimedean link by

$$\mathrm{Val}_{X,x} = \{\text{valuations of } k(X) \text{ which centered at } x\}.$$

Then Li defines the following function $\mathrm{Val}_{X,x} \to \mathbb{R}_{\geq 0} \bigcup \{+\infty\}$:

$$\widehat{\mathrm{vol}} : v \to A^n_X(v) \cdot \mathrm{vol}(v) \quad \text{if } A_X(v) < \infty \qquad \text{and} \qquad v \to +\infty \quad \text{if } A_X(v) = \infty,$$

and proposes to study the invariant

$$\widehat{\mathrm{vol}}(x \in X) := \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}(v)$$

and the minimizer of $\widehat{\mathrm{vol}}(v)$.

We have the following results, which were all conjectured in [3].

Theorem (Blum, [1])**.** *A minimizer of* $\widehat{\mathrm{vol}}(v)$ *in* $\mathrm{Val}_{X,x}$ *exists.*

Theorem (Xu, [8])**.** *Any minimizer of* $\widehat{\mathrm{vol}}(v)$ $(v \in \mathrm{Val}_{X,x})$ *is quasi-monomial.*

Theorem (Xu-Zhuang, [9])**.** *The minimizer of* $\widehat{\mathrm{vol}}(v)$ $(v \in \mathrm{Val}_{X,x})$ *is unique up to a rescaling.*

For $v \in \mathrm{Val}_{X,x}$, we denote by Φ its value monoid, i.e. the image of $v\colon R \to \mathbb{R}$. Then we denote by $\mathrm{gr}_v(R) = \bigoplus_{k\in\Phi} \mathfrak{a}_k/\mathfrak{a}_{>k}$ the associated graded ring. The following conjecture in [3] remains open for now.

Conjecture (Higher Rank Finite Generation Conjecture, local)**.** *If* v *is a minimizer of* $\widehat{\mathrm{vol}}(\cdot)$*, then* $\mathrm{gr}_v(R)$ *is finitely generated.*

The following theorem completes the (conjectural) local stability picture.

Theorem (Li-Xu, [5, 6])**.** *Assuming a minimizer* v *of* $\widehat{\mathrm{vol}}(\cdot)$ *satisfies that* $\mathrm{gr}_v(R)$ *is finitely generated. Then* $\big(o \in X_0 := \mathrm{Spec}(\mathrm{gr}_v(R))\big)$ *is klt, and* $(o \in X_0, \xi_v)$ *is K-semistable as a Fano cone where* ξ_v *is the vector induced by* $v\colon \Phi \to \mathbb{R}$.

Conversely, if a valuation $v \in \mathrm{Val}_{X,x}$ *satisfies that* v *is quasi-monomial,* $\mathrm{gr}_v(R)$ *is finitely generated,* $\big(o \in X_0 := \mathrm{Spec}(\mathrm{gr}_v(R))\big)$ *is klt, and* $(o \in X_0, \xi_v)$ *is K-semistable as a Fano cone. Then* v *is a minimizer of* $\widehat{\mathrm{vol}}(\cdot)$.

Therefore, up to the above higher rank finite generation conjecture, a minimizer $v \in \mathrm{Val}_{X,x}$ precisely corresponds to the unique degeneration of $(x \in X)$ to a K-semistable Fano cone singularity. Moreover, we have the following further degeneration result.

Theorem (Li-Wang-Xu, [4])**.** *Any K-semistable Fano cone singularity* (X_0, ξ) *has a unique K-polystable Fano cone degeneration* (Y, ξ).

Example: Let $x \in X = \mathrm{Spec}(R)$ be an n-dimensional smooth point. It is a nontrivial result to know that the canonical blow up E computes the minimizer of $\widehat{\mathrm{vol}}(\cdot)$. Then the associated graded ring $\mathrm{gr}_E R \cong k[x_1, ..., x_r]$ with the grading given by the total degree. In particular, we can view the degeneration as a cone over $\mathbb{P}^{n-1}$, which is K-polystable.

Future direction: The above picture gives a stable degeneration for any singularity $(x \in X)$. It is also interesting to consider the family version, namely constructing the simultaneous stable degeneration of a family of singularities $(S \subset X) \to S$, under the assumption that $s \to \widehat{\mathrm{vol}}(s \in X_s)$ $(s \in S)$ is a local constant. However, the condition of *invariance of volumes* is subtler to formulate when the base S

is a general scheme. Moreover, there should be a robust moduli theory of K-(semi,poly)stable Fano cone singularities, analogue to K-moduli theory of Fano varieties, which serves as the target of this simultaneous degeneration theory.

References

[1] H. Blum, *Existence of valuations with smallest normalized volume*, Compos. Math. **154** (2018), no. 4, 820–849.

[2] S. Donaldson, S. Sun, *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry, II.*, J. Differential Geom. **107** (2017), no. 2, 327–371.

