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Graph Theory

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Abstract. Graph theory is a quickly developing area of mathematics, with an increasing number of connections to various parts of mathematics and computer science. The workshop aimed at bringing together a broad range of researchers at various career stages to discuss recent exciting developments, in particular, the Product Structure Theorem and progress towards the resolution of Hadwiger's Conjecture. While the workshop was impacted by the COVID pandemic, it still offered many interesting talks, which updated its participants on recent developments covering the whole breadth of graph theory, and collaboration opportunities.

Mathematics Subject Classification (2010): 05C.

Introduction by the Organizers

Graph theory has seen many substantial developments in the last two years, most of which have been presented and discussed at the workshop. In particular, the Tuesday afternoon was devoted to discussing the Product Structure Theorem and its applications. The workshop was attended by 58 participants, 16 attending on-site and 42 remotely. Many attendees switched from on-site participation to remote one during the Christmas break (as of December 14, there were 33 participants registered to attend on-site), in particular, because of the new travel restrictions for those arriving from the UK imposed by Germany on December 19 and the general anticipation of the coming Omicron wave. We have been pleased that the workshop attendees included many early career researchers and females; the diversity of the workshop attendees was also reflected in the selection of those giving the talks during the workshop. The geographic diversity of the participants with many remote attendees connecting from distant time zones put additional constraints on the workshop schedule, however, making the recordings of the talks available during the workshop helped those participating from distant locations to follow the workshop. To facilitate collaboration and communication among on-site and remote participants, we also set up a Discord server where workshop announcements were posted and channels to discuss particular research problems were provided.

The schedule of the workshop spanned the whole week, starting on Monday at 9:00am and concluding on Friday at 6:15pm. Each day started with a 50-minute talk discussing a major recent development. On Monday, the talk was followed by 5-minute minipresentations given by all on-site participants; the minipresentations were supposed to introduce them and their research areas to others. On the remaining days of the workshop, the long talk was followed by a single 25-minute talk and the rest of the morning was devoted to collaborative work. This arrangement was made to enable the remote participants based in Canada and the US, who formed the majority of remote participants, to contribute actively to most of the discussion following of the scheduled talks. Each of the afternoons, except Wednesday, started with a 50-minute talk. On Monday, the talk was followed by an open problem session, which many on-site and remote participants contributed to. Many of the problems presented in the session were discussed and some even solved during the following week. In particular, the counterexample constructed by James Davies to a conjecture posted by Nicolas Trotignon is presented further in the report. On Tuesday, Thursday and Friday, the afternoon program contained additional three 25-minute talks.

One of the scientific highlight of the workshop was a series of talks on Tuesday afternoon, which concerned the Product Structure Theorem and its applications. The Product Structure Theorem asserts every planar graph is subgraph of a product of a tree-like graph and a path, a remarkable result giving new insights in the structure of planar graphs and yielding a completely new way to approach some notoriously difficult problems concerning planar graphs. The series was started by Vida Dujmović, who presented the Theorem and its proof in her 50-minute talk. Some of many applications of the Theorem, both in mathematics and computer science, were covered in the subsequent talks by Gwenaël Joret and Piotr Micek.

The remaining seven 50-minute talks updated the workshop participants on major recent developments in graph theory in addition to the Product Structure Theorem. Marthe Bonamy presented her recent proof on Gallai's Path Decomposition Conjecture for planar graphs. Maria Chudnovsky talked about her fascinating work on tree-width of graphs with forbidden induced subgraphs. Reinhard Diestel introduced a general canonical way of decomposing graphs that capture their global structure. Jacob Fox updated the participants on his recent results on size Ramsey numbers of graphs. Tom Kelly presented a solution of Erdős-Faber-Lovász conjecture, a breakthrough result that attracted attention of many mathematicians, e.g., Tim Gowers commented on the solution of the conjecture in his tweet as follows: *It's one of those lovely statements that sounds as though it should be either easyish to prove or false, but that turns out to be neither.* Stephan

Kreutzer discussed in his talk extensions of the theory of graph minors to directed graphs, an intriguing research direction that keeps attracting attention of graph theorists for over two decades, and indeed, the topics from his talk led to several follow up discussions during the workshop. Finally, Luke Postle presented his recent substantial steps towards proving Hadwiger's Conjecture that he contributed to.

While the workshop was severely impacted by the pandemic, the impact was mitigated to the largest possible extent by excellent technical facilities for hybrid meetings that the Oberwolfach Research Institute offers. We would like to particularly thank the Oberwolfach staff for running the Institute's program so smoothly despite all challenges related to the pandemic. We are also indebted to Samuel Mohr, one of the participants, who kindly agreed to help with operating technical facilities provided by the Institute and greatly contributed to make the workshop run so smoothly, and to Carla Groenland for her help with preparing this report.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Bojan Mohar and Bhargav Narayanan in the "Simons Visiting Professors" program at the MFO.

Workshop: Graph Theory

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Abstracts

Gallai's path-decomposition conjecture

Marthe Bonamy
(joint work with Alexandre Blanché and Nicolas Bonichon)

Given a graph G, a *k-path decomposition* of G is a partition of edges of G in k paths. In 1968 [2], Gallai stated this simple but surprising conjecture: every finite undirected connected graph on n vertices admits a $\lceil \frac{n}{2} \rceil$-path decomposition. Gallai's conjecture is still unsolved as of today, and has only been confirmed on very specific classes of graphs.

An *odd semi-clique* is obtained from a clique on $2k+1$ vertices by deleting at most $k-1$ edges. Bonamy and Perrett asked the following question [1, Question 1.1]: Does every connected graph G that is not an odd semi-clique admit a $\lfloor \frac{n}{2} \rfloor$-path decomposition?

We answer this question positively for planar graphs. Only two odd semi-cliques are planar: the triangle K_3 and K_5 minus one edge, which we denote by K_5^- (see Figure 1). We can therefore state the result as follows:

Theorem 1. *Every connected planar graph G on n vertices except K_3 and K_5^- can be decomposed in $\lfloor \frac{n}{2} \rfloor$ paths.*

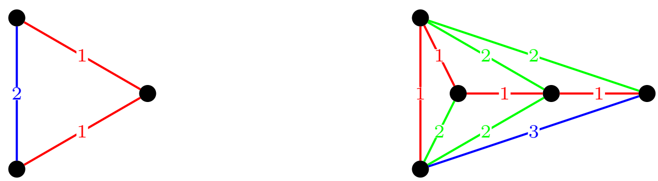

Figure 1. On the left a 2-path decomposition of K_3 and on the right a 3-path decomposition of K_5^-

To prove this theorem we first show that any minimal counter example must satisfy a list of properties, then argue that there is no such graph.

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Clique minors in graphs with a forbidden subgraph

Matija Bucić
(joint work with Jacob Fox and Benny Sudakov)

A graph Γ is said to be a *minor* of a graph G if for every vertex v of Γ we can choose a connected subgraph G_u of G, such that subgraphs G_u are vertex disjoint and G contains an edge between G_v and $G_{v'}$ whenever v and v' make an edge in Γ. The notion of graph minors is one of the most fundamental concepts of modern graph theory and has found many applications in topology, geometry, theoretical computer science and optimisation. Many of these applications have their roots in the celebrated Robertson-Seymour theory of graph minors, developed over more than two decades and culminating in the proof of Wagner's conjecture [13]. One of several equivalent ways of stating this conjecture is that every family of graphs, closed under taking minors can be characterised by a finite family of excluded minors. A forerunner to this result is Kuratowski's theorem [9], one of the most classical results of graph theory dating back to 1930. In a reformulation due to Wagner it postulates that a graph is planar if and only if neither K_5 nor $K_{3,3}$ are its minors.

Another cornerstone of graph theory is the famous 4-colour theorem dating back to 1852 which was finally settled with the aid of computers in 1976 by [1]. It states that every planar graph G has chromatic number at most four. In light of Kuratowski's theorem, Wagner has shown that in fact the 4-colour theorem is equivalent to showing that every graph without K_5 as a minor has $\chi(G) \leqslant 4$. In 1943 Hadwiger proposed a natural generalisation, namely that every graph with $\chi(G) \geqslant r$ has K_r as a minor. Hadwiger's conjecture is known for $r \leqslant 5$ (for the case of $r = 5$ see [14]) but despite receiving considerable attention over the years it is still widely open for $r \geqslant 6$, see [15] for the current state of affairs.

Here we study the question of how large a clique minor can one guarantee to find in a graph G which belongs to a certain restricted family of graphs. A prime example of this type of problems is Hadwiger's conjecture itself. Another natural example asks what happens if instead of restricting the chromatic number we assume a lower bound on the average degree. Note that $\chi(G) \geqslant r$ implies that G has a subgraph of minimum degree at least $r - 1$. So the restriction in this problem is weaker than in Hadwiger's conjecture and we are interested in how far can this condition take us. This question, first considered by Mader [11] in 1968, was answered in the 80's independently by Kostochka [8] and Thomason [16] who show that a graph of average degree r has a clique minor of order $\Theta(r/\sqrt{\log r})$. This is best possible up to a constant factor as can be seen by considering a random graph with appropriate edge density (whose largest clique minor was analysed by Bollobás, Catlin and Erdős in [3]).

This unfortunately means that this approach is not strong enough to prove Hadwiger's conjecture for all graphs. For almost four decades, bounding the chromatic number through average degree and using the Kostochka-Thomason theorem gave the best known lower bound on the clique minor given the chromatic number. Very recently, Norin, Postle, and Song [12] got beyond this barrier and following a series of works Delcourt and Postle [4] obtained the currently best result, by showing that every graph of chromatic number r has a clique minor of size $\Omega(r/\log\log r)$. This still falls short of proving Hadwiger's conjecture for all graphs.

However, if we impose some additional restrictions on the graph it turns out we can do much better. One of the most natural restrictions, frequently studied in combinatorics, is to require our graph G to be H-free for some other, small graph H. This problem was first considered by Kühn and Osthus [10] who showed that given $s \leqslant t$ every $K_{s,t}$-free graph with average degree r has a clique minor of order $\Omega\left(r^{1+2/(s-1)}/\log^3 r\right)$. The polylog factor in this result was subsequently improved by Krivelevich and Sudakov who obtained in a certain sense the best possible bound. They also obtain tight results, for the case of C_{2k}-free graphs. These results show Hadwiger's conjecture holds in a stronger form for any H-free graph, provided H is bipartite. On the other hand, if H is not bipartite then taking G to be a random bipartite graph shows that the bound of Kostochka [8] and Thomason [16] can not be improved.

A natural next question is whether we can do better if we assume a somewhat stronger condition than a bound on the average degree or the chromatic number. A natural candidate is an upper bound on $\alpha(G)$, the size of a largest independent set in G. Indeed, the chromatic number of a graph G is at least $n/\alpha(G)$. An old conjecture, which is implied by Hadwiger's conjecture, (see [15]) states that if $\alpha(G) \leqslant r$ then G has a clique minor of order n/r. Duchet and Meyniel [5] showed in 1982 that this conjecture holds within a factor of 2, following a number of slight improvements Fox [7] gave the first improvement of the multiplicative constant 2. Building upon the ideas of [7], Balogh and Kostochka [2] obtain the best known bound to date.

In light of these results, Norin asked whether in this case assuming additionally that G is triangle-free allows for a better bound. This question was answered in the affirmative by Dvořák and Yepremyan [6] who show that for n/r large enough, a triangle-free n-vertex graph with $\alpha(G) \leqslant r$ has a clique minor of order $(n/r)^{1+\frac{1}{26}}$. They naturally ask if the same holds if instead of triangle-free graphs we consider K_s-free graphs. We show that this is indeed the case.

Theorem 1. *Let $s \geqslant 3$ be an integer. Every K_s-free n-vertex graph G with $\alpha(G) \leqslant r$ has a clique minor of order at least $(n/r)^{1+\frac{1}{10(s-2)}}$, provided n/r is large enough.*

For the case of $s = 3$ our result has a simpler proof and gives a better constant in the exponent compared to that in [6]. As an additional illustration we use our strategy to obtain a short proof of a result of Kühn and Osthus [10] about finding clique minors in $K_{s,t}$-free graphs. The above-mentioned two examples put quite different restrictions on the structure of the underlying graph, nevertheless our approach performs well in both cases. This leads us to believe that our strategy, or minor modifications of it, could provide a useful tool for finding clique minors in graphs under other structural restrictions as well.

Method Given a graph G our strategy for finding minors of large average degree goes as follows:

(1) We independently colour each vertex of G red with probability p and blue otherwise.
(2) Each blue vertex chooses independently one of its red neighbours (if one exists) uniformly at random.

This decomposes the graph into stars, either centred at a red vertex with leaves being the blue vertices, which have chosen the central vertex as their red neighbour or being isolated blue vertices which had no red neighbours to choose from. We obtain our random minor $\mathcal{M}(G,p)$ by contracting each star into a single vertex and deleting the isolated blue vertices.

We note that similar strategies were employed by both Kühn and Osthus [10], and Dvořák and Yepremyan [6]. Our strategy above streamlines their approaches for finding dense minors. This helps us to develop a new way of analysing the outcome, allowing us to answer the above question of Dvořák and Yepremyan [6] as well as to obtain simpler proofs of the results of both [10] and [6].

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Local Separators

JOHANNES CARMESIN

Tree-decompositions and corresponding methods to split graphs along separators are a central tool in algorithmic as well as structural graph theory. It is a natural step to consider local separators of graphs, vertex sets that split graphs only locally.

Similar to the fact that tree-decompositions can be viewed as a set of non-crossing genuine separators, a set of non-crossing local separators can be subsumed as a decomposition of the graph along a genuine decomposition graph. This extends tree-decompositions by allowing genuine decomposition graphs. In recent years we have used these ideas to solve the following problems:

- extend Whitney's planarity criterion from the plane to arbitrary surfaces
- a duality theorem characterising graphs containing bounded subdivisions of wheels
- a local strengthening of the block-cutvertex theorem
- a local strengthening of the 2-separator theorem
- a criterion to detecting normal subgroups in finite groups
- a decomposition theorem for graphs without cycles of intermediate length

We expect that these methods will be applied further in parallel computing for large networks.

One of the big challenges in Graph Theory today is to develop methods and algorithms to study sparse large networks; that is, graphs where the number of edges is about linear in the number of vertices, and the number of vertices is so large that algorithms whose running time is estimated in terms of the vertex number are not good enough. Important contributions that provide partial results towards this big aim include the following.

(1) **Bejamini-Schramm limits of graphs.** Bejamini and Schramm introduced a notion of convergence of sequences of graphs that is based on neighbourhoods of vertices of bounded radius in [3]. Applications of these methods include: testing for minor closed properties [4] by Benjamini, Schramm and Shapira or the proof of recurrence of planar graph limits by Gurel-Gurevich and Nachmias [8].

(2) **From Graphons to Graphexes.** Graphons have turned out to be a useful tool to study dense large networks [10, 11]. Motivated by these successes, analogues for sparse graph limits are proposed in [5, 6, 9].

(3) **Graph Clustering.** The spectrum of the adjacency matrix of a graph can be used to identify large clusters, see the surveys [19] or [17].

(4) **Nowhere dense classes of graphs.** In their book [12], Nešetřil and Ossana de Mendez systematically study a whole zoo of classes of sparse graphs and the relation between these classes.

(5) **Refining tree-decompositions techniques.** Empirical results by Adcock, Sullivan and Mahoney suggest that some large networks do have tree-like structure [1]. In [2], these authors say that: 'Clearly, there is a need to develop Tree-Decompositions heuristics that are better-suited for

the properties of realistic informatics graphs'. And they set the challenge to develop methods that combine the local and global structure of graphs using tree-decompositions methods.

Much of sparse graph theory – in particular of graph minor theory – is built upon the notion of a separator, which allows to cut graphs into smaller pieces, solve the relevant problems there and then stick together these partial solutions to global solutions. These methods include: tree-decompositions [14], the 2-separator theorem and the block-cutvertex theorem, Seymour's decomposition theorem for regular matroids [18], as well as clique sums and rank width decompositions [13]. Understanding the relevant decomposition methods properly is fundamental to recent breakthroughs such as the Graph Minor Theorem [15] or the Strong Perfect Graph Theorem [16]. As whether a given vertex set is separating depends on each vertex individually. So in the context of large networks it is unfeasible to test whether a set of vertices is separating. We believe that in order to extend such methods from sparse graphs to large networks, it is key to answer the following question. What are local separators of large networks?

Here, we answer this question. Indeed, we provide an example demonstrating that the naive definition of local separators misses key properties of separators. Then we introduce local separators of graphs that lack this defect. Our new methods have the following applications.

(1) A unique decomposition theorem for graphs along their local 2-separators analogous to the 2-separator theorem;
(2) an exact characterisation of graphs with no bounded subdivision of a wheel. This connects to direction (4) outlined above;
(3) an analogue of the tangle-tree theorem of Robertson and Seymour, where the decomposition-tree is replaced by a general graph. This connects to direction (5).

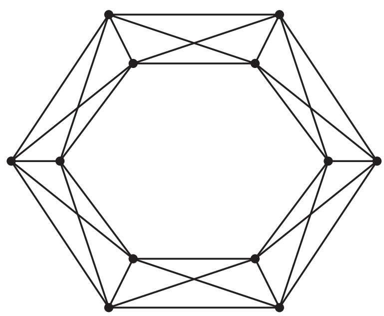

FIGURE 1. The graph $C_6 \boxtimes K_1$.

Example 1. *What is the structure of the graph in 1? According to the 2-separator theorem, this graph is 3-connected and hence a basic graph that cannot be decomposed further. In this paper, however, we consider finer decompositions and according to our main theorem, the structure of this graph is: a family of complete graphs K_4 glued together in a cyclic way.*

Our results. The 2-separator theorem[1] (in the strong form of Cunningham and Edmonds [7]) says that every 2-connected graph has a unique minimal tree-decomposition of adhesion two all of whose torsos are 3-connected or cycles. We work with the natural extension of 'tree-decompositions' where the decomposition-tree is replaced by an arbitrary graph. We refer to them as 'graph-decompositions'.

Addressing the challenge set by Adcock, Sullivan and Mahoney, our main result is the following local strengthening of the 2-separator theorem.

Theorem 2. *For every $r \in \mathbb{N} \cup \{\infty\}$, every connected r-locally 2-connected graph G has a graph-decomposition of adhesion two and locality r such that all its torsos are r-locally 3-connected or cycles of length at most r.*

Moreover, the separators of this graph-decomposition are the r-local 2-separators of G that do not cross any other r-local 2-separator.

