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Quantum Groups – Algebra, Analysis and Category Theory (hybrid meeting)

Organized by
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Abstract. The meeting was devoted to discussing the state of the art of different branches of tensor categories and quantum groups, with emphasis on the exchange of ideas between the purely algebraic and operator algebraic sides of these theories.

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Introduction by the Organizers

The workshop *Quantum Groups - Algebra, Analysis and Category Theory* was organized by Masaki Izumi (Kyoto), Sergey Neshveyev (Oslo), Dmitri Nikshych (Durham) and Adam Skalski (Warsaw). There were 54 participants, 32 of which were present at the institute and 22 participated online.

The program consisted of 28 talks (19 delivered by speakers present in Oberwolfach, 9 by online participants) on a variety of topics, from modular categories to Nichols algebras, quantum probability and quantization of Lie groups. One of the goals of the workshop was to give a new impulse to interactions between purely algebraic and analytic (operator algebraic) sides of the theory. To better familiarize the participants with the current state of different branches, Stefaan Vaes, Pavel Etingof and Nicolas Andruskiewitsch were asked to give one hour overview talks on the first day of the workshop, describing the modern state of art regarding operator algebraic, categorical and Hopf algebraic aspects of theory of quantum groups. All other talks were 45 minutes long. On the second day in the evening we also had an informal session discussing open problems.

Practically every session of the meeting was set up so that the speakers represented different research background and directions. The diversity of the topics and participants stimulated a lot of discussions. For many participants this was also the first offline meeting after almost two years of lockdowns and travel restrictions, and the ability to discuss mathematics with colleagues in person was greatly appreciated. In addition to the exciting scientific program, on Wednesday we made the traditional afternoon hike to St. Roman.

The following are the abstracts of the talks, in the order in which they were presented.

Workshop (hybrid meeting): Quantum Groups – Algebra, Analysis and Category Theory

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Abstracts

Discrete quantum groups, subfactors and tensor categories: a survey of analytic aspects and questions

Stefaan Vaes

As a general introduction to some of the themes of the workshop, this talk is providing an overview of the following three topics and their interactions: compact quantum groups, finite index subfactors and rigid C*-tensor categories. Throughout, several open problems are presented.

1. Compact quantum groups

As defined by Woronowicz in 1987, a *compact quantum group* (A, Δ) is a pair consisting of a unital C*-algebra A and a unital $*$-homomorphism Δ from A to the minimal tensor product $A \otimes_{\min} A$ satisfying the co-associativity relation $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$, and satisfying the "cancelation laws" given by the requirement that $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ have a dense linear span in $A \otimes_{\min} A$.

The dual object of any discrete group Γ is a compact quantum group, where A could be the reduced group C*-algebra $C_r^*(\Gamma)$ or the universal C*-algebra $C^*(\Gamma)$. Both C*-algebras contain the canonical unitary operators $(u_g)_{g \in \Gamma}$ and the comultiplication is given by $\Delta(u_g) = u_g \otimes u_g$. Already this example shows that the C*-algebra of a compact quantum group is an interesting object of study. Many basic questions, like simplicity of the reduced C*-algebra, calculation of K-theory, etc, remain open for even the easiest examples of compact quantum groups.

Compact quantum groups have a lot of properties in common with compact groups. In particular, Woronowicz proved that they have a unique Haar state $h : A \to \mathbb{C}$ and that their finite-dimensional unitary representation theory behaves exactly as Peter-Weyl theory. Taking the GNS Hilbert space of h, we can associate to (A, Δ) the von Neumann algebra A'' acting on $L^2(A, h)$. This is the quantum analogue of the group von Neumann algebra $L(\Gamma)$. One says that (A, Δ) is *of Kac type* if the Haar state is a trace. Then, A'' is a finite von Neumann algebra, with a canonical faithful tracial state. Again, even for basic examples of compact quantum groups, many questions remain open. When is A'' a factor? Can some of them be classified up to isomorphism?

To every matrix $F \in M_n(\mathbb{C})$ with $F\overline{F} = \pm 1$, Wang and Van Daele associated in 1995 the *universal orthogonal quantum group* $A_o(F)$, generated by the coefficients of a unitary $n \times n$ matrix U with the property that $U = F\overline{U}F^{-1}$. When $n = 2$, one recovers Woronowicz' famous $\mathrm{SU}_q(2)$ quantum group.

The von Neumann algebra M of $A_o(F)$ is a very interesting object. In [7], it was proven that if F is sufficiently close to the identity matrix, then M is a factor. For general F, this problem is open. When $F = I_n$ is the $n \times n$ matrix, it was gradually shown that the II_1 factor M shares a lot of qualitative properties with the free group factors $L(\mathbb{F}_k)$. Quite surprisingly, it could be proven in [2] that M is not isomorphic to a free group factor.

2. Rigid C*-tensor categories

The category of finite-dimensional unitary representations of a compact quantum group is a *rigid C*-tensor category*, which is equipped with a canonical functor to the category of finite-dimensional Hilbert spaces Hilb. Conversely, Woronowicz' Tannaka-Krein theorem shows the converse: to any fiber functor from a rigid C*-tensor category $\mathcal{C}$ to Hilb corresponds a unique compact quantum group.

This makes questions of existence and classification of such fiber functors relevant and interesting. For instance, one may classify all fiber functors on the *Temperley-Lieb-Jones category* $\mathrm{Rep}(\mathrm{SU}_q(2))$ and they give rise to exactly the universal orthogonal quantum groups $A_o(F)$ mentioned above.

For $n \geq 3$, the classification of fiber functors on $\mathrm{Rep}(\mathrm{SU}_q(n))$ is wide open. It would be particularly interesting to decide whether these tensor categories admit a *dimension preserving* fiber functor. Indeed, by [1], we know that for $n \geq 3$, the rigid C*-tensor category $\mathrm{Rep}(\mathrm{SU}_q(n))$ has property (T). If we can find a dimension preserving fiber functor on a property (T) tensor category, then the resulting compact quantum group is of Kac type, its dual discrete quantum group has property (T) and also the associated finite von Neumann algebra has property (T). This would thus be a source of highly intriguing property (T) von Neumann algebras.

We note here that in [6], the first "genuinely quantum" examples of property (T) discrete quantum groups were given. The data for the construction is a *triangle presentation* in the sense of [3]: a finite set F and a subset $T \subset F \times F \times F$ with the following properties. The set T is cyclically invariant: if $(a, b, c) \in T$, then $(b, c, a) \in T$. Given $a, b \in F$, there is at most one $c \in F$ such that $(a, b, c) \in T$. If this is the case, we say that b is a successor of a and we say that a is a predecessor of b. We further impose that all distinct elements $a, b \in F$ have exactly one common predecessor and exactly one common successor. We finally impose that there is an integer $q \geq 2$, called the order of T, such that every $a \in F$ has exactly $q + 1$ successors and exactly $q + 1$ predecessors. One then defines the Kac type compact quantum group $\mathbb{G}_T$ generated by the coefficients of a unitary representation $(U_{ab})_{a,b\in F}$ subject to the relation that $t_T = \sum_{(a,b,c)\in T} e_a \otimes e_b \otimes e_c$ is an invariant vector for the 3-fold tensor power of U.

The above construction is motivated by the triangle groups of [3]: the group Γ_T with generators $(g_a)_{a\in F}$ subject to the relations $g_a g_b g_c = e$ for all $(a, b, c) \in T$. By [3], these are exactly the groups that act freely and transitively on the vertices of an $\widetilde{A}_2$-building. In the most regular cases, i.e. when this building is Bruhat-Tits, we prove in [6] that the dual of $\mathbb{G}_T$ has Kazhdan's property (T). We do this by proving that $\mathrm{Rep}(\mathbb{G}_T)$ has property (T) as a rigid C*-tensor category, which we show by establishing a Żuk type spectral gap criterion for tensor categories.

The groups Γ_T and the quantum groups $\mathbb{G}_T$ depend, up to isomorphism, heavily on the concrete choice of T. In [6], we however conjecture that the representation category $\mathrm{Rep}(\mathbb{G}_T)$ only depends on the order q. This conjecture can be formulated as a purely combinatorial statement about the number of specific colorings of trivalent bipartite planar graphs, which has been checked numerically for graphs with a small number of vertices.

3. Finite index subfactors

There is another crucial interplay between rigid C*-tensor categories and *Jones' subfactors* [4]. To every finite index subfactor $N \subset M$, Jones associated his *standard invariant.* First, the *basic construction* provides a new subfactor $M \subset M_1$ of equal index $[M : N] = [M_1 : M]$, where M_1 is generated by M and the *Jones projection* e_0. Iterating this construction, one gets the *Jones tower* $N \subset M \subset M_1 \subset M_2 \subset \cdots$ together with the Jones projections e_n, which belong to $M_i' \cap M_j$ whenever $i < n < j$. These relative commutants $M_i' \cap M_j$ form a lattice of multimatrix algebras containing the canonical projections e_n that satisfy the Jones relations: $e_n e_m = e_m e_n$ if $|m-n| \geq 2$ and $e_n e_{n\pm 1} e_n = \lambda e_n$, where $\lambda = [M : N]^{-1}$.

The standard invariant of a subfactor can be axiomatized in several ways: as a λ-lattice in the sense of Popa, as a subfactor planar algebra in the sense of Jones, but also in a tensor category language. The iterated relative tensor products $M \otimes_N M \otimes_N \cdots \otimes_N M$ can be viewed as M-M-bimodules, but also as N-M-, or M-N-, or N-N-bimodules. This gives rise to a rigid C*-2-category $\mathcal{C}$ with only two 0-objects, say 0 and 1, with $\mathcal{C}_{00}$ consisting of N-N-bimodules, $\mathcal{C}_{01}$ consisting of N-M-bimodules, etc. Also, there is a given generating object in $\mathcal{C}_{01}$, namely M. In this way, the standard invariant of a subfactor can be equivalently axiomatized as a rigid C*-2-category $\mathcal{C}$ with two 0-objects and a given generator in $\mathcal{C}_{01}$.

A subfactor is then "nothing else" than a fully faithful unitary functor from $\mathcal{C}$ to the rigid C*-2-category of II_1 factors, where the 0-objects are the II_1-factors and the 1-morphisms are the finite index bimodules between II_1-factors.

To what extent is the standard invariant a complete invariant? Popa's groundbreaking theorem [5] says that an amenable standard invariant arises from exactly one hyperfinite subfactor! Here, a subfactor $N \subset M$ is called hyperfinite if N and M are isomorphic with the hyperfinite II_1 factor R.

Beyond amenability, almost all basic questions remain open. Can every rigid C*-tensor category "act" freely on the hyperfinite II_1 factor? Are there, in the nonamenable case, infinitely many such actions up to "cocycle conjugacy"? What about the Temperley-Lieb-Jones categories at index values > 4? Any progress on one of these questions would be an important step forward.

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Quantum $SL(2,\mathbb{R})$

Kenny De Commer
(joint work with Joel Right Dzokou Talla)

Consider $SL(2,\mathbb{C})$ with its associated $*$-structure $g \mapsto g^*$ given by complex conjugate transpose, and let $SU(2)$ be its associated maximal compact subgroup of unitaries. Let T be the maximal torus of diagonal operators in $SU(2)$, and let $T_{\mathbb{C}} \subseteq SL(2,\mathbb{C})$ be its complexification. We can choose two natural real forms (= anti-holomorphic involutive group automorphisms commuting with $*$) on $SL(2,\mathbb{C})$ which are conjugate and globally $T_{\mathbb{C}}$-preserving:

$$\theta_V(g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} (g^*)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \theta_S(g) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (g^*)^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Their fixed point subgroups lead respectively to the real Lie groups $SU(1,1)$ (the *Vogan form*) and $SL(2,\mathbb{R})$ (the *Satake form*). The maximal compact subgroups of these are then respectively T and $K = SO(2)$.

Consider now the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ of $SL(2,\mathbb{C})$. Its universal Lie algebra $U(\mathfrak{sl}(2,\mathbb{C}))$ can be *quantized* into a Hopf algebra $U_q(\mathfrak{sl}(2,\mathbb{C}))$ depending on a generic complex number q [Jim85, Dri86]: concretely, its underlying algebra is the universal algebra generated by elements $k^{\pm 1}, e, f$ with

$$ke = q^2 ek, \qquad kf = q^{-2} fk, \qquad ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

This quantization depends essentially on our choice of maximal complex torus $T_{\mathbb{C}}$ in $SL(2,\mathbb{C})$ as well as a choice of positive root system. This entails that quantization of a real form is a rather delicate matter, strongly sensitive to the interaction with the maximal torus of the particular representative for the real form in its conjugacy class.

For the compact real form, a quantization of the associated $*$-structure turns out to exist when q is real, resulting in a Hopf $*$-algebra $U_q(\mathfrak{su}(2))$ with $k^* = k$ and $e^* = fk$. Its $*$-representation theory on Hilbert spaces is well-behaved, and very closely related to that of $SU(2)$. For q real, one can also create immediately a quantized enveloping Hopf $*$-algebra $U_q(\mathfrak{su}(1,1))$ by putting $k^* = k$ and $e^* = -fk$, but the representation theory of this $*$-algebra is much less well-behaved [Wor91, Koe03]. For $\mathfrak{sl}(2,\mathbb{R})$, it was up till now assumed that one needs to have $|q| = 1$ to define the Hopf $*$-algebra $U_q(\mathfrak{sl}(2,\mathbb{R}))$.

With our work, we show that the $*$-algebra $U_q(\mathfrak{sl}(2,\mathbb{R}))$ also makes sense when q is real and positive. The catch is that $U_q(\mathfrak{sl}(2,\mathbb{R})$ will no longer be a Hopf $*$-algebra, but a coideal $*$-subalgebra in a larger Hopf $*$-algebra.

To explain this, we recall first that, dual to $U_q(\mathfrak{su}(2))$, one has the Hopf $*$-algebra $\mathcal{O}_q(SU(2))$, the quantized function algebra of $SU(2)$ [Wor87]. It is the universal $*$-algebra generated by elements α, γ such that the 2-by-2-matrix $U = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$ is unitary. The natural pairing between $\mathcal{O}_q(SU(2))$ and $U_q(\mathfrak{su}(2))$ allows one to create a new Hopf $*$-algebra, the *Drinfeld double*, generated by $\mathcal{O}_q(SU(2))$ and

$U_q(\mathfrak{su}(2))$ with specific interchange relations. One can consider this Hopf $*$-algebra as a quantization $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ of the enveloping algebra of $\mathfrak{sl}(2,\mathbb{C})$ as a *real* Lie algebra. Indeed, with $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \oplus \mathfrak{a} \oplus \mathfrak{n}$ the Iwasawa decomposition with respect to the given positive root system, one may interpret also $\mathcal{O}_q(SU(2)) = U_q(\mathfrak{a} \oplus \mathfrak{n})$, and one finds indeed close connections between unitary representations of $SL(2,\mathbb{C})$ and $*$-representations of $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ [Pus93].

We now construct $U_q(\mathfrak{sl}(2,\mathbb{R}))$ as a coideal $*$-subalgebra of $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$,

$$\Delta : U_q(\mathfrak{sl}(2,\mathbb{R})) \to U_q(\mathfrak{sl}(2,\mathbb{R})) \otimes U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}).$$

Indeed, the Satake form is such that the Iwasawa decomposition of $\mathfrak{sl}(2,\mathbb{C})$ restricts to the one of $\mathfrak{sl}(2,\mathbb{R})$, so

$$\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0 = (\mathfrak{su}(2) \cap \mathfrak{sl}(2,\mathbb{R})) \oplus (\mathfrak{a} \cap \mathfrak{sl}(2,\mathbb{R})) \oplus (\mathfrak{n} \cap \mathfrak{sl}(2,\mathbb{R})).$$

Here $\mathfrak{k}$ is the Lie algebra of $SO(2)$. Since the work of Koornwinder [Koo93] it is known that the universal enveloping algebra of $\mathfrak{k}$ quantizes into a left coideal $*$-subalgebra $U_q(\mathfrak{k})$ of $U_q(\mathfrak{su}(2))$, concretely given as

$$U_q(\mathfrak{k}) = \mathbb{C}\langle B\rangle \subseteq U_q(\mathfrak{su}(2)), \qquad B = q^{-1/2}(e - fk).$$

Dually, the set of elements in $\mathcal{O}_q(SU(2))$ invariant under infinitesimal left translations by $U_q(\mathfrak{k})$ forms a right coideal $*$-subalgebra $\mathcal{O}_q(K\backslash SU(2)) \subseteq \mathcal{O}_q(SU(2))$, known as the (equatorial) Podleś sphere [Pod87]. It is the universal $*$-algebra generated by elements X, Y, Z with Z selfadjoint, $X^* = Y$ and

$$XZ = q^2 ZX, \qquad YZ = q^{-2} ZY, \tag{1}$$

$$XY = 1 - q^2 Z^2, \qquad YX = 1 - q^{-2} Z^2. \tag{2}$$

Under the interpretation $\mathcal{O}_q(SU(2)) = U_q(\mathfrak{a} \oplus \mathfrak{n})$, the $*$-algebra $\mathcal{O}_q(K\backslash SU(2))$ corresponds to a quantized enveloping $*$-algebra $U_q(\mathfrak{a}_0 \oplus \mathfrak{n}_0)$ for $\mathfrak{a}_0 \oplus \mathfrak{n}_0$. We thus define $U_q(\mathfrak{sl}(2,\mathbb{R})$ as the $*$-algebra generated by $U_q(\mathfrak{k})$ and $U_q(\mathfrak{a}_0 \oplus \mathfrak{n}_0)$ inside $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$. Equivalently, $U_q(\mathfrak{sl}(2,\mathbb{R})$ becomes the universal $*$-algebra generated by generators $B = B^*$, $X, Y = X^*$ and $Z = Z^*$ satisfying (1), (2) and

$$BX = q^2 XB + (1+q^2)Z, \qquad BY = q^{-2} YB + (1+q^{-2})Z, \tag{3}$$

$$BZ = ZB - (X+Y). \tag{4}$$

In this setup, the representation theory of $U_q(\mathfrak{sl}(2,\mathbb{R})$ can be studied. We look at *admissible* $*$-representations of $U_q(\mathfrak{sl}(2,\mathbb{R})$ on pre-Hilbert spaces, meaning that the pre-Hilbert space decomposes as a direct sum of eigenspaces of B. In this way, making use of the central self-adjoint element

$$\Omega = iq^{-1}X + (q - q^{-1})iZB - iqY \in U_q(\mathfrak{sl}(2,\mathbb{R}),$$

one can give a complete list of irreducible $*$-representations of $U_q(\mathfrak{sl}(2,\mathbb{R})$ [DCDz21a].