[3] C. Li, *Minimizing normalized volumes of valuations*, Math. Z. **289** (2018), no. 1-2, 491–513.

[4] C. Li, X. Wang, C. Xu, *Algebraicity of the Metric Tangent Cones and Equivariant K-stability*, J. Amer. Math. Soc. **34** (2021) no. 4, 1175–1214.

[5] C. Li, C. Xu, *Stability of Valuations: Higher Rational Rank*, Peking Math. J. **1** (2018) no. 1, 1–79.

[6] C. Li, C. Xu, *Stability of Valuations and Kollár Components*, J. Euro. Math. Soc. **22** (2020) no. 8, 2573–2627.

[7] D. Martelli, J. Sparks, S. Yau, *Sasaki-Einstein manifolds and volume minimisation*, Comm. Math. Phys. **280** (2008), no. 3, 611–673.

[8] C. Xu, *A minimizing valuation is quasi-monomial*, Ann. of Math. (2) **191** (2020), no. 3, 1003–1030.

[9] C. Xu, Z. Zhuang, *Uniqueness of the minimizer of the normalized volume function*, Camb. J. Math. **9** (2021), no. 1, 149–176.

Reporter: Tomasz Pełka

Participants

Prof. Dr. Norbert A'Campo
Departement Mathematik und
Informatik Universität Basel
Spiegelgasse 1
4051 Basel
SWITZERLAND

Prof. Dr. André Belotto da Silva
8, rue Charles Fourier
75013 Paris
FRANCE

Dr. Maciej Borodzik
Instytut Matematyki
Uniwersytet Warszawski
ul. Banacha 2
02-097 Warszawa
POLAND

Prof. Dr. Igor Burban
Institut für Mathematik
Universität Paderborn
Warburger Strasse 100
33098 Paderborn
GERMANY

Prof. Dr. Raf Cluckers
Laboratoire Paul Painlevé, UMR 8524
CNRS, Université de Lille
Cité Scientifique
Bâtiment M2
59655 Villeneuve d'Ascq Cedex
FRANCE

Prof. Dr. Jose Ignacio Cogolludo-Agustin
Departamento de Matemáticas - IUMA
Universidad de Zaragoza
Facultad de Ciencias
c/Pedro Cerbuna 12
50009 Zaragoza
SPAIN

Dr. Javier F. de Bobadilla de Olazabal
Basque Center of Mathematical Sciences
Alameda Mazarredo, 14
48009 Bilbao, Bizkaia
SPAIN

Prof. Dr. Alexandru Dimca
Laboratoire J.-A. Dieudonne
Université de Nice Sophia Antipolis
Parc Valrose
06108 Nice Cedex 2
FRANCE

Dr. Eleonore Faber
School of Mathematics
University of Leeds
Leeds LS2 9JT
UNITED KINGDOM

Dr. Alexandre Fernandes
Departamento de Matematica
Universidade Federal de Ceara
Campus do Pici - Bloco 914
Av. Humberto Monte
Fortaleza, CE 60.455-760
BRAZIL

Dr. Mauricio D. Garay
23, Avenue Briand
91440 Bures-sur-Yvette
FRANCE

Prof. Dr. Evgeny Gorskiy
Department of Mathematics
University of California, Davis
1, Shields Avenue
Davis, CA 95616-8633
UNITED STATES

Prof. Dr. Victor Goryunov
Department of Mathematical Sciences
University of Liverpool
Peach Street
Liverpool L69 7ZL
UNITED KINGDOM

Prof. Dr. Dr. h.c. Gert-Martin Greuel
Fachbereich Mathematik
Technische Universität Kaiserslautern
67653 Kaiserslautern
GERMANY

Prof. Dr. Sabir M. Gusein-Zade
Department of Geometry and Topology
Faculty of Mechanics and Mathematics
Moscow State University
Leninskie Gory
Moscow 119 992
RUSSIAN FEDERATION

Dr. Immanuel Halupczok
Mathematisches Institut
HHU Düsseldorf
Universitätsstraße 1
40225 Düsseldorf
GERMANY

Prof. Dr. Shihoko Ishii
Graduate School of Mathematical
Sciences of University of Tokyo
Hai-Dien District / 3-8-1 Komaba
Tokyo 167-8585
JAPAN

Dr. Ailsa Keating
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WA
UNITED KINGDOM

Prof. Dr. Anatoly Libgober
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 S. Morgan Street
Chicago, IL 60607-7045
UNITED STATES

Prof. Dr. Francois Loeser
Institut de Mathématiques de Jussieu
Sorbonne Université
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Michael Lönne
Mathematisches Institut
Universität Bayreuth
Postfach 101251
95447 Bayreuth
GERMANY

Prof. Dr. Ignacio Luengo
Facultad de Matemáticas
Departamento de Álgebra y Geometría
Universidad Complutense de Madrid
Plaza de Ciencias, 3
28040 Madrid
SPAIN

