Mathematisches Forschungsinstitut Oberwolfach

Report No. 56/2021

DOI: 10.4171/OWR/2021/56

Mini-Workshop: (Anosov)^3 (hybrid meeting)

Organized by
Colin Guillarmou, Orsay
Benjamin Delarue (born Küster), Paderborn
Beatrice Pozzetti, Heidelberg
Tobias Weich, Paderborn

5 December – 11 December 2021

Abstract. Three different active fields are subsumed under the keyword *Anosov theory*: Spectral theory of Anosov flows, dynamical rigidity of Anosov actions, and Anosov representations. In all three fields there have been dynamic developments and substantial breakthroughs in recent years. The mini-workshop brought together researchers from the three different communities and sparked a joint discussion of current ideas, common interests, and open problems.

Mathematics Subject Classification (2010): Primary: 37D20, 22E46, 58D19; Secondary: 20F67, 37D40, 51M10, 58J50, 53C35, 81Q20, 58J40 37C40, 37C30, 37D30, 37C15, 37C85, 53C24, 57S30, 22E40, 20H10.

Introduction by the Organizers

The term *Anosov flow* for a uniformly hyperbolic flow goes back to the work of Anosov from the 1960's, who proved structural stability under perturbations as well as ergodicity of such flows on compact manifolds. Analogous results were proven for *Anosov maps* on smooth manifolds. The first active field of research represented at the conference aims at studying the spectral theory of Anosov flows, in view of applications to dynamical zeta functions, counting periodic orbits or mixing properties. *Anosov actions* are natural generalizations of Anosov flows and maps: the groups $\mathbb{R}$ (in the case of flows) or $\mathbb{Z}$ (in the case of maps) whose actions define the dynamics are replaced by either an abelian Lie group A or an arbitrary Lie group G, depending on the level of generality. Such Anosov actions have been studied since the 70's by Zimmer, Katok, Spatzier and others, leading to striking rigidity results, but some of the main conjectures on that aspect still remain open. The notion of an *Anosov representation* is more recent: Labourie introduced it in 2006 for holonomy representations of fundamental groups of compact negatively curved Riemannian manifolds and Guichard-Wienhard generalized the definition to arbitrary hyperbolic groups in 2012. Since then the field had fast paced developments.

The hybrid workshop *(Anosov)*^{3}, organised by Colin Guillarmou (Orsay), Benjamin Küster (Paderborn), Beatrice Pozzetti (Heidelberg) and Tobias Weich (Paderborn) was, despite the pandemic situation, well attended with 14 participants in locus and 6 online participants with broad geographic representation. This workshop was a nice blend of researchers with various backgrounds coming from the three different fields. The program featured 3 mini courses (3×60 minutes) and 10 research talks of 60 minutes. The remaining time was used for intense discussions among the participants as well as an open problem session on Thursday afternoon.

Spectral theory of Anosov flows. The geodesic flow on the unit tangent bundle of any Riemannian locally symmetric space of rank one (compact or not) is a canonical example of an Anosov flow in the sense that the Anosov property can be read off directly once the flow is described in the Lie-theoretic language available for such spaces. More generally, it has been known since the 1960's thanks to works of Hadamard and É. Cartan that the geodesic flow on the unit tangent bundle of any negatively curved compact Riemannian manifold is Anosov. This gives a rich class of interesting examples.

Originally, the study of Anosov flows (and maps) concentrated on their behavior as dynamical systems; this leads to establishing stability, transitivity, ergodicity, and mixing properties. In the 1980's, Ruelle and Pollicott observed that the rate of the exponential mixing of an Anosov flow on a compact manifold, as well as the quantitative behavior of correlation functions of observables evolving according to such a flow, are encoded in a discrete set of numbers in the complex plane, the so-called *Pollicott-Ruelle resonances*. This uncovered a deep and extremely fruitful connection between the fields of dynamical systems and spectral analysis. Pollicott-Ruelle resonances of an Anosov flow can often be treated as eigenvalues of the generator of the flow on anisotropic Banach and Hilbert spaces. The first mini-course of the workshop, by Yannick Bonthonneau, introduced the audience to this topic and explained how the spectral theory of Pollicott-Ruelle resonances is helpful to meromorphically continue dynamical zeta functions or to encode topological information of the underlying manifold. In his talk Joachim Hilgert explained how Pollicott-Ruelle resonances of the geodesic flow of locally symmetric spaces can be related to a quantum spectrum (spectrum of some Laplacian). Some aspects of stable ergodicity of Anosov flows were addressed in the talk of Gerhard Knieper. Livio Flaminio addressed decay of correlation of the horocycle flow which can be related to an Anosov flow, but which is itself not Anosov anymore.

Anosov actions. Natural generalizations of hyperbolic flows are transversally hyperbolic actions of groups of higher rank on manifolds. A canonical example is the Weyl chamber flow on a higher rank locally symmetric space. In contrast to the rank one case, there are very strong rigidity properties for higher rank Anosov actions such as measure rigidity or cocycle rigidity. As a consequence Weyl chamber flows of higher rank cannot be deformed, while for rank one geodesic flows one can easily perturb the Riemannian metric on the locally symmetric space to obtain a different Anosov flow. It is conjectured that all smooth higher rank Anosov actions without rank one factors are conjugate to algebraic actions after passing to finite covers. The second minicourse, by Ralf Spatzier, described a recent breakthrough towards this conjecture. Two other aspects of Anosov actions were addressed in the talks: Thi Dang presented results on topological mixing of positive diagonal flows and Jialun Li presented new results about the counting and equidistribution of torus orbits for Weyl chamber flows on finite volume locally symmetric spaces of rank $k > 2$.

Anosov representations. Anosov representations provide a rich class of non-compact locally symmetric spaces of higher rank, that are by now understood as a good generalization of convex co-compact manifolds of higher rank. As already mentioned, the field stems out of pioneering work of Labourie and Guichard-Wienhard, and has undergone fast and exciting development in the past ten years. We now have equivalent characterizations of the holonomy representations of these manifolds from geometric, Lie theoretic, dynamical and projective geometric viewpoints. The third minicourse, by Fanny Kassel, gave a great introduction to these various aspects. In her talk Anna Wienhard addressed questions about the geometry of these locally symmetric spaces and Lasse Wolf addressed in his talk the question how such geometric properties can be exploited to prove absence of embedded eigenvalues on such locally symmetric spaces. Francois Ledrappier discussed recent advances in the study of the Hausdorff dimension of the limit sets of Anosov representations, and relations with the Falconer dimension of limit sets.

Anosov representations can be partially encoded in reparametrizations of the geodesic flow of the abstract Gromov hyperbolic group, and as a result there are uniformly hyperbolic flow(s) underlying Anosov representations. This powerful viewpoint makes them amenable to the thermodynamical formalism and lead to the definition of Riemannian metrics on their moduli space, the so called pressure metrics. In his talk Andres Sambarino introduced various pressure metrics (associated with different linear functionals), and discussed their (non)-degeneracy. Xian Dai discussed a different pressure metric on the Hitchin component in $\mathrm{SL}(3, \mathbb{R})$, which can be studied with tools coming from microlocal analysis.

Mini-Workshop (hybrid meeting): (Anosov)^3

Table of Contents

Abstracts

Mini Course on Ruelle resonances and microlocal analysis

Yannick Guedes Bonthonneau

Ruelle resonances are discrete sets of complex numbers that describe the *characteristic frequencies* of hyperbolic dynamical systems. In these three lectures, the first was dedicated to explaining their classical definition in the case of uniformly expanding maps, emphasizing the role of regularity in the definition.

To deal with hyperbolic systems, it is necessary to implement functional spaces of mixed regularity. After motivating this observation of the 90's, a framework to do exactly this was presented. The purpose of the second presentation was to introduce the necessary tools from microlocal analysis, and in the third talk these tools were put in practice, culminating in the now classical result

Theorem 1. *Let $\phi_t : M \to M$ be a smooth Anosov flow on a compact manifold. Then for any $C > 0$, there exists an anisotropic space $C^\infty(M) \subset H_C \subset \mathcal{D}'(M)$, on which ϕ_t acts as a bounded semi-group, and so that X the generator of ϕ_t has discrete spectrum in $\{z \in \mathbb{C} | \ \Re z > -C\}$.*

Additionally, the spectral datum does not depend on the choice of space H_C, nor does it on C.

Some consequences and extensions regarding zeta functions, Anosov actions, and hyperbolic dynamics on non-compact spaces were also discussed.

References

[DZ16] Semyon Dyatlov and Maciej Zworski. Dynamical zeta functions for Anosov flows via microlocal analysis. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(3):543–577, 2016.

[FS11] Frédéric Faure and Johannes Sjöstrand. Upper bound on the density of Ruelle resonances for Anosov flows. *Comm. Math. Phys.*, 308(2):325–364, 2011.

[Ru89] David Ruelle. The thermodynamic formalism for expanding maps. *Communications in Mathematical Physics volume*, 125 :239–262, 1989.

[GLP13] Paolo Giulietti, Carlangelo Liverani, Mark Pollicott. Anosov flows and dynamical zeta functions *Annals of mathematics*, 178(2) :687–773, 2013.

[BKL02] M. Blank, G. Keller, and C. Liverani. Ruelle-Perron-Frobenius spectrum for Anosov maps. *Nonlinearity*, 15 :1905–1973, 2002.

[BL07] O. Butterley and C. Liverani. Smooth Anosov flows: correlation spectra and stability. *J. Mod. Dyn.*, 1 :301–322, 2007.

Mini Course on Anosov representations

Fanny Kassel

Anosov representations are discrete, faithful (or finite-kernel) representations of Gromov hyperbolic groups into noncompact real semisimple Lie groups, with strong dynamical properties. They were introduced by Labourie [10] for fundamental groups of closed negatively-curved manifolds, and generalized by Guichard–Wienhard [5]. They have been much studied in the past few years, and play an important role in higher Teichmüller–Thurston theory and in recent developments in the theory of discrete subgroups of Lie groups (see [8, 11]).

In the first lecture, we started with some historical motivation: given a closed, oriented hyperbolic surface S with fundamental group $\pi_1(S)$, we discussed *Hitchin representations* of $\pi_1(S)$ into $\mathrm{PSL}(n,\mathbb{R})$, namely continuous deformations of the representation obtained by composing the holonomy $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ of S with the irreducible representation of $\mathrm{PSL}(2,\mathbb{R})$ into $\mathrm{PSL}(n,\mathbb{R})$. Choi–Goldman (for $n = 3$), Fock–Goncharov and Labourie (for general n) proved that these representations are all faithful with discrete image, and this is what led Labourie to introduce the notion of an Anosov representation. Hitchin representations define a connected component of the character variety of $\pi_1(S)$ into $\mathrm{PSL}(n,\mathbb{R})$, the *Hitchin component*, which was proved by Hitchin to be homeomorphic to $\mathbb{R}^{(n^2-1)(2g-2)}$, where $g \geq 2$ is the genus of S. The Hitchin component is the prototype of a *higher Teichmüller space*, with striking analogies to the classical Teichmüller space of S.

In the rest of the first lecture, we discussed Anosov representations of Gromov hyperbolic groups into semisimple Lie groups G of real rank one: these coincide with convex cocompact representations into G. We stated several characterizations of these representations, and sketched the proofs of the equivalences.

In the second lecture, we gave the general definition of an Anosov representation into a noncompact real semisimple Lie group G of arbitrary real rank, in terms of a pair of continuous boundary maps and of a relative contraction property (*dominated splitting*) for a certain natural flow on a bundle. Given a semisimple Lie group G, there are several (finitely many) types of Anosov representations into G, determined by the choice of a conjugacy class of parabolic subgroups P of G. We focused particularly on the case of P_i-Anosov representations into $G = \mathrm{SL}(n,\mathbb{K})$ or $\mathrm{PGL}(n,\mathbb{K})$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and P_i is the stabilizer of an i-plane in $\mathbb{K}^n$, for $1 \leq i \leq n-1$. We explained some important properties of Anosov representations, e.g. that they form an open subset of the representation space $\mathrm{Hom}(\Gamma, G)$. We gave several examples of families of Anosov representations, and briefly discussed higher Teichmüller theory in relation to Hitchin and maximal representations.

