Mathematisches Forschungsinstitut Oberwolfach

Report No. 5/2022

DOI: 10.4171/OWR/2022/5

Non-Archimedean Geometry and Applications

Organized by
Peter Schneider, Münster
Peter Scholze, Bonn
Michael Temkin, Jerusalem
Annette Werner, Frankfurt

30 January – 5 February 2022

Abstract. The workshop focused on recent developments in non-Archimedean analytic geometry with various applications to other fields. The topics of the talks included foundational results on analytic spaces as well as applications to the local Langlands conjecture, birational geometry, p-adic cohomology theories, Shimura varieties and the non-Archimedean Simpson correspondence.

Mathematics Subject Classification (2020): 11xx, 13xx, 14xx.

Introduction by the Organizers

The workshop on Non-Archimedean Analytic Geometry and Applications was organized by Peter Schneider (Münster), Peter Scholze (Bonn), Michael Temkin (Jerusalem) and Annette Werner (Frankfurt).

Non-Archimedean Geometry is a central area of Geometry with particular relevance for Arithmetic Geometry and numerous applications to other fields. Among the crucial problems are the famous Langlands program and the huge field of p-adic Hodge theory. Basic frameworks to attack these questions are the non-Archimedean analytic spaces introduced by Vladimir Berkovich and Roland Huber. Recently, Peter Scholze has added perfectoid spaces as a powerful new tool to attack deep problems in p-adic Hodge theory and representation theory. This has led to a rich edifice of deep theories connecting characteristic zero and characteristic p geometry in an unexpected way. Recent developments in this direction include diamonds and prismatic cohomology.

The workshop had 34 on-site participants, and 19 online participants. Altogether we had 20 one hour talks. A summary of the topics can be found below. Several participants explained work in progress or new conjectures or promising techniques to attack open conjectures. The workshop provided a lively platform to discuss these new ideas with other experts.

During the workshop we learned about the Geometrization of the local Langlands conjecture by Laurent Fargues and Peter Scholze. For a general reductive group G over a nonarchimedean local field E, the local Langlands correspondence predicts a relation between the representation theory of $G(E)$ and so-called L-parameters, which are 1-cocycles of the Weil group W_E with values in the Langlands dual group $\hat{G}$. In one direction, Fargues and Scholze constructed this correspondence by using the moduli space of G-bundles on the Fargues–Fontaine curve. Technically, all geometric objects here are not schemes or built from schemes, but rather certain complicated stacks built from perfectoid spaces, which are very nonnoetherian p-adic analytic spaces. The technical underpinnings of the theory include strong descent theorems like pro-étale or v-descent for vector bundles or étale cohomology. Much of this foundational work has now been put to use in various other questions in p-adic geometry, as was seen in other talks given at the workshop.

Motivated by the work of Fargues–Scholze, Johannes Anschütz developed a Fourier transform for Banach–Colmez spaces, with the idea of extending the approach of Drinfeld and Laumon for constructing Hecke eigensheaves. As a confirmation that this is on the right track, they showed that their Fourier transform produces vector spaces of the dimension predicted by the Jacquet-Langlands correspondence for GL_2.

Ben Heuer explained a contribution to the p-adic Simpson correspondence relating vector bundles for the v-topology (which correspond to Faltings' generalized representations) and Higgs bundles in a sheafified way. In order to define Higgs bundles in the relative setting, he introduced a new category of smoothoid spaces combining smooth adic and perfectoid spaces. Matti Würthen proved in his talk that in the good reduction case Higgs bundles are associated to genuine representations of the fundamental group if their reduction satisfies a condition by Lan-Shen-Zuo. Since this condition can be checked in interesting cases, these results are an important step towards the fundamental open question which Higgs bundles correspond to genuine representations in a p-adic Simpson correspondence. Hélène Esnault discussed a result ensuring that rigid local systems on varieties over number fields are crystalline at almost all places, which is used in a critical way in the recent full proof of the André–Oort conjecture. The proof makes use of the p-adic Simpson correspondence.

A few talks considered the interplay between various geometric questions and non-archimedean spaces. A fascinating application of geometry over valuation rings to the classical birational geometry was discussed in a talk of Johannes Nicaise. He constructed a specialization map for stably birational types, which can be viewed as a birational analogue of the limit mixed Hodge structure, and used it to obtain numerous new examples of projective varieties of low dimension which are provably not stably rational. Andreas Mihatsch introduced a formalism of piecewise smooth δ-forms on Berkovich spaces, which extends the formalism of Chambert-Loir and Ducros, defined the Green current δ-form of closed subspaces given by vanishing of a regular sequence of functions, and illustrated the theory with an application of Green currents to intersection theory on models of Lubin-Tate spaces. Katharina Hübner described a non-archimedean approach to bounding wild ramification locus from below. As an application a question about equivalence of various definitions of tameness of finite morphisms between algebraic varieties is finally settled independently of existence of resolution of singularities. Piotr Achinger constructed a specialization map between de Jong's fundamental group of a non-archimedean analytic space and the pro-étale fundamental group of its formal model, and explained why for η-normal models this map has a dense image.

A series of talks discussed duality and operations on cohomologies of p-adic analytic spaces with p-adic and $\mathbb{F}_p$-coefficients. The talks of Bogdan Zavyalov and Lucas Mann presented two different proofs of Poincaré duality for $\mathbb{F}_p$-cohomology of analytic spaces over $\mathbb{C}_p$ (or any algebraically closed and complete extension of $\mathbb{Q}_p$). In both cases, the starting point was the comparison theorem of Scholze which reduces the question to the almost duality for cohomology with $\mathcal{O}^+/p$-coefficients. Zavyalov reduced this latter question to a computation in a polystable case by establishing a new (locally equivariant) version of a local altered semistable reduction. Mann established a whole 6-functor formalism for coherent almost modules modulo a pseudo-uniformizer and used it to prove Poincaré duality; his method makes essential use of the formalism of condensed mathematics and solid modules recently developed by Dustin Clausen and Peter Scholze. Dustin Clausen explained an application of the theory of solid modules to a proof of the Grothendieck–Hirzebruch–Riemann–Roch theorem for proper smooth rigid-analytic varieties, which is a new result. Pierre Colmez discussed duality for p-adic pro-étale cohomology of open spaces over both p-adic fields and $\mathbb{C}_p$. In the p-adic case a number of conjectures were suggested and discussed during the conference. It might be the case that the conjectures are a bit overoptimistic and a naive topological duality should be corrected using the condensed approach, though a proof in the case of dimension one was presented.

Eugen Hellmann explained work on moduli spaces of Galois-equivariant vector bundles on the Fargues–Fontaine curve. These take the role of spaces of Langlands parameters in the p-adic local Langlands correspondence, and more precisely they are expected to appear in a formulation of the p-adic local Langlands correspondence for locally analytic representations. In particular, Hellmann showed evidence that these moduli spaces are rigid-analytic Artin stacks, and how they are related to the Emerton–Gee formal algebraic stack of (ϕ, Γ)-modules. Also on the subject of the p-adic local Langlands correspondence, David Hansen explained an application of Lucas Mann's 6-functor formalism to Scholze's functor from p-adic $\mathrm{GL}_n(K)$-representations to p-adic representations of the units $D^\times$ in the central division algebra D/K of invariant $1/n$, in particular on a version of Poincaré duality in this setting.

Vladimir Berkovich studied analytic spaces over $\mathbb{C}((t))$. Interesting cohomology theories in this case are related to the classical cohomology of complex analytic spaces. Berkovich constructed "étale" cohomology with coefficients in $\mathbb{Z}$ and de Rham cohomology, showed that they combine into a canonial mixed Hodge structure and established Poincaré duality for the latter. In particular, this subsumes the classical limit MHS construction.