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Induced subgraphs and tree decompositions

Maria Chudnovsky
(joint work with Tara Abrishami, Sepehr Hajebi and Sophie Spirkl)

For a graph $G = (V(G), E(G))$, a *tree decomposition* (T, χ) of G consists of a tree T and a map $\chi : V(T) \to 2^{V(G)}$ with the following properties:

(i) For every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \chi(t)$.
(ii) For every $v_1v_2 \in E(G)$, there exists $t \in V(T)$ such that $v_1, v_2 \in \chi(t)$.
(iii) For every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in \chi(t)\}$ is connected.

For each $t \in V(T)$, we refer to $\chi(t)$ as a *bag of* (T, χ). The *width* of a tree decomposition (T, χ), denoted by width(T, χ), is $\max_{t\in V(T)} |\chi(t)| - 1$. The *treewidth* of G, denoted by tw(G), is the minimum width of a tree decomposition of G.

Treewidth, first introduced by Robertson and Seymour in their monumental work on graph minors, is an extensively studied graph parameter, mostly due to the fact that graphs of bounded treewidth exhibit interesting structural [6] and algorithmic [3] properties. Accordingly, one would naturally desire to understand the structure of graphs with large treewidth, and in particular the unavoidable substructures emerging in them. For instance, for each k, the $(k \times k)$-*wall*, denoted by $W_{k\times k}$, is a planar graph with maximum degree three and with treewidth k (see Figure 1; a precise definition can be found in [2]). Every subdivision of $W_{k\times k}$ is also a graph of treewidth k. The unavoidable subgraphs of graphs with large treewidth are fully characterized by the Grid Theorem of Robertson and Seymour, the following.

Theorem 1 ([5]). *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph of treewidth at least $f(k)$ contains a subdivision of $W_{k\times k}$ as a subgraph.*

Following the same line of thought, our motivation is to study the unavoidable induced subgraphs of graphs with large treewidth. Together with subdivided walls mentioned above, complete graphs and complete bipartite graphs are easily observed to have arbitrarily large treewidth: the complete graph K_{t+1} and the complete bipartite graph $K_{t,t}$ both have treewidth t. Line graphs of subdivided

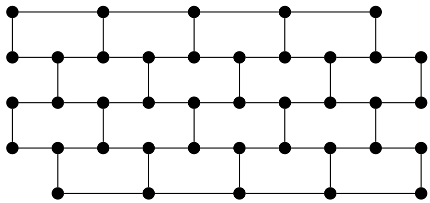

FIGURE 1. $W_{5\times 5}$

walls form another family of graphs with unbounded treewidth, where the *line graph* $L(F)$ of a graph F is the graph with vertex set $E(F)$, such that two vertices of $L(F)$ are adjacent if the corresponding edges of G share an end. One may ask whether these graphs are all we have to exclude as induced subgraphs to obtain a constant bound on the treewidth:

Question 2. *Is it true that for all k, t, there exists $c = c(k, t)$ such that every graph with treewidth more than c contains as an induced subgraph either a K_t, or a $K_{t,t}$, or a subdivision of $W_{k\times k}$ or the line graph of a subdivision of $W_{k\times k}$?*

Sintiari and Trotignon [7] provided a negative answer to this question. To describe their result, we require a few more definitions. Let H be a graph. We say G *contains* H if G has an induced subgraph isomorphic to H. We say G is H*-free* if G does not contain H. For a family $\mathcal{H}$ of graphs we say that G is $\mathcal{H}$-free if G is H-free for every $H \in \mathcal{H}$. Given a graph G, a *path in* G is an induced subgraph of G that is a path. A *hole* in a graph is an induced cycle of length at least four. The *length* of a hole or a path is the number of edges in it.

A *theta* is a graph consisting of two non-adjacent vertices a, b and three paths P_1, P_2, P_3 from a to b of length at least two, such that P_1^*, P_2^*, P_3^* are pairwise disjoint and anticomplete to each other. Note that the complete bipartite graph $K_{2,3}$ is a theta. Also, it is readily seen that for large enough k, all subdivisions of $W_{k\times k}$ contain thetas, and of course line graphs of subdivisions of $W_{k\times k}$ contain triangles. So the following theorem provides a negative answer to Question 2.

Theorem 3 ([7])**.** *For every integer $\ell \geqslant 1$, there exists a (theta, triangle)-free graph G_ℓ such that* $\mathrm{tw}(G_\ell) \geqslant \ell$.

However, it was immediately observed that graphs in 3 contain vertices of arbitrarily large degrees. In [1] it was conjectured conjectured that for every k, every graph with bounded maximum degree and sufficiently large treewidth contains either a subdivision of the $(k \times k)$-wall or the line graph of a subdivision of the $(k\times k)$-wall as an induced subgraph. In [2] two theorems supporting this conjecture are proved; we will describe them next.

First let us define two more types of graphs. A *pyramid* is a graph consisting of a vertex a and a triangle $\{b_1, b_2, b_3\}$, and three paths P_i from a to b_i for $1 \leqslant i \leqslant 3$ of length at least one, such that P_1^*, P_2^*, P_3^* are pairwise disjoint, for $i \neq j$ the only edge between $P_i \setminus \{a\}$ and $P_j \setminus \{a\}$ is b_ib_j, and at most one of P_1, P_2, P_3 has length exactly one.

A *prism* is a graph consisting of two triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, and three paths P_i from a_i to b_i for $1 \leqslant i \leqslant 3$, all of length at least one, and such that for $i \neq j$ the only edges between P_i and P_j are $a_i a_j$ and $b_i b_j$. A *pinched prism* is a graph consisting of a hole H of length at least six, together with a vertex b_1 such that $N_H(b_1)$ is an induced two-edge matching. (This graph is often called a 'line wheel', but here we choose to emphasize its similarity to a prism with a "pinched" path). A *generalized prism* is a graph that is either a prism or a pinched prism.

For $t \geqslant 2$, a *t-theta* is theta where each of the three paths has length at least t. A *t-pyramid* is defined similarly. The first result of [2] is:

Theorem 4 ([2])**.** *For all k, t, every graph of bounded maximum degree and sufficiently large treewidth contains a t-theta, a t-pyramid, or the line graph of a subdivision of the $(k \times k)$-wall as an induced subgraph.*

This result affirmatively answers a question of [4] asking whether every graph of bounded maximum degree and sufficiently large treewidth contains either a theta or a triangle as an induced subgraph (where a *theta* means a t-theta for some $t \geqslant 2$).

A *subdivided cubic caterpillar* is a tree of maximum degree at most three all of whose vertices of degree three lie on a path. The second result of [2] is:

Theorem 5. [2] *For every subdivided cubic caterpillar T, every graph with bounded maximum degree and sufficiently large treewidth contains either a subdivision of T or the line graph of a subdivision of T as an induced subgraph.*

Additionally the authors of [7] observed that the number of vertices of the graphs G_ℓ from Theorem 3 is exponential in their treewidth, while for walls and their line graphs, the number of vertices is polynomial in the treewidth. This radical difference lead to the following conjecture.

Conjecture 6 ([7])**.** *There exists a constant c such that if G is a (theta, triangle)-free graph, then $\mathrm{tw}(G) \leqslant c \log(|V(G)|)$.*

We prove this conjecture, and in fact its generalization, as follows. Let $\mathcal{C}$ be the class of (theta, pyramid, generalized prism)-free graphs. Also, for every integer $t \geqslant 1$, let $\mathcal{C}_t$ be the class of (theta, pyramid, generalized prism, K_t)-free graphs. We show:

Theorem 7. *For every $t \geqslant 1$ there exists a constant c_t such that every $G \in \mathcal{C}_t$ has treewidth at most $c_t \log(|V(G)|)$.*

The following more general conjecture of Trotignon was disproved by James Davies during the workshop:

Conjecture 8. *For all $t \geqslant 0$ there exists $c = c(t)$ such that if G is a graph with no K_t, no $K_{t,t}$, no subdivision of $W_{t\times t}$ and no line graph of a subdivision of $W_{t\times t}$ as induced subgraphs, then $\mathrm{tw}(G) \leqslant c \log(|V(G)|)$.*

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Pivot-minor-closed classes are χ-bounded

James Davies

Local complementation at a vertex v in a graph G replaces the induced subgraph on the neighbourhood $N(v)$ of v by its complement graph. *Pivoting* an edge uv in a graph G is the act of performing local complementation at u, then v, and then u again in order. A graph H is a *vertex-minor* of a graph G if H can be obtained from G by a sequence of vertex deletions and local complementations. *Pivot-minor* is defined similarly with the operation of local complementation replaced by the operation of pivoting. So pivot-minors generalize vertex-minors. We discuss two natural examples of graphs closed under vertex-minors, and one more that is closed under pivot-minors but not vertex-minors.

A *circle graph* is an intersection graph of chords on a circle where two vertices are adjacent whenever their corresponding chords intersect. *Rank-width* is a width parameter analogues to that of tree-width but for dense classes of graph and dense minor-like relations such as vertex-minors and pivot-minors. Likewise circle graphs can be thought of as the analogue of planar graphs for vertex and pivot-minors. This analogy is best shown by comparing the classical grid theorem of Robertson and Seymour [12] to a recent grid theorem for vertex-minors of Geelen, Kwon, McCarty and Wollan [7], which states that graphs of sufficiently large rank-width contain every circle graph as a vertex-minor. Bipartite graphs are closed under pivot-minors but not vertex-minors. Via their fundamental graphs, pivot-minors of bipartite graphs essentially generalize minors for binary matroids, and thus graph minors.

Dense classes of graphs often have unbounded chromatic number for the trivial reason that they often contain cliques of unbounded size. This introduces the notion of *χ-boundedness*. A class of graphs $\mathcal{G}$ is χ-bounded if graphs in $\mathcal{G}$ with bounded clique number also have bounded chromatic number. All discussed classes closed under pivot-minors are known to be χ-bounded. Gyárfás [8] proved that circle graphs are χ-bounded, and Dvořák and Král' [6] proved that classes of graphs with bounded rank-width are χ-bounded. More generally for vertex-minors, Geelen [6] conjectured that every proper vertex-minor-closed class of graphs is χ-bounded, which was proved by Davies [4]. Most generally Choi, Kwon and Oum [2] conjectured that every proper pivot-minor-closed class of graphs is χ-bounded. Our main result is a proof of this conjecture

Theorem 1. *Every proper pivot-minor-closed class of graphs is χ-bounded.*

The proof uses the strategy laid out by the notion ρ-control, and we discuss this strategy. This strategy has been used to prove a number of χ-boundedness theorems [13]. For a graph G and a positive integer ρ, a *ρ-ball* is an induced subgraph formed by a vertex v and the vertices at distance at most ρ from v. We let $\chi^{(\rho)}(G)$ denote the maximum chromatic number of a ρ-ball contained in a graph G, and we say that a class of graphs $\mathcal{G}$ is *ρ-controlled* if there exists a function f such that $\chi(G) \leqslant f(\chi^{(\rho)}(G))$ for all $G \in \mathcal{G}$. This notion can naturally split the problem of proving that a class of graphs $\mathcal{G}$ is χ-bounded into three subproblems. The first is to show that for some $\rho \geqslant 2$, $\mathcal{G}$ is ρ-controlled. The next is to reduce ρ and show that $\mathcal{G}$ is 2-controlled. The final subproblem is to make use of the fact that $\mathcal{G}$ is 2-controlled to prove χ-boundedness.

The next natural step would be improving the χ-bounding function. Kim and Oum [9] made the following conjecture.

Conjecture 2. *Every proper pivot-minor-closed class of graphs is polynomially χ-bounded.*

The discussed natural classes closed under pivot-minors are all polynomially χ-bounded. For circle graphs this was proved by Davies and McCarty [5], and furthermore it is now know that they have a χ-bounding function of $\Theta(\omega \log \omega)$ [3, 11]. Bonamy and Pilipczuk [1] proved that graphs of bounded rank-width are polynomially χ-bounded. We also report on an extension of Vizing's theorem that pivot-minors of line graphs (a class distinct from the class of lines graphs and the class of all graphs) with clique number ω are $\omega + 1$ colourable.

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Global-local decompositions of graphs via coverings

Reinhard Diestel

We show that every finite connected graph G decomposes canonically into local parts whose relative structure is reflected by a simpler graph H. Both H and the associated decomposition $\mathcal{H} = (V_h)_{h \in H}$ of G depend only on G and an integer parameter $r > 0$, which we may choose to set our intended degree of local focus.

H and $\mathcal{H}$ are obtained as quotients of the canonical tree of tangles [3, 4], and its associated tree-decomposition, of the covering of G whose characteristic subgroup in $\pi_1(G)$ is generated by the cycles in G of length at most r.

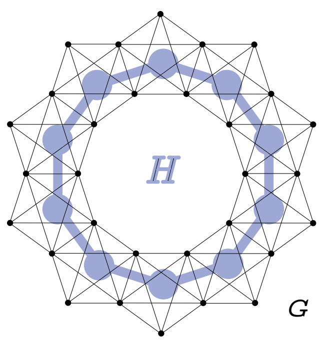

Figure 1. The r-global structure of G for $3 \leqslant r \leqslant 9$ is displayed by a cycle. Its r-local parts are K^5s.

Our main result was inspired by the intuition behind [1]. It reads as follows:

Theorem 1. [2] *For every integer $r > 0$, every finite graph G has a canonical decomposition modelled on another graph $H = H(G, r)$ that displays its r-global structure.*

The theorem extends to some infinite graphs too; these include all Cayley graphs of finitely generated groups (with finite generating sets).

Here, a *decomposition* of G *modelled on* H, or H-*decomposition* of G for short, is a family $(V_h)_{h\in H}$ of sets V_h of vertices of G, the *parts* of this decomposition, that are associated with the nodes h of H in such a way that

- $\bigcup_{h\in H} G[V_h] = G$;
- for every vertex $v \in G$, the subgraph of H induced by $\{ h \in H \mid v \in V_h \}$ is connected.

The decompositions of a graph that are modelled on a tree are thus precisely its tree-decompositions.

However while tree-decompositions have been used and studied widely, they do not fit all graphs well: some graphs G have tree-decompositions into small, dense, or otherwise interesting parts, but others do not. Such other graphs G, however, may well have their own informative coarser overall structure: one expressible not by a tree but by some other graph H. Our theorem finds the optimal such H for every given graph G, given the desired degree r of local focus.

Our decompositions and their models $H = H(G, r)$ are *canonical* in that they commute with graph isomorphisms: if $(V_h)_{h\in H}$ and $(V_{h'})_{h'\in H'}$ are the decompositions which our proof constructs for graphs G and G', then any graph isomorphism $\sigma\colon G \to G'$ maps the parts V_h of G to the parts $V_{h'}$ of G' in such a way that $h \mapsto h'$ is a graph isomorphism $H \to H'$. In particular, the graph H on which Theorem 1 models a given graph G is unique, up to isomorphism, for every r.

The formal definition of H 'displaying the r-global structure' of G comprises a large amount of information and will be given in [2]. The construction of $H = H(G, r)$, however, is not hard to describe. It works as follows.

Given G and r, let S_r denote the subgroup of the fundamental group $\pi_1(G, v_0)$ of G based at a vertex v_0 that is generated by the elements represented by a walk in G from v_0 to a cycle of length at most r, round it, and back along the access path. This S_r is a normal subgroup; let G_r denote the normal covering of G with S_r as characteristic subgroup.

By our choice of r, this covering G_r reflects all the short cycles of G, as well as those of its longer cycles that are generated in $\pi_1(G)$ by the short ones. The other longer cycles of G are usually unfolded to 2-way infinite paths, or *double rays*. (One can construct examples where G_r covers G with finitely many sheets, but those are rare.) Thus, G_r mirrors G in its 'r-local' aspects, but not in its 'r-global' aspects, where it is simply tree-like.

As a consequence of this tree-likeness of the global aspects of G_r, tree-decompositions will be a better fit for G_r than they were for G, whose global aspects could include long cycles that would not fittingly be captured by tree-decompositions. We exploit this by applying to G_r, not to G, the tree-decompositions that represent the state of the art from the theory of graph minors: those that distinguish all the maximal blocks, tangles and ends of G efficiently, and are canonical in the sense described earlier.

Since those tree-decompositions $(V_t)_{t \in T_r}$ are canonical, the group D_r of deck-transformations of G_r over G acts on their decomposition trees T_r. The canonical model H for our r-local decomposition of G is then obtained as the orbit space T_r/D_r. The parts V_h of the H-decomposition of G modelled on H are obtained as the projections to G of the parts V_t of the canonical tree-decomposition of G_r. The bulk of our proof consists of showing that T_r/D_r is indeed graph.

Potential applications

Since $H = H(G, r)$ and the associated H-decomposition of G are canonical, we can study how graph invariants of G interact with those of H as r ranges between 1 and $|G|$. The question to what extent standard graph invariants—the chromatic number, say, or connectivity—are of local or global character, and how their local and global aspects interact, drives much of the research in graph theory both structural and extremal.

Theorem 1 now gives such studies a precise formal basis.

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Product Structure Theorem

Vida Dujmović

Paths, trees, and bounded treewidth graphs are some of the most well understood families of graphs. The central aim of the results presented in this abstract is to answer the following question: Can planar graphs be "factored" into these simpler graphs? It turns out the the answers is yes (in some form at least), as detailed below.

The statment of the main result includes *strong product* and *treewidth* thus we first define these two terms. The *strong product* $A \boxtimes B$ of two graphs A and B is the graph whose vertex set is the Cartesian product $V(A \boxtimes B) := V(A) \times V(B)$ and in which two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if:

(1) $x_1x_2 \in E(A)$ and $y_1y_2 \in E(B)$; or
(2) $x_1 = x_2$ and $y_1y_2 \in E(B)$; or
(3) $x_1x_2 \in E(A)$ and $y_1 = y_2$.

A *tree-decomposition* of a graph G consists of a collection $\{B_x \subseteq V(G) : x \in V(T)\}$ of subsets of $V(G)$, called *bags*, indexed by the vertices of a tree T, and with the following properties:

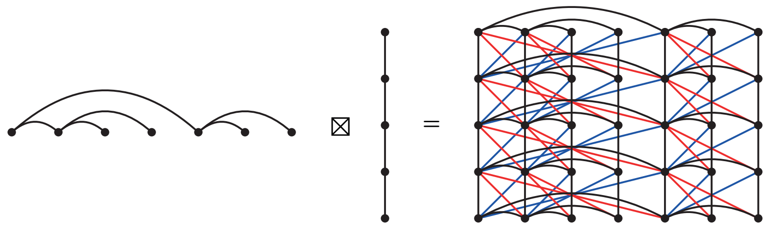

Figure 1. The strong product $H \boxtimes P$ of a tree H and a path P.