Prof. Dr. Mark McLean
Department of Mathematics
Stony Brook University
Math. Tower
Stony Brook, NY 11794-3651
UNITED STATES

Prof. Dr. David Mond
Mathematics Institute
University of Warwick
Zeeman Building
Coventry CV4 7AL
UNITED KINGDOM

János Nagy
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
P.O.Box 127
1364 Budapest
HUNGARY

Prof. Dr. András Némethi
Alfred Renyi Institute of Mathematics
P.O.Box 127
1364 Budapest
HUNGARY

Dr. Hong Duc Nguyen
Fachbereich Mathematik
T.U. Kaiserslautern
Erwin-Schrödinger-Straße
67653 Kaiserslautern
GERMANY

Dr. Kien Huu Nguyen
Departement Wiskunde
Faculteit der Wetenschappen
Katholieke Universiteit Leuven
Celestijnenlaan 200B
3001 Leuven
BELGIUM

Prof. Dr. Tomohiro Okuma
Department of Mathematical Sciences
Yamagata University
1-4-12 Kojirakawa-machi
Yamagata 990-8560
JAPAN

Prof. Dr. Karol Palka
Institute of Mathematics of the
Polish Academy of Sciences
ul. Sniadeckich 8
00-656 Warszawa
POLAND

Dr. Irma Pallarés Torres
Departement Wiskunde
Faculteit der Wetenschappen
Katholieke Universiteit Leuven
Celestijnenlaan 200B
3001 Leuven
BELGIUM

Dr. Tomasz Pełka
Basque Center for Applied Mathematics
Alameda de Mazarredo 14
48009 Bilbao, Bizkaia
SPAIN

Maria Pe Pereira
Instituto de Matemática
Interdisciplinar-Facultad de Ciencias
Matemáticas
Universidad Complutense de Madrid
Plaza de Ciencias s/n
28040 Madrid
SPAIN

Prof. Dr. Anne Pichon
FRUMAM FR 2291, CNRS
Aix Marseille Université
Case 39
3, place Victor Hugo
13331 Marseille Cedex 9
FRANCE

Prof. Dr. Patrick Popescu-Pampu
Département de Mathématiques
Université de Lille
Bât. M2
59655 Villeneuve d'Ascq Cedex
FRANCE

Richard Rimanyi
Department of Mathematics
University of North Carolina
at Chapel Hill
Phillips Hall
Chapel Hill, NC 27599-3250
UNITED STATES

Dr. Agustín Romano Velázquez
Department of Mathematics
Tata Institute of Fundamental Research
Mumbai 400 005
INDIA

Prof. Dr. Ana Ros Camacho
School of Mathematics, Cardiff
University
21-23 Senghennydd Rd
CF24 4AG Cardiff CF24 4AG
UNITED KINGDOM

Professor emeritus Kyoji Saito
Research Institute for Mathematical
Sciences
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

Dr. Morihiko Saito
Research Institute for Mathematical
Sciences
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

Dr. Jose Edson Sampaio
Universidade Federal do Ceará
Rua Campus do Pici, s/n, Bloco 914,
Pici
60714665 Fortaleza
BRAZIL

Prof. Dr. Jörg Schürmann
Mathematisches Institut
Universität Münster
Einsteinstraße 62
48149 Münster
GERMANY

Dr. Baldur Sigurðsson
Vietnam Academy of Science and
Technology
Institute of Mathematics
15 Hoang Quoc Viet road
Hanoi 10000
VIETNAM

Prof. Dr. Jan Stevens
Department of Mathematics
Chalmers University of Technology
412 96 Göteborg
SWEDEN

Dr. László Tamás
Faculty of Mathematics and Computer
Science
Babes-Bolyai University
M. Kogalniceanu Nr. 1
400084 Cluj-Napoca
ROMANIA

Prof. Dr. Duco van Straten
Institut für Mathematik
Johannes-Gutenberg Universität Mainz
Staudingerweg 9
55128 Mainz
GERMANY

Prof. Dr. Wim Veys
Department of Mathematics
University of Leuven
Celestijnenlaan 200B
3001 Leuven (Heverlee)
BELGIUM

Prof. Dr. Jonathan M. Wahl
Department of Mathematics
University of North Carolina
Phillips Hall
Chapel Hill NC 27599-3250
UNITED STATES

Dr. Botong Wang
School of Mathematical Sciences
Beijing Normal University
Beijing 100875
CHINA

Dr. Andrzej Weber
Institute of Mathematics
University of Warsaw
ul. Banacha 2
02-097 Warszawa
POLAND

Prof. Dr. Chenyang Xu
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Dr. Matthias Zach
Institut fuer Algebraische Geometrie
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover
GERMANY

Prof. Dr. Mikhail Zaidenberg
Laboratoire de Mathématiques
Université de Grenoble I
Institut Fourier
B.P. 74
38402 Saint-Martin-d'Hères Cedex
FRANCE