Ana Caraiani explained a new proof of a theorem of Ohta on the ordinary part of completed cohomology of the modular curve, relating it to Hida's spaces of ordinary p-adic modular forms; the proof makes use of the Hodge-Tate period map and the Bruhat stratification on the flag variety, as well as some comparison theorems in integral p-adic Hodge theory. It is likely that the approach can be generalized to higher-dimensional Shimura varieties. Yujie Xu explained new results on the geometry of integral models of Shimura varieties (of Hodge type); in particular, she showed that even in the special fibre, the map to the Siegel moduli space is a closed immersion, for well-chosen level structures.

Bhargav Bhatt obtained striking applications of techniques from p-adic Hodge theory and prismatic cohomology to long-standing open problems in commutative algebra and birational geometry, in particular on the Cohen-Macaulayness of absolute integral closures, and Kodaira vanishing up to finite covers. These results have been critical in recent progress on the minimal model program for 3-folds in mixed characteristic.

Andreas Bode explained a theory of rigid-analytic $\hat{D}$-modules and a theory of 6 functors in this setup, using the theory of bornological vector spaces. In particular, he explained a notion of holonomicity in this setup and proved stability under some important operations.

Due to the hybrid format the schedule was adjusted so that on the first three days talks took place only in the afternoons, leaving the mornings for joint work and discussions among the on-site participants. The organizers made a specific effort to invite PhD students and postdocs. For many of them it was the first Oberwolfach workshop they ever attended. More than half of the talks, namely 11, were given by Phd students, postdocs and younger untenured faculty. The organizers also identified possible female invitees, thus ensuring that among the participants of the workshop there were ten women mathematicians.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows".

Workshop: Non-Archimedean Geometry and Applications

Table of Contents

Abstracts

Rigid analytic moduli stacks of (φ, Γ)-modules

Eugen Hellmann
(joint work with Valentin Hernandez, Benjamin Schraen)

Let F be a finite extension of $\mathbb{Q}_p$ and denote by $G_F = \mathrm{Gal}(\bar{F}/F)$ the absolute Galois group of F, where $\bar{F}$ is a fixed algebraic closure of F. We consider the Fargues-Fontaine curve $X = Y/\varphi^{\mathbb{Z}}$ constructed in [4] and which comes equipped with a continuous G_F-action. Here

$$Y = \mathrm{Spa}(W(\mathcal{O}_{\mathbb{C}_p^\flat}))\backslash\{p[\varpi] = 0\},$$

where $\mathbb{C}_p^\flat$ is the tilt of the completion $\mathbb{C}_p$ of $\bar{F}$ and $\varpi \in \mathcal{O}_{\mathbb{C}_p^\flat}$ is the choice of a pseudo-uniformizer. The space Y by construction comes equipped with the action of a Frobenius φ and a continuous G_F-action commuting with φ.

We aim to study families of G_F-equivariant vector bundles on X, parametrized by rigid analytic spaces. More precisely, we consider the category fibered in groupoids

$$\mathcal{X}_d : S \longmapsto \{G_F\text{-equivariant vector bundles of rank } d \text{ on } S \times_{\mathrm{Sp}(\mathbb{Q}_p)} X\}$$

on the category of rigid analytic spaces over $\mathbb{Q}_p$. Obviously there are many similarities with the theory of vector bundles on an algebraic curve. One main difference which makes the study of $\mathcal{X}_d$ harder is the fact that there is no analogue of an ample line bundle in the equivariant context: given an equivariant vector bundle $\mathcal{V}$ on X it is not possible to twist $\mathcal{V}$ by a (power of an equivariant) line bundle such that its higher cohomology vanishes.

By the work of Fargues-Fontaine and Berger, equivariant vector bundles on X admit an equivalent description in terms of (φ, Γ)-modules over the Robba ring $\mathcal{R}_F$ of F. This equivalence in fact generalizes to families parametrized by rigid analytic spaces and we will freely use this equivalence in the following. As (φ, Γ)-modules over $\mathcal{R}_F$ have a model over some annulus in the open unit disc, descent for vector bundles on rigid analytic spaces implies:

Proposition. *The category fibered in groupoids $\mathcal{X}_d$ is a stack for the Tate-fpqc topology on the category of rigid analytic spaces over $\mathbb{Q}_p$.*

In general we expect the following:

Conjecture. *The stack $\mathcal{X}_d$ is a rigid analytic Artin stack of dimension*

$$d^2[F : \mathbb{Q}_p](= \dim \mathrm{Res}_{F/\mathbb{Q}_p} \mathrm{GL}_d)$$

and a local complete intersection.

Towards this conjecture we have the following partial result:

Theorem.

(i) *The diagonal of $\mathcal{X}_d$ is representable.*
(ii) *For a finite extension L of $\mathbb{Q}_p$ and an L-valued point $x \in \mathcal{X}_d$, there exists an open neighborhood $\mathcal{U}_x \subset \mathcal{X}_d$ of x such that $\mathcal{U}_x$ is a rigid analytic Artin stack of the expected dimension and a local complete intersection.*

Example. In the case $d = 1$ we can explicitly describe $\mathcal{X}_1$ as follows: by a result of Kedlaya-Pottharst-Xiao [5] every (φ, Γ)-module over $\mathcal{R}_F \widehat{\otimes}_{\mathbb{Q}_p} A$, for an affinoid algebra A, is of the form $\mathcal{R}_F(\delta) \otimes \mathcal{L}$ for a uniquely determined character $\delta : F^\times \to A^\times$ and a uniquely determined line bundle $\mathcal{L}$ on $\operatorname{Sp} A$. Hence we find that $\mathcal{X}_1 = \mathcal{T}/\mathbb{G}_m$, where $\mathcal{T}$ is the space of continuous characters of $F^\times$, on which $\mathbb{G}_m$ acts trivially. It should be noted that $\mathcal{X}_d$ does not coincide with the generic fiber $\mathfrak{X}_d^{\mathrm{rig}}$ of the Emerton-Gee stack $\mathfrak{X}_d$ of (φ, Γ)-modules [3] (defined on p-power torsion $\mathbb{Z}_p$-schemes). In the one-dimensional case we can write

$$\mathcal{X}_1 = \mathcal{W} \times \mathbb{G}_m/\mathbb{G}_m,$$
$$\mathfrak{X}_d^{\mathrm{rig}} = \mathcal{W} \times (\widehat{\mathbb{G}}_m)^{\mathrm{rig}}/(\widehat{\mathbb{G}}_m)^{\mathrm{rig}},$$

where $\mathcal{W}$ denotes the space of continuous characters of $\mathcal{O}_F^\times$ and $(\widehat{\mathbb{G}}_m)^{\mathrm{rig}}$ is the rigid analytic generic fiber of the formal multiplicative group (that does not coincide with the rigid analytic space $\mathbb{G}_m$).

Finally we study B-structures of equivariant vector bundles, for a fixed Borel subgroup $B \subset \mathrm{GL}_d$. In the classical context of vector bundles on an algebraic curve $\mathbb{X}$, Drinfeld (see [1]) defines a compactification $\overline{\mathrm{Bun}}_B$ of the stack Bun_B of B-bundles on $\mathbb{X}$ that comes equipped with projections to Bun_d, the stack of rank d vector bundles on $\mathbb{X}$, and to Bun_T, the stack of T-bundles (for a fixed choice of a maximal torus $T \subset B$). The literal translation of this construction to the context of G_F-equivariant vector bundles in the Fargues-Fontaine curve X yields stacks

$$\overline{\mathcal{X}}_B^{\mathrm{naive}}, \mathcal{X}_B \text{ and } \mathcal{X}_T$$

on the category of rigid analytic spaces over $\mathbb{Q}_p$.

Theorem.