- for every vertex v of G, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty (connected) subtree of T, and
- for every edge vw of G, there is a vertex $x \in V(T)$ for which $v, w \in B_x$.

The *width* of such a tree-decomposition is $\max\{|B_x| : x \in V(T)\}-1$. The *treewidth* of a graph G is the minimum width of a tree-decomposition of G.

A tree-decomposition $\{B_x \subseteq V(G) : x \in V(T)\}$ of a graph G is k-*simple*, for some integer k, if it has width at most k, and for every set S of k vertices in G, we have $|x \in V(T) : S \subset B_x| \leqslant 2$. The *simple treewidth* of a graph G is the minimum integer k such that G has a k-simple tree-decomposition

Theorem 1 (Dujmović et al.[5]). *Every planar graph G is a subgraph of $H \boxtimes P$ for some planar graph H of (simple) treewidth at most 8 and for some path P.*

Theorem 1 can be generalized (replacing 8 with a larger constant) to bounded genus graphs, and more generally to apex-minor free graphs [5]. Dujmović, Morin, and Wood [6] gave analogous product structure theorems for some non-minor closed families of graphs including k-planar graphs, map graphs, powers of bounded-degree planar graphs, and k-nearest-neighbour graphs of points in the plane. Dujmović, Esperet, Morin, Walczak, and Wood [4] proved that a similar product structure theorem holds for graphs of bounded degree from any (fixed) proper minor-closed class.

The proof of Theorem 1 also gives the following variant of the theorem.

Theorem 2 ([5]). *Every planar graph is a subgraph of $H \boxtimes P \boxtimes K_3$ for some planar graph H of treewidth at most 3 and for some path P.*

The treewidth 3 in the above theorem is best possible even if K_3 is replaced with any constant size complete graph [5]. That leaves the question of whether the treewidth 8 in Theorem 1 can be reduced.

The proof of Theorem 1 (and Theorem 2) is based on earlier work of Pilipczuk and Siebertz[9]. The proof has inductive step involving a cycle comprised of at most 6 shortest paths in the given planar graph G. The proof variant that leads to the proof of Theorem 1 had an additional property that these (at most 6) shortest parts are all a vertex to its ancestor paths in a given breath first search tree of G.

Ueckerdt et al.[10] managed to replaced "6 shortest paths" in the proof with 5. That, together with another technique, allowed them to reduced the bound in Theorem 1 to 6.

Theorem 3 (Ueckerdt et al.[10]). *Every planar graph G is a subgraph of $H \boxtimes P$ for some planar graph H of (simple) treewidth at most 6 and for some path P.*

Closing the gap between the upper bound 6 in Theorem 3 and the lower bound 3 is an interesting open problem.

Since their discovery, Theorem 1 and Theorem 2 have been used to solve a number of long standing open problems, including: queue number of planar graphs [5], non-repetitive chromatic number of planar graphs [3] and adjacency labelling (universal graphs) for planar graphs [2, 7]. It also lead to the best known bounds on centered colourings [8] and l-vertex ranking [1].

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Sketches for distances in graphs

Louis Esperet

(joint work with Nathaniel Harms, Gwenaël Joret and Andrey Kupavskii)

Let $C \geqslant 1$ be a constant, and $s : \mathbb{N} \to \mathbb{N}$ be a function. A family of graphs $\mathcal{G}$ is (C, s)-*sketchable* if for any $r > 0$ there is a mapping $D = D_r : \{0,1\}^{s(r)} \times \{0,1\}^{s(r)} \to \{0,1\}$, and for any graph $G \in \mathcal{G}$ there is a probability distribution over mappings $\mathrm{sk} = \mathrm{sk}_{G,r} : V(G) \to \{0,1\}^{s(r)}$ such that for any $x, y \in V(G)$ with $d(x,y) \leqslant r$,

$$\mathbb{P}[D(\mathrm{sk}(x), \mathrm{sk}(y)) = 1] \geqslant \tfrac{2}{3};$$

And for any $x, y \in V(G)$ with $d(x,y) > C \cdot r$,

$$\mathbb{P}[D(\mathrm{sk}(x), \mathrm{sk}(y)) = 0] \geqslant \tfrac{2}{3}.$$

A family of graphs $\mathcal{G}$ is *sketchable* if it is (C, s)-sketchable, for some constant $C \geqslant 1$ and for some *bounded* function $s : \mathbb{N} \to \mathbb{N}$ (that is, s is such that there is a constant $S \geqslant 0$ such that for any $r \geqslant 0$, $s(r) \leqslant S$).

The notion of sketchability was introduced in the wider context of general metric spaces, and there is now a good understanding of which finite dimensional *normed spaces* are sketchable [1]. However this does not say much about graph metrics, and the following question arises naturally.

Question. Which families of graphs are sketchable?

It is perhaps interesting to first consider the simpler case $C = 1$. Here it can be checked that if the class of all paths is $(1, s)$-sketchable, then the function s cannot be bounded by a constant. Thus it still makes sense to understand which classes are $(1, s)$-sketchable for some arbitrary (unbounded) function s. It turns out to be tightly connected with the notion of bounded expansion. Given a graph G and an integer $r \geqslant 0$, a *depth-r minor* of G is a graph obtained by contracting pairwise disjoint connected subgraphs of radius at most r in a subgraph of G. For any function f, we say that a class of graphs $\mathcal{G}$ has *expansion* at most f if any depth-r minor of a graph of $\mathcal{G}$ has average degree at most $f(r)$ (see [2] for more details on this notion). Note that every proper minor-closed family has constant expansion. We say that a class $\mathcal{G}$ has *bounded expansion* if there is a function f such that $\mathcal{G}$ has expansion at most f.

We prove the following.

Theorem 1. *For any class $\mathcal{G}$ of bounded expansion, there is a function s such that $\mathcal{G}$ is $(1, s)$-sketchable.*

A class $\mathcal{G}$ is *monotone* if any subgraph of a graph of $\mathcal{G}$ is also in $\mathcal{G}$. We have the following partial converse of Theorem 1.

Theorem 2. *If a monotone class $\mathcal{G}$ is $(1, s)$-sketchable for some function s, then $\mathcal{G}$ has bounded expansion.*

It turns out that we can indeed replace the constant 1 in Theorem 2 by 5. Moreover, assuming the validity of a conjecture of Thomassen [3], we can replace 1 by any fixed constant $C \geqslant 1$.

We now turn to the general case $C \geqslant 1$, and recall that in the definition of a *sketchable class* above, the size of the labels is bounded by an absolute constant, unlike in Theorems 1 and 2 above.

We prove the following.

Theorem 3. *Any proper minor-closed class $\mathcal{G}$ is sketchable, while the class of graphs of maximum degree 3 is not sketchable.*

Note that the class of graphs of maximum degree 3 has bounded expansion, so there is a clear difference with the setting of Theorem 1.

Again, we have a partial converse of Theorem 3 similar to Theorem 2 (but its proof is very different from that of Theorem 2).

Theorem 4. *Any monotone sketchable class has bounded expansion.*

We conjecture that the following stronger result holds.

Conjecture 5. *If for some monotone class $\mathcal{G}$, there is a constant $C \geqslant 1$ and a function s such that $\mathcal{G}$ is (C, s)-sketchable, then $\mathcal{G}$ has bounded expansion.*

Note that this would imply both Theorems 2 and 4. As alluded to above, Conjecture 5 would be implied by a conjecture of Thomassen [3], stating that every graph of large average degree contains a subgraph of large girth and average degree.

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On Size Ramsey Numbers

Jacob Fox

(joint work with David Conlon and Yuval Wigderson)

Given two graphs G and H, we say that G *is Ramsey for* H if every two-coloring of the edges of G contains a monochromatic copy of H. Graph Ramsey theory is mainly concerned with determining which graphs G are Ramsey for a given H. In particular, of central concern is the *Ramsey number* $r(H)$ of H, defined as the minimum number of vertices in a graph G which is Ramsey for H.

We study the *size Ramsey number* $\hat{r}(H)$, defined as the minimum number of edges in a graph G which is Ramsey for H. The size Ramsey number was introduced by Erdős, Faudree, Rousseau, and Schelp [6] in 1978. They proved several bounds on size Ramsey numbers, noting, for example, the basic inequality

$$\hat{r}(H) \leqslant \binom{r(H)}{2}$$

and presenting a proof, due to Chvátal, that this bound is tight when H is a complete graph. They ended their paper with four questions, asking for the asymptotic order of $\hat{r}(H)$ as H ranges over four specific families of graphs. We fully resolve two of these questions and make substantial progress on a third. The fourth question, about the size Ramsey number of paths, was resolved by Beck [1], who proved the surprising result that $\hat{r}(P_n) = \Theta(n)$ for the path P_n with n vertices. This breakthrough inspired many of the subsequent developments in the field.

The first question asked by Erdős, Faudree, Rousseau, and Schelp [6] was about $\hat{r}(K_{s,t})$ for $s \leqslant t$. They proved the bounds

$$\Omega(st2^s) \leqslant \hat{r}(K_{s,t}) \leqslant O(s^2t2^s),$$

with the lower bound only holding for $t = \Omega(s^2)$. However, in a later paper [7], Erdős and Rousseau proved the lower bound $\hat{r}(K_{s,t}) = \Omega(st2^s)$ for all $s \leqslant t$. More recently, Pikhurko [8] found an asymptotic formula for $\hat{r}(K_{s,t})$ for all fixed s and $t \to \infty$.

Our first main result is an improved lower bound on $\hat{r}(K_{s,t})$.

Theorem 1. *For all $s \leqslant t$,*

$$\hat{r}(K_{s,t}) = \Omega\left(s^{2-\frac{s}{t}}t2^s\right).$$

In particular, if $t \geqslant (1+\delta)s$ for any fixed $\delta > 0$, then we get a power saving over the earlier lower bound of $\Omega(st2^s)$. Moreover, once $t = \Omega(s \log s)$, the bound is tight up to a constant factor.

Corollary 2. *If $t = \Omega(s \log s)$, then*

$$\hat{r}(K_{s,t}) = \Theta(s^2t2^s).$$

The second question raised by Erdős, Faudree, Rousseau, and Schelp [6] concerned book graphs. Given positive integers k and n, the *book graph* $B_n^{(k)}$ consists of n copies of K_{k+1}, glued along a common K_k. Equivalently, it can be described as the join of a K_k and an independent set of order n. This K_k is called the *spine* and the vertices of the independent set are called *pages*. Ramsey numbers of book graphs play a central role in Ramsey theory, because all known techniques for proving upper bounds on the diagonal Ramsey numbers $r(K_t)$ rely on induction schemes that repeatedly use bounds on $r(B_n^{(k)})$ for appropriately chosen $k < t$ and n. These Ramsey numbers have received considerable attention of late, beginning with work of Conlon [2], who asymptotically determined $r(B_n^{(k)})$ for n sufficiently large in terms of k, and continuing with work of the authors [3, 4] giving alternative proofs and exploring variations of the basic question. Regarding size Ramsey numbers, Erdős, Faudree, Rousseau, and Schelp [6] proved that

$$\Omega(k^2n^2) \leqslant \hat{r}(B_n^{(k)}) \leqslant O(16^kn^2)$$

for n sufficiently large in terms of k. Thus, while they were able to prove that the dependence on n is quadratic, there was a massive gap between the lower and upper bounds for the dependence on k. Our second main result closes this gap, determining $\hat{r}(B_n^{(k)})$ up to a constant factor for n sufficiently large in terms of k.

Theorem 3. *For every fixed $k \geqslant 2$ and all sufficiently large n,*

$$\hat{r}(B_n^{(k)}) = \Theta(k2^kn^2).$$

The third question raised in [6] concerns graphs which we call starburst graphs (they appear to have not been previously named in the literature). For positive integers k and n, the *starburst graph* $S_n^{(k)}$ is obtained from K_k by adding n pendant edges to every vertex of K_k; thus, it has $kn+k$ vertices. Erdős, Faudree, Rousseau, and Schelp [6] proved that if k is fixed and n is sufficiently large, then

$$\Omega(k^3n^2) \leqslant \hat{r}(S_n^{(k)}) \leqslant O(k^4n^2).$$

Thus, in this case, there is only a $\Theta(k)$ gap between the upper and lower bounds. Our final main result shows that the lower bound is tight up to the constant factor.

Theorem 4. *For every fixed $k \geqslant 2$ and all sufficiently large n,*

$$\hat{r}(S_n^{(k)}) = \Theta(k^3n^2).$$

The proofs of our main theorems are all relatively short, but they employ a surprising array of different techniques. In Theorem 1, the main new idea is to use a random coloring with a hypergeometric distribution between certain vertices and their higher degree neighbors, rather than the uniform distribution that is usually used in Ramsey-theoretic lower bound constructions. The lower bound in Theorem 3 uses a degree-based random coloring, where the probability an edge is red depends on the degrees of its endpoints, while the upper bound uses some of the regularity techniques that were recently developed for studying the ordinary Ramsey numbers of books [2, 3, 4]. Finally, Theorem 4 is proved by examining the properties of an appropriate random graph.

The full version of this paper is [5].

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Subgraphs of large connectivity and chromatic number

António Girão
(joint work with Bhargav Narayanan)

Many of the central open problems in graph theory concern structures that are unavoidable in graphs of large chromatic number, Hadwiger's Conjecture [1] being perhaps the most notable example. Here, we shall address two related questions that arise in the study of Hadwiger's conjecture and its list colouring analogue.

Our starting point is the following well-known fact: every graph of chromatic number at least $4k+1$ contains a subgraph of connectivity at least k, as follows from a classical result of Mader [2] asserting that every graph of minimum degree at least $4k$ contains a k-connected subgraph. It is natural to then ask if a graph of large chromatic number must contain a subgraph of both large connectivity and large chromatic number; this non-trivial problem was answered by Alon, Kleitman, Thomassen, Saks and Seymour [3] who showed for each $k \in \mathbb{N}$ that there exists a minimal $f(k) \in \mathbb{N}$ such that every graph with chromatic number at least $f(k)+1$ contains a subgraph whose connectivity and chromatic number are both at least k, and that $f(k) = O(k^3)$. This was improved by Chudnovsky, Penev, Scott and Trotignon [4] who (amongst other things) showed that $f(k) = O(k^2)$.

The results described above have since found many applications in the study of graphs of large chromatic number. Motivated by applications to the study of Hadwiger's conjecture, Norin asked if the aforementioned results could be sharpened to show an essentially best-possible estimate of $f(k) = O(k)$; our result answers this question affirmatively. We point out that our result has already proved crucial in improving the state of the art on Hadwiger's Conjecture developed in a series of papers of Norin, Song, Postle and Delcourt. (see e.g. [5, 6, 7])

Theorem 1. *For each $k \in \mathbb{N}$, every graph G with chromatic number at least $7k+1$ contains a subgraph H with both connectivity and chromatic number at least k.*

In other words, Theorem 1 asserts that $f(k) \leqslant 7k$, and from below, Alon, Kleitman, Thomassen, Saks and Seymour [3] showed that $f(k) \geqslant 2k-3$. While these bounds are not too far apart, we make no particular effort to optimise the multiplicative constant in our result since it seems unlikely that this will completely bridge the gap between the upper and lower bounds.

Finally, it is worth mentioning that all of [3, 4] treat the more general 'asymmetric' problem of finding a subgraph of connectivity at least k and chromatic number at least m. Our arguments also yield asymmetric analogues of Theorems 1 with essentially optimal bounds.

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Further applications of the product structure theorem

Gwenaël Joret

(joint work with Vida Dujmović, Louis Esperet, Cyril Gavoille, Piotr Micek and Pat Morin)

The recent product structure theorem for planar graphs [6] states that every planar graph G is contained as a subgraph in the strong product of a graph H of treewidth 8 and a path P. Piotr Micek's talk already covered a number of applications of this theorem, including to queue numbers, nonrepetitive colorings, and p-centered colorings of planar graphs. These applications were typically quick and smooth, requiring only short arguments.

In this talk I will describe two further applications of the product structure theorem for planar graphs. By contrast to the other applications mentioned above, the use of the product structure theorem is less straightforward here and somewhat more technical. The two problems under consideration are the following ones:

(1) What is the minimum number of edges in a graph containing all n-vertex planar graphs as subgraphs? The best known bound is $O(n^{3/2})$, due to Babai, Chung, Erdös, Graham, and Spencer [2].

(2) What is the minimum number of *vertices* in a graph containing all n-vertex planar graphs as *induced* subgraphs? Here Bonamy, Gavoille, and Pilipczuk [3] recently established a $O(n^{4/3})$ bound.

Using the product structure theorem for planar graphs, we show that a bound of $n^{1+o(1)}$ can be achieved for these two problems [7, 5].

We conclude this abstract by mentioning two directions for future research. First, it would be interesting to refine the near-linear bounds into asymptotically exact bounds for the two problems. For problems (1) and (2) specialized to n-vertex trees, the right asymptotics are respectively $\Theta(n \log n)$ (Chung and Graham [4]) and $\Theta(n)$ (Alstrup, Dahlgaard, and Knudsen [1]). For all we know, this could be the correct answer for planar graphs too.

Second, it would be interesting to establish near-linear bounds more generally for n-vertex K_t-minor free graphs instead of n-vertex planar graphs. These graphs do not satisfy a product structure theorem in general.

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A proof of the Erdős–Faber–Lovász conjecture

Tom Kelly
(joint work with Dong Yeap Kang, Daniela Kühn, Abhishek Methuku and Deryk Osthus)

In 1972, Erdős, Faber, and Lovász conjectured the following equivalent statements. Let $n \in \mathbb{N}$.

(1) If $G_1, \dots, G_n$ are complete graphs, each on at most n vertices, such that every pair shares at most one vertex, then $\chi(\bigcup_{i=1}^n G_i) \leqslant n$.
(2) Every n-vertex linear hypergraph has chromatic index at most n.

Here the *chromatic index* $\chi'(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the minimum number of colors need to color the edges of $\mathcal{H}$ so that no two edges of the same color share a vertex. A hypergraph $\mathcal{H}$ is *linear* if every two distinct edges of $\mathcal{H}$ intersect in at most one vertex.

Erdős considered this to be 'one of his three most favorite combinatorial problems'. The simplicity and elegance of its formulation initially led the authors to believe it to be easily solved. However, as the difficulty became apparent Erdős offered successively increasing rewards for a proof of the conjecture, which eventually reached \$500.