In the third and last lecture, we gave several characterizations of Anosov representations, which generalize some of the classical dynamical characterizations of convex cocompactness in rank one, based on work of Kapovich–Leeb–Porti [6, 7], Guéritaud–Guichard–Kassel–Wienhard [4], Bochi–Potrie–Sambarino [1], and

Kassel–Potrie [9]. We explained how Anosov representations are also characterized in terms of some geometric notion of convex cocompactness, not in the Riemannian symmetric space of G, but in the setting of convex projective geometry, following Danciger–Guéritaud–Kassel [2, 3] and Zimmer [12].

References

[1] J. Bochi, R. Potrie, A. Sambarino. Anosov representations and dominated splittings. *J. Eur. Math. Soc.*, 21:3343–3414, 2019.

[2] J. Danciger, F. Guéritaud, F. Kassel. Convex cocompactness in pseudo-Riemannian hyperbolic spaces. *Geom. Dedicata*, 192:87–126, 2018, special issue *Geometries: A Celebration of Bill Goldman's 60th Birthday.*

[3] J. Danciger, F. Guéritaud, F. Kassel. Convex cocompact actions in real projective geometry. arXiv:1704.08711.

[4] F. Guéritaud, O. Guichard, F. Kassel, A. Wienhard. Anosov representations and proper actions. *Geom. Topol.*, 21:485–584, 2017.

[5] O. Guichard, A. Wienhard. Anosov representations : Domains of discontinuity and applications. *Invent. Math.*, 190:357–438, 2012.

[6] M. Kapovich, B. Leeb, J. Porti. Anosov subgroups: Dynamical and geometric characterizations. *Eur. J. Math.*, 3:808–898, 2017.

[7] M. Kapovich, B. Leeb. Discrete isometry groups of symmetric spaces. In *Handbook of group actions*, vol. IV, p. 191–290, International Press, 2018.

[8] F. Kassel. Geometric structures and representations of discrete groups. In *Proceedings of the International Congress of Mathematicians 2018 (ICM 2018)*, p. 1113–1150, World Scientific, 2019.

[9] F. Kassel and R. Potrie. Eigenvalue gaps for hyperbolic groups and semigroups. *J. Mod. Dynamics*, 22, 2022, to appear.

[10] F. Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.*, 165:51–114, 2006.

[11] A. Wienhard. An invitation to higher Teichmüller theory. In *Proceedings of the International Congress of Mathematicians 2018 (ICM 2018)*, p. 1007–1034, World Scientific, 2019.

[12] A. Zimmer. Projective Anosov representations, convex cocompact actions, and rigidity. *J. Differential Geom.*, to appear; arXiv:1704.08582.

Mini Course on Higher Rank Anosov Actions

Ralf Spatzier

Dynamical systems with extra symmetry turn out to be surprisingly rigid. For instance generically a diffeomorphism cannot have infinite index in its centralizer in the diffeomorphism group, as conjectured by Smale [10] and proved by Bonatti, Crovisier and Wilkinson for C^1-diffeomorphisms of a compact manifold [1]. For hyperbolic systems one can hope to go one step further, and make the following question (posed as a question in [3] and as a conjecture in [6, Conjecture 16.8]).

Conjecture. (Katok-Spatzier) *All C^∞ Anosov actions of $\mathbb{R}^k$ or $\mathbb{Z}^k$, $k \geq 2$ (higher rank), on any compact manifold without C^∞ rank one factors are C^∞ conjugate to an algebraic action after passing to finite covers.*

Here we call an action by a group G on a compact manifold M *Anosov* if G contains an element g which acts normally hyperbolically w.r.t. the orbit foliation $\mathcal{O}$ of G. This simply means that there is a continuous splitting of the tangent bundle $TM = E^s_g \oplus E^u_g \oplus T\mathcal{O}$ such that E^s_g gets contracted uniformly by g in forward time, and E^u_g in backward time. We call an action totally Anosov if the set of Anosov elements in G is dense in G.

A rank 1 factor of the action is a factor action of $\mathbb{R}^k$ or $\mathbb{Z}^k$ which is trivial on a corank1 subgroup. This condition is clearly necessary as one can always perturb rank 1 factors.

The simplest examples of such actions are commuting linear maps on tori, or the action by the diagonal subgroup of $SL(n, \mathbb{R})$ on $SL(n, \mathbb{R})/\Gamma$ where Γ is a uniform lattice.

This conjecture was motivated by the Higher Rank Rigidity theorems in Riemannian geometry (flats correspond to abelian groups), Margulis' superrigidity theorem and Zimmer's Program. It got strong support by the works of Hurder, Katok and Lewis, Katok and this author and others who proved local rigidity theorems for homogeneous higher rank Anosov actions [8]. For higher rank $\mathbb{Z}^k$ Anosov actions on tori (or nilmanifolds) the conjecture was proved by Rodriguez Hertz and Wang [9, 4] assuming the linearization of the action (on first homology in the case of a torus) does not admit rank 1 factors (i.e. factors such that a corank 1 subgroup acts trivially).

For a special class of hyperbolic actions, the so-called totally Cartan actions, this was proved recently by Vinhage and Spatzier [11]. These are Anosov actions such that maximal intersections of unstable spaces for different Ansoov elements define one-dimensional foliations. These are called coarse Lyapunov foliations. Our results do not depend on the structure of the underlying manifold.

Most importantly, we introduce a novel way of providing a homogeneous structure to a system coming from actions of free products of Lie groups, a technique introduced by Vinhage in his thesis [12] in which he applied this technique in a local rigidity setting.

Theorem. (S-Vinhage)*Let $\mathbb{R}^k \times \mathbb{Z}^\ell$ be a $C^{1,\theta}$ transitive, totally Cartan action, for some $\theta \in (0, 1)$. Assume no finite cover of the action has a non-Kronecker $C^{1,\theta}$ rank one factor. Then the action is $C^{1,\theta}$-conjugate to an affine homogeneous action (up to finite cover). Moreover, if the action is C^∞, so is the conjugacy.*

Besides establishing the above conjecture in this setting, we can also allow rank 1 factors and determine fairly precisely how those interact with each other and the higher rank components without further rank 1 factors. In addition, we do not need to assume a volume preserving assumption, just certain transitivity.

As mentioned, the main conjecture was motivated in part by the Zimmer program on actions of higher rank semisimple Lie groups and their lattices. And indeed, similar techniques combined with Zimmer cocycle superrigidity allow us to prove the following classification result for such groups:

Theorem. (Butler-Damjanovic-S-Vinhage-Xu) *Suppose that every simple factor of a real semisimple group G has real rank at least 2. Let G act C^∞ on a compact manifold X. Assume the restriction of the action to some split Cartan subgroup*

$A \subset G$ is totally Anosov and preserves an invariant volume. Then the action is smoothly conjugate to an algebraic G-action.

In the mini course, we will carefully describe the various definitions and results, introduce the main tools and explain how we use them We will also speculate on the promise and outlook for future work.

One Key ingredient. Suppose we are given a totally Cartan action. Then earlier work by Kalinin and the speaker [7] provides a metric along every 1-dimensional coarse Lyapunov foliation $\mathcal{W}^\lambda$ which expands and contracts precisely according by a constant $\lambda(a)$ where $\lambda : \mathbb{R}^k \to \mathbb{R}$ is a linear functional. This allows to introduce one parameter groups which acts on X by moving a point x along its coarse Lyapunov manifold $\mathcal{W}^\lambda(x)$. Then the free product of these actions together with $\mathbb{R}^k$ acts transitively on X (by local product structure of stable manifolds), intertwined by the various $e^{\lambda(a)}$ for $a \in \mathbb{R}^k$.

The free product naturally has the structure of a topological group, is locally connected and path connected, and connected but is far from locally compact. However if we can prove constancy of the stabilizers at different point (constant "cycle" structure) it turns out that the image of this group in the homeomorphisms of X, is a Lie group, thanks to a very general result by Geason and Palais [5] on topological transformation groups.

So it all comes down to proving constancy of cycles. This is done by analyzing geometric commutators which can be defined using the local product structure of coarse Lyapunov foliations. The latter generate all cycles. The proofs are long and complicated but quite geometric.

References

[1] C. Bonatti, S. Crovisier, A. Wilkinson. The C^1 generic diffeomorphism has trivial centralizer. *Publ. Math. Inst. Hautes Études Sci.*, 109 (2009), 185–244.

[2] C. Butler and D. Damjanovic and R. Spatzier and K. Vinhage and D. Xu. Anosov actions: classification and the Zimmer program, in preparation.

[3] K. Burns and A. Katok. Manifolds with nonpositive curvature. *Ergodic Theory Dynam. Systems*, 5(2):307–317, 1985.

[4] D. Fisher, B. Kalinin, and R. Spatzier. Global rigidity of higher rank Anosov actions on tori and nilmanifolds. *J. Amer. Math. Soc.*, 26(1):167–198, 2013. With an appendix by James F. Davis.

[5] A. M. Gleason and R. S. Palais, On a class of transformation groups, *Amer. J. Math.*, 79 (1957), 631–648.

[6] Boris Hasselblatt. Problems in dynamical systems and related topics. In *Dynamics, ergodic theory, and geometry*, volume 54 of Math. Sci. Res. Inst. Publ., pages 273–324. Cambridge Univ. Press, Cambridge, 2007.

[7] Boris Kalinin and Ralf Spatzier. On the classification of Cartan actions. *Geom. Funct. Anal.*, 17(2):468–490, 2007.

[8] A. Katok and R. J. Spatzier. Differential rigidity of Anosov actions of higher rank abelian groups and algebraic lattice actions. Tr. Mat. Inst. Steklova, 216(Din. Sist. i Smezhnye Vopr.):292–319, 1997.

[9] F. Rodriguez Hertz and Z. Wang. Global rigidity of higher rank abelian Anosov algebraic actions. *Invent. Math.*, 198(1):165–209, 2014.

[10] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73 (1967), 747– 817.

[11] R. J. Spatzier and K. Vinhage. Cartan Actions of Higher Rank Abelian Groups and their Classification. arXiv:1901.06559v3.

[12] K. Vinhage. On the rigidity of Weyl chamber flows and Schur multipliers as topological groups. *J. Mod. Dyn.*, 9:25–49, 2015.

Pressure metric in the space of Riemannian metrics

Xian Dai

(joint work with Nikolaos Eptaminitakis)

In this talk, we investigate the pressure metric defined in the space of negatively curved Riemannian metrics proposed by Guillarmou, Knieper and Lefeuvre. We motivate the talk from introducing the classical Weil-Petersson metric on the Teichmuller space. We generalize the Weil-Petersson metric to the pressure metric on the space of negatively curved Riemannian metrics using Thermodynamic formalism. Then we focus our study of the pressure metric on the Blaschke locus that contains the Teichmüller space. The Blaschke locus contains all Blaschke metrics arisen from affine geometry and is naturally related to the Hitchin component in $\mathrm{PGL}(3, \mathbb{R})$. Finally, we show that, with respect to the pressure metric, a special family of geodesics in this locus have infinite lengths. This is joint work in progress with Nikolaos Eptaminitakis.

References

[ASS20] Daniele Alessandrini and Gye-Seon Lee and Florent Schaffhauser. Hitchin components for orbifolds. *arXiv e-prints:* 1811.05366, 2020.

