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Enveloping Algebras and Geometric Representation Theory (hybrid meeting)

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Abstract. The workshop brought together experts investigating algebraic Lie theory from the geometric and categorical viewpoints.

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Introduction by the Organizers

This workshop continues a series of conferences on enveloping algebras, as the first part of the title suggests, but the focus of these meetings and also the organisers have changed over the years to reflect the newest developments in the field of algebraic Lie theory. This year the main focus was on geometric and categorical methods, with an eye to explicit and combinatorial formulas.

This time, because of the pandemic, the workshop had to be held in hybrid format. 31 participants came to Oberwolfach and attended the meeting in person, while 25 participants took part online. Among a total of 25 talks (20 long talks, 5 short talks), 15 were given by in-person participants, and 10 by online participants. The high quality of the new equipment of MFO allowed for a smooth communication between in-person and online participants. Wednesday afternoon was reserved for a walk to Oberwolfach.

A major theme of the conference focussed on Coulomb branches, affine Grassmannians and the geometric Satake correspondence, with lectures by Finkelberg, Braverman, Nakajima, Kamnitzer, and Patimo. Connections with the geometry of curves and surfaces were prominent with the work of Davison and Schiffmann on Cohomological Hall Algebras. The classical topic of Springer theory and affine

Hecke algebras is always very active, as could be seen from the talks of Vasserot, Stroppel, Liu, and Bittmann. Quantum groups, quantum affine algebras and their generalizations continues to be a core subject of algebraic Lie theory, with various viewpoints presented by Hernandez, Negut, Fujita, Schiffmann, and Kim, as well as, of course, representation theory of Lie algebras and algebraic groups, with beautiful talks by Serganova, Achar, Losev, and Kumar. Williamson gave a remarkable lecture on his recent progress on the conjecture of combinatorial invariance of Kazhdan-Lusztig polynomials.

All participants in Oberwolfach were happy to be able to interact again during breaks and evenings, after many months of online communication only. We all hope that for the next workshop it will be possible to go back to the traditional Oberwolfach format.

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Workshop (hybrid meeting): Enveloping Algebras and Geometric Representation Theory

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Abstracts

The perverse filtration on the preprojective stack and the enveloping algebra of the BPS Lie algebra

Ben Davison

Setup. Throughout, Q will denote a finite quiver, i.e. a pair of finite sets Q_1 and Q_0 of arrows and vertices (respectively), along with two morphisms $s, t\colon Q_1 \to Q_0$ between them. We form the doubled quiver $\overline{Q}$ by adding an arrow a^* for each arrow $a \in Q_1$, with a^* given the opposite orientation to a. Let $\mathbf{f} \in \mathbb{N}^{Q_0}$ be a dimension vector. We define the *framed quiver* $Q_{\mathbf{f}}$ by setting $(Q_{\mathbf{f}})_0 = Q_0 \cup \{\infty\}$, and then adding $\mathbf{f}_i$ arrows from ∞ to i for each $i \in Q_0$. We write dimension vectors for $Q_{\mathbf{f}}$ in the form $(\mathbf{d}, n)$, where $\mathbf{d}$ is a dimension vector for Q and $n \in \mathbb{N}$.

For a quiver Q', we define the preprojective algebra

$$\Pi_{Q'} = \mathbb{C}\overline{Q'}/\langle \sum_{a\in Q'_1} [a, a^*]\rangle.$$

We denote by $X^{\zeta}_{(\mathbf{d},1)} \coloneqq X^{\zeta}_{(\mathbf{d},1)}(\Pi_{Q_{\mathbf{f}}})$ the moduli space of stable $(\mathbf{d}, 1)$-dimensional $\Pi_{Q_{\mathbf{f}}}$-modules ρ, where stability is the condition that the one-dimensional vector space associated to the vertex ∞ generates ρ. This variety is smooth. We denote by $X_{(\mathbf{d},1)} \coloneqq X_{(\mathbf{d},1)}(\Pi_{Q_{\mathbf{f}}})$ the coarse moduli space of $(\mathbf{d}, 1)$-dimensional $\Pi_{Q_{\mathbf{f}}}$-modules. We denote by $q\colon X^{\zeta}_{(\mathbf{d},1)} \to X_{(\mathbf{d},1)}$ the natural (GIT) morphism. Finally we denote by $X_{\mathbf{d}} \coloneqq X_{\mathbf{d}}(\Pi_Q)$ the coarse moduli space of $\mathbf{d}$-dimensional Π_Q-modules. This is an affine variety, whose points are in bijection with semisimple $\mathbf{d}$-dimensional Π_Q-modules. Where an object $\mathcal{Y}_{\mathbf{d}}$ is defined with respect to a dimension vector $\mathbf{d}$ for the fixed quiver Q, we denote by $\mathcal{Y}_{\bullet}$ the coproduct across all $\mathbf{d} \in \mathbb{N}^{Q_0}$.

We consider the monoidal structure $\mu\colon X_{\bullet} \times X_{\bullet} \to X_{\bullet}$, taking pairs of semisimple Π_Q-modules to their direct sum. The morphism μ is finite, meaning that the convolution tensor product on the category $\mathrm{Perv}(X_{\bullet})$ defined by $\mathcal{F} \boxtimes_{\mu} \mathcal{G} = \mu_*(\mathcal{F} \boxtimes \mathcal{G})$ is bi-exact, making $\mathrm{Perv}(X_{\bullet})$ into a tensor category. The infinite union of varieties $X_{(\bullet,1)}$ is a module for $X_{\bullet}$, again via the morphism that sends a pair of modules to their direct sum. Again, this morphism is finite, so that $\mathrm{Perv}(X_{(\bullet,1)})$ is a module category for the tensor category $\mathrm{Perv}(X_{\bullet})$. An *algebra object* in $\mathrm{Perv}(X_{\bullet})$ is a triple $(\mathcal{F} \in \mathrm{Perv}(X_{\bullet}), m\colon \mathcal{F} \boxtimes_{\mu} \mathcal{F} \to \mathcal{F}, \eta\colon \mathbb{Q}_{X_0} \to \mathcal{F})$ satisfying the usual axioms, while a module in $\mathrm{Perv}(X_{(\bullet,1)})$ over a given algebra object $\mathcal{F}$ is a pair $(\mathcal{G} \in \mathrm{Perv}(X_{(\bullet,1)}), \mathcal{F} \boxtimes_{\mu} \mathcal{G} \to \mathcal{G})$ satisfying the usual axioms.

For each $\mathbf{d} \in \mathbb{N}^{Q_0}$ there is a $0_{\mathbf{d}} \in X_{\mathbf{d}}$ given by letting all arrows in $\overline{Q}$ act by zero. We denote by $\iota\colon 0_{\bullet} \to X_{\bullet}$ the inclusion of monoids, and by $s\colon X_{\bullet} \to 0_{\bullet}$ the projection. Since ι is finite, we obtain a tensor functor $\iota_*\colon \mathrm{Perv}(0_{\bullet}) \to \mathrm{Perv}(X_{\bullet})$, a right inverse to the (derived) functor $\mathrm{H} = s_*\colon \mathrm{Perv}(X_{\bullet}) \to \mathcal{D}(\mathrm{Vect}_{\mathbb{N}^{Q_0}})$.

Examples. **(1)** For X smooth, we denote by $\mathbb{Q}_{X,\mathrm{vir}}$ the constant sheaf on X, shifted so that it is perverse. Assume for now that Q has no loops. Then by [5], $\mathrm{H}(X^{\zeta}_{(\bullet,1)}, \mathbb{Q}_{\mathrm{vir}})$ carries a $\mathfrak{n}^-_Q$-action, which lifts to a $\iota_*\mathfrak{n}^-_Q$-action on $q_*\mathbb{Q}_{X^{\zeta}_{(\bullet,1)},\mathrm{vir}} =:$

$\mathcal{F}$. Moreover, $\mathcal{F}$ is a semisimple perverse sheaf, the action of $\iota_*\mathfrak{n}_Q^-$ preserves the dimensions of the supports, and the submodule of $\mathcal{F}$ with zero-dimensional supports is equal to $\iota_* L_{\mathbf{f}}$, the highest weight module determined by the framing vector $\mathbf{f}$.

(2) Let Q be the Jordan quiver, i.e. the quiver with one vertex and one loop. Then $X^\zeta_{(d,1)} \cong \mathrm{Hilb}_d(\mathbb{A}^2)$, $X_d \cong \mathrm{Sym}^d(\mathbb{A}^2)$, and $\mathrm{H}(X^\zeta_{(\bullet,1)}, \mathbb{Q}_{\mathrm{vir}}) \cong \mathrm{Sym}(\bigoplus_{d\geq 1} \mathbb{Q}[2])$ is a free $\mathrm{U}(\mathbf{heis}^+)$-module, where $\mathbf{heis}^+ = \bigoplus_{d\geq 1} \mathbb{Q}[2]$ is made into a Lie algebra by giving it the zero Lie bracket. By a result of Göttsche and Soergel [4], there is an isomorphism

$$q_*\mathbb{Q}_{X^\zeta_{(\bullet,1)},\mathrm{vir}} \cong \mathrm{Sym}_{\boxtimes_\mu}\left(\bigoplus_{d\geq 1} \Delta_{d,*}\mathbb{Q}_{\mathbb{A}^2,\mathrm{vir}}\right); \qquad \Delta_d \colon \mathbb{A}^2 \hookrightarrow \mathrm{Sym}^d(\mathbb{A}^2).$$

Furthermore there is a lift of the $\mathbf{heis}^+$ action to an action of $\bigoplus_{d\geq 1} \Delta_{d,*}\mathbb{Q}_{\mathbb{A}^2,\mathrm{vir}}$ (with the trivial bracket) on $q_*\mathbb{Q}_{X^\zeta_{(\bullet,1)},\mathrm{vir}}$.

Hall algebras. Let $\mathfrak{M}_{\mathbf{d}}(\Pi_Q)$ denote the stack of $\mathbf{d}$-dimensional Π_Q-modules, and denote by $\mathtt{JH}\colon \mathfrak{M}_\bullet(\Pi_Q) \to X_\bullet(\Pi_Q)$ the semisimplification morphism. In the above examples, the action is provided by picking certain classes in $\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_\bullet(\Pi_Q), \mathbb{Q}_{\mathrm{vir}})$ and letting them act via correspondences. Via [6] the object $\mathcal{A}_{\Pi_Q} := \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_\bullet(\Pi_Q), \mathbb{Q}_{\mathrm{vir}})$ itself carries a Hall algebra structure. The algebras that we recalled above, acting on the cohomology of Nakajima quiver varieties, are subalgebras of these Hall algebras. So a natural question is: what is the structure of the entire algebra $\mathcal{A}_{\Pi_Q}$? Or, more ambitiously, can we describe the algebra object $\underline{\mathcal{A}}_{\Pi_Q} := \mathtt{JH}_*\mathbb{Q}_{\mathrm{vir}}$? This is an algebra object in $\mathcal{D}^+(\mathrm{Perv}(X_\bullet))$, which lifts the object $\mathcal{A}_{\Pi_Q}$ in the sense that there is an isomorphism of $\mathbb{N}^{Q_0}$-graded algebras $\mathrm{H}(\underline{\mathcal{A}}_{\Pi_Q}) \cong \mathcal{A}_{\Pi_Q}$.

Theorem 1. *Let $\mathfrak{L}^0\mathcal{A}_{\Pi_Q} \subset \mathcal{A}_{\Pi_Q}$ be the subalgebra of "perverse-exact raising operators" (to be introduced shortly). Then $\mathfrak{L}^0\mathcal{A}_{\Pi_Q} \cong \mathrm{U}(\mathfrak{g}_{\tilde{Q},\tilde{W}})$ where $\mathfrak{g}_{\tilde{Q},\tilde{W}}$ is the BPS Lie algebra* [3] *for the tripled quiver $\tilde{Q}$ obtained by adding a loop ω_i at each vertex of the doubled quiver $\overline{Q}$, and $\tilde{W} = \sum_{a\in Q_1}[a,a^*]\sum_{i\in Q_0}\omega_i$. Furthermore the $\mathfrak{g}_{\tilde{Q},\tilde{W}}$-action on $\mathrm{H}^{\mathrm{BM}}(X^\zeta_{(\bullet,1)}, \mathbb{Q}_{\mathrm{vir}})$ lifts to a $\underline{\mathfrak{g}}_{\Pi_Q}$ action on $q_*\mathbb{Q}_{X^\zeta_{(\bullet,1)},\mathrm{vir}}$, where $\underline{\mathfrak{g}}_{\Pi_Q}$ is a Lie algebra object in $\mathrm{Perv}(X_\bullet)$ satisfying*

(0) *There is an isomorphism of Lie algebras $\mathrm{H}(\underline{\mathfrak{g}}_{\Pi_Q}) \cong \mathfrak{g}_{\tilde{Q},\tilde{W}}$.*
(1) *The perverse sheaf $\underline{\mathfrak{g}}_{\Pi_Q}$ is semisimple (and lifts to a pure weight zero mixed Hodge module).*
(2) *The Lie subalgebra of $\underline{\mathfrak{g}}_{\Pi_Q}$ given by summands with zero-dimensional support is isomorphic to $\iota_*\mathfrak{n}^-_{Q'}$, where Q' is obtained from Q by removing all vertices supporting loops.*
(3) *There is an isomorphism $\mathrm{H}^0(\underline{\mathfrak{g}}_{\Pi_Q}) \cong \mathfrak{n}^-_{Q'}$.*

(4) *There is an equality* $\sum_{i\in\mathbb{Z}}\dim(\mathrm{H}^i(\underline{\mathfrak{g}}_{\Pi_Q}))q^{i/2} = \mathtt{a}_{Q,\mathbf{d}}(q^{-1})$*, where* $\mathtt{a}_{Q,\mathbf{d}}(q)$ *is the Kac polynomial, counting isomorphism classes of absolutely indecomposable* $\mathbf{d}$*-dimensional* $\mathbb{F}_qQ$*-modules.*

The decomposition theorem for 2-Calabi–Yau categories. To define $\mathfrak{L}^0\mathcal{A}_{\Pi_Q}$ we appeal to the following general result from [2]:

Theorem 2. *Let* $\mathscr{C}$ *be a "nice" 2-Calabi–Yau category (examples: semistable Higgs bundles of fixed slope, Gieseker semistable coherent sheaves on a K3 surface* S *with fixed slope, modules over a fixed preprojective algebra* Π_Q*). Let* $p\colon \mathfrak{M}_{\mathscr{C}} \to X_{\mathscr{C}}$ *be the morphism from the stack of objects in* $\mathscr{C}$ *to the good moduli space. Then*

$$p_*\mathbb{D}\mathbb{Q}_{\mathrm{vir}} \cong \bigoplus_{i\in 2\cdot\mathbb{Z}_{\geq 0}} {}^{\mathfrak{p}}\mathcal{H}^i(p_*\mathbb{D}\mathbb{Q}_{\mathrm{vir}})[-i]$$

and each cohomology sheaf is semisimple.

It follows that ${}^{\mathfrak{p}}\tau^{\leq 0}\mathtt{JH}_*\mathbb{Q}_{\mathrm{vir}}$ is an algebra object in $\mathrm{Perv}(X_\bullet)$. In fact, it is isomorphic to the universal enveloping algebra object $\mathrm{U}(\underline{\mathfrak{g}}_{\Pi_Q})$, and by definition, $\mathfrak{L}^0\mathcal{A}_{\Pi_Q} = \mathrm{H}({}^{\mathfrak{p}}\tau^{\leq 0}\mathtt{JH}_*\mathbb{Q}_{\mathrm{vir}})$. Theorem 1 is proved by realising the category of Π_Q-modules as the "dimensional reduction" of the category of $\Pi_Q[x]$-modules, and then studying the cohomological DT theory of this $\underline{3}$CY category: see [1] for details.

Generators. Our sheaf-theoretic analysis of the algebra $\mathcal{A}_{\Pi_Q}$ enables us to say something concrete about the generators of $\mathfrak{g}_{\Pi_Q}$. Expanding upon the semisimplicity statement of Theorem 2, we have the following theorem:

Theorem 3. *Let* $\mathbf{d} \in \mathbb{N}^{Q_0}$ *be such that there exists a simple* $\mathbf{d}$*-dimensional* Π_Q*-module. Then there is a canonical decomposition*

$$\underline{\mathfrak{g}}_{\Pi_Q,\mathbf{d}} \cong \mathcal{IC}_{X_{\mathbf{d}}}(\mathbb{Q}_{\mathrm{vir}}) \oplus C$$

and the Lie bracket $\underline{\mathfrak{g}}_{\Pi_Q,\mathbf{d}'} \boxtimes_\mu \underline{\mathfrak{g}}_{\Pi_Q,\mathbf{d}''} \to \underline{\mathfrak{g}}_{\Pi_Q,\mathbf{d}}$ *for* $\mathbf{d}'+\mathbf{d}''=\mathbf{d}$ *factors through the inclusion of* C*. We deduce that there is a canonical decomposition*

$$\mathfrak{g}_{\Pi_Q,\mathbf{d}} \cong \mathrm{IC}(X_{\mathbf{d}}) \oplus \mathrm{H}(C)$$

and the Lie bracket for the BPS Lie algebra factors through the inclusion of $\mathrm{H}(C)$*.*

In words, the theorem states that the intersection cohomology of $X_{\mathbf{d}}$ forms a canonical subspace of a space of primitive generators in degree $\mathbf{d}$ for $\mathfrak{g}_{\tilde{Q},\tilde{W}}$. We conjecture that $\mathrm{IC}(X_{\mathbf{d}})$ is a *complete* space of primitive degree $\mathbf{d}$ generators.

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Representations of shifted quantum groups (or quantized Coulomb branches) and Baxter polynomials

David Hernandez

We explain the application of polynomiality of Q-operators to representations of truncated shifted quantum affine algebras (quantized K-theoretical Coulomb branches). The Q-operators are transfer-matrices associated to prefundamental representations of the Borel subalgebra of a quantum affine algebra, via the standard R-matrix construction. In a joint work with E. Frenkel, we have proved that, up to a scalar multiple, they act polynomially on simple finite-dimensional representations of a quantum affine algebra. This establishes the existence of Baxter polynomial in a general setting. In the framework of the study of K-theoretical Coulomb branches, Finkelberg-Tsymbaliuk introduced remarkable new algebras, the shifted quantum affine algebras and their truncations. We propose a conjectural parametrization of simple modules of a non simply-laced truncation in terms of the Langlands dual quantum affine Lie algebra (this has various motivations, including the symplectic duality relating Coulomb branches and quiver varieties). We prove that a simple finite-dimensional representation of a shifted quantum affine algebra descends to a truncation as predicted by this conjecture. This is derived from the existence of Baxter polynomial.

Shifted quantum affine algebras and their truncations arose [FT] in the study of quantized K-theoretic Coulomb branches of 3d $N = 4$ SUSY quiver gauge theories in the sense of Braverman-Finkelberg-Nakajima which are at the center of current important developments. A presentation of (truncated) shifted quantum affine algebras by generators and relations was given by Finkelberg-Tsymbaliuk.

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra, and $\hat{\mathfrak{g}}$ the corresponding untwisted affine Kac-Moody algebra, central extension of the loop algebra $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$. Drinfeld and Jimbo associated to each complex number $q \in \mathbb{C}^*$ a Hopf algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ called a quantum affine algebra. Shifted quantum affine algebras $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ can be seen as variations of $\mathcal{U}_q(\hat{\mathfrak{g}})$, but depending on two coweights μ_+, μ_- of the underlying simple Lie algebra $\mathfrak{g}$. These coweights corresponding to shifts of formal power series in the Cartan-Drinfeld elements (that is quantum analogs of the $t^r h \in \mathcal{L}\mathfrak{g}$, with $r \in \mathbb{Z}$ and $h \in \mathfrak{h}$ in the Cartan subalgebra of $\mathfrak{g}$). In particular $\mathcal{U}_q^{0,0}(\hat{\mathfrak{g}})$ is a central extension of the ordinary quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$. Up to isomorphism, $\mathcal{U}_q^{\mu_+,\mu_-}(\hat{\mathfrak{g}})$ only depends on $\mu = \mu_+ + \mu_-$ and will be denoted simply by $\mathcal{U}_q^{\mu}(\hat{\mathfrak{g}})$.

The truncations are quotients of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ and depend on additional parameters, including a dominant coweight λ and additional "flavour" parameters (which are complex numbers). In this context, these parameters λ and μ can be interpreted as parameters for generalized slices of the affine Grassmannian $\overline{\mathcal{W}}_\mu^\lambda$ (usual slices when μ is dominant).

In [H], we develop the representation theory of shifted quantum affine algebras. We establish several analogies with the representation theory of ordinary quantum affine algebras, but our approach is also based on several techniques. In particular, we relate these representations to representations of the Borel subalgebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$. Consider a category $\mathcal{O}_\mu$ of representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ which is an analog of the ordinary category $\mathcal{O}$. We obtain induction/restriction functors to the category $\mathcal{O}$ of $\mathcal{U}_q(\hat{\mathfrak{b}})$-modules.

For general untwisted types, the category $\mathcal{O}$ of representations of the quantum affine Borel algebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ was introduced and studied in [HJ]. Some representations in this category extend to a representation of the whole quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$, but many do not, including the prefundamental representations constructed in [HJ] and whose transfer-matrices have remarkable properties for the corresponding quantum integrable systems [FH].

Let us now discuss truncated shifted quantum affine algebras, quotients of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$. For simply-laced types, simple representations of truncated shifted Yangians have been parametrized in terms of Nakajima monomial crystals [KTWWY]. See the Introduction of [H] for comments on related results in [NW].

We will use Baxter polynomiality in quantum integrable systems for an approach to general types. Let us recall that to each representation V of $\mathcal{U}_q(\hat{\mathfrak{b}})$ in the category $\mathcal{O}$, is attached a transfer-matrix $t_V(z)$ which is a formal power series in a formal parameter z with coefficients in $\mathcal{U}_q(\hat{\mathfrak{g}})$. Given another simple finite-dimensional representation W of $\mathcal{U}_q(\hat{\mathfrak{g}})$, we get a family of commuting operators on $W[[z]]$. This is a quantum integrable model generalizing the XXZ-model. It is established in [FH], the spectrum of this system, that is the eigenvalues of the transfer-matrices, can be described in terms of certain polynomials, generalizing Baxter's polynomials associated to the XXZ-model. These Baxter's polynomials are obtained from the eigenvalues of transfer-matrices associated to prefundamental representations of $\mathcal{U}_q(\hat{\mathfrak{b}})$. Moreover, this Baxter polynomiality implies the polynomiality of certain series of Cartan-Drinfeld elements acting on finite-dimensional representations [FH]. We relate this result to the structures of representations of truncated shifted quantum affine algebras. In particular, we give in [H] a uniform proof of the finiteness of the number of simple isomorphism classes for truncations.