The following three infinite families of hypergraphs are extremal for this conjecture:

- finite projective planes of order k (known to exist when k is a prime power), which are $(k+1)$-uniform, linear, intersecting hypergraphs on n vertices with n edges where $n = k^2 + k + 1$;
- degenerate planes, also called 'near pencils', which are linear, intersecting hypergraphs on n vertices with n edges for any $n \in \mathbb{N}$ consisting of one edge of size $n-1$ and $n-1$ edges of size two; and
- complete graphs on n vertices where $n \in \mathbb{N}$ is odd (as well as some 'local' modifications of these).

We prove the Erdős–Faber–Lovász conjecture for every large n, as follows.

Theorem 1 ([12])**.** *For every sufficiently large n, every linear hypergraph on n vertices has chromatic index at most n.*

Let us overview previous progress leading up to these results. Predating the Erdős–Faber–Lovász conjecture, in 1948 de Bruijn and Erdős [3] showed that every intersecting n-vertex linear hypergraph has at most n edges. Equivalently, the line graph of an n-vertex linear hypergraph contains no clique of size greater than n. Seymour [16] proved that every n-vertex linear hypergraph $\mathcal{H}$ contains a matching of size at least $|\mathcal{H}|/n$, which implies the de Bruijn-Erdős theorem, as an intersecting hypergraph has matching number one. Kahn and Seymour [11] strengthened this result by proving that every n-vertex linear hypergraph has fractional chromatic index at most n. Chang and Lawler [1] proved that every n-vertex linear hypergraph has chromatic index at most $\lceil 3n/2 - 2 \rceil$. A breakthrough of Kahn [8] in 1992 yielded an approximate version of the conjecture, by showing that every n-vertex linear hypergraph has chromatic index at most $n + o(n)$. Recently Faber and Harris [6] proved the conjecture for linear hypergraphs whose edge sizes range between 3 and $cn^{1/2}$ for a small absolute constant $c > 0$. More background and earlier developments related to the Erdő–Faber–Lovász conjecture are detailed in the surveys of Kahn [9, 10] and of Kayll [15]. See also the recent survey by the authors [13].

In 1977, Erdős [4, Problem 9] asked the following equivalent questions (see also [2, Problem 95] and [5]). Let $n, t \in \mathbb{N}$.

(1) If $G_1, \ldots, G_n$ are complete graphs, each on at most n vertices, such that every pair shares at most t vertices, what is the maximum possible value of the chromatic number $\chi\left(\bigcup_{i=1}^{n} G_i\right)$?

(2) If $\mathcal{H}$ is an n-vertex hypergraph of maximum degree at most n and codegree at most t, what is the maximum possible value of the chromatic index $\chi'(\mathcal{H})$?

The case $t = 1$ corresponds to the Erdős–Faber–Lovász conjecture. Building on the ideas of [12], we answer the question for all $2 \leqslant t < \sqrt{n}$ and sufficiently large n. The range when t is larger is already covered asymptotically by an observation of Horák and Tuza [7].

Theorem 2 ([14])**.** *There exists $n_0 \in \mathbb{N}$ such that the following holds for all $n, t \in \mathbb{N}$ where $n \geqslant n_0$ and $t \geqslant 2$. If $G_1, \ldots, G_n$ are complete graphs, each on at most n vertices, such that $|V(G_i) \cap V(G_j)| \leqslant t$ for all distinct $i, j \in [n]$, then*

$\chi(\bigcup_{i=1}^n G_i) \leqslant tn$. Moreover, for infinitely many $k \in \mathbb{N}$, if $n = k^2 + k + 1$ and $t \leqslant k$, then there exist such $G_1, \ldots, G_n$ such that $\bigcup_{i=1}^n G_i$ has tn vertices and is complete (and in particular has chromatic number tn).

In the dual setting of edge-coloring hypergraphs, this result implies the following for $t \geqslant 2$ and n sufficiently large: If $\mathcal{H}$ is an n-vertex hypergraph with maximum degree at most n and codegree at most t, then $\chi'(\mathcal{H}) \leqslant tn$. We actually prove the following result, which is stronger in three respects. Firstly, we allow the maximum degree of $\mathcal{H}$ to be at most $(1-\varepsilon)tn$ for any $\varepsilon > 0$. Secondly, we prove that the result actually holds more generally for list coloring. Thirdly, we characterize the hypergraphs for which equality holds in the bound; in particular, $\chi'(\mathcal{H}) = tn$ holds only if $\mathcal{H}$ is a *t-fold projective plane*.

Theorem 3 ([14]). *For every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that the following holds for all $n, t \in \mathbb{N}$ where $n \geqslant n_0$. If $\mathcal{H}$ is an n-vertex hypergraph with codegree at most t and maximum degree at most $(1-\varepsilon)tn$, then $\chi'_\ell(\mathcal{H}) \leqslant tn$. Moreover, if $\chi'_\ell(\mathcal{H}) = tn$, then there exists $k \in \mathbb{N}$ such that $n = k^2 + k + 1$ and $\mathcal{H}$ is a t-fold projective plane of order k.*

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Digraph Minors

Stephan Kreutzer
(joint work with Archontia Giannopoulou, Ken-ichi Kawarabayashi and O-Joung Kwon)

At the core of the theory of graph minors developed by Robertson and Seymour in their graph minors series [9] is a powerful structure theorem which describes, for any fixed graph H, the common structure of all finite graphs not containing H as a minor ([13]). Roughly it states that every graph G with no H-minor can be decomposed into pieces each of which is *almost embeddedable* into a surface whose genus is bounded by a function of H, and which can be glued together in a tree structure. A particularly simple form of this structure theorem applies when the excluded minor H is planar: in that case, the parts fit together in a tree-structure and they have bounded size, i.e., G has a *tree-decomposition* of bounded *(tree) width*. This is exactly the *Excluded Grid Theorem*, first proved by Robertson and Seymour in [10] (see also [1]). It states that there is a function $f(k)$ such that every graph of tree width (or, equivalently, which contains a tangle of order) at least $f(k)$ contains a $k \times k$-grid as a minor. This is the best possible outcome in at least two respects. Not only is there no such integer when H is not planar, but no graph of tree width k has a minor isomorphic to the $(k+1) \times (k+1)$-grid.

The case when H is non-planar is much harder. In order to motivate the structure to capture this case, we need to introduce a few structural tools.

The Two Disjoint Paths Problem and Flatness. Let C be a cycle in a graph G. We say that a *C-cross* in G is a pair of disjoint paths P_1, P_2 with ends s_1, t_1 and s_2, t_2, respectively, such that s_1, s_2, t_1, t_2 occur on C in the order listed, and the paths are otherwise disjoint from C. This concept is very related to the famous Two Disjoint Paths Problem. To this end let $s_1, s_2, t_1, t_2 \in V(G)$. The Two Disjoint Paths Problem asks whether or not there exist two disjoint paths P_1, P_2 in G such that P_i has ends s_i and t_i. To relate this to C-crosses assume that G has a cycle C with $\{s_1, s_2, t_1, t_2\}$ in the order listed. Note that the edges of C can be added without changing the problem. It follows that the Two Disjoint Paths Problem can be answered affirmatively if, and only if, the graph G has a C-cross.

It is well-known that if G can be drawn in the plane with C bounding a face, then it has no C-cross. Complementing this observation, the well-known result by Jung [5], Seymour [14], Shiloach [15], and Thomassen [16], says that if G does not have a C-cross, then G can be *nearly* drawn in the plane with the outer face boundary C containing s_1, t_1, s_2, t_2 in this order. The definition of *nearly* is too complicated to mention here, but if we require G to be 4-connected, then we can replace *nearly drawn* by drawn. In this case we call G *C-flat*.

The Flat Wall Theorem. We are now ready to describe a weaker version of the structure theorem (excluded K_t theorem) of Robertson and Seymour [12, Theorem 9.8], known as the Flat Wall Theorem. Let W be a large wall in a graph G with no K_t minor. The Flat Wall Theorem asserts that there exist a set of vertices $A \subseteq V(G)$ of bounded size (that only depends on t) and a big subwall W' of W that is disjoint from A with the following property. Let C' be the outer cycle of W'. Then C' separates the graph $G - A$ into two subgraphs (we call C' *separating* in $G - A$), and the subgraph containing W', say H, can be *nearly* drawn in the plane with C' bounding the outer face boundary (again, the definition of *nearly* is too complicated to describe here). This is equivalent to saying that H is C'-flat in $G - A$.

This theorem is an important step towards the full excluded minor theorem of Robertson and Seymour [13] mentioned above.

Towards a Directed Structure Theory. As a first step towards a structure theory specifically for directed graphs, Reed [7] and Johnson, Robertson, Seymour and Thomas [8] proposed a concept of *directed tree width* and *directed tangle*, and conjectured a directed analogue of the Excluded Grid Theorem. The conjecture had been open for nearly 20 years. It was solved in 2015 by Kawarabayashi and Kreutzer [4]. This result provides a significant first step towards generalising the theory of graph minors to directed graphs, a project we are currently undertaking. Another step towards this goal is the proof of a directed analogue of the tangle decomposition theorem in [11] by Giannopoulou, Kawarabayashi, Kreutzer, and Kwon in [3].

In this talk I will focus on a third important step towards our goal of proving a directed structure theorem for digraph minors, the generalisation of the *Flat Wall Theorem* explained above to directed graphs proved in [2].

Roughly speaking, the Directed Flat Wall Theorem states the following. Given a digraph G containing a sufficiently large cylindrical wall W in G, either we can find a tournament of order t as a butterfly minor or there exists a set of vertices $A \subseteq V(G)$ of bounded size (bounded by a function of t) and a reasonably big cylindrical subwall W' of W that is disjoint from A and has the following properties. Let C' be the outer cycle of W'. Then C' is separating in $G - A$, and the strong component of $G - A - C$ containing W', say H, is C'-flat in $G - A$. Indeed, for each face F of W' there are components (bridges) B attached to F, and $B \cup F$ is F-flat in $G - A$ as well. Moreover, $B \cup F$ is of bounded directed tree width. Thus, every digraph of sufficiently large directed tree-width which excludes some tournament as a butterfly minor contains a "flat cylindrical wall" after deleting a bounded number of apex vertices.

In the directed setting, the theorem comes in two variants, depending on whether we exclude a tournament or a bidirected clique as a butterfly minor. The directed flat wall theorem can be extended to digraphs excluding a bidirected clique as butterfly minor at the expense of weakening the notion of *flatness* of the resulting wall.

While the flat wall theorems in the directed setting may at first sight appear to be very similar to the undirected flat wall theorem, there are significant differences in the notion of flatness that can be achieved. The main obstacle is that the directed version of the two path theorem mentioned above does not hold for directed graphs. We will discuss these differences at the end of the talk and also discuss the impact they have on a potential directed structure theorem.

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A unified Erdős-Pósa theorem for cycles in graphs labelled by multiple abelian groups

O-joung Kwon
(joint work with Pascal Gollin, Kevin Hendrey, Sang-il Oum and Youngho Yoo)

Erdős and Pósa [2] proved in 1965 that every graph contains either k pairwise vertex-disjoint cycles, or a set of $\mathcal{O}(k \log k)$ vertices that hits all cycles of the graph. This breakthrough result sparked extensive research on finding hitting-packing dualities for various graph families.

In particular, cycles with modularity constraints have been considered. For example, Thomassen [4] showed that for every positive integer m, an analogue of the Erdős-Pósa theorem holds for the family of cycles of length 0 modulo m, and Thomas and Yoo [3] proved that for every integer ℓ and every odd prime power m, an analogue of the Erdős-Pósa theorem holds for the family of cycles of length ℓ modulo m. However, this property does not hold for all types of cycles of length ℓ modulo m; Lovász and Schrijver (see [4]) found a class of graphs which shows that such a duality does not exist for the family of odd cycles. Dejter and Neumann-Lara [1] found infinitely many pairs (ℓ, m) for which an analogue of the Erdős-Pósa theorem does not hold for the family of cycles of length ℓ modulo m.

Dejter and Neumann-Lara [1] asked to find all pairs (ℓ, m) of integers for which an analogue of the Erdős-Pósa theorem holds for the family of cycles of length ℓ modulo m. We completely answer this question and extend our result to a more general setting on group-labelled graphs.

For an abelian group Γ and a graph G, a function $\gamma\colon E(G) \to \Gamma$ is called a Γ-*labelling* of G. The γ-*value* of a subgraph H of G is the sum of $\gamma(e)$ over all edges e in H. Cycles of length ℓ modulo m can be naturally encoded in the setting of $\mathbb{Z}_m$-labelled graphs, where each edge has value 1 and the target cycles have values exactly ℓ. Given a set S of vertices in G, cycles containing a vertex of S, called S-cycles, can be encoded as non-zero cycles with respect to the $\mathbb{Z}$-labelling which assigns value 1 to edges incident with vertices in S and 0 to all other edges. We will discuss more examples later. Using multiple abelian groups, we may encode cycles satisfying several properties together.

In a simpler form, our result can be stated as follows. For every pair of positive integers m and ω, there is a function $f_{m,\omega}\colon \mathbb{N} \to \mathbb{N}$ satisfying the following property. Let $\Gamma = \prod_{i\in[m]} \Gamma_i$ be a product of m abelian groups, and for each $i \in [m]$, let Ω_i be a subset of Γ_i with $|\Omega_i| \leqslant \omega$. Let A be the set of all elements $g \in \Gamma$ such that $\pi_i(g) \in \Gamma_i \setminus \Omega_i$ for all $i \in [m]$, and suppose that

(1) for all $a \in A$, we have $\langle 2a\rangle \cap A \neq \emptyset$,
(2) for all $a, b, c \in \Gamma$ with $\langle a, b, c\rangle \cap A \neq \emptyset$, we have $(\langle a, b\rangle \cup \langle b, c\rangle \cup \langle a, c\rangle) \cap A \neq \emptyset$.

Let G be a Γ-labelled graph with Γ-labelling γ and let $\mathcal{O}$ be the set of all cycles of G whose γ-value is in A. Then for all $k \in \mathbb{N}$ there exists either a set of k vertex-disjoint cycles in $\mathcal{O}$, or a hitting set for $\mathcal{O}$ of size at most $f_{m,\omega}(k)$. On the other hand, if A does not satisfy at least one of (1) and (2), then for every positive integer t, there is a graph G with a Γ-labelling γ such that for the set $\mathcal{O}$ of cycles of G with values in A, there are no two vertex-disjoint cycles in $\mathcal{O}$ and there is no hitting set for $\mathcal{O}$ of size at most t.

This yields a corollary about cycles with modularity constraints. Let ℓ and m be integers with $m \geqslant 2$, and let $m = p_1^{a_1} \cdots p_n^{a_n}$ be the prime factorization. Then there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every graph G and every integer k, G contains either k vertex-disjoint cycles of length ℓ modulo m or a set of at most $f(k)$ vertices hitting all such cycles, if and only if the following conditions are satisfied:

(1) If $p_i = 2$ for some $i \in [n]$, then $\ell \equiv 0 \pmod{p_i^{a_i}}$.
(2) There do not exist three distinct $i_1, i_2, i_3 \in [n]$ such that $\ell \not\equiv 0 \pmod{p_{i_j}^{a_{i_j}}}$ for each $j \in [3]$.

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Homomorphism counts in robustly sparse graphs

Chun-Hung Liu

Determine the maximum number of edges of a graph with a given property is a central problem in extremal graph theory. For example, Turán's theorem determines the maximum number of edges of an n-vertex graph with no K_t-subgraph. Since the number of edges equals the number of K_2-subgraphs, we can consider the following more general question.

Question 1. *Let $\mathcal{G}$ be a graph class, and let H be a graph. What is the maximum number* $\mathrm{ex}(H, \mathcal{G}, n)$ *of H-subgraphs contained in an n-vertex graph in $\mathcal{G}$?*

Question 1 has been extensively studied for various graph classes $\mathcal{G}$. In this talk, we consider classes of graphs with a robustly sparse property. Our main result shows that an obvious lower bound for $\mathrm{ex}(H, \mathcal{G}, n)$ determines $\mathrm{ex}(H, \mathcal{G}, n)$ up to a constant factor.

To state the obvious lower bound, we need some definitions. A *separation* of a graph H is an ordered pair (A, B) of subsets of $V(H)$ such that $A \cup B = V(H)$, and there exists no edge between $A - B$ and $B - A$. The *order* of (A, B) is $|A \cap B|$. A collection $\mathcal{C}$ of separations of H is *independent* if for every $(X, Y) \in \mathcal{C}$, $X - Y \neq \emptyset$, and for distinct $(A, B), (C, D) \in \mathcal{C}$, $A \subseteq D$ and $C \subseteq B$.

Let H be a graph. Let $Z \subseteq V(H)$. Let k be a positive integer. We define $H \wedge_k Z$ to be the graph obtained from a union of k disjoint copies of H by for each $z \in Z$, identifying the k copies of z into a vertex. We say an independent collection $\mathcal{C}$ of separations of H is *$\mathcal{G}$-duplicable* if $H \wedge_\ell \mathcal{C} \in \mathcal{G}$ for infinitely many positive integers ℓ. It is easy to show the following lower bound for $\mathrm{ex}(H, \mathcal{G}, n)$.

Proposition 2. *For any graph H and graph class $\mathcal{G}$, $\mathrm{ex}(H, \mathcal{G}, n) = \Omega(n^k)$, where k is the maximum size of a $\mathcal{G}$-duplicable independent collection of separations of H.*

A graph class $\mathcal{G}$ has *bounded expansion* if there exists a function f such that for every graph G in $\mathcal{G}$ and nonnegative integer r, every graph obtained from a subgraph of G by contracting disjoint subgraphs of G of radius at most r has average degree at most $f(r)$. Note that the graphs in a bounded expansion class are robustly sparse in the sense that contracting disjoint subgraphs with bounded radius cannot create arbitrarily dense graphs. A graph class is *hereditary* if it is closed under deleting vertices. The following is our main theorem.

Theorem 3. *For every hereditary class $\mathcal{G}$ with bounded expansion and for every graph $H \in \mathcal{G}$, $\mathrm{ex}(H, \mathcal{G}, n) = \Theta(n^k)$, where k is the maximum size of a $\mathcal{G}$-duplicable independent collection of separations of H.*

Theorem 3 solves a number of open questions. For example, proper minor-closed families are known to have bounded expansion, so Theorem 3 implies the following result, solving a question of Eppstein [1].

Corollary 4. *If $\mathcal{G}$ is a proper minor-closed family and H is a graph in $\mathcal{G}$, then $\mathrm{ex}(H, \mathcal{G}, n) = \Theta(n^k)$ for some integer k.*

Győri et al. [2] proposed a variant of Eppstein's question.