[GKL21] Colin Guillarmou and Gerhard Knieper and Thibault Lefeuvre. Geodesic stretch, pressure metric and marked length spectrum rigidity. *Ergodic Theory and Dynamical Systems*, 1-49. doi:10.1017/etds.2021.75.

[Lab07] François Labourie. Flat projective structures on surfaces and cubic holomorphic differentials *Pure and Applied Mathematics Quarterly*, 3(4, Special Issue: In honor of Grigory Margulis. Part 1):1057–1099, 2007.

[Lof01] John Loftin. Affine spheres and convex $\mathbb{RP}^n$-manifolds. *American Journal of Mathematics*, 123(2):255–274, 2001.

Topological mixing of positive diagonal flows

Thi Dang

A dynamical system consisting of a flow ϕ^t acting on a locally compact topological space Ω is *topologically mixing* if for every open non-empty sets U, V, there exists $T > 0$ such that for every $t > T$, the intersection $\phi^t U \cap V$ is non-empty.

Let G be a connected, real linear, semisimple Lie group without compact factors, let $\Gamma < G$ be a Zariski dense discrete subgroup. Consider $\mathfrak{a}$ Cartan subspace in $Lie(G)$, denote by $A = \exp(\mathfrak{a})$ the associated maximal $\mathbb{R}$-split torus, and choose $\mathfrak{a}^{++}$ an open positive Weyl chamber. *Positive diagonal flows* are the one-parameter flow of the form $\phi_Y^t(\Gamma g) = \Gamma g e^{tY}$ where $Y \in \mathfrak{a}^{++}$ is non zero. They act by right multiplication on $\Gamma \backslash G$. Let M be the compact subgroup such that AM is the centralizer of A in G. *Regular Weyl chamber flows* are the one-parameter flow of the form $\phi_Y^t(\Gamma g M) = \Gamma g e^{tY} M$ where $Y \in \mathfrak{a}^{++}$ is non zero. Positive diagonal flows factor regular Weyl chamber flows.

In the case of lattices (they are Zariski dense by Borel density Theorem), topological mixing of positive diagonal flows is due to Howe–Moore [HM79].

Isometry group of hyperbolic n-spaces. Assume $G = \mathrm{SO}(n,1)_0$ where $n \geq 3$ and $\Gamma < G$ discrete, torsion free, the space $\Gamma\backslash G$ corresponds to the frame bundle over the hyperbolic orbifold $\Gamma\backslash\mathbb{H}^n$. In this case, positive diagonal flows correspond to the geodesic frame flow, which factors the geodesic flow. The latter corresponds to the regular Weyl chamber flow.

The non-wandering set for the geodesic frame flow Ω_G is the lift in $\Gamma\backslash G$ of the non-wandering set of the geodesic flow $\Omega \subset \Gamma\backslash G/M$. Mixing with respect to the lift of the Bowen–Margulis–Sullivan (BMS) measure to Ω_G were obtained for discrete subgroups admitting a finite BMS measure by Winter [Win15]. Exponential rate of mixing was obtained for convex cocompact groups ([Win16]), geometrically finite subgroups with large critical exponent (Mohammadi–Oh [MO15], then Edwards–Oh [EO21]). Finally, Sarkar–Winter [SW20] extended the exponential mixing results to all geometrically finite subgroups.

In the general case of Zariski dense subgroups, topological mixing of the geodesic frame flow is due to Maucourant–Schapira [MS19].

Regular Weyl chamber flows. Assume G is higher rank i.e. $\dim \mathfrak{a} \geq 2$, for example $G = \mathrm{SO}(3,1)_0 \times \mathrm{SO}(3,1)_0$. For Zariski dense subgroup, Olivier Glorieux and the author obtained a necessary and sufficient condition for topological mixing. It involved the two following objects.

Let $\Gamma < G$ be a Zariski dense subgroup. The *limit cone* of Γ, denoted by $\mathcal{B}(\Gamma)$ is a closed cone of $\overline{\mathfrak{a}^{++}}$ introduced by Benoist in [Ben97]. He proved that when Γ is Zariski dense, $\mathcal{B}(\Gamma)$ is convex of non-empty interior. In fact, $\mathcal{B}(\Gamma) \cap \mathfrak{a}^{++}$ is the relative closed cone containing all non zero parameters $Y \in \mathfrak{a}^{++}$ such that ϕ_Y^t has a periodic orbit. The *non-wandering set* Ω was introduced by Conze–Guivarc'h [CG00] for $\mathrm{SL}(n,\mathbb{R})$. It is the smallest closed subset of $\Gamma\backslash G/M$ containing all the A-orbit generated by periodic orbits of regular Weyl chamber flows.

Theorem 1 (N-T. D-0. Glorieux [DG20])**.** *Let G be a connected, real linear, semisimple Lie group without compact factors. Let Γ be a Zariski dense, discrete subgroup of G. Let $Y \in \mathfrak{a}^{++}$.*

The regular Weyl chamber flow ϕ_Y^t is topologically mixing on Ω if and only if Y is in the interior of the limit cone $\mathcal{B}(\Gamma)$.

Thirion [Thi09] for Ping-Pong groups, Sambarino [Sam14] for the image of Hitchin representations and Edwards–Lee–Oh [ELO20] for Borel Anosov groups proved that ϕ_Y^t when $Y \neq 0$ is in the interior of the limit cone is measurably mixing for a family of higher rank BMS measure that all charge Ω. Our result provides an obstruction for measurable mixing of regular Weyl chamber flows ϕ_Y^t when $Y \neq 0$ is in $\partial\mathcal{B}(\Gamma) \cap \mathfrak{a}^{++}$.

Positive diagonal flows. Denote by Ω_G the lift in $\Gamma\backslash G$ of Ω. In the case when M is abelian and connected, for instance when $G = \mathrm{S0}(3,1)_0 \times \mathrm{SO}(3,1)_0$, the author obtained a generalization of the previous theorem.

Theorem 2 (N-T. D. [Dan20])**.** *Let G be a connected, real linear, semisimple Lie group without compact factors. Assume that M is* abelian and connected*. Let Γ be a Zariski dense, discrete subgroup of G. Let $Y \in \mathfrak{a}^{++}$.*

The positive diagonal flow ϕ_Y^t is topologically mixing on Ω_G if and only if Y is in the interior of the limit cone $\mathcal{B}(\Gamma)$.

For positive diagonal flows and in the case of Borel Anosov groups, Lee–Oh [LO20] recently obtained an A-ergodic decomposition of the lifted higher rank BMS measures, then Chow–Sarkar [CS21] proved local mixing of ϕ_Y^t for these measures, for all non-zero $Y \in \mathfrak{a}^{++}$ in the interior of the limit cone.

I gave a sketch of the proof of the obstruction for measurable mixing of positive diagonal flows ϕ_Y^t and of regular Weyl chamber flows when $Y \in \mathfrak{a}^{++}$ is non-zero and in the boundary of the limit cone.

References

[Ben97] Y. Benoist. Propriétés asymptotiques des groupes linéaires. *Geom. Funct. Anal.*, 7(1):1–47, 1997.

[CG00] J.-P. Conze and Y. Guivarc'h. Limit sets of groups of linear transformations. *Sankhyā Ser. A*, 62(3):367–385, 2000. Ergodic theory and harmonic analysis (Mumbai, 1999).

[CS21] M. Chow and P. Sarkar. Local mixing of one-parameter diagonal flows on Anosov homogeneous spaces, 2021.

[Dan20] N.-T. Dang. Topological mixing of positive diagonal flows, Accepted for publication in *Israël J. Math.*, 2020.

[DG20] N.-T. Dang and O. Glorieux. Topological mixing of weyl chamber flows. *Ergodic Theory and Dynamical Systems*, page 1–27, 2020.

[ELO20] S. Edwards, M. Lee, and H. Oh. Anosov groups: local mixing, counting, and equidistribution, 2020.

[EO21] S. Edwards and H. Oh. Spectral gap and exponential mixing on geometrically finite hyperbolic manifolds. *Duke Math. J.*, 170(15):3417–3458, 2021.

[HM79] R. E. Howe and C. C. Moore. Asymptotic properties of unitary representations. *J. Functional Analysis*, 32(1):72–96, 1979.

[LO20] M. Lee and H. Oh. Ergodic decompositions of Burger-Roblin measures on Anosov homogeneous spaces, 2020.

[MO15] A. Mohammadi and H. Oh. Matrix coefficients, counting and primes for orbits of geometrically finite groups. *J. Eur. Math. Soc. (JEMS)*, 17(4):837–897, 2015.

[MS19] F. Maucourant and B. Schapira. On topological and measurable dynamics of unipotent frame flows for hyperbolic manifolds. *Duke Math. J.*, 168(4):697–747, 2019.

[Sam14] A. Sambarino. Hyperconvex representations and exponential growth. *Ergodic Theory Dynam. Systems*, 34(3):986–1010, 2014.

[SW20] P. Sarkar and D. Winter. Exponential mixing of frame flows for convex cocompact hyperbolic manifolds, 2020.

[Thi09] X. Thirion. Propriétés de mélange du flot des chambres de Weyl des groupes de ping-pong. *Bull. Soc. Math. France*, 137(3):387–421, 2009.

[Win15] D. Winter. Mixing of frame flow for rank one locally symmetric spaces and measure classification. *Israel J. Math.*, 210(1):467–507, 2015.

[Win16] D. Winter. Exponential mixing of frame flow for convex cocompact hyperbolic manifolds, 2016.

Correlations functions for horocycle flows on Abelian covers of compact Riemann Surfaces

Livio Flaminio
(joint work with Davide Ravotti)

We establish an asymptotic expansion of the correlations for the horocycle flow on $\mathbb{Z}^d$-covers of compact Riemann Surfaces . Also it is possible to prove that the spectra around 0 of the Casimir operators on any increasing sequence of finite Abelian covers equidistribute with respect to an absolutely continuous measure.

1. Notation

Let Γ be a co-compact lattice in $G = \mathrm{PSL}(2,\mathbb{R})$, and let $M = \Gamma\backslash G$. We can identify M with the unit tangent bundle of the compact orientable hyperbolic surface $S = \Gamma\backslash\mathbb{H}$. We denote by $g \geq 2$ its genus.

Abelian covers $S_0 = \Gamma_0\backslash\mathbb{H}$ (respectively, $M_0 = \Gamma_0\backslash G$) of S (respectively, of M) are in one-to-one correspondence with intermediate subgroups $[\Gamma,\Gamma] \leq \Gamma_0 \leq \Gamma$. We shall assume that the Galois quotient group $\mathscr{G} := \Gamma/\Gamma_0$ is a free Abelian group of rank d, where $0 < d \leq 2g$.

The horocycle flow $\{h_t\}_{t\in\mathbb{R}}$ on M_0 (or on any quotient of G) is the homogeneous flow generated by $U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R})$, namely the flow given by

$$h_t(\Gamma_0 g) = \Gamma_0 g \exp(tU).$$

2. Infinite mixing asymptotics

Our main result, Theorem 1, provides an asymptotic expansion of the correlations for the horocycle flow between smooth observables with compact support. The first term in the expansion is usually called *Krickeberg* (or *local*) *mixing*. To the best of our knowledge, Theorem 1 is the first infinite mixing result for parabolic flows.

Theorem 1. *There exists a constant $\sigma(\Gamma_0) > 0$ such that the following holds. Let $v, w \in \mathscr{C}_c^\infty(M_0)$. There exist $(c_j)_{j\in\mathbb{N}}$ such that for all $t \geq 1$ we have*

$$\langle w, v \circ h_t\rangle \sim \left(\frac{g-1}{2}\right)^{\frac{d}{2}} \sigma(\Gamma_0) \frac{\mathrm{vol}(u)\,\mathrm{vol}(v)}{(\log t)^{\frac{d}{2}}} + \sum_{j=1}^{\infty} \frac{c_j}{(\log t)^{j+\frac{d}{2}}}.$$

The constant σ in Theorem 1 is the determinant of an explicit period matrix of harmonic 1-forms. In the case where $\Gamma_0 = [\Gamma,\Gamma]$, we have $\mathscr{G} = H_1(M,\mathbb{Z}) = \mathbb{Z}^{2g}$, and $\sigma(\Gamma_0) = 1$.