We mention two salient features of the above construction method.

- Using the theory of quantum symmetric pairs as developed by G. Letzter [Let99], the above considerations generalize, allowing the construction of a coideal quantization of *any* semisimple real Lie algebra.

- In the above constructions, the $*$-algebra $U_q(\mathfrak{su}(2))$ can be replaced by the (non-unital) $*$-algebra $\mathbb{C}_q[SU(2)]$, quantizing the $*$-algebra of polynomial functions on $SU(2)$ with the *convolution* $*$-algebra structure. One can construct the Drinfeld double $\mathbb{C}_q[SL(2,\mathbb{C})]$ of $\mathcal{O}_q(SU(2))$ with $\mathbb{C}_q[SU(2)]$, leading to a $*$-algebraic quantum group [VD98] that can be integrated immediately within the C*-algebraic or von Neumann algebraic framework of locally compact quantum groups [KV00, KV03], see e.g. [VY20] for more information. Similarly, one can construct the quantized group $*$-algebra $\mathbb{C}_q[SL(2,\mathbb{R})]$ as a coideal within (the multiplier $*$-algebra of) $\mathbb{C}_q[SL(2,\mathbb{C})]$, leading to an algebraic theory with immediate access to analytic techniques. For example, in [DCDz21b] it is shown that $\mathbb{C}_q[SL(2,\mathbb{R})]$ admits a quasi-invariant trace, leading straightforwardly to the notion of regular representation associated to quantum $SL(2,\mathbb{R})$.

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Quantum symmetric pairs and deformed Chebyshev polynomials

Stefan Kolb
(joint work with Milen Yakimov)

This talk is based on the recent paper [KY21]. In this paper we give a complete conceptual description of quantum symmetric pair coideal subalgebras for generalized Satake diagrams of Kac-Moody type in terms of generators and relations. The main tool to obtain the defining relations is the star-product method devised in [KY20]. The defining relations are expressed in terms of continuous q-Hermite polynomials and a new family of deformed Chebyshev polynomials of the second kind.

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On pointed Hopf algebras with finite GK-dimension

Nicolás Andruskiewitsch

This is a report on recent progress in the classification of pointed Hopf algebras with finite Gelfand-Kirillov dimension over the field $\mathbb{C}$, specifically within the method proposed long ago by H.-J. Schneider and the speaker.

Let G be a finitely-generated group; by a celebrated result of Gromov, we may assume that G is nilpotent-by-finite. Recall that a Yetter-Drinfeld module over the group algebra $\mathbb{C}G$ is a G-module V provided with a G-grading: $V = \oplus_{g\in G} V_g$ such that $h \cdot V_g = V_{hgh^{-1}}$ for all $g, h \in G$. Hence the support of V is a union of conjugacy classes. Let ${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ be the braided tensor category of Yetter-Drinfeld modules over $\mathbb{C}G$. Thus we have the notion of Hopf algebras in ${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$; if R is such an object, then $R\#\mathbb{C}G$ is a genuine Hopf algebra.

The first step in the method consists in addressing the following question.

Question 1. *Classify (or at least characterize) all coradically graded connected Hopf algebras* $\mathcal{E} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{E}^n$ *in* ${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ *such that* $\operatorname{GK-dim} \mathcal{E}\#\mathbb{C}G < \infty$.

In the situation of Question 1, if $V := \mathcal{E}^1 \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$, then $\mathcal{E}$ is a *post-Nichols algebra* of V. Indeed, the subalgebra of $\mathcal{E}$ generated by V is isomorphic to the Nichols algebra $\mathscr{B}(V)$. See the survey [1]. The core of Question 1 is the following.

Question 2. *Classify all* $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ *such that* $\operatorname{GK-dim} \mathscr{B}(V)\#\mathbb{C}G < \infty$.

Dually, a graded connected Hopf algebra $\mathcal{R} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{R}^n$ in ${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ generated by $V \simeq \mathcal{R}^1$ is a *pre-Nichols* algebra of V. (The only pre- and post-Nichols algebra of V is $\mathscr{B}(V)$). Once Question 2 is solved, Question 1 essentially reduces to:

Question 3. *Classify all pre-Nichols algebras* $\mathcal{R}$ *of those* $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ *with* $\operatorname{GK-dim} \mathscr{B}(V)\#\mathbb{C}G < \infty$ *such that* $\operatorname{GK-dim} \mathcal{R}\#\mathbb{C}G < \infty$. .

Let us focus on Question 2. Using a Theorem of Malcev, we proved recently:

Theorem 1. **[2]** If the support of V contains an infinite conjugacy class, then $\operatorname{GK-dim} \mathscr{B}(V)\#\mathbb{C}G = \infty$.

Let G be a finitely-generated torsion-free nilpotent group. It is well-known that every non-central conjugacy class is infinite. Hence for any $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$, if $\operatorname{GK-dim} \mathscr{B}(V)\#\mathbb{C}G < \infty$, then $\sup V \subset Z(G)$, so V 'comes from the abelian case'.

To proceed with the analysis of Nichols algebras over finitely-generated nilpotent groups with torsion, we need the notion of finite rack *of type C*: see [6]. Let G be a finite group. In [6] we proved that given a conjugacy class $\mathcal{O}$ of type C, we have $\dim \mathscr{B}(V) = \infty$ for any $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ such that $\sup V = \mathcal{O}$. The proof relies on results of I. Heckenberger, H.-J. Schneider and L. Vendramin, based in turn on the theory of the Weyl groupoid [11]. Along this line it seems natural to propose:

Conjecture 1. **[2]** Let G be a group and let $\mathcal{O}$ be a finite conjugacy class of type C. Then $\operatorname{GK-dim} \mathscr{B}(V) = \infty$ for any $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ such that $\sup V = \mathcal{O}$.

The usefulness of the Conjecture stems from the next result, see [2]: Let G be a finitely-generated nilpotent group whose non-trivial torsion subgroup has odd order. Then a finite conjugacy class $\mathcal{O}$ of G is either abelian or else of type C.

The preceding discussion shows that Nichols algebras with abelian support occupy a central place in the theory. *Assume from now on that G is abelian.* Let $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ be finite-dimensional. Fix a decomposition $V = V_1 \oplus \cdots \oplus V_\theta$ where the V_i's are indecomposable objects in ${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$. We consider three kind of indecomposable objects. We say that an indecomposable $U \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ is

- a *point* if $\dim U = 1$;
- a *block* if $\dim U > 1$ and U is indecomposable as braided vector space;
- a *pale block* if $\dim U > 1$ and U is decomposable as braided vector space.

Accordingly we consider various possibilities for V. First V is of diagonal type if $\dim V_i = 1$ for all i, that is, if it is a direct sum of points. This is studied through the theory of (generalized) root systems and Weyl groupoids [11].

Conjecture 2. **[3]** Let V be of diagonal type. Then $\operatorname{GK-dim} \mathscr{B}(V) < \infty$ iff V has finite root system (and then the classification follows from [12]).

The evidence for Conjecture 2 (*that we assume from now on*) is strong [4, 10].

Second, if V is a direct sum of blocks and points, then the classification is complete [3]. For instance, the only Nichols algebras of blocks with finite GK-dim are the well-known Jordan plane and another algebra with 2 generators and 2 relations coined the super Jordan plane. Finally, if V has at least one component that is a pale block, then the classification is complete when $\dim V$ is 3 or 4 [3, 5].

A final word on Question 3. Let $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ with $\operatorname{GK-dim} \mathscr{B}(V) < \infty$. The class of pre-Nichols algebras of V with finite GK-dim is a partially ordered set bounded above by $\mathscr{B}(V)$. A pre-Nichols algebra is *eminent* if it is a minimum in this poset. Eminent pre-Nichols algebras do not always exist but surprisingly large classes of V of diagonal type do admit an eminent pre-Nichols algebra that, even more, turns out to be the distinguished pre-Nichols algebra introduced in [8]. See [7, 9] for details.

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Frobenius Exact symmetric tensor categories

Pavel Etingof, Kevin Coulembier, Victor Ostrik

A fundamental theorem of P. Deligne (2002) states that a pre-Tannakian category over an algebraically closed field of characteristic zero admits a fiber functor to the category of supervector spaces (i.e., is the representation category of an affine proalgebraic supergroup) if and only if it has moderate growth (i.e., the lengths of tensor powers of an object grow at most exponentially). We prove a characteristic p version of this theorem. Namely we show that a pre-Tannakian category over an algebraically closed field of characteristic $p > 0$ admits a fiber functor into the Verlinde category Ver_p (i.e., is the representation category of an affine group scheme in Ver_p) if and only if it has moderate growth and is Frobenius exact. This implies that Frobenius exact pre-Tannakian categories of moderate growth admit a well-behaved notion of Frobenius-Perron dimension. It follows that any semisimple pre-Tannakian category of moderate growth has a fiber functor to Ver_p (so in particular Deligne's theorem holds on the nose for semisimple pre-Tannakian categories in characteristics 2,3). This settles a conjecture of Ostrik from 2015.

In particular, this result applies to semisimplifications of categories of modular representations of finite groups (or, more generally, affine group schemes), which gives new applications to classical modular representation theory. For example, it allows us to characterize, for a modular representation V, the possible growth rates of the number of indecomposable summands in $V^{\otimes n}$ of dimension prime to p.

Root of unity quantum cluster algebras: discriminants, Cayley-Hamilton algebras, and Poisson orders

Milen Yakimov

(joint work with Shengnan Huang, Thang T. Q. Lê, Bach Nguyen, Kurt Trampel)

In this talks, we described a theory of root of unity quantum cluster algebras, which includes various families of algebras from Lie theory and topology.

For each exchange matrix $\widetilde{B}$ and a primitive ℓ-th root of unity $\varepsilon^{1/2}$, we define a root of unity quantum cluster algebra $\mathbf{A}_\varepsilon(\widetilde{B})$ and a root of unity upper quantum cluster algebra $\mathbf{U}_\varepsilon(\widetilde{B})$, and prove the Laurent phenomenon inclusion $\mathbf{A}_\varepsilon(\widetilde{B}) \subseteq \mathbf{U}_\varepsilon(\widetilde{B})$. Inside both algebras, we contruct central subalgebras $\mathbf{C}_\varepsilon(\widetilde{B})$ and $\mathbf{Z}_\varepsilon(\widetilde{B})$, which are shown to be isomorphic to the underlying cluster algebra and upper cluster algebra extended from $\mathbb{Z}$ to $\mathbb{Z}[\varepsilon^{1/2}]$. This is done under the mild assumption that ℓ is odd and coprime to the skew-symmetrizing integers for the principal part of $\widetilde{B}$.

The following were the main results presented at the talk:

Theorem.

(1) Each root of unity upper quantum cluster algebra $\mathbf{U}_\varepsilon(\widetilde{B})$ is a maximal order in a central simple algebra.

(2) If ℓ is odd and coprime to the skew-symmetrizing integers for the principal part of $\widetilde{B}$, then there is a trace function $\mathrm{tr} : \mathbf{U}_\varepsilon(\widetilde{B}) \to \mathbf{Z}_\varepsilon(\widetilde{B})$ that makes the triple $(\mathbf{U}_\varepsilon(\widetilde{B}), \mathbf{Z}_\varepsilon(\widetilde{B}), \mathrm{tr})$ a Cayley–Hamilton algebra in the sense of Procesi.

(3) Under the assumptions of the previous part, if $\mathbf{U}_\varepsilon(\widetilde{B})$ is free over $\mathbf{Z}_\varepsilon(\widetilde{B})$, then its discriminant with respect to the trace function tr equals

$$\ell^{N\ell^N} f_1^{\ell a_1} \dots f_n^{\ell a_n},$$

where N is the rank of $\mathbf{U}_\varepsilon(\widetilde{B})$, $f_1, \dots, f_n$ are its frozen variables and $a_1, \dots, a_n$ are nonnegative integers. In concrete cases, the values of the integers $a_1, \dots, a_n$ are determined by filtration or grading arguments.

(4) For all symmetrizable Kac–Moody algebras $\mathfrak{g}$, Weyl group elements w, and primitive ℓ-th roots of unity $\varepsilon^{1/2}$ such that ℓ is odd and coprime to the symmetrizing integers of the Cartan matrix of $\mathfrak{g}$, the integral quantum Schubert cell algebra $U_\varepsilon(\mathfrak{n}_-(w))_{\mathbb{Z}[\varepsilon]}$ is isomorphic to a root of unity quantum cluster algebra $\mathbf{A}_\varepsilon(\widetilde{B}) \cong \mathbf{U}_\varepsilon(\widetilde{B})$. Under this isomorphism, the

De Concini–Kac–Procesi central subalgebra of $U_\varepsilon(\mathfrak{n}_-(w))_{\mathbb{Z}[\varepsilon]}$ corresponds to $\mathbf{C}_\varepsilon(\widetilde{B}) \cong \mathbf{Z}_\varepsilon(\widetilde{B})$.

(5) Under the assumptions of the previous part, the discriminant of the integral quantum Schubert cell algebra $U_\varepsilon(\mathfrak{n}_-(w))_{\mathbb{Z}[\varepsilon]}$ over its De Concini–Kac–Procesi central subalgebra is given by

$$\ell^{N\ell^N} \prod_i \Delta_{\omega_i, w\omega_i}^{(\ell-1)\ell^{N-1}},$$

where N is the length of w and $\Delta_{\omega_i, w\omega_i}$ are generalized minors of $\mathfrak{g}$ viewed as functions on the Schubert cell $B_+ w B_+/B_+$, identified with the spectrum of the De Concini–Kac–Procesi central subalgebra.

(6) If ℓ is odd and coprime to the skew-symmetrizing integers for the principal part of $\widetilde{B}$, and $\mathbf{U}_\varepsilon(\widetilde{B})$ is a strict root of unity upper quantum cluster algebra, then there is a canonical structure of a Poisson order on the pair $(\mathbf{U}_\varepsilon(\widetilde{B}), \mathbf{Z}_\varepsilon(\widetilde{B}))$ (in the sense of Brown–Gordon) for which the Poisson structure on $\mathbf{Z}_\varepsilon(\widetilde{B})$ is nonzero and equals the Gekhtman–Shapiro–Vainshtein Poisson structure on the underlying classical cluster algebra.

On the monoidal invariance of the cohomological dimension of Hopf algebras

Julien Bichon

In this talk, mainly based on [2], I discussed the following question:

Question 1. *If A and B are Hopf algebras having equivalent linear tensor categories of comodules, do we have* $\mathrm{cd}(A) = \mathrm{cd}(B)$*?*

Here $\mathrm{cd}(A)$ denotes the global dimension of A, which coincides as well with the Hochschild cohomological dimension.

The following list summarizes, to the best of my knowledge, the cases where the answer to Question 1 is known to be positive.

(1) A, B have bijective antipode and are smooth [2].
(2) A, B are cosemisimple and their antipodes satisfy $S^4 = \mathrm{id}$ [1].
(3) A, B are cosemisimple and $\mathrm{cd}(A)$, $\mathrm{cd}(B)$ are finite [2].
(4) A, B are finite-dimensional, and the characteristic of the base field is zero, or satisfies $p > d^{\frac{\varphi(d)}{2}}$, where $d = \dim(A)$ [2].
(5) A, B are finite-dimensional and A^* is unimodular [2].

A general method to tackle Question 1 is based on the fact that if $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$ as above, results by Schauenburg [6] ensures that there exists an A-B-bi-Galois object R, and then on proving that $\mathrm{cd}(A) = \mathrm{cd}(R) = \mathrm{cd}(B)$.