Conjecture 5 ([2])**.** *For any finite set of graphs $\mathcal{F}$ and for any graph H, if $\mathcal{G}$ is the set of all planar graphs with no subgraph isomorphic to any member in $\mathcal{F}$, and $H \in \mathcal{G}$, then $\mathrm{ex}(H, \mathcal{G}, n) = \Theta(n^k)$ for some integer k.*

Note that the class $\mathcal{G}$ mentioned in Conjecture 5 is not a minor-closed family. But this class is a subclass of a proper minor-closed family, so it still has bounded expansion. Therefore, Theorem 3 immediately implies Conjecture 5.

Corollary 6. *Conjecture 5 holds.*

When more information of the graph class $\mathcal{G}$ is provided, we can described $\mathcal{G}$-duplicable independent collections more concretely. For an independent collection $\mathcal{C}$ of separations of H, the *central torso* is the graph obtained from $H[\bigcap_{(A,B)\in\mathcal{C}} B]$ by adding edges such that $A \cap B$ is a clique for every $(A, B) \in \mathcal{C}$; a *peripheral torso* is a graph obtained from $H[X]$ for some $(X, Y) \in \mathcal{C}$ by adding edges such that $X \cap Y$ is a clique.

The following corollary of Theorem 3 disproves a conjecture of Huynh and Wood [4].

Corollary 7. *Let s, t be positive integers with $s \leqslant t$. Let $\mathcal{G}$ be the class of graphs containing no $K_{s,t}$-minor. Then for every graph $H \in \mathcal{G}$, $\mathrm{ex}(H, \mathcal{G}, n) = \Theta(n^k)$, where k is the maximum size of an independent collection $\mathcal{C}$ of separations of H of order at most $s - 1$ such that every torso of $\mathcal{C}$ is $K_{s,t}$-minor free.*

Theorem 3 also implies the following result, answering a question of Huynh and Wood [4].

Corollary 8. *Let t be a positive integer. Let $\mathcal{G}$ be the class of graphs of path-width at most t. Then for every graph H of path-width at most t, $\mathrm{ex}(H, \mathcal{G}, n) = \Theta(n^k)$, where k is the maximum size of an independent collection $\mathcal{C}$ of separations of H of order at most t such that $H \wedge_{\binom{t}{2}+2t+3} \mathcal{C}$ has path-width at most t.*

Colin de Verdière parameter $\mu(G)$ of a graph G is the largest corank of certain matrices associated with G. Many graphs with certain topological properties can be characterized by using this algebraic parameter. For example, $\mu(G) \leqslant 1$ if and only if G is a disjoint union of paths; $\mu(G) \leqslant 2$ if and only if G is outerplanar; $\mu(G) \leqslant 3$ if and only if G is planar; $\mu(G) \leqslant 4$ if and only if G is linkless embeddable. For any integer k, the class of graphs with $\mu \leqslant k$ is minor-closed. So Theorem 3 implies the following result, improving an earlier result of Huynh and Wood [4] who proved the case when H is a tree.

Corollary 9. *Let t be a positive integer. Let $\mathcal{G}$ be the class of all graphs G with $\mu(G) \leqslant t$. Let H be a graph with $\mu(H) \leqslant t$. Then $\mathrm{ex}(H, \mathcal{G}, n) = \Theta(n^k)$, where k is the maximum size of an independent collection of separations of H of order at most $t-1$ such that every its torso L satisfies $\mu(L) \leqslant t$.*

For a surface Σ, the *Σ-crossing number* of a graph G is the minimum integer t such that G can be drawn in Σ with at most t crossings. Any class of graphs with bounded Σ-crossing number is a topological minor-closed family, and hence has bounded expansion. So Theorem 3 implies the following.

Corollary 10. *Let Σ be a surface. Let t be a nonnegative integer. Let $\mathcal{G}$ be the class of graphs with Σ-crossing number at most t. Let $H \in \mathcal{G}$. Then $\mathrm{ex}(H, \mathcal{G}, n) = \Theta(n^k)$, where k is the maximum size of an independent collection of separations of H of order at most 2 whose every peripheral torso is planar and whose central torso can be drawn in Σ with at most t crossings such that every peripheral edge does not contain any crossing.*

Note that the case $t = 0$ of Corollary 10 implies the earlier result about graphs with bounded Euler genus in [4].

Another example of bounded expansion classes is the class of graphs admitting a book embedding with a bounded number of pages. So Theorem 3 applies to this class and solves another question of Eppstein [1].

Counting the number of H-subgraphs is equivalent to counting the number of injective homomorphisms from H, up to a constant factor. So Theorem 3 can be equivalently stated as counting the number of injective homomorphisms. Being injective is a property that is consistent with respect to isomorphisms and taking induced subgraphs. In fact, we can prove the following stronger form of Theorem 3, which counts the number of homomorphisms satisfying any given property, as long as this property is "consistent with respect to isomorphisms and taking induced subgraphs".

Theorem 11. *For any hereditary class $\mathcal{G}$ with bounded expansion, graph H, and set $\mathcal{S}$ of homomorphisms from H to members in $\mathcal{G}$ "consistent with respect to isomorphisms and taking induced subgraphs", we have*

$$\max_{G \in \mathcal{G}, |V(G)|=n} |\{f \in \mathcal{S} : f : V(H) \to V(G)\}| = \Theta(n^k),$$

where k is the maximum size of a "$(\mathcal{G}, \mathcal{S})$-duplicable" independent collection of separations of H.

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Quick applications of the product structure theorem

Piotr Micek

The product structure theorem for planar graphs (and beyond) has been proven to be amazingly applicable in a very short period of time from its first informal publication in April 2019. Within my presentation, I covered three independent results closing long-standing open problems constituing their own research areas. The main aim though was to present a simple strategy to prove statements for planar graphs when analogous statements are known for graphs of bounded treewidth.

Let me start with queue layouts of planar graphs. A *queue layout* of a planar graph consists of a total ordering on its vertices and an assignment of its edges to *queues* such that no two edges in a single queue are nested. The minimum number of queues needed in a queue layout of a graph G is called its *queue number* and denoted by $\mathrm{qn}(G)$.

In 1992 Heath, Leighton, and Rosenberg [1] asked (and later conjectured) if the queue number of planar graphs is bounded. This was a tantalizing open problem and attracted a lot of research around it. The only non-trivial bounds for the queue number of n-vertex planar graph were growing with n. Using the Lipton-Tarjan separator theorem one can easily show that the queue number of an n-vertex planar graph is $O(\sqrt{n})$. Di Battista et al. [2] proved a first breakthrough on this topic, by showing that every n-vertex planar graph has queue number $O(\log^2 n)$. Dujmović [3] improved this bound to $O(\log n)$ with a simpler proof. Within [4], so in the paper introducing the product structure theorem itself, we proved that $\mathrm{qn}(G) \leqslant 49$, for every planar graph G. The strategy of the proof is captured by the following chain of inequalities

$$\begin{aligned}
\mathrm{qn}(G) &\leqslant \mathrm{qn}(H \boxtimes P \boxtimes K_3) \\
&\leqslant 3 \cdot 3 \cdot \mathrm{qn}(H) + \left\lfloor \frac{3}{2} \cdot 3 \right\rfloor \\
&\leqslant 3 \cdot 3 \cdot 5 + \left\lfloor \frac{3}{2} \cdot 3 \right\rfloor \\
&= 49,
\end{aligned}$$

where H is a graph with $\mathrm{stw}(H) \leqslant 3$, P is a path, and K_3 is a complete graph on 3 vertices. The first inequality follows from (1) the product structure theorem, i.e. $G \subseteq H \boxtimes P \boxtimes K_3$ and (2) the monotonicity of the qn() operator. The second inequality is the technical contribution of the paper. We argue that we can take out K_3 and P from the operator paying a small multiplicative constant. Finally, we are using a result by Alam et al. [5], i.e., if $\mathrm{stw}(H) \leqslant 3$, we have $\mathrm{qn}(H) \leqslant 5$.

The second example is a breakthrough result on the nonrepetitive chromatic number of planar graphs. Consider a word w over some alphabet of symbols. A *square* of a word w is a word $w^2 = ww$. E.g. $(ab)^2 = abab$. A word is *nonrepetitive* if it contains no squares. A coloring of vertices of a graph is *nonrepetitive* if for every path in G, the color sequence along the path is nonrepetitive. The nonrepetitive chromatic number of a graph, denoted by $\pi(G)$, is the least integer k such that G admits a nonrepetitve coloring with k colors. Already in 1906, Thue has proved that $\pi(P) \leqslant 3$ for every path P. In 2002, Alon et al. [6] conjectured that planar graphs have bounded nonrepetitve chromatic number. This was a central open problem in a very lively area of research. The product structure theorem allows a simple proof for the conjecture. Indeed, Dujmović et al. [7] proved that $\pi(G) \leqslant 768$ for every planar graph G. Their proof strategy is captured within the following lines:

$$\begin{aligned}
\pi(G) &\leqslant \pi(H \boxtimes P \boxtimes K_3) \\
&\leqslant \pi^*(H \boxtimes P \boxtimes K_3) \\
&\leqslant \pi^*(H \boxtimes P) \cdot 3 \\
&\leqslant \pi^*(H) \cdot 4 \cdot 3 \\
&\leqslant 4^3 \cdot 4 \cdot 3 = 768.
\end{aligned}$$

The first inequality follows from (1) the product structure theorem, i.e., $G \subseteq H \boxtimes P \boxtimes K_3$ and (2) the monotonicity of the $\pi()$ operator. Within the second inequality a new operator π^* is introduced. We have $\pi(G) \leqslant \pi^*(G)$ for all graphs G. Moreover, $\pi^*()$ works better with the strong product as we can see in the third and fourth inequality. Finally, $\pi^*(H) \leqslant 4^k$ for all graphs H with $\mathrm{tw}(H) \leqslant k$ follows by [8].

The third and last example is on p-centered chromatic numbers of planar graphs. A vertex coloring ϕ of a graph G is *p-centered* if for every connected subgraph H of G either ϕ uses more than p colors on H or there is a color that appears exactly once on H. Centered colorings form one of the families of parameters that allow to capture notions of sparsity of graphs. The *p-centered chromatic number* $\chi_p(G)$ of G is the minimum integer k such that there is a p-centered coloring of G using k colors.

In 2016, Dvořák asked if $\chi_p(G)$ is bounded by polynomial in p for G being planar. This was proved by Mi. Pilipczuk and Siebertz [9]. Together with Debski, Felsner, and Schröder [10] we proved that $\chi_p(G) = O(p^3 \log p)$ for all planar graphs

G. The proof is captured within the following lines:

$$\begin{aligned}\chi_p(G) &\leqslant \chi_p(H \boxtimes P \boxtimes K_3)\\ &\leqslant \chi_p(H \boxtimes P) \cdot \overline{\chi_p}(K_3)\\ &\leqslant \chi_p(H) \cdot \overline{\chi_p}(P) \cdot \overline{\chi_p}(K_3)\\ &\leqslant \chi_p(H) \cdot (p+1) \cdot 3\\ &= O(p^2 \log p) \cdot (p+1) \cdot 3\\ &= O(p^3 \log p).\end{aligned}$$

Again, the first inequality is by the product structure theorem. The second and third inequalities are technical contributions of our paper and introduce yet another auxiliary parameter $\overline{\chi_p}(G)$. The rest follows from the bound for $\chi_p(H) = O(p^k \log p)$ whenever $\mathrm{stw}(H) \leqslant k$.

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Obstructions for matroids of path-width at most k and graphs of linear rank-width at most k

Sang-il Oum
(joint work with Mamadou Moustapha Kanté, Eun Jung Kim and O-joung Kwon)

The class of graphs of path-width at most k is minor-closed and therefore the list of excluded minors for the class of graphs of path-width at most k is finite for each k by the theorem of Robertson and Seymour [13]. In 1998, Lagergren [10] proved that each excluded minor for the class of graphs of path-width at most k has at most $2^{O(k^4)}$ edges.

We aim to prove analogous theorems for the class of matroids of path-width at most k and for the class of graphs of linear rank-width at most k. For a matroid M on the ground set $E(M)$, we define its connectivity function λ_M by

$$\lambda_M(X) = r_M(X) + r_M(E(M) - X) - r(M) \quad \text{for } X \subseteq E(M),$$

where r_M is the rank function of M. The path-width of a matroid M is defined as the minimum *width* of linear orderings of its elements, called *path-decompositions* or *linear layouts*, where the width of a path-decomposition $e_1, e_2, \ldots, e_n$ is defined as the maximum of the values $\lambda_M(\{e_1, e_2, \ldots, e_i\})$ for all $i = 1, 2, \ldots, n$.

For matroid path-width, we do not yet know whether there are only finitely many excluded minors for the class of matroids of path-width at most k. Previously, Koutsonas, Thilikos, and Yamazaki [9] showed a lower bound, proving that the number of excluded minors for the class of matroids of path-width at most k is at least $(k!)^2$.

Geelen, Gerards, and Whittle [4] proved that for each finite field $\mathbb{F}$, $\mathbb{F}$-representable matroids of bounded branch-width are well-quasi-ordered under taking minors. This implies that for each finite field $\mathbb{F}$, there are only finitely many $\mathbb{F}$-representable excluded minors for the class of matroids of path-width at most k. However, their theorem does not provide any method of constructing the list of $\mathbb{F}$-representable excluded minors. We are now ready to state our main theorem, showing an explicit upper bound of the size of every $\mathbb{F}$-representable excluded minor.

Theorem 1. *For a finite field $\mathbb{F}$ and an integer k, each $\mathbb{F}$-representable excluded minor for the class of matroids of path-width at most k has at most $2^{|\mathbb{F}|^{O(k^2)}}$ elements.*

Thus, by Theorem 1, we 'have' an algorithm to construct a monadic second-order formula $\varphi_k^{\mathbb{F}}$ to decide whether an $\mathbb{F}$-representable matroid has path-width at most k and we 'have' a fixed-parameter algorithm to decide whether an input $\mathbb{F}$-represented matroid has path-width at most k. Note that there is a subtle difference between "have" and "there exist"; by Geelen, Gerards, and Whittle [4], we knew that there exists $\varphi_k^{\mathbb{F}}$, but we did not know how to construct it, because their proof is non-constructive. By Theorem 1 we can enumerate all matroids of small size to find the list of all $\mathbb{F}$-representable excluded minors and therefore we can finally construct $\varphi_k^{\mathbb{F}}$.

We remark that Geelen, Gerards, Robertson, and Whittle [3] showed an analogous theorem for branch-width of matroids; for each $k \geqslant 1$, every excluded minor for the class of matroids of branch-width at most k has at most $(6^{k+1}-1)/5$ elements.[1]

By extending our method slightly, we also prove a similar theorem for the linear rank-width of graphs as follows.

Theorem 2. *Each excluded pivot-minor for the class of graphs of linear rank-width at most k has at most $2^{2^{O(k^2)}}$ vertices.*

Since every vertex-minor obstruction is also a pivot-minor obstruction, we deduce the following.

Corollary 3. *Each excluded vertex-minor for the class of graphs of linear rank-width at most k has at most $2^{2^{O(k^2)}}$ vertices.*

The situation is very similar to that of matroids representable over a fixed finite field. Oum [11] showed that graphs of bounded rank-width are well-quasi-ordered under taking pivot-minors, which implies that the list of excluded pivot-minors for the class of graphs of linear rank-width at most k is finite. Again its proof is non-constructive and therefore it provides no algorithm to construct the list. Jeong, Kwon, and Oum [6, 7] proved that any list of excluded pivot-minors characterizing the class of graphs of linear rank-width at most k has at least $2^{\Omega(3^k)}$ graphs.

Corollary 3 answers an open problem of Jeong, Kwon, and Oum [7] on the number of vertices of each excluded vertex-minor for the class of graphs of linear rank-width at most k. Adler, Farley, and Proskurowski [1] characterized excluded vertex-minors for the class of graphs of linear rank-width at most 1. Theorem 6.1 of Kanté and Kwon [8] implies that distance-hereditary excluded vertex-minors for the class of graphs of linear rank-width at most k have at most $O(3^k)$ vertices.

Previously, we only knew the existence of a modulo-2 counting monadic second-order formula Φ_k testing whether a graph has linear rank-width at most k. This is due to the theorem of Courcelle and Oum [2] stating that for each graph H, there is a modulo-2 counting monadic second-order formula to decide whether a graph has a pivot-minor isomorphic to H. As there is a polynomial-time algorithm to decide a modulo-2 counting monadic second-order formula for graphs of bounded rank-width (see [2, Proposition 5.7]), we can conclude that there 'exists' a polynomial-time algorithm to decide whether an input graph has linear rank-width at most k. However, this algorithm is based on the existence of Φ_k, and we did not know how to construct Φ_k. Finally, by Theorem 2, we know how to construct Φ_k algorithmically.

[1] In [3], the connectivity function of matroids is defined to have +1, which makes $(6^k-1)/5$.

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Recent Progress on Hadwiger's Conjecture

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(joint work with Michelle Delcourt)

In 1943 Hadwiger [3] made the following famous conjecture.

Conjecture 1 (Hadwiger's Conjecture)**.** *For every integer $t \geqslant 1$, every graph with no K_t minor is $(t-1)$-colorable.*

Hadwiger's Conjecture is widely considered among the most important problems in graph theory and has motivated numerous developments in graph coloring and graph minor theory. For an overview of major progress on Hadwiger's Conjecture, we refer the reader to [8], and to the recent survey by Seymour [9] for further background.

The following is a natural weakening of Hadwiger's Conjecture.

Conjecture 2 (Linear Hadwiger's Conjecture)**.** *There exists a constant $C > 0$ such that for every integer $t \geqslant 1$, every graph with no K_t minor is Ct-colorable.*

For many decades, the best general bound on the number of colors needed to properly color every graph with no K_t minor had been $O(t\sqrt{\log t})$, a result obtained independently by Kostochka [4, 5] and Thomason [10] in the 1980s. These results bound the "degeneracy" of graphs with no K_t minor. Recall that a graph G is *d-degenerate* if every non-empty subgraph of G contains a vertex of degree at most d. A standard inductive argument shows that every d-degenerate graph is $(d+1)$-colorable. Thus the following bound on the degeneracy of graphs with no K_t minor gives a corresponding bound on their chromatic number.