With our method it is possible to obtain an analogous asymptotic expansion for the correlations of the geodesic flow. In that case, the result has been proved recently by Dolgopyat, Nandori and Pène [2], with different methods. See also the previous result by Oh and Pan [5].

3. Outline of the proof

Mixing rates of the one-parameter subgroup $\{\exp(tU)\}$ acting on an irreducible unitary representation of $\mathrm{PSL}_2(\mathbb{R})$ have been studied by several authors. The gist of the matter is that *mixing rates are uniform for all representations in the unitary dual* $\widehat{\mathrm{PSL}_2(\mathbb{R})}$ *lying outside a (Fell) neighbourhood of the trivial representation.* In fact, following the lines of Ratner's work [7], denoting by H_λ the Hilbert space of a unitary representation of $\mathrm{PSL}_2(\mathbb{R})$ of Casimir parameter $\lambda = \lambda(\nu) := \frac{1-\nu^2}{4}$ and by $W^3(H_\lambda) \subset H_\lambda$ the subspace of Sobolev vectors we have

Theorem 2. *There exists a constant $C \geq 1$ such that the following holds. Let $W^3(H_\lambda)$ be the Sobolev space of an irreducible representation of $\mathrm{PSL}_2(\mathbb{R})$ of Casimir parameter $0 \leq \lambda < 1/4$. For all $w, v \in W^3(H_\lambda)$, there exist $A = A(w, v)$, with*

$$|A(w,v)| \leq C\|w\|_3\,\|v\|_3,$$

such that for all $t \geq 1$ we have

$$\left|\langle w, v\circ h_t\rangle - A(w,v)t^{-1+|\nu(\lambda)|}\right| \leq C\|w\|_3\,\|v\|_3\,t^{-1}.$$

Moreover, assume that $\lambda \mapsto w_\lambda \in W^3(H_\lambda)$ and $\lambda \mapsto v_\lambda \in W^3(H_\lambda)$ are parameterised families of vectors. If $\lambda \mapsto \langle w_\lambda, U^j v_\lambda \circ h_t\rangle$ is of class $\mathscr{C}^k$ for $j = 0, 1, 2$ and for all $t \in \mathbb{R}$, then $\lambda \mapsto A(w_\lambda, v_\lambda)$ is of class $\mathscr{C}^k$. If $\lambda = 0$, then (clearly) $A(w, v) = \langle w, v\rangle = \mathrm{vol}(w)\,\mathrm{vol}(v)$.

Unitary representations with Casimir parameter $\lambda \notin [0, 1/4)$ yield faster decay for the correlations of horocycle flows and therefore are not a matter of concern for us. Thus we may limit ourselves to consider the representations with $\lambda \in [0, 1/4)$. In particular we may disregard the discrete series of $\mathrm{PSL}_2(\mathbb{R})$ and limit our considerations to those representations that are generated by a $K = \mathrm{PSO}(2)$-invariant vector.

The above theorem shows that the difficulty of the analysis arises from the fact that the trivial representation is not isolated in the $\mathrm{PSL}_2(\mathbb{R})$-spectrum of M_0. This is equivalent to say that the spectrum of the Laplace-Beltrami operator on the infinite surface S_0 is continuous all the way to zero.

The analysis of the work of Phillips ans Sarnak [6] overcomes this difficulty. Indeed, since the action of Galois group $\mathscr{G}$ on M_0 by deck transformations commutes with the regular representation of G on $L^2(M_0)$, these two action are simultaneously "diagonalisable". This means that we can decompose the Hilbert space $L^2(M_0)$ as a Hilbert direct integral of irreducible unitary representations $\chi \otimes \pi$ of $\mathscr{G} \times G$ (here χ is a character of $\mathscr{G}$ and π an irreducible unitary representations of G).

Indeed, for any $\chi \in \widehat{\mathscr{G}}$, the space $L^2(M, \chi)$ of functions on M_0 with finite square norm $\int_M |f(x)|^2 \mathrm{d}x$ and satisfying $f(\gamma^{-1}x) = \chi([\gamma])f(x)$, for all $(x, \gamma) \in M_0 \times \Gamma$, is precisely the space of the L^2 sections of a smooth line bundle M_χ over M. An immediate application of Parseval identity implies

Lemma 3. *We have the following G-equivariant isometric direct integral decomposition:*

$$L^2(M_0) = \int_{\widehat{\mathscr{G}}} L^2(M,\chi)\,\mathrm{d}\chi, \qquad f = \int_{\widehat{\mathscr{G}}} \pi_\chi(f)\,\mathrm{d}\chi.$$

where $\pi_\chi(f)(x) = \sum_{[\gamma]\in\mathscr{G}} f([\gamma]x)\chi([\gamma])$, for all $x \in M_0$ and all $\chi \in \widehat{\mathscr{G}}$.

Furthermore, for any $f \in C_c^r(M_0)$ and any $\chi \in \widehat{\mathscr{G}}$, the function $\pi_\chi(f)$ is a C^r section of the line bundle M_χ.

We have (see also [1]):

Lemma 4. *The family of line bundles M_χ depends analytically on $\chi \in \widehat{\mathscr{G}}$.*

As the spaces $L^2(M,\chi)$ are G-invariant, and since the base M is compact, each of these spaces decompose into a direct sum of irreducible unitary sub-representations of G. To understand this decomposition, we observe that the $\mathrm{PSL}_2(\mathbb{R})$-spectrum of $L^2(M,\chi)$ is determined by the eigenvalues of the Casimir operator. Furthermore, since we are disregarding the irreducible sub-representations of the discrete series, the interesting irreducible sub-representations $H_{\mu_j(\chi)}$ occurring in $L^2(M,\chi)$ are determined by the eigenvalues $\mu_j(\chi)$ of the Casimir operator on PSO(2)-invariant vectors. Equivalently they are determined by the spectrum $(\mu_j(\chi))$ of the Laplace-Beltrami operator on the space $L^2(S,\chi) = L^2(M,\chi)/\mathrm{PSO}(2)$. The space $L^2(S,\chi)$ is the space of L^2 section of a line bundle S_χ over the compact surface S. We may identify smooth sections of this bundle to smooth functions in the interior of a nice fundamental domain F of the $\mathscr{G}$-action on S_0 which furthermore satisfy on ∂F boundary conditions given by the character χ. Thus, if we let

$$0 = \nu_0 < \nu_1 \le \nu_2 \le \cdots$$

be the spectrum of the Laplace-Beltrami operator on F with Neumann boundary condition, by the min-max principle, the spectrum $\{\mu_j(\chi)\}$ of the Laplace-Beltrami operator on $L^2(S,\chi)$ satisfies the inequalities

$$0 = \nu_0 \le \mu_0(\chi) \le \nu_1 \le \mu_1(\chi) \le \nu_2 \le \mu_2(\chi) \le \cdots$$

The previous discussion and the inequality $\nu_1 > 0$ imply that all the representations corresponding to eigenvalues $\mu_j(\chi)$ with $j \ge 1$ and any χ have fractional polynomial decay of correlation functions. Thus we concentrate our attention of the function $\chi \mapsto \mu_0(\chi)$. Clearly for the trivial character we $\chi_0 = 1$ have $\mu_0(1) = \nu_0 = 0$. We have

Lemma 5 (see [6])**.** *The bottom eigenvalue $\mu_0(\chi)$ of the Laplace-Beltrami operator on $L^2(S,\chi)$ satisfies the following properties:*

1. *$\mu_0(\chi) \ge 0$ and $\mu_0(\chi) = 0$ if and only if $\chi = 1$;*
2. *$\mu_0(\xi)$ is real analytic in the variable $\chi \in \widehat{\mathscr{G}}$;*
3. *The critical point $\chi = 1$ is non degenerate:*

$$\left(\mathrm{d}^2\mu_0\right)_{\chi=1} = \frac{2\pi}{g-1}Q,$$

where the matrix Q is positive definite. If $\Gamma_0 = [\Gamma,\Gamma]$, then $\det(Q) = 1$.

The above lemma and the previous discussion imply that for any open neighbourhood $\mathcal{U}$ of the trivial character $\chi = 1$, all the representations corresponding to eigenvalues $\mu_0(\chi)$ with $\chi \notin \mathcal{U}$ have fractional polynomial decay of correlation functions. Thus the major contribution arises from the components in $L^2(S, \chi)$ in the subspace $H_{\mu_0(\chi)}$ with $\chi \in \mathcal{U}$.

Now Theorem 1 follow by the application of Theorem 2 and lemma 5 using a stationary phase development.

References

[1] A. Deitmar, *Benjamini-Schramm and spectral convergence*, Enseign. Math., **64**(3/4):371–394, 2018.

[2] D. Dolgopyat, P. Nandori, F. Pène, *Asymptotic expansion of correlation functions for $\mathbb{Z}^d$ covers of hyperbolic flows*, preprint arXiv:1908.11504, 2019.

[3] L. Flaminio, G. Forni, *Invariant distributions and time averages for horocycle flows*, Duke Math. J., **119**(3):465–536, 2003.

[4] D. Jakobson, F. Naud, L. Soares, *Large degree covers and sharp resonances of hyperbolic surfaces*, Ann. Inst. Fourier (Grenoble), **70**(2):523–596, 2020.

[5] H. Oh, W. Pan, *Local mixing and invariant measures for horospherical subgroups on abelian covers*, IMRN, **19**:6036–6088, 2019.

[6] R. Phillips, P. Sarnak, *Geodesics in homology classes*, Duke Math. J., **55**(2):287–297, 1987.

[7] M. Ratner, *The rate of mixing for geodesic and horocycle flows*, Ergodic Theory Dynam. Systems **7**:267–288, 1987.

Quantum-classical correspondences

Joachim Hilgert

In this talk, I reported on recent work (with C. Arends, Y. Guedes Bonthonneau, C. Guillarmou, T. Weich and L. Wolf; [GHW18, GHW21, BGHW20, HWW21, AH21]) describing relations between the spectral theories of Anosov type actions and their quantizations.

Historically the subject starts with the Poisson summation formula and the Selberg trace formula for compact locally symmetric spaces. In those cases "spectral" on the classical side refers to the length spectrum of closed geodesics, whereas on the quantum side one has the spectrum of the Laplacian which can be viewed as the quantization of the geodesic flow. About twenty years ago Lewis-Zagier [LZ01] and Flaminio-Forni [FF03] observed correspondences for hyperbolic surfaces where also on the classical side one considers eigenvalues of operators related to the geodesic flow. Our approach was inspired by the work [DFG15] of Dyatlov-Faure-Guillarmou, which uses microlocal analysis and anisotropic Sobolev spaces to establish Pollicott-Ruelle resonant states for the geodesic flow on compact hyperbolic spaces, and relates them to the Laplace eigenfunctions.

Our framework is as follows: let G be a non-compact simple real Lie group and G/K be the corresponding Riemannian symmetric space. Then the algebra $\mathbb{D}(G/K)$ of G-invariant differential operators on G/K is commutative and contains the Laplacian Δ, i.e. the Laplace-Beltrami operator. If G has real rank 1 the algebra $\mathbb{D}(G/K)$ is generated by Δ. Given a discrete subgroup Γ of G, the double coset space $\Gamma\backslash G/K$ is a locally symmetric space and $\mathbb{D}(G/K)$ factors to $\Gamma\backslash G/K$ in the sense that Γ-invariant functions on G/K get mapped to Γ-invariant functions.