In the smooth case this is achieved by following closely arguments of Yu [7]. In general one notices furthermore that

$$\mathrm{cd}(A) = \mathrm{pd}_{{}_R\mathcal{M}_R^B}(R) \geq \mathrm{pd}_{{}_R\mathcal{M}_R}(R) = \mathrm{cd}(R)$$

where $\mathrm{pd}_{{}_R\mathcal{M}^B_R}(R)$ is the projective dimension of R in the category of R-bimodules inside B-comodules, and hence the main question then is to compare $\mathrm{pd}_{{}_R\mathcal{M}^B_R}(R)$ and $\mathrm{pd}_{{}_R\mathcal{M}_R}(R) = \mathrm{cd}(R)$. The main ingredient in this comparison in the cosemisimple case is a twisted averaging trick, leading to introduce the concept of twisted separable functor in [2], a generalization of the notion of separable functor from [5].

After having presented this general strategy, I discussed the following example. Let $p, n \geq 1$ and let $A_h^p(n)$ be the algebra presented by generators u_{ij}, $1 \leq i, j \leq n$, and relations

$$\sum_{j=1}^{n} u_{ij}^p = 1 = \sum_{j=1}^{n} u_{ji}^p, \quad u_{ij}u_{ik} = 0 = u_{ji}u_{ki}, \text{ for } k \neq j,$$

The algebra $A_h^p(n)$ has a natural Hopf algebra structure, and using (3) in the above list together with monoidal equivalences constructed by Lemeux-Tarrago [4] and Fima-Pittau [3], one proves that for $n \geq 4$, one has $\mathrm{cd}(A_h^p(n)) = 3$.

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Tensor categories generated by one object

Alexei Davydov

Let $\mathcal{C}$ be a (strict) monoidal category such that $End_{\mathcal{C}}(I) = k$, where I denotes the unit object of $\mathcal{C}$. Given an object X of $\mathcal{C}$, the sequence A_* with $A_n = End_{\mathcal{C}}(X^{\otimes n})$ is *multiplicative* (see [5]) with respect to the homomorphisms $\mu_{m,n}$ given by the tensor product on morphisms

$$End_{\mathcal{C}}(X^{\otimes m}) \otimes End_{\mathcal{C}}(X^{\otimes n}) \to End_{\mathcal{C}}(X^{\otimes m+n}).$$

That is for any $l, m, n \geqslant 0$ the following diagram commutes:

$$\begin{array}{ccc} A_l \otimes A_m \otimes A_n & \xrightarrow{\mu_{l,m}\otimes I} & A_{l+m} \otimes A_n \\ \downarrow{\scriptstyle I\otimes\mu_{m,n}} & & \downarrow{\scriptstyle \mu_{l+m,n}} \\ A_l \otimes A_{m+n} & \xrightarrow{\mu_{l,m+n}} & A_{l+m+n}. \end{array}$$

Moreover, any multiplicative sequence can be obtained in this way. Indeed, starting with a multiplicative sequence A_*, define the category $\mathcal{C}(A_*)$ (the *Schur-Weyl category* of [5]) with objects $[n]$ parameterized by natural numbers, with no morphisms between different objects and with the endomorphism algebras $End_{\mathcal{C}(A_*)}([n]) = A_n$. Define tensor product on the objects of $\mathcal{C}(A_*)$ by $[m]\otimes[n] = [m+n]$. The multiplicative structure of the sequence A_* yields the tensor product on morphisms.

One important example of Schur-Weyl category is the free symmetric tensor category $\S(A)$ generated by one object X with $End_{\S}(X) = A$.

Possible monoidal structures on a given tensor category naturally form an object of an algebro-geometric nature (the moduli space). The tangent space to the moduli space of tensor structures is computed by the third cohomology of a certain complex, the deformation complex of the tensor category [2, 9].

In [4] we explore the internal organisation of the deformation complex of a Schur-Weyl category. This complex comes equipped with a natural decreasing filtration. Components of the associated graded complex (called horizontal complexes) have a uniform structure. Horizontal complexes can be presented as cochain complexes associated with diagrams of vector spaces in the form of a higher-dimensional cube (called cubic diagrams here). Cubic diagrams coming from free symmetric tensor categories are diagrams of invariants of symmetric group representations. Using the relation with the cohomology of simplicial cubes we compute the cohomology of cubic diagrams of invariants. This allows us to describe the deformation cohomology of free symmetric tensor categories. Let $\S = \S(k)$ be the free symmetric tensor category generated by one object whose endomorphisms are scalars k. We showed (assuming that the ground field is of zero characteristic) that the deformation cohomology $H^*(\mathcal{S})$ is the exterior algebra $\Lambda(e_1, e_3, e_5, ...)$ on odd degree generators $deg(e_{2i-1}) = 2i-1$ [4, theorem 4.11]. Let $\S(A)$ be the free symmetric tensor category generated by one object whose endomorphism algebra A is a domain. Then (assuming that A is a commutative algebra, which is more than one dimensional) the deformation cohomology of the free symmetric category $\mathcal{S}(A)$ is the exterior algebra of the first cohomology $H^1(\mathcal{S}(A)) = A$, i.e. $H^*(\mathcal{S}(A)) \simeq \Lambda^*(A)$ [4, theorem 4.15].

The free symmetric tensor category $\mathcal{S}(A)$ can be thought of as the limiting case of the representation category $\mathcal{R}ep(\mathfrak{gl}(V)\otimes A)$ of the general linear Lie algebra $\mathfrak{gl}(V)\otimes A$, when the dimension of the vector space V goes to infinity. The deformation cohomology of the representation category $\mathcal{R}ep(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ can be identified with the adjoint $\mathfrak{g}$-invariants of the exterior algebra $\Lambda^*(\mathfrak{g})^{\mathfrak{g}}$ [2]. Here we assume that the characteristic of the ground field is zero. Classical invariant theory says that $\Lambda^*(\mathfrak{gl}(V))^{\mathfrak{gl}(V)}$ is the exterior algebra $\Lambda(x_1, x_3, ..., x_{2d-1})$ with generators of degree $deg(x_{2i-1}) = 2i-1$ and with $d = dim(V)$ (see e.g. [7, 8]). We use the Schur-Weyl duality functor $SW : \mathcal{S} \to \mathcal{R}ep(\mathfrak{gl}(V))$, sending the generator to the vector representation V, to relate the deformation cohomology of $\S$ and of $\mathcal{R}ep(\mathfrak{gl}(V))$. Note that the functoriality property of the deformation cohomology is not straightforward and is similar to the functoriality of the centre, or the

Hochschild cohomology (see [3]): a tensor functor $F : \mathcal{C} \to \mathcal{D}$ gives rise to a cospan of homomorphisms of graded algebras

$$\begin{array}{ccc} & H^*(F) & \\ \nearrow & & \nwarrow \\ H^*(\mathcal{C}) & & H^*(\mathcal{D}) \end{array}$$

where $H^*(F)$ is the deformation cohomology of tensor functor F. We show that the deformation cohomology of the Schur-Weyl functor SW is the exterior algebra $H^*(SW) = \Lambda(e_1, e_3, ..., e_{2d-1})$, the homomorphism $H^*(\mathcal{S}) \to H^*(SW)$ is the quotioning by the ideal generated by $e_s, s > 2d-1$, and the homomorphism $H^*(\mathcal{R}ep(\mathfrak{gl}(V))) \to H^*(SW)$ is an isomorphism sending x_m to $((m-1)!)^{-1}e_m$ [4, theorem 5.7]. This in particular gives a precise formulation of the intriguing connection between the combinatorics of partitions (giving the answer for $H^*(\mathcal{S})$) and the exterior invariants of $\mathfrak{gl}(V)$ observed by Kostant in [8]. We also relate the deformation cohomology of $\mathcal{S}(A)$ and of $\mathcal{R}ep(\mathfrak{gl}(V \otimes A))$ under the assumption that A is a commutative domain algebra of dimension higher than one over an algebraically closed field. Then the deformation cohomology of the Schur-Weyl functor $SW : \mathcal{S}(A) \to \mathcal{R}ep(\mathfrak{gl}(V \otimes A))$ (sending the generator to the vector representation $V \otimes A$) is the exterior algebra $H^*(SW) = \Lambda^*(A)$ and the homomorphisms

$$H^*(\mathcal{S}(A)) \;\to\; H^*(SW) \;\leftarrow\; H^*(\mathcal{R}ep(\mathfrak{gl}(V \otimes A)))$$

are isomorphisms [4, theorem 5.12].

The one-dimensional cohomology $H^3(\mathcal{S})$ suggests that the moduli space of tensor structures of $\mathcal{S}$ is (locally) one-dimensional. This was shown to be true globally in [2]. The detailed analysis of this moduli space and its relation to the one-parameter family of Hecke categories from [5] is the subject of an ongoing work.

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The Langlands duality conjecture for skein modules

David Jordan
(joint work with David Ben-Zvi, Sam Gunningham, Pavel Safronov)

Skein modules of 3-manifolds were introduced by Prztzycki and Turaev in the late 1980's, and today are understood to capture collections of Wilson line operators on the Hilbert space of a 3-manifold, in the 4-dimensional topological field theory attached to a ribbon braided tensor category A. In concrete terms, the skein module is a quotient by a free vector space of A-labelled ribbon graphs, by "skein relations" – local relations taking place in some ball of the 3-manifold M, where we evaluate ribbon graphs into morphisms in A, and usher in all relations which arise in this way.

A famous conjecture of the physicist Edward Witten was recently proved by Sam Gunningham, Pavel Safronov and myself [1]: this states that if M is a closed oriented 3-manifold, and if the braided tensor category A is the representation category $\mathrm{Rep}_q(G)$ of a quantum group at generic parameters q, then the skein module of M is a finite-dimensional vector space. This was considered a surprising conjecture because it is well-known that at $q = 1$ the skein module becomes the algebra of functions on the G-character variety of M (a moduli space of representations of $\pi_1(M) \to G$. This variety is typically positive-dimensional, and hence its algebra of functions is infinite-dimensional.

The resolution of Witten's conjecture allows us to formulate a new conjecture, concerning the dimension of skein modules. Given the group G, let us denote by $\mathrm{Sk}_G(M)$ the skein module attached to $\mathrm{Rep}_q(G)$. Let G^L denote the *Langlands dual group* to G.

Conjecture. We have an equality of integers,

$$\dim \mathrm{Sk}_G(M) = \dim \mathrm{Sk}_{G^L}(M),$$

for any closed oriented 3-manifold M.

In the talk I have explained the proof of the following

Theorem (Gunningham, J, Safronov)**.** The conjecture holds in the case $G = SL_2$ (so that $G^L = PGL_2$), and $M = \Sigma_g \times S^1$, for any genus g.

The proof relies on the notion of 1-form symmetries in TQFT. We give a relation between the skein module for a simply connected group G^{sc} and its adjoint for $G^{ad} = G^{sc}/Z(G^{sc})$. This involves a construction of "twisted skein modules", where we use the $Z(G)$ action to twist the skein module, and thereby obtain a class of twisted skein modules indexed by $H_1(M, Z(G))$, each carrying a natural action of $H_1(M, Z(G)^\vee)$. Since SL_2 and PGL_2 are not only Langlands dual, but also related in the above fashion, we can compute PGL_2-skeins using known results about SL_2-skeins. We also use in a crucial way that the classical $q = 1$ limit of PGL_2-skeins is a (quotient of) a twisted character variety, and that twisted character varieties are smooth, unlike the untwisted counterparts.

We believe the general resolution of our conjecture will be very difficult, but that even incremental progress will shed light on the geometric Langlands conjectures, and especially on its correct formulation for 3-manifolds.

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The Bost-Connes system and C^*-categories

Christian Voigt
(joint work with Jamie Antoun)

The Bost-Connes (BC) system [2] is a well-studied quantum statistical mechanical system with intimate connections to number theory. It consists of the Hecke C^*-algebra $A = C^*(G, H)$ associated to the $ax+b$-group $G = P^+(\mathbb{Q})$ and its almost normal subgroup $H = P^+(\mathbb{Z})$, where

$$P^+(R) = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \in GL(2, R) \mid a > 0 \right\}$$

for a subring $R \subset \mathbb{R}$, together with the time evolution induced from the modular automorphism group of the von Neumann algebra generated by the canonical representation of A on $l^2(G/H)$.

From the point of view of physics, important information about the system is contained in the structure of its KMS-states [3]. A key feature of the BC-system is the intertwining property between the Galois action on extremal KMS_∞-states, evaluated on a natural rational subalgebra of A, and the symmetries of the system. In fact, some of the structure of the BC-system is visible not only rationally but at the integral level [4], and even over the field with one element [5].

It is known from other contexts that the existence of integral structure hints at underlying categorified information; compare for instance the theory of KLR-algebras and categorification of quantum groups [6]. In this talk we discuss work in progress on a categorical version of the BC-system, based on the rigid C^*-tensor category $\mathbf{A}$ associated to its underlying Hecke pair (G, H), following [8], [1]. Working with group actions on C^*-categories one can interpret this indeed as a direct analogue of the Hecke algebra construction, and we review some background in this regard. This construction is different from the approach to categorifying Bost-Connes type systems taken by Marcolli-Tabuada [7].

We give an explicit description of the structure of our categorical BC-system in terms of double cosets and representations of their stabiliser groups. The fusion ring $\hat{\mathcal{A}}$ of $\mathbf{A}$ can be presented in terms terms of generators X_n for $n \in \mathbb{N}^*$ and $X^{\beta,\gamma}$ for $\beta, \gamma \in \mathbb{Q}/\mathbb{Z}$, satisfying the following relations:

a) $X_m X_n = X_n X_m$ for all $m, n \in \mathbb{N}^*$.
b) $(X_n)^* X_m = X_m (X_n)^*$ if $m, n \in \mathbb{N}^*$ are coprime.
c) $(X^{\beta,\gamma})^* = X^{-\beta,-\gamma}$ and $X^{\alpha,\beta} X^{\gamma,\delta} = X^{\alpha+\gamma,\beta+\delta}$ for all $\alpha, \beta, \gamma, \delta \in \mathbb{Q}/\mathbb{Z}$.

d) $X_n X^{0,n\beta} = X^{0,\beta} X_n$ for all $n \in \mathbb{N}^*$ and $\beta \in \mathbb{Q}/\mathbb{Z}$.
e) $X_n X^{\alpha,0} = X^{n\alpha,0} X_n$ for all $n \in \mathbb{N}^*$ and $\alpha \in \mathbb{Q}/\mathbb{Z}$.
f) $(X_n)^* X^{\beta,\gamma} X_n = \sum_{n\rho=\beta} X^{\rho,n\gamma}$ for all $n \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{Q}/\mathbb{Z}$.
g) $X_n X^{\beta,\gamma} (X_n)^* = \sum_{n\rho=\gamma} X^{n\beta,\rho}$ for all $n \in \mathbb{N}^*$ and $\beta, \gamma \in \mathbb{Q}/\mathbb{Z}$.

From this presentation it is easy to see that the fusion ring $\hat{\mathcal{A}}$ admits a non-split surjection onto the integral BC-algebra $\mathcal{A} \subset A$. However, the former contains strictly more information than the latter, and an interesting feature is that $\hat{\mathcal{A}}$ has additional symmetries which are not visible in $\mathcal{A}$. In particular, $\hat{\mathcal{A}}$ admits the "reflection" automorphism which exchanges X_n and $(X_n)^*$, and maps $X^{\beta,\gamma}$ to $X^{\gamma,\beta}$. The representation theory of $\hat{\mathcal{A}}$ differs from the one of the BC-algebra as well. For instance, in contrast to the BC-algebra, the extended BC-algebra $\hat{\mathcal{A}}$ has a wealth of finite dimensional representations.

The C^*-tensor category $\mathbf{A}$ is amenable, and the complexification of $\hat{\mathcal{A}}$ admits a canonical C^*-completion $\hat{A}$. More importantly, the C^*-algebra $\hat{A}$ is equipped with a natural time evolution, compatible with the time evolution in the BC-system. The fact that the C^*-subalgebra of $\hat{A}$ generated by the operators X_n has a lot more irreducible representations then its counterpart in the BC-system is the main reason why the structure of KMS-states for $\hat{A}$ is more complicated than for A. Roughly speaking, the extremal KMS_β-states for the latter appear at the boundary of the space of extremal KMS_β-states for $\hat{A}$ for all $\beta \geq 0$, but the precise structure still needs to be worked out.

We observe that the KMS_∞-states of the BC-system can be realised naturally at the level of categories by looking at dimension functions on the category of $\mathbb{Q}/\mathbb{Z}$-graded Hilbert spaces. However, at present it remains open whether further information about the structure of KMS-states can be derived from purely categorical considerations.

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Constructing non-semisimple modular categories with relative monoidal centers

Chelsea Walton

The focus of our work is the construction non-semisimple modular categories. We establish when Müger centralizers inside non-semisimple modular categories are also modular. As a consequence, we obtain conditions under which relative monoidal centers give (non-semisimple) modular categories, and we also show that examples include representation categories of small quantum groups. We further derive conditions under which representations of more general quantum groups, braided Drinfeld doubles of Nichols algebras of diagonal type, give (non-semisimple) modular categories. This is joint work with Robert Laugwitz.