Theorem 3 ([4, 5, 10])**.** *Every graph with no K_t minor is $O(t\sqrt{\log t})$-degenerate.*

It has been shown that there exist graphs with no K_t minor and minimum degree $\Omega(t\sqrt{\log t})$. Thus the bound in Theorem 3 is tight. Until very recently $O(t\sqrt{\log t})$ remained the best general bound for the chromatic number of graphs with no K_t minor when Norin, Song and I [8] improved this with the following theorem.

Theorem 4. *For every $\beta > \frac{1}{4}$, every graph with no K_t minor is $O(t(\log t)^{\beta})$-colorable.*

Delcourt and I [1] made the following further improvement to Theorem 4.

Theorem 5. *Every graph with no K_t minor is $O(t \log\log t)$-colorable.*

Theorem 5 is in fact a corollary of the following more technical main result.

Theorem 6. *There exists an integer $C = C_6 \geqslant 1$ such that the following holds: Let $t \geqslant 3$ be an integer. Let G be a graph and let*

$$f(G,t) := \max_{H \subseteq G} \left\{ \frac{\chi(H)}{a} :\ a \geqslant \frac{t}{\sqrt{\log t}},\ \mathrm{v}(H) \leqslant C a \log^4 a,\ H \text{ is } K_a\text{-minor-free} \right\}.$$

If G has no K_t minor, then

$$\chi(G) \leqslant C \cdot t \cdot (1 + f(G,t)).$$

Theorem 6 has a number of interesting corollaries. As mentioned, a first corollary of Theorem 6 is Theorem 5. This follows straightforwardly by using the best-known bounds on the chromatic number of small K_t-minor-free graphs. A second corollary is that Linear Hadwiger's Conjecture reduces to small graphs as follows.

Corollary 7. *There exists an integer $C = C_7 \geqslant 1$ such that the following holds: If for every integer $t \geqslant 3$ we have that every K_t-minor-free graph H with $\mathrm{v}(H) \leqslant Ct\log^4 t$ satisfies $\chi(H) \leqslant Ct$, then for every integer $t \geqslant 3$ we have that every K_t-minor-free graph G satisfies $\chi(G) \leqslant C^2 t$.*

Proof. Follows from Theorem 6 by setting $C = 2C_6$. □

A third corollary of Theorem 6 shows that Linear Hadwiger's Conjecture holds if the clique number of the graph is small as a function of t.

Corollary 8. *There exists $C = C_8 \geqslant 1$ such that the following holds: Let $t \geqslant 3$ be an integer. If G is a K_t-minor-free graph with $\omega(G) \leqslant \frac{\sqrt{\log t}}{(\log\log t)^2}$, then $\chi(G) \leqslant Ct$.*

In 2003, Kühn and Osthus [6] proved that Hadwiger's Conjecture holds for graphs of girth at least five provided that t is sufficiently large. In 2005, Kühn and Osthus [7] extended this result to the class of $K_{s,s}$-free graphs for any fixed positive integer $s \geqslant 2$. Along this line, we have the following corollary of Theorem 6.

Corollary 9. *Linear Hadwiger's Conjecture holds for the class of K_r-free graphs for every fixed r.*

Note that the constant in Corollary 9 depends on r; adding the assumption that t is sufficiently large with respect to r permits a constant that does not depend on r. In 2017, Dvořák and Kawarabayashi [2] showed that there exist triangle-free graphs of tree-width at most t (and hence K_{t+2}-minor-free) and chromatic number at least $\lceil \frac{t+3}{2} \rceil$. Hence the result in Corollary 9 is tight up to the multiplicative constant.

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Burling graphs revisited

Nicolas Trotignon

(joint work with Pegah Pournajafi)

The *Burling sequence* is a sequence of triangle-free graphs of increasing chromatic number. Any graph which is an induced subgraph of a graph in this sequence is called a Burling graph. These graphs have attracted some attention because they have geometric representations and because they provide counter-examples to several conjectures about bounding the chromatic number in classes of graphs.

The goal of this talk is to provide new definitions of Burling graphs. Three of them are geometrical : they characterize Burling graphs as intersection graphs of various geometrical objects (line segments of the plane, frames of the plane, axis-aligned boxes of $\mathbb{R}^3$). All these representations of Burling graphs were known, mostly from [1] and [2]. Our contribution is to add restrictions to the configurations of the geometrical objects so that there is an equivalence between the intersection graphs and the Burling graphs, see [3].

Among our new equivalent definitions of Burling graphs, one is of a more combinatorial flavour. It says how any Burling graph can be derived from a tree with some specific rules. This definition is convenient decide whether some given graph is Burling or not. We use it to give several generic examples of Burling graphs or rules to find edges whose subdivision preserves being a Burling graph. We also use it to find examples of graphs that are not Burling, see [4]. Among several consequences of all this, one is that graphs that do not contain any subdivision of K_5 as an induced subgraph have unbounded chromatic number, see [5].

It turns out that this talk was already given several time in 2021 to audiences more or less overlapping Oberwolfach's attendees. Therefore, the focus is on open questions that we explain now and that were not advertised previously.

- Could it be that for all Burling graph H, graphs with no subdivisions of H are χ-bounded? This is probably far too much, but disproving it would be interesting.
- Let H be a Burling graph. Is the chromatic number of Burling graphs that do not contain H as an induced subgraph bounded? If yes, it shows that Burling graphs a kind of best possible or "minimal". If not, this would interestingly provide new triangle-free graphs of high chromatic number.
- Describe more precisely Burling graphs. Possibly characterize them by excluding induced subgraphs, but the description will be certainly messy. Or maybe just recognize them in polytime which seem to be possible.
- Is there a polytime algorithm to compute a maximum stable set in an input Burling graph? The question might seem artificial but it would maybe provide an answer to a conjecture or Thomassé, Trotignon and Vušković : if a class of graph is closed under taking induced subgraphs and admits a polytime algorithm to compute a maximum stable set, then it is χ-bounded.
- Can we put triangles back in Burling graphs? The question might seem strange. But it turns out that while triangle graphs of high chromatic number are well studied, almost nothing is known about graphs of high chromatic number that do contain triangles but no K_4. In particular, it is not known whether the chromatic number of such graphs can be unbounded while the chromatic number if its triangle-free induced subgraphs is bounded.

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Short proofs of rainbow matching results

Benny Sudakov
(joint work with David Munhá Correia and Alexey Pokrovskiy)

Research regarding rainbow matchings in graphs dates back to the work of Euler on various problems about transversals in Latin squares. A Latin square of order n is an $n \times n$ array filled with n different symbols, where no symbol appears in the same row or column more than once. A transversal in a Latin square of order n is a set of m entries such that no two entries are in the same row, same column, or have the same symbol. A transversal is said to be *full* if $m = n$ and *partial* otherwise. Despite the fact that not every Latin square contains a full transversal, it is plausible to ask whether every Latin square contains a large partial transversal. Indeed, the celebrated conjecture of Ryser, Brualdi and Stein [10, 5, 11] states that every Latin square contains a transversal which uses all but at most one symbol.

Conjecture 1. *Every Latin square of order n contains a transversal of size $n-1$.*

There is a bijective correspondence between Latin squares of order n and proper edge-colourings of the complete bipartite graph $K_{n,n}$ with n colours. Indeed, let a Latin square S have $\{1, 2, \ldots, n\}$ as its set of symbols and let $S_{i,j}$ denote the symbol at the entry (i, j). To S we associate an edge-colouring of $K_{n,n}$ with the colours $\{1, 2, \ldots, n\}$ by setting $V(K_{n,n}) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and letting the edge between x_i and y_j receive colour $S_{i,j}$. Note that this colouring is proper, and moreover, each colour consists of a matching of size n. It is now easy to see that transversals of size m in S correspond to rainbow matchings of size m in the coloured $K_{n,n}$. Therefore, the Ryser-Brualdi-Stein conjecture states - every properly edge-colouring of $K_{n,n}$ with n colours has a rainbow matching of size $n-1$.

The Ryser-Brualdi-Stein conjecture is just one thread of the research on rainbow matchings and rainbow subgraphs more broadly. There are many other interesting conjectures, some of them motivated by strengthening Ryser-Brualdi-Stein, others motivated by other branches of mathematics. As an example, consider the following conjecture of Aharoni-Berger [1].

Conjecture 2. *Let G be a properly edge-coloured bipartite multigraph with n colours having at least $n+1$ edges of each colour. Then G has a rainbow matching using every colour.*

The motivation for this conjecture is the Ryser-Brualdi-Stein conjecture which it strengthens (to see this, consider a properly coloured $K_{n,n}$ as in Conjecture 2; delete one colour to obtain a graph satisfying the Aharoni-Berger conjecture). Given the difficulty of the Ryser-Brualdi-Stein conjecture, much of the effort has been put into proving asymptotic versions of Conjecture 2. There are two natural approaches one can take in proving weakenings of this conjecture, which we will refer to as a *weak asymptotic* and a *strong asymptotic*. The weak asymptotic asks for rainbow matchings which uses nearly all colours.

Weak asymptotic: *Let G be a properly edge-coloured bipartite multigraph with n colours having at least $n+1$ edges of each colour. Then G has a rainbow matching of size $n - o(n)$.*
A weak asymptotic version of the Aharoni-Berger conjecture was proved by Barat-Gyárfás-Sarkozy [4] who prove the above with error term $o(n) = \sqrt{n}$. Their proof was very short and elegant, using the method of alternating paths.

Another direction is to prove *qualitatively stronger* asymptotic results. For us "strong asymptotic" will mean a result, which guarantees matchings using all the colours in the graph, at the cost of having slightly more edges of each colour.

Strong asymptotic: *Let G be a properly edge-coloured bipartite multigraph with n colours of size at least $n + o(n)$ each. Then G has a rainbow matching using every colour.*
The reason we call the above statement as a "strong" asymptotic, is that it implies the previously mentioned weak asymptotic. Indeed suppose we have a properly edge-coloured bipartite multigraph G with n colours having at least $n+1$ edges of each colour. Delete $o(n)$ colours in order to obtain a new graph G' with $n' = n - o(n)$ colours and each colour having $n' + o(n) + 1$ edges. The strong asymptotic applies to this to give a rainbow matching using every colour. This gives a rainbow matching of size $n' = n - o(n)$ in the original graph. Moreover, note that we can choose which $o(n)$ colours we want to miss. This simple argument shows that the "strong asymptotic with error term $o(n)$ implies the weak asymptotic with error term $o(n)$.

It was believed that the strong asymptotic is fundamentally more difficult than the weak one. Indeed, it took much longer for the strong asymptotic to be proved, and the proof methods involved were considerably more difficult. It is easy to see that if there are $2n$ edges of each colour has a rainbow matching of size n. Indeed, if the largest matching M in such a graph had size $\leqslant n-1$, then one of the $2n$ edges of the unused colour would be disjoint from M, and we could get a larger matching by adding it. This simple bound has been successively improved by many authors. Aharoni, Charbit, and Howard [2] proved first that matchings of size $\lfloor 7n/4 \rfloor$ are sufficient to guarantee a rainbow matching of size n. Kotlar and Ziv [7] improved this to $\lfloor 5n/3 \rfloor$. The third author then proved that $\phi n + o(n)$ is sufficient, where $\phi \approx 1.618$ is the Golden Ratio [8]. Clemens and Ehrenmüller [6] showed that $3n/2 + o(n)$ is sufficient. Aharoni, Kotlar, and Ziv [3] showed that having $3n/2 + 1$ edges of each colour in an n-edge-coloured bipartite multigraph guarantees a rainbow matching of size n. Finally, the strong asymptotic, as stated above, was proved by the third author in [9]. This proof was much longer and more difficult than Barat-Gyárfás-Sarkozy's proof of the weak asymptotic. It also gave a considerably weaker error term.

Now, we've already seen that "if the strong asymptotic is true, then the weak asymptotic is true". The main idea of this paper is a very short trick, that we call "the sampling trick", which allows one to prove the converse statement. This trick will allow us to prove results like "suppose the weak asymptotic is true with $o(n) = n/f(n)$; then the strong asymptotic is true with $o(n) = 3n/\sqrt{f(n)}$". Combining this with the Barat-Gyárfás-Sarkozy result, we obtain the strong asymptotic version of the Aharoni-Berger conjecture with a much improved error term.

Theorem 3. *Let G be a properly edge-coloured bipartite multigraph with n colours having at least $n+n^{3/4}$ edges of each colour. Then G has a rainbow matching using every colour.*

Our approach, in addition to giving a polynomial error term, vastly simplifies the original 40 page proof (a full proof will now take less than 2 pages). The "sampling trick" is very versatile and applies to many other problems and conjectures. In all our applications, it allows us to either prove a strong asymptotic for the first time, or to greatly simplify an existing proof of the strong asymptotic.

- We give the first asymptotic proof of the "non-bipartite" Aharoni-Berger conjecture, solving two conjectures of Aharoni, Berger, Chudnovsky and Zerbib.
- We give a very short asymptotic proof of Grinblat's conjecture (first obtained by Clemens, Ehrenmüller, and Pokrovskiy). Furthermore, we obtain a new asymptotically tight bound for Grinblat's problem as a function of edge multiplicity of the corresponding multigraph.
- We give the first asymptotic proof of a 30 year old conjecture of Alspach.

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Common graphs with arbitrary chromatic number

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(joint work with Dan Král' and Fan Wei)

Ramsey's Theorem [24], one of the most well-known results in combinatorics, asserts that for every graph H there exists a number N such that any 2-edge-coloring of the complete graph with N vertices contains a monochromatic copy of H. Determining the smallest such N, which is known as the Ramsey number $r(H)$ of a graph H, is a famous open problem even in the case H is a complete graph despite a recent progress on upper bounds [4, 25]. In fact, Erdős offered \$100 for determining whether the limit $r(K_n)^{1/n}$ exists, and another \$250 for computing its value. In our work, we are concerned with a more general problem of how many monochromatic copies of a graph H necessarily exist in any 2-edge-coloring of the n-vertex complete graph.

Goodman's Theorem [16] states that the number of monochromatic copies of the triangle K_3 is asymptotically minimized by the random 2-edge-coloring of a complete graph. We say that a graph H is *common* if the number of monochromatic copies of H is asymptotically minimized by the random 2-edge-coloring of a complete graph. In particular, K_3 is common and more generally every cycle is common [27]. In 1962, Erdős [11] conjectured that every complete graph is common, and later Burr and Rosta [2] conjectured that every graph is common. Both of these conjectures turned out to be false: in the late 1980s, Sidorenko [26, 27] showed that a triangle with a pendant edge is not common, and Thomason [30] showed that K_4 is not common. More generally, any graph containing K_4 is not common [20] (and thus almost every graph is not common), and there are graphs H and 2-edge-colorings of complete graphs with the number of monochromatic copies of H being sublinear in the number of monochromatic copies of H in the random 2-edge-coloring [3, 13].

A characterization of the class of common graphs is an intriguing open problem and there is even no conjectured description of the class. This is also very closely related to the famous conjecture of Sidorenko [29] and of Erdős and Simonovits [12], which asserts every bipartite graph has the Sidorenko property. Since every graph with the Sidorenko property is common, the conjecture, if true, would imply that all bipartite are common. Families of bipartite graphs proven to have the Sidorenko property [1, 27, 28, 5, 8, 7, 9] provide examples of bipartite graphs that are common. Common graphs that are not bipartite, i.e., their chromatic number is larger than two, are scarce. In particular, Jagger, Štovíček and Thomason asked whether there exists a common graph with chromatic number at least four. While odd cycles and even wheels [20, 27] are examples of 3-chromatic common graphs, also see [17], the existence of a common graph with chromatic number at least four was open until about 10 years ago when the 5-wheel was shown to be common [19]. The question whether there exist common graphs with arbitrary large chromatic number has been reiterated in [19], and Conlon, Fox and Sudakov list the following problem in the survey paper "Recent developments in graph Ramsey theory" [6, Problem 2.28]:

Problem. *Do there exist common graphs of all chromatic numbers?*

Our main result is the positive answer to this problem.

Theorem 1. *For every $\ell \in \mathbb{N}$, there exists a connected common graph with chromatic number ℓ.*

We treat common graphs using methods from the theory of graph limits, which allows us applying spectral tools from the operator theory. In order to prove Theorem 1, we establish the following stronger statement: every graph of sufficiently large girth can be embedded in a graph H such that any 2-edge-coloring of a complete graph has asymptotically at least as many monochromatic copies of H as the random 2-edge-coloring.

Our proof of Theorem 1 is split into two cases based on whether the considered 2-edge-coloring is close to the random coloring or not; we refer to the two cases as the local regime and the non-local regime. The core of the proof is formed by the arguments related to the local regime, which is described by the existence of a dominant eigenvalue of the operator associated with the 2-edge-coloring. While both the Sidorenko property and common graphs have been studied in the local regime [22, 15, 14, 10, 18] before, the proof of Theorem 1 required developing new spectral based techniques to control (in)dependence of monochromatic embeddings of different parts of H in the host 2-colored complete graph.

Our techniques extend to the setting of so-called k-common graphs introduced in [20]: a graph H is *k-common* if the random k-edge-coloring of a complete graph asymptotically minimizes the number of monochromatic copies of H among all k-edge-colorings. This notion provides another link to the Sidorenko property: a graph H has the Sidorenko property if and only if H is k-common for all k [21]. If H is k-common, then it is k'-common for all $k' \leqslant k$, and thus k-common graphs for $k \geqslant 3$ are even more rare than common graphs. In fact, the question of Jagger, Štovíček and Thomason [20] about the existence of a non-bipartite k-common graph for any given $k \geqslant 3$ has been resolved only recently in [21]. Using the techniques developed to prove Theorem 1, we can also prove the following.

Theorem 2. *For every $k \geqslant 2$ and $\ell \in \mathbb{N}$, there exists a connected k-common graph with chromatic number ℓ.*

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Rainbow clique subdivisions and blow-ups

Liana Yepremyan

This talk was based on two recent papers [6, 5]. In [6] we show that for any integer $m \geqslant 2$, every properly edge-coloured graph on n vertices with more than $n^{1+o(1)}$ edges contains a rainbow subdivision of K_m. This is a rainbow analogue of some classical results on clique subdivisions and extends some results on rainbow Turán numbers. Our method relies on the framework introduced by Sudakov and Tomon [16] which we adapt to find robust expanders in the coloured setting.

Consequently we improved this bound in [5]. We show that for every integer $m \geqslant 2$ and large n, every properly edge-coloured graph on n vertices with at least $n(\log n)^{60}$ edges contains a rainbow subdivision of K_m. Using the same framework, in [5] we also prove a result about the extremal number of r-blow-ups of subdivisions. We show that for integers $r, m \geqslant 2$ and large n, every graph on n vertices with at least $n^{2-\frac{1}{r}}(\log n)^{\frac{60}{r}}$ edges has an r-blow-up of a subdivision of K_m. These results are sharp up to a polylogarithmic factor. Our proofs in the second paper use the connection between mixing time of random walks and expansion in graphs.