In this framework the quantum side is the spectral theory of $\mathbb{D}(G/K)$ on $\Gamma\backslash G/K$. If $\Gamma\backslash G/K$ happens to be a compact manifold (Γ co-compact and torsion free) this spectral theory is presented in great detail in [DKV79].

To describe also the classical side we view the (co-)tangent bundle $T(G/K) \cong T^*(G/K)$ as the homogeneous bundle $G \times_K \mathfrak{p} \cong G \times_K \mathfrak{p}^*$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition for the Lie algebra $\mathfrak{g}$ of G. Then the space $C^\infty(T^*(G/K))^G$ of G-invariant Hamiltonian functions is Poisson-commutative. If $\mathfrak{a} \subseteq \mathfrak{p}$ is a maximal abelian subspace and $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is the corresponding Weyl group, restriction yields an isomorphism $C^\infty(T^*(G/K))^G = C^\infty(G \times_K \mathfrak{p}^*)^G \cong C^\infty(\mathfrak{a}^*)^W$. Using the symplectic structure on $T^*(G/K)$ a G-invariant Hamiltonian function yields a G-invariant Hamiltonian vector field H_f. The classical side of our spectral correspondences is the theory of Ruelle-Taylor resonances for the commutative family $\{H_f \mid f \in C^\infty(T^*(G/K))^G\}$ and their flows Φ_f^t.

Using the surjection $\varphi : G/M \times \mathfrak{a} \to G \times_K \mathfrak{p}$, $(gM, X) \mapsto [g, X]$ with $M = Z_K(\mathfrak{a})$, in view of the intertwining property $\varphi(ge^{t\nabla f|_{\mathfrak{a}}(X)}M) = \Phi_f^t(\varphi(gM))$, [Hil05], one relates the Hamiltonian flows with the partially hyperbolic Weyl chamber flow $T(G/M) = G \times_M (\mathfrak{n}_+ + \mathfrak{a} + \mathfrak{n}_-) \curvearrowleft A = \exp(\mathfrak{a})$. Here $\mathfrak{n}_\pm$ is the sum of root spaces for the positive, resp. negative restricted roots associated with $(\mathfrak{g}, \mathfrak{a})$. In real rank 1 this flow reduces to the geodesic flow.

If $\Gamma\backslash G/K$ is a compact manifold, according to [BGHW20] the classical spectral theory can be analyzed via the space $\mathcal{D}'_+(\Gamma\backslash G/K)$ of distributions on $\Gamma\backslash G/K$ with wavefront set annihilated by $\Gamma\backslash G \times_M (\mathfrak{n}_+ + \mathfrak{a})$. In fact, the space $\mathrm{Res}(\lambda)$ of resonant states for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ are the $u \in \mathcal{D}'_+(\Gamma\backslash G/K)$ satisfying $(H + \lambda(H))u = 0$ for all $H \in \mathfrak{a}$. The subspace $\mathrm{Res}^0(\lambda)$ of resonant states annihilated by the smooth sections of $G \times_M \mathfrak{n}_-$ are called first-band resonant states.

Recall (e.g. from [DKV79]) that the Harish-Chandra isomorphism associates to each $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ a character χ_λ of the algebra $\mathbb{D}(G/K)$. If λ is generic, then according to [HWW21] the canonical projection $\pi : \Gamma\backslash G/M \to \Gamma\backslash G/K$ induces a linear bijection $\pi_* : \mathrm{Res}^0(\lambda) \to {}^\Gamma E_{\chi_\lambda}$, where the latter is the space of Γ-invariant joint $\mathbb{D}(G/K)$-eigenfunctions with character χ_λ. The generic resonant states are closely related with irreducible spherical principal series representations and the associated Poisson transforms are crucial in establishing the spectral correspondence. For applications of such spectral correspondences we refer to [GHW21, BGHW20, HWW21, BGW21].

As in the theory of scalar Poisson transforms, "generic" in our context means that for all positive restricted roots α, we have $\frac{2\langle \lambda+\rho, \alpha\rangle}{\langle \alpha, \alpha\rangle} \notin \mathbb{N}_0$. Here ρ is the usual half-sum of positive restricted roots and $\langle \cdot, \cdot \rangle$ is the bilinear form on $\mathfrak{a}^*_{\mathbb{C}}$ induced by the Killing form. In real rank 1 the spectral correspondence can be extended to all λ, [AH21] ([FF03, GHW18] in the case of surfaces). To do that one has to use reducible spherical principal series and vector valued Poisson transforms.

For non-compact locally symmetric spaces there are only partial results: For real hyperbolic spaces one can deal with the convex co-compact case, [GHW18, Had20]. First steps have been taken also for hyperbolic spaces with cusps, [GW17].

References

[AH21] C. Arends and J. Hilgert, *Spectral correspndences for rank* 1 *locally symmetric spaces – the case of exceptional parameters*, arXiv:2112.11073 (2021).

[BGHW20] Y. Guedes Bonthonneau, C. Guillarmou, J. Hilgert, and T. Weich, *Ruelle-Taylor resonances of Anosov actions*, arXiv:2007.14275 (2020).

[BGW21] Y. Guedes Bonthonneau, C. Guillarmou, and T. Weich, *SRB measures for Anosov actions*, arXiv:2103.12127 (2021).

[DFG15] S. Dyatlov, F. Faure, and C. Guillarmou, *Power spectrum of the geodesic flow on hyperbolic manifolds*, Anal. PDE **8** (2015), 923–1000.

[DKV79] J. J. Duistermaat, J. A. C. Kolk, and V. S. Varadarajan, *Spectra of compact locally symmetric manifolds of negative curvature*, Invent. Math. **52** (1979), no. 1, 27–93. MR 532745

[FF03] L. Flaminio and G. Forni, *Invariant distributions and time averages for horocycle flows*, Duke Math. J. **119** (2003), 465–526.

[GHW18] C. Guillarmou, J. Hilgert, and T. Weich, *Classical and quantum resonances for hyperbolic surfaces*, Math. Ann. **370** (2018), 1231–1275.

[GHW21] ———, *High frequency limits for invariant Ruelle densities*, Ann. H. Lebesgue **4** (2021), 81–119.

[GW17] Y. Guedes Bonthonneau and T. Weich, *Ruelle-Pollicott resonances for manifolds with hyperbolic cusps*, arXiv:1712.07832 (2017).

[Had20] Ch. Hadfield, *Ruelle and quantum resonances for open hyperbolic manifolds*, Int. Math. Res. Not. IMRN (2020), 1445–1480.

[Hil05] Joachim Hilgert, *An ergodic Arnold-Liouville theorem for locally symmetric spaces*, Twenty years of Bialowieza: a mathematical anthology, World Sci. Monogr. Ser. Math., vol. 8, World Sci. Publ., Hackensack, NJ, 2005, pp. 163–184. MR 2181551

[HWW21] J. Hilgert, T. Weich, and L. L. Wolf, *Higher rank quantum-classical correspondence*, arXiv:2103.05667 (2021).

[LZ01] J. Lewis and D. Zagier, *Period functions for Maass wave forms. I*, Ann. of Math. **153** (2001), 191–258.

Anosov geodesic flows, closed geodesics and stable ergodicity

Gerhard Knieper
(joint work with Benjamin Schulz)

In this talk we have shown that Finsler Anosov flows can be characterized by stable hyperbolicity of its closed geodesic. The same is true for Riemannian Anosov flows on closed surfaces. As an application we deduced that for closed surfaces C^2 stably ergodic geodesic flows are Anosov. These results are based on joint work with Benjamin Schulz.

Theorem A. *(Knieper, Schulz) Let M be a closed manifold and denote by $\mathcal{F}_{\mathrm{hyp}}(M)$ the set of Finsler metrics for whom all closed geodesic are hyperbolic. Then the C^2 interior of $\mathcal{F}_{\mathrm{hyp}}(M)$ are exactly the set of Finsler metrics for which the geodesic flow is Anosov. For closed surfaces an analogous result holds for Riemannian metrics as well.*

Conjecture. We conjecture that Theorem A also holds for Riemannian metrics on closed manifolds of arbitrary dimensions.

The proof of Theorem A in the Finsler case uses the following important result of Newhouse [N77].

Theorem 1. *Let (V, ω) be a symplectic manifold and $H_0 : V \to \mathbb{R}$ a smooth Hamiltonian with a compact regular energy level $H_0^{-1}(1)$. Then there exists a C^2 neighborhood $U \subset C^\infty(V)$ of H_0 and a dense set $Q \subset U$ such that for all $H \in Q$ either the Hamiltonian flow ϕ_H^t on $H^{-1}(1)$ is Anosov or it has a closed non-hyperbolic orbit.*

The proof of Newhouse's Theorem is based on a C^2-closing Lemma for Hamiltonian systems.
Another ingredient in the proof of Theorem A is the following characterization of geodesic Anosov flows for Finsler metrics [CIS98] (resp. Riemannian metrics [Ru91]).

Theorem 2. *Let (M, F_0) be a closed Finsler manifold. Then $\phi_{F_0}^t : SM \to SM$ is an Anosov flow if and only if F_0 is contained in a C^2 open neighborhood of Finsler metrics without conjugate points. An analogous result holds for Riemannian metrics.*

Sketch of the proof of Theorem A: Let M be a closed manifold and U a C^2 open neighborhood of Finsler metrics for whom all closed geodesics are hyperbolic. Using Theorem 1 of Newhouse one can choose for each $F \in U$, a sequence F_k of Finsler metrics in U for which the geodesic flow is Anosov that converges in the C^2 topology to F. Since the Anosov property is an open but not closed condition one cannot deduce directly that the geodesic flow of F is Anosov. However, Theorem 2 implies that the metrics F_k do not have conjugate points. But since the set of metrics without conjugate points is C^2 closed one obtains that F has no conjugate points as well. Hence, we have proved that all metrics in U do not have conjugate points and, therefore, by the Theorem 2 above, each $F_0 \in U$ is a metric for which the geodesic flow is Anosov. □

Remark 3. Note that it is not possible to deduce Theorem A from Newhouse's Theorem in the Riemannian case. However, for surfaces there is a replacement by the following Theorem of Contreras and Mazzucchelli which is proved among other things in [CM21].

Theorem 4. *Let M be a closed surface. Then exists a C^2-dense set of Riemannian metrics Q such that for all $g \in Q$ either its geodesic flow is Anosov or the surface contains a non-hyperbolic closed orbit.*

Remark 5. For Riemannian metrics which do not contain contractible closed geodesics this result has been also obtained by Schulz [Sch21].

As an application of Theorem A we obtain:

Theorem B. *(Knieper, Schulz) Let M be a closed surface and $\mathcal{F}_{\mathrm{erg}}(M)$ (resp. $\mathcal{R}_{\mathrm{erg}}(M)$) be the set of Finsler (resp. Riemannian) metrics for which the geodesic flow is ergodic with respect to the Liouville measure. Then the C^2 interior of $\mathcal{F}_{\mathrm{erg}}(M)$ (resp. $\mathcal{R}_{\mathrm{erg}}(M)$) consists of metrics for which the geodesic flow is Anosov.*

Remark 6. One can replace the set of ergodic Finsler (resp. Riemannian ergodic) metrics by the larger set of metrics for which the geodesic flow is topologically transitive. Then the C^2 interior of this set consists of metrics for which the geodesic flow is Anosov as well.

Besides Theorem A the proof of this result uses the following Theorem which is a consequence of results by Moser [Mo77] and Siegel and Moser [SM71].