In non-simply-laced types, we propose a parametrization of these simple representations. We use a limit obtained from interpolating (q,t)-characters. The latter were defined by Frenkel and the author as an incarnation of Frenkel-Reshetikhin deformed $\mathcal{W}$-algebras interpolating between q-characters of a non simply-laced quantum affine algebra and its Langlands dual. They lead to the definition of an interpolation between the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}}))$ of finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ (at $t=1$) and the Grothendieck ring $\mathrm{Rep}(\mathcal{U}_t({}^L\hat{\mathfrak{g}}))$ of finite-dimensional representations of the Langlands dual algebra quantum affine algebra $\mathcal{U}_t({}^L\hat{\mathfrak{g}})$ (at $q = \epsilon$ a certain root of 1) :

$$\begin{array}{ccc} & \mathfrak{K}_{q,t} & \\ {\scriptstyle t=1} \swarrow & & \searrow {\scriptstyle q=\epsilon} \\ \mathrm{Rep}(\mathcal{U}_q(\hat{\mathfrak{g}})) & & \mathrm{Rep}(\mathcal{U}_t({}^L\hat{\mathfrak{g}})) \end{array}$$

Here $\mathfrak{K}_{q,t}$ is the ring of interpolating (q, t)-characters.

To describe our parametrization, we found it is relevant to use a different specialization of interpolating (q, t)-characters that we call Langlands dual q-characters (with $t = 1$ for variables but $q = \epsilon$ for coefficients).

The interpolating (q, t)-characters are closely related to the deformed $\mathcal{W}$-algebras which appeared in the context of the geometric Langlands program. Note also that the parametrization in [KTWWY] for simply-laced types can be understood in the context of symplectic duality (more precisely from the equivariant version of the Hikita conjecture for the symplectic duality formed by an affine Grassmannian slice and a quiver variety). Hence the statement of our conjecture is also motivated by the symplectic duality and the Langlands duality.

Conjecture [H] *The simple modules of a truncation are explicitly parametrized by monomials in the Langlands dual q-character of a finite-dimensional representation of the Langlands dual quantum affine algebra.*

Based on results by Nakajima, Nakajima-Weekes [NW] gave a bijection between more general simple representations of a non simply-laced quantized Coulomb branch and those for simply-laced types and so a parametrization of simple representations in category $\mathcal{O}$ of truncated non simply-laced shifted Yangians (and quantum affine algebras). After using the comparison between simply-laced and twisted q-characters by the author, one can consider a possible relation between the two parametrizations. In small examples discussed in a correspondence between Nakajima, this different method seems to give the same parametrization as our result.

A main evidence for the Conjecture is the following, obtained as a consequence of the Baxter polynomiality.

Theorem [H] *A finite-dimensional simple representation in* $\mathcal{O}_\mu$ *descends to a certain explicit truncation as predicted by the Conjecture.*

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Lyndon words and quantum loop groups

Andrei Negut
(joint work with Alexander Tsymbaliuk)

Fix a finite type root system, with simple roots $\{\alpha_i\}_{i\in I}$ and set of positive roots Δ^+. Take the positive half of the corresponding quantum group:

$$U_q(\mathfrak{g}) \supset U_q(\mathfrak{n}^+) = \mathbb{Q}(q)\langle e_i\rangle_{i\in I}\Big/(q\text{-Serre relation})$$

By [3, 8, 10], one may define the following shuffle algebra, built on "words" $[i_1 \dots i_k]$ where $i_1, \dots, i_k \in I$ (so the elements $i \in I$ are considered the "letters"):

$$\mathcal{F} = \bigoplus_{i_1,\dots,i_k \in I} \mathbb{Q}(q) \cdot [i_1 \dots i_k]$$

with multiplication given by:

$$[i_1 \dots i_k] * [i_{k+1} \dots i_{k+l}] = \sum_{\text{shuffles } \sigma} [i_{\sigma(1)} \dots i_{\sigma(k+l)}] \cdot q^{\text{some integer}}$$

where the word "shuffle" refers to those permutations $\sigma \in S(k+l)$ which have the property that $\sigma^{-1}(a) < \sigma^{-1}(b)$ if $1 \le a < b \le k$ or $k < a < b \le k+l$. The motivation behind this construction is that one has an algebra homomorphism:

$$\Phi : U_q(\mathfrak{n}^+) \hookrightarrow \mathcal{F}, \qquad e_i \mapsto [i], \ \forall i \in I \tag{1}$$

This homomorphism encapsulates a number of features of quantum groups. Specifically, by work of [4], there exists a bijection:

$$\ell : \Delta^+ \xrightarrow{\sim} \{\text{standard Lyndon words}\} \tag{2}$$

(we will not review the properties of a word being "standard" or "Lyndon"). By [5, 9], this bijection can be recovered via the shuffle algebra incarnation (1): for any positive root β, there is a unique (up to scalar multiple) element:

$$e_\beta \in U_q(\mathfrak{n}^+) \tag{3}$$

such that $\Phi(e_\beta)$ has minimal leading word (here we must fix a total order on I, and work with the induced lexicographic order on words) among all elements of Im Φ has degree β. Then *loc. cit.* show that this leading word is precisely $\ell(\beta)$, and e_β is a renormalization of Lusztig's root vector (defined via braid group actions, [6]).

In joint work with Alexander Tsymbaliuk, we provide a "loop version" of the results above, i.e. one pertaining to the positive half of the quantum loop group:

$$U_q(L\mathfrak{g}) \supset U_q(L\mathfrak{n}^+) = \mathbb{Q}(q)\langle e_{i,d}\rangle_{i\in I, d\in\mathbb{Z}}\Big/(\text{relations})$$

In [7], we consider "letters" as being $[i^{(d)}]$ for all $i \in I$ and $d \in \mathbb{Z}$, and define:

$$\mathcal{F}^L = \bigoplus_{\substack{i_1,\dots,i_k \in I \\ d_1,\dots,d_k \in \mathbb{Z}}} \mathbb{Q}(q) \cdot [i_1^{(d_1)} \dots i_k^{(d_k)}]$$

with multiplication given by:

$$[i_1^{(d_1)} \dots i_k^{(d_k)}] * [i_{k+1}^{(d_{k+1})} \dots i_{k+l}^{(d_{k+l})}] =$$
$$= \sum_{\substack{\text{shuffles } \sigma \\ \pi_1+\dots+\pi_{k+l}=0}} [i_{\sigma(1)}^{(d_{\sigma(1)}+\pi_1)} \dots i_{\sigma(k+l)}^{(d_{\sigma(k+l)}+\pi_{k+l})}] \cdot (\text{some scalar } \in \mathbb{Q}(q))$$

The shuffle algebra above is defined so that we have an algebra homomorphism:

$$\Phi^L : U_q(L\mathfrak{n}^+) \hookrightarrow \mathcal{F}^L, \qquad e_{i,d} \mapsto [i^{(d)}], \ \forall i \in I, d \in \mathbb{Z} \tag{4}$$

We also prove analogues of the bijection (2) to the loop setup, as well as characterizations of root vectors in the quantum loop group in terms of the homomorphism (4), by analogy with the discussion involving (3).

The main application of the homomorphism (4) is to connect the shuffle algebra $\mathcal{F}^L$ with another type of shuffle algebra, which was related to quantum loop groups in [1] (and motivated by the elliptic algebras defined by [2]):

$$\mathcal{A}^+ \hookrightarrow \bigoplus_{\{k_i\}_{i\in I} \in \mathbb{N}^I} \mathbb{Q}(q)(\dots, z_{i1}, \dots, z_{ik_i}, \dots)^{\text{sym}}$$

$$\mathcal{A}^+ = \left\{ \frac{r(\dots, z_{i1}, \dots, z_{ik_i}, \dots)}{\prod_{1 \le i < i' \le I} \prod_{a \le k_i, a' \le k_{i'}} (z_{ia} - z_{i'a'})} \right\}$$

where above, r goes over all symmetric (in the variables z_{ia} for each i separately) Laurent polynomials which satisfy the wheel conditions:

$$r\Big|_{(z_{i1},\dots,z_{i,1-a_{ij}})=z_{j1}(q^{a_{ij}},\dots q^{a_{ij}+2},\dots,q^{-a_{ij}-2},q^{-a_{ij}})} = 0$$

for all $i \neq j$ in I, where $\{a_{ij}\}_{i,j\in I}$ is the Cartan matrix corresponding to our finite type root system. The multiplication in $\mathcal{A}^+$ is given by the formula:

$$R(\dots, z_{i1}, \dots, z_{ik_i}, \dots) * R(\dots, z_{i1}, \dots, z_{ik'_i}, \dots) =$$

$$\text{Sym}\left[\frac{R(\dots, z_{i1}, \dots, z_{ik_i}, \dots)R'(\dots, z_{i,k_i+1}, \dots, z_{i,k_i+k'_i}, \dots)}{\prod_{i\in I} k_i! k'_i!} \prod_{\substack{i,i' \in I \\ a \le k_i, a' > k'_{i'}}} \frac{z_{ia} - q^{-(\alpha_i, \alpha_{i'})} z_{i'a'}}{z_{ia} - z_{i'a'}}\right]$$

(above, $(\cdot,\cdot)$ denotes the scalar product in our given root system). As before, the multiplication above is designed so that one has an algebra homomorphism:

$$\Upsilon : U_q(L\mathfrak{n}^+) \to \mathcal{A}^+, \qquad e_{i,d} \mapsto z_{i1}^d, \ \forall i \in I, d \in \mathbb{Z} \tag{5}$$

whose injectivity was proved in [11], but whose surjectivity was until now an open problem. Besides, one may ask what is the connection between (4) and (5). To answer these questions, in [7] we show that the assignment:

$$R \mapsto \sum_{\substack{i_1,\dots,i_k \in I \\ d_1,\dots,d_k \in \mathbb{Z}}} [i_1^{(d_1)} \dots i_k^{(d_k)}] \int_{|x_1| \ll \dots \ll |x_k|} \frac{R(x_1,\dots,x_k) x_1^{-d_1} \dots x_k^{-d_k}}{\prod_{1 \le a < b \le k} \frac{x_a - q^{-(\alpha_{i_a}, \alpha_{i_b})} x_b}{x_a - x_b}} \prod_{a=1}^{k} \frac{dx_a}{2\pi i x_a}$$

(in the right-hand side, we plug each x_a in place of an argument of the form $z_{i_a \bullet}$ of R, where the choice of bullet is immaterial due to the symmetry of R) yields an algebra homomorphism:

$$\iota : \mathcal{A}^+ \to \mathcal{F}^L$$

which is compatible with (4) and (5) in the sense that:

$$\iota \circ \Upsilon = \Phi^L$$

Using this algebra homomorphism, we prove in [7] the following.

Theorem: The map (5) is an isomorphism.

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Green correspondence for supergroups

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(joint work with Inna Entova-Aizenbud, Alexander Sherman)

Let G be an algebraic supergroup over $\mathbb{C}$ with Lie superalgebra $\mathfrak{g}$. We call G quasireductive if the underlying group G_0 is reductive. The category $\operatorname{Rep} G$ of finite-dimensional representations of a quasireductive group G has enough projective and injective objects, [3]. By $\overline{\operatorname{Rep}}G$ we denote the category of all representations of G.

If $K \subset G$ is a quasireductive supergroup then the induction functor

$$\operatorname{Ind}_K^G : \operatorname{Rep} K \to \overline{\operatorname{Rep}}G$$

is exact since G/K is an affine supervariety. We call a quasireductive $K \subset G$ *splitting* if the natural map $i : \mathbb{C} \to \operatorname{Ind}_K^G \mathbb{C}$ splits, i.e. there exists a morphism $p : \operatorname{Ind}_K^G \mathbb{C} \to \mathbb{C}$ of G-modules such that $p \circ i = \operatorname{Id}$.

It is not difficult to see that the following conditions on a quasireductive $K \subset G$ are equivalent:

(1) K is a splitting subgroup of G.
(2) For any G-module, the restriction morphism:

$$\operatorname{Ext}^1_G(M, \mathbb{C}) \to \operatorname{Ext}^1_K(M, \mathbb{C})$$

is injective.
(3) For any pair of G-modules M, M', where M' is finite-dimensional, the restriction morphism

$$\operatorname{Ext}^i_G(M, M') \to \operatorname{Ext}^i_K(M, M')$$

is injective for all i.

The following properties of splitting subgroups are straightforward.

Lemma 1. (a) If $H \subset K$ and $K \subset G$ are both splitting then H is splitting in G.
(b) $M \in \operatorname{Rep} G$ is projective iff $\operatorname{Res}_K M$ is projective in $\operatorname{Rep} K$.
(c) The restriction functor $\operatorname{Res}_K : \operatorname{Rep} G \to \operatorname{Rep} K$ maps negligible morphisms to negligible morphisms.

Example 1. Let us assume that G is a finite group and the ground field has a positive characteristic p. Then a subgroup $K \subset G$ is splitting iff K contains a Sylow p-subgroup of G.

We get the following necessary geometric condition for splitting. We call an odd element $x \in \mathfrak{g}_1$ semisimple if $[x, x] \in \mathfrak{g}_0$ is semisimple.

Proposition 1. Let K be splitting in G. Then the G_0-orbit of every odd semisimple $x \in \mathfrak{g}_1$ intersects $\mathfrak{k}_1$.

The key ingredient in the proof of Proposition 1 is a generalization of so called DS-functor. Let $M \in \operatorname{Rep} G$, x be an odd semisimple element in $\mathfrak{g}$, set

$$DS_x M := \ker x_M / (\operatorname{im} x_M \cap \ker x_M).$$

The correspondence $M \to DS_x M$ defines a symmetric monoidal functor from $\operatorname{Rep} G$ to the category of super vector spaces. If $G_0 x \cap \mathfrak{k}_1 = \emptyset$ then the odd vector field x on G/K does not have zeros. In this case $DS_x \mathcal{O}(G/K) = 0$. On the other hand, $\mathcal{O}(G/K) = \mathbb{C} \oplus N$ implies $DS_x \mathcal{O}(G/K) \neq 0$ and we obtain a contradiction.

Description of splitting subgroups of G seems to be important for finding a geometric projectivity criterion and building theory of support for quasireductive supergroups. We found a nice splitting subgroup for $G = GL(m|n)$ which can be consider as an analogue of Sylow subgroup in Example 1.

Theorem 1. Let $G = GL(m|n)$ with $m \geq n$ and $H = SL(1|1)^n \subset G$ be the natural embedding by block matrices. Then H is splitting in G.

Furthermore, we conjecture that every splitting subgroup K of $GL(m|n)$ contains a subgroup conjugate to H.

The proof of Theorem 1 uses a few ingredients. Firstly, we use the transitivity property Lemma 1(a) to reduce the proof to the following question. We fix $p \leq m$ and $q \leq n$ and consider $K = GL(p|q) \times GL(m-p|n-q) \subset G$. When is K is splitting?

To answer this question we use the compact form (unitary supergroup) $U(m|n)$ of $GL(m|n)$ and the supergrassmannian

$$Gr(p|q, m|n) = U(m|n)/(U(p|q) \times U(m-p|n-q))$$

of $(p|q)$-dimensional subspaces in $\mathbb{C}^{m|n}$. Note that $Gr(p|q, m|n)$ is a compact real subsupermanifold in G/K. It is not difficult to see that $Gr(p|q, m|n)$ admits $U(m|n)$-invariant volume form ω. The map $p : \mathcal{O}(G/K) \to \mathbb{C}$ given by

$$p(f) = \int_{Gr(p|q,m|n)} f\omega,$$

where $\int$ stands for the Berezin integral, is a morphism of $GL(m|n)$-modules. Furthermore, $p \circ i \neq 0$ if and only if the volume of $Gr(p|q, m|n)$ is not zero. We use the Schwarz–Zaboronsky localization formula, [4], to obtain the following.

Theorem 2. The volume of $Gr(p|q, m|n)$ iff $p - q \leq m - n$ and $(m-p) - (n-q) \leq m - n$ (as before, we assume $m \geq n$).

Theorem 2 confirms the conjecture of T. Voronov, [5].

Our next result gives the following analogue of the Green correspondence for finite groups in positive characteristics, see for example [1], 2.12. Let $H \subset G$ be as in Theorem 1 and K denote the normalizer of H in G. Note that K is isomorphic to a semidirect product of the symmetric group S_n and $GL(m-n) \times GL(1|1)^n$. For an arbitrary symmetric monoidal rigid category $\mathcal{C}$ let $I^+(\mathcal{C})$ denote the set of isomorphism classes of indecomposable objects of non-zero categorical dimension.

Conjecture. (a) If $M \in I^+(\operatorname{Rep} G)$ then $\operatorname{Res}_K M$ has a unique up to isomorphism indecomposable summand $M' \in I^+(\operatorname{Rep} K)$. The correspondence $M \mapsto M'$ establishes the bijection between $I^+(\operatorname{Rep} G)$ and $I^+(\operatorname{Rep} K)$.

Recall that if $\mathcal{C}$ is a symmetric monoidal rigid category then its semisimplification $\mathcal{C}_{ss}$ is the quotient of $\mathcal{C}$ by the tensor ideal of negligible morphisms. The category $\mathcal{C}_{ss}$ is a semisimple symmetric monoidal rigid category and isomorphism classes of simple objects of $\mathcal{C}_{ss}$ are in bijection with $I^+(\mathcal{C})$, see [2] for details. Then Lemma 1(c) and Theorem 3 imply the following

Corollary. The symmetric monoidal rigid categories $\operatorname{Rep} G_{ss}$ and $\operatorname{Rep} K_{ss}$ are equivalent.

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The center of the small quantum group and cohomology of an affine Springer fiber

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(joint work with Roman Bezrukavnikov, Pablo Boixeda Alvarez, Peng Shan)

Let $\mathrm{u}_\zeta \subset \mathrm{U}_\zeta$ be the small and the big (Lusztig) quantum group for a quasi-simple, connected, simply connected linear group $\check{G}$ at a root of unity ζ of order ℓ, where ℓ is odd, greater than the Coxeter number, and coprime to the determinant of the Cartan matrix.

Both u_ζ and U_ζ split as a direct sum indexed by elements of $\check{\Lambda}/W_{\ell,\mathrm{af}}$ where Λ is the weight lattice, let u_ζ^ω, U_ζ^ω denote the corresponding summands. Let $Z(\mathrm{u}_\zeta^\omega)$ be the center of u_ζ^ω. The Morita class of u_ζ^ω depends only on the stabilizer of ω in the affine Weyl group, hence so does the ring $Z(\mathrm{u}_\zeta^\omega)$. If J is a proper subset in the set Σ_{af} of vertices in the affine Dynkin graph of G, we set $Z(\mathrm{u}_\zeta^J) = Z(\mathrm{u}_\zeta^\omega)$ if the stabilizer of ω is generated by simple reflections s_α with $\alpha \in J$.

Notice that in view of Lusztig's Frobenius homomorphism (in the version by Beck and Kirillov) the group $\check{G}$ acts on $Z(\mathrm{u}_\zeta^\omega)$ for all ω. It follows from the definitions that $Z(\mathrm{u}_\zeta^\omega)^G$ is the center of the big quantum group U_ζ intersected with u_ζ^ω.

Let $\mathcal{F}\ell = G(\mathbb{C}((t)))/\mathcal{I}$ denote the affine flag variety of the Langlands dual group G. For each set $J \subset \Sigma_{\mathrm{af}}$ of affine simple reflections, let P^J be the corresponding (standard) parahoric and $\mathcal{F}\ell^J = G(\mathbb{C}((t)))/P^J$ be the corresponding partial affine flag variety. In particular, we let $\mathcal{G}\mathfrak{r} = \mathcal{F}\ell^{J_{-\check{\rho}}}$ be the affine Grassmannian, where $J_{-\check{\rho}}$ is the set of vertices of the finite Dynkin graph, and $\mathcal{F}\ell = \mathcal{F}\ell^{J_0}$ be the affine flag variety, where $J_0 = \emptyset$.

We fix an element $\gamma = ts$ where $s \in \mathfrak{g}$ is regular semisimple. Let $\mathcal{F}\ell_\gamma \subset \mathcal{F}\ell$ be the corresponding affine Springer fiber and $\mathcal{F}\ell_\gamma^J \subset \mathcal{F}\ell^J$ be the corresponding affine Spaltenstein fiber, i.e., $\mathcal{F}\ell_\gamma^J$ is the set of points $x \in \mathcal{F}\ell^J$ such that γ lies in the pro-unipotent radical ${}^0Stab(x)$ of $Stab(x)$. The coweight lattice Λ of G is identified with a subgroup in the centralizer of γ, so it acts on $\mathcal{F}\ell_\gamma^J$ and hence on its cohomology. This action is called the left action.

The left action on $H^\bullet(\mathcal{F}\ell_\gamma)$ and $H^\bullet(\mathcal{F}\ell_\gamma^J)$ extends to an action of the affine Weyl group W_{af} where the finite Weyl group acts by monodromy via the map $\pi_1(\mathfrak{g}^{rs}) \to W$.

Conjecture. *We have canonical isomorphisms*

$$H^\bullet(\mathcal{F}\ell_\gamma^J)^{\check{\Lambda}} \cong Z(\mathrm{u}_\zeta^J)^{\check{T}} \quad , \quad H^\bullet(\mathcal{F}\ell_\gamma^J)^{W_{\mathrm{af}}} \cong Z(\mathrm{u}_\zeta^J)^{\check{G}}.$$

Corollary. *Consider the $\mathbb{C}^\times$-action on $\mathcal{G}\mathfrak{r}$ by loop rotation. Then we have a canonical isomorphism $H^\bullet({}^0\mathcal{G}\mathfrak{r}^\zeta_{\gamma_\ell})^\Lambda \cong Z(\mathrm{u}_\zeta)^T$ and $H^\bullet({}^0\mathcal{G}\mathfrak{r}^\zeta_{\gamma_\ell})^{W_{\mathrm{af}}} \cong Z(\mathrm{u}_\zeta)^G$.*

We will check (based on results of the second author with Losev) that

$$\dim(H^\bullet(\mathcal{F}\ell_\gamma)^{W_{\mathrm{af}}}) = (h+1)^r$$

where h is the Coxeter number and r is the rank and also that for G of type A we have $H^\bullet(\mathcal{F}\ell_\gamma)^{W_{\mathrm{af}}} = H^\bullet(\mathcal{F}\ell_\gamma)^{\check{\Lambda}}$. Thus the conjecture implies that the action of $\check{G}$ on the center is trivial if $\check{G}$ is of type A and that conjectures [2, conj. 4.9], [3, conj. 3.14] hold.

We now state our results in the direction of the Conjecture. In particular, we define an injective map from the LHS to the RHS of both isomorphisms.

1.1. **The deformed setting.** A standard way to compute $H^\bullet(\mathcal{F}\ell_\gamma)$ is based on considering the *equivariant* cohomology $H^\bullet_T(\mathcal{F}\ell_\gamma)$. We now describe a variation of the conjecture involving $H^\bullet_T(\mathcal{F}\ell_\gamma)$. We'll focus on the case $J = \emptyset$, hence $\mathcal{F}\ell^J = \mathcal{F}\ell$.