Crossed product equivalence of quantum automorphism groups

Michael Brannan
(joint work with Samuel J. Harris)

Given a finite-dimensional C*-algebra B equipped with a faithful state ψ, Wang constructed in [10] the *quantum automorphism group* $G^+(B,\psi)$ of the finite measured quantum space (B,ψ). By construction, $G^+(B,\psi)$ is a C*-algebraic compact quantum group whose underlying Hopf $*$-algebra $\mathcal{O}(G^+(B,\psi))$ is defined to be the universal $*$-algebra generated by the coefficients of a unital $*$-homomorphism $\rho : B \to B \otimes \mathcal{O}(G^+(B,\psi))$ satisfying the ψ-invariance condition: $(\psi \otimes 1)\rho(x) = \psi(x)1$ for all $x \in B$. The Hopf $*$-algebra structure is uniquely determined by requiring that B be a (fundamental) comodule over $\mathcal{O}(G^+(B,\psi))$. (That is, the coproduct Δ is determined by the identity $(\rho \otimes 1)\rho = (1 \otimes \Delta)\rho$, and so on). In general, $G^+(B,\psi)$ can be regarded as a non-commutative analogue of the compact group of $*$-automorphisms $\mathrm{Aut}(B)$. More precisely, we call $\alpha \in \mathrm{Aut}(B)$ *ψ-preserving* if $\psi \circ \alpha = \psi$ and let $G(B,\psi) < \mathrm{Aut}(B)$ denote the subgroup of all ψ-preserving automorphisms, then the algebra of cooordinate functions $\mathcal{O}(G(B,\psi))$ is precisely the abelianization of $\mathcal{O}(G^+(B,\psi))$. In the following discussion, we will always take ψ to be the canonical "Plancharel" trace on B (that is, the unique $\mathrm{Aut}(B)$-invariant tracial state on B) so that $G^+(B,\psi)$ can truly be regarded as the quantum analogue of $\mathrm{Aut}(B)$. With this in mind, we shall suppress the ψ-dependence in our notation and simply write $G^+(B) = G^+(B,\psi)$ for the remainder.

The main examples to keep in mind in this discussion are the two extreme cases where $B = \mathbb{C}^n$ is abelian, and where $B = M_n(\mathbb{C})$ is a full matrix algebra. In the former case, $G^+(B) = S_n^+$ is the well-known quantum permutation group. In the latter case, $G^+(B) = G^+(M_n(\mathbb{C}))$ can be identified with $\mathbb{P}U_n^+$, the projective version of the free unitary quantum group U_n^+ [1]. (This generalizes the well-known classical result that the conjugation action of U_n on M_n induces an isomorphism $\mathrm{Aut}(M_n) \cong \mathbb{P}U_n$.)

In the world of operator algebraic quantum groups, the structure of the reduced quantum group C*-algebras $C_r(G^+(B))$ and quantum group von Neumann algebras $L^\infty(G^+(B))$ are of great interest. In general, if $\dim B \geq 5$, it is known that $C_r(G^+(B))$ is non-nuclear, exact, simple, with unique trace, and possesses the complete metric approximation property, and $L^\infty(G^+(B))$ is a non-injective, weakly amenable, a-T-menable, strongly solid II_1-factor [2, 5, 7, 8]. It was also shown in [3] that $\mathcal{O}(S_n^+)$ is always residually finite-dimensional and $L^\infty(S_n^+)$ always has the Connes Embedding Property.

A natural question that follows from the above results are: How much do the operator algebras $C_r(G^+(B))$ and $L^\infty(G^+(B))$ actually depend on the initial data of B? One of the key tools in proving many of the above results (e.g., strong solidity, and approximation properties) is the fact that at the quantum groups $G^+(B_1)$ and $G^+(B_2)$ are monoidally equivalent if and only if $\dim B_1 = \dim B_2$ [6]. This suggests at an informal level that the operator algebras $C_r(G^+(B))$ or $L^\infty(G^+(B))$ may be closely related (possibly isomorphic) as we range over B with $\dim B$ fixed. In particular, it is natural to ask if monoidal equivalence can be used to transfer Connes embeddability from $L^\infty(S_n^+)$ to all $L^\infty(G^+(B))$. At the C*-algebra level, Voigt [9] showed using K-theory methods that on-the-nose algebra isomorphisms are not always possible. For example, $K_0(C_r(G^+(M_n))) = \mathbb{Z} \oplus \mathbb{Z}_n$ while $K_0(C_r(S_{n^2}^+)) = \mathbb{Z}^{(n^2-1)^2-1}$. Nonetheless, it is an interesting question to ask to what degree the algebras above differ (on either the C*-/von Neumann level.) The following result provides some major progress in this direction.

Theorem A *Let $B = \bigoplus_{r=1}^m M_{k_r}(\mathbb{C})$, $n = \dim B \geq 4$, and $d = \prod_{r=1}^m k_r$. Then there exist (Haar) trace-preserving embeddings*

$$\mathcal{O}(G^+(B)) \hookrightarrow M_d \otimes M_d \otimes \mathcal{O}(S_n^+)$$
$$\mathcal{O}(S_n^+) \hookrightarrow M_d \otimes M_d \otimes \mathcal{O}(G^+(B))$$

The key tool in proving this result is a seemingly unrelated result obtained by the authors together with P.Ganesan [4], which computes the quantum chromatic number of the so called quantum complete graphs $\mathcal{K}(B)$. There it was shown that a certain quantum colouring of $\mathcal{K}(B)$ gives rise to a representation $\pi : \mathcal{A} \to M_d$, where $\mathcal{A}$ is the linking algebra associated to the monoidal equivalence between $G^+(B)$ and S_n^+.

An almost immediate consequence of the above result is the following.

Theorem B *Let B be any finite dimensional C*-algebra and ψ any faithful tracial state. Then $\mathcal{O}(G^+(B,\psi))$ is residually finite-dimensional and $L^\infty(G^+(B,\psi))$ is Connes embeddable.*

The proof of Theorem B goes as follows: Using a free product decomposition due to the first author (see also [5, Proposition 21]), together with standard results on the stability of RFDness and Conne embeddability with respect to free products, it suffices to consider the case where ψ is the Plancharel trace. In this case, the result follows immediately from the trace-preserving embeddings in Theorem A and the fact that the desired approximation properties area already known to hold for S_n^+.

Finally, let us conclude with a strengthening of Theorem A, which says that the embeddings given in that theorem can actually be extended to algebra isomorphisms by adding in some group actions and considering iterated crossed products. In the following, we use the same notation and symbols as in Theorem A.

Theorem C *Let* $\Gamma = \prod_{r=1}^{m} \mathbb{Z}_{k_r} \times \mathbb{Z}_{k_r}$. *Then there are commuting actions* α_1, α_2 *of* Γ *on* $\mathcal{O}(G^+(B))$ *(resp. commuting actions* β_1, β_2 *of* Γ *on* $\mathcal{O}(S_n^+)$*) which induce isomorphisms of iterated crossed products*

$$(\mathcal{O}(G^+(B)) \rtimes \alpha_1\Gamma) \rtimes_{\alpha_2} \Gamma \cong M_d \otimes M_d \otimes \mathcal{O}(S_n^+)$$

$$(O(S_n^+) \rtimes \beta_1\Gamma) \rtimes_{\beta_2} \Gamma \cong M_d \otimes M_d \otimes \mathcal{O}(G^+(B))$$

We note that the above isomorphisms also exist at the full C*-level, reduced C*-level, and at the von Neumann algebra level. Let us conclude these results with an open problem: *Can Theorem C be used to somehow show that the isomorphism class of* $L^\infty(G^+(B))$ *only depends on* $n = \dim B$*?* Note that this question only makes sense in the regime $n \geq 5$, because at $n = 4$ the answer is known to be "no". In this case, $L^\infty(S_4^+)$ is known to be noncommutative while on the other hand $L^\infty(G^+(M_2)) \cong L^\infty(SO(3))$.

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Generalized parabolic categories over quantum conjugacy classes of finite order

Andrey Mudrov

Let G be a simple complex algebraic group of type A, B, C, D, G_2 with a maximal torus T. Fix a Gauss decomposition of G relative to T Denote by $U_q(\mathfrak{g})$ the quantum group with the deformation parameter q not a root of unity. For every point t in T of finite order we consider a full subcategory $\mathcal{O}(t)$ of the $U_q(\mathfrak{g})$ -category $\mathcal{O}$ that is stable under tensor product with finite dimensional quasi-classical $U_q(\mathfrak{g})$-modules. We prove that $\mathcal{O}(t)$ is semi-simple for all q away from a finite set and equivalent to the category of finitely generated equivariant projective modules over the quantized conjugacy class of t. Its objects are "faithful representations" of quantized equivariant vector bundles on the conjugacy class of t.

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Quantum Affine Algebras and their Representations

Vyjayanthi Chari

The representation theory of affine and quantum affine algebras has been studied intensively for forty years. There have been many remarkable developments stemming from the study of highest weight representations of affine Lie algebras: the construction of the monster group, the connection with the Rogers-Ramanujam identities, the theory of vertex algebras to name a few.

The study of the category of finite dimensional representations of the affine Lie algebra (or rather its commutator subalgebra) is also very rich; this is because the category is not semisimple. The irreducible representations are easily described for the Lie algebra; however in the quantum case the irreducible modules are very complicated and many approached have been developed to study them.

In this talk we shall survey some of the results in the classical case [3], some of the early results in the quantum case [4, 6]. We then discuss some ongoing work [1,2] on particular families of modules which arise from the deep connections of this theory with cluster algebras, [7, 8].

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Square roots in the Witt group and higher central charges

Eric C. Rowell

(joint work with Eric C. Rowell, Siu-Hung Ng, Yuze Ruan, Yilong Wang, Qing Zhang)

When studying mathematical structures such as topological spaces, finite groups, Lie algebras and braided tensor categories a natural (albeit vague) question to ask is: *are all of the examples constructed using standard tools from some natural families?* To make sense of this one needs to have precise notions of "natural families" and "standard tools." Finite simple groups that do not fit into standard families are called *sporadic*, finite dimensional complex simple Lie algebras not arising from orthogonal, symplectic, or special linear groups are *exceptional* and smooth manifolds homeomorphic but not diffeomorphic to a sphere are called *exotic*. For braided fusion categories, candidates for natural families are categories $Rep(D^\omega G)$ associated with finite groups, pointed categories $\mathcal{C}(A,q)$ associated with metric groups and those obtained from quantum groups at roots of unity, denoted by $\mathcal{C}(\mathfrak{g},\ell)$. Standard tools would surely include the Deligne product, G-(de-)equivariantization and G-extensions. A version of this question can be found in [5]. A possible counter-example was explored in [4], but refuted in [3].

There are somewhat parallel notions in condensed matter physics: symmetry gauging and anyon condensation. These are regarded as topological phase transitions. Roughly symmetry gauging extends a global (group) symmetry to a local symmetry, while anyon condensation is somewhat like a quotient–one "condenses" some anyons by identifying them with the vacuum anyon. Therefore it is of interest in physics to understand the topological phases of matter that differ only by symmetry gauging/anyon condensation.

The Witt group $\mathcal{W}$ for non-degenerate braided fusion categories (NDBFCs) was introduced in [1] and developed further in [2]. It allows for a precise definition of "exotic" for modular categories, and simultaneously a description of topological phases that differ by one of the phase transitions mentioned above. Generalizing the usual Witt group for non-degenerate quadratic forms on finite abelian groups, $\mathcal{W}$ is itself an infinite rank abelian group. The elements of $\mathcal{W}$ are classes $[\mathcal{C}]$ of NDBFCs "modulo centers", i.e. two NDBFCs are Witt-equivalent if they differ by Deligne products with Drinfeld centers $\mathcal{Z}(\mathcal{F}_i)$ of fusion categories (which represent the unit in $\mathcal{W}$). The product in the category is the usual Deligne product $\boxtimes$ and the inverse of $[\mathcal{C}]$ is represented by $[\mathcal{C}^{rev}]$ obtained by reversing the braiding on $\mathcal{C}$.

One may conveniently define a modular category $\mathcal{C}$ to be *exotic* if

$$\mathcal{C} \notin \langle \mathcal{C}(\mathfrak{g}, \ell) : \mathfrak{g}, \ell \rangle$$

that is, if $\mathcal{C}$ is not in the subgroup of $\mathcal{W}$ generated by quantum group categories.

$\mathcal{W}$ has a torsion part and a free part. The torsion part is quite interesting: it has exponent 32, and contains all of the classes of pointed NDBFCs as well as the celebrated Ising categories $\mathcal{I}^\nu$, $\nu = 2k+1$ for $0 \leq k \leq 7$, related to $SU(2)_2$. The categories $\mathcal{I}^\nu$ have order 16 in the Witt group, while pointed categories have orders dividing 8. A first step towards finding exotic modular categories might be to understand $Tor(\mathcal{W})$.

A number of questions were presented in [1], one of which is whether the categories $\mathcal{I}^\nu$ each have infinitely many Witt classes $[\mathcal{C}]$, modulo the subgroup generated by pointed classes and the Ising classes, such that $[\mathcal{C}^{\boxtimes 2}] = [\mathcal{I}^\nu]$. As the pointed part $\mathcal{W}_{pt}$ is infinite, it is easy to construct infinitely many such classes from a single class, but whether this could be done modulo $\mathcal{W}_{pt}$ was left open.

In [6] we show that each of the categories $SO(N)_N$ for N odd satisfy

$$[SO(N)_N]^{\boxtimes 2} = [\mathcal{I}^\nu]$$

for some ν (this was known previously), and that the quotient by the subgroup generated by pointed classes and the Ising classes of the subgroup G of $\mathcal{W}$ generated by these $SO(N)_N$ is an infinite rank exponent 2 group. The methods we developed to achieve this result employ the higher central charges defined in [7], and a generalization known as the *signature* of a fusion category $\mathcal{F}$. The idea is to consider the dimensions $\dim(\mathcal{F})$, which are totally positive algebraic integers. Their square roots then are totally real algebraic integers, so that for any $\sigma \in Gal(\overline{\mathbb{Q}})$ the sign of $\sigma(\sqrt{\dim(\mathcal{F})})$ is well-defined. This provides a useful invariant of Witt classes of pseudo-unitary NDBFCs (which have canonical spherical, and hence modular, structures): for any $\mathcal{C}$ representing its Witt class, $\epsilon_{[\mathcal{C}]} : Gal(\overline{\mathbb{Q}}) \to \{\pm 1\}$ is well-defined and does not depend on the representative $\mathcal{C}$, and these homomorphisms can be used to distinguish Witt classes.

A natural related problem is to study $SO(N)_N$ for N even. In [8] we used similar methods to find that the Witt classes of $SO(N)_N$ for N even yield infinitely many square roots of the trivial Witt class $[Vec]$, modulo $\mathcal{W}_{pt}$. Interestingly, this yielded the first verified example of a modular category $\mathcal{D}_4$ that is its own Witt-inverse, is simple and contains no condensable anyons (i.e. completely anisotropic). This settles another question: [1, Question 6.8].

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The Kohno–Drinfeld theorem for symmetric pairs

Makoto Yamashita
(joint work with Kenny De Commer, Sergey Neshveyev, Lars Tuset)

The Kohno–Drinfeld theorem relates braided monoidal categories that appear as deformation of the category of finite dimensional representations of a simple compact Lie algebra (or a simple complex Lie algebra). As a corollary, it provides equivalence of braid group representations, one coming from monodromy of Knizhnik–Zamolodchikov equations, and another from universal R-matrix of Drinfeld–Jimbo quantized universal enveloping algebras.

In this talk, we present an analogue for compact symmetric pairs, that is, for pairs of Lie algebras $\mathfrak{k} < \mathfrak{u}$, where $\mathfrak{u}$ is a simple compact Lie algebra and $\mathfrak{k}$ is the fixed point of an involutive automorphism of $\mathfrak{u}$. The geometric structure controlling such deformation is the compact symmetric space U/K, where U and K are compact Lie groups integrating $\mathfrak{u}$ and $\mathfrak{k}$.

On the side of KZ equations are the 2-cyclotomic KZ equations [EE05]. This leads to a twisted variant of ribbon braided module category structure on $\operatorname{Rep} K$ over Drinfeld's braided tensor category from KZ equations. We obtain a deformation quantization of U/K by the action of associativity structure morphisms of this category, combined with a Drinfeld twist that appears in the KD theorem. The equations allow modification by the center of $\mathfrak{k}$ (which is at most 1-dimensional), that corresponds to the Poisson pencil structure on U/K with respect to the Poisson action of U.

On the side of quantized universal enveloping algebras are the Letzter coideals [L99], and universal K-matrix by Kolb and Balagović [BK19]. These again define ribbon twisted braided module categories with nontrivial parameters appearing when $\mathfrak{k}$ has a nontrivial center.

In [DCNTY19], we proposed a problem of comparing these constructions, with unitary structures when the parameters belong to a "real axis". One motivation comes from a similar work of Brochier [B12] on the Cartan subalgebra of $\mathfrak{u}$. In [DCNTY20], we settle this problem over formal power series. Concretely, we prove a rigidity of quasi-coactions on the multiplier model of $\mathcal{U}(\mathfrak{k})[[h]]$ by Drinfeld's quasi-bialgebra $\mathcal{U}(\mathfrak{u})[[h]]$, and show that the Letzter coideals can be put in this framework. Our proof can be regarded as an analogue of [DS97] for symmetric pairs, and when resolving cohomological obstructions we combine insights from the theory of formality, and a concrete presentation of the associator from 2-cyclotomic KZ equations that correspond to dynamical r-matrices.