Theorem 7. *Let M be a closed surface with a Finsler metric F which has an elliptic closed orbit. Then there exists a local symplectic coordinate system such that the Poincaré map P is in a local neighborhood $0 \in \mathbb{R}^2$ of the form (Birkhoff normal form)*

$$P(x) = A_{\alpha+\beta\cdot\|x\|^2}(x) + \mathcal{O}(\|x\|^4)$$

where A_φ is the rotation in $\mathbb{R}^2$ with angle $\varphi \in \mathbb{R}$. The constants α and β are symplectically invariant and called Birkhoff invariants. The elliptic orbit is called of twist type if $\beta \neq 0$. If the elliptic orbit is of twist type then there is a flow-invariant torus which divides the phase space into two flow-invariant open sets of positive Liouville measure and, therefore, the geodesic flow is not topologically transitive and in particular not ergodic.

Sketch of the proof of Theorem B: Suppose U is a C^2 open neighborhood of Finsler- or Riemannian metrics for which the geodesic flow is ergodic. Then the set U consists of metrics for whom all geodesics are hyperbolic, otherwise there would exist a metric in U that has a non-hyperbolic closed geodesic. By a sufficiently small perturbation of this metric one obtains another metric in U with a closed elliptic orbit of twist type. Then Theorem 7 implies that this metric has a non-ergodic geodesic flow which contradicts the assumption made for U. Finally, the hyperbolicity of closed geodesics for all metrics in U yields by Theorem A that each metric in U is Anosov.

References

[CIS98] G. Contreras, R. Iturriaga and H. Sánchez-Morgado. On the creation of conjugate points for Hamiltonian systems. *Nonlinearity* 11(2): 335–361, 1998.

[CM21] G. Contreras and M. Mazzucchelli. Proof of the C^2-stability conjecture for geodesic flows of closed surfaces. arXiv:2190.10704v3, 10 Nov. 2021.

[Mo77] J. Moser. Proof of a generalized form of a fixed point theorem due to G.D. Birkhoff. In *Lecture Notes in Mathematics*, 1977, 464–494.

[N77] S. Newhouse. Quasi-elliptic periodic points in conservative dynamical systems. *Amer. J. Math.* 99(5): 1061–1077, 1977.

[Ru91] R. Ruggiero. On the creation of conjugate points. *Math. Zeitschrift*, 208(1): 41–55, 1991.

[Sch21] B. Schulz. Characterization of Anosov metrics on closed surfaces and stable ergodicity. PhD. thesis, Ruhr-Universität Bochum, 2021.

[SM71] C.L. Siegel and J. Moser. Lectures on celestial mechanics Berlin, Heidelberg: Springer 1971.

Hausdorff dimension of the limit set for some Anosov representations

François Ledrappier
(joint work with Pablo Lessa)

We relate the Hausdorff dimension of the limit set for the action on the space of flags of Anosov representations in $\mathrm{PSL}_d(\mathbb{R})$ of a $CAT(-1)$ finitely presented group to the Falconer dimensions of the representation. This is joint Work In Progress with Pablo Lessa (Montevideo).

For a matrix $g \in \mathrm{SL}_d(\mathbb{R})$, write $\sigma_1(g) \geq \ldots \geq \sigma_d(g)$ for the logarithms of the *singular values* of g and write $0 \leq \lambda_1(g) \leq \ldots \leq \lambda_{d(d-1)/2}(g)$ for the differences $\lambda_k(g) := \sigma_{i_k}(g) - \sigma_{j_k}(g)$, for all $i_k < j_k$.

Let $G \subset \mathrm{SL}_d(\mathbb{R})$ be finitely generated, denote by $|\cdot|$ the word length associated to a finite fixed symmetric generator Σ. The group is called *Anosov* if there exist $c, \zeta > 0$ such that for all $p = 1, \ldots, d-1$, all $g \neq Id, \sigma_{p+1}(g) - \sigma_p(g) \leq -\zeta|g| + c$. Let ∂G be the Gromov boundary of the group G and $\mathcal{F}$ the space of flags in $\mathbb{R}^d$. The space ∂G is endowed with a Gromov distance, the space $\mathcal{F}$ with a Riemannian metric invariant under rotations. There is a Hölder continuous equivariant mapping $\xi : \partial G \to \mathcal{F}$ such that for $t \neq t' \in \partial G$, $\xi(t)$ and $\xi(t')$ are in general position. The image $\Lambda := \xi(\partial G)$ is called the *limit set* of the action of G.

Let G be a discrete subgroup of $\mathrm{SL}_d(\mathbb{R})$. Following Falconer (see [F88], [PSW19]), we define, for $r \in [p-1, p], p = 1, \ldots, d(d-1)/2$,

$$\begin{aligned}
\Phi_G(r) &:= \sum_{g \in G} \exp - \left(\sum_{k=1}^{p-1} \lambda_k(g) + (r-p+1)\lambda_p(g) \right), \\
\underline{\Phi}_G(r) &:= \sum_{g \in G} \exp - \left(\sum_{k=1}^{p-1} \lambda_{d(d-1)/2-k+1}(g) + (r-p+1)\lambda_{d(d-1)/2-p+1}(g) \right).
\end{aligned}$$

Let G be a discrete subgroup of $\mathrm{SL}_d(\mathbb{R})$. The (resp. lower) Falconer dimension of G is the critical value (*resp.* $\underline{\mathrm{Dim}}_F(G)$) $\mathrm{Dim}_F(G)$

$$\begin{aligned}
\mathrm{Dim}_F(G) &:= \sup\{r : 0 \leq r \leq d(d-1)/2, \Phi_G(r) = +\infty\}, \\
\underline{\mathrm{Dim}}_F(G) &:= \sup\{r : 0 \leq r \leq d(d-1)/2, \underline{\Phi}_G(r) = +\infty\}.
\end{aligned}$$

In dimension $d = 2$, $\mathrm{Dim}_F(G) = \underline{\mathrm{Dim}}_F(G)$ is the critical exponent of the series $\sum e^{-r\lambda(g)}$, the Poincaré exponent of the group. Sullivan observed that the Hausdorff dimension and the Minkowski (or box-)dimension of the limit set of a convex cocompact group of isometries of a hyperbolic space coincide with the Poincaré exponent ([Sul79]). Our main result is

Main theorem. *Let G be a finitely generated Anosov subgroup of $\mathrm{SL}_d(\mathbb{R})$, Λ the limit set of G for its action on $\mathcal{F}$. The Hausdorff dimension $Dim_H(\Lambda)$, the Minkowski dimension $Dim_M(\Lambda)$ and the Falconer dimensions $\underline{Dim}_F(G), Dim_F(G)$ satisfy*

$$\underline{\mathrm{Dim}}_F(G) \leq \mathrm{Dim}_H(\Lambda) \leq \mathrm{Dim}_M(\Lambda) \leq \mathrm{Dim}_F(G).$$

See [PSW19] (see also [GMT19]) for a more precise result for the action on the projective space instead of the whole flag space, and for more general groups. For several finite dimensional families of contracting C^1 IFS or C^1 repellers, the maximum $\mathrm{Dim}_F(G)$ is achieved for a positive measure set of parameters (see [F88], [FS21] and the references therein).

The first step in the proof is to show $\mathrm{Dim}_M(\Lambda) \leq \mathrm{Dim}_F(G)$. The proof follows the schemes of the proofs of the similar results in [Z97], [BCH10], [PSW19], Theorem B and [FS20].

The rest of the proof uses random walks on Γ. With the notations of [LL21], let $\mathcal{M}(G)$ be the space of probability measures μ on G such that $\sum_{g\in G} |g|\mu(g) < +\infty$ and $\bigcup_{n\geq 1} \mathrm{supp}(\mu^{(n)}) = G$. Let $\mu \in \mathcal{M}$. There is a unique *stationary* probability measure ν on $\mathcal{F}$. By [LL21], the measure ν is exact-dimensional and the dimension $\delta(\nu)$ satisfies

$$\delta(\nu) \ \leq \ \mathrm{Dim}_H(\Lambda) \ \leq \ \mathrm{Dim}_M(\Lambda).$$

The first inequality follows from [Y82] since ν is supported by Λ and the second one is a general property of dimension theory. We want to construct ν such that $\delta(\nu) \geq \underline{\mathrm{Dim}}_F(G)$. In dimension $d = 2$, for a convex cocompact G, this is a property of the Patterson-Sullivan measure.

In higher dimensions, we still can construct an analog of the Patterson-Sullivan measure. For $\xi \in \mathcal{F}, \xi = \{\{0\} \subset U_1'(\xi) \subset \ldots \subset U_{d-1}'(\xi) \subset \mathbb{R}^d\}$, $j = 1\ldots, d-1$, and $g \in G$, we define the expansion coefficients $\sigma_i(g,\xi)$ by

$$\sum_{i\leq j} \sigma_i(g,\xi) \ = \ \log|\ \det_{U_j'(\xi)}(g)|,$$

where, for any subspace U in $\mathbb{R}^d$, $|\ \det_U(g)|$ is the Jacobian of the linear mapping from U to gU, both endowed with the Euclidean metric.

Order the differences $\rho_k(g,\xi) := \sigma_{i_k}(g,\xi) - \sigma_{j_k}(g,\xi)$, for all $i_k < j_k$, in such a way that $\rho_1(g,\xi) \geq \ldots \geq \rho_{d(d-1)/2}(g,\xi) \geq 0$. We define the *type* $\mathcal{T}(g,\xi)$ the corresponding order on the pairs $(i,j), i < j$. There is a finite set of possible types. Since the set of types is finite, there is (at least) one type $\mathcal{T}_0$ such that the restricted function $\underline{\Phi}_{G,\mathcal{T}_0}$

$$\underline{\Phi}_{G,\mathcal{T}_0}(r) \ := \sum_{g\in G, \mathcal{T}(g)=\mathcal{T}_0} e^{-\left(\sum_{k=1}^{p-1}\rho_k(g)+(r-p+1)\rho_p(g)\right)}$$

has $\underline{\mathrm{Dim}}_F(G)$ as critical value. We fix such a type $\mathcal{T}_0$ realizing $\underline{\mathrm{Dim}}_F(G)$.

Define, for $g \in G, \xi \in \mathcal{F}$, a type $\mathcal{T}$ and a real $r, 0 \leq p-1 \leq r \leq p \leq d(d-1)/2$,

$$\beta_r^{\mathcal{T}}(g,\xi) := \left(\sum_{k=1}^{p-1} \rho_k(g,\xi) + (r-p+1)\rho_p(g,\xi)\right),$$

where the order of the indices $k, 1 \leq k \leq d(d-1)/2$, corresponds to the type $\mathcal{T}$. We have, following [Q02],

Proposition. *Fix a type $\mathcal{T}, r, 0 \leq r \leq d(d-1)/2$. The function $(g,s) \mapsto \beta_r^{\mathcal{T}}(g, \xi(s))$ is Hölder continuous in s and defines a cocycle on $G \times \partial G$. There exists a finite measure ν_0 on ∂G such that*

$$\frac{dg_*\nu_0}{d\nu_0}(s) \;=\; e^{\left(\beta^{\mathcal{T}_0}_{\dim_F(G)}(g, g^{-1}\xi(s))\right)}. \tag{1}$$

Since the Radon-Nykodym cocycle in (1) is Hölder continuous, we can use a result by Connell and Muchnik ([CM07]) for $CAT(-1)$ groups. With our notations, it reads as

Proposition.(([CM07])) *There exists a probability measure $\mu_0 \in \mathcal{M}(G)$ such that the measure ν_0 is the only μ_0-stationary measure on ∂G.*

We now know by [LL21] that the measure $\nu = \xi_*(\nu_o)$ is exact dimensional on $\mathcal{F}$. Moreover, [LL21] yields a formula for the dimension $\delta(\nu)$ in terms of local dimensions of conditional measures. Comparing with (1) finally gives:

$$\delta(\nu) \;\geq\; \underline{\mathrm{Dim}}_F(G).$$

References

[BCH10] Jungchao Ban, Yongluo Cao and Huyi Hu. The dimension of a non-conformal repeller and an average conformal repeller *Trans. Amer. Math. Soc.*, 362: 727–751, (2010).