We proceed to define a flat Λ-graded algebra $A_\mathfrak{t}$ over $R = \mathrm{k}[\mathfrak{t}]$ such that $A = A_\mathfrak{t} \otimes_R (R/\mathfrak{t}R)$ is Morita equivalent (as a Λ-graded algebra) to u_ζ^0. We sketch two ways to define $A_\mathfrak{t}$.

1) Let $\mathfrak{U}_\zeta$ be the Kac-De Concini quantum group. Recall that the center of $\mathfrak{U}_\zeta$ contains $\mathrm{k}[\check{G}^*]$ where $\check{G}^*$ is the Poisson dual group to $\check{G}$. We have $\mathfrak{U}_\zeta \otimes_{\mathrm{k}[\check{G}^*]} \mathrm{k}[\{1\}] = \mathrm{u}_\zeta$. Consider the completion of $\mathfrak{U}_{\zeta,\mathfrak{t}} = \mathfrak{U}_\zeta \otimes_{\mathrm{k}[G^*]} \mathrm{k}[\check{T}]$ at the central character corresponding to $1 \in \check{T} \subset \check{G}^*$. It splits as a direct sum indexed by the summands of u_ζ, we let $\hat{A}_\mathfrak{t}$ be the summand corresponding to a regular weight λ. This is an algebra over the completion $\hat{R}$ of R at $0 \in \mathfrak{t}$, which we identify with the completion of $\check{T}$ at 1. Next, one can show that $\hat{A}_\mathfrak{t}$ has an equivariant structure for the dilation action of $\mathbb{G}_m$ on $\hat{R}$ and let $A_\mathfrak{t}$ be the algebra of $\mathbb{G}_m$-finite elements in $\hat{R}$.

2) Let $\mathbf{A}$ be the noncommutative Springer resolution. This is a ring over $\mathrm{k}[\check{\mathfrak{t}}]$ together with a $\check{G}$-action, it comes equipped with a derived equivalence $D^b(\mathbf{A}\text{ mod }) \cong D^b(Coh(\tilde{\mathfrak{g}}))$. We set $A_\mathfrak{t} = \mathbf{A} \otimes_{\mathcal{O}(\check{\mathfrak{t}})} \mathcal{O}(\check{\mathfrak{g}})$.

Let $Z_\mathfrak{t}$ denote the center of $A_\mathfrak{t}$. Let $Z_\mathfrak{t}$ and $Z(\mathrm{u}_\zeta^0)$ be the center of the categories of Λ-graded modules over $A_\mathfrak{t}$ and A (or equivalently u_ζ^0) respectively.

Theorem. *We have $Z_\mathfrak{t} \cong H^\bullet_T(\mathcal{F}\ell_\gamma)$ and $Z \cong H^\bullet_T(\mathcal{F}\ell_\gamma)^\Lambda$.*

Here the second equation follows directly from the first one. The latter is one of the main results of this paper, it will be deduced from the GKM description of equivariant cohomology and considerations of the action of central elements on the deformed baby Verma modules.

One can check that $H^\bullet(\mathcal{F}\ell_\gamma) \cong H^\bullet_T(\mathcal{F}\ell_\gamma) \otimes_{H^\bullet_T} \mathrm{k}$ while the tensor product $Z_t \otimes_R k$ admits an injective map to $Z(\mathrm{u}^0_\zeta)$. Thus we get

Corollary. *We have an injective map* $H^\bullet(\mathcal{F}\ell_\gamma) \to Z(\mathrm{u}^0_\zeta)$.

It is easy to see that $Z(\mathrm{u}^0_\zeta)^{\check{\Lambda}} \cong Z(\mathrm{U}_\zeta)^{\hat{0},\check{T}}$, thus we get an injective map

$$H^\bullet(\mathcal{F}\ell_\gamma)^{\check{\Lambda}} \to Z(\mathrm{U}_\zeta)^{\hat{0},\check{T}}. \tag{1}$$

1.2. **Sheaves on $\mathcal{G}\mathfrak{r}$.** Let $Rep(\mathrm{U}_\zeta)^0$ be a regular block of integrable modules. We have a map

$$Z(\mathrm{U}_\zeta)^{\hat{0},\check{G}} \subset End(Id_{Rep(\mathrm{U}_\zeta)^0}). \tag{2}$$

Recall the equivalence $Rep(\mathrm{U}_\zeta)^0 \cong Perv_{^0\mathcal{I}}(\mathcal{G}\mathfrak{r})$ in [1]. The pull-back functor is a full embedding $Perv_{^0\mathcal{I}}(\mathcal{G}\mathfrak{r}) \to Perv_{^0\mathcal{I}}(\mathcal{F}\ell)$ inducing a homomorphism

$$End(Id_{Perv_{^0\mathcal{I}}(\mathcal{F}\ell)}) \to End(Id_{Perv_{^0\mathcal{I}}(\mathcal{G}\mathfrak{r})}) = End(Id_{Rep(\mathrm{U}_\zeta)^0}). \tag{3}$$

Using (the proof of) Koszul duality [BY] one gets an isomorphism

$$H^\bullet(\mathcal{F}\ell) \cong End(Id_{Perv_{^0\mathcal{I}}(\mathcal{F}\ell)}). \tag{4}$$

Proposition. *The image of the map* $H^\bullet(\mathcal{F}\ell) \to End(Id_{Rep(\mathrm{U}_\zeta)^0})$ *obtained by composing* (3), (4) *is contained in the image of* (2).

Checking this amounts to checking compatibility of the action of $H^\bullet(\mathcal{F}\ell)$ with the tensor action of the Satake category $Perv_{G(\mathbb{C}[[t]])}(\mathcal{G}\mathfrak{r})$ on $Perv_{^0\mathcal{I}}(\mathcal{G}\mathfrak{r})$. The Claim yields a map

$$H^\bullet(\mathcal{F}\ell) \to Z(\mathrm{U}_\zeta)^{\hat{0},\check{G}}. \tag{5}$$

Proposition. *The map* (5) *coincides with the composition of* (1) *with the restriction map* $H^\bullet(\mathcal{F}\ell) \to H^\bullet(\mathcal{F}\ell_\gamma)$.

The restriction map $H^\bullet(\mathcal{F}\ell) \to H^\bullet(\mathcal{F}\ell_\gamma)$ lands in W_{af}-invariants. It follows from the results of the second author with Losev that the map $H^\bullet(\mathcal{F}\ell) \to H^\bullet(\mathcal{F}\ell_\gamma)^{W_{\mathrm{af}}}$ is onto. Thus we get an injective map from $H^\bullet(\mathcal{F}\ell_\gamma)^{W_{\mathrm{af}}}$ to $Z(\mathrm{U}_\zeta)^{\hat{0},\check{G}}$.

1.3. **An alternative description of the map.** The map (5) admits an alternative description which will be used in the proof of the Proposition above. Let Z_{HC} be the center of the quantum group $\mathfrak{U}_q$ viewed as a ring over $\mathbf{A}$ given by the standard generators and relations. We have $Z_{HC} \otimes \mathbf{F} \cong \mathbf{F}[\mathfrak{t}]^W$. The map $Z_{HC}/(q-\zeta) \to Z(\mathrm{U}_\zeta)^{\hat{0},\check{G}}$ factors through the completion at the regular point $\lambda \in \mathfrak{t}^*/W$. This completion is isomorphic to $\mathrm{k}[[\mathfrak{t}]]$. The map above factors further through a map $\mathrm{k}[[\mathfrak{t}]]/(\mathrm{k}[[\mathfrak{t}]]^W_+) \cong H^\bullet(G/B) \to Z(\mathrm{U}_\zeta)^{\hat{0},\check{G}}$.

Recall that $H^\bullet(\mathcal{F}\ell) = H^\bullet(G/B) \otimes H^\bullet(\mathcal{G}\mathfrak{r})$. The map (5) restricted to the first factor coincides with the map described in the previous paragraph. To describe the restriction of (5) to the second factor recall that $H^\bullet(\mathcal{G}\mathfrak{r})$ is a completion of polynomial ring whose generators are in bijection with generators of the ideal $\mathrm{k}[[\mathfrak{t}]]^W_+ \subset \mathrm{k}[[\mathfrak{t}]]$. Let $z \in (\mathrm{k}[[\mathfrak{t}]]^W_+)$ be a generator and $h_z \in H^\bullet(\mathcal{G}\mathfrak{r})$ the corresponding generator in $H^\bullet(\mathcal{G}\mathfrak{r})$. Let $\tilde{z} \in Z_{HC}$ whose specialization to $q = \zeta$ maps to z under the isomorphism between the completion of $Z_{HC}/(q-\zeta)$ and $\mathrm{k}[[\mathfrak{t}]]$. We claim that the element $\tilde{z}/(q-\zeta)$ has a well defined reduction in the Lusztig quantum group U_ζ. Moreover, this reduction lies in $Z(\mathrm{U}_\zeta)^0$. The resulting central element is the image of h_z under (5).

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Gaiotto conjecture on quantum geometric Satake for quantum supergroups

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(joint work with Alexander Braverman, Roman Travkin)

1.1. **Geometric Satake equivalence and FLE.** Let $\mathbf{F} = \mathbb{C}(\!(t)\!) \supset \mathbb{C}[\![t]\!] = \mathbf{O}$. Let G be a connected reductive group over $\mathbb{C}$. Let $\mathrm{Gr}_G = G(\mathbf{F})/G(\mathbf{O})$ be the affine Grassmannian of G. One can consider the category $\mathrm{Perv}_{G(\mathbf{O})}(\mathrm{Gr}_G)$ of $G(\mathbf{O})$-equivariant perverse sheaves on Gr_G. This is a tensor category over $\mathbb{C}$. The geometric Satake equivalence identifies this category with the category $\mathrm{Rep}(G^\vee)$ of finite-dimensional representations of the Langlands dual group $G^\vee$.

The above equivalence is very important for many applications (e.g. it is in some sense the starting point for the geometric Langlands correspondence) but at the same time it has two serious drawbacks:

1) It does not hold on the derived level. In fact, the derived Satake equivalence [1] does provide a description of the derived category $D_{G(\mathbf{O})}(\mathrm{Gr}_G)$ in terms of $G^\vee$ but the answer is certainly not the derived category of $\mathrm{Rep}(G^\vee)$.

2) For many reasons it would be nice to generalize the above equivalence so that the category $\mathrm{Rep}(G^\vee)$ gets replaced with the category $\mathrm{Rep}_q(G^\vee)$ — the category of finite-dimensional representations of the corresponding quantum group. But it seems that it is impossible to find such a generalization.

On the other hand, J. Lurie and D. Gaitsgory found a replacement of the geometric Satake equivalence (called the Fundamental Local Equivalence, or FLE) where both of the above problems disappear.

Namely, let U be a maximal unipotent subgroup of G, and let $\overline{\chi}\colon U \to \mathbb{G}_a$ be its generic character. Let $\chi\colon U(\mathbf{F}) \to \mathbb{G}_a$ be given by the formula $\chi(u(t)) = \mathrm{Res}_{t=0}\overline{\chi}(u(t))$. Let now $\mathrm{Whit}(\mathrm{Gr}_G)$ be the derived category of $(U(\mathbf{F}),\chi)$-equivariant sheaves on Gr_G. Then this category is equivalent to $D(\mathrm{Rep}(G^\vee))$. Moreover, this statement can be generalized to an equivalence between the category $D(\mathrm{Rep}_q(G^\vee))$ and the corresponding category $\mathrm{Whit}_q(\mathrm{Gr}_G)$ (sheaves twisted by the corresponding complex power of a certain determinant line bundle on Gr_G). This equivalence preserves the natural t-structures on both sides.

1.2. **Gaiotto conjectures.** Let now $G = \mathrm{GL}(N)$. In this case D. Gaiotto constructed certain series of subgroups of G endowed with an additive character, which in many respects resemble the pair $(U, \overline{\chi})$ above. Namely, fix $M < N$. Consider the natural embedding of $\mathrm{GL}(M)$ into $\mathrm{GL}(N)$. Then one can construct (see [2, §2]) a unipotent subgroup $U_{M,N}$ of $\mathrm{GL}(N)$ that is normalized by $\mathrm{GL}(M)$, and a character $\overline{\chi}_{M,N}\colon U_{M,N} \to \mathbb{G}_a$ which fixed by the adjoint action of $\mathrm{GL}(M)$ such that conjecturally for generic q we have an equivalence (the notation is explained below):

$$SD_{\mathrm{GL}(M,\mathbf{O})\widetilde{\ltimes}(U_{M,N}(\mathbf{F}),\chi_{M,N}),q}(\overset{\bullet}{\mathcal{D}}) \simeq D(\mathrm{Rep}_q(\mathrm{GL}(M|N)), \tag{1}$$

that respects the natural t-structures on both sides, i.e. it should induce an equivalence

$$S\mathrm{Perv}_{\mathrm{GL}(M,\mathbf{O})\widetilde{\ltimes}(U_{M,N}(\mathbf{F}),\chi_{M,N}),q}(\overset{\bullet}{\mathcal{D}}) \simeq \mathrm{Rep}_q(\mathrm{GL}(M|N)). \tag{2}$$

Here the notations are as follows: a) $\mathcal{D}$ stands for certain determinant line bundle on $\mathrm{Gr}_{\mathrm{GL}(N)}$ and $\overset{\bullet}{\mathcal{D}}$ is the total space of this line bundle with zero section removed.

b) $SD_{\mathrm{GL}(M,\mathbf{O})\widetilde{\ltimes}(U_{M,N}(\mathbf{F}),\chi_{M,N}),q}(\overset{\bullet}{\mathcal{D}})$ stands for the derived category of $\mathrm{GL}(M,\mathbf{O})\widetilde{\ltimes}(U_{M,N}(\mathbf{F}),\chi_{M,N})$-equivariant q-monodromic sheaves $\overset{\bullet}{\mathcal{D}}$ with coefficients in super-vector spaces; $S\mathrm{Perv}_{\mathrm{GL}(M,\mathbf{O})\widetilde{\ltimes}(U_{M,N}(\mathbf{F}),\chi_{M,N}),q}(\overset{\bullet}{\mathcal{D}})$ stands for the corresponding category of perverse sheaves.

c) $\mathrm{GL}(M,\mathbf{O})\widetilde{\ltimes}U_{M,N}(\mathbf{F})$ stands for the semidirect product.

d) $\mathrm{GL}(M|N)$ is the super group of automorphisms of the super vector space $\mathbb{C}^{M|N}$ and $\mathrm{Rep}_q(GL(M|N))$ is the category of finite-dimensional representations of the corresponding quantum group.

Let us note that the above formulation is for generic q; a similar formulation should hold for all q, but one has to be more careful about the precise form of the corresponding quantum super group over $\mathbb{C}[q,q^{-1}]$.

1.3. **What is actually done?** In this talk we deal with the case $M = N-1$ for generic q. The advantage of the $M = N-1$ assumption is that in this case the group $U_{M,N}$ is trivial, so $\mathrm{GL}(M,\mathbf{O})\widetilde{\ltimes}U_{M,N}(\mathbf{F})$ is just equal to $\mathrm{GL}(N-1,\mathbf{O})$ (and the character χ is trivial as well). The current work should be thought of as a sequel to [2]. There we consider (among other things) the case $q = 1$. As was noted above, one has to be careful about specializing to non-generic q. It turns out that for $q = 1$ the correct statement is as follows.

Consider a degenerate version $\underline{\mathfrak{gl}}(N-1|N)$ where the supercommutator of the even elements (with even or odd elements) is the same as in $\mathfrak{gl}(N-1|N)$, while the supercommutator of any two odd elements is set to be zero. In other words, the even part $\underline{\mathfrak{gl}}(N-1|N)_{\bar{0}} = \mathfrak{gl}_{N-1} \oplus \mathfrak{gl}_N$ acts naturally on the odd part $\underline{\mathfrak{gl}}(N-1|N)_{\bar{1}} = \mathrm{Hom}(\mathbb{C}^{N-1}, \mathbb{C}^N) \oplus \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^{N-1})$, but the supercommutator $\underline{\mathfrak{gl}}(N-1|N)_{\bar{1}} \times \underline{\mathfrak{gl}}(N-1|N)_{\bar{1}} \to \underline{\mathfrak{gl}}(N-1|N)_{\bar{0}}$ equals zero.

The category of finite dimensional representations of the corresponding supergroup $\underline{\mathrm{GL}}(N-1|N)$ is denoted $\mathrm{Rep}(\underline{\mathrm{GL}}(N-1|N))$. In [2] we construct a tensor equivalence from the abelian category $S\mathrm{Perv}_{\mathrm{GL}(N-1,\mathbf{O})}(\mathrm{Gr}_{\mathrm{GL}_N})$ of equivariant perverse sheaves with coefficients in super vector spaces to $\mathrm{Rep}(\underline{\mathrm{GL}}(N-1|N))$. Here the monoidal structure on $S\mathrm{Perv}_{\mathrm{GL}(N-1,\mathbf{O})}(\mathrm{Gr}_{\mathrm{GL}_N})$ is defined via the fusion product (nearby cycles in the Beilinson-Drinfeld Grassmannian). This equivalence is reminiscent of the classical geometric Satake equivalence $\mathrm{Perv}_{\mathrm{GL}(N,\mathbf{O})}(\mathrm{Gr}_{\mathrm{GL}_N}) \cong \mathrm{Rep}(\mathrm{GL}_N)$, but as was noted above it should rather be thought of as analog of FLE. In particular, in [2] we also prove the corresponding derived equivalence

$$SD_{\mathrm{GL}(N-1,\mathbf{O})}(\mathrm{Gr}_{\mathrm{GL}_N}) \simeq D(\mathrm{Rep}(\underline{\mathrm{GL}}(N-1|N)))$$

(in fact, we first prove the derived equivalence and then show that it is compatible with the t-structures on both sides).

The main purpose of our work is to prove (2) for $M = N-1$. In other words, assuming that q is transcendental[1] we prove a braided monoidal equivalence between abelian categories $S\mathrm{Perv}_{\mathrm{GL}(N-1,\mathbf{O}),q}(\overset{\bullet}{\mathcal{D}})$ and $\mathrm{Rep}_q(\mathrm{GL}(N-1|N))$. The braided tensor structure on the geometric side is again defined via the fusion product. Contrary to the case $q = 1$, we use the abelian equivalence (2) to derive the derived equivalence (1). It follows from an equivalence $D(S\mathrm{Perv}_{\mathrm{GL}(N-1,\mathbf{O}),q}(\overset{\bullet}{\mathcal{D}})) \simeq SD_{\mathrm{GL}(N-1,\mathbf{O}),q}(\overset{\bullet}{\mathcal{D}})$.

Let us note that the $q = 1$ case discussed above is a special case of a very general set of conjectures due to D. Ben-Zvi, Y. Sakellaridis and A. Venkatesh; those conjectures were in fact motivated by known results about automorphic L-functions. However, to the best of our knowledge, it is not known how to extend those general conjectures to the "quantum" (i.e. general q) case. Thus in some sense at the moment the only motivation for the equivalences (1) and (2) comes from mathematical physics.

1.4. **Outline of the proof of the main theorem.** Our argument follows the scheme of D. Gaitsgory's proof [4] of the FLE for generic q. We use the Lurie-Gaitsgory generalization of [3]: a braided tensor equivalence between $\mathrm{Rep}_q(\mathrm{GL}(N-1|N))$ and an appropriate category FS of factorizable sheaves on configuration

[1] Probably the assumption that q is not a root of unity should suffice but for our current proof we need to assume that q is transcendental for certain technical reasons.

spaces of a smooth projective curve C. We actually construct a braided tensor equivalence $F\colon S\mathrm{Perv}_{\mathrm{GL}(N-1,\mathbf{O}),q}(\dot{\mathcal{D}}) \xrightarrow{\sim} \mathrm{FS}$.

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Knot homology, tautological classes and $\mathfrak{sl}_2$

Anton Mellit

(joint work with Eugene Gorsky, Matt Hogancamp)

Khovanov-Rozansky homology of a link represented as a closure of a braid $\beta = \sigma_a^{\pm 1}\sigma_b^{\pm 1}\sigma_c^{\pm 1}\cdots$ is defined by

$$H^*(HH^*(F_a^{\pm 1} \otimes F_b^{\pm 1} \otimes F_c^{\pm 1} \otimes \cdots)).$$

It is expected to be related to the cohomology of the so-called braid variety, which can be thought of as the variety parametrizing local systems on the disk with certain reduction on the boundary corresponding to the braid. In [2] the cohomology of classical character varieties was related to the cohomology of braid varieties. These considerations suggests that the Khovanov-Rozansky homology should have an action of certain tautological classes, which in geometry corresponds to the integration of Chern classes of the tautological bundle over the disk.

In [1] we construct this action on the y-ified version of the Khovanov-Rozansky homology from [3]. The starting point of the y-ification construction is the observation that on each complex

$$F_\beta = F_a^{\pm 1} \otimes F_b^{\pm 1} \otimes F_c^{\pm 1} \otimes \cdots$$

the action of the variables $x_1, \ldots, x_n$ on the left is homotopic to the action on the right. To obtain the tautological classes we have to go one step further. Consider any symmetric function $f \in \mathbb{C}[x_1, \ldots, x_n]^{S_n}$. Then the action of f on the left is homotopic to the action on the right by the above. On the other hand, the action of any symmetric function on the left *equals* to the action on the right. Thus the homotopy is closed, and it turns out is itself homotopic to zero. These new homotopies are what we need. More precisely, consider the dg-algebra

$$\mathcal{A} = \frac{\mathbb{C}[x_1, \ldots, x_n;\, x_1', \ldots, x_n';\, \xi_1, \ldots, \xi_n;\, u_1, \ldots, u_n]}{(f(\mathbf{x}) = f(\mathbf{x}')\ \forall f \in \mathbb{C}[x_1, \ldots, x_n]^{S_n})},$$

with the differential

$$dx_i = dx'_i = 0,\ d\xi_i = x_i - x'_i,\ du_k = \sum_{i=1}^{n} \frac{x_i^k - x_i'^k}{x_i - x'_i}\xi_i.$$

Then we prove

Theorem. *Any object of the form F_β admits and action of the dg-algebra $\mathcal{A}$. The resulting $\mathcal{A}$-module is a braid invariant.*

The above result holds with respect to an appropriate notion of equivalence of $\mathcal{A}$-modules. Namely, we localize the category of $\mathcal{A}$-modules by the class of morphisms $M \to M'$ which become *homotopy equivalences* when restricted to the subalgebra generated by x_i and x'_i.