Our classification of ribbon twisted braided module category structures can be made concrete. Namely, the deformation parameter (which appears when $\mathfrak{k}$ has nontrivial center) is reflected in the eigenvalues of ribbon braid, which is the extra generator for the type B braid group representation associated with such structures. This allows us to write a concrete formula for the correspondence between the KZ side and the coideal side in the case of AIII type inclusions.

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On the structure of Nichols algebras

Istvan Heckenberger

Nichols algebras arise as a basic tool in the Lifting Method of Andruskiewitsch and Schneider to study and classify pointed Hopf algebras. They are fundamental examples of Hopf algebras in braided monoidal categories, and appear very naturally in the context of Hopf algebra triples.

A group G is a semidirect product of two subgroups if and only if there are group homomorphisms $\gamma : H \to G$ and $\pi : G \to H$ such that $\pi\gamma = \mathrm{id}_H$. This generalizes to Hopf algebras. A triple (A, π, γ) is called a *Hopf algebra triple over a Hopf algebra H*, if A is a Hopf algebra (with comultiplication Δ), and $\pi : A \to H$ and $\gamma : H \to A$ are Hopf algebra maps such that $\pi\gamma = \mathrm{id}_H$. Let

$$A^{\mathrm{co}\,H} = \{a \in A \,|\, (\mathrm{id} \otimes \pi)\Delta(a) = a \otimes 1\}$$

(the H-coinvariants). Then A is the *bosonization*

$$A \cong A^{\mathrm{co}\,H} \# H,$$

where # is a vector space tensor product, but multiplication and comultiplication are usually not componentwise. Unfortunately, $A^{\mathrm{co}\,H}$ is not a usual Hopf algebra, but a Hopf algebra in the braided monoidal category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules over H. Luckily, one can define Hopf algebra triples over braided Hopf algebras, and then the coinvariants still form a braided Hopf algebra, see [1, Ch. 3]. This justifies to study braided Hopf algebras. For convenience, one usually works in ${}^H_H\mathcal{YD}$.

Nichols algebras are $\mathbb{N}_0$-graded braided Hopf algebras $\mathfrak{B}$ which are trivial in degree 0, are generated as an algebra in degree one, and all primitive elements (those with $\Delta(x) = 1 \otimes x + x \otimes 1$) are in degree one. For each object $V \in {}^H_H\mathcal{YD}$ there is up to isomorphism a unique Nichols algebra $\mathfrak{B}(V)$ having degree one part V. However, generally there is no satisfactory information on the algebra and the coalgebra structures of $\mathfrak{B}(V)$.

The functor $\mathfrak{B}$ is compatible with bosonization: For any pair $\gamma : U \to V$ and $\pi : V \to U$ in ${}^H_H\mathcal{YD}$ with $\pi\gamma = \mathrm{id}_U$ the coinvariants $\mathfrak{B}(V)^{\mathrm{co}\,\mathfrak{B}(U)}$ are a Nichols algebra in ${}^{\mathfrak{B}(U)\#H}_{\mathfrak{B}(U)\#H}\mathcal{YD}$.

If V is the direct sum of (at least two) one-dimensional objects, then V is of *diagonal type*, and in reasonable cases (e.g. if $\mathfrak{B}(V)$ has finite dimension) one can attach to V a generalized root system which gives rise to a PBW basis and a very good understanding of $\mathfrak{B}(V)$. Much of this structure is preserved if V is more generally the direct sum of at least two simple objects.

The basis of the role of generalized root systems in the context of Nichols algebras are reflection functors. For a dual pair A, B of braided Hopf algebras, a reflection functor is roughly an equivalence

$$ {}^A_A\mathcal{YD} \to {}^B_B\mathcal{YD} $$

of braided monoidal categories, where the module (comodule) structure over B is determined by the comodule (module) structure over A. Such functors are applied efficiently in the context of Nichols algebras by identifying first a Hopf subalgebra A and a Hopf algebra triple over A, then applying a reflection functor, and finally bosonize by B. This way one creates from a Nichols algebra typically a significantly different one.

In order to explain the structure of Nichols algebras in terms of generalized root systems, one studies sequences of right (equivalently, left) coideal subalgebras. E.g. enlarging a right coideal subalgebra works by looking for a Hopf algebra triple for the Nichols algebra, where the coideal subalgebra is in the left coinvariants, and then applying the reflection functor and the bosonization. Combinatorially, such sequences of right coideal subalgebras correspond to elements of the Weyl groupoid of the generalized root system, see [1].

Nichols algebras of simple Yetter-Drinfeld modules have a trivial root system. There are interesting observations and developments around left coideal subalgebras of Nichols algebras in the category of H-comodules. (Coinvariants of left coideal subalgebras in the above context are such objects.) E.g. in the finite-dimensional case these are Frobenius algebras. For the known finite-dimensional Nichols algebras over non-abelian groups it was observed that there are not many such left-coideal subalgebras. On the other hand, it is very rare that all of them are generated in degree one. A better understanding of the lattice of such left coideal subalgebras is expected to become very useful.

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Furstenberg boundary for discrete quantum groups

Roland Vergnioux

(joint work with Mehrdad Kalantar, Paweł Kasprzak, Adam Salski)

The concept of boundary actions in topological dynamics was introduced by Furstenberg in 1950s. It was also Furstenberg who noted the existence of a universal boundary action for any locally compact group G; nowadays this action, as well as the relevant space, is called the Furstenberg boundary of G. In the last decade, beginning with work of Kalantar and Kennedy, this notion found unexpected and groundbreaking applications in the study of operator algebras associated with discrete groups. In particular it was shown by Breuillard, Kalantar, Kennedy and Ozawa that the reduced group C^*-algebra $C^*_r(\Gamma)$ of a discrete group Γ is simple if and only if the action of Γ on its Furstenberg boundary is free, and that $C^*_r(\Gamma)$ admits unique trace if and only if this action is faithful, if and only if the amenable radical of Γ is trivial.

Another source of operator algebras sharing many properties with these related to discrete groups is provided by the theory of compact (equivalently, discrete) quantum groups, as initiated by Woronowicz. Of particular interest is the class of orthogonal free quantum groups $FO(Q)$ ($Q \in GL_N(\mathbb{C})$, $Q\bar{Q} = \pm I_N$) introduced by Wang and Van Daele, which leads to operator algebras behaving in many ways as the ones associated with the classical free groups, see e.g. works by Banica; Vaes and Vergnioux; Brannan; De Commer, Freslon and Yamashita. Note however that the corresponding von Neumann algebras are not isomorphic to free group factors as shown more recently by Brannan and Vergnioux.

Already in the classical case, the notion of the Furstenberg boundary fits naturally with Hamana's work on injective envelopes, so that it is suitable for noncommutative generalizations. This leads us to the following definition in the quantum framework, where Γ is a discrete *quantum* group: a unital Γ-C^*-algebra A is a Γ-boundary if every unital completely positive (UCP) Γ-map $T : A \to B$ to any other Γ-C^*-algebra B is automatically completely isometric (CI). In other words, working in the category of Γ-C^*-algebras with UCP Γ-maps as morphisms and UCI Γ-maps as extensions, the extension $\mathbb{C} \subset A$ is essential.

Following work of Hartman and Kalantar, we provide the following tool to show that a given unital Γ-C^*-algebra A is a Γ-boundary. Fix a state $\mu \in S(c_0(\Gamma))$ and assume that A admits a *unique* state $\nu \in S(A)$ such that $\mu * \nu = \nu$ (stationarity). Assume moreover that the corresponding Poisson map $P_\nu : A \to \ell^\infty(\Gamma)$, $a \mapsto \nu * a$ is completely isometric. Then A is a Γ-boundary. This sufficient condition applies in particular to the quantum Gromov boundary $\partial_G FO(Q)$ constructed by Vaes and Vergnioux, which is equipped with a natural stationary state ω. It was already known that P_ω is completely isometric, and we show that ω is uniquely stationary.

We also investigate faithfulness of boundary actions. Quantum group actions in general do not admit 'kernels' viewed as quantum subgroups, but rather 'co-kernels', understood as quantum subgroups of the dual quantum group. More precisely, given a Γ-C^*-algebra A with coaction α we consider the associated Baaj-Vaes subalgebra $N_\alpha \subset \ell^\infty(\Gamma)$ obtained as the weak closure of the subspace $\{\nu * a; a \in A, \nu \in A^*\}$ and we say that α is faithful if $N_\alpha = \ell^\infty(\Gamma)$. When A is the universal Furstenberg boundary, i.e. the injective envelope of $\mathbb{C}$ in the previously mentioned category, we show that N_α is the unique minimal relatively amenable Baaj-Vaes subalgebra of $\ell^\infty(\Gamma)$, i.e. the quantum counterpart of the amenable radical from the classical theory.

We moreover show that if a unimodular discrete quantum group Γ acts faithfully on a Γ-boundary A, then the corresponding reduced C^*-algebra $C^*_r(\Gamma)$ admits a unique trace, thus extending the previously mentioned classical result. On the other hand, in the non-unimodular case, existence of a faithful Γ-boundary action entails the absence of KMS state with respect to the scaling group: this is a new feature of the quantum theory. Note that the reverse implications remain open in the quantum case. Finally, we show that the boundary action of $FO(Q)$ on its quantum Gromov boundary is faithful. In the unimodular case ($Q^*Q = I_N$) we recover in this way the unique trace property already obtained by Vaes and Vergnioux for $C^*_r(FO(Q))$. Our proof of faithfulness of the action is however quite involved and it would certainly be interesting to acquire a better understanding of the dynamical properties of $\partial_G FO(Q)$, including e.g. a quantum counterpart for the freeness of the action.

A geometric point of view on approximation of Nichols algebras

Giovanna Carnovale

(joint work with Francesco Esposito, Lleonard Rubio y Degrassi)

Nichols(=shuffle) algebras are a family of graded Hopf algebras in a braided monoidal category $\mathcal{V}$. Notable examples are: symmetric and exterior algebras, and the positive parts of quantized enveloping algebras. Even though they can be described as quotients of tensor algebras, it is extremely hard to determine a presentation, and even to verify when they are finitely presented. A presentation of finite dimensional Nichols algebras when $\mathcal{V}$ is the category of (complex) Yetter-Drinfeld modules over a finite abelian group has been obtained in [2], some further examples have been computed, and there is vast literature on Nichols algebras of finite Gelfand-Kirillov dimension, however a full comprehension of the general case is far from being achieved, see [1] and references therein.

Technically speaking, for V an object in $\mathcal{V}$, one can equip the tensor algebra $T_!(V) = \bigoplus_{j\geq 0} V^{\otimes j}$ and the cotensor (big shuffle) algebra $T_*(V)$ with a graded bialgebra structure in $\mathcal{V}$. Then, the Nichols algebra $T_{!*}(V)$ of V is the image of a (uniquely determined) graded bialgebra morphism $\Omega\colon T_!(V) \to T_*(V)$ extending the identity on V.

In order to have a hold on a presentation of $T_{!*}(V)$, it is important to answer the following questions:

(a) For a given V, is $T_{!*}(V)$ finitely presented? Equivalently: is there an $N \in \mathbb{N}$ such that

$$T_{!*}(V) = T_!(V)/(\mathrm{Ker}\Omega \cap \bigoplus_{2\leq j\leq N} V^{\otimes j}),$$

that is: does $T_{!*}(V)$ coincide with its N-th approximation for some N?

(b) In case of an affirmative answer to (a) for a given V, how to determine the minimal possible N?

An example. The Fomin-Kirillov algebras FK_n for $n \geq 3$ form a family of quadratic, non-commutative algebras. They have been introduced in [5] in order to study Schubert calculus on the flag variety of $\mathrm{G}L_n(\mathbb{C})$. The degree 1 term V_n in FK_n has the structure of a Yetter-Drinfeld module over $\mathbb{S}_n$ and $FK_n = T_!(V_n)/(\mathrm{Ker}\Omega \cap V^{\otimes 2})$ for every n, [10]. It was proven in [6, 10], with the contribution of Graña, that $FK_n = T_!(V_n)$ for $n = 3, 4, 5$. Under this restriction these algebras are finite-dimensional, [5, 6, 11]. At present, the dimension of these algebras for $n \geq 6$ is unknown, and we do not know if they coincide with $T_{!*}(V_n)$, despite many efforts by several researchers.

All bialgebras mentioned up to now have a common feature: they are primitive (= connected and coconnected) bialgebras, that is, their degree 0 term is the trivial object. Kapranov and Schechtman have constructed in [9, §3] an equivalence of categories L between the category $PB(\mathcal{V})$ of primitive balgebras in $\mathcal{V}$ and the category $\mathcal{FPS}(\mathrm{Sym}(\mathbb{C}), \mathcal{V})$ of factorizable perverse sheaves on $\mathrm{Sym}(\mathbb{C})$ with values in $\mathcal{V}$. Here $\mathrm{Sym}(\mathbb{C})$ is the space of monic polynomials, with infinitely many connected components $\mathrm{Sym}^n(\mathbb{C})$, for $n \geq 0$ given by monic polynomials of degree n, stratified in terms of mutiplicities of roots. A sequence of perverse sheaves on each $\mathrm{Sym}^n(\mathbb{C})$ is factorizable if it satisfies a technical condition ensuring compatibility with the monoid structure on $\mathrm{Sym}(\mathbb{C})$. The restriction of a factorizable perverse sheaf on $\mathrm{Sym}(\mathbb{C})$ to the open stratum $\mathrm{Sym}_{\neq}(\mathbb{C})$ consisting of multiplicity-free polynomials is a sequence of local systems on each $\mathrm{Sym}_{\neq}(\mathbb{C}) \cap \mathrm{Sym}^n(\mathbb{C})$, that is, a representation of the fundamental group of each $\mathrm{Sym}_{\neq}(\mathbb{C}) \cap \mathrm{Sym}^n(\mathbb{C})$, which is the braid group B_n. Factorizability forces these representations to be compatible, i.e., they should be the representations $V^{\otimes n}$ for V an object in $\mathcal{V}$, with action coming from the braiding. We denote such a local system by $\mathcal{L}(V)$. For $N \geq 0$ we consider the dense open subsets $\mathrm{Sym}_{\leq N}(\mathbb{C})$ of $\mathrm{Sym}(\mathbb{C})$ consisting of polynomials with root multiplicities not exceeding N, and the corresponding open inclusions:

$$\alpha_N\colon \mathrm{Sym}_{\leq N}(\mathbb{C}) \to \mathrm{Sym}(\mathbb{C}), \quad j\colon \mathrm{Sym}_{\neq}(\mathbb{C}) \to \mathrm{Sym}(\mathbb{C}).$$

They give rise to restriction and extension functors, following Grothendieck 6-functors formalism, [3]. Under Kapranov and Schechtman's equivalence we have:

$$L(T_!(V)) = j_!\mathcal{L}(V), \quad L(T_*(V)) = j_*\mathcal{L}(V), \quad L(T_{!*}(V)) = j_{*!}\mathcal{L}(V).$$

The extension functor $\alpha_{N!}$ does not preserve perverse sheaves in general, and it needs truncation ${}^p\tau_{\geq 0}$ to preserve them.

Theorem.

(1) For any $N \in \mathbb{N}$ there exists a functor $F_N\colon PB(\mathcal{V}) \to PB(\mathcal{V})$ extending the N-th approximation construction to all primitive bialgebras.
(2) The endofunctor ${}^p\tau_{\geq 0}\alpha_{N!} \circ \alpha_N^*$ on perverse sheaves on $\mathrm{Sym}(\mathbb{C})$ preserves factorizable perverse sheaves and satisfies ${}^p\tau_{\geq 0}\alpha_{N!} \circ \alpha_N^* \circ L = F_N \circ L$.

Corollary. For an object $V \in \mathcal{V}$ we have $T_{!*}(V) = T_!(V)/(\mathrm{Ker}\Omega \cap \bigoplus_{j\leq N} V^{\otimes j})$ if and only if ${}^p\tau_{\geq 0}\alpha_{N!} \circ \alpha_N^* \circ j_{!*}\mathcal{L}(V) = j_{!*}\mathcal{L}(V)$. In particular, $T_{!*}(V) = T_!(V)$ if and only if ${}^p\tau_{\geq 0}j_!\mathcal{L}(V) = j_{!*}\mathcal{L}(V)$.

The geometric point of view might shed light on what the difficulties in the algebraic framework mean geometrically and viceversa, and there is hope that for better known local systems (=representations of the braid group), it might lead to an answer for the algebraic questions we started with.

1. Possible directions

(1) The extensions j_* and $j_{!*}$ of local systems could be studied by looking at the covering of $\mathrm{Sym}_{\neq}(\mathbb{C})$ given by Hurwitz spaces. One may hope to obtain a simpler local system on a (possibly) more complicated space.
(2) The category of perverse sheaves on $\mathrm{Sym}^n(\mathbb{C})$ can be described in terms of (bi)representations of quiver with relations, [8]. One should be able to provide a generalisation of this interpretation to perverse sheaves with values in $\mathcal{V}$, and to see what kind of compatibilities on the representations of the quivers at different degrees encode factorizability. This way one could obtain a combinatorial description of factorizable perverse sheaves on $\mathrm{Sym}(\mathbb{C})$ with values on $\mathcal{V}$.
(3) One can hope to exploit asymptotic calculations along the lines of [4] to prove finite presentation for Nichols algebras.