[CM07] Chris Connell and Adam Muchnik. Harmonicity of Gibbs measures *Duke Math. J.* 137: 461–509, (2007).

[F88] Kenneth J. Falconer. The Hausdorff dimension of self-affine fractals *Math. Proc.Cambridge Phil. Soc.* 103(2): 339–350, (1988).

[FS20] De-Jun Feng and Károly Simon. Dimension estimates for C^1 Iterated Function Systems and repellers, Part I *preprint*, ArXiv.math: 2007.15320.

[FS21] De-Jun Feng and Károly Simon. Dimension estimates for C^1 Iterated Function Systems and repellers, Part II *Erg. Th. & Dynam. Sys.*, to appear.

[GMT19] Olivier Glorieux, Daniel Monclair and Nicolas Tholozan. Hausdorff dimension of limit sets for projective Anosov representations, *preprint* arXiv:1902.01844.

[LL21] François Ledrappier and Pablo Lessa. Exact dimension of Furstenberg measures ArXiv:2105.11712

[PSW19] Beatrice Pozzetti, Andrés Sambarino and Anna Wienhard. Anosov representations with Lipschitz limit set *preprint*, ArXiv 1910.06627.

[Q02] Jean-François Quint. Mesures de Patterson-Sullivan en rang supérieur *Geom. Funct. Anal.*, 12: 776–809, (2002).

[Sul79] Dennis Sullivan. The density at infinity of a discrete group of hyperbolic motions *Publ. Math. de l'I.H.E.S.*, 50:171–202, (1979).

[Y82] Lai-Sang Young. Dimension, Entropy and Lyapunov exponents *Ergodic Theory Dynam. Systems.* 2: 109–124, (1982).

[Z97] Yingjie Zhang. Dynamical upper bounds for Hausdorff measure of invariant sets *Ergodic Theory Dynam. Systems.* 17: 739–756, (1997)

Counting and equidistribution of periodic diagonal orbits

Jialun Li
(joint work with Thi Dang)

Let G be $\mathrm{SL}_d(\mathbb{R})$, denote by M the sign group. Let Γ be any finite subgroup of the lattice $\mathrm{SL}_d(\mathbb{Z})$ that acts freely on G/M. I will talk about the counting and equidistribution of compact periodic orbits of the diagonal group A on the space $\Gamma\backslash G/M$. Counting periodic diagonal orbits is a natural generalization of the prime geodesic theorem on compact hyperbolic surfaces, dating back to Huber, Margulis and Bowen. In the $\mathrm{SL}_2(\mathbb{Z})$ case, it is the topic of Sarnak's thesis, and it is connected to the counting of real quadratic orders.

Let $C(A)$ be the set of periodic compact diagonal orbits on $\Gamma\backslash G/M$. For each periodic compact diagonal orbit F, we can define a lattice $\Lambda(F)$ in the Lie algebra $\mathfrak{a}$ of the diagonal group A. An element Y in $\mathfrak{a}$ is in $\Lambda(F)$ if for any point x in F, we have $x\exp(Y)=x$. Let $\mathfrak{a}^{++}$ be the positive Weyl chamber, then the main counting result (a version of the prime geodesic theorem in the higher rank case): there exits $\epsilon>0$ such that as t goes to $+\infty$,

$$\sum_{F\in C(A)} \#(\Lambda(F)\cap B(0,t)\cap\mathfrak{a}^{++})vol(F) = vol(D_t)+O(e^{-\epsilon t}).$$

Here $B(0,t)$ is the ball of radius t in $\mathfrak{a}$ with respect to the Killing norm and D_t is the ball of radius t in the Riemannian symmetric space G/K. We only count periodic diagonal orbits with their systole less than t. The multiplicity is given by the number $\#(\Lambda(F)\cap B(0,t)\cap\mathfrak{a}^{++})$. For the hyperbolic surface case, this multiplicity is exactly the number of closed geodesics of length less than t over the same base F. The equidistribution result is similar with $vol(F)$ replaced by an A-invariant measure on F.

The counting result follows directly from the equidistribution result. I will sketch a proof. For the equidistribution on the compact part: We first use Hopf coordinates to separate the Haar measure. We generalize an idea of Roblin from the hyperbolic case to the higher rank case. It consists in approximating fixed points on the Furstenberg boundary of loxodromic elements by their angular part. Then we use the angular distribution of lattices points [GN12] to obtain equidistribution in the Furstenberg boundary. For the non-compact part, we need to prove the non-escape of mass of the periodic diagonal orbits. We use ideas from homogeneous dynamics, such as the systole and the Siegel domain. The talk is based on recent joint work with Thi Dang.

References

[Hub59] G, Huber. Zur analytischen Theorie hyperbolischen Raumformen und Bewegungsgruppen. *Mathematische Annalen*, 138:1–26, 1959.

[Bow72] R. Bowen. The Equidistribution of Closed Geodesics. *American Journal of Mathematics*, 94(2):413–423, 1972. Publisher: Johns Hopkins University Press.

[GN12] A. Gorodnik and A. Nevo. Counting lattice points. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2012(663):127–176, February 2012.

Pressure metric degenerations (are Lie-theoretic)

Andrés Sambarino

The Hitchin component of $\mathsf{PSL}_d(\mathbb{R})$ is a special connected component of the character variety

$$\mathfrak{X}\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{R})\big) = \hom\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{R})\big)/\mathsf{PSL}_d(\mathbb{R}),$$

where S is a closed connected oriented surface of genus greater than 2 and $\mathsf{PSL}_d(\mathbb{R})$ is the Lie group of volume preserving $d \times d$ matrices up to a scalar multiple, defined as follows. Let $\tau_d : \mathsf{SL}_2(\mathbb{R}) \to \mathsf{SL}_d(\mathbb{R})$ be the unique (up to conjugation) morphism that acts irreducibly[1] on $\mathbb{R}^d$. A representation $\rho : \pi_1 S \to \mathsf{PSL}_d(\mathbb{R})$ is called *Fuchsian* if it factors as

$$\rho : \pi_1 S \to \mathsf{PSL}_2(\mathbb{R}) \xrightarrow{\tau_d} \mathsf{PSL}_d(\mathbb{R}),$$

where the first arrow is discrete and faithful. A *Hitchin component* $\mathcal{H}_d(S)$ is then a connected component of $\mathfrak{X}\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{R})\big)$ that contains a Fuchsian representation. For each d, there are either one or two Hitchin components $\mathcal{H}_d(S)$ according to wether d is, respectively, odd or even.

As it is proved by Hitchin [4], $\mathcal{H}_d(S)$ is analytically diffeomorphic to $\mathbb{R}^{(2g-2)(d^2-1)}$. It is, in particular, a differentiable manifold and its tangent space at every point can be described as the first co-homology group with twisted coefficients

$$\mathsf{T}_\rho \mathcal{H}_d(S) = H^1_{\mathrm{Ad}\,\rho}\big(\pi_1 S, \mathfrak{sl}_d(\mathbb{R})\big). \tag{1}$$

Let us denote by $\mathfrak{a} = \big\{(a_1, \cdots, a_d) \in \mathbb{R}^d : \sum a_i = 0\big\}$ a Cartan subspace of $\mathfrak{sl}_d(\mathbb{R})$, $\mathfrak{a}^+ = \{a \in \mathfrak{a} : a_1 \geq \cdots \geq a_d\}$ a Weyl chamber and its *dual cone* by

$$(\mathfrak{a}^+)^* = \{\varphi \in \mathfrak{a}^* : \varphi|\mathfrak{a}^+ \geq 0\}.$$

For every $\varphi \in \mathbb{P}(\mathfrak{a}^+)^*$ a procedure from Bridgeman-Canary-Labourie-S. [2], that involves the thermodynamical formalism over hyperbolic systems developed in the 70's and 80's (see Bowen-Ruelle [1] and Parry-Pollicott [7]), equips the Hitchin component with a $\mathrm{Out}(S)$-invariant semi-definite bi-linear form $\mathbf{P}^\varphi$, called a *pressure form.*

Deciding which linear forms φ induce Riemannian metrics on $\mathcal{H}_d(S)$ is still an unsolved issue and in this report we focus on the *simple roots*

$$\Delta = \big\{\sigma_i \in \mathfrak{a}^* : \sigma_i(a) = a_i - a_{i+1}\big\}.$$

The main motivation for studying these pressure forms is the following.

Consider $\mathcal{H}_d(S)$ as a subset of $\mathfrak{X}\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{C})\big)$, the latter equipped the complex structure J induced by the complex structure of $\mathsf{PSL}_d(\mathbb{C})$. It follows from the Anosov property, established by Labourie [6], that there exists a neighborhood $\mathcal{U}$ of $\mathcal{H}_d(S)$ inside $\mathfrak{X}\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{C})\big)$ such that for every $\rho \in \mathcal{U}$ and every $p \in \{1, \cdots, d-1\}$ there exists a continuous equivariant map $\xi^{\sigma_p}_\rho : \partial\pi_1 S \to \mathcal{G}_i(\mathbb{C}^d)$.

[1] i.e. has no non-trivial invariant subspaces,

Fix a(ny) Riemannian metric on the Grassmanian $\mathcal{G}_i(\mathbb{C}^d)$ of p-dimensional subspaces of $\mathbb{C}^d$ and denote by

$$\mathrm{Hff}_{\sigma_p}(\rho) = \dim_{\mathrm{Hff}}\left(\xi_\rho^{\sigma_p}(\partial\pi_1 S)\right)$$

the corresponding Hausdorff dimension.

For every $\sigma \in \Delta$ the function Hff_σ is analytic on a (possibly smaller) neighborhood of $\mathcal{H}_d(S)$ and critical at $\mathcal{H}_d(S)$ (Pozzetti-S.-Wienhard [8]). Its Hessian is thus well defined and one has the following relation with the pressure forms for the simple roots:

Theorem 1 (Bridgeman-Pozzetti-S.-Wienhard [3])**.** *For every $v \in \mathsf{T}\mathcal{H}(S, \mathsf{G}_\mathbb{R})$ and every $\sigma \in \Delta$ one has*

$$\mathrm{Hess}\,\mathrm{Hff}_\sigma(Jv) = \mathbf{P}^\sigma(v).$$

In light of equation (1), natural decompositions of $\mathsf{T}_\rho\mathcal{H}_d(S)$ arise when ρ is not Zariski dense. To simplify their description we introduce the following definition.

Definition 2. *Consider $\rho \in \mathcal{H}_d(S)$, then a ρ-*adjoint factor *is an irreducible factor of the representation* $\mathrm{Ad}\circ\rho : \pi_1 S \to \mathsf{GL}(\mathfrak{sl}_d(\mathbb{R}))$*. If V is such a factor let us denote by $V^0 = V \cap \mathfrak{a}$.*

The adjoint factors depend only on the Zariski closure of ρ, and when ρ is Fuchsian one needs to look at the decomposition of $\mathfrak{sl}_d(\mathbb{R})$ as an $\mathfrak{sl}_2(\mathbb{R})$-module via $\mathrm{ad}\circ d_{\mathrm{id}}\tau_d$. This was carried out by Kostant [5]: there are $d-1$ factors,

$$\mathfrak{sl}_d(\mathbb{R}) = \bigoplus_{e=1}^{d-1} V_e$$

say, and V_e has dimension $2e+1$. The corresponding 0-space V_e^0 will be denoted by k^e and called a *Kostant line*, since they are in fact one-dimensional.