Then we analyse the action of the Hochschild homology functor. Let c be the number of connected components of the closure of β. We have a dg-algebra

$$\mathcal{CA} = \mathbb{C}[x_1, \ldots, x_c;\ \xi_1, \ldots, \xi_c;\ u_1, u_2 \ldots]$$

with the differential

$$dx_i = 0,\ d\xi_i = 0,\ du_k = \sum_{i=1}^{c} kx_i^{k-1}\xi_i.$$

Theorem. *Any complex of the form $HH^i(F_\beta)$ admits and action of the dg-algebra $\mathcal{CA}$. The resulting $\mathcal{CA}$-module is a link invariant.*

In particular, passing to the y-ified homology, which is a certain form of a Koszul duality, we obtain

Theorem. *The y-ified Khovanov-Rozansky homology of any link admits an action of a commuting family of operators $F_1, F_2, \ldots$ defined by*

$$F_k = \sum_i kx_i^{k-1}\frac{\partial}{\partial y_i} + u_k.$$

This action is a link invariant.

As an application of our construction, we restrict our attention to the operator F_2 and show

Theorem. *For any link, the operator $e = F_2$ is a part of an $\mathfrak{sl}_2$ triple e, f, h. In particular, the Poincaré polynomial of the y-ified Khovanov-Rozansky homology is symmetric. In the case of a knot, the Poincaré polynomial of the ordinary Khovanov-Rozansky homology is symmetric as conjectured by Dunfield-Gukov-Rasmussen* [4].

We end with the following question:

Question. *What are the relations in the algebra that acts on the y-ified Khovanov-Rozansky homology generated by x_i, y_i, F_k and the $\mathfrak{sl}_2$-operators f, h?*

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Unipotent Harish-Chandra bimodules

Ivan Losev

The talk is based on the joint work with Lucas Mason-Brown and Dmytro Matvieievskyi, arXiv:2108.03453.

Let G be a real reductive Lie group. One of the most basic unsolved problems in representation theory and abstract Harmonic analysis is the classification of the set $\mathrm{Irr}_u(G)$ of irreducible unitary G-representations. Although this problem remains unsolved, some general patterns have emerged. For each group G there should be a finite set of 'building blocks' $\mathrm{Unip}(G) \subset \mathrm{Irr}_u(G)$ called *unipotent representations* with an array of distinguishing properties. Every representation $\mathcal{B} \in \mathrm{Irr}_u(G)$ should be obtained through one of several procedures (like parabolic induction) from a unipotent representation $\mathcal{B}_L \in \mathrm{Unip}(L)$ of a suitable Levi subgroup $L \subset G$. Furthermore, the representations in $\mathrm{Unip}(G)$ should be indexed by nilpotent co-adjoint G-orbits and their equivariant covers. This roadmap has emerged over many decades (see [V2, Chps 7-13] for an overview) and is supported by numerous successes in important special cases (e.g. [V1] and [Ba]). A crucial problem with this approach is that the set $\mathrm{Unip}(G)$ has not yet been defined in the appropriate generality. The main contribution of this paper is a definition of 'unipotent' in the case when G is complex. Our definition generalizes the notion of *special unipotent*, due to Barbasch-Vogan and Arthur ([BV],[Ar]).

The representations we define arise from finite equivariant covers of nilpotent co-adjoint G-orbits. To each such cover $\widetilde{\mathbb{O}}$, we attach a distinguished filtered algebra $\mathcal{A}_0$ equipped with a graded Poisson isomorphism $\mathrm{gr}(\mathcal{A}_0) \simeq \mathbb{C}[\widetilde{\mathbb{O}}]$. The existence of this algebra follows from the theory of filtered quantizations of conical symplectic singularities, see [L]. The algebra $\mathcal{A}_0$ receives a distinguished homomorphism from the universal enveloping algebra $U(\mathfrak{g})$, and the kernel of this homomorphism is a completely prime primitive ideal in $U(\mathfrak{g})$ with associated variety $\overline{\mathbb{O}}$. A *unipotent ideal* is any ideal in $U(\mathfrak{g})$ which arises in this fashion. A *unipotent representation* is an irreducible Harish-Chandra bimodule which is annihilated (on both sides) by a unipotent ideal.

Our definitions are vindicated by the many favorable properties which these ideals and bimodules enjoy. First of all, we show that both unipotent ideals and bimodules have nice geometric classifications. Unipotent ideals are classified by certain geometrically-defined equivalence classes of covers of nilpotent orbits. Unipotent bimodules are classified by irreducible representations of certain finite groups. For classical groups, we show that all unipotent ideals are maximal and all unipotent bimodules are unitary. In addition, we show that all unipotent bimodules are, as G-representations, of a very special form, proving a conjecture of Vogan ([V3]). Finally, we show that all special unipotent bimodules are unipotent. The final assertion is proved using a certain refinement of Barbasch-Vogan-Lusztig-Spaltenstein duality, inspired by the symplectic duality of [BLPW]. Along the way, we establish combinatorial algorithms (in classical and exceptional types) for computing the infinitesimal characters of unipotent ideals.

Our definition of 'unipotent' admits an obvious generalization to real reductive Lie groups. If G is such a Lie group, it is natural to call 'unipotent' any irreducible Harish-Chandra module which is annihilated by a unipotent ideal. This set includes, as a proper subset, all special unipotent representations in the sense of [ABV, Sec 27]. Unfortunately, some of the representations in this larger set are not unitary. We conjecture that this problem disappears if we impose the condition $\operatorname{codim}(\partial\mathbb{O}, \overline{\mathbb{O}}) \geq 4$. This condition is satisfied by all rigid nilpotent orbits.

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Cacti, crystals, and categorical braid group actions

Iva Halacheva

(joint work with T. Licata, I. Losev, O. Yacobi)

Let $\mathfrak{g}$ denote a simply-laced semisimple Lie algebra with Dynkin diagram I, weight lattice X, dominant weights X_+, and w_0 the longest element in the Weyl group W of $\mathfrak{g}$. In a sequence of several papers, Chuang–Rouquier and Khovanov–Lauda construct a categorification of the representations of $\mathfrak{g}$, as well as the representations of its quantum group $\mathcal{U}_q(\mathfrak{g})$ [4, 11, 8, 9]. Let $V = \bigoplus_\mu V_\mu$ denote an integrable representation of $\mathcal{U}_q(\mathfrak{g})$ together with its weight space decomposition, and let $\mathcal{C} = \bigoplus_\mu \mathcal{C}_\mu$ along with the Chevalley functors $E_i, F_i, i \in I$, be a categorification of V, where each $\mathcal{C}_\mu$ is a graded abelian category.

Braid group actions. There is an action of the braid group $Br_{\mathfrak{g}}$ on V defined by Lusztig [10], giving the "quantum Weyl group symmetries" of V. Let 1_μ denote the projection onto the μ weight space. Then for every $i \in I$, the action of each generator $t_i : V \to V$ is defined as (where $\mu_i := (\mu, \alpha_i)$ is the pairing with the simple root α_i):

$$t_i 1_\mu = \sum_{-a+b=\mu_i} (-q)^{-b} E_i^{(a)} F_i^{(b)} 1_\mu$$

Chuang and Rouquier [4] define functors $\Theta_i \mathbb{1}_\mu : D^b(\mathcal{C}) \to D^b(\mathcal{C})$ called the **Rickard complexes** which categorify the operators $t_i 1_\mu$. Each such functor is induced from a complex of functors $\Theta_i \mathbb{1}_\mu$ supported in non-positive cohomological degrees whose components are, for $n \geq 0$:

$$(\Theta_i \mathbb{1}_\mu)^{-n} = \begin{cases} E^{(-\mu_i+n)} F_i^{(n)} \mathbb{1}_\mu \langle -n \rangle & \text{if } \mu_i \leq 0 \\ F_i^{(\mu_i+n)} E_i^{(n)} \mathbb{1}_\mu \langle -n \rangle & \text{if } \mu_i \geq 0 \end{cases}$$

where $E_i^{(k)}, F_i^{(k)}$ are the divided powers of E_i and F_i, $\langle \cdot \rangle$ is the grading shift, and the differential $d^n : (\Theta_i \mathbb{1}_\mu)^{-n} \to (\Theta_i \mathbb{1}_\mu)^{-n+1}$ is constructed from the counits and adjunctions for E_i and F_i. Cautis and Kamnitzer [2] show that these Rickard complexes satisfy the braid relations, and hence define an action $Br_{\mathfrak{g}} \curvearrowright D^b(\mathcal{C})$.

Perverse equivalences. Consider two graded triangulated categories $\mathcal{T}$ and $\mathcal{T}'$ with translation functor $[\,\cdot\,]$ and grading shift $\langle \cdot \rangle$, with t-structures t and t', and filtrations $\mathcal{T}_\bullet, \mathcal{T}'_\bullet$ compatible with these t-structures:

$$0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \ldots \subset \mathcal{T}_r = \mathcal{T}, \quad 0 = \mathcal{T}'_0 \subset \mathcal{T}'_1 \subset \ldots \subset \mathcal{T}'_r = \mathcal{T}'$$

as well as a function $p : \{0, 1, \ldots, r\} \to \mathbb{Z}$. As defined by Chuang and Rouquier [3], a graded equivalence $F : \mathcal{T} \to \mathcal{T}'$ is called a **perverse equivalence** with respect to $(\mathcal{T}_\bullet, \mathcal{T}'_\bullet, p)$ if $F(\mathcal{T}_i) = \mathcal{T}'_i$ for all i, and the induced equivalence $F[-p(i)] : \mathcal{T}_i/\mathcal{T}_{i-1} \to \mathcal{T}'_i/\mathcal{T}'_{i-1}$ is t-exact. We prove the following theorem, which generalizes a result of Chuang and Rouquier for the case $\mathfrak{g} = \mathfrak{sl}_2$.

Theorem 1 ([6]). *Let $\mathcal{C}$ denote a categorical $\mathcal{U}_q(\mathfrak{g})$-representation as above.*

(1) *If $\mathcal{C}$ is an isotypic categorification of type $\lambda \in X_+$, and $\mu \in X$, then $\Theta_{w_0} \mathbb{1}_\mu [-ht(\mu - w_0(\lambda))] : D^b(\mathcal{C}_\mu) \to D^b(\mathcal{C}_{w_0(\mu)})$ is a t-exact equivalence.*
(2) *For general $\mathcal{C}$, $\Theta_{w_0} \mathbb{1}_\mu : D^b(\mathcal{C}_\mu) \to D^b(\mathcal{C}_{w_0(\mu)})$ is a perverse equivalence with respect to the filtrations induced from a Jordan–Hölder filtration on $\mathcal{C}$, and perversity function $p(i) = ht(\mu - w_0(\lambda_i))$, where the λ_i are the highest weights for the successive quotients of the Jordan–Hölder filtration.*

Cactus group actions. To the Lie algebra $\mathfrak{g}$, one can associate another group related to the braid group, called the cactus group $C_{\mathfrak{g}}$. It has generators c_J indexed by connected subdiagrams $J \subseteq I$ and three types of relations:

(1) $c_J^2 = 1$ for all $J \subseteq I$,
(2) $c_J c_K = c_K c_J$ if the graph $J \cup K$ is disconnected,

(3) $c_J c_K = c_K c_{\tau_K(J)}$ if $J \subseteq K$, where τ_K is the Dynkin diagram automorphism coming from the longest parabolic Weyl group element w_0^K.

For any integrable representation V of $\mathcal{U}_q(\mathfrak{g})$, Kashiwara [7] constructs a crystal B_V in the setting of "$q = 0$", which is a set coming from a basis of V, as well as operators $\tilde{e}_i, \tilde{f}_i : B_V \to B_V \cup \{0\}$, for any $i \in I$, coming from the Chevalley generators of $\mathfrak{g}$. There is an "internal" action on such a crystal, as a set, by the cactus group $C_\mathfrak{g}$ which is analogous to the action of the braid group $Br_\mathfrak{g}$ on V [5].

If $\mathcal{C}$ is a categorification of V, then the set $\mathrm{Irr}(\mathcal{C})$ of simple objects of $\mathcal{C}$ up to shift has a crystal structure isomorphic to the crystal B_V ([4, Proposition 5.20],[12],[1]). For each connected subdiagram $J \subseteq I$, we know from Theorem 1 that the Rickard complex $\Theta_{w_0^J} \mathbb{1}_\mu$ is a perverse equivalence, where $\mu \in X$. Hence, we obtain an induced bijection $\theta_J : \mathrm{Irr}(\mathcal{C}) \to \mathrm{Irr}(\mathcal{C})$. We prove the following theorem.

Theorem 2 ([6])**.** *The assignment $c_J \mapsto \theta_J$ defines an action of $C_\mathfrak{g}$ on* $\mathrm{Irr}(\mathcal{C})$ *which coincides with the internal cactus group action on B_V.*

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Charges via the affine Grassmannian

Leonardo Patimo

Kostka–Foulkes polynomials are q-analogue of the weight multiplicities, and they can be defined by a q-deformation of the famous Kostant multiplicity formula. Let X be the weight lattice and Φ be the root system of a complex semisimple algebraic group G. Let W be the corresponding Weyl group. For every pair of dominant weights μ, λ with $\mu \leq \lambda$, the Kostka–Foulkes polynomials $K_{\lambda,\mu}(q) \in \mathbb{Z}[q]$ are defined as

$$K_{\lambda,\mu}(q) = \sum_{w \in W} (-1)^{\ell(w)} \mathrm{kpf}_q(w(\lambda + \rho) - \mu - \rho))$$

where ρ is the half-sum of positive roots and $\mathrm{kpf}_q : X \to \mathbb{Z}[q]$ is the q-analogue of the Kostant partition function, i.e. the coefficient of q^k in $\mathrm{kpf}_q(\nu)$ is the number of ways in which ν can be written as a sum of k positive roots in Φ^+.

Charge Statistics. Kostka–Foulkes polynomials have positive coefficients. In type A, Lascoux and Schützenberger [LS78] gave a combinatorial meaning to the coefficients of $K_{\lambda,\mu}(q)$ by defining a *charge statistic* ch such that

$$K_{\lambda,\mu}(q) = \sum q^{ch(T)},$$

where the sum runs over all semistandard Young tableaux of shape λ and weight μ. The original definition of ch is based on an operation on the set of tableaux called *cyclage*.

However, beyond type A, the question of finding combinatorial interpretations for the coefficients of $K_{\lambda,\mu}(q)$ is still open, and only in type C there is a general conjecture available [Lec05]. (On the other hand, several combinatorial interpretations exist in any type for the weight multiplicities, i.e. for $K_{\lambda,\mu}(1)$, see e.g. [Lit94].)

This question motivated us to look for a more intrinsic construction of the charge statistic, which avoids the combinatorics of semistandard tableaux. Our construction is based instead on the geometry of the affine Grassmannian. In fact, Kato [Kat82] showed that $K_{\lambda,\mu}(q)$ are Kazhdan–Lusztig polynomials for the affine Grassmannian $Gr_G^\vee$ of the Langlands dual group $G^\vee$ of G. This means that $K_{\lambda,\mu}(q)$ can be obtained as the Poincaré polynomials of the stalk of the intersection cohomology sheaves IC_λ of Schubert varieties in $Gr_G^\vee$.

Hyperbolic Localization on the Affine Grassmannian. A convenient way of computing stalks is using Braden's hyperbolic localization [Bra03]. Let $\hat{T} = T^\vee \times \mathbb{C}^*$ be the augmented torus, where $T^\vee$ is a maximal torus in $G^\vee$ and $\mathbb{C}^*$ is the group of loop rotations. Then, every cocharacter $\eta \in X_\bullet(\hat{T})$ determines a $\mathbb{C}^*$-action on $Gr_G^\vee$. For any $\mu \in X$, we can define the hyperbolic localization for η at the $\hat{T}$-fixed point $\mu \in Gr_G^\vee$ as

$$\mathrm{HL}_\mu^\eta(\mathcal{F}) = H_c^\bullet(Y_\mu^+, \mathcal{F})$$

where Y_μ^+ is the attractive set of μ with respect to η.

If η is dominant with respect of the affine root system, i.e. if $(\eta, \alpha) > 0$ for any positive affine real root α (we call this the *KL region*), then the attractive sets are isomorphic to affine spaces $\mathbb{C}^\ell$. In this case, the hyperbolic localization of an equivariant sheaf coincides (up to a shift) with its stalk in μ. Hence, the graded dimension $\mathrm{hl}_{\mu,\lambda}^\eta(q) := \mathrm{grdim}\mathrm{HL}_\mu^\eta(IC_\lambda)$ coincides (up to a renormalization) with the Kostka–Foulkes polynomials $K_{\lambda,\mu}(q)$.

If η belongs instead to the sublattice $X_\bullet(T^\vee) \subset X_\bullet(\hat{T})$ and it is dominant with respect all positive roots in the (finite) root system Φ (we call this the *MV region*), then the attractive set Y_μ^+ are the *semi-infinite orbits* (cf. [MV07]). In this case hyperbolic localization in μ is an exact functor, called the *weight functor*, which correspond via geometric Satake to μ-weight space.

Remarkably, in the MV region the hyperbolic localization is concentrated in a single degree. Hence, we can recover the grading in $K_{\lambda,\mu}(q)$ by considering a family of cocharacters $\eta(t)$ going from the MV region to the KL region and recording how hyperbolic localization changes along the way. The graded dimension $\mathrm{hl}^\eta_{\mu,\lambda}(q) := \mathrm{grdim}\mathrm{HL}^\eta_\mu(IC_\lambda)$ only changes when we cross a wall

$$H_\beta = \{\eta \in X_\bullet(\hat{T}) \mid (\eta, \beta) = 0\} \subseteq X_\bullet(T)$$

corresponding to a real affine root β. In this case, if $t_1 < t_2$ are such that $\eta(t_1)$ and $\eta(t_2)$ are on the opposite sides of the wall H_β, we have

$$\mathrm{hl}^{\eta(t_2)}_{\mu,\lambda}(q) = \begin{cases} q^{-2} \cdot \mathrm{hl}^{\eta(t_1)}_{\mu,\lambda}(q) & \text{if } \mu > s_\beta(\mu), \\ \mathrm{hl}^{\eta(t_1)}_{\mu,\lambda}(q) + (1-q^{-2})\mathrm{hl}^{\eta(t_1)}_{s_\beta(\mu),\lambda}(q) & \text{if } \mu < s_\beta(\mu), \end{cases} \tag{1}$$

where s_β is the affine reflection corresponding to β.

Wall Crossing and Crystal Graphs. To obtain a charge statistic, one needs to translate the wall crossing in hyperbolic localization (1) into a combinatorial operation. We achieve this in the setting of Kashiwara's crystals.

Let $\mathcal{B}(\lambda)$ be the crystal corresponding to the irreducible representation of G of highest weight λ and let $\mathcal{B}(\lambda)_\mu$ denote the subsets of elements of weight μ. Assume we have a *recharge statistic* in $\eta(t_1)$, i.e. a function $r(\eta(t_1), -)$ such that

$$\mathrm{hl}^{\eta(t_1)}_{\mu,\lambda}(q) = \sum_{T \in \mathcal{B}(\lambda)_\mu} q^{r(\eta(t_1),T)}.$$

To obtain a recharge statistic for $\eta(t_2)$ one needs a *swapping function*, i.e. a function $\psi : \mathcal{B}(\lambda)_\mu \to \mathcal{B}(\lambda)_{s_\beta(\mu)}$ for $\mu < s_\beta(\mu)$ such that

$$r(\eta(t_1), \psi(T)) = r(\eta(t_1), T) - 1.$$

As the name suggests, a recharge for $\eta(t_2)$ can be obtained by swapping the values of $r(\eta(t_1), -)$ as indicated by ψ.

Swapping function exists in any type. Unfortunately, only in type A (and for a specific family of cocharacters) we are able to construct them explicitly. In this case, we find that swapping functions can be defined using the modified crystal operators. If $\alpha = \alpha_i + \alpha_{i+1} + \ldots + \alpha_j$ is a positive root in Φ^+, then

$$e_\alpha = w e_i w^{-1} \qquad \text{and} \qquad f_\alpha = w f_i w^{-1},$$

where e_i, f_i are the "ordinary" crystal operators, and $w = s_{i+1}s_{i+2}\ldots s_j$.

Proposition. *If $s_\beta(\mu) = \mu + k\alpha$, then*

$$e^k_\alpha : \mathcal{B}(\lambda)_\mu \to \mathcal{B}(\lambda)_{s_\beta(\mu)}$$

is a swapping function between $\eta(t_1)$ and $\eta(t_2)$.

We remark that modified crystal operators have been studied in [LL21] and applied to define the *atomic decomposition* of crystals. The atomic decomposition is in fact crucial in the proof of the Proposition since it allows us to restrict to a multiplicity one situation. Thanks to the swapping functions, we can now translate (1), and this process leads to a new construction (independent from [LS78]) of the charge statistic in type A.

Theorem ([Pat21]). *For $T \in \mathcal{B}(\lambda)$ we have*

$$ch(T) = \sum_{\alpha \in \Phi^+} \epsilon_\alpha(T).$$

where $\epsilon_\alpha(T) = \max\{k \mid e_\alpha(T) \neq 0\}$.

We hope to extend in future works our results beyond type A.

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Symmetric bow varieties and Coulomb branches of quiver gauge theories of classical affine types

Hiraku Nakajima

For a given pair of a complex reductive group $\mathbf{G}$ and its finite dimensional representation $\mathbf{N}$, we associate the Coulomb branch $\mathcal{M}_C \equiv \mathcal{M}_C(\mathbf{G}, \mathbf{N})$ [2]. It is an affine algebraic Poisson variety. We are interested in a particular case when $(\mathbf{G}, \mathbf{N})$ arises from a framed representation of a quiver.

Let $Q = (Q_0, Q_1)$ be a quiver with the set of vertices Q_0 and the set of oriented edges Q_1. We have two maps $\mathrm{o}, \mathrm{i}\colon Q_1 \to Q_0$ given by outgoing and incoming vertices of edges. We choose two finite dimensional Q_0-graded complex vector spaces $V = \bigoplus_{i \in Q_0} V_i$, $W = \bigoplus_{i \in Q_0} W_i$. Then we take

$$\mathbf{G} = \prod_{i \in Q_0} \mathrm{GL}(V_i), \quad \mathbf{N} = \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{\mathrm{o}(h)}, V_{\mathrm{i}(h)}) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}(W_i, V_i),$$

where $\mathbf{G}$ acts on $\mathbf{N}$ by conjugation.

Let us suppose that Q has no edge loops, and consider a symmetric Kac-Moody Lie algebra $\mathfrak{g}$ whose Dynkin diagram is Q with orientation forgotten. Let $\alpha_i^\vee$, $\varpi_i^\vee$ be the i-th simple and fundamental coroots respectively. We define two coweights λ, μ from V, W as $\lambda = \sum \dim W_i \varpi_i^\vee$, $\mu = \lambda - \sum \dim V_i \alpha_i^\vee$.