2. Open questions

(1) Is it possible to give a geometric interpretation of bosonization and of the reflection functors from I. Heckenberger's talk?
(2) When $\mathcal{V}$ is the category of Yetter-Drinfeld modules over a finite abelian group, the condition for a Nichols algebra to be a free algebra, that is, the $N = 1$ case, has been translated in terms of nonvanishing of infinitely many polynomials in [7]. Can these polynomials be interpreted in geometric terms?

(3) Could one describe more explicitly the geometry behind the fact that $FK_n \simeq T_{!*}(V_n)$ for $n = 3, 4, 5$?
(4) What can we say about the local systems and sheaves for which the corresponding Nichols algebra is known to be of finite/finite GK dimension?

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Quantization of $\mathbf{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ (and some of its relatives)

Victor Gayral

(joint work with P. Bieliavsky, S. Neshveyev and L. Tuset)

The aim of this talk is to present an explicit construction of a dual unitary 2-cocycle for a class of locally compact groups which share many common features with the affine group $\mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$.

Recall that for a locally compact (second countable) group G, a dual unitary 2-cocycle for G is an element $\Omega \in W^*(G \times G)$ which satisfies the relation:

$$(\Omega \otimes 1)(\hat{\Delta} \otimes \mathrm{Id})(\Omega) = (1 \otimes \Omega)(\mathrm{Id} \otimes \hat{\Delta})(\Omega).$$

Recall also that by a result of De Commer [1], the von Neumann bialgebra

$$(W^*(G), \Omega\hat{\Delta}(.)\Omega^*)$$

defines a locally compact quantum group (that is, it possesses left and right invariant weights).

Let V be an abelian locally compact group. We are concerned with semi-direct products $G = H \ltimes V$ satisfying the following dual orbit condition:

Definition 1. We say that a locally compact group $G = H \ltimes G$ satisfies the *dual orbit condition of depth* 1 if there exists an element $\xi_0 \in \hat{V}$ such that the map

$$H \to \hat{V}, \quad q \mapsto q^\flat \xi_0,$$

is a measure class isomorphism.

We say that G satisfies the *dual orbit condition of depth* $\ell \in \mathbb{N} \setminus \{0, 1\}$ if:
(1) there exists $\xi_0 \in \hat{V}$ whose H-orbit is of full measure,
(2) the little group of the orbit $G' := \mathrm{Stab}_G(\xi_0)$ is of the form $H' \ltimes V'$, with V' Abelian, and satisfies the dual orbit condition of depth $\ell - 1$,
(3) there exists another closed subgroup Q of H such that (Q, G') forms a matched pair for H,
(4) H' normalizes Q (in H).

Concrete examples of groups satisfying the dual orbit condition of depth ℓ are given by:

- $\mathrm{GL}_{\ell k}(\mathbb{K}) \ltimes \mathrm{Mat}_{\ell k,k}(\mathbb{K})$
- $\big(\mathrm{SL}_{\ell k+1}(\mathbb{K}) \times (\mathbb{K}^*)^k\big) \ltimes \mathrm{Mat}_{\ell k+1,k}(\mathbb{K})$
- $\big(\mathrm{SL}_k(\mathbb{K}) \times \mathrm{GL}_{\ell k}(\mathbb{K})\big) \ltimes \mathrm{Mat}_{k,\ell k}(\mathbb{K})$
- $\big(\mathrm{SL}_k(\mathbb{K}) \times \mathrm{GL}_{\ell k+1}(\mathbb{K})\big) \ltimes \mathrm{Mat}_{k,\ell k+1}(\mathbb{K})$,

where $k \in \mathbb{N}^*$ and $\mathbb{K}$ is a nondiscrete locally compact field (of arbitrary characteristic and possibly skew).

An important point here is that such groups possess a very poor (reduced) representation theory:

Proposition 2. *Let G satisfying the dual orbit condition of depth ℓ. Then $W^*(G)$ is a type I factor and G possesses a unique (class of) irreducible square-integrable representation, inductively given by*

$$\pi_G := \mathrm{Ind}_{G' \ltimes V}^{G}(\pi_{G'} \otimes \xi_0).$$

In [2], we gave a general formula for a dual unitary 2-cocycle for all locally compact groups for which the group von Neumann algebras are type I factors and which are endowed with an equivariant unitary quantization map:

Definition 3. Let $(\pi, \mathcal{H}_\pi)$ be a square-integrable irreducible unitary representation of a locally compact group G. A π-equivariant unitary quantization for G is a map

$$\mathrm{Op} \in \mathcal{U}\big(L^2(G), \mathrm{HS}(\mathcal{H}_\pi)\big),$$

such that for all $g \in G$:

$$\mathrm{Ad}_{\pi(g)} \circ \mathrm{Op} = \mathrm{Op} \circ \lambda_g.$$

In detail, we have:

Theorem 4. *Let G be a locally compact group such that $W^*(G)$ is a type I factor and which is endowed with a π-equivariant unitary quantization map* Op. *Then the following defines a dual unitary 2-cocycle for G:*

$$\Omega := (J\hat{J} \otimes J\hat{J})\mathcal{G}^*(1 \otimes J\hat{J})\hat{W}.$$

In this formula, J and $\hat{J}$ are the modular conjugations of $L^\infty(G)$ and of $W^*(G)$, $\hat{W}$ is the fundamental unitary, $\mathcal{G} \in \mathcal{U}(L^2(G \times G))$ is the Galois map of the I-factorial G-Galois object $(\mathcal{B}(\mathcal{H}_\pi), \mathrm{Ad}_{\pi(.)})$ transported to $L^2(G)$ via the quantization map Op*:*

$$\mathcal{G}(f_1 \otimes f_2)(g,h) = \tilde{\Lambda}\Big(\big(\mathrm{Ad}\pi(g)\big)(\mathrm{Op}(f_1)K^{-1/2})\mathrm{Op}(f_2)K^{-1/2}\Big)(h),$$

where K is Duflo-Moore's formal dimension operator of the representation π, that is the densely defined operator with densely defined inverse, uniquely determined by

$$\int_G |\langle\varphi_1, \pi(g)\varphi_2\rangle|^2 dg = \|\varphi_1\|^2 \|K^{1/2}\varphi_2\|^2,$$

for all $\varphi_1 \in \mathcal{H}_\pi$ and $\varphi_2 \in \mathrm{Dom}(K^{1/2})$, and $\tilde{\Lambda}$ is the GNS-map uniquely determined by

$$\tilde{\Lambda}(\mathrm{Op}(f)K^{-1/2}) := f \quad \textit{for} \quad f \in L^2(G) \quad \textit{such that} \quad \mathrm{Op}(f)K^{-1/2} \in B(H).$$

For a group $G = H \ltimes V$ which satisfies the dual orbit condition of depth ℓ, it thus remains to construct an equivariant unitary quantization map:

Theorem 5. *Let $G_\ell = H_\ell \ltimes V_\ell$ satisfying the dual orbit condition of depth ℓ. Set inductively $\mathrm{Stab}_{G_\ell}(\xi_0) = G_{\ell-1} = H_{\ell-1} \ltimes V_{\ell-1}$ and Q_ℓ be such that $(Q_\ell, G_{\ell-1})$ forms a matched pair for H_ℓ. Last, let T be the Radon measure on $(Q_\ell \times \cdots \times Q_1)^2$ given by*

$$T(\phi) = \int_{Q_\ell \times \cdots \times Q_1} \phi(e, \cdots, e; q_\ell, \cdots, q_1)\, \omega_\ell(q_\ell, \cdots, q_1) \frac{d_{Q_\ell}(q_\ell)}{|q_\ell|_{V_\ell}} \cdots \frac{d_{Q_1}(q_1)}{|q_1|_{V_1}},$$

where ω_ℓ is the continuous function on $Q_\ell \times \cdots \times Q_1$ given in therm of the modular functions by:

$$\omega_\ell = \Delta_{H_\ell}^{1/2}\Delta_{Q_\ell}^{-1/2} \otimes \cdots \otimes \Delta_{H_1}^{1/2}\Delta_{Q_1}^{-1/2}.$$

Then, the following sesquilinear form on $C_c(Q_\ell \times \cdots \times Q_1)$:

$$\langle\varphi_1, \mathrm{Op}(f)\varphi_2\rangle = \int_G f(g)\, T\big(J\pi(g)^*\varphi_1 \otimes \pi(g)^*\varphi_2\big)\, dg,$$

extends for $\ell = 1, 2$ or 3, to a equivariant unitary quantization map.

The generic case $\ell \geq 4$ is still under study. The explicit formulas for the associated dual unitary 2-cocycles, will be published somewhere else.

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Quasi-polynomial generalisations of Macdonald polynomials

Jasper V. Stokman

(joint work with Siddhartha Sahi and Vidya Venkateswaran)

There exists a rich interplay between representation theory of (quantum group analogs of) reductive groups and (basic) hypergeometric functions associated to root systems. In [3, §5] this is beautifully described in terms of a hierarchy of multivariate hypergeometric functions. The top level of the hierarchy contains the Macdonald polynomials, which naturally arise as zonal spherical functions on quantum symmetric pairs (see [4]). They reduce to Whittaker functions in the p-adic limit, and to Heckman-Opdam polynomials in the classical limit. Representation theory of Hecke type algebras provides a very effective tool to study these classes of multivariate special functions. In case of Macdonald polynomials this involves representation theory of Cherednik's [1] double affine Hecke algebra (DAHA).

Whittaker functions on *n-fold metaplectic covers* of reductive p-adic groups have been extensively studied in recent years. They occur as the local parts of Weyl group multiple Dirichlet series (see [5]). They admit an explicit expression in terms of metaplectic variants of Demazure-Lusztig operators (see [2]). The same is true for the metaplectic analogs of Iwahori-Whittaker functions (see [6]).

The involvement of Demazure-Lusztig type operators is establishing a concrete link between metaplectic (Iwahori-)Whittaker functions and representation theory of affine Hecke algebras, which is explored in [7]. The talk reports on the generalisation of the theory of metaplectic (Iwahori-)Whittaker functions to the level of Macdonald polynomials using DAHA representation theory, based on [8].

The relevant DAHA-modules are realised on spaces of quasi-polynomials, with the double affine Hecke algebra acting by multiplication operators and truncated, twisted Demazure-Lusztig operators. The spaces of quasi-polynomials are spanned by monomials x^y ($y \in \mathcal{O}$) with $\mathcal{O}$ an affine Weyl group orbit in the ambient Euclidean space of the root system. The DAHA-module reduces to Cherednik's [1] basic representation when $\mathcal{O}$ is the orbit containing the origin.

The double affine Hecke algebra contains the group algebras of two lattices. On the space of quasi-polynomials one group algebra acts by multiplication operators, the other by generalised Cherednik operators. The quasi-polynomial generalisations of the non-symmetric Macdonald polynomials are the simultaneous quasi-polynomial eigenfunctions of the generalised Cherednik operators. The metaplectic Iwahori-Whittaker functions are recovered as their p-adic limit. Hecke algebra symmetrisation leads to quasi-polynomial generalisations of the symmetric Macdonald polynomials.

A natural open question is the following: are there natural quantum group interpretations of the quasi-polynomial generalisations of the symmetric Macdonald polynomials?

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Comparing different constructions of modular categories

Julia Plavnik

Modular categories arise naturally in many areas of mathematics, such as conformal field theory, representations of braid groups, quantum groups, and Hopf algebras, low dimensional topology. These highly structured algebraic objects have important applications in condensed matter physics, for example modeling certain topological phases of matter, and are useful for topological quantum computation.

Despite recent progress on the classification of modular categories, we are still in the early stages of this theory and the general landscape remains largely unexplored. One important step towards deepening our understanding of modular categories is to have well-studied constructions. The simplest one is the Deligne product of modular categories $\mathcal{C} \boxtimes \mathcal{B}$, which is like a direct product of the modular categories $\mathcal{C}$ and $\mathcal{B}$. Another well-known construction is the Drinfeld center, categorifying the notion of the center of a monoid, which gives a modular category when input a spherical fusion category. Müger and Bruguières introduced independently the notion of modularization; given a premodular category with Tannakian Müger (or symmetric) center $\mathrm{Rep}(G)$, one can de-equivariantize by the group G and get a modular category. A generalization of this idea, it is the relative tensor product or condensation of premodular categories, see for example [2]. In this case, condensation is implemented by considering local (or dyslectic) modules of a connected ètale algebra in the input premodular category getting a new modular category as a result.

Gauging is a well-known procedure in physics to promote a global symmetry to a local one. With my collaborators Cui, Galindo, and Z. Wang, we gave a mathematical description of this process in terms of higher categories [1]. Gauging is a 2-step process: given a modular category with a categorical action by a group G, one first extend to get a G-crossed braided fusion category, using the techniques developed in [5], and then G-equivariantize getting a modular category. Gauging preserves central charge and Witt classes but in general is hard to find the modular data and the braid group representations of the resulting category. A new construction related to extension theory is the so-called ribbon zesting [3]. One starts with a G-graded modular category and "twist" the tensor product by an invertible that depends on the grading, giving rise to a 2-cocycle and 3-cochain data to change the associativity constraint. In a similar way, one can modify the braid and the twist of the original category. If the grading is the Universal grading one get a modular category as a result. The big advantage of this procedure is that there are explicit formulas for the modular data, braid image, and links and knots invariants ([4]) of the new modular category. Even if this construction is not the most general, since it is a particular case of extension theory, it has already proven to be useful to construct examples and for categorification of fusion rules and modular data.

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Frobenius exact symmetric tensor categories

Victor Ostrik

(joint work with K. Coulembier, P. Etingof)

In this talk we consider *pre-Tannakian categories*, that is categories $\mathcal{C}$ with the following properties:

1) $\mathcal{C}$ is rigid symmetric tensor category;

2) $\mathcal{C}$ is abelian and length $l(X)$ of any object $X \in \mathcal{C}$ is finite;

3) $\mathcal{C}$ is linear over some field k, with finite dimensional Hom spaces, and endomorphism algebra of the unit object reduced to k.

Let us assume in addition that the field k is algebraically closed of characteristic $p > 0$. An example of pre-Tannakian category is so called *Verlinde category* $\mathrm{V}er_p$ defined as a semisimplification (i.e. quotient by the negligible morphisms) of the category of representations of the cyclic group of order p. This is semisimple category with $p-1$ isomorphism classes of simple objects.

For $X \in \mathcal{C}$ we set $Fr_+(X) \in X \in \mathcal{C}$ to be the image of divided $p-$th power of X (i.e. invariants of the symmetric group S_p acting on $X^{\otimes p}$) in the symmetric $p-$th power of X (i.e. coinvariants of the symmetric group S_p acting on $X^{\otimes p}$). It is easy to see that the assignment $X \mapsto Fr_+(X)$ is an additive functor; however this functor is not necessarily exact. We say that $\mathcal{C}$ is *Frobenius exact* if the functor Fr_+ is exact. For example any semisimple category is Frobenius exact.

We say that a pre-Tannakian category $\mathcal{C}$ is of *moderate growth* if for any object $X \in \mathcal{C}$, the sequence $l(X^{\otimes n})$ grows no faster than some geometric progression (depending on X). Also for any $X \in \mathcal{C}$ we define its *alternating power* $A^n X$ as the image of the antisymmetrizer acting on $X^{\otimes n}$.

Main Theorem A pre-Tannakian category $\mathcal{C}$ admits an exact symmetric tensor functor $\mathcal{C} \to \mathrm{V}er_p$ if and only if one of the following equivalent conditions is satisfied:

1) $\mathcal{C}$ is Frobenius exact and of moderate growth;

2) $\mathcal{C}$ is Frobenius exact and for any $X \in \mathcal{C}$ we have $A^n X = 0$ for sufficiently large n (depending on X).

This theorem should be compared with well known theorems by Deligne giving a characterization of Tannakian and super Tannakian categories in characteristic zero.

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Subfactors, fusion categories, and symmetric quadrilaterals

Pinhas Grossman

(joint work with Masaki Izumi, Noah Snyder, Scott Morrison, David Penneys, Emily Peters)

A basic problem in subfactor theory is to classify the intermediate subfactors of a subfactor $N \subseteq M$ (in other words, factors P such that $N \subseteq P \subseteq M$). In 1960 Nakamura and Takeda proved the following Galois-type correspondence: for a fixed-point subfactor of an outer action of a finite group G on a factor M, the intermediate subfactors of the fixed-point subfactor $M^G \subseteq M$ are precisely the fixed-point algebras M^H for subgroups H of G [9]. More generally, one can define the Galois group of a subfactor $N \subseteq M$ as the group of automorphisms of M which fix N pointwise.

Sano and Watatani initiated the study of quadrilaterals of subfactors, which are pairs of intermediate subfactors P and Q such that $P \wedge Q = N$ and $P \vee Q = M$ [10].

Such a quadrilateral is said to commute if the four factors form a commuting square with respect to the conditional expectation, and there is also a dual notion of cocommutativity. For the fixed point subfactor of an outer action of a finite group G on M, a quadrilateral coming from a pair of subgroups H and K of G always cocommutes, but only commutes if $HK = G$.