(i) *The stacks $\mathcal{X}_B$ and $\overline{\mathcal{X}}_B^{\mathrm{naive}}$ are rigid analytic Artin stacks.*
(ii) *There is a well defined "scheme-theoretic" image $\overline{\mathcal{X}}_B$ of $\mathcal{X}_B$ in $\overline{\mathcal{X}}_B^{\mathrm{naive}}$ which is a rigid analytic Artin stack.*

We point out that there are some important differences with the classical theory of vector bundles on curves:

- The stack $\mathcal{X}_B$ is not dense in $\overline{\mathcal{X}}_B^{\mathrm{naive}}$, hence the closure procedure in (ii) is necessary.
- The projection $\overline{\mathcal{X}}_B \to \mathcal{X}_d$ is neither proper nor surjective. It rather should be expected that the induced map

$$(*) \qquad \overline{\mathcal{X}}_B \longrightarrow \mathcal{X}_d \times \mathcal{X}_T$$

is proper, and its image is equi-dimensional of dimension $\frac{d(d+1)}{2}[F : \mathbb{Q}_p]$ which is the dimension of the Weil restriction from F to $\mathbb{Q}_p$ of a Borel subgroup in GL_d. The pullback of the scheme-theoretic image of $(*)$ to the generic fibers of (framed) Galois deformation rings conjecturally coincides with the trianguline variety of [2].

References

[1] A. Braverman, D. Gaitsgory, *Geometric Eisenstein series*, Inventiones Math. **150** (2002), 287–384.

[2] C. Breuil, E. Hellmann, B. Schraen, *Une interprétation modulaire de la variété trianguline*, Math. Annalen **367** (2017), 1587–1645.

[3] M. Emerton, T. Gee, *Moduli stacks of étale (φ, Γ)-modules and the existence of crystalline lifts*, arXiv:1908.07185, preprint 2019.

[4] L. Fargues, J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, Astérisque **406** (2008).

[5] K. Kedlaya, J. Pottharst, L. Xiao, *Cohomology of arithmetic families of (φ, Γ)-modules*, Journal of the AMS **27** (2014), 1043–1115.

Variation of stable birational type and bounds for complete intersections

Johannes Nicaise

(joint work with John Christian Ottem)

This talk is based on [NO19]. We explain how the specialization map for stable birational types can be combined with Shinder's principle of variation in families to establish new bounds for stable rationality for complete intersections.

1. Introduction

Let F be an algebraically closed field of characteristic zero.

Definition. *Let X and Y be F-schemes of finite type. We say that X and Y are* stably birational *if $X \times \mathbb{P}^{\ell}_F$ is birational to $Y \times \mathbb{P}^m_F$ for some $\ell, m \geq 0$. We say that X is* stably rational *if it is stably birational to* $\operatorname{Spec} F$*; equivalently, if $X \times \mathbb{P}^{\ell}_F$ is rational for sufficiently large ℓ.*

It is obvious from the definition that rationality implies stable rationality. The converse implication is true in dimension at most 2 but breaks down in higher dimensions.

It is a fundamental problem in algebraic geometry to determine for which values of $n, d > 0$ a very general degree d hypersurface in $\mathbb{P}^{n+1}_F$ is (stably) rational. Here, *very general* means that we may exclude a countable union of Zariski closed strict subsets of the parameter space $|\mathcal{O}_{\mathbb{P}^{n+1}_F}(d)|$. The best known result is due to Stefan Schreieder [Sch19]: for $n \geq 3$ and $d \geq \log_2 n + 2$, a very general hypersurface of degree d in $\mathbb{P}^{n+1}_F$ is *not* stably rational. This logarithmic bound is a significant improvement of earlier linear bounds, but probably still far from sharp. It seems to be expected that smooth hypersurfaces of degree $d \geq 4$ are never stably rational, but there is no value for d for which such a result is known in all dimensions.

The aim of this talk is to deduce analogous results for complete intersections. Taking as input a result for hypersurfaces (for instance, Schreieder's bound), our method produces a similar bound for complete intersections. It is based on the specialization map for stable birational types [NS19] and Shinder's principle of variation of stable birational type in maximally degenerating families.

2. The specialization map

We denote by SB_F the set of stable birational equivalence classes $\{X\}$ of integral F-schemes X of finite type. If X is any F-scheme of finite type, we can still define a class $\{X\}$ by setting

$$\{X\} = \{X_1\} + \ldots + \{X_r\} \in \mathbb{Z}[\mathrm{SB}_F]$$

where $X_1, \ldots, X_r$ are the irreducible components of X and $\mathbb{Z}[\mathrm{SB}_F]$ is the free abelian group on the set SB_F. This group carries a natural ring structure, uniquely characterized by the multiplication rule $\{X\} \cdot \{Y\} = \{X \times Y\}$ for all F-schemes X and Y of finite type.

The ring $\mathbb{Z}[\mathrm{SB}_F]$ keeps track of stable birational types in a rather naive way, but, surprisingly, still lends itself to some non-trivial calculus. In [NS19], we defined a specialization map for stable birational types that attaches a "limit" type to any degenerating one-parameter family of smooth and proper F-schemes. It should be viewed as a birational analog of the nearby cycles functor or limit mixed Hodge structure.

To define the specialization map, we first set up some notation. We denote by k an algebraically closed field of characteristic zero, and we set $R = \cup_{n>0} k[[t^{1/n}]]$. This is a valuation ring with quotient field $K = \cup_{n>0} k((t^{1/n}))$, the field of Puiseux series over k.

Theorem (Nicaise–Shinder [NS19])**.** *There exists a unique ring morphism*

$$\mathrm{sp}\colon \mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_k]$$

such that, for every strictly toroidal proper flat R-scheme $\mathcal{X}$, we have

$$\mathrm{sp}(\{\mathcal{X}_K\}) = \sum_S (-1)^{\mathrm{codim}(S)} \{S\}$$

where the sum runs over the strata S in the special fiber $\mathcal{X}_k$ (that is, the connected components of intersections of sets of irreducible components of $\mathcal{X}_k$).

The strictly toroidal R-schemes in this statement generalize the class of strictly semistable R-schemes, and are more flexible in applications. The expression for $\mathrm{sp}(\{\mathcal{X}_K\})$ should be viewed as a "virtual stable birational type" for $\mathcal{X}_k$; it is similar to the calculation of the nearby cycles/limit mixed Hodge structure by means of the weight spectral sequence.

The theorem can be deduced from Hrushovski and Kazhdan's theory of motivic integration, or from the weak factorization theorem. A special case of the formula is already noteworthy: if $\mathcal{X}$ is smooth and proper over R, then the formula collapses to $\mathrm{sp}(\{\mathcal{X}_K\}) = \{\mathcal{X}_k\}$. It is highly non-trivial that the stable birational type of $\mathcal{X}_k$ only depends on the stable birational type of $\mathcal{X}_K$, but not on $\mathcal{X}_K$ itself. In fact, this settled a long-standing problem in birational geometry.

Corollary. Stable rationality specializes in smooth and proper families in characteristic zero.

Indeed, if $\mathcal{X}$ is smooth and proper over R and $\mathcal{X}_K$ is stably rational, then $\{\mathcal{X}_K\} = \{\mathrm{Spec}\, K\}$ so that

$$\{\mathcal{X}_k\} = \mathrm{sp}(\{\mathcal{X}_K\}) = \mathrm{sp}(\{\mathrm{Spec}\, K\}) = \{\mathrm{Spec}\, k\}$$

which means that $\mathcal{X}_k$ is also stably rational. This specialization result was further upgraded to rationality by Kontsevich and Tschinkel in [KT19].

By contraposition, the specialization map can also be used as an obstruction to stable rationality: if $\mathcal{X}$ is a strictly toroidal proper flat R-scheme and

$$\sum_S (-1)^{\mathrm{codim}(S)}\{S\} \neq \{\mathrm{Spec}\, k\}$$

in $\mathbb{Z}[\mathrm{SB}_k]$, then $\mathcal{X}_K$ is not stably rational. To apply this obstruction to concrete examples, it is important to rule out accidental cancellations in the alternating sum. For this purpose, the following generalization of a theorem due to Shinder is often convenient.