When Q is of finite ADE type, the corresponding Coulomb branch $\mathcal{M}_C$ was determined in [3]. The answer is given in terms of the affine Grassmannian Gr_G for the adjoint type group G with the Lie algebra $\mathfrak{g}$ as follows. Recall that Gr_G is described as $G(\mathcal{K})/G(\mathcal{O})$, where $\mathcal{K} = \mathbb{C}((z))$, $\mathcal{O} = \mathbb{C}[[z]]$. We have an action of $G(\mathcal{O})$ on Gr_G and orbits are parametrized by dominant integral coweights. Since λ is dominant by its definition, we have the corresponding $G(\mathcal{O})$-orbit, denoted by Gr_G^λ. Let us denote its closure by $\overline{\mathrm{Gr}}_G^\lambda$. If μ is also dominant, we can consider another orbit Gr_G^μ which is contained in $\overline{\mathrm{Gr}}_G^\lambda$ as we have $\mu \leq \lambda$ by construction. Then $\mathcal{M}_C$ is a transversal slice to Gr_G^μ in $\overline{\mathrm{Gr}}_G^\lambda$, constructed in [1]. If μ is not necessarily dominant, one defines a *generalized slice*, which recovers the transversal slice above for dominant μ. See [3] for detail.

The definition of Coulomb branches arising a quiver can be generalized to the case of *valued* quivers [9]. It corresponds to a symmetrizable Kac-Moody Lie algebra. Moreover the above description of a Coulomb branch for finite type by a generalized slice remains to be true.

Generalized slices play important role in geometric Satake correspondence. They contain the so-called Mirkovic-Vilonen cycles as lagrangian subvarieties. When μ is dominant, this was more or less known. When μ is not dominant, it was proved in [5]. Irreducible components of Mirkovic-Vilonen cycles give a base of a weight space of a finite dimensional irreducible representation of the Langlands dual group of G. Therefore Mirkovic-Vilonen cycles are important objects in geometric representation theory.

Since $\mathcal{M}_C$ makes sense for an arbitrary valued quiver, it is natural to expect that we could hope a generalization of geometric Satake correspondence to arbitrary symmetrizable Kac-Moody Lie algebra. This was formulated in [3] and showed in the affine type A in [7].

The proof in [7] uses crucially a description of $\mathcal{M}_C$ as a Cherkis bow variety. See [8]. If we hope to take a similar approach for other cases, we need to introduce analog of a Cherkis bow variety. Bow varieties themselves are interesting varieties to explore, so their variants will be also worth studying.

Suppose the valued quiver Q is of finite classical type. As we explained above, $\mathcal{M}_C$ is the generalized slice associated with a classical group G. Note that a classical group is a fixed point locus of an involuation on a general linear group GL_n, if we ignore a difference between SO_n and O_n. Therefore it is natural to expect that $\mathcal{M}_C$ is a fixed point locus of an involution on the Coulomb branch of type A. In fact, $\mathcal{M}_C$ is the moduli space of monopoles on $\mathbb{R}^3$ when Q is finite type with $\lambda = 0$, and such a description for classical types was given in [4]. Here the moduli space of type A monopoles is described as the moduli space of solutions of Nahm's equation, an ancestor of the bow variety. The involution corresponds to taking dual monopoles.

When a valued quiver is of classical affine type, it is natural to consider the fixed point locus of an involution on the bow variety. The fixed point locus is the first step towards a definition of a *symmetric bow variety.*

Cherkis bow variety, in the description in [8], is a symplectic reduction of a product of two types of symplectic manifolds. One is a symplectic vector space $\mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_2, V_1)$ with an action of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$. Another is

- $T^*\mathrm{GL}(V_1) \times V_1 \times V_1^*$ if $\dim V_1 = \dim V_2$, or
- $\mathrm{GL}(V_1) \times S$, where S is Slodowy slice to a nilpotent orbit of hook type $(\dim V_1 - \dim V_2, 1^{\dim V_2})$ if $\dim V_1 > \dim V_2$

with an action of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$. (We exchange V_1 and V_2 if $\dim V_1 < \dim V_2$.) This pair of symplectic manifolds with $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$-action is an example of 'dual pairs' discussed in Braverman's talk.

In order to define an involution on a bow variety, it is enough to define it on the product of one or two copies of either of the above building block. For example, let us consider a case when we use a single copy of $\mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_2, V_1)$. Let us consider type D for brevity. Then we put a symplectic form on $V_1 \oplus V_2$ so that V_1, V_2 are lagrangian. Then the involution is given by sending $(C, D) \in \mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_2, V_1)$ to (C^*, D^*) where the adjoint $*$ is given by the symplectic form. Therefore the fixed point locus consisting of (C, D) with $C^* = C$, $D^* = D$. The group $\mathrm{GL}(V_1)$ acts both on V_1, V_2, where V_2 is regarded as the dual representation.

If we use two copies, we take V_2 in common: $\mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_2, V_1)$ and $\mathrm{Hom}(V_3, V_2) \oplus \mathrm{Hom}(V_2, V_3)$. Then we take a symplectic form on $V_1 \oplus V_3$ as above, and also a symplectic form on V_2. Then $C \in \mathrm{Hom}(V_1, V_2)$ is sent to $C^* \in \mathrm{Hom}(V_2, V_3)$. Other components are sent in a similar manner. The fixed point locus is $\mathrm{Hom}(V_1, V_2) \oplus \mathrm{Hom}(V_2, V_1)$ with the group action of $\mathrm{GL}(V_1) \times \mathrm{Sp}(V_2)$.

This construction works in some cases, but not in general. There are two problems in these fixed point loci.

(1) $\mathcal{M}_C$ is known to have deformation parametrized by $\sum \dim W_i - 1$ parameters, called flavor symmetry. We cannot produce all deformation as fixed point loci.
(2) $\mathcal{M}_C$ is known to be an irreducible variety. Fixed point loci are in general non-irreducible.

The first problem is already illustrated in the above example: we cannot deform the moment map for the $\mathrm{Sp}(V_2)$-action, as $\mathrm{Sp}(V_2)$ only has a discrete center. On the other hand, the number of copies corresponds to $\dim W_i$.

Let us illustrate a method to overcome these difficulties in the second case above using two copies. In general, we need to consider the product of ℓ copies ($\ell \in \mathbb{Z}_{>0}$), but the following construction can be generalized. We take the symplectic reduction with respect to $\mathrm{GL}(V_2)$ with a generic stability condition $\zeta_{\mathbb{R}}$ and generic moment map level $\zeta_{\mathbb{C}}$. Then we get a type A_1 quiver variety $\mathfrak{M}_\zeta$ ($\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$), which is smooth. Then one can define an involution σ on $\mathfrak{M}_\zeta$ by using the so-called reflection functor [6]. Then the fixed point locus $\mathfrak{M}_\zeta^\sigma$ has several connected components. We can find a *correct* connected component, which can be identified with a deformed or partially resolved version of $\mathcal{M}_C$. If we want to describe the genuine $\mathcal{M}_C$, we take $\zeta_{\mathbb{C}} = 0$ and the affinization of $\mathfrak{M}^\sigma_{(\zeta_{\mathbb{R}},0)}$.

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Cohomological construction of translation functors for trigonometric double affine Hecke algebras

Wille Liu

Let $(\mathfrak{h}, R)$ be a irreducible finite root system and let $\mathrm{H}^{\mathrm{DAHA}}$ be the double affine Hecke algebra attached to $(\mathfrak{h}, R)$. Extending the K-theoretic construction of affine Hecke algebras due to Ginzburg, Vasserot constructed an isomorphism from $\mathrm{H}^{\mathrm{DAHA}}$ to the equivariant K-theoretical convolution ring of the affine Steinberg variety and used it to give a classification of simple $\mathrm{H}^{\mathrm{DAHA}}$-modules in the category $\mathcal{O}$ of $\mathrm{H}^{\mathrm{DAHA}}$. This construction works equally well for the trigonometric degeneration $\mathrm{H}^{\mathrm{trig}}$.

The trigonometric double affine Hecke algebra $\mathrm{H}^{\mathrm{trig}}$ is a "filtered quantisation" of the skewed tensor product $\mathbb{C}W\#(\mathbb{C}[\mathfrak{h}_{\mathbb{C}}] \times \mathbb{C}Q^\vee)$, where $Q^\vee \subset \mathfrak{h}$ is the coroot lattice, and it depends on a parameter $c \in \mathbb{C}$. A general principle of quantisation suggests the existence of an equivalence of the bounded derived category

$$\mathrm{D}^{\mathrm{b}}(\mathrm{H}^{\mathrm{trig}}_c\text{−mod}) \xrightarrow{\cong} \mathrm{D}^{\mathrm{b}}(\mathrm{H}^{\mathrm{trig}}_{c'}\text{−mod})$$

whenever the parameters $c, c' \in \mathbb{C}$ satisfy $c - c' \in \mathbb{Z}$. We construct such a functor by modifying the K-theoretic construction of Vasserot: let $\mathrm{L}\dot{\mathfrak{g}} \to \mathrm{L}\mathfrak{g}$ be the affine Springer resolution, where $\mathrm{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[\![\varpi]\!]$ is the loop Lie algebra; let $B_{c',c} = \mathrm{H}_{\bullet}^{LG \rtimes \mathbb{C}^\times \times \mathbb{C}^\times}(\mathrm{L}\dot{\mathfrak{g}} \times_{\mathrm{L}\mathfrak{g}} \varpi^{c'-c}\mathrm{L}\dot{\mathfrak{g}})_c$ be a specialisation of the equivariant Borel–Moore homology; there is a natural $(\mathrm{H}_c^{\mathrm{trig}}, \mathrm{H}_{c'}^{\mathrm{trig}})$-bimodule structure on $B_{c',c}$, which yields a functor

$$B_{c,c'} \otimes^{\mathrm{L}} -\colon \mathrm{D}^{\mathrm{b}}(\mathrm{H}_c^{\mathrm{trig}}-\mathrm{mod}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{H}_{c'}^{\mathrm{trig}}-\mathrm{mod}).$$

We show that this functor is an equivalence of category by applying the technique of equivariant localisation for the affine Springer resolution as well as sheaf-theoretic methods.

A Schur-Weyl duality between DAHA and quantum groups

Léa Bittmann

(joint work with Alex Chandler, Anton Mellit, Chiara Novarini)

In [4], Suzuki and Vazirani considered a class of representations of the type A Double Affine Hecke Algebra, called X-semisimple representations. They gave a classification of all irreducible X-semisimple representations of the DAHA $\ddot{H}_{q,t}(m)$, away from roots of unity, using particular combinatorial objects. By Schur-Weyl duality considerations, Jordan and Vazirani [1] constructed one of these X-semisimple representations, the *rectangular representation* out of representations of the quantum group $U_\nu(\mathfrak{gl}_N)$, where $\nu^2 = q$ and m is a multiple of N. We focus here on the root of unity case.

One of the ways to consider representations of the quantum group $U_\nu(\mathfrak{gl}_N)$ at roots of unity is by studying the *fusion category* (at level K), obtained as a quotient of the category of tilting modules by its subcategory of negligible modules. The resulting category is semisimple and ribbon.

Inspired by the approach of [4] and [1], we constructed, for a, b such that $m = aN - bK$, a module $W_{(N,K,a,b)}$ using new combinatorial object called *Doubly Periodic Tableaux.* We show that this module $W_{(N,K,a,b)}$ can be as a certain hom-space of the fusion category. By the quantum affine Schur-Weyl duality presented in [3], $W_{(N,K,a,b)}$ can be endowed with a action of the affine Hecke algebra, but we can extend this action to the DAHA using ribbon calculus.

Indeed, $W_{(N,K,a,b)}$ can be realized as a *ribbon skein module*, by considering ribbons in a solid torus with one marked point. Combining this approach with the combinatorics of Doubly Periodic Tableaux, we obtain the following result.

Theorem 1. *The module $W_{(N,K,a,b)}$ is an X-semisimple graded irreducible representation of the Double Affine Hecke Algebra.*

Moreover, similarly to the results of [4], the $W_{(N,K,a,b)}$ can be used to classify representations of the DAHA at roots of unity.

Theorem 2. *The $W_{(N,K,a,b)}$ exhaust all X-semisimple graded irreducible representations of the Double Affine Hecke Algebra at roots of unity.*

Let W be the direct sum of the $W_{(N,K,a,b)}$, where for fixed N, K we take one pair $a, b \geq 0$:

$$W = \bigoplus_{m=aN+bK} W_{(N,K,a,b)}.$$

Then we can show that W is a faithful representation of $\ddot{H}_{q,t}(m)$.

As an application, we obtain a proof of a conjecture of Morton and Samuelson. In [2], Morton and Samuelson constructed a surjective homomorphism φ from the DAHA $\ddot{H}_{q,t}(m)$ to the *tangle skein algebra* on the thickened torus and conjectured it was an isomorphism.

By the ribbon calculus considerations, we can show that the action of $\ddot{H}_{q,t}(m)$ on W factors through φ, and the faithfulness of this action gives the result.

Theorem 3. *The map φ is an isomorphism.*

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Proof of the relative Humphreys conjecture

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(joint work with William Hardesty)

Let G be a connected reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic p, and let G_1 be its first Frobenius kernel. Assume that $p > h$, where h is the Coxeter number for G. It is a classical result [4, 5] that there is a G-equivariant isomorphism of rings

$$\mathrm{Ext}^{\bullet}_{G_1}(\Bbbk, \Bbbk) \cong \Bbbk[\mathcal{N}],$$

where $\mathcal{N}$ denotes the variety of nilpotent elements in the Lie algebra of G, and $\Bbbk[\mathcal{N}]$ is its coordinate ring. Moreover, this is an isomorphism of graded rings, where we equip $\Bbbk[\mathcal{N}]$ with the grading induced by making the multiplicative group $\mathbb{G}_{\mathrm{m}}$ act on $\mathcal{N}$ with weight -2. Thus, for any G-module M, the vector spaces $\mathrm{Ext}^{\bullet}_{G_1}(M, M)$ and $\mathrm{Ext}^{\bullet}_{G_1}(\Bbbk, M)$ can be regarded as $G \times \mathbb{G}_{\mathrm{m}}$-equivariant (quasi-)coherent sheaves on $\mathcal{N}$. The *Humphreys conjecture* [6] predicts the supports of these sheaves in the case where M is an indecomposable tilting G-module.

We now introduce notation needed to give a precise statement for $\mathrm{Ext}^{\bullet}_{G_1}(\Bbbk, M)$. (This case is known as the *relative* Humphreys conjecture. See [2] for a discussion of how this is related to the conjecture on $\mathrm{Ext}^\bullet_{G_1}(M,M)$.) Let $\mathbf{X}$ be the weight lattice for a maximal torus of G, and let $\mathbf{X}^+$ be the set of dominant weights. Let W be the Weyl group, and let $W_{\mathrm{ext}} = W \ltimes \mathbf{X}$ be the extended affine Weyl group. For $\lambda \in \mathbf{X}$, let t_λ denote the corresponding element in W_{ext}, and let w_λ denote the shortest element of the right coset Wt_λ. For $\mu \in \mathbf{X}^+$, let $\mathsf{T}(\mu)$ denote the indecomposable tilting G-module of highest weight μ. It is easy to show that $\mathrm{Ext}^\bullet_{G_1}(\Bbbk, \mathsf{T}(\mu))$ vanishes unless μ is of the form $w_\lambda \cdot_p 0$ with $\lambda \in -\mathbf{X}^+$. (Here "$\cdot_p$" denotes the p-dilated dot action.) Suppose now that $\mu = w_\lambda \cdot_p 0$ for some $\lambda \in -\mathbf{X}^+$. The element w_λ belongs to some 2-sided Kazhdan–Lusztig cell $\mathbf{c} \subset W_{\mathrm{ext}}$. The set of 2-sided Kazhdan–Lusztig cells is in bijection with the set of nilpotent orbits [7]. Let $C_{\mathbf{c}} \subset \mathcal{N}$ be the orbit corresponding to $\mathbf{c}$. The relative Humphreys conjecture predicts that

$$\operatorname{supp} \mathrm{Ext}^\bullet_{G_1}(\Bbbk, \mathsf{T}(w_\lambda \cdot_p 0)) = \overline{C_{\mathbf{c}}}.$$

In the present work [1], the authors prove this conjecture. The main tool is the notion of a "silting subcategory" of the derived category $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$. In previous joint work with S. Riche [2], the authors had constructed a collection of objects

$$\mathfrak{S}_\lambda \in D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N}) \qquad \text{for } \lambda \in \mathbf{X}^+$$

with the property that

$$\mathrm{Ext}^k_{G_1}(\Bbbk, \mathsf{T}(w_{w_0\lambda} \cdot_p 0)) \cong \bigoplus_{i+j=k} R^i\Gamma(\mathcal{N}, \mathfrak{S}_\lambda)_j.$$

Moreover, it was shown in [2] that if $w_{w_0\lambda} \in \mathbf{c}$, then

$$\operatorname{supp} \mathfrak{S}_\lambda \supset \overline{C_{\mathbf{c}}}, \tag{1}$$

with equality for $p \gg 0$. Here is a brief outline the proof of the relative Humphreys conjecture in [1].

Step 1. First, one shows that the full subcategory of $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ consisting of direct sums of objects of the form $\mathfrak{S}_\lambda[n]\langle -n\rangle$ is a *silting subcategory.* This means that this category is Karoubian; it generates $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ as a triangulated category; and its objects satisfy a certain positive Ext-vanishing property. Denote this category by $\mathrm{Silt}_1(\mathcal{N})$. The $\mathfrak{S}_\lambda[n]\langle -n\rangle$ are precisely the indecomposable objects in $\mathrm{Silt}_1(\mathcal{N})$.

Step 2. In view of (1), one can reduce the relative Humphreys conjecture to the problem of showing that $\mathrm{Silt}_1(\mathcal{N})$ contains "enough objects with small support." Here is a more precise statement. Given an orbit $C \subset \mathcal{N}$, let $D^{\mathrm{b}}_{\overline{C}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ denote the full subcategory of $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ consisting of objects supported (set-theoretically) on $\overline{C}$. The relative Humphreys conjecture holds if and only if for every orbit C, $D^{\mathrm{b}}_{\overline{C}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ is generated as a triangulated category by $D^{\mathrm{b}}_{\overline{C}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N}) \cap \mathrm{Silt}_1(\mathcal{N})$.

Step 3. Separately, [1] contains a construction of a new silting subcategory $\mathrm{Silt}_2(\mathcal{N})$ whose indecomposable objects are parametrized (up to shift by $[n]\langle -n\rangle$)

by pairs

$$\left\{ (C, \mathcal{T}) \,\middle|\, \begin{array}{c} C \subset \mathcal{N} \text{ a nilpotent orbit, and} \\ \mathcal{T} \in \mathrm{Coh}^G(C) \text{ an indecomposable tilting vector bundle} \end{array} \right\}$$

For each such pair $(C, \mathcal{T})$, the corresponding indecomposable object $\mathcal{S}(C, \mathcal{T})$ is supported on $\overline{C}$. In particular, $\mathrm{Silt}_2(\mathcal{N})$ manifestly has "enough objects with small support."

Step 4. To prove the relative Humphreys conjecture, it is enough to show that $\mathrm{Silt}_1(\mathcal{N}) = \mathrm{Silt}_2(\mathcal{N})$. This claim holds because both silting subcategories are preserved by Serre–Grothendieck duality, and an easy lemma shows that there is at most one silting subcategory preserved by such a duality functor.

As a side effect of the proof, some conjectures from [3] are also established.

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Towards combinatorial invariance for Kazhdan-Lusztig polynomials

Geordie Williamson

Let (W, S) be a Coxeter group. To any pair of elements $x, y \in W$ one may associate the Kazhdan-Lusztig polynomial

$$P_{x,y} \in \mathbb{Z}[q].$$

This polynomial is at the heart of many problems, and participants working in Geometric Representation Theory need no convincing of the importance of Kazhdan-Lusztig polynomials.

In the 1980s, Lusztig and Dyer independently posed the following:

Combinatorial invariance conjecture: $P_{x,y}$ only depends on the isomorphism class of the Bruhat interval $[x, y]$.

It is a very striking conjecture. Kazhdan-Lusztig polynomials are extremely complicated invariants. The conjecture suggests that they are providing some fine invariants of Bruhat order. However, I still don't really know what this conjecture "means". For more on this fascinating conjecture, see [1].

In this lecture I described a new formula for Kazhdan-Lusztig polynomials for symmetric groups. Traditionally, one computes Kazhdan-Lusztig polynomials by induction over the length: one begins at the identity, and works one's way "out" in the Coxeter group, gradually generating more Kazhdan-Lusztig polynomials. A key feature of the new formula is that it relies on induction on the rank (the n in S_n) rather than the length. Thus, whilst traditional methods for computing Kazhdan-Lusztig polynomials involve elements which are not in the Bruhat interval, the new formula involves only the calculation of Kazhdan-Lusztig polynomails $P_{u,v}$ where u, v both belong to $[x, y]$. Thus it is very much in the spirit of the combinatorial invariance conjecture.

Let us describe the formula in vague terms.[1] Consider permutations $x, y \in S_{n+1}$, which we view as permutations of $0, 1, \ldots, n$. Suppose that $x \leq y$ in Bruhat order. The set of elements

$$\{z \in S_{n+1} \mid x(n) = z(n)\}$$

has a unique maximum c. Moreover, the interval $[x, c]$ consists exactly of the intersection of the coset $S_n \cdot x$ with $[x, y]$, where S_n consists of permutations of $0, 1, \ldots, n-1$. The formula says:

$$\partial P_{x,y} = I_{x,y,c} + H_{x,y,c}$$

Here:

(1) $\partial P_{x,y}$ is the "q-derivative" of the Kazhdan-Lusztig polynomial. It is defined by the formula

$$\partial P_{x,y} = \frac{P_{x,y}(q) - q^{\ell(y)-\ell(x)} P_{x,y}(q^{-1})}{1-q}.$$

(This polynomial is rather standard in the theory, and the Kazhdan-Lusztig polynomial $P_{x,y}$ can be recovered from it, thanks to the degree bounds for Kazhdan-Lusztig polynomials.)

(2) $I_{x,y,c}$ is the "inductive piece", which depends only on the (inductively known) Kazhdan-Lusztig polynomials $P_{z,y}$ and $P_{x,z}$ for for $z \in (x, c]$.