The simplest quadrilaterals are those whose sides (meaning the inclusions $N \subseteq P$, $N \subseteq Q$, $P \subseteq M$, $Q \subseteq M$) have “minimal extra structure”, a condition which can be expressed using the notion of supertransitivity. A subfactor is said to be k-supertransitive iff its principal graph does not have a branch point (i.e. a vertex with valence at least three) within a distance of k from the initial vertex. For a subgroup subfactor $M^G \subseteq M^H$, k-supertransivity for $k \leq 3$ coincides with k-transitivity of the action of G on the coset space G/H. In particular, if $G = S_n$ is the permutation group of a set of size n and H is the stabilizer of a singelton subset, then G acts 3-transitively on G/H, and therefore the corresponding subgroup subfactor is 3-supertransitive. However, a subgroup subfactor of index greater than 3 is never k-supertransitive for $k > 3$.

In [5] and [4] it was show than that there are only two examples of noncommuting irreducible quadrilaterals whose sides are all 4-supertransitive. One of these is also non-cocommuting, and the other one is the fixed point subfactor of an outer S_3 action. On the other hand, there is an infinite family of noncommuting quadrilaterals whose sides are all 3-supertransitive, given by the following construction. Let $G = S_n$ be the permutation group of $\{1, ..., n\}$, and let H, K, and L be the stabilizers of $\{1\}$, $\{2\}$, and $\{1, 2\}$, respectively. Then for an outer action of G on M, the subgroup quadrilateral $M^G \subseteq M^H, M^K \subseteq M^L$ is a noncommuting, cocomuting quadrilateral all of whose sides are 3-supertransitive.

This motivates the following definition:

Definition. A symmetric quadrilateral is a noncommuting, cocomuting irreducible quadrilateral all of whose sides are 3-supertransitive.

The following result was shown in [4]:

Theorem. Let $N \subseteq P, Q \subseteq M$ be a symmetric quadrilateral. Then we have $[P : N] = [Q : N] = [M : P] + 1 = [M : Q] + 1$, and the Galois group of $N \subseteq M$ is a subgroup of S_3; with equality only for the specific example $M^{S_3} \subseteq M$.

Remark. In fact, 3-supertransitivity of the sides is more than is needed; the result was shown under the weaker assumption that all sides of the quadrilateral are 2-supertransitive, and $N \subseteq P$ has trivial second cohomology in the sense of Izumi-Kosaki [8].

Thus aside from the S_3 example, there are exactly three possibilities for the Galois group of a symmetric quadrilateral: namely, it is trivial, $\mathbb{Z}_2$, or $\mathbb{Z}_3$. The purpose of this talk is to describe a coincidence of these three possibilities with certain small-index subfactors.

The principal graphs of subfactors with index less than 4 are simply laced Dynkin diagrams. In the 1990s Haagerup initiated a program to classify subfactors with index slightly larger than 4 [7]. Asaeda and Haagerup discovered two new "exotic" subfactors, known as the Haagerup subfactor (index $\frac{5+\sqrt{13}}{2}$) and the Asaeda-Haagerup subfactor (index $\frac{5+\sqrt{17}}{2}$) [2]. A third exotic subfactor, the Extended Haagerup subfactor, whose index lies in a cubic number field, was subsequently constructed in [3].

The main result in this talk is that each of the Haagerup, Asaeda-Haagerup, and Extended Haagerup subfactors gives rise to a symmetric quadrilateral. Moreover, these quadrilaterals correspond to the three possibilities for the Galois group ($\mathbb{Z}_3$ for the Haagerup quadrilateral, $\mathbb{Z}_2$ for the Asaeda-Haagerup quadrilateral, and trivial for the Extended Haagerup quadrilateral).

A symmetric quadrilateral whose two upper sides are both the Haagerup subfactor was found in [4]. A symmetric quadrilateral are both the Asaeda-Haagerup subfactor was then found in [1]. The Extended Haagerup subfactor, on the other hand, doesn't itself appear as a side of a symmetric quadrilateral. But it turns out that there is a symmetric quadrilateral which is related to the Extended Haagerup subfactor in the following way.

For a finite-index, finite-depth subfactor $N \subseteq M$, the category of N-N-bimodules tensor-generated by ${}_N M_N$ is a fusion category. There is also a corresponding dual fusion category of M-M-bimodules. These two fusion categories are called the even parts of the subfactor, and the N-M-bimodule ${}_N M_M$ generates an invertible bimodule category between the two even parts, which is called a Morita equivalence. Conversely, every object in a Morita equivalence between two unitary fusion categories is realized by a subfactor in this way. Thus a finite-index, finite-depth subfactor corresponds to an object in a Morita equivalence between two fusion categories.

Given a subfactor, it is then natural to ask: what is the entire Morita equivalence class of its even parts, what are all the Morita equivalences between them, and what are all the subfactors associated with these Morita equivalences?

The full Morita equivalence class of the Extended Haagerup subfactor was recently computed in [6]. It turns out that there are four fusion categories in this Morita equivalence class - two corresponding to the even parts of the Extended Haagerup subfactor, and two new ones, constructed using planar algebra methods; and a unique Morita equivalence between each pair. Moreover, we can explicitly describe the multifusion ring which gives the fusion rules between objects in Morita equivalences between these four fusion categories. From this ring, we can extract complete information about intermediate subfactors in this class as follows: an irreducible intermediate subfactor occurs when a simple object in one of the Morita equivalences admits a nontrivial factorization; quadrilaterals occur when the same object admits two different factorizations. In particular, we find a symmetric quadrilateral in the Extended Haagerup class, as well as another quadrilateral with 2-supertransitive sides but otherwise similar properties.

It is striking that the three possibilities for the Galois group of a symmetric quadrilateral are realized by the three exotic subfactors with index less than 5.

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Towards a higher genus Kohno–Drinfeld theorem

Adrien Brochier

Let G be a reductive algebraic group over $\mathbb{C}$. Two ribbon categories over the ring $\mathbb{C}[[\hbar]]$ of formal power series in $\hbar$ are attached to this data:

- The category $\mathrm{Rep}_\hbar\, G$ of integrable representations of the quantum group associated with G. This category has a ribbon structure given by explicit formulas.
- The category $\mathrm{Rep}_\Phi\, G$ whose objects are of the form $V[[\hbar]]$ for an integrable representation V of G and morphisms are morphisms of those, whose tensor product is the usual one but with a highly non trivial ribbon structure induced by a Drinfeld associator Φ.

A Drinfeld associator is a formal power series satisfying a set of algebraic equations modelled on Mac Lane's pentagon equation for monoidal categories. Drinfeld [Dri90, Dri89] shows the set of associator is non-empty using analytic techniques, by constructing a particular associator Φ_{KZ} from the monodromy of the so-called Knizhnik–Zamolodchikov equation.

A fundamental result in quantum algebra is the following:

Theorem (Drinfeld)**.** There is an equivalence of ribbon categories

$$\mathrm{Rep}_\hbar\, G \simeq \mathrm{Rep}_\Phi\, G.$$

This recovers a theorem of Kohno [Koh87] stating that the monodromy of the KZ equation can be computed using quantum groups. In the other direction, this gives a geometric justification of the very existence of the ribbon category $\mathrm{Rep}_\hbar G$. Nowadays, associators are key to the deep relationship between deformation-quantization and higher algebra and topological operads.

To any ribbon category $\mathcal{A}$ is associated a 2d topological field theory that attaches categories to surfaces, carrying actions of the mapping class group and the braid group of the underlying surface. This construction attaches to the disc the category $\mathcal{A}$ itself and can be defined using the general formalism of factorization homology [AF15, CS, Lur09], which in the particular case at hand is closely related to Walker's notion of "skein category" [Wal, Coo19]. In the case $\mathcal{A} = \mathrm{Rep}_\hbar G$, this provides canonical quantizations of character varieties, i.e. canonical deformations of the categories of quasi-coherent sheaves on moduli stacks of G-representations of fundamental groups of surfaces, in the direction of the Atiyah–Bott–Goldman Poisson bracket. These categories were computed by Ben-Zvi–Jordan and myself as categories of modules over well-known and explicit quantum algebras [BZBJ18b, BZBJ18a].

The main goal of this talk is to sketch a construction of higher genus associators, i.e. of certain categories carrying representations of mapping class groups and braid groups of surfaces defined by combinatorial formulas using an associator, and to relate those to the above mentioned construction. It might seem that one should just run the same construction with the category $\mathcal{A} = \mathrm{Rep}_\Phi G$ instead, but it doesn't quite work for an interesting reason. While factorization homology has to do with quantizations of character varieties, the combinatorial construction using associator is related to quantization of moduli stacks of sufficiently regular meromorphic connection on Riemann surfaces. Via the Riemann-Hilbert correspondence, those are Poisson isomorphic to the character varieties, but this isomorphism is non trivial (and only analytic/formal, not algebraic). Therefore, the main technical step in our construction is a quantization of a certain combinatorial version of the Riemann-Hilbert map introduced by Alekseev–Malkin–Meinrenken.

The KZ equation has an analog over any Riemann surface. In genus one, its monodromy has been computed explicitly using associators by Calaque–Enriquez–Etingof. Our construction recovers their formula in that case.

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Lévy-Khinchine decomposition for $SU_q(n)$

Anna Kula

(joint work with Uwe Franz, Martin Lindsay, Michael Skeide)

1. From Lévy processes to Schürmann triples. In the classical probability, Lévy processes form one of the best known clasess of stochastic processes. This class includes Wiener process (Brownian motion), Poisson process, compound Poisson process and many others. A (classical) Lévy process on a given probability space Ω with values in $\mathbb{R}^n$ consists of measurable functions $X_t : \Omega \to \mathbb{R}^n$ $(t \geq 0)$, which are stochastically continuous, whose increments $X_t - X_s$ are independent and stationary, and which start at 0 ($X_0 = 0$ almost everywhere); see for instance [2] for details. If one wants to generalize this definition to processes taking values in more general spaces, the structure of group (or a semi-group at least) on Ω is necessary, to make sense of composition of elements of the process $(X_t + (-X_s))$ and to fix the starting point (the neutral element). A further generalization of the notion of Lévy process to noncommutative setting uses the standard dualization, where one considers functions on a space instead of points, and mappings between spaces become morphisms between algebras over the spaces (with the arrow reversed). All this leads to the definition of noncommutative Lévy process (LP, for short) on a $*$-bialgebra $\mathcal{B}$ with values in a noncommutative probability space $\mathcal{P}$, originally stated in [1], as a family of $*$-homomorphisms from $\mathcal{B}$ to $\mathcal{P}$, satisfying (noncommutative counterpart of) independence and stationarity of increments, together with the weak continuity and the increment property. The latter states that the process starts in the counit and that one can compose increments using the convolution product of bialgebra maps. (More on LPs on $*$-bialgebras can be found for example in [6].)

To any LP one can associate a continuous convolution semigroup of states and its generating functional (GF) ψ, which turns out to satisfy the following three conditions: normalization $\psi(1_{\mathcal{B}}) = 0$, hermitianity $\psi(b^*) = \overline{\psi(b)}$ for $b \in \mathcal{B}$ and conditional positivity $\psi(b^*b) \geq 0$ for $b \in \ker \varepsilon$ ($1_{\mathcal{B}}$ is the unit of $\mathcal{B}$ and ε denotes the counit in $\mathcal{B}$). In 1993, Schürmann [9] showed that to any GF, via a GNS-type construction, one can associate a triple consisting of

- a $*$-representation $\pi : \mathcal{B} \to L(H)$ on some pre-Hilbert space,
- a 1-π-ε-cocycle $\eta : \mathcal{B} \to H$, i.e a linear mapping satisfying $\eta(ab) = \pi(a)\eta(b) + \eta(a)\varepsilon(b)$
- a linear functional $\psi : \mathcal{B} \to \mathbb{C}$ satisfying $\psi(ab) = \varepsilon(a)\psi(b) + \langle \eta(a^*), \eta(b)\rangle + \psi(a)\varepsilon(b)$.

Such triples, called Schürmann triples, are said to be *surjective* if $\eta(\mathcal{B}) = H$.

2. Classification problem and the Lévy-Khinchine decomposition. It turns out that surjective Schürmann triples actually parametrize Lévy processes up to stochastic equivalence. So if one wants to describe all LPs on a given $*$-bialgebra $\mathcal{B}$, one needs to: (1) understand the representation theory of $\mathcal{B}$, which can be a difficult problem (we will not address it here) (2) describe all associated 1-π-ε-cocycles, which bounds to study the first Hochschild cohomology $H^1(\mathcal{B}, {}_\pi H_\varepsilon)$, (3) choose for which pairs (π, η) there exists (at least one) associated GF, (4) study in how many different ways the completion of (π, η) by a ψ can be performed. As for (4), it is known that whenever a pair (π, η) admits a generating functional ψ, all other possible completions are of the form $\psi + \delta$, where δ is a *drift term.* Drifts are particularly easy generating functionals, satisfying $\delta(ab) = \varepsilon(a)\delta(b) + \delta(a)\varepsilon(b)$ (so the associated cocycle is zero), and are classified by $H^1(\mathcal{B},_\varepsilon \mathbb{C}_\varepsilon)$. They are noncommutative counterparts of generators of deterministic Lévy processes. Among the steps (2)-(4) of the classification scheme, the most difficult is the third one. This is due to the potential conflicts caused by relations in $\mathcal{B}$. More precisely, given a representation π and a cocycle η it is necessary to define $\psi(1_\mathcal{B}) = 0$ and $\psi(ab) = \langle \eta(a^*), \eta(b)\rangle$ for $a, b \in \ker\varepsilon$. Such mapping is always normalized, cond. positive and hermitian on $\mathrm{Span}\,(\ker\varepsilon)^2$. But whenever $ab = cd$ for some $a, b, c, d \in \ker\varepsilon$ it must follow that $\langle \eta(a^*), \eta(b)\rangle = \psi(ab) = \psi(cd) = \langle \eta(c^*), \eta(d)\rangle$. If this is not the case, the pair (π, η) does not admits a GF (see [11, Example 2.1]).

An idea how to approach the problem comes from the classical probability: the Lévy-Khinchine formula (and its generalization: the Hunt formula) states that any generator of classical LPs decomposes into the sum of Gaussian part and the jump part, and each of these parts can be parametrized independently. In the noncommutative setting, the notion of gaussian GFs was introduced by Schürmann [8] in order to deal with the question whether a similar decomposition would hold for $*$-bialgebras. Unfortunately, we know since 2015 ([5], see also [4]) that there are $*$-bialgebras for which generating functionals fail to decompose in this way. On the other hand, a list of cases where the (noncommutative analogue of) Lévy-Khinchine decomposition holds include: commutative $*$-bialgebras and the Brown-Glockner-von Waldenfels algebra [8], $SU_q(2)$ [10], the quantum permutation group S_n^+, the quantum reflexion groups H_n^{p+}, the quantum automorphism group of graphs [3], and the universal quantum groups U_F^+ and O_F^+ with F^*F having eigenvalues of multiplicity one [4]. Recently, we showed that also $SU_q(n)$ and $U_q(n)$ with $n \geq 3$ admit the Lévy-Khinchine decomposition and we parametrized all GFs on $SU_q(n)$. Still, it remains open to understand which $*$-bialgebras also have the property called (LK), that any GF on it decomposes into the gaussian part and the remaining (purely non-gaussian) term. Let us note that neither the property (LK) nor its negations transfer to quantum subgroups (quotients of algebras).

3. LK-decomposition in terms of Schürmann triples. Let us first explain that the Schürmann triple associated to a gaussian GF always consists of the representation $\pi_G = \varepsilon(.)\mathrm{id}_H$ and a 1-ε-ε-cocycle η_G (which we call *gaussian* cocycle). Now, if (π, η, ψ) is a Schürmann triple, and H is its representation space, then we can always define its *maximal Gaussian subspace* of H as $H_{\mathrm{G}} := \bigcap_{a \in \ker \varepsilon} \ker \pi(a)$. This is the maximal subspace of H such that $\pi|_{H_G}(a) = \varepsilon(a)\mathrm{id}_{H_G}$. It is reducing for π, hence yields the decomposition $\pi = \pi_{\mathrm{G}} \oplus \pi_{\mathrm{R}}$. Furthermore, if we define P_{G} to be the orthogonal projection onto H_{G}, then $\eta_{\mathrm{G}} := P_{\mathrm{G}} \circ \eta$ is a gaussian cocycle (with values in H_G) and its complement $\eta_{\mathrm{R}} = (I - P_{\mathrm{G}}) \circ \eta$ is a cocycle too, called (*purely non-Gaussian*), and $\eta = \eta_{\mathrm{G}} \oplus \eta_{\mathrm{R}}$. So the LK-decomposition always holds for pairs representation-cocycle: $(\pi, \eta) = (\pi_{\mathrm{G}}, \eta_{\mathrm{G}}) \oplus (\pi_{\mathrm{R}}, \eta_{\mathrm{R}})$.

It is the next step which is crucial in our consideration. If we can define the GFs ψ_{G} and ψ_{R} to complete the two pairs above into Schürmann triples, then we have the LK-decomposition. But we know that in general this can fail. In fact, it is enough to know how to define just one of the ψ's, as the other one will pop up as $\psi - \psi_{\mathrm{x}}$. Because of that, one of possible approaches to the study of the LK-problem relies on showing that either only gaussian or only non-gaussian pairs (π, η) admit completion to Schürmann triples. The first strategy (treating the gaussian pairs) turned out to be fruitful in the case of $SU_q(2)$ [10], whereas the other one was necessary for $SU_q(n)$, $n \geq 3$.