1. Introduction

The *Turán number* of a graph H, denoted $\mathrm{ex}(n, H)$, is the maximum possible number of edges in an n-vertex graph that does not contain a copy of H. A *proper edge-colouring* of a graph is an assignment of colours to its edges so that edges that share a vertex have distinct colours. A *rainbow* subgraph of an edge-coloured graph is a subgraph whose edges have distinct colours. The *rainbow Turán number* of a graph H, denoted $\mathrm{ex}^*(n, H)$, is the maximum possible number of edges in a properly edge-coloured graph on n vertices with no rainbow copy of H. This notion was introduced by Keevash, Mubayi, Sudakov and Verstraëte [8] as a rainbow variant of Turán numbers. One can define $\mathrm{ex}(n, \mathcal{H})$ and $\mathrm{ex}^*(n, \mathcal{H})$ analogously for a family of graphs $\mathcal{H}$.

It was shown in [8] that $\mathrm{ex}^*(n, H) = (1 + o(1))\,\mathrm{ex}(n, H)$ for non-bipartite H. Just like in the regular Turán problems far less is known about rainbow Turán numbers of bipartite graphs. The authors of [8] raised two problems concerning rainbow Turán numbers of even cycles, one concerning an even cycle of fixed length $2k$ and the other concerning the family $\mathcal{C}$ of all cycles. For all $k \geqslant 2$, they showed that $\mathrm{ex}^*(n, C_{2k}) = \Omega(n^{1+1/k})$ and conjectured that $\mathrm{ex}^*(n, C_{2k}) = \Theta(n^{1+1/k})$. The authors of [8] verified the conjecture for $k \in \{2,3\}$. Following further progress on the conjecture by Das, Lee and Sudakov [2], Janzer [4] recently resolved the conjecture.

Regarding the rainbow Turán number of the family $\mathcal{C}$ of all cycles, Keevash, Mubayi, Sudakov and Verstraëte [8] showed that $\mathrm{ex}^*(n,\mathcal{C}) = \Omega(n\log n)$, by considering a naturally defined proper edge-colouring of the hypercube Q_k, where $k = \lfloor \log n \rfloor$. The vertices of Q_k are binary vectors of length k and for any two vectors which differ exactly at ith coordinate there is an edge of colour i. It is not hard to see that in such a colouring of Q_k there is no rainbow cycle. They also showed that $\mathrm{ex}^*(n,\mathcal{C}) = O(n^{4/3})$ and asked if $\mathrm{ex}^*(n,\mathcal{C}) = O(n^{1+o(1)})$ and furthermore, if $\mathrm{ex}^*(n,\mathcal{C}) = O(n\log n)$. Das, Lee and Sudakov [2] answered the first question affirmatively, by showing that $\mathrm{ex}^*(n,\mathcal{C}) \leqslant ne^{(\log n)^{\frac{1}{2}+o(1)}}$. Recently, Janzer [4] improved this bound by establishing that $\mathrm{ex}^*(n,\mathcal{C}) = O\big(n(\log n)^4\big)$, which is tight up to a polylogarithmic factor. Together with Jiang and Methuku [6] we proved the following generalisation of Das, Lee and Sudakov [2] on $\mathrm{ex}^*(n,\mathcal{C})$.

Theorem 1 (Jiang, Methuku, Yepremyan [6])**.** *For every integer $m \geqslant 2$ there exists a constant $c > 0$ such that for every integer $n \geqslant m$ the following holds. If G is a properly edge-coloured graph on n vertices with at least $ne^{c\sqrt{\log n}}$ edges, then G contains a rainbow subdivision of K_m, where each edge is subdivided at most $1300\log^2 n$ times.*

The method used in [6] utilises robust expanders in the coloured setting together with a density increment argument, inspired in part by the method introduced by Sudakov and Tomon [16].

Later, in [5] we lowered the $e^{O(\sqrt{\log n})}$ error term in Theorem 1 to a polylogarithmic term, which in conjunction with the above-mentioned $\Omega(n\log n)$ lower bound on $\mathrm{ex}^*(n,\mathcal{C})$ determines the rainbow Turán number of the family of K_m-subdivisions up to a polylogarithmic factor.

Theorem 2. *Fix an integer $m \geqslant 2$ and let n be sufficiently large. Suppose that G is a properly edge-coloured graph on n vertices with at least $n(\log n)^{60}$ edges. Then G contains a rainbow subdivision of K_m, where each edge is subdivided at most $(\log n)^6$ times.*

Theorem 2 provides the rainbow analogue of a fundamental (and highly influential) result of Mader [13] stating that for every integer $m \geqslant 2$, there exists $d = d(m)$ such that every graph with average degree at least d contains a subdivision of K_m. Research on this problem has a long history, see e.g., Mader [14], Komlós and Szemerédi [9, 10], and Bollobás and Thomason [1].

Our proof of Theorem 2 exploits the connection between mixing time of random walks and edge expansion. This connection is used in conjunction with counting lemmas developed by Janzer in [4] regarding homomorphisms of cycles in graphs. We also prove a strengthening of Theorem 2, regarding 'rooted' rainbow subdivisions of K_m in expanders. For this stronger version, in addition to the ingredients used for proving Theorem 2, we use the framework of [6] and an additional idea used by Letzter in [11].

We also obtain a generalisation of the following result of Janzer [4] concerning the Turán number of blow-ups of cycles. For an integer $r \geqslant 1$ and a graph F, the *r-blow-up* of F, denoted $F[r]$, is the graph obtained by replacing each vertex of F with an independent set of size r and each edge of F by a $K_{r,r}$. Let $\mathcal{C}_0[r] := \{C_{2k}[r] : k \geqslant 2\}$. Answering a question of Jiang and Newman [7], Janzer [4] proved the following.

Theorem 3 (Janzer [4])**.** *Let $r \geqslant 1$ be a fixed integer. Then* $\mathrm{ex}(n, \mathcal{C}_0[r]) = O\big(n^{2-1/r}(\log n)^{7/r}\big)$.

This bound is tight up to a polylogarithmic factor as random graphs show that $\mathrm{ex}(n, \mathcal{C}_0[r]) = \Omega(n^{2-1/r})$. We generalise Theorem 3 by proving the following.

Theorem 4. *Let $r, m \geqslant 1$ be fixed integers and let n be sufficiently large. Suppose that G is a graph on n vertices with at least $n^{2-\frac{1}{r}}(\log n)^{\frac{60}{r}}$ edges. Then G contains an r-blow-up of a subdivision of K_m, where each edge is subdivided at most* $(\log n)^6$ *times.*

This bound is also tight up to a polylogarithmic factor as shown by random graphs. The proof of Theorem 4 follows the proof of our first main result, Theorem 2, with the additional use of a 'balanced supersaturation' result, due to Morris and Saxton [15]. Such a result gives us a collection of $K_{r,r}$'s in a sufficiently dense graph such that no copy of $K_{1,r}$ is contained in too many copies of $K_{r,r}$ (the result in [15] is more general but this condition suffices for our purposes). This sort of result is usually used in conjunction with the container method in order to upper bound the number of H-free graphs. So it is quite interesting that we use this 'balanced supersaturation' result for $K_{r,r}$'s in a new setting.

2. Main ideas

Our method in [6] builds on the method used by Sudakov and Tomon in [16] together with some new ideas. We incorporate the minimality notion commonly used in the study of graph minors and adapt the notion of "expander" conveniently to our setting. For a graph G, we denote by $d(G)$ the average degree of G.

Definition 5. *A graph G is said to be d-*minimal *if $d(G) \geqslant d$ but $d(H) < d$ for every proper subgraph $H \subseteq G$. Given $d \geqslant 1$, $\eta \in (0,1)$ and $\varepsilon \in (0, \frac{1}{2}]$, an n-vertex graph G is called a (d, η, ε)-*expander *if G is d-minimal, and for every subset $S \subseteq V(G)$ of size at most $(1-\varepsilon)n$, we have $d(S) \leqslant (1-\eta)d$.*

We show that most of the edges of a sufficiently dense graph can be covered by edge-disjoint expanders. In a properly edge-coloured expander, from any given vertex v, we can reach almost all of the other vertices by a rainbow path of poly-logarithmic length avoiding a given set of vertices and colours. Additionally, the notion of minimality ensures that the set of these "reachable" vertices induces most of the edges in the expander. This additional feature of the expander is used to show the existence of a common large intersection of reachable vertices for a pair of vertices x and y in a general graph (not necessarily an expander). Eventually we are either able to join any pair of vertices by a rainbow path of poly-logarithmic length avoiding a bounded set of colours and vertices (thus allowing us to build a copy of the desired rainbow K_t-subdivision) or find a much denser subgraph. We then complete the proof via a density increment argument as in [16].

In the subsequent paper [5], the main novelty is the use of the connection between the mixing time of random walks and our notion of expansion. It is a well-known and very useful fact that so-called 'large conductance' implies 'small' mixing time (see, e.g., Lovász [12]). Moreover, our notion of expansion implies that our expanders have large conductance. Using these facts we show that if additionally expanders are almost regular then long enough walks are close to being uniformly distributed. We also use two counting lemmas of Janzer from [4]. In a properly edge-coloured graph, say that a closed walk is *degenerate* if it is either not rainbow or visits a vertex more than once. The first lemma from [4] implies that in a properly edge-coloured graph which is close to being regular, the number of degenerate closed $2k$-walks is significantly smaller than the number of closed $2k$-walks, provided that k is sufficiently large. Given two vertices x and y, a closed $2k$-walk W is said to *be hosted* by x and y if it starts at x and reaches y after k steps. We call a pair of vertices (x, y) *good* if the number of degenerate closed $2k$-walks hosted by x and y is significantly smaller than the number of closed $2k$-walks hosted by x and y. The second lemma from [4] that we use shows that if a pair (x, y) is good then there are many short pairwise colour-disjoint and internally vertex-disjoint k-paths from x to y.

Using results about random walks on graphs, which relate mixing time to expansion, we show that in an expander G on n vertices which is close to being regular, for k suitably large (at least polylogarithmic in n), the numbers of closed $2k$-walks hosted by any two pairs of vertices are within a suitable polylogarithmic factor (in n) of each other. This, combined with the fact that the number of degenerate closed $2k$-walks is small compared to the total number of closed $2k$-walks (due to the first lemma above), implies that almost all pairs of vertices are good. Thus, using Turán's theorem, we find a copy of K_m in the graph formed by good pairs. This, together with the fact that there are many short colour-disjoint and internally vertex-disjoint rainbow paths between any good pair of vertices (due to the second lemma above) allows us to greedily build the desired rainbow-subdivision of K_m.

We also prove a stronger version of Theorem 2 asserting that in an expander G which is close to being regular and whose average degree is large enough, for any set S of m vertices, there exists a rainbow K_m-subdivision with the vertices of S being the branching vertices. The main step in this proof shows that for any two vertices x and y in G there is a short rainbow x, y-path avoiding a prescribed small set C of vertices and colours. By iterating this over all pairs of vertices in S, we can build the desired rainbow K_m-subdivision.

To show that there is a short rainbow x, y-path in G, we first apply tools due to Jiang, Methuku and Yepremyan [6] and Letzter [11] to show that there is a set of vertices U of size $\Omega(n)$ such that for each $v \in U$ there is such a short rainbow x, v-path $P(v)$ and a short rainbow y, v-path $Q(v)$, both of which avoid C, such that no colour is used on too many of these paths $P(v)$ and $Q(v)$. It easily follows that for almost all pairs (u, v) with $u, v \in U$, the paths $P(u)$ and $Q(v)$ are colour-disjoint. This, combined with the fact that most pairs in U are good (in the sense mentioned earlier), implies that there exists at least one good pair (u, v) for which $P(u)$ and $Q(v)$ are colour-disjoint. This allows us to find a suitable short rainbow u, v-path L such that $P(u) \cup L \cup Q(v)$ is a rainbow x, y-walk which contains the desired rainbow x, y-path.

To prove Theorem 4 about r-blow-ups of subdivisions of cliques, we use similar arguments as the ones for proving Theorem 2, albeit tailored to the setup of r-blow-ups. Recall that we are given a graph G on n vertices with at least $n^{2-\frac{1}{r}}(\log n)^{\frac{60}{r}}$ edges. A supersaturation result due to Erdős and Simonovits [3] implies that G has many copies of $K_{r,r}$. Our approach is to take a large collection $\mathbb{F}$ of copies of $K_{r,r}$ in G and consider an auxiliary graph $\mathcal{H}$ whose vertices are r-sets of vertices in G, and whose edges correspond to copies of $K_{r,r}$ in $\mathbb{F}$. To find a blow-up of a K_m-subdivision in G, it would suffice to find a 'clean' subdivision of K_m in $\mathcal{H}$, denoted $\mathcal{K}$, where the r-sets in G corresponding to the vertices of $\mathcal{K}$ are pairwise disjoint. However, in order for our framework to be applicable, we need a crucial additional property of $\mathbb{F}$ that any r-set of vertices A in G and vertex $u \notin A$ do not lie in too many $K_{r,r}$-copies in this collection $\mathbb{F}$. Fortunately, the existence of such a collection $\mathbb{F}$ is guaranteed by a 'balanced supersaturation' result due to Morris and Saxton [15].

3. Open questions

To conclude, by our results it follows that there is a constant $c \leqslant 60$ such that any n-vertex properly edge-coloured graph G with at least $n(\log n)^c$ edges contains a rainbow subdivision of K_m. On the other hand, an immediate lower bound is given by the best known lower bound from [8] on $\mathrm{ex}^*(n, \mathcal{C})$, which is $\Omega(n \log n)$. This shows that our bound is tight up to a polylogarithmic factor. We pose the following question.

Question 6. *Fix $m \geqslant 2$. What is the smallest c such that for all sufficiently large n the following holds: if G is a properly edge-coloured graph on n vertices with at least $\Omega(n(\log n)^c)$ edges, then it contains a rainbow subdivision of K_m? In particular, is $c = 1$?*

From our results it also follows that any n-vertex graph with at least $n^{2-\frac{1}{r}}(\log n)^{\frac{60}{r}}$ edges contains an r-blow-up of a subdivision of K_m. This bound is tight up to $(\log n)^{\frac{60}{r}}$ factor, due to the following proposition, which was mentioned in [4] as a remark. We pose the following question, which strengthens Question 6.2 in [4].

Question 7. *Fix $m \geqslant 2$. Is it true that if G is an n-vertex graph with at least $\Omega(n^{2-\frac{1}{r}})$ edges then it contains an r-blow-up of a subdivision of K_m?*

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Problem Session

Louis Esperet (chair)

1. Well-linked sets in directed graphs (Ken-ichi Kawarabayashi)

Two sets S and T with $|S| = |T|$ are (S,T)-*well-linked* in a directed graph if for any $A \subseteq S$ and $B \subseteq T$ with $|A| = |B|$, there are $|A| = |B|$ disjoint paths (linkage) from A to B. Note: S, T may not be disjoint. Its order is $|S| + |T|$

Motivation:

- Sparsest directed cut (for general graphs, no good approximation, but for planar graphs, $O(\log^3 n)$ approx. by Sidiropoulos and KK in FOCS'21).

  More precisely, directed graph decomposition based on sparsest cuts. (For undirected graphs, this leads to a well-linked set decomposition, and this method leads to the polynomial grid theorem).
- Disjoint paths problem (in planar graphs) $S = \{s_1, s_2, s_3 \dots, s_k\}$, $T = \{t_1, t_2, t_3, \dots, t_k\}$. See below.

- A generalization of "well-linked" set (this (S,T)-well linked set can be defined even for DAG).

  Indeed, we are interested in DAG when there is no such a (S,T)-well linked set for any S,T. Maybe in some cases, some algorithmic questions can be faster?

Some problems:

- Okamura-Seymour for directed planar graphs:

  More precisely, for a directed planar graph G with the outer boundary C, if all vertices in $S \cup T$ are in C, and S,T satisfy a $(S.T)$-well-linked set, there are paths P_i with source node s_i and terminal node t_i, for $i = 1,\dots,k$, such that each vertex in G is used in at most two (or any constant number) of the paths.
- Polynomial acyclic grid theorem:

  If (S,T) is well-linked of order $f(k)$, there is an acyclic grid W of order k (i.e., W consists of two linkages X,Y of order k, such that X is from top to bottom, and Y is from left to right), as a minor.

  Moreover f is a polynomial function of k

  We are actually interested in a more relaxed form: G contains either W or biclique of order k as a minor. We are even interested in the case when G is a DAG (or G has no k disjoint cycles).

  If this kind of a form is true, we have a very good chance to show the polynomial bound for Erdős-Posa for directed disjoint cycles (i.e., a polynomial version of Younger's conjecture)

2. Linear rank-width of graphs excluding some tree as a vertex-minor (O-joung Kwon)

For a linear ordering $L = v_1, v_2, \dots, v_n$ of vertices of a graph G, the *width* of L is defined as the maximum rank of the matrices $A(G)[\{v_1 \dots v_i\}, \{v_{i+1}, \dots v_n\}]$, where $A(G)$ denotes the adjacency matrix, and the rank is computed over the binary field. The *linear rank-width of* G is the minimum width over all linear orderings of G.

Local complementation at a vertex v is the operation that replaces the subgraph induced by $N(v)$ with its complement. H is a *vertex-minor* of a graph G if H can be obtained from G by local complementations and vertex deletions.

Problem: For every tree T, does the class of graphs having no T vertex-minor have bounded linear rank-width?

Related papers:

- The grid theorem for vertex-minors JCTB accepted (Geelen, Kwon, McCarty, and Wollan)
- Obstructions to bounded rank-depth and shrub-depth JCTB 2021 (Kwon, McCarty, Oum, and Wollan)
- Tree pivot-minors and linear rank-width SIDMA 2021 (Dabrowski, Dross, Jeong, Kanté, Kwon, Oum, and Paulusma)

3. Twin-width of graphs (Sang-il Oum)

Question. What is the maximum twin-width of an n-vertex graphs?

It was proved in [Ahn, Kevin Hendrey, Donggyu Kim, Sang-il Oum, Bounds for the Twin-width of Graphs, arXiv:2110.03957] that Payley graphs have twin-width equal to $\frac{n-1}{2}$. The paper also contains an upper bound for general graphs (of order $\frac{n}{2} + O(\sqrt{n \log n})$).