(3) $H_{x,y,c}$ is the "hypercube cluster piece" which only depends on the (inductively known) Kazhdan-Lusztig polynomials $P_{u,y}$ for u in the set

$$\mathcal{H}_{x,y,c} = \{tx \in [x, y] \mid tx \notin [x, c]\}.$$

(That is, these are precisely the vertices neighbouring x in the Bruhat graph, which do not lie in $[x, c]$.) More precisely, its definition is as follows. First consider the polynomial

$$\widetilde{H}_{x,y,c} = \sum_{\emptyset \neq I \subset \mathcal{H}_{x,y,c}} (q-1)^{|I|} P_{\theta(I),y}$$

[1] The details are too involved for an Oberwolfach report, but will appear soon on the arxiv.

where $\theta(I)$ is defined to be the lcm (in the sense of posets) of all the elements of I. (Note that the existence of this lcm is non-trivial: Bruhat order rarely forms a lattice.) Then we define:

$$\mathcal{H}_{x,y,c} = q^{\ell(y)-\ell(x)-1}\widetilde{H}_{x,y,c}(q^{-1}).$$

Where does this formula come from? Consider a slice $S_{x,y}$ to the orbit BxB/B to the Schubert variety $\overline{ByB/B}$ inside the flag variety G/B for $G = GL_{n+1}(\mathbb{C})$. It is a standard result that

$$\partial P_{x,y} = \sum_{i\geq 0} \dim IH^i(\mathbb{P}\dot{S}_{x,y}, \mathbb{Q})q^{i/2}$$

where $\mathbb{P}\dot{S}_{x,y}$ denotes the quotient by an attractive torus action of the $S_{x,y} := S_{x,y} - \{x\}$. By considering a torus action which fixes $GL_n/B \subset GL_{n+1}/B$ and is non-trivial on the rest, one can produce an open/closed decomposition

$$\mathbb{P}\dot{S}_{x,y} = U \sqcup \mathbb{P}^m$$

such that U retracts onto $\mathbb{P}\dot{S}_{x,c}$. This allows one to express $IH^i(\mathbb{P}\dot{S}_{x,y}, \mathbb{Q})$ in terms of the cohomology of a sheaf on $\mathbb{P}\dot{S}_{x,c}$, and a T-equivariant sheaf on $\mathbb{P}^m$. This the the origin of the inductive and hypercube cluster pieces above.

Finally, the most exciting aspect of this formula is that it conjecturally solves combinatorial invariance for symmetric groups. If we do not know the labelling of the vertices in the Bruhat interval, we cannot recover c. However we can abstract the properties of the sub-interval $[x, c]$ inside $[x, y]$. The most important of these properties appear to be the fact that $[x, c]$ is "diamond complete" (a notion introduced by Patimo [2]) and the fact that the edges leaving $[x, c]$ from any vertex $v \in [x, c]$ "span a hypercube cluster". (Basically, one asks for the existence of lcms, so that the above formula for the hypercube cluster piece makes sense.) We conjecture that this formula holds for any $c' \in [x, y]$ such that $[x, c']$ satisfies these two properties. We have checked this conjecture on over a million Bruhat intervals.

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Cohomological Hall algebras of sheaves on surfaces and affine Yangians (in progress)

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(joint work with E. Diaconescu, M. Porta, F. Sala and E. Vasserot)

Let S be a smooth quasi-projective complex surface, and let $j : C \to S$ be a closed and reduced curve, with at most normal crossing singularities. We denote by $\mathcal{C}oh_c(S)$ and $\mathcal{C}oh_C(S)$ the stacks classifiying coherent sheaves on S with proper support, and the stack classifying coherent sheaves on S with set-theoretical support included in C. Note that while $\mathcal{C}oh_c(S)$ is an Artin stack which is locally of finite type, $\mathcal{C}oh_C(S)$ is only an Ind-Artin stack. Let us set

$$\mathbb{Y}_S = H_*(\mathcal{C}oh_c(S), \mathbb{Q}), \qquad \mathbb{Y}_{S,C} = H_*(\mathcal{C}oh_C(S), \mathbb{Q})$$

where H_* stands for Borel-Moore homology.

Theorem 1. *The following holds :*

1. *The spaces $\mathbb{Y}_S, \mathbb{Y}_{S,C}$ may be naturally equipped with graded associative, topological algebras structure;*
2. *The closed embedding $j : C \to S$ induces a natural continuous algebra homomorphism $\mathbb{Y}_{S,C} \to \mathbb{Y}_S$;*
3. *Let $k : C' \to C$ be a closed embedding. Then the map k induces a natural continuous algebra homomorphism $\mathbb{Y}_{S,C'} \to \mathbb{Y}_{S,C}$;*
4. *The algebra $\mathbb{Y}_{S,C}$ only depends on the formal neighborhood $\widehat{C}_S$ of C inside S; i.e. if $j : C \to S, j' : C \to S'$ are closed embedding and $a : \widehat{C}_S \xrightarrow{\sim} \widehat{C}_{S'}$ then a induces an isomorphism $\mathbb{Y}_{S,C} \xrightarrow{\sim} \mathbb{Y}_{S',C}$.*

Theorem 1 has an obvious analog if we replace Borel-Moore homology by K-theory (more precisely, G-theory). One can also hope to upgrade this at various categorical levels. One can also incorporate the action of a torus (or a reductive algebraic group) in the picture and consider equivariant versions. Finally, one can replace the curve C by a zero-dimensional (reduced) subscheme $P \subset S$ and consider the algebra $\mathbb{Y}_{S,P}$. Theorem 1 uniformizes and generalizes several previously known constructions, see [Mi], [SS] (when $S = T^*C$), [KV], [Zh] (for zero-dimensional subschemes) and [PS] (categorical versions).

A first set of interesting examples arises when S is a surface and C is a union of transversally intersecting $\mathbb{P}^1$s, whose intersection matrix is a (generalized) Dynkin diagram (in particular, every irreducible component of C is of self-intersection -2); such configurations arise for instance for resolutions of Kleinian singularities (in which case the Dynkin diagram is of finite type) or for elliptic surfaces (in which case the Dynkin diagram is of affine type). Using Theorem 1, (3) and (4) one is tempted to reduce the study of $\mathbb{Y}_{S,C}$ in such situation to the case of a single $\mathbb{P}^1$ (i.e. $\mathbb{P}^1 \subset T^*\mathbb{P}^1$), and two intersecting $\mathbb{P}^1$s (i.e. the case of a resolution of a Kleinian singularity of type A_2). This is indeed possible in many cases thanks to the following result.

Theorem 2. *Let S, C be as above. Assume that C is a union of transversally intersecting (-2)-rational curves, and that there exists a collection of closed subsets $D_1 \subset D_2 \subset \cdots \subset D_n = C$ such that $D_i \backslash D_{i-1}$ is irreducible and cohomologically pure for all i. Set $C_i = \overline{D_i \backslash D_{i-1}}$. Then*

(1) *the algebra $\mathbb{Y}_{S,C}$ is topologically generated by the collection of subalgebras $\mathbb{Y}_{S,C_i}$;*

(2) *the multiplication map*

$$\mathbb{Y}_{S,C_1} \otimes_{\mathbb{Y}_{S,C_1 \cap C_2}} \mathbb{Y}_{S,C_2} \otimes_{\mathbb{Y}_{S,C_2 \cap C_3}} \cdots \otimes_{\mathbb{Y}_{S,C_{n-1} \cap C_n}} \mathbb{Y}_{S,C_n} \longrightarrow \mathbb{Y}_{S,C}$$

is a dense embedding.

The heuristic meaning of statement (2) is that $\mathbb{Y}_{S,C}$ is generated by the collection of subalgebras $\mathbb{Y}_{S,C_i}$ modulo some *local* relations, i.e. relations involving only a pair of intersecting $\mathbb{P}^1$s. In that direction, we have the following result

Theorem 3. *Assume that $S = T^*\mathbb{P}^1$, $C = \mathbb{P}^1$. Consider the action of $T = (\mathbb{C}^*)^2$ on S given by loop rotation and dilation along the fiber. Let $\{0, \infty\}$ be the two fixed points. Then*

(1) *$\mathbb{Y}_{S,\{0\}}, \mathbb{Y}_{S,\{\infty\}}$ are isomorphic to (suitably specializations of) the affine Yangian of $\mathfrak{gl}(1)$ introduced in* [SV]*;*

(2) *$\mathbb{Y}_{S,C}$ is a (limit of) subquotient algebra(s) of the (two-parameter) affine Yangian of $\mathfrak{gl}(2)$, see e.g.* [Ko].

More generally, the same techniques allow one prove that for $S = \widetilde{\mathbb{C}^2/\Gamma}$ a resolution of Kleinian singularity of type $\Gamma = \mathbb{Z}/N\mathbb{Z}$, and $C \subset S$ the exceptional fiber, the algebra $\mathbb{Y}_{S,C}$ is a (limit of) subquotient algebra(s) of the (two-parameter) affine Yangian of $\mathfrak{gl}(N)$. However, the precise description of the subalgebras in question is more involved.

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Monoidal Jantzen filtrations for quantum affine algebras

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(joint work with David Hernandez)

Let $U_q(\widehat{\mathfrak{g}})$ be the (untwisted) quantum affine algebra associated with a complex simple Lie algebra $\mathfrak{g}$ and a generic quantum parameter $q \in \mathbb{C}^\times$. This is a Hopf algebra over $\mathbb{C}$. Its finite-dimensional modules form an interesting abelian monoidal category $\mathscr{C}$. For example, the category $\mathscr{C}$ is neither semisimple as an abelian category, nor braided as a monoidal category. In particular, the tensor product $V \otimes W$ is not isomorphic to its opposite $W \otimes V$ for general simple modules $V, W \in \mathscr{C}$. Nevertheless, their Jordan-Hölder factors coincide up to reordering. In other words, we have $[V \otimes W] = [W \otimes V]$ in the Grothendieck ring $K(\mathscr{C})$ for any simple modules $V, W \in \mathscr{C}$, and hence $K(\mathscr{C})$ is commutative. Indeed, this commutativity follows from the injectivity of the so-called q-character homomorphism $\chi_q \colon K(\mathscr{C}) \to \mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1} \mid i \in I, a \in \mathbb{C}^\times]$ due to Frenkel-Reshetikhin [3], where I is an index set of the simple roots of $\mathfrak{g}$.

By the classification result due to Chari-Pressley [1], the set of simple modules in $\mathscr{C}$ (modulo isomorphism) is in bijection with the set $\mathcal{M} \subset \mathcal{Y}$ of monomials in the variables $Y_{i,a}$. For each $m \in \mathcal{M}$, the corresponding simple module $L(m)$ is of highest weight m, namely $\chi_q(L(m))$ has m as its highest term. The problem to compute $\chi_q(L(m))$ for all $m \in \mathcal{M}$ is of fundamental importance. At the present moment, a closed formula (like the Weyl character formula) is not known.

One possible strategy is to find an algorithm to compute $\chi_q(L(m))$ recursively, analogous to the Kazhdan-Lusztig algorithm. Let us explain this idea briefly. For each $x \in I \times \mathbb{C}^\times$, the q-character of the simple module $V_x \coloneqq L(Y_x)$ (called a fundamental module) can be computed by an algorithm due to Frenkel-Mukhin [2]. For each monomial $m = Y_{x_1} \cdots Y_{x_d} \in \mathcal{M}$, if $(x_1, \dots, x_d)$ is ordered suitably, the corresponding tensor product of the fundamental modules $M(m) \coloneqq V_{x_1} \otimes \cdots \otimes V_{x_d}$ has a simple head isomorphic to $L(m)$. Moreover, there exists a partial ordering of $\mathcal{M}$ (called the Nakajima partial ordering) such that we have

$$[M(m)] = [L(m)] + \sum_{m' < m} P_{m,m'}[L(m')]$$

in $K(\mathscr{C})$. The module $M(m)$ is called a standard module. Since we know $\chi_q(M(m))$, it is enough to compute the multiplicities $P_{m,m'}$. For this purpose, we consider a one-parameter (non-commutative) deformation of $K(\mathscr{C})$, called the quantum Grothendieck ring. It was introduced by Nakajima [6] and by Varagnolo-Vasserot [7] for simply-laced $\mathfrak{g}$, and by Hernandez [5] for general $\mathfrak{g}$. Recently, it was also studied in relation with quantum cluster algebras and derived Hall algebras. The quantum Grothendieck ring $K_t(\mathscr{C})$ is an $\mathbb{Z}[t^{\pm 1/2}]$-subalgebra of a quantum torus $\mathcal{Y}_t$ deforming $\mathcal{Y}$, stable under a natural anti-involution $y \mapsto \bar{y}$ of $\mathcal{Y}_t$, and comes with a standard $\mathbb{Z}[t^{\pm 1/2}]$-basis $\{M_t(m)\}_{m \in \mathcal{M}}$. Under the specialization $t \to 1$, $M_t(m)$ goes to $[M(m)]$. We can prove (see [6, 5]) that there exists the canonical basis $\{L_t(m)\}_{m\in\mathcal{M}}$ satisfying $\overline{L_t(m)} = L_t(m)$ and

$$M_t(m) = L_t(m) + \sum_{m'<m} P_{m,m'}(t)L_t(m')$$

for some $P_{m,m'}(t) \in t\mathbb{Z}[t]$. This characterization enables us to compute the polynomials $P_{m,m'}(t)$ recursively. When $\mathfrak{g}$ is simply-laced, the following result was obtained by using perverse sheaves on quiver varieties.

Theorem 1 (Nakajima [6], Varagnolo-Vasserot [7]). *When $\mathfrak{g}$ is simply-laced, the following properties hold:*

(P1) *Analog of Kazhdan-Lusztig conjecture: under the specialization $t \to 1$, $L_t(m)$ goes to $[L(m)]$, or equivalently, we have $P_{m,m'}(1) = P_{m,m'}$.*
(P2) *Positivity: for any $m' < m$, we have $P_{m,m'}(t) \in \mathbb{Z}_{\geq 0}[t]$.*

Later, Hernandez [5] conjectured that these properties hold for general $\mathfrak{g}$. Very recently, we obtained some pieces of evidence of this conjecture.

Theorem 2 (F.-Hernandez-Oh-Oya [4]). *The property* (P1) *also holds when $\mathfrak{g}$ is of type B. The property* (P2) *holds for general $\mathfrak{g}$.*

Having these results, we propose the following question.

Question 3. *What is representation-theoretic meaning of $K_t(\mathscr{C})$ or $P_{m,m'}(t)$?*

Here we try to answer this question by introducing "monoidal Jantzen filtrations" for any tensor products of fundamental modules. For any sequence $\underline{x} = (x_1, \dots, x_d)$ of elements of $I \times \mathbb{C}^\times$, let $V_{\underline{x}} := V_{x_1} \otimes \cdots \otimes V_{x_d}$ be the corresponding tensor product, which is not necessarily a standard module. By using R-matrices, we are going to define a $U_q(\widehat{\mathfrak{g}})$-modules filtration

$$V_{\underline{x}} \supset \cdots \supset F_{-1} \supset F_0 \supset F_1 \supset \cdots \tag{1}$$

satisfying $F_{\ll 0} = V_{\underline{x}}$ and $F_{\gg 0} = \{0\}$. Then we define an element $[V_{\underline{x}}]_t$ of the t-deformed Grothendieck group $K(\mathscr{C})_t := K(\mathscr{C}) \otimes \mathbb{Z}[t^{\pm 1/2}]$ by

$$[V_{\underline{x}}]_t := \sum_{n\in\mathbb{Z}} [\mathrm{gr}_n^F V_{\underline{x}}] \otimes t^n.$$

Observe that the sets $\{[L(m)]_t := [L(m)] \otimes 1\}_{m\in\mathcal{M}}$ and $\{[M(m)]_t\}_{m\in\mathcal{M}}$ both form $\mathbb{Z}[t^{\pm 1/2}]$-bases of $K(\mathscr{C})_t$. Then we define a $\mathbb{Z}[t^{\pm 1/2}]$-bilinear map $*\colon K(\mathscr{C})_t \times K(\mathscr{C})_t \to K(\mathscr{C})_t$ by

$$[M(m)]_t * [M(m')]_t := t^{\gamma(m,m')}[M(m) \otimes M(m')]_t,$$

where $\gamma\colon \mathcal{M} \times \mathcal{M} \to \frac{1}{2}\mathbb{Z}$ is a skew-symmetric bilinear form on $\mathcal{M}$ related to the structure constants of the quantum torus $\mathcal{Y}_t$. Also, $K(\mathscr{C})_t$ is endowed with a natural involution $\overline{X \otimes f(t)} = X \otimes f(t^{-1})$. Now we propose the following :

Conjecture 4. *The pair $(K(\mathscr{C})_t, *)$ defines a $\mathbb{Z}[t^{\pm 1/2}]$-algebra with anti-involution, and it is isomorphic to the quantum Grothendieck ring $K_t(\mathscr{C})$ identifying the standard basis $\{M_t(m)\}_{m\in\mathcal{M}}$ with the basis $\{[M(m)]_t\}_{m\in\mathcal{M}}$.*

Remark 5.

(1) The associativity of the map $*$ is unclear from the definition.
(2) Conjecture 4 implies the above properties (P1) & (P2).

At the present moment, we have the following theorem as a piece of evidence of Conjecture 4. Our proof also uses perverse sheaves on quiver varieties.

Theorem 6 (F.-Hernandez)**.** *Conjecture 4 is true when $\mathfrak{g}$ is simply-laced.*

In the remaining part, we explain how to construct the filtration (1). Let $\mathcal{O} := \mathbb{C}[\![u]\!] \subset \mathcal{K} := \mathbb{C}(\!(u)\!)$. For each fundamental module V_x $(x \in I \times \mathbb{C}^\times)$, we can define its formal spectral parameter deformation $\widehat{V}_x$ over $\mathcal{O}$, which is a $U_q(\widehat{\mathfrak{g}})_{\mathcal{O}}$-module, in a suitable way so that, after the localization $\mathcal{O} \to \mathcal{K}$, we have a unique $U_q(\widehat{\mathfrak{g}})_{\mathcal{K}}$-linear isomorphism :

$$\left(\widehat{V}_x \otimes_{\mathcal{O}} \widehat{V}_y\right)_{\mathcal{K}} \simeq \left(\widehat{V}_y \otimes_{\mathcal{O}} \widehat{V}_x\right)_{\mathcal{K}} \tag{2}$$

respecting highest weight vectors, for any $x, y \in I \times \mathbb{C}^\times$. The isomorphism (2) is called the normalized R-matrix. For a sequence $\underline{x} = (x_1, \ldots, x_d)$, we consider the corresponding $\mathcal{O}$-deformed tensor product $\widehat{V}_{\underline{x}} := \widehat{V}_{x_1} \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} \widehat{V}_{x_d}$. We regard $\widehat{V}_x$ as an $\mathcal{O}$-lattice of its localization $(\widehat{V}_{\underline{x}})_{\mathcal{K}}$. For a fixed $\underline{x}$, let $\underline{x}_0$ denote another sequence obtained from $\underline{x}$ by reordering so that $V_{\underline{x}_0}$ is a standard module, and let $\underline{x}'_0$ denote the sequence opposite to $\underline{x}_0$. Then we have the $U_q(\widehat{\mathfrak{g}})_{\mathcal{K}}$-linear isomorphisms $R\colon (\widehat{V}_{\underline{x}_0})_{\mathcal{K}} \to (\widehat{V}_{\underline{x}})_{\mathcal{K}}$ and $R'\colon (\widehat{V}_{\underline{x}'_0})_{\mathcal{K}} \to (\widehat{V}_{\underline{x}})_{\mathcal{K}}$ by composing the normalized R-matrices. Finally, for each $n \in \mathbb{Z}$, we define

$$F_n := \pi\left(\widehat{V}_{\underline{x}} \cap \sum_{k,l \in \mathbb{Z}; k+l=n} R(u^k \widehat{V}_{\underline{x}_0}) \cap R'(u^l \widehat{V}_{\underline{x}'_0})\right),$$

where $\pi\colon \widehat{V}_{\underline{x}} \to V_{\underline{x}}$ is the specialization $u \to 0$.

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Root Components for Tensor Product of Affine Kac–Moody Lie algebra modules

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(joint work with Samuel Jeralds)

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody Lie algebra, and fix two dominant integral weights $\lambda,\ \mu \in \mathscr{P}^+$. To these, we can associate the integrable, highest weight (irreducible) representations $V(\lambda)$ and $V(\mu)$. Then, the content of the tensor decomposition problem is to express the product $V(\lambda) \otimes V(\mu)$ as a direct sum of irreducible components; that is, to find the decomposition

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu \in \mathscr{P}^+} V(\nu)^{\oplus m^{\nu}_{\lambda,\mu}},$$

where $m^{\nu}_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$ is the multiplicity of $V(\nu)$ in $V(\lambda) \otimes V(\mu)$. While this is a classical problem with a straightforward statement, determining the multiplicities $m^{\nu}_{\lambda,\mu}$ exactly–or even determining when $m^{\nu}_{\lambda,\mu} > 0$–is a challenging endeavor. Various algebraic, geometric, and combinatorial methods have been developed to understand the tensor decomposition problem; see [Ku4] for a survey in the case of (finite dimensional) semisimple Lie algebras.

While having a complete description for the components of a tensor product is desirable, many significant results in the literature demonstrate the existence of "families" of components; that is, components that are uniformly described and exist for tensor product decompositions regardless of $\mathfrak{g}$. One such example is given by the *root components* $V(\lambda + \mu - \beta)$ for a positive root β.

We show the existence of root components for affine Lie algebras generalizing the corresponding result in the finite case (i.e., when $\mathfrak{g}$ is a semisimple Lie algebra) as in [Ku1]. Recall that by a *Wahl triple* (introduced in [Ku1] though not christened as *Wahl triple* there) we mean a triple $(\lambda, \mu, \beta) \in (\mathscr{P}^+)^2 \times \Phi^+$ such that

(P1) $\lambda + \mu - \beta \in \mathscr{P}^+$, and
(P2) If $\lambda(\alpha_i^\vee) = 0$ or $\mu(\alpha_i^\vee) = 0$, then $\beta - \alpha_i \notin \Phi \sqcup \{0\}$,

where Φ (resp. Φ^+) is the set of all the roots (resp., positive roots).

The main representation theoretic result of the paper is the following theorem.

Theorem I. *For any affine Kac-Moody Lie algebra* $\mathfrak{g}$ *and Wahl triple* $(\lambda, \mu, \beta) \in (\mathscr{P}^+)^2 \times \Phi^+$,

$$V(\lambda + \mu - \beta) \subset V(\lambda) \otimes V(\mu).$$

We construct the proof in three parts. First, notice that the conditions (P1) and (P2) are invariant under adding δ; that is, if (λ, μ, β) is a Wahl triple, then so is $(\lambda, \mu, \beta + k\delta)$ for any $k \in \mathbb{Z}_{\geq 0}$. This allows us to make use of the Goddard-Kent-Olive construction of the Virasoro algebra action on the tensor product $V(\lambda)\otimes V(\mu)$ and we explore its action on the subspaces $W^{\lambda+\mu-\beta}$. This reduces the problem to certain 'maximal root components'. In the second part, closely following the construction of root components for simple Lie algebras as in [Ku1], we show the existence of the bulk of the maximal root components. Finally, in the third part, we construct the remaining maximal root components that are excluded from the previous methods explicitly using familiar, but ad hoc, constructions from the general tensor decomposition problem using the PRV components.