Corollary (Variation of stable birational type)**.** Let $f\colon \mathcal{Y} \to \mathrm{Spec}\, k[t]$ be a proper and flat morphism with connected fibers such that $\mathcal{X} = \mathcal{Y} \times_{k[t]} R$ is strictly toroidal and such that every stratum of $\mathcal{X}_k$ is stably rational. Then either all smooth geometric fibers of f are stably rational, or there are at most countably many closed fibers of fixed stable birational type.

In the latter case, we can change the stable birational type of a closed fiber by slightly perturbing it in the family. This variation principle is again a direct consequence of the existence of the specialization map. Assume that W is a connected smooth and proper k-scheme such that f has uncountably many closed fibers that are stably birational to W. A standard Hilbert scheme argument then implies that $W \times_k K$ is stably birational to $\mathcal{X}_K$. It follows that

$$\{W\} = \mathrm{sp}(\{W \times_k K\}) = \mathrm{sp}(\{\mathcal{X}_K\}) = \sum_S (-1)^{\mathrm{codim}(S)}\{S\}$$

in $\mathbb{Z}[\mathrm{SB}_k]$, where S runs over the strata of $\mathcal{X}_k$. Since each of these strata is stably rational by assumption, the right hand side of this chain of equalities is an integer multiple of $\{\mathrm{Spec}\, k\}$. It follows that $\{W\} = \{\mathrm{Spec}\, k\}$, and thus that W, and therefore $\mathcal{X}_K$, are stably rational. Specialization of stable rationality now implies that all smooth geometric fibers of f are stably rational.

3. Bounds for complete intersections

Theorem (Nicaise–Ottem [NO19]). *Let n, r and $d_1 \leq \ldots \leq d_r$ be positive integers. Assume that*

$$n + r \geq \sum_{i=1}^{r-1} d_i + 2.$$

Suppose moreover that there exists a non-stably rational smooth hypersurface of degree d_r in $\mathbb{P}_k^{n+r-\sum_{i=1}^{r-1} d_i}$. Then a very general complete intersection of degree $(d_1, \ldots, d_r)$ in $\mathbb{P}_k^{n+r}$ is not stably rational.

For instance, the theorem implies that a very general complete intersection of a quadric and a quartic in $\mathbb{P}_k^m$ is not stably rational for $m = 6, 7, 8$, because a very general quartic in $\mathbb{P}_k^{m-2}$ is not stably rational.

To prove the theorem, we consider very general homogeneous polynomials $F_1, \ldots,$
F_r of degrees $d_1, \ldots, d_r$ in $n+r+1$ variables, and we degenerate $F_1, \ldots, F_{r-1}$ into products of disjoint sets of variables: the equations

$$\begin{cases} tF_1 - \prod_{j=1}^{d_1} z_{1j} & = & 0 \\ \ldots & & \\ tF_{r-1} - \prod_{j=1}^{d_{r-1}} z_{r-1,j} & = & 0 \\ F_r & = & 0 \end{cases}$$

define a strictly toroidal closed subscheme $\mathcal{X}$ in $\mathbb{P}_R^{n+r}$.

The generic fiber $\mathcal{X}_K$ is a smooth complete intersection of degree $(d_1, \ldots, d_r)$ in $\mathbb{P}_K^{n+r}$. We will show that $\mathcal{X}_K$ is not stably rational; specialization of stable rationality then implies the analogous result for very general complete intersections of degree $(d_1, \ldots, d_r)$ in $\mathbb{P}_k^{n+r}$.

The strata of $\mathcal{X}_k$ are of the form $Z(F_r) \cap L$ where $Z(F_r)$ is the hypersurface in $\mathbb{P}_k^{n+r}$ defined by $F_r = 0$ and L is a linear subspace of $\mathbb{P}_k^{n+r}$ defined by the vanishing of a subset of the coordinates $z_{11}, \ldots, z_{r-1,d_{r-1}}$. There is a unique minimal stratum $Z(F_r) \cap L_0$ where L_0 is a linear subspace of $\mathbb{P}_k^{n+r}$ of codimension $d_1 + \ldots + d_{r-1}$. By assumption, $Z(F_r) \cap L_0$ is not stably rational.

When applied to the strictly toroidal R-scheme $\mathcal{X}$, the formula for the specialization map takes the form

$$\mathrm{sp}(\{\mathcal{X}_K\}) = \pm\{Z(F_r) \cap L_0\} + \sum_{L \neq L_0} \pm\{Z(F_r) \cap L\}.$$

Since $Z(F_r) \cap L_0$ is not stably rational, we know that $\{Z(F_r) \cap L_0\} \neq \{\mathrm{Spec}\, k\}$. To conclude that $\mathrm{sp}(\{\mathcal{X}_K\}) \neq \{\mathrm{Spec}\, k\}$, and thus that $\mathcal{X}_K$ is not stably rational, we must argue that the contribution of $\{Z(F_r) \cap L_0\}$ does not cancel out with other terms in the alternating sum. For this purpose, we can apply the variation principle: using techniques from tropical geometry, one can put each of the varieties $Z(F_r) \cap L$ with $L_0 \subsetneq L$ in a family over $\mathrm{Spec}\, k[t]$ such that the stable birational type of $Z(F_r) \cap L$ varies with t but the subvariety $Z(F_r) \cap L_0$ remains constant in the family. For a detailed account of this construction, we refer to [NO19].

References

[KT19] M. Kontsevich and Yu. Tschinkel, *Specialization of birational types*, Invent. Math. **217(2)** (2019), 415–432.

[NO19] J. Nicaise and J.C. Ottem, *Tropical degenerations and stable rationality*, to appear in Duke Math. J.

[NS19] J. Nicaise and E. Shinder, *The motivic nearby fiber and degeneration of stable rationality*, Invent. Math. **217(2)** (2019), 377–413.

[Sch19] S. Schreieder, *Stably irrational hypersurfaces of small slopes*, J. Amer. Math. Soc. **32(4)** (2019), 1171–1199.

Prismatization

Bhargav Bhatt

(joint work with Jacob Lurie)

This talk covers some results of [1, 2]. Most of the results discussed were also independently discovered by Drinfeld [3]

1. Introduction

In previous joint work [4] with Scholze, we introduced the following object:

Definition 1.1. For any (bounded) p-adic formal scheme X, the absolute prismatic site X_{Δ} is the category of all bounded prisms (A, I) equipped with a map $\mathrm{Spf}(A/I) \to X$, topologized via (say) the flat topology.

The paper [4] mostly focussed on the relative variant of this construction. In this talk, we explain how to understand the above absolute version via stacks.

2. The classical analog (following Simpson)

Fix a field k of characteristic 0; work with presheaves on k-algebras. We explain how to interpret de Rham cohomology of k-varieties as the $\mathcal{O}$-cohomology of an associated sheaf.

Construction 2.1 (The ring sheaf $\mathbf{G}_a^{dR}$)**.** Let $\widehat{\mathbf{G}_a} = \mathrm{Spf}(k[\![t]\!]) \subset \mathbf{G}_a = \mathrm{Spec}(k[t])$ be the formal completion of $\mathbf{G}_a$ at 0, so $\widehat{\mathbf{G}_a}(R) = \mathrm{Nil}(R)$.