In general, since the Zariski closure of a Hitchin representation is reductive (see for example Hitchin [4]) the Lie algebra $\mathfrak{sl}_d(\mathbb{R})$ decomposes as a sum of adjoint factors and hence so does the cohomology

$$H^1_{\mathrm{Ad}\,\rho}\big(\Gamma, \mathfrak{sl}_d(\mathbb{R})\big) = \bigoplus_{V \text{ adjoint factor}} H^1_{\mathrm{Ad}\,\rho}(\Gamma, V).$$

Moreover, the vector subspace $V^0 \subset \mathfrak{a}$ is the zero-restricted-weight-space of the representation

$$\big(\mathrm{Ad}_{\mathsf{PSL}_d(\mathbb{R})}|\mathsf{H}\big)|\mathsf{V} : \mathsf{H} \to \mathsf{GL}(\mathsf{V})$$

of the Zariski closure H of ρ.

Finally, recall that the character variety $\mathfrak{X}\big(\pi_1 S, \mathsf{PSL}_d(\mathbb{R})\big)$ is equipped with a natural involution $\underline{\mathrm{i}}$, induced by the external[2] automorphism of $\mathsf{PSL}_d(\mathbb{R})$, $g \mapsto (g^{-1})^{\mathrm{t}}$. Denote also by $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ the associated opposition involution $\mathrm{i}(a_1, \ldots, a_d) = (-a_d, \cdots, -a_1)$.

The purpose of this report is to advertise the following ongoing work.

[2]when $d \geq 3$

Theorem 3 (S.). *Consider a Hitchin representation $\rho \in \mathcal{H}_d(S)$ and fix a ρ-adjoint factor V. Consider also a simple root $\sigma \in \Delta$. Then the pressure form $\mathbf{P}^\sigma$ is degenerate at $H^1_{\operatorname{Ad}\rho}(\pi_1 S, V)$, in which case $\mathbf{P}^\sigma$ will vanish identically, if and only if either*

- *ρ is Fuchsian and the associated Kostant line k^e lies in the kernel of σ.*
- *ρ is fixed by $\underline{\mathrm{i}}$, $\mathrm{i}|V^0 = -\operatorname{id}$ and σ is i-invariant.*

References

[1] Bowen, R. and Ruelle, D., The ergodic theory of Axiom A flows, *Invent. Math.*, **29**, 1975, 181–202.

[2] Bridgeman, M. and Canary, R. and Labourie, F. and Sambarino, A., The pressure metric for Anosov representations, *Geom. Funct. Anal.*, **25**(4), 2015, 1089–1179.

[3] Bridgeman, M. and Pozzetti, B. and Sambarino, A. and Wienhard, A., Hessian of Hausdorff dimension on purely imaginary directions, to appear in *Bull. Lond. Math. Soc.* http://arxiv.org/abs/2010.16308, 2020.

[4] Hitchin, N. J., Lie groups and Teichmüller space, *Topology*, **31**(3), 1992, 449–473.

[5] Kostant, B., The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, *Amer. J. Math.*, **81**, 1959,973–1032.

[6] Labourie, F., Anosov Flows, Surface Groups and Curves in Projective Space, *Invent. Math.*, **165**, 2006, 51–114.

[7] Parry, W. and Pollicott, M., *Zeta Functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque Vol. 187-188, 1990.

[8] Pozzetti, B. and Sambarino, A. and Wienhard, A., Conformality for a robust class of non-conformal attractors, *J. Reine Angew. Math.*, **2021**(774), 2021, 1–51.

Dynamics of Anosov representations – in the interior and at the boundary

Anna Wienhard

In this talk I discussed some aspects of the dynamics of images of Anosov representations on symmetric spaces, with a special view towards orbit growth, the limit set at infinity, the geometry of the locally symmetric spaces and their compactifications.

The first part of the talk focussed on geometry. I explained how the locally symmetric spaces of infinite volume, that arise as quotients of Anosov representations can be compactified, and how these compactifications are related to domains of discontinuity for the Anosov representations in compactifications of symmetric spaces and flag varieties. It would be interesting to use these compactifications to say something about spectral properties of the Anosov representation and/or the quotients.

The second part of the talk focussed on dynamics. I reviewed some recent results on orbit growth and entropy of Anosov representations, and on how they are related to the Hausdorff dimension of the limit set of the Anosov representation. I shortly adressed relations to counting problems (see work of Sambarino, Carvajales, Hee Oh et. al.), and raised some open questions.

Absence of embedded eigenvalues for higher rank locally symmetric spaces

Lasse L. Wolf
(joint work with Tobias Weich)

In [Pat75] Patterson showed that for a geometrically finite hyperbolic surface of infinite volume all eigenvalues of the Laplacian are smaller than 1/4, i.e. there are no eigenvalues embedded in the essential spectrum $[1/4, \infty)$. This result has been extended to hyperbolic manifolds of higher dimension by Lax-Phillips [LP82] which they use to obtain estimates for lattice point counting in hyperbolic manifolds. In the case of surfaces the infinite volume assumption is equivalent to the geometric condition that the surface has a funnel. This geometric assumption has consequences on the spectral property of the Laplacian. It implies that the latter has no embedded eigenvalues. Our goal is to establish similar connections between the geometry and the spectral analysis for more general symmetric spaces, in particular for higher rank symmetric spaces. Let $X = G/K$ be a symmetric space of non-compact type and Γ a torsion-free subgroup of G. In contrast to hyperbolic surfaces where only funnels and cusps occur as non-compact ends, the classification of the ends of $\Gamma\backslash X$ in the general setting remains a mystery. Hence, the geometry at infinity is far more complicated in this case. Nevertheless, we provide a geometric condition on Γ that implies the absence of embedded eigenvalues. It is motivated by the following observation: a funnel occurs if and only if there is a boundary point $x_\infty \in \partial\mathbb{H}$ and a neighborhood U of x_∞ such that U is contained in a fundamental domain for Γ acting on $\mathbb{H}$. In particular, x_∞ is a wandering point for the action of Γ on the compactification $\overline{\mathbb{H}}$ of $\mathbb{H}$, i.e. there is a neighborhood U of x_∞ such that γU intersects U non-trivially only for finitely many $\gamma \in \Gamma$. Our geometric condition is that there exists a wandering point x_∞ for the action of Γ on the maximal Satake compactification $\overline{X}^S$ of X with $x_\infty \in \partial\overline{X}^S$. (Obviously every point in X is wandering.) Note that (in contrast to the case of symmetric spaces of rank one) there are multiple different G-compactifications for a higher rank symmetric space X. We presented the following

Theorem 1. *Let $X = G/K$ be a symmetric space of non-compact type and Γ a torsion-free subgroup of G such that there exists a wandering point x_∞ for the action of Γ on $\overline{X}^S$ with $x_\infty \in \partial\overline{X}^S$. Then $\Gamma\backslash X$ has no embedded eigenvalues.*

To define embedded eigenvalues consider the algebra $\mathbb{D}(X)$ of G-invariant differential operators on X. Let $G = KAN$ be the Iwasawa decomposition and χ_λ the Harish-Chandra character of $\mathbb{D}(X)$ for $\lambda \in \mathrm{Hom}(\mathrm{Lie}(A), \mathbb{C})$. Then λ is a joint eigenvalue if there is an L^2-function on $\Gamma\backslash X$ that is an eigenfunction for each $D \in \mathbb{D}(X)$ with eigenvalue $\chi_\lambda(D)$. We call λ an embedded eigenvalue if in addition $\lambda \in \mathrm{Hom}(\mathrm{Lie}(A), i\mathbb{R})$ holds. This generalizes the two results mentioned above.

An interesting example is obtained for Anosov representations.

Corollary 2. *Let* Γ *be the image of a* P*-Anosov representation that is torsion-free and non-cocompact. Then* $\Gamma\backslash X$ *has no embedded eigenvalues.*

This corollary can be obtained using the works of Kapovich-Leeb [KL15] (and partially Guichard-Kassel-Wienhard [GKW15]). They provide a compactification of the locally symmetric space $\Gamma\backslash X$ modeled on the maximal Satake compactification, i.e. there is a region of proper discontinuity $\Omega \subseteq \overline{X}^S$ for the action of Γ on $\overline{X}^S$ such that $X \subseteq \Omega$ and the action of Γ on Ω is cocompact. Since every point in a region of proper discontinuity is wandering and X is a proper subset of Ω we have a wandering point that is in $\partial\overline{X}^S$. We deduce the Corollary by applying the theorem.

References

[Bor16] D. Borthwick. Spectral Theory of Infinite-Area Hyperbolic Surfaces. Progress in Mathematics, vol. 318, 2nd edn. Birkhäuser/Springer, 2016.

[GKW15] O. Guichard, F. Kassel, and A. Wienhard. Tameness of Riemannian locally symmetric spaces arising from Anosov representations. arXiv preprint arXiv:1508.04759, 2015.

[KL15] M. Kapovich and B. Lee. Finsler bordifications of symmetric and certain locally symmetric spaces. *Geometry & Topology* 22(5), 2533–2646, 2018.

[LP82] P. Lax and R.S. Phillips. The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces. *J. Funct. Anal.* 46, 280–350, 1982.

[Pat75] S.J. Patterson. The Laplacian operator on a Riemann surface. *Compositio Mathematica* 31(1):83–107, 1975.

Reporter: Thi Dang

Participants

Dr. Xian Dai
Mathematisches Institut
Ruprecht-Karls-Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Dr. Nguyen Thi Dang
Mathematisches Institut
Ruprecht-Karls-Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Prof. Dr. Livio Flaminio
Département de Mathématiques
Université de Lille
Cité Scientifique - Bâtiment M2
59655 Villeneuve d'Ascq Cédex
FRANCE

Dr. Yannick Guedes Bonthonneau
Département de Mathématiques
Université Paris 13
99, Avenue Jean-Baptiste Clement
93430 Villetaneuse Cedex
FRANCE

Prof. Dr. Colin Guillarmou
Département de Mathématiques
Faculté des Sciences d'Orsay
Université Paris-Saclay
91405 Orsay Cedex
FRANCE

Prof. Dr. Joachim Hilgert
Fakultät EIM - Elektrotechnik,
Informatik und Mathematik
Universität Paderborn
Warburger Strasse 100
33098 Paderborn
GERMANY

Dr. Mitul Islam
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Prof. Dr. Fanny Kassel
I.H.E.S.
Le Bois Marie
35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

Prof. Dr. Gerhard Knieper
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum
GERMANY

Dr. Benjamin Delarue (born Küster)
Institut für Mathematik
Universität Paderborn
Warburger Straße 100
33098 Paderborn
GERMANY

Prof. Dr. Francois M. Ledrappier
LPSM
Sorbonne Université
4 Place Jussieu
75252 Paris Cedex
FRANCE

Dr. Thibault Lefeuvre
Institut de Mathématiques de
Jussieu-Paris Rive Gauche
Sorbonne Université - Campus Pierre et
Marie Curie
4, place Jussieu
75252 Paris Cedex
FRANCE

Dr. Jialun Li
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
8057 Zürich
SWITZERLAND

Dr. Daniel Monclair
Université Paris-Saclay
IMO, Bâtiment 307, Rue Michel Magat
F-91405 Orsay Cedex
FRANCE

Jun-Prof. Dr. Maria Beatrice Pozzetti
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Prof. Dr. Federico Rodriguez Hertz
Penn State
105 McAllister building
Montevideo 16803
URUGUAY

Dr. Andrés Sambarino
CNRS
Sorbonne Université
Box 169
4 Place Jussieu
75252 Paris Cedex
FRANCE

Prof. Dr. Ralf J. Spatzier
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Prof. Dr. Tobias Weich
Institut für Mathematik
Universität Paderborn
Warburger Straße 100
33098 Paderborn
GERMANY

Prof. Dr. Anna Katharina Wienhard
Mathematisches Institut
Ruprecht-Karls-Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Lasse Lennart Wolf
Institut für Mathematik
Universität Paderborn
Warburger Strasse 100
33098 Paderborn
GERMANY