We next prove the corresponding geometric results. Let $\mathcal{G}$ be the 'maximal' Kac-Moody group associated to the symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ and let $\mathcal{P}$ be a standard parabolic subgroup of $\mathcal{G}$ corresponding to a subset S of the set of simple roots $\{\alpha_1, \cdots, \alpha_\ell\}$, i.e., S is the set of simple roots of the Levi group of $\mathcal{P}$. In the sequel, we abbreviate S as the subset of $\{1, \cdots, \ell\}$. In particular, for $S = \emptyset$, we have the standard Borel subgroup $\mathcal{B}$ (corresponding to the Borel subalgebra $\mathfrak{b}$). Let W be the Weyl group of $\mathfrak{g}$ and let $W'_{\mathcal{P}}$ be the set of smallest length coset representatives in $W/W_{\mathcal{P}}$, where $W_{\mathcal{P}}$ is the subgroup of W generated by the simple reflections $\{s_k\}_{k\in S}$. For any pro-algebraic $\mathcal{P}$-module M, by $\mathscr{L}(M)$ we mean the corresponding homogeneous vector bundle on $\mathcal{X}_{\mathcal{P}} := \mathcal{G}/\mathcal{P}$ associated to the principal $\mathcal{P}$-bundle: $\mathcal{G} \to \mathcal{G}/\mathcal{P}$ by the representation M of $\mathcal{P}$.

For any integral weight $\lambda \in \mathfrak{h}^*$ (where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$ with the corresponding standard maximal torus $\mathcal{H}$ of $\mathcal{G}$), such that $\lambda(\alpha_k^\vee) = 0$ for all $k \in S$, the one dimensional $\mathcal{H}$-module $\mathbb{C}_\lambda$ (given by the character λ) admits a unique $\mathcal{P}$-module structure (extending the $\mathcal{H}$-module structure); in particular, we have the line bundle $\mathscr{L}(\mathbb{C}_\lambda)$ on $\mathcal{X}_{\mathcal{P}}$. We abbreviate the line bundle $\mathscr{L}(\mathbb{C}_{-\lambda})$ by $\mathscr{L}(\lambda)$ and its restriction to the Schubert variety $X_w^{\mathcal{P}} := \overline{\mathcal{B}w\mathcal{P}/\mathcal{P}}$ by $\mathscr{L}_w(\lambda)$ (for any $w \in W'_{\mathcal{P}}$). Given two line bundles $\mathscr{L}(\lambda)$ and $\mathscr{L}(\mu)$, we can form their external tensor product to get the line bundle $\mathscr{L}(\lambda \boxtimes \mu)$ on $\mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}$. A dominant integral weight μ is called *S-regular* if $\mu(\alpha_k^\vee) = 0$ if and only if $k \in S$. The set of such weights is denoted by $\mathscr{P}_S^{+^o}$.

Then, we prove that, for any $\mu \in \mathscr{P}_S^{+^o}$ and $w \in W'_{\mathcal{P}}$ such that $X_w^{\mathcal{P}}$ is $\mathcal{P}$-stable under the left multiplication:

$$H^p(X_w^{\mathcal{P}}, \mathscr{I}_e^k \otimes \mathscr{L}_w(\mu)) = H^p(X_w^{\mathcal{P}}, (\mathscr{O}_w/\mathscr{I}_e^k) \otimes \mathscr{L}_w(\mu)) = 0, \text{ for all } p > 0, k = 1, 2,$$

where $\mathscr{I}_e$ is the ideal sheaf of $X_w^{\mathcal{P}}$ at the base point e and $\mathscr{O}_w$ denotes the structure sheaf of $X_w^{\mathcal{P}}$. We further explicitly determine $H^0(X_w^{\mathcal{P}}, \mathscr{I}_e^2 \otimes \mathscr{L}_w(\mu))$ and $H^0(X_w^{\mathcal{P}}, (\mathscr{O}_w/\mathscr{I}_e^2) \otimes \mathscr{L}_w(\mu))$.

For any $w \in W'_{\mathcal{P}}$ such that the Schubert variety $X_w^{\mathcal{P}}$ is $\mathcal{P}$-stable, define the $\mathcal{G}$-Schubert variety:

$$\hat{\mathcal{X}}_w^{\mathcal{P}} := \mathcal{G} \times^{\mathcal{P}} X_w^{\mathcal{P}}.$$

Consider the isomorphism:

$$\delta : \mathcal{G} \times^{\mathcal{P}} \mathcal{X}_{\mathcal{P}} \simeq \mathcal{X}_{\mathcal{P}} \times \mathcal{X}_{\mathcal{P}}, \ [g, x] \mapsto (g\mathcal{P}, gx), \text{ for } g \in \mathcal{G} \text{ and } x \in \mathcal{X}_{\mathcal{P}}.$$

We have the canonical embedding

$$\hat{\mathcal{X}}_w^{\mathcal{P}} \hookrightarrow \mathcal{G} \times^{\mathcal{P}} \mathcal{X}_{\mathcal{P}}.$$

In particular, we can restrict the line bundle $\mathscr{L}(\lambda \boxtimes \mu)$ via the above isomorphism δ to $\hat{\mathcal{X}}_w^{\mathcal{P}}$ to get the line bundle denoted $\mathscr{L}_w(\lambda \boxtimes \mu)$. Then, for $\lambda, \mu \in \mathscr{P}_S^{+^o}, k = 1, 2, w \in W'_{\mathcal{P}}$ as above and $p \geq 0$, we determine:

$$H^p(\hat{\mathcal{X}}_w^{\mathcal{P}}, \hat{\mathscr{I}}_e^k \otimes \mathscr{L}_w(\lambda \boxtimes \mu)) \text{ and } H^p(\hat{\mathcal{X}}_w^{\mathcal{P}}, (\hat{\mathscr{O}}_w/\hat{\mathscr{I}}_e^k) \otimes \mathscr{L}_w(\lambda \boxtimes \mu)),$$

in terms of the cohomology of the partial flag variety $\mathcal{G}/\mathcal{P}$ with coefficients in explicit homogeneous vector bundles, where $\hat{\mathscr{I}}_e$ denotes the ideal sheaf of $\widehat{\mathcal{X}}_e^{\mathcal{P}}$ in $\hat{\mathcal{X}}_w^{\mathcal{P}}$ and $\hat{\mathscr{O}}_w$ is the structure sheaf of $\hat{\mathcal{X}}_w^{\mathcal{P}}$. In fact, we show that

$$H^p(\hat{\mathcal{X}}_w^{\mathcal{P}}, \hat{\mathscr{I}}_e \otimes \mathscr{L}_w(\lambda \boxtimes \mu)) = 0, \text{ for all } p > 0.$$

Let $\Phi^+ \subset \mathfrak{h}^*$ be the set of positive roots, $\Phi_S^+ := \Phi^+ \cap (\oplus_{k\in S} \mathbb{Z}_{\geq 0}\alpha_k)$ and $\Phi^+(S) := \Phi^+ \setminus \Phi_S^+$.

We next study the vanishing of the first cohomology $H^1(\hat{\mathcal{X}}_w^{\mathcal{P}}, \hat{\mathscr{I}}_e^2 \otimes \mathscr{L}_w(\lambda \boxtimes \mu))$ and prove the following crucial result:

Proposition I. *Let $\mathfrak{g}$ be an affine Kac-Moody Lie algebra. Then, for any $\lambda, \mu \in \mathscr{P}_S^{+^o}$ (where S is an arbitrary subset of the simple roots of $\mathfrak{g}$) and any $w \in W'_{\mathcal{P}}$ such that the Schubert variety $X_w^{\mathcal{P}}$ is $\mathcal{P}$-stable, consider the following two conditions:*

(a) $H^1(\hat{\mathcal{X}}_w^{\mathcal{P}}, \hat{\mathscr{I}}_e^2 \otimes \mathscr{L}_w(\lambda \boxtimes \mu)) = 0$.

(b) For all the real roots $\beta \in \Phi^+(S)$, satisfying $S \subset \{0 \leq i \leq \ell : \beta - \alpha_i \notin \Phi^+ \sqcup \{0\}\}$ and $\lambda+\mu-\beta \in \mathscr{P}^+$, there exists a $f_\beta \in \operatorname{Hom}_{\mathfrak{b}}(\mathbb{C}_{\lambda+\mu-\beta} \otimes V(\lambda)^\vee, V(\mu))$ such that

$$X_\beta(f_\beta(\mathbb{C}_{\lambda+\mu-\beta} \otimes v_\lambda^*)) \neq 0, \textit{ for } X_\beta \neq 0 \in \mathfrak{g}_\beta,$$

where $V(\lambda)^\vee$ is the restricted dual of $V(\lambda)$, and $v_\lambda^* \neq 0 \in [V(\lambda)^\vee]_{-\lambda}$.

Then, the condition (b) implies the condition (a).

Further, we show that under the assumptions of the above proposition, the condition (b) of the proposition is satisfied in all the cases except possibly $\mathring{\mathfrak{g}}$ of type F_4 or G_2. The proof of the above proposition relies on explicit constructions of root components obtained in the earlier sections including that of the GKO operator and the following lemma:

Lemma II. *Let (λ, μ, β) be a Wahl triple for a real root β and let $V(\lambda+\mu-\beta) \subset V(\lambda) \otimes V(\mu)$ be a δ-maximal root component. Observe that $\beta \in \mathring{\Phi}^+$ or $\beta = \delta - \gamma$ for $\gamma \in \mathring{\Phi}^+$. Then, for $\beta \in \mathring{\Phi}^+$, the validity of condition (b) of Proposition I for $V(\lambda+\mu-\beta)$ implies its validity for $V(\lambda+\mu-\beta-k\delta)$ for any $k \geq 0$.*

Moreover, if $\beta = \delta - \gamma$ for $\gamma \in \mathring{\Phi}^+$, then we have an identity connecting the condition (b) of Proposition I for $V(\lambda+\mu-\delta+\gamma)$ with that of $V(\lambda+\mu-\beta)$.

Using the above Proposition I, we obtain the following main geometric result of the paper:

Theorem III. *Let $\mathfrak{g}$ be an affine Kac-Moody Lie algebra and let $w \in W'_{\mathcal{P}}$ be such that the Schubert variety $X_w^{\mathcal{P}}$ is $\mathcal{P}$-stable. Then, for any $\lambda, \mu \in \mathscr{P}_S^{+^o}$ (where S is an arbitrary subset of the simple roots) such that the condition (b) of Proposition I is satisfied for all the Wahl triples (λ, μ, β) for any real root $\beta \in \Phi^+$,*

$$H^p(\hat{\mathcal{X}}_w^{\mathcal{P}}, \hat{\mathscr{I}}_e^2 \otimes \mathscr{L}_w(\lambda \boxtimes \mu)) = 0, \textit{ for all } p > 0.$$

In particular, the canonical Gaussian map

$$H^0(\hat{\mathcal{X}}_w^{\mathcal{P}}, \hat{\mathscr{I}}_e \otimes \mathscr{L}_w(\lambda \boxtimes \mu)) \to H^0(\hat{\mathcal{X}}_w^{\mathcal{P}}, (\hat{\mathscr{I}}_e/\hat{\mathscr{I}}_e^2) \otimes \mathscr{L}_w(\lambda \boxtimes \mu))$$

is surjective.

In particular, the theorem holds for any simply-laced $\mathring{\mathfrak{g}}$ *and* $\mathring{\mathfrak{g}}$ *of types* B_ℓ, C_ℓ*. Moreover, it also holds for* $\mathring{\mathfrak{g}}$ *of type* F_4 *in the case* $\mathcal{P}$ *is the Borel subgroup* $\mathcal{B}$*.*

As a fairly straight forward corollary of the above theorem, taking inverse limits, we get the following:

Corollary IV. *Under the notation and assumptions of the above theorem, the canonical Gaussian map*

$$H^0(\mathcal{X}_\mathcal{P} \times \mathcal{X}_\mathcal{P}, \mathscr{I}_D \otimes \mathscr{L}(\lambda \boxtimes \mu)) \to H^0(\mathcal{X}_\mathcal{P} \times \mathcal{X}_\mathcal{P}, (\mathscr{I}_D/\tilde{\mathscr{I}}_D^2) \otimes \mathscr{L}(\lambda \boxtimes \mu))$$

is surjective, where $\mathscr{I}_D$ *is the ideal sheaf of the diagonal* $D \subset \mathcal{X}_\mathcal{P} \times \mathcal{X}_\mathcal{P}$ *and* $\tilde{\mathscr{I}}_D^2$ *is defined as* $\varprojlim_w \hat{\mathscr{I}}_e(w)^2$.

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Affine Grassmannian slices and categorification

Joel Kamnitzer

(joint work with Ben Webster, Alex Weekes, Oded Yacobi)

Under the geometric Satake equivalence, slices in the affine Grassmannian give a geometric incarnation of dominant weight spaces in representations of reductive groups. These affine Grassmannian slices carry a natural Poisson structure and are quantized by algebras known as truncated shifted Yangians [3]. From this perspective, we expect to categorify these weight spaces using category $\mathcal{O}$ for these truncated shifted Yangians.

The slices in the affine Grassmannian and truncated shifted Yangians can also be defined as special cases of the Coulomb branch construction of Braverman-Finkelberg-Nakajima [1]. From this perspective, we find many insights. First, we can generalize affine Grassmannian slices to the case of non-dominant weights and arbitrary symmetric Kac-Moody Lie algebras. Second, we establish a link with modules for KLRW algebras [2]. Finally, in forthcoming work, we define a categorical $\mathfrak{g}$-action on the categories $\mathcal{O}$, using Hamiltonian reduction between Coulomb branch algebras [4]. In particular, we define restriction and induction functors which generalize the Bezrukavnikov-Etingof restriction/induction functors for Cherednik algebras.

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Parabolic Hilbert schemes and rational Cherednik algebra

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(joint work with Eugene Gorsky, José Simental)

In this talk we explicitly construct an action of the rational Cherednik algebra $H_{1,m/n}(\mathcal{S}_n, \mathbb{C}^n)$ corresponding to the permutation representation of $\mathcal{S}_n$ on the $\mathbb{C}^*$-equivariant homology of parabolic Hilbert schemes of points on the plane curve singularity $\{x^m = y^n\}$ for coprime m and n. The authors use this in [5] to construct actions of quantized Gieseker algebras on parabolic Hilbert schemes on the same plane curve singularity, and actions of the Cherednik algebra at $t = 0$ on the equivariant homology of parabolic Hilbert schemes on the non-reduced curve $\{y^n = 0\}$. Our main tool is the study of the combinatorial representation theory of the rational Cherednik algebra (RCA) via the subalgebra generated by Dunkl-Opdam elements. We use an alternative presentation of $H_{1,m/n}(\mathcal{S}_n, \mathbb{C}^n)$.

Fix coprime positive integers m and n, and let $C := \{x^m = y^n\}$ be a plane curve singularity in $\mathbb{C}^2$. Note that for every ideal $I \subseteq \mathcal{O}_C = \mathbb{C}[[x,y]]/(x^m - y^n)$ we have that $\dim(I/xI) = n$. We consider the *parabolic Hilbert scheme* $\mathrm{PHilb}_{k,n+k}(C)$ that is the following moduli space of flags

$$\mathrm{PHilb}_{k,n+k}(C) := \{\mathcal{O}_C \supset I_k \supset I_{k+1} \supset \cdots \supset I_{k+n} = xI_k\} \tag{1}$$

where I_s is an ideal in the ring of functions $\mathcal{O}_C$ of codimension s. Moreover, we set $\mathrm{PHilb}^x(C) := \sqcup_k \mathrm{PHilb}_{k,n+k}(C)$. The natural $\mathbb{C}^*$ action on C naturally lifts to $\mathrm{PHilb}^x(C)$. Since m and n are coprime, the fixed points are precisely the flags of monomial ideals. In particular, the classes of these fixed points form a basis for the localized equivariant cohomology. The first main result is the following.

Theorem 1. *There is a geometric action of $H_{1,m/n}(\mathcal{S}_n, \mathbb{C}^n)$ on the localized $\mathbb{C}^*$-equivariant homology of $\mathrm{PHilb}^x(C)$. Moreover, with this action $H_*^{\mathbb{C}^*}(\mathrm{PHilb}^x(C))$ gets identified with the simple highest weight module $L_{m/n}(\mathrm{triv})$.*

Recall that the rational Cherednik algebra $H_{t,c} := H_{t,c}(\mathcal{S}_n, \mathbb{C}^n)$ contains the trivial idempotent $e := \frac{1}{n!}\sum_{p \in \mathcal{S}_n} p$, and we can form the spherical subalgebra $eH_{t,c}e$. As a consequence of Theorem 1 we get that the spherical subalgebra acts on the equivariant homology of the Hilbert scheme $\mathrm{Hilb}(C) := \sqcup_k \mathrm{Hilb}_k(C)$.

Corollary 2. *There is an action of the spherical RCA $eH_{1,m/n}(\mathcal{S}_n, \mathbb{C}^n)e$ on the localized $\mathbb{C}^*$-equivariant homology of $\mathrm{Hilb}(C)$. Moreover, with this action $H_*^{\mathbb{C}^*}(\mathrm{Hilb}(C))$ gets identified with $eL_{m/n}(\mathrm{triv})$.*

Remark 3. By [7, 8] the homology of the Hilbert schemes of singular curves is closely related to the homology of the corresponding compactified Jacobian, equipped with a certain "perverse" filtration. By [4, 9, 10, 11] the latter homology carries an action of the spherical trigonometric Cherednik algebra. Furthermore by [9, 10] the associated graded space admits a natural action of the spherical rational Cherednik algebra corresponding to the *reflection representation* of $\mathcal{S}_n$ (also known as spherical rational Cherednik algebra of $\mathfrak{sl}_n$). The construction of this action uses global Springer theory developed by Yun [13].

The main advantage of our proof of Theorem 1 is that it does not use compactified Jacobians or perverse filtration at all. The generators of $H_{1,m/n}(\mathcal{S}_n, \mathbb{C}^n)$ are identified with certain explicit operators in the homology of $\mathrm{PHilb}^x(C)$.

The idea behind the proof of Theorem 1 is to identify a basis in $L_{m/n}(\mathrm{triv})$ that corresponds to the fixed-point basis in $H_*^{\mathbb{C}^*}(\sqcup_k \mathrm{PHilb}_{k,n+k}(C))$. Our main tool to construct this basis is a presentation of the RCA $H_{t,c}(\mathcal{S}_n, \mathbb{C}^n)$ that is better-suited for this purpose than the usual presentation. To lighten notation, we write $H_c = H_{1,c}(\mathcal{S}_n, \mathbb{C}^n)$. Recall that, in its usual presentation, the algebra H_c has generators x_i, y_i $(i = 1, \ldots, n)$ and $\mathcal{S}_n$. It is naturally graded, with x_i of degree 1, y_i of degree -1 and $\mathcal{S}_n$ in degree zero. Dunkl and Opdam [2] constructed a family of commuting operators $u_1, \ldots, u_n$ of degree 0 in H_c. The algebra H_c is, in fact,

y^2	xy^2	x^2y^2	1						
y	xy	x^2y	x^3y	x^4y	3				
1	x	x^2	x^3	x^4	x^5	x^6	2		

FIGURE 1. A flag of monomial ideals in $\mathrm{PHilb}_{15,15+3}(x^4 = y^3)$: $I_{15} = \langle x^3y^2, x^5y\rangle, I_{16} = \langle x^4y^2, x^5y, x^7\rangle, I_{17} = \langle x^4y^2, x^5y\rangle$.

generated by the group algebra of $\mathcal{S}_n$, two additional generators

$$\tau := x_1(12\cdots n), \lambda := (12\cdots n)^{-1}y_1,$$

and optionally the u_i where $u_i := x_iy_i - c\sum_{j<i}(ij)$. It is clear that τ, λ and $\mathcal{S}_n$ already generate the algebra since one can obtain x_1 and y_1 (and hence all x_i and y_i) using them. In the interest of space, we will skip giving the list of relations between τ, λ, u_i and the generators of $\mathcal{S}_n$ here. This presentation of the algebra H_c has already appeared in the more complicated cyclotomic setting in the work of Griffeth [6] and Webster [12]. We use this presentation of the algebra H_c when c is a rational number with denominator precisely n to simultaneously diagonalize the u_i on the polynomial representation $\Delta_c(\mathrm{triv})$ and give an explicit combinatorial description of the weights. We prove that the action of the operators τ and λ sends a u_k-weight vector to a multiple of another weight vector, and describe the action of $\mathcal{S}_n$ on an weight basis explicitly.

The proof of Theorem 1 is based, roughly speaking, on the comparison of the basis of fixed points in $H^*(\sqcup_k \mathrm{PHilb}_{k,n+k}(C))$ with a basis comprised of u-weight vectors.

We have a geometric analogue of the shift operator τ. Given a flag $I_k \supset I_{k+1} \supset \cdots \supset I_{k+n} = xI_k$, we can consider the flag $I_{k+1} \supset \cdots \supset I_{k+n} = xI_k \supset I_{k+n+1} = xI_{k+1}$. This defines a map $T : \mathrm{PHilb}_{k,n+k} \to \mathrm{PHilb}_{k+1,n+k+1}$. It is slightly more subtle to define the operator Λ corresponding to λ.

We define the line bundles $\mathcal{L}_i$, $1 \le i \le n$ on the parabolic flag Hilbert scheme as follows. The fiber of $\mathcal{L}_i$ over the flag $I_k \supset I_{k+1} \supset \cdots \supset I_{k+n} = xI_k$ is I_{k+i-1}/I_{k+i}. Below we give a more detailed version of Theorem 1.

Theorem 4. *(a) The total localized equivariant homology*

$$U = \bigoplus_{k=0}^{\infty} H_*^{\mathbb{C}^*}(\mathrm{PHilb}_{k,n+k})$$

carries an action of the rational Cherednik algebra $H_{m/n}$. The action of $\mathcal{S}_n$ is the usual Springer action, $u_{n+1-i} + m(n-1)$ correspond to capping with $c_1(\mathcal{L}_i)$ and the action of τ and λ correspond to the operators T and Λ on U.

(b) The representation U is irreducible and isomorphic to $L_{m/n}(\mathrm{triv})$. Under this isomorphism, fixed points of $\mathbb{C}^*$ action correspond to a suitably normalized

u-weight basis which is parameterized by sequences $\mathbf{a} = (a_1, \dots, a_n)$ *such that* $|a_i - a_j| \leq m$ *for all* i, j *and if* $a_i - a_j = m$ *then* $i < j$*. Under this bijection, the weight of* $\mathcal{L}_i$ *at a fixed point corresponds to that of the operator* $nu_{n+1-i} + m(n-1)$*.*

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Monoidal categorification of cluster algebras and quantum affine algebras

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(joint work with Masaki Kashiwara, Se-jin Oh, Euiyong Park)

Cluster algebras are special commutative rings introduced by Fomin and Zelevinsky in the early 2000s. For a given quiver Q, the cluster algebra $A(Q)$ is defined as the subring of $\mathbb{Q}(x_i; i \in Q_0)$ generated by special elements called *cluster variables* in the field of rational functions. The process of creating a new cluster variable from given cluster variables is called a *mutation*. While studying finite-dimensional representations of quantum affine algebras, Hernandez and Leclerc introduced the notion of monoidal categorification of cluster algebra([1]). A monoidal categorification of a given cluster algebra means that the Grothendieck ring is isomorphic to the cluster algebra and that special elements called cluster monomials correspond to simple objects.