$\rightsquigarrow \widehat{\mathbf{G}_a} \to \mathbf{G}_a$ gives an "ideal" in $\mathbf{G}_a$, so $\mathbf{G}_a^{dR} := \mathbf{G}_a/\widehat{\mathbf{G}_a}$ is naturally an étale sheaf of rings on k-algebras; explicitly, $\mathbf{G}_a^{dR}(R) = R_{red}$

Using $\mathbf{G}_a^{dR}$, we obtain:

Definition 2.2 (The de Rham space)**.** Let X/k be a smooth variety. Let X^{dR} be the presheaf on k-algebras given by

$$X^{dR}(R) = X\left(\mathbf{G}_a^{dR}(R)\right) = X(R_{red}).$$

Theorem 2.3 (Simpson)**.** *There is a natural isomorphism $R\Gamma(X^{dR}, \mathcal{O}) \simeq R\Gamma_{dR}(X)$. Moreover, there is an analogous statement with coefficients: vector bundles on X^{dR} identify with vector bundles on X equipped with a flat connection, and this identification is compatible with cohomology.*

One has a variant of this object for crystalline cohomology as well: given a smooth p-adic formal scheme $X/\mathbf{Z}_p$ say, one has

$$R\Gamma(X^{crys}, \mathcal{O}) \simeq R\Gamma_{crys}(X/\mathbf{Z}_p),$$

where X^{crys} is the functor on p-nilpotent rings defined by

$$X^{crys}(R) = X((\mathbf{G}_a/\mathbf{G}_a^\sharp)(R)),$$

where $\mathbf{G}_a^\sharp$ is the PD-hull of 0 in $\mathbf{G}_a$. We remark that the quotient $\mathbf{G}_a/\mathbf{G}_a^\sharp$ is interpreted now as a ring stack (as the map $\mathbf{G}_a^\sharp \to \mathbf{G}_a$ need not be injective on R-valued points); in particular, $(\mathbf{G}_a/\mathbf{G}_a^\sharp)(R)$ is a 1-truncated animated ring, and the right side above is interpreted as a mapping space in derived algebraic geometry. (A similar comparison holds true for coefficients.)

3. Relative prismatization

Let (A, I) be a bounded prism, and let $X/(A/I)$ be a smooth p-adic formal A/I-scheme. Recall that for any (p, I)-complete A-algebra R, the structure map $A \to R$ refines uniquely to a δ-map $A \to W(R)$; write $I_{W(R)} \to W(R)$ for the base change of $I \to A$, regarded as a generalized Cartier divisor on $\mathrm{Spec}(W(R))$.

Definition 3.1 (The relative prismatization)**.** Define a presheaf $\mathrm{WCart}_{X/A}$ of groupoids on (p, I)-complete A-algebras R via

$$\mathrm{WCart}_{X/A}(R) = X\left(W(R)/I_{W(R)}\right),$$

where $W(R)/I_{W(R)}$ is interpreted as a 1-truncated animated ring. Similarly, define

$$\mathrm{WCart}^{HT}_{X/A} = \mathrm{WCart}_{X/A} \times_{\mathrm{Spf}(A)} \mathrm{Spf}(A/I)$$

to be the restriction of $\mathrm{WCart}_{X/A}$ to a functor on p-nilpotent A/I-algebras.

The stack $\mathrm{WCart}_{X/A}$ plays the role of Simpson's de Rham space for relative prismatic cohomology. In particular, its $\mathcal{O}$-cohomology will be the relative prismatic cohomology (Theorem 3.4). Granting that comparison, the following two results follow from known results for relative prismatic cohomology. However, they admit much more direct proofs using the stacky approach.

Proposition 3.2 (Hodge-Tate comparison)**.** *There is a natural map* $\mathrm{WCart}^{HT}_{X/A} \to X$ *over* A/I*. Moreover, this map is a* $T^\sharp_{X/(A/I)}\{1\}$*-gerbe.*

Proposition 3.3 (Crystalline comparison)**.** *There is a natural identification of ring stacks*

$$W/^L p := \mathrm{Cone}\left(W \xrightarrow{p} W\right) \simeq \mathbf{G}_a/\mathbf{G}_a^\sharp.$$

on p*-nilpotent rings.*

Theorem 3.4. *There is a natural identification* $R\Gamma((X/A)^{\Delta}, \mathcal{O}) \simeq R\Gamma_{\Delta}(X/A)$*; a similar statement holds true for coefficients.*

4. Absolute prismatization

To geoemtrize the absolute prismatic site of a p-adic formal scheme X, we need to geometrize the notion of a prism (corresponding to the case $X = \mathrm{Spf}(\mathbf{Z}_p)$). This is accomplished as follows:

Definition 4.1. The presheaf WCart (called the *Cartier–Witt stack*) of groupoids on p-nilpotent rings is given by setting $\mathrm{WCart}(R)$ to be the groupoid of maps $\alpha : I \to W(R)$, where I is an invertible $W(R)$-module and α is a $W(R)$-linear map such that for any local generator $d \in I$, the Witt vector $\alpha(d) = (a_0, a_1, a_2, ...) \in W(R)$ is distinguished, i.e., a_0 is nilpotent and a_1 is a unit.

The stack WCart can be written as the quotient stack W_{dist}/W^*, where $W_{dist} \subset W$ is the subfunctor of the Witt vector functor spanned by distinguished elements (so W_{dist} is the formal completion of an affine open in W), and the quotient is formed in (say) the etale topology. Generalizing to arbitrary X is now straightforward given our previous work:

Definition 4.2 (The Cartier-Witt stack)**.** For a bounded p-adic formal scheme X, define a presheaf WCart_X of groupoids on p-nilpotent rings via

$$\mathrm{WCart}_X(R) := \{\left(I \xrightarrow{\alpha} W(R), \eta \in X\left(W(R)/I\right)\right) \mid \alpha \in \mathrm{WCart}(R)\}$$

Theorem 4.3. *For quasi-syntomic X, we have $R\Gamma(\mathrm{WCart}_X, \mathcal{O}) \simeq R\Gamma_{\Delta}(X)$; a similar statement holds true for coefficients.*

These results are useful as they convert the potentially unwieldy notions (such as that of a crystal of vector bundles on the absolute prismatic site) to much more finitistic ones (such as that of a vector bundle on a relatively concrete stack). Some applications include:

(1) Absolute prismatic cohomology via q-de Rham cohomology: For quasi-syntomic X and $p > 2$, we have a homotopy pullback square

$$\begin{array}{ccc} R\Gamma_{\Delta}(X) & \longrightarrow & R\Gamma_{qdR}(X)^{h\mathbf{Z}_p^*} \\ \downarrow & & \downarrow \\ R\Gamma_{dR}(X) & \longrightarrow & R\Gamma_{dR}(X)^{h\mathbf{Z}_p^*}, \end{array}$$

where the top right denotes the q-de Rham cohomology of X, the left vertical map is the de Rham comparison and the right vertical map is obtained by setting $q = 1$.

(2) Sen theory and Deligne–Illusie (due to Drinfeld): If X is a smooth scheme over a perfect field k equipped with a lift to $W_2(k)$, then the de Rham complex $F_*\Omega^*_X$ of X admits a natural endomorphism Θ (called the Sen operator) that acts on $\mathcal{H}^i(F_*\Omega^*_X) \simeq \Omega^i_{X^{(1)}}$ by $-i$. In particular, if $\dim(X) < p$, then the generalized eigenvalue decomposition for Θ splits the conjugate filtration on $F_*\Omega^*_X$, thus yielding a new proof of the Deligne–Illusie theorem on the degeneration of the conjugate and Hodge-to-de Rham spectral sequences.

(3) Drinfeld's refined prismatization via the Nygaard filtration: Using Theorem 4.3, we expect to show in forthcoming work that Drinfeld's stack $X^{\mathbb{\Delta}'}$ from [3] is the Rees stack for the Nygaard filtration when X is the spectrum of a quasiregular semiperfectoid ring.