4. LK-decomposition on $SU_q(n)$. The twisted $SU_q(n)$, $q \in (0,1)$, – in its algebraic version – is just the universal unital $*$-algebra generated by the elements of $u = (u_{jk})_{j,k=1}^n$ which satisfy the unitarity relation and the twisted determinant condition, see [12]. Studying the problem of the existence of the LK-decomposition on $SU_q(n)$ for $n \geq 3$ we first showed that, contrary to the case $SU_q(2)$, there exists gaussian pairs (π_G, η_G) on $SU_q(n)$ which do not admit GF. For $SU_q(n)$, we show that gaussian cocycles are necessarily of the form $\eta = \sum_{j=2}^n \varepsilon_j'(.)h_j$, where (for $j = 2, \ldots, n$) h_j are vectors in H and ε_j' are special functionals being like 'a derivation of the counit in the direction of u_{jj}'. Now, a necessary and sufficient condition for η to admit a GF is that $\langle h_j, h_k \rangle \in \mathbb{R}$, and an example for which this does not hold can be easily constructed. This means that if, for a given ψ, we decompose the associated Schürmann pair (π, η) into $(\pi_{\mathrm{G}}, \eta_{\mathrm{G}}) \oplus (\pi_{\mathrm{R}}, \eta_{\mathrm{R}})$, *a priori* we can end up with η_G not admitting GFs.

In order to show that the (LK) property holds for $SU_q(n)$, we focus on the pair $(\pi_{\mathrm{R}}, \eta_{\mathrm{R}})$, which we decompose further into

$$(\pi|_{H_n}, \eta_n) \oplus \cdots \oplus (\pi|_{H_2}, \eta_2),$$

where each $(\pi|_{H_j}, \eta_j)$ lives on the quantum subgroup of smaller size $SU_q(j)$. For the first step we take $H_n := \ker(I - \pi(u_{nn}))^\perp$ and then proceed by induction. To define a GF associated to $(\pi|_{H_j}, \eta_j)$, we use approximation by coboundaries. Coboundaries are 1-π-ε-cocycles which are defined as $\eta(a) = [\pi(a) - \varepsilon(a)I]v$ for some $v \in H$. From our point of view they are particularly nice in the sence that they always admit GFs, the latter being necessarily of the form $\psi(a) = \langle v, [\pi(a) - \varepsilon(a)I]v\rangle$. In our paper we show that each η_j is determined by $\eta(u_{jj}) \in H_j$ via

$$\eta_j(a) = \lim_{p\to 1} \big(\pi(a) - \varepsilon(a)I\big) \underbrace{(I - p\pi(u_{jj}))^{-1}\eta_j(u_{jj})}_{f_{j,p}},$$

so it is a limit of coboundaries. Hence $\psi_j(a) := \langle \lim_{p\to 1}\langle f_{j,p}, [\pi(a) - \varepsilon(a)I]f_{j,p}\rangle$ is a well-defined generating functional, associated to η_j.

Suprisingly (and contrary to the case $n = 2$), not every vector in H may occur as the value of $\eta(u_{nn})$ for a cocycle, but there exists a dense subspace H_0 of H such that any element $f \in H_0$ determines the cocycle η_n (then η_n is necessarily a coboundary). A careful study of this phenomenon yields a parametrization of all GFs on $SU_q(n)$ by the quadruples: (r, B, π, f), where $r \in \mathbb{R}^{n-1}$ and a positive definite matrix $B \in M_{n-1}(\mathbb{R})$ describe the gaussian part of the functional, and the remaining term is uniqely defined by $\pi : \mathrm{Pol}(SU_q(n)) \to B(H)$, a representation with $\pi(u_{nn} - 1)$ injective on H, and $f = (f_2, \dots, f_n)$, a $(n-1)$-tuple of vectors, each f_j in a closure of H_j with respect to the norm $\|f\|_j = \sqrt{\sum_{s=2}^{j}(I - \pi_j(u_{ss})f}$.

A similar method works also for the quantum group $U_q(n)$, and we conjecture that it can be further extend to q-deformations $\mathbb{G}_q$ of other simple compact Lie groups.

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Partition actions of partition quantum groups

Amaury Freslon
(joint work with Franck Taipe and Simeng Wang)

The connection between compact quantum groups (in the sense of [6]) and the combinatorics of partitions has been known since the founding works of T. Banica on the representation theory of the free orthogonal and unitary quantum groups. It was later formalized by T. Banica and R. Speicher in the seminal paper [2], under the name of *easy quantum groups*. We report here on a work in progress to extend the combinatorial description to the level of ergodic actions of quantum groups, providing a unified framework of several families of such actions.

Let us start by recalling the fundamental construction. A *category of partitions* is a collection $\mathcal{C}$ of partitions of finite sets which is stable under several operations defined through a graphical representation : horizontal concatenation (denoted by $\otimes$), vertical concatenation (denoted by $\circ$), reflection (denoted by $*$) and rotation. To make sense of these, we draw the partitions with two parallel horizontal rows of points with strands connecting points if an only if they belong to the same subset of the partition. Then, we denote by $P(k, \ell)$ the set of all partitions with k upper points and ℓ lower points. Given any integer N, there is a canonical way to turn a category of partitions $\mathcal{C}$ into a concrete rigid C*-tensor category, and the machinery of Tannaka-Krein duality [7] then produces a compact quantum group $\mathbb{G}_N(\mathcal{C})$.

An ergodic action of a compact quantum group $\mathbb{G}$ on a unital C*-algebra is a coaction map $\alpha : A \to A \otimes C(\mathbb{G})$ such that

$$\{x \in A \mid \alpha(x) = x \otimes 1\} = \mathbb{C}.1_A.$$

Such a map naturally yields a functor on the representation category of $\mathbb{G}$, sending an irreducible representation v to is isotypical component in A. Moreover, this functor has some compatibility with the tensor product operation on representations, making it a *weak tensor functor*. It was proven by C. Pinzari and J. Robert in [5] (see also [4]) that the data of such a functor is enough to recover the action.

Based on the previous result, we can now build actions through functors by working at the level of partitions. Let us briefly sketch two different constructions of that type. First, recall that a partition p is said to be *projective* if $p.p = p = p^*$. Given a category of partitions $\mathcal{C}$, we call a set of projective partitions $\mathcal{P}$ a *module of projective partitions over* $\mathcal{C}$ if it satisfies the following assumptions :

(1) $\mathcal{P}_k \otimes \mathcal{P}_\ell \subset \mathcal{P}_{k+\ell}$ for all $k, \ell \in \mathbb{N}$ (here $\mathcal{P}_n$ is the subset of elements of $\mathcal{P}$ with n points in each row) ;
(2) $r \circ p \circ r^* \in \mathcal{P}_\ell$ for all $p \in \mathcal{P}_k$ and $r \in \mathcal{C}(k, \ell)$.

One of our main results is that given a category of partitions $\mathcal{C}$, a module of projective partitions $\mathcal{P}$ over $\mathcal{C}$ and an integer N, there exists a canonical associated weak tensor functor yielding an ergodic action of $\mathbb{G}_N(\mathcal{C})$. Moreover, if $\mathcal{C}$ consists of non-crossing partitions and if $\mathcal{P}$ is the set of all projective partitions in $\mathcal{C}$, then the action is the one corresponding to the first row space of $\mathbb{G}_N(\mathcal{C})$ (see [3] for details on this notion).

Another construction uses partitions lying on one line instead of projective ones. More precisely, a *module of line partitions over* $\mathcal{C}$ is a set of partitions $\mathcal{L}$ with only lower points such that

- $\mathcal{L}_k \otimes \mathcal{L}_\ell \subset \mathcal{L}_{k+\ell}$ for all $k, \ell \in \mathbb{N}$;
- $r \circ p \in \mathcal{L}_\ell$ for all $p \in \mathcal{L}_k$ and $r \in \mathcal{C}(k, \ell)$.

Once again, this produces an action and we prove that for any inclusion of categories of partitions (without any assumption on crossings) $\mathcal{C} \subset \mathcal{C}'$, setting

$$\mathcal{L}_k = \mathcal{C}'(0, k)$$

yield the action of $\mathbb{G}_N(\mathcal{C})$ on the homogeneous space $\mathbb{G}_N(\mathcal{C})/\mathbb{G}_N(\mathcal{C}')$. This result can be proven through a theorem of [5], but we provide another, more elementary proof using induction for actions of compact quantum groups.

Surprisingly, modules of line partitions can also be "shifted" in some cases, meaning that $\mathcal{L}_k$ will consist in partitions with more than k points. The main problem then is to define the weak tensor structure. We are able to do this in some cases, thereby recovering some finite-dimensional actions of certain quantum permutation groups, as well as inductions thereof.

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Open Problems Section

Actions of free orthogonal quantum groups

Kenny De Commer

For $q \in \mathbb{R} \setminus \{0\}$, let $\mathscr{F}_q$ be the set of finite square matrices F satisfying $F\overline{F} = -\mathrm{sgn}(q)$ and $\mathrm{Tr}(F^*F) = q + q^{-1}$, and write N_F for the dimension of the matrix F. Construct for $F, G \in \mathscr{F}_q$ the unital $*$-algebra [BDRV06]

$$\mathcal{O}(\mathbb{F}O_{F,G}) = \{u_{ij} \mid 1 \leq i \leq N_F, 1 \leq j \leq N_G, U \text{ unitary and } F\overline{U}G^{-1} = U\}.$$

There are unital $*$-homomorphisms

$$\Delta^H_{F,G} : \mathcal{O}(\mathbb{F}O_{F,G}) \to \mathcal{O}(\mathbb{F}O_{F,H}) \otimes \mathcal{O}(\mathbb{F}O_{H,G}), \qquad u_{ij} \mapsto \sum_{k=1}^{N_H} u_{ik} \otimes u_{kj},$$

The $\mathcal{O}(\mathbb{F}O_F) = \mathcal{O}(\mathbb{F}O_{F,F})$ are the Hopf $*$-algebras corresponding to free orthogonal quantum groups [VDW96, Ban96], and for $F = \begin{pmatrix} 0 & -\mathrm{sgn}|q|^{1/2} \\ |q|^{1/2} & 0 \end{pmatrix}$ we get the Hopf $*$-algebra of quantum $SU(2)$ [Wor87].

The $*$-algebras $\mathcal{O}(\mathbb{F}O_{F,G})$ admit a unique state $h_{F,G} : \mathcal{O}(\mathbb{F}O_{F,G}) \to \mathbb{C}$ such that

$$(\mathrm{id} \otimes h_{F,G})(\Delta^F_{F,G}(x)) = (h_{F,G} \otimes \mathrm{id})(\Delta^G_{F,G}(x)) = h_{F,G}(x)1, \qquad x \in \mathcal{O}(\mathbb{F}O_{F,G}).$$

Denote $L^\infty(\mathbb{F}O_{F,G})$ for the von Neumann algebra completion of $\mathcal{O}(\mathbb{F}O_{F,G})$ inside its GNS-representation with respect to $h_{F,G}$. The comultiplication maps above extend to normal unital $*$-homomorphisms

$$\Delta^H_{F,G} : L^\infty(\mathbb{F}O_{F,G}) \to L^\infty(\mathbb{F}O_{F,H}) \overline{\otimes} L^\infty(\mathbb{F}O_{H,G}).$$

If $\alpha_F : M_F \to M_F \overline{\otimes} L^\infty(\mathbb{F}O_F)$ is a coaction of $L^\infty(\mathbb{F}O_F)$ on a von Neumann algebra M_F, we obtain a coaction $\alpha_G = (\mathrm{id} \otimes \Delta^G_{F,G})_{|M_G}$ of $L^\infty(\mathbb{F}O_G)$ on the von Neumann algebra

$$M_G := \{z \in M_F \overline{\otimes} L^\infty(\mathbb{F}O_{F,G}) \mid (\alpha_F \otimes \mathrm{id})z = (\mathrm{id} \otimes \Delta^F_{F,G})z\},$$

giving natural equivalences between categories of coactions [DRVV06], preserving the ergodicity condition

$$\alpha(z) = z \otimes 1 \qquad \Leftrightarrow \qquad z \in \mathbb{C}1.$$

An ergodic coaction (M_F, α_F) of $L^\infty(\mathbb{F}O_F)$ is *embeddable* if we have a normal equivariant embedding $\iota : M_F \subseteq L^\infty(\mathbb{F}O_F)$. Then $\iota(M_F) \subseteq L^\infty(\mathbb{F}O_F)$ is a right coideal von Neumann subalgebra.

The ergodic coactions of the $L^\infty(\mathbb{F}O_F)$ were classified in [DCY15] in terms of certain weighted graphs, extending the classification of A. Wassermann for classical $SU(2)$ [Was88]. Here are two questions concerning these coactions.

- If (M_F, α_F) is an ergodic coaction of $L^\infty(\mathbb{F}O_F)$, does there exist $G \in \mathscr{F}_q$ such that M_G is equivariantly W*-Morita equivalent with a coideal von Neumann subalgebra of $L^\infty(\mathbb{F}O_G)$? If not, can such ergodic coactions be nicely characterized? Note that given a concrete pair of ergodic coactions, checking if one embeds into the other can be reduced to solving a series of quadratic equations [DCY15, Section 6]. In terms of the associated tensor C*-categories, we are asking which connected module C*-categories for the Temperley-Lieb tensor C*-category $\mathcal{T}_q$ factor through a fiber functor for $\mathcal{T}_q$.
- If (M, α) is an ergodic coaction of $L^\infty(SU_q(2))$ and M is a type I von Neumann algebra, is M equivariantly W*-Morita equivalent with a coideal von Neumann subalgebra of $L^\infty(SU_q(2))$? Note that the question has a positive answer if formulated on the level of the associated C*-algebras [DCY15, Proposition 7.2]. Since one knows explicitly the structure of the von Neumann algebras of coideals of quantum $SU(2)$ [Tom08], this would classify all projective unitary representations of quantum $SU(2)$, that is, coactions of $L^\infty(SU_q(2))$ on type I-factors $B(\mathcal{H})$ [DC11], which is currently also still an open problem.

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Open problem

Makoto Yamashita

Give a universal algebraic construction of coideals: in [T], Tamarkin gave a new construction of Hopf algebraic deformation of universal enveloping algebras quantizing Lie bialgebras based on formality of the E_2-operad. Can we give a similar construction of coideals, that correspond to the Letzter coideals, based on the structure of Swiss cheese operad, or some modification thereof? (It is too naive to expect formality for the latter operad as M.Y. learned from A. Brochier, and we should try to combine some geometric structure of U/K to further simplify cohomological structures.)

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Residual nilpotence of the augmentation ideal

Amaury Freslon
(joint work with Uwe Franz and Adam Skalski)

Let A be a Hopf algebra and let I be its augmentation ideal. If I^n denotes the n-th power of I, we denote by

$$I^\infty = \bigcap_{n\in\mathbb{N}} I^n$$

the nilpotent residual of I. This is a Hopf ideal in A and the general question is:

Question : For which A does $I^\infty = \{0\}$? Or equivalently, what is A/I^∞ in terms of A ?

Here are some facts we know concerning this question

- If $A = \mathcal{O}(G)$ for a classical compact group G, then $I^\infty = \{0\}$ if and only if G is connected.
- If $A = \mathbb{C}[\Gamma]$ for a discrete group Γ, then $I^\infty = \{0\}$ if and only if Γ is residually torsion-free nilpotent.
- If A has non-trivial idempotents, then $I^\infty \neq \{0\}$.

We are in fact mainly interested in the case where A is the Hopf $*$-algebra associated with a compact quantum group (see [3]). Then, one can prove that A/I^∞ is of Kac type which means that for q-deformations all the "q-deformed" relations should be killed by the quotient. This suggests to focus on free quantum groups (see [2]), with the main question being

Question : What is I^∞ for $\mathcal{O}(O_N^+)$?

For U_N^+, we have $I^\infty = \{0\}$ because the quantum group is topologically generated (see [1]) by U_N (which is connected) and the dual of the free group $\widehat{\mathbb{F}}_N$ (which is residually torsion free nilpotent). This leads us to the final problem:

Question : Does there exist a discrete group Γ which is residually torsion-free nilpotent and such that O_N^+ is topologically generated by SO_N and $\widehat{\Gamma}$?

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Open problem

Nicolás Andruskiewitsch

Question. (N.A.) Find a genuine Nichols algebra over a non-abelian group whose support is not abelian, it has infinite dimension and finite GK-dimension. (Here genuine means excluding any trivial example). For instance, take the rack of transpositions in the symmetric groups in 3 letters. Taking the constant cocycle equal to -1, the corresponding Nichols algebra has dimension 12 (It is the Fomin-Kirillov algebra FK_3). What is the Nichols algebra corresponding to the constant cocycle equal to 1?

Open problems

Eric C. Rowell

1. Let R be a *unitary* solution to the (constant) Yang-Baxter equation, and $\rho^R : B_n \to GL(V^{\otimes n})$ the braid group representation obtained from R. Conjecture: the image $\rho^R(B_n)$ is a virtually abelian group (i.e. abelian-by-finite).
2. (due to Czenky and Plavnik) Let $\mathcal{C}$ be an odd dimensional (integral) modular category. Then $\mathcal{C}$ has a non-trivial invertible object.

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