We provide a family of monoidal subcategories $\mathcal{C}^{[a,b]}$ $(a, b \in \mathbb{Z}, a \leq b)$ of the category of finite-dimensional representations of quantum affine algebras such that $\mathcal{C}^{[a,b]}$ is a monoidal categorification of its Grothendieck ring. First, we find a criterion for monoidal categorification: if a family of simple representations forms a nice monoidal seed (called an Λ-*admissible seed*) in a monoidal subcategory $\mathcal{C}$, then $\mathcal{C}$ is a monoidal categorification of its Grothendieck ring([3]). A key property of an Λ-admissible seed is that at each mutable vertex, there is a simple representation corresponding to the mutation of the cluster variable at the vertex. This property, together with some other conditions, guarantees that there exists a simple representation corresponding to *any* cluster variable. So it suffices to find an Λ-admissible seed for each $\mathcal{C}^{[a,b]}$. For this, we develop a combinatorial model, called *admissible chains of i-boxes*, which allows us to obtain a large family of Λ-admissible seeds([4]). These results provide an affirmative answer to the monoidal categorification conjecture([2]) for the Hernandez-Leclerc category $\mathcal{C}^{-} = \mathcal{C}^{[-\infty,0]}$.

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Semicuspidal categories of affine KLR algebras

Alexandre Minets

(joint work with Ruslan Maksimau)

1. KLR algebras

Let $Q = (I, E)$ be a quiver, and $\alpha \in \mathbb{Z}^I_{\geq 0}$ a dimension vector, $|\alpha| := \sum_{i \in I} \alpha_i$. Denote by $\mathsf{Rep}_\alpha Q$ the moduli stack of representations of Q over $\mathbb{C}$ with dimension vector α. Similarly, denote by $\mathsf{Fl}_\alpha Q$ the moduli stack of full flags of representations of Q:

$$\mathsf{Fl}_\alpha Q = \left\{ 0 = V_0 \subset V_1 \subset \ldots \subset V_{|\alpha|} \mid V_k \in \mathsf{Rep}_{\beta^{(k)}} Q, |\beta^{(k)} - \beta^{(k-1)}| = 1, \beta^{(|\alpha|)} = \alpha \right\}$$

Note that $\mathsf{Fl}_\alpha Q$ is not connected; more precisely, its connected components are parameterized by words in I whose letters sum up to α. We have a proper morphism $\pi : \mathsf{Fl}_\alpha Q \to \mathsf{Rep}_\alpha Q$, which sends a flag $(V_0 \subset V_1 \subset \ldots \subset V_{|\alpha|})$ to $V_{|\alpha|}$.

Definition 1. *Let* $\mathbf{k}$ *be either* $\mathbb{Z}$ *or a field. The KLR algebra*[1] $R(\alpha)$ *is defined to be the Ext-algebra* $\operatorname{Ext}^*(\pi_* \underline{\mathbf{k}}, \pi_* \underline{\mathbf{k}})$, *where* $\underline{\mathbf{k}} \in D^b_c(\mathsf{Fl}_\alpha Q)$ *is the constant sheaf.*

[1] also known as quiver Hecke algebra

Historically, KLR algebras were introduced by Khovanov, Lauda and Rouquier in a more combinatorial way. The equivalent definition above was given by Varagnolo and Vasserot. When Q is a quiver of ADE type, the categories of graded projective modules over KLR algebras provide a categorification of the negative half of the quantum group $U_q(\mathfrak{g}_Q)$. In particular, various questions about bases of $U_q(\mathfrak{g}_Q)$ categorify to the study of categories of $R(\alpha)$-modules.

2. Stratification of algebras

In order to study a given algebra, we might want to cut it up into more manageable pieces. One way to do this is provided by the following definition.

Definition 2. *Let A be an algebra. A proper stratification of A is a finite chain of two-sided ideals*

$$A \supset I_1 \supset I_2 \supset \dots$$

satisfying the following conditions:

(1) *I_k/I_{k+1} is a projective A/I_{k+1}-module;*
(2) $\operatorname{Hom}_{A/I_{k+1}}(I_k/I_{k+1}, A/I_k) = 0$*;*
(3) *I_k/I_{k+1} is finitely generated and flat over $A_k := \operatorname{End}_{A/I_{k+1}}(I_k/I_{k+1})$.*

When all $A_k = \mathbf{k}$, Cline-Parshall-Scott theorem implies that A-mod is a highest weight category. When A is a graded algebra and all A_k are finitely generated and graded, A-mod is an affine highest weight category, as defined by Kleshchev [2]. In general, the category of finitely generated representations of a properly stratified algebra enjoys properties similar to highest weight categories, such as the existence of standard/costandard modules and BGG reciprocity.

We have a couple of situations where KLR algebras are known to admit proper stratifications. By the work of Kato [1], $R(\alpha)$-mod is an affine highest weight category for Q of ADE type. More generally, suppose Q is an affine type quiver. Let $\theta \in \mathbb{Z}^I$ be a stability function, and μ the corresponding slope. For any full flag $(V_\bullet) \in \mathsf{Fl}_\alpha Q$ set $\mu(V_\bullet) = \max_k \mu(V_k)$. This function is locally constant. Define

$$I_t = R(\alpha) \cdot \operatorname{Ext}^*(\pi_* \underline{\mathbf{k}}_\mu, \pi_* \underline{\mathbf{k}}_\mu) \cdot R(\alpha),$$

where $\underline{\mathbf{k}}_\mu$ is the constant sheaf on the union of connected components of $\mathsf{Fl}_\alpha Q$ where $\mu(V_\bullet) > t$. For a certain choice of θ, this chain of ideals is known to define a proper stratification of $R(\alpha)$ [3].[2] Moreover, all composition factors A_k are either polynomial rings, or *semicuspidal algebras* $C(n\delta) = R(n\delta)/I_{\mu(n\delta)}$. When $\mathbf{k}$ is a field of big characteristic, $C(n\delta)$ is known to be Morita equivalent to the affine wreath product algebra $\mathcal{A}_n(Z_{Q_0})$, where Z_{Q_0} is the zigzag algebra of the ADE type quiver Q_0 associated to Q.

In our work [4], we compute $C(n\delta)$ in any characteristic, when Q is the Kronecker quiver. The main idea of our computation is to restrict all sheaves to the semistable locus, where the ideal $I_{\mu(n\delta)}$ tautologically vanishes.

[2] this is a heavy paraphrasing of the original result, which was obtained by algebraic methods

3. KLR and Schur algebras of quivers and curves

Let X be a smooth curve over $\mathbb{C}$, and $\mathsf{Tor}_n(X)$ the moduli stack of coherent torsion sheaves on X of length n. Similarly to quiver case, we can define the stack of partial flags $\mathsf{PFl}_n(X)$ and of full flags $\mathsf{Fl}_n(X)$ of torsion sheaves, which admit proper maps $\pi : \mathsf{PFl}_n(X) \to \mathsf{Tor}_n(X)$, $\pi' : \mathsf{Fl}_n(X) \to \mathsf{Tor}_n(X)$.

Definition 3. *The Schur algebra of X is given by $\mathcal{S}_n(X) = \mathrm{Ext}^*(\pi_*\underline{\mathbf{k}}, \pi_*\underline{\mathbf{k}})$. The KLR algebra of X is given by $\mathcal{R}_n(X) = \mathrm{Ext}^*(\pi'_*\underline{\mathbf{k}}, \pi'_*\underline{\mathbf{k}})$.*

Theorem 4 ([4])**.** *We have $\mathcal{R}_n(X) \simeq \mathcal{A}_n(H^*(X))$.*

The Schur algebra $\mathcal{S}_n(X)$ doesn't have such an explicit description. Still, we show that it can be described as an algebra of differential operators on a certain polynomial representation, and provide an explicit diagrammatic basis for it.

Let $X = \mathbb{P}^1$. It is a classical result that for a certain stability function the semistable locus $\mathsf{Rep}_{n\delta}Q$ for the Kronecker quiver is isomorphic to $\mathsf{Tor}_n\mathbb{P}^1$. One can check that restriction to the semistable locus gives rise to a map of algebras $\Phi_n : C(n\delta) \to \mathcal{S}_n(\mathbb{P}^1)$.

Theorem 5 ([4])**.** *Φ_n is an isomorphism in characteristic zero, and injective over $\mathbf{k} = \mathbb{Z}$. The image of Φ_n, denoted by $\widetilde{\mathcal{S}}_n(\mathbb{P}^1)$, is generated by certain geometric classes. In particular, for $\mathbf{k} = \mathbb{F}_q$ we have $C(n\delta) \simeq \widetilde{\mathcal{S}}_n(\mathbb{P}^1) \otimes_{\mathbb{Z}} \mathbb{F}_q$.*

Note that for $\mathbf{k} = \mathbb{F}_q$, the map Φ_n is neither injective or surjective in general.

4. Other quivers

For any Frobenius algebra F, we can define Schur algebra $\mathcal{S}_n(F)$ as an algebra of differential operators, in such a way that $\mathcal{S}_n(X) = \mathcal{S}_n(H^*(X))$. Moreover, one can make sense of $\widetilde{\mathcal{S}}_n(F)$ in this generality, too. Note that in characteristic zero $\widetilde{\mathcal{S}}_n(F) = \mathcal{S}_n(F)$, and $\mathcal{S}_n(F)$ is Morita equivalent to $\mathcal{A}_n(F)$.

Conjecture 6. *For any quiver of affine type Q and $\mathbf{k} = \mathbb{F}_q$, we have $C(n\delta) \simeq \widetilde{\mathcal{S}}_n(Z_{Q_0}) \otimes_{\mathbb{Z}} \mathbb{F}_q$.*

Unfortunately, our previous approach fails to work. The reason is that while restriction to the semistable locus always provides a quotient of $R(n\delta)$, for other affine quivers its kernel is strictly bigger than $I_{\mu(n\delta)}$. One way to circumvent this is to think of $\pi_*\underline{\mathbf{k}}$ as a sheaf over $T^*\,\mathsf{Rep}\,Q$ via microlocalization. In this setting, we can introduce a finer quotient by throwing out singular supports of the sheaves appearing in the definition of $I_{\mu(n\delta)}$. This leaves us with the semistable locus $\mathsf{Rep}\,\Pi_Q$ of representations of the *preprojective algebra* Π_Q, which is known to be related to torsion sheaves on the resolution of Kleinian singularity corresponding to Q. While we are unable to follow this line of reasoning through for now, we have the following result in a related direction.

Theorem in progress. There exists a Steinberg-type variety Z over the semistable locus of $\mathsf{Rep}\,\Pi_Q$, such that $H_*^{BM}(Z)$ is equipped with a convolution product, and we have an isomorphism of algebras $H_*^{BM}(Z) \simeq \mathcal{S}_n(Z_{Q_0})$.

I'd like to end by stressing that the convolution product above is not of Chriss-Ginzburg type, but rather similar to the one appearing in [5].

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String cones and cluster varieties

Bea Schumann

(joint work with Gleb Koshevoy)

Let $\mathfrak{g}$ be a finite dimensional simple simply-laced Lie algebra over $\mathbb{C}$ and U_q^+ the upper half of the corresponding quantized enveloping algebra.

For every reduced expression $w_0 = s_{i_1} s_{i_2} \dots s_{i_N}$ there exists a polyhedral parametrization of Lusztig's canonical basis of U_q^+ by the integer points of a rational polyhedral cone $\mathcal{S}_{\mathbf{i}} \subset \mathbb{R}^N$. Here $\mathbf{i} = (i_1, i_2, \dots i_N)$ is the word of a reduced expression $s_{i_1} s_{i_2} \dots s_{i_N}$ of the longest Weyl group element of $\mathfrak{g}$.

If we consider two reduced words $\mathbf{i_1}$ and $\mathbf{i_2}$, then there is a piecewise linear bijection $\Psi^{\mathbf{i_1}}_{\mathbf{i_2}} : \mathbb{R}^N \to \mathbb{R}^N$ such that $\Psi(\mathcal{S}_{\mathbf{i}_1}) = \mathcal{S}_{\mathbf{i}_2}$ which can be used to compute the inequalities of all string cones. The motivation of this project is to study the facets of $\mathcal{S}_{\mathbf{i}}$ for any reduced word $\mathbf{i}$. As a first step we obtain the following sufficient criterion for a set of inequalities to contain no redundancies.

Theorem 1 ([3])**.** *Let $f_1, \dots, f_k : \mathbb{R}^N \to \mathbb{R}$ be linear maps such that*

$$\mathcal{S}_{\mathbf{i}} = \{x \in \mathbb{R}^N \mid f_j(x) \geq 0 \forall j \in \{1, \dots, k\}.$$

If all coefficients of f_j are in the set $\{-1, 0, 1\}$ then the set of inequalities is minimal, i.e. each f_j is facet defining.

The theorem is applicable in many situations, i.e. the inequalities of $\mathcal{C}_{\Sigma_{\mathbf{i}}}$ are non-redundant if $\mathfrak{g}$ is of type A or if $\mathfrak{g}$ is arbitrary and $\mathbf{i}$ is a "nice" word in the sense of Littelmann ([4]).

We conjecture that the above theorem is indeed an equivalence. In the rest of these notes we explain our approach to prove the above theorem also leading to a stronger conjecture.

Let G be a simple simply-connected algebraic group with Lie algebra $\mathfrak{g}$ and $U \subset B \subset G$ the unipotent radical of a Borel subgroup. By duality the integer points of the string cones $\mathcal{S}_{\mathbf{i}}$ parametrize the dual canonical bases of $\mathbb{C}[U]$.

Since U is a so-called partial compactification of a cluster variety, we can apply the machinery of Gross-Hacking-Keel-Kontsevich [1] to U (up to some technical conditions) giving a basis for $\mathbb{C}[U]$ together with many parametrizations of this basis by rational polyhedral cones $\mathcal{C}_\Sigma$ (Σ a possibly infinite index set).

The string cones appear as a subset of these polyhedral cones, as the following theorem shows.

Theorem 2 ([2])**.** *The string cones are, up to change of coordinates, a subset of the parametrizations $\mathcal{C}_\Sigma$, i.e. for any reduced expression $\mathbf{i}$ there exists an index $\Sigma_\mathbf{i}$ and a unimodular bijection*

$$\mathcal{C}_{\Sigma_\mathbf{i}} \to \mathcal{S}_\mathbf{i}$$

(and the technical conditions are satisfied here).

Let us look closer at the definition of the cones $\mathcal{C}_\Sigma$. To each index Σ is associated an open torus $T_\Sigma^\vee = (\mathbb{C}^*)^N$ in the dual cluster variety to U, denoted by $\mathcal{X}$. There exists a regular function $W : \mathcal{X} \to \mathbb{C}$ (called potential) such that Gross-Hacking-Keel-Kontsevich's basis for $\mathbb{C}[U]$ is parametrized by the integer points of

$$\mathcal{C}_\Sigma = \{x \in \mathbb{R}^N \mid [W|_{T_\Sigma^\vee}]_{trop}(x) \geq 0 \text{ for a } \Sigma \ (\iff \text{ for any } \Sigma)\}.$$

Here $W|_{T_\Sigma^\vee} \in \mathbb{C}[T_\Sigma^\vee]$, hence $W|_{T_\Sigma^\vee} \in \mathbb{C}[x_k^{\pm 1} \mid 1 \leq k \leq N]$ is a Laurent polynomial. The function $[W|_{T_\Sigma^\vee}]_{trop} : \mathbb{R}^N \to \mathbb{R}$ is the piecewise linear map we get when we replace multiplication by addition and addition by taking the minimum.

By cluster magic magic the potential $W|_{T_\Sigma^\vee} \in \mathbb{C}[T_\Sigma^\vee]$ has only non-positive exponents and our theorem in this language reads as follows.

Theorem 3 ([3])**.** *If each exponent of any variable in $W|_{T_\mathbf{i}^\vee}$ is in the set $\{-1, 0\}$, then the inequalities of $\mathcal{C}_{\Sigma_\mathbf{i}}$ (and hence $\mathcal{S}_\mathbf{i}$) are non-redundant.*

Note that the place of the coefficients is now taken by the exponents since we consider the tropicalization of $W|_{T_\mathbf{i}^\vee}$. However, the coefficients of the potential seem to lead to a complete description of the facet-defining inequalities.

Conjecture 4. *A term of $W|_{T_\mathbf{i}^\vee}$ leads to a redundant inequality after tropicalization if and only if its coefficient is less or equal to* 1.

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Motivic Springer Theory

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(joint work with Jens Eberhardt)

Many interesting algebras appearing in representation theory arise geometrically as *convolution algebras* in homology theories, mostly in Borel-Moore homology. In this talk we will reformulate some of these constructions in terms of Chow rings and (generalisations) of Chow motives. The main result is then a formality statement describing an equivalence between the the category of graded perfect complexes over this convolution algebra and the category of equivariant Springer motives.

The setup. We fix as a ground ring of our variteties an algebraically closed field k. The space of coefficients of sheaves is $\mathcal{Q}$.

Let $Y_i, i \in I$ be a (for simplicity finite) collection of smooth algebraic varieties with a proper maps $\mu_i : Y_i \to Y$ to some not necessarily smooth variety Y. We can then consider the *generalised Steinberg varieties* $Z_{i,j} = Y_i \times)YY_j$ for each pair $i, j \in I$ and their disjoint union $\dot{\cup}_{i,j} Z_{i,j}$. These varieties generalise the classical Steinberg varieties of triples $(x, \mathfrak{b}, \mathfrak{b}')$ with $x \in \mathfrak{b} \cap \mathfrak{b}'$ of triples consisting of a nilpotent element x and Borels $\mathfrak{b}$, $\mathfrak{b}'$ in a semisimple Lie algebra.

We consider the Chow ring Z of Z with *its convolution product* of cycles given by the following correspondences involving the obvious projection/diagonal maps:

$$\begin{array}{ccccc} Z_{i,j} & & & & \\ & \nwarrow & & & \\ & & Z_{j,k} \times Z_{j,k} \longleftarrow Y_i \times_Y \times Y_j \times_Y Y_k \longrightarrow Z_{i,k} & & \\ & \swarrow & & & \\ Z_{j,k} & & & & \end{array}$$

Mostly we have in addition a compatible action of an affine algebraic group G and consider *equivariant Chow groups* with the convolution product.

Important examples of convolution algebras.

(1) The most important eample is the *Springer resolution* $\mu : \tilde{\mathcal{N}} \to Y = \mathcal{N}$ of the the nilpotent cone $\mathcal{N}$ of a simple algebraic group. In this case $|I| = 1$. Note that in this case the degree zero of the convolution algebra is the group algebra of the corresponding Weyl group.

(2) Given a type ADE or affine type A quiver we can consider the variety of representations $Y = \mathrm{Rep}_d$ of a fixed dimension vector d and the variety of triples consisting of $x \in \mathrm{Rep}_d$ and two filtrations with semisimple subquotients. That is the space of pairs of flagged representations over Y. In case one takes full flags only, the convolution algebra is the *Quiver Hecke algebra* of [VV11]; for general flag varieties one obtains the *Quiver Schur algebra* from [SW14].

(3) Following Lusztig [Lus89], one might construct graded Hecke algebras as convolution algebras.

(4) The T-equivariant Bott-Samulson resolutions of the Schubert varieties of a flag variety gives rise to a convolution algebra which can be identified with the endomorphism algebra of corresponding sum of Soergel bimodules.

Chow motives and weight structures. Let now $k = \overline{\mathbb{F}_p}$. Consider the *category of correspondences over a variety* S. That means objects are formal symbols $M(X/S)$ for $X \to S$ proper. The morphism space from $M(X/S)$ to $M(X'/S)$ is given by the Chow groups $\mathrm{Ch}(X \times_S X')$ with convolution as composition. The *category of Chow motives* $\mathrm{Chow}(S)$ is given by passing to the Karoubian envelope and fomal inversion of powers of the Lefschetz motive. By a result of Bondarko [Bon10], this category of Chow motives is the heart of a weight structure in the *triangulated category* $\mathcal{C}(S)$ *of mixed motives.* In the special case $S = \mathrm{Spec}(k)$ this category is the original Voevodsky category of mixed motives.

We explain the notion of a *tilting family* in an arbitrary triangulated category and their connection with weight structures. Following Bondarko, [Bon10] we obtain a weight complex functor from $\mathcal{C}(S)$ to the heart $\mathrm{Chow}(S)$ and give a characterisation when this is an equivalence. The existence of a tilting family in our setup is given by the following:

Proposition 1. *There is a tilting family in* $\mathcal{C}(S)$ *whose objects we call Springer motives.*

The construction of this family is similar to the construction of semisimple perverse sheaves in the more classical setups.

We describe famous functors as examples of tilting families and weight complex functors. For instance, as the name suggests, tilting objects and the *Ringel duality* in the context of highest weight categories provide a tilting family with a weight functor as equivalence. Moreover, the simple objects and the *Koszul duality functor* in the context of the bounded derived categories of graded modules over a fixed Koszul algebra provide another example of a tilting family with a weight complex functor.

Formality result. We say that the setup is *pure Tate*, abbreviated (PT), if the motive of the fiber of μ_i for any point x is in the additive Karoubian closed category generated by the $\mathcal{Q}(n)[2n]$ for $n \in \mathbb{Z}$. Here $[\_]$ denotes the homological, shift functor and (n) the Tate twist functor. This property is crucial for our result:

Theorem 2. *Assume* (PT) *and that* $\mu_i(Y_i) \subset Y$ *has finitely many* G*-orbits for any* $i \in I$*. Then the weight complex functor provides an equivalence*

$$D_{\mathrm{perf}}(\mathrm{Ch}(\mathrm{Z})) \cong \{\textit{Springer motives}\}$$

between the perfect derived category of graded modules over the convolution algebra $\mathrm{Ch}(Z)$ *and the subcategory of Springer motives (in the derived category of* G*-equivariant Chow motives).*

Remark 3. The theorem allows to work with ordinary algebras and modules and does not require some higher structures. This is why we call it the *formality theorem.* Note moreover that the motivic setting has built in the Tate shift and thus the grading. Thus in this approach the grading is involved conceptually. The rather cumbersome and demanding usage of mixed Hodge modules in the classical setting of D-modules and perverse sheaves can be avoided for our purposes.

Theorem 4. *The pure Tate condition* (PT) *and the finite orbits assumptions hold in the above listed cases. Thus the formality result hold for all these examples.*

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