References

[1] B. Bhatt and J. Lurie, Absolute prismatic cohomology
[2] B. Bhatt and J. Lurie, The prismatization of p-adic formal schemes
[3] V. Drinfeld, *Prismatization*
[4] B. Bhatt and P. Scholze, *Prisms and prismatic cohomology*

Arithmetic properties of rigid local systems

Hélène Esnault
(joint work with Michael Groechenig)

1. Introduction

The aim of the lecture, which relies on [3], is to generalize to the quasi-projective non-projective case the theorem described in the introduction and Theorem 5.4 of [2] stating that irreducible complex rigid local systems, while restricted to a p-adic formal scheme with good reduction, for p large, underlie the structure of a Fontaine-Laffaille module and define p-adic local systems.

2. Assumption and preliminary geometric facts

Assumption 1 (Throughout)**.** X smooth quasi-projective $/\mathbb{C}$; all *irreducible* complex local systems $\mathbb{L}$ of rank r with *unipotent monodromies* at ∞ are *strongly cohomologically rigid* i.e. $H^1(X, \mathcal{E}nd(\mathbb{L})) = 0$. (*Verified for Shimura varieties of real rank* ≥ 2 *by Margulis superrigidity.*)

Facts 2 (Preliminary geometry)**.**
1) For $j : X \hookrightarrow \bar{X}$ a good compactification, has

$$H^1(\bar{X}, j_{!*}\mathcal{E}nd(\mathbb{L})) \xrightarrow{\text{inj}} H^1(X, \mathcal{E}nd(\mathbb{L}))$$

so (*strongly cohomologically rigid*) $\Longrightarrow$ (*cohomologically rigid*) (fixing unipotent conjugacy classes at ∞ and the determinant) $\Longrightarrow$ (*rigid*) (fixing unipotent conjugacy classes at ∞ and the determinant).

2) E Deligne's extension of $(\mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}^{\text{an}}, 1 \otimes d)$ with $E \to \Omega^1_{\bar{X}}(\log \infty) \otimes_{\mathcal{O}_{\bar{X}}} E$ with nilpotent residues. So

$$H^1(\bar{X}, \Omega^\bullet(\log \infty) \otimes \mathcal{E}nd(E, \nabla)) = H^1(X, \mathcal{E}nd(\mathbb{L})) = 0.$$

Atiyah class computation + Hodge theory ([4, Appendix B])$\Longrightarrow$

$$0 = c_i(E) \in H^{2i}(\bar{X}, \mathbb{Q}), \ i \geq 1.$$

3) (E,∇) necessarily semi-stable as saturated sub $(E',\nabla') \subset (E,\nabla)$ has also nilpotent residues, so $(E',\nabla') \subset (E,\nabla)$ locally split outside of a codimension 2 subset $\Sigma \subset \infty \subset \bar{X}$ in $\bar{X}$, and for $j : \bar{X} \setminus \Sigma \hookrightarrow \bar{X}$, has $(E',\nabla') = j_* j^*(E',\nabla') \subset (E,\nabla) = j_* j^*(E,\nabla)$. Thus (E',∇') is Deligne's extension as it is determined outside of codimension 2. Thus $0 = c_i(E') \in H^{2i}(\bar{X},\mathbb{Q}),\ i \geq 1$.

4) Langer moduli ([9, Theorem 1.1]) $M_{dR}(r),\ M_{Dol}(r)$ of *stable* log-objects to the Hilbert polynom $P(E) = P(\oplus_1^r \mathcal{O}_{\bar{X}})$ are defined over some S smooth $/\mathbb{Z}$, $M_{dR}(r)_S\ M_{Dol}(r)_S \to S$ flat and $X_S, \bar{X}_S \to S$ relative NCD and base change for $H^1(\bar{X}_S, \Omega^\bullet(\log \infty) \otimes \mathcal{E}nd(E_S,\nabla_S))$.

5) The characteristic polynomials of the residues $E_S \to \Omega^1(\log \infty_S) \otimes E_S$ at ∞_S are regular functions on $M_{dR}(r)_S$, so the nilpotent residues condition defines closed subs $M^\circ_{dR}(X)(r)_S \subset M_{dR}(r)_S$, which after shrinking are flat $/S$. $M^\circ_{dR}(X)(r)_S(\mathbb{C})$ consists precisely of the Deligne's extensions of the underlying (strongly cohomologically rigid) *irreducible* local systems. So it is 0-dimensional, say of cardinality N. By taking an étale cover of S, we may assume that $M^\circ_{dR}(X)(r)_S$ consists of N-sections.

6)

Theorem 3 ([1], Theorem 1.1)**.** *All $\mathbb{L}$ are integral. (Finiteness of $\{\mathbb{L}\}$ implies all $\mathbb{L}$ defined over one $\mathcal{O}_L$, L number field).*

7) Mochizuki ([10, Theorem 10.5]): any (E,∇) with nilpotent residues deforms real analytically to a polarized $\mathbb{C}$-VHS, so rigidity implies all the (E,∇) underlie a polarized $\mathbb{C}$-VHS. So with 6)

Claim 4. Assumption 1 $\Longrightarrow$ all $\mathbb{L}$ underlie a polarized $\bar{\mathbb{Z}}$-VHS.

8) Boundedness of possible Hodge filtrations.

Definition 5 (Good model)**.** $S/\mathbb{Z}$ smooth, condition 5), plus: all (E,∇,Fil) defined over S, Fil locally split $/S$. So $(gr^{Fil}(E), KS)$ stable Higgs, locally free $/S$ nilpotent. We assume also for any $\mathrm{Spec}(W(\mathbb{F}_q)) \to S$, char $\mathbb{F}_q > 2r+2$.

Theorem 6 ([3], Theorem A.4)**.** *Assumption 1 $\Longrightarrow$ on a good model, for any $\mathrm{Spec}(W) \to S$, with $W = W(\mathbb{F}_q)$, the formal connection $(\hat{E}_W, \hat{\nabla}_W)/\widehat{\overline{X}_W}$ carries the structure of a locally free Fontaine-Laffaille module.*

(Standard definition of a Fontaine-Laffaille module right now irrelevant as we shall work with an equivalent definition).

3. Sketch of Proof of Theorem 6

s closed point of $\mathrm{Spec}(W)$. Has $M^\circ_{dR}(r)_s =: (dR)^\circ_s$ consisting of N s-points, and $(Dol)^\circ_s$ defined as the *set* of *stable* rank r log Higgs bundles (V,θ) with the residues of θ being nilpotent and Hilbert polynomial $P(V) = P(\oplus_1^r \mathcal{O}_{\bar{X}})$.

Claim 7. C^{-1} (Ogus-Vologodksy [11, Theorem 2.8]) : $(Dol)^\circ_s \to (dR)^\circ_s$ and is injective (in particular $(Dol)^\circ_s$ is finite).

Proof. C^{-1} defined for $p > r+1$ plus lift to W_2, preserves stability, total Chern classes and nilpotency of the residues at ∞. □

Claim 8. $H^1(\bar{X}_s, \mathcal{E}nd(C^{-1}(V,\theta))) = H^1(\bar{X}_s, \mathcal{E}nd(V,\theta)) = 0.$

Proof. C^{-1} defined for $2r$ rank objects, preserves cohomology, $LHS = 0$ by 4). □

By Claim 7, $|(Dol)_s^\circ| = M \leq N = |(dR)_s^\circ|$. Let $M' \leq M$ be the number of objects in $(Dol)_s^\circ$ of the shape $(V,\theta) = (gr^{Fil}(E), KS)$.