4. Treewidth of hereditary classes (Nicolas Trotignon)

The following conjecture was made by several people:

Conjecture: For every integer ℓ, there exists $C_\ell > 0$ such that if a graph G contains none of the following as an induced subgraph:

- subdivision of an $\ell \times \ell$ wall
- line graph of a subdivision of an $\ell \times \ell$ wall
- $K_{\ell,\ell}$
- K_ℓ

then, treewidth$(G) \leqslant C_\ell \log |V(G)|$.

Remarks:

- It would be a nice "induced subgraph" version of the celebrated Robertson and Seymour grid theorem, and maybe too much to believe. So, particular cases would be interesting. Also, trying to disprove it would be interesting.
- A weaker statement is proved by Tara Abrishami, Maria Chudnovsky, Sepehr Hajebi, and Sophie Spirkl in *Induced subgraphs and tree-decompositions III. Three-path-configurations and logarithmic tree-width*, available in arxiv 2109.01310.
- The logarithm in the conclusion is needed, as shown by a construction of Ni Luh Dewi Sintiari and Nicolas Trotignon described in *(Theta, triangle)-free and (even hole, K4)-free graphs. Part 1 : Layered wheels*, available in arxiv 1906.10998.

Variant proposed by S. Thomassé: under the same conditions, but with the $t \times t$-wall replaced by any fixed cubic graph H, the graph G has bounded twin-width.

Counterexample to the conjecture by James Davies

We present a simple construction of a graph with large girth, tree-width polynomial in the number of vertices that contains no induced subdivision of a 5×5 wall and no induced line graph of a subdivision of a 5×5 wall. This disproves the conjectures and adds to the list of graphs that would appear in a possible induced version of the grid theorem. Other known graphs that must appear in such a list include layered wheels as constructed by Ni Luh Dewi Sintiari and Nicolas Trotignon.

For each $j \in [n]$ let P_j be a path contain disjoint subpaths $P_{1,j}, \ldots, P_{n,j}$ in order. Let G be a graph consisting of disjoint anti-complete paths $P_1, \ldots, P_n$ and vertices $v_1, \ldots, v_n$ such that for each $i \in [n]$, the neighbourhood of v_i is contained in $V(P_{i,1}) \cup \cdots \cup V(P_{i,n})$ and for each $j \in [n]$, at least one neighbour of v_i is contained in $V(P_{i,j})$.

The graph G has tree-width at least n since it contains K_n as a minor by contracting the connected induced subgraphs $G[V(P_i) \cup \{v_i\}]$ down into individual vertices. By appropriately choosing the neighbours of each v_i, the graph G can be chosen to have large girth. If we bound the length of the paths $P_1, \ldots P_n$, then G will have at most $O(n^2)$ vertices. In particular this means that the tree-width of G will be $\Omega(\sqrt{|V(G)|})$.

A key property of the construction is that for $i \in \{2, \ldots, n-1\}$, the closed neighbourhood $N[v_i]$ of v_i forms a cutset of G. Using this feature it can be carefully shown that G does not contain an induced subdivision of a 5×5 wall or a line graph of a subdivision of a 5×5 wall. Maria Chudnovsky and Nicolas Trotignon further observed that for appropriate choice of G (choosing each v_i to have exactly one neighbour in each $P_{i,j}$), the graph G contains no induced wheel.

5. Hitting all maximum independent sets (Noga Alon)

For a graph $G = (V, E)$ on n vertices let $\alpha(G)$ denote its independence number, and let $h(G)$ denote the minimum cardinality of a set S of vertices that intersects all maximum independent sets of G (that is, $\alpha(G - S) < \alpha(G)$).

Conjecture (Bollobás, Erdős and Tuza, 1991): If $\alpha(G) = \Omega(n)$ then $h(G) = o(n)$.

A relaxed conjecture: If $\chi(G) = O(1)$ then $h(G) = o(n)$. (Open even for $\chi(G) = 3$.)

Remarks:

- Hajnal (1965): If $\alpha(G) > n/2$ then $h(G) = 1$.
- There are graphs $G = G_n$ on n vertices with $\alpha(G) > n/4$, $\chi(G) \leqslant 8$ and $h(G) > \sqrt{n}/2$, and there are graphs $G = G_n$ on n vertices with $\alpha(G) = (1/2 - o(1))n$ and $h(G) > (\log n)^{0.999}$. This settles questions of Friedgut, Kalai and Kindler, and of Dong and Wu.
- If G is regular and $\alpha(G) > 0.250001n$ then $h(G) \leqslant O(\sqrt{n \log n})$. In particular this holds for regular 3-colorable graphs.

6. Beyond Hadwiger in F-free graphs (Matija Bucić)

Question (B., Fox and Sudakov). For which graphs F does the following hold: G does not contain F as a subgraph $\implies$ $\exists$ a clique minor of size $(\chi(G))^{1+c}$ for $c = c(F) > 0$?

- Kuhn-Osthus, 2005: true if F is bipartite and ask the question for $F = K_s$.
- Dvořák and Kawarabayashi, 2017: not true if F contains a triangle.
- Delcourt and Postle 2021: showed there is a linear sized clique minor $\forall F$
- B., Fox and Sudakov 2021: true $\forall F$ with Hall ratio in place of χ.

7. Grids and connectivity in digraphs (Stephan Kreutzer)

Societies. A *society in a digraph* is pair (G, Ω) where G is a digraph and Ω is a cyclic ordering of some set $\Omega(G) \subseteq V(G)$.

A *cross* in (G, Ω) is a pair P_1, P_2 of disjoint directed paths such that the endpoints of P_i are s_i, t_i and s_1, s_2, t_1, t_2 occur in Ω in this order.

Question. If (G, Ω) is a cross-free society, then is it true that there always is a planar digraph H with $\Omega(G) \subseteq V(H)$ such that H has an embedding into the plane with the vertices of $\Omega(H)$ appearing in the outer face in the order specified by Ω such that H allows for exactly the same connectivity between vertices of $\Omega(G)$ as G?

What should be true. We can replace $G \setminus \Omega(G)$ by a suitable grid to get at least the same connectivity as in G but we may get more.

8. Compressibility of acyclic digraphs (Andrzej Grzesik)

For an acyclic oriented graph F define the *compressibility* of F as the smallest integer k such that F is homomorphic to any tournament on k vertices.

One can observe that the compressibility of F determines the asymptotic answer to the Turán problem asking for the maximum number of edges in a graph not containing F as a subgraph, because an oriented equivalent of Erdős-Stone theorem holds with the compressibility instead of the chromatic number.

Question: For what graphs F the compressibility of F is linear/polynomial in the length of the longest directed path in F?

It is known to be linear for powers of paths and orientations of trees and cycles, polynomial for 2-outdegenerated graphs, and exponential for transitive tournaments.

9. Does Erdős-Posá hold for vertex minors? (Paul Wollan)

Conjecture: For every circle graph H, $\exists f$ such that $\forall k$ and G, either

- G has kH as a vertex-minor, or
- $\exists$ a *rank $f(k)$ pertubation* G^* of G such that G^* has no H vertex minor.

Remarks:

- The role of circle graphs in vertex minor structure is the analog of the role of planar graphs in graph minor structure
- Grid theorem for vertex minors says it suffices to prove that in a graph of bounded rank width, either we have
  - kH vertex minor, or
  - a bounded rank perturbation of G has no H vertex minor

During the Oberwolfach meeting Xiaoyu He and Yuval Wigderson noted that the proof of Theorem 1.4 in their paper `https://arxiv.org/abs/2105.02383` with Jacob Fox can be adapted to this setting in order to obtain that compressibility is quasipolynomial in the length of the longest path for bounded degree graphs.

10. Coloring of planar graphs with vertex pairings (Johannes Carmesin)

A *pairing* of a graph is a partition of its vertex set into sets of size two of non-adjacent vertices.

Question (C, Kurkofka, Mihaylov, Nevinson) Given a planar graph G with a pairing, can G be coloured with 11 colours such that paired vertices receive the same colour?

Remarks:

- originated from trying to extend the 4-colour theorem to 3D;
- **The answer to this question is 'no'**. This was pointed out by Noga Alon as well as Michal Pilipczuk and Lukasz Bozyk, see `https://www.jstor.org/stable/24966248?seq=1\#metadata_info_tab_contents` and `https://mathworld.wolfram.com/EmpireProblem.html`
- upper bound of 12. This bound is tight, see the above references

11. 2-well-quasi-order of planar graphs (Nathan Bowler)

Question: Are planar graphs *2-well-quasi-ordered* under the minor relation?

That is, can we rule out the existence of a family of planar graphs $(G_{ij})_{i<j\in\mathbb{N}}$ such that there are no $i < j < k \in \mathbb{N}$ for which G_{ij} is a minor of G_{jk}?

12. An inequality for the symmetric group (Bhargav Narayanan)

Let $w : E(K_n) \to \mathbb{R}_{\geqslant 0}$ be any non-negative weighting of the edges of the complete graph on $[n]$ vertices. Given w, we associate two quantities to any permutation $\pi \in S_n$. First, the order-weight of π is given by

$$\mathrm{ord}(\pi) = \prod_{i=1}^{n-1} w(\pi(i), \pi(i+1));$$

this comes from looking at π as an 'ordering' of $[n]$, and then multiplying the weights on the edges of the Hamilton path corresponding to π. Second, the cycle-weight of π is given by

$$\mathrm{cyc}(\pi) = \prod_{i=1}^{n} w(i, \pi(i)),$$

where $w(j, j)$ is taken to be 1 for all $j \in [n]$; this comes from looking at π as a product of cycles, and then multiplying the weights on the edges in the cycle decomposition of π (with multiplicity, and with fixed-points contributing weight 1).

Here is a rather intriguing conjectural inequality: for all w as above, we have

$$\sum_{\pi \in S_n} \mathrm{cyc}(\pi) \geqslant \sum_{\pi \in S_n} \mathrm{ord}(\pi),$$

with equality only holding for w identically 1 on each edge.

For $n = 2$ with a single weight $x \geqslant 0$ on the lone edge of K_2, the inequality reads $1 + x^2 \geqslant 2x$, which is trivially true. For $n = 3$ with weights $x, y, z \geqslant 0$ on the three edges of K_3, the inequality reads $1 + x^2 + y^2 + z^2 + 2xyz \geqslant 2xy + 2yz + 2zx$, and this can be verified with a little effort. Finally, for $n = 4$, my computer has verified the conjecture, but I know of no nice proof.

This came from some joint work with Lisa Sauermann on counting spanning trees where we just wanted this inequality for 0/1-valued w, i.e., for graphs. In this special case, the inequality says that the number of Hamilton paths in any graph G (counted twice, once for each orientation) is at most the number of permutations of the vertex set where each vertex is sent either to itself or to one of its neighbours. I do not know how to prove this in general either.

I do know the conjecture to be true when all weights are $\geqslant 1$, but this is rather simple. When all the weights are equal, the inequality follows from a suitable application of Jensen's inequality to the random variable tracking the number of fixed points of a (uniformly) random permutation.

13. Geometric reconstruction (Alex Scott)

Let S be a set of n points in $\mathbb{R}^d$. The *k-deck* of S is the multiset of all k-point subsets of S, given up to isometry. For example, the 2-deck of S is equivalent to knowing how many times each distance occurs in S. We say that a set S is *reconstructible from its k-deck* if every set with the same k-deck as S is isometric to S.

How large does k need to be so that every set of n points is reconstructible from its k-deck?

In one dimension, it is not hard to see that $k = 4$ is enough (i.e. every finite set of $\mathbb{R}$ is reconstructible from its 4-deck). But in two dimensions, the problem is more difficult. [N. Alon, Y. Caro, I. Krasikov and Y. Roditty, Combinatorial reconstruction problems, *J. Combin. Theory Ser. B* **47** (1989), 153–161] raised the question, and showed that every set of n points in
$mathbbR^2$ can be reconstructed from its $(\log_2 n + 1)$-deck. [L. Pebody, A. J. Radcliffe and A. D. Scott, All finite subsets of the plane are 18-reconstructible, *SIAM J. Discrete Math.* **16** (2003), 262–275] showed that there is a constant k that will do for all finite sets (in fact $k = 36$ is enough).

In three or more dimensions, much less is known. The arguments of Alon, Caro, Krasikov and Roditty show that logarithmic size is enough, but there is no non-constant lower bound. With Jamie Radcliffe, I conjecture the following.

Conjecture: There is some $k \in \mathbb{N}$ such that every finite subset of $\mathbb{R}^3$ is determined up to isometry by its k-deck.

14. A variant of the Erdős–Faber–Lovász conjecture (Tom Kelly)

The Erdős–Faber–Lovász conjecture is the following: *If $G_1, \ldots, G_n$ are complete graphs, each on at most n vertices, such that every pair shares at most one vertex, then $\chi(\bigcup_{i=1}^{n} G_i) \leqslant n$.*

In joint work with Dong Yeap Kang, Daniela Kühn, Abhishek Methuku, and Deryk Osthus from last year, we proved this conjecture for all sufficiently large n. There are still several variations and possible generalizations that remain open. One such example is the following:

Problem. *Let $G_1, \ldots, G_n$ be graphs, each of chromatic number at most $n-1$, such that every pair shares at most one vertex. What is the largest possible chromatic number of $\chi(\bigcup_{i=1}^{n} G_i)$?*

Erdős asked a related problem in 1981 that turned out to be trivial, but this is probably what he really wanted to ask.

In joint work with Daniela Kühn and Deryk Osthus, we proved an upper bound of $2n - 3$. It is possible that the answer is simply n, which would actually imply the Erdős–Faber–Lovász conjecture.

Update: Luke Postle provided a construction which gives a lower bound of $7n/6$.

15. Structure and coloring of 3-connected graphs with no large odd holes (Xingxing Yu)

Definition: Let $\mathcal{G}$ denote the class of all graphs G with the following properties:

- G is 3-connected and internally 4-connected,
- the girth of G is 5, and
- G contains no odd holes of length at least 7.

Question (Robertson 2010; Plummer and Zha 2012): Find a structural characterization of graphs in $\mathcal{G}$.

Conjecture (Plummer and Zha 2012): All graphs in $\mathcal{G}$ are 3-colorable.

Maria Chudnovsky and Paul Seymour proved the conjecture above during the workshop (see `https://arxiv.org/pdf/2201.11505.pdf`).

16. Querying for Subgraphs (Xiaoyu He)

Suppose G is an infinite hidden Erdős-Rényi random graph $G(\mathbb{N}, p)$, $p > 0$ very small.

Let H be a fixed target graph we would like to find in G, e.g. $H = K_4$.

Problem: Let $f(H, p)$ be the number of adjacency queries needed to reveal a copy of H in G with probability at least 1/2. What is the growth rate of $f(H, p)$ as $p \to 0^+$?

For cliques, [Conlon, Fox, Grinshpun, H. '19] proved that

- $f(K_3, p) \asymp p^{-\frac{3}{2}}$,
- $f(K_4, p) \asymp p^{-2}$,
- $f(K_5, p) \asymp p^{-\frac{8}{3}}$
- $p^{-(2-\sqrt{2})n+O(1)} \leqslant f(K_n, p) \leqslant p^{-2/3n-O(1)}$

Problem: What about K_6? Know $p^{-\frac{13}{4}} \ll f(K_6, p) \ll p^{-\frac{10}{3}}$.

For degenerate graphs [Alweiss, Ben Hamida, H., Moreira '20] proved that if H is d-degenerate ($d \geqslant 2$), then $f(H, p) = o(p^{-d})$. However, there exists a 2-degenerate H with $\frac{p^{-2}}{\log^4(1/p)} \ll f(H, p) \ll \frac{p^{-2}}{\log(1/p)}$.

Problem. For which H is $f(H, p) \asymp p^{-c}$ for some constant c?

17. Asymptotic dimension of embedded graphs (Chun-Hung Liu)

The *asymptotic dimension* of a graph class $\mathcal{F}$ is the minimum d such that there exists a function f such that $\forall G \in \mathcal{F}$ and $\forall r \in \mathbb{N}$, $V(G)$ can be colored with $d+1$ colors so that $\forall x, y \in V(G)$, if they are connected by a monochromatic path in G^r, then the distance between x, y in G is $\leqslant f(r)$.

Theorem (Gromov):

(1) The asymptotic dimension of the class of d-dimensional grids $= d$.
(2) Any infinite class of bounded degree expanders has infinite asymptotic dimension.

Theorem (Bonamy, Bousquet, Esperet, Groenland, L., Pirot, Scott):

(1) Any proper minor-closed family has asymptotic dimension $\leqslant 2$.
(2) The class of (g, k)-planar graphs has asymptotic dimension $= 2$.

Question: Does every graph class consisting of "essentially d-dimensional objects" have asymptotic dimension $\leqslant d$? $\leqslant g(d)$? $\geqslant d$? $\geqslant g(d)$?

Question: Does the class of graphs admitting book embeddings with k pages have asymptotic dimension 2?

Note added. It appears from remarks of Noga Alon and Vida Dujmović during the session that the answer to this second question is negative, as there exist bounded degree expanders with bounded page number.

18. Ramsey's theorem for matroid lines (Jim Geelen)

Conjecture. For $r \gg \ell$, if we 2-color the elements of a simple rank-r matroid M with no lines of length $> \ell$, then there is a monochromatic line.

- true for binary matroids
- true for $\mathbb{R}$-representable matroids
- true for $\ell \leqslant 3$
- natural extension to higher rank flats is also open

19. Planar graphs that are far from being 3-colorable (Louis Esperet)

For $\epsilon > 0$, a graph G is *ϵ-far* from a property $\mathcal{P}$ if one needs to delete at least $\epsilon|V(G)|$ edges from G to obtain a graph from $\mathcal{P}$.

It was proved in [A. Czumaj, M. Monemizadeh, K. Onak, C. Sohler, *Planar Graphs: Random Walks and Bipartiteness Testing*, arxiv 1407.2109] that if an n-vertex planar graph is ϵ-far from being bipartite, then G contains $\Omega(n)$ edge-disjoint odd cycles.

Question (Sohler): Is it true that if an n-vertex planar graph G is ϵ-far from being 3-colorable, then G contains $\Omega(n)$ edge-disjoint non-3-colorable subgraphs?

If true, it would imply that 3-colorability of planar graphs is testable in the sparse model, i.e., that we only need constantly many queries in this model to decide wether a planar graph is 3-colorable, or ϵ-far from being 3-colorable, with good probability.

Note added. The problem was solved by Sergey Norin during the workshop. Norin proved that the question has a positive answer, not only for 3-colorability of planar graphs, but for any monotone property of a proper minor-closed class.

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