Corollary 9. *Given $(V,\theta) \in (Dol)_s^\circ$ there is at most one possible (E, ∇, Fil) with $(V,\theta) = (gr^{Fil}(E), KS)$.*

Proof. Given Fil on (E,∇), then Rees $(\oplus_{i\in\mathbb{Z}}(Fil^i E)t^{-i}, \nabla_t)$ on $X[t,t^{-1}]$ has fibre $(gr^{Fil}E, KS)$ at $t=0$ and (E,∇) at $t=\infty$. Deformation of $(gr^{Fil}E, KS)$ from $\mathbb{F}_q[t]/(t^n)$ to $\mathbb{F}_q[t]/(t^{n+1})$ is computed by $H^1(\bar{X}_s, \mathcal{E}nd(V,\theta)) = 0$. As any (E,∇) is endowed with at least one Fil (by 7) and good model), has $N \leq M'$. □

Corollary 10. *1) $M' = M = N$; C is bijective; the p-curvature of any $(E,\nabla) \in (dR)_s^\circ$ is nilpotent;*
2) any $(E,\nabla) \in (dR)_s^\circ$ carries precisely one Fil;
3) $gr : (dR)_s^\circ \to (Dol)_s^\circ$, $(E,\nabla) \mapsto (gr^{Fil}E, KS)$ is well defined and bijective.

Proof. Ad 1): $N = M' \leq M \leq N$ (first inequality from Corollay 9, last inequality by Claim 7). Thus $M' = M = N$. C^{-1} sends nilpotent Higgs to nilpotent p-curvature dR.

Ad 2): any $(E,\nabla) \in (dR)_s^\circ$ carries at least one Fil by the good model and more would imply $M > N$ by Corollary 9. 3) follows. □

Corollary 11. *$\sigma := C^{-1} \circ gr$ is a permutation of $(dR)_s^\circ$ and has finite order $f|N!$.*

Definition 12. The chain

$$(E_0, \nabla_0, Fil_0, \phi_0 : C^{-1}(gr^{Fil_0}E_0, KS) \cong (E_1, \nabla_1),$$
$$E_1, \nabla_1, Fil_1, \ldots, E_{f-1}, \nabla_{f-1}, Fil_{f-1}, \phi_{f-1} : C^{-1}(gr^{Fil_{f-1}}E_{f-1}, KS) \cong (E_0, \nabla_0))$$

is called a f-periodic Higgs-de Rham flow. ([7], [8]).

Proposition 13. *1) The f-periodic Higgs-de Rham flow lifts to $\widehat{\overline{X}_W}$ in what is still a f-'periodic Higgs-de Rham flow' over $\widehat{\overline{X}_W}$.*
2) The operator σ becomes the Frobenius on the isocrystals $(\hat{E}_W, \hat{\nabla}_W)_K$.

Here $W = W(\mathbb{F}_q)$, $K = \mathrm{Frac}(W)$ and recall that the p-curvatures of the mod p-reduction are nilpotent so we have isocrystals with a Frobenius structure.

Proof. The (E,∇) in $(dR)_s^\circ$ lift by definition to $\widehat{\overline{X}_W}$ together with their Hodge filtration. So gr is defined on $\widehat{\overline{X}_W}$ yielding some $(\hat{V}_W, \hat{\theta}_W)$ so (V_K, θ_K) which in addition are stable. C^{-1} is defined on $\widehat{\overline{X}_W}$ by Ogus-Vologodsky. As the lift $(\hat{E}_W, \hat{\nabla}_W)$ is uniquely determined by its reduction to $s \implies f$-periodicity.

Remark 14. Claim 8 $\Longrightarrow$ (semi-continuity of coherent cohomology)

$$H^1(\bar{X}_K, \mathcal{E}nd(V_K, \theta_K)) = 0 \Longrightarrow (V_K, \theta_K) \in M^\circ_{Dol}(r)_S(K).$$

If we define $M^\circ_{Dol}(X)(r)_S \subset M_{dR}(r)_S$, so $(V,\theta) \in M^\circ_{Dol}(\mathbb{C})$ if the residues at ∞ are nilpotent and $H^1(\bar{X}, \mathcal{E}nd(V,\theta)) = 0$, and we assume by étally shrinking S in the good model that in addition $M^\circ_{Dol}(X)(r)_S(S)$ consists of different (finitely many) S-sections, we see that in fact we have N such and they all come from the Higgs-de Rham flow. If we had a log-Simpson correspondence at the level of moduli $/\mathbb{C}$, we would know this and could shorten the argument.

□

Proposition 15. *Lan-Sheng-Zuo, Lan-Sheng-Yang-Zuo: 1) Fully faithful functor: (f-periodic Higgs-de Rham flow with nilpotent residues level $\le p-1$) $\to$ (log-Fontaine-Laffaille modules with $Frob^f$-structure, with nilpotent residues level $\le p-1$).*

2) Generalization of Fontaine-Laffaille-Faltings [5] *Theorem 2.6* and p.43 i) applied to $\oplus_{i=0}^{f-1} Frob^i(object)$: fully faithful functor (log-Fontaine-Laffaille modules with $Frob^f$-structure, with nilpotent residues, level $\le p-1$) $\to$ (crystalline local systems on X_K with values in $GL_r(\mathbb{Z}_{p^f})$).*

Remark 16. Crystalline here is defined by Fontaine on K, Faltings on 'small' opens defined by a ring R, étale over the Tate algebra of $\mathbb{G}_m^d$ over W, then on their rings by admissibility of a $B_{crys}(R)$, then gluing. Generalization by Tan-Tong, and just a few days ago Du-Lu-Moon-Shimizu. *All those concepts restricted to* Spec$(W) \to X_W$ *yield the same definition, which is Fontaine's one.* Matti Würthen told us that he can construct directly a prismatic F-crystal in the sense of Bhatt-Scholze out of a Fontaine-Laffaille module. Granted this in the log-version, one could enhance a bit Theorem 6 to

Theorem 17 (?)**.** *Assumption 1 $\Longrightarrow$ on a good model, for any* Spec$(W) \to S$*, the formal connection $(\hat{E}_W, \hat{\nabla}_W)/(\widehat{\overline{X}_W} \setminus \infty_W)$ carries the structure of a prismatic F-crystal.*

4. Étale theorem

Has $\mathcal{O}(S) \subset \mathbb{C}$ and choose $W \subset \mathbb{C}$ for Spec$(W) \to S$ as in Theorem 6. This defines $\bar{K} \subset \mathbb{C}$ with (Grothendieck) $\pi_1(X_\mathbb{C}) \xrightarrow{\cong} \pi_1(X_{\bar{K}})$. By Theorem 3, under Assumption 1, each $\mathbb{L}$ is integral. A p-place $\mathfrak{p}$ of $\mathcal{O}_L$, $p =$char$(\mathbb{F}_q)$, defines $\mathbb{L}_\mathfrak{p}$ on $X_\mathbb{C}$. Keep the same letter $\mathbb{L}_\mathfrak{p}$ for the $\overline{\mathcal{O}_{L_\mathfrak{p}}}$ local system on $X_{\bar{K}}$.

Theorem 18 ([3], Theorem A.21)**.** *Assumption 1 $\Longrightarrow \mathbb{L}_\mathfrak{p}$ defined on $X_{\bar{K}}$ descends to a crystalline local system on X_K with values in $GL_r(\mathbb{Z}_{p^f})$ for some $f|N!$.*

Proof. By the compatibitliy of Faltings p-adic Simpson correspondence on $\widehat{\overline{X}_W}$ [6, Theorem 5] with his Fontaine-Laffaille functor [5] *loc.cit*, calling π_i, $i = 1, \ldots, N$ the $GL_r(\mathbb{Z}_{p^f})$ local systems on X_K defined Proposition 15, where f is a l.c.m. of the periods, which divides $N!$, the $\pi_i/X_{\bar{K}}$ correspond to the Higgs bundles $(V,\theta)_{\bar{K}}$ which are stable, so in particular $\pi_i/X_{\bar{K}}$ is irreducible. Likewise

$$\dim_{\mathbb{C}_p} \mathrm{Hom}(X_{\mathbb{C}_p}, \pi_1, \pi_2) \le \dim_{\mathbb{C}_p} \mathrm{Hom}(X_{\mathbb{C}_p}, (V_1,\theta_1), (V_2,\theta_2)) = 0$$

as Faltings functor from small p-adic local systems to Higgs bundles is faithful. So the number of isomorphism classes of $\pi_i/X_{\bar{K}}$ is the same as the one of π_i/X_K which is N. But by the complex Riemann-Hilbert correspondence, there are precisely N isomorphism classes of $\mathbb{L}_{\mathfrak{p}}$.

□

Remark 19. Theorem 18 via Remark 16 is the way Pila-Shankar-Tsimerman use our work for Shimura varieties of real rank ≥ 2 in their proof of the André-Oort conjecture for those [12, Theorem 1.2].

References

[1] H. Esnault, M. Groechenig, *Cohomologically rigid connections and integrality*, Selecta Mathematica **24** (5) (2018), 4279–4292.
[2] H. Esnault, M. Groechenig, *Rigid connections and F-isocrystals*, Acta Math. **225** 1 (2020), 103–158.
[3] H. Esnault, M. Groechenig, *Frobenius structures and unipotent monodromy at infinity*, Appendix to [12], `https://arxiv.org/abs/2109.08788`
[4] H. Esnault, E. Viehweg, *Logarithmic de Rham complexes and vanishing theorems*, Inventiones math. **86** (1986), 161–194.
[5] G. Faltings, *Crystalline cohomology and p-adic Galois-representations*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 25–80, Johns Hopkins Univ. Press, Baltimore, MD, 1989.
[6] G. Faltings, *A p-adic Simpson correspondence*, Advances in Math. **198** (2005), 847–862.
[7] G. Lan, M. Sheng, K. Zuo, *Semistable Higgs bundles, periodic Higgs bundles and representations of algebraic fundamental groups*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 10, 3053–3112.
[8] G. Lan, M. Sheng, Y. Yang, K. Zuo, *Uniformization of p-adic curves via Higgs-de Rham flows*, Journal für die reine und angewandte Mathematik (Crelles Journal) **747** (2019), 63–108.
[9] A. Langer, *Semistable modules over Lie algebroids in positive characteristic*, Doc. Math. **19** (2014), 509–540.
[10] T. Mochizuki, *Kobayashi-Hitchin correspondence for tame harmonic bundles and an application*, Astéerisque textbf309 (2006), viii+117.
[11] A. Ogus, V. Vologodksy, *Nonabelian Hodge theory in characteristic p*, Publ. math. Inst. Hautes Études Sci. **106** (2007), 1–138.
[12] J. Pila, Ananth Shankar, J. Tsimerman, *Canonical Heights on Shimura Varieties and the André-Oort Conjecture*, `https://arxiv.org/abs/2109.08788`

Hodge theory for non-Archimedean analytic spaces

Vladimir Berkovich

Let K be a non-Archimedean field with a nontrivial discrete valuation whose ring of integers K° contains the field of complex numbers $\mathbf{C}$ which maps isomorphically onto the residue field $\widetilde{K}$, the quotient of K° by the maximal ideal $K^{\circ\circ}$. In work in progress on complex analytic vanishing cycles for formal schemes, I defined integral "etale" cohomology and de Rham cohomology without and with compact support for so-called restricted and bounded K-analytic spaces. In this talk I present de Rham and Hodge theoretic results relating both cohomology.

1. Restricted and bounded K-analytic spaces

The category of restricted K-analytic spaces $K\text{-}\widehat{\mathcal{A}n}$ is the localization of the category of quasicompact special formal schemes flat over K° with respect to proper morphisms that induce isomorphisms between their generic fibers. If $\widehat{X}$ is such a space represented by a formal model $\mathfrak{X}$, the generic fiber $\mathfrak{X}_\eta$ of the latter does not depend on $\mathfrak{X}$ and is the underlying K-analytic space X of $\widehat{X}$. The correspondence $\widehat{X} \mapsto X$ is a faithful functor $K\text{-}\widehat{\mathcal{A}n} \to K\text{-}\mathcal{A}n$ to the category of K-analytic spaces.

Objects of the category of bounded K-analytic spaces $K\text{-}\breve{\mathcal{A}n}$ are pairs (X, Y) consisting of a separated compact strictly K-analytic space X and an open subset $Y \subset X$ such that its complement $X \backslash Y$ is a strictly analytic subdomain of X. A strong morphism $(X', Y') \to (X, Y)$ is a morphism $\varphi : X' \to X$ with $\varphi(Y') \subset Y$. It is called a quasi-isomorphism if it induces an isomorphism $Y' \stackrel{\sim}{\to} Y$. The category $K\text{-}\breve{\mathcal{A}n}$ is the localization of the category of such pairs with strong morphisms between them with respect to quasi-isomorphisms. The correspondence $\breve{Y} = (X, Y) \mapsto Y$ is a faithful functor $K\text{-}\breve{\mathcal{A}n} \to K\text{-}\mathcal{A}n$. By Raynaud theory, one can construct a faithful functor $K\text{-}\breve{\mathcal{A}n} \to K\text{-}\widehat{\mathcal{A}n} : \breve{X} \mapsto \widehat{X}$, and the correspondence $X \mapsto (X, X)$ gives rise to fully faithful functors from the category of separated *compact* strictly K-analytic spaces to both categories.

2. Integral "étale" cohomology

Integral étale cohomology of a restricted or bounded K-analytic space depend on the choice of a universal covering of the complex analytic space $\mathbb{K}_1^\star = (K^{\circ\circ}/(K^{\circ\circ})^2) \backslash\{0\}$, which is isomorphic to the punctured complex plane. Moreover, this space is a classifying space of universal coverings of itself (in a certain precise sense). This allows one to construct a natural faithful functor $\Pi(\mathbb{K}_1^\star) \to \mathrm{G}(K) : \varpi \mapsto K^{(\varpi)}$ from the fundamental groupoid of $\mathbb{K}_1^\star$ to the étale fundamental groupoid of K that takes a point ϖ to an algebraic closure $K^{(\varpi)}$ of K. For example, for any K-analytic space X one can construct local systems $\mathcal{H}^n_{\text{ét}}(\overline{X}, \mathbf{Z}_l) : \varpi \mapsto H^n_{\text{ét}}(X \widehat{\otimes} \widehat{K}^{(\varpi)}, \mathbf{Z}_l)$ and $\mathcal{H}^n(|\overline{X}|, \mathbf{Z}) : \varpi \mapsto H^n(|X \widehat{\otimes} \widehat{K}^{(\varpi)}|, \mathbf{Z})$.

Fact 2.1. One can construct, for every restricted (resp. bounded) K-analytic space $\widehat{X}$ (resp. $\breve{X}$), quasi-unipotent local systems of finitely generated abelian groups $\mathcal{H}^n(\overline{\widehat{X}}, \mathbf{Z})$ (resp. $\mathcal{H}^n_c(\overline{\breve{X}}, \mathbf{Z})$) on $\mathbb{K}_1^\star$, functorial in $\widehat{X}$ (resp. $\breve{X}$) and such that

$$\mathcal{H}^n(\overline{\widehat{X}}, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_l \stackrel{\sim}{\to} \mathcal{H}^n_{\text{ét}}(\overline{X}, \mathbf{Z}_l) \text{ (resp. } \mathcal{H}^n_c(\overline{\breve{X}}, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_l \stackrel{\sim}{\to} \mathcal{H}^n_{c,\text{ét}}(\overline{X}, \mathbf{Z}_l)\text{)} .$$

Functoriality of cohomology with compact support (here and below) has the usual meaning, i.e., it is covariant with respect to open immersions and contravariant with respect to proper morphisms.

3. de Rham theory

Let $\omega^1_{K^\circ}$ be the module of one log differential forms on the formal scheme $\mathrm{Spf}(K^\circ)$ provided with the canonical log structure $K^\circ\backslash\{0\} \hookrightarrow K^\circ$. One has $\omega^1_{K^\circ} = K^\circ d\log(\varpi)$ for any generator ϖ of $K^{\circ\circ}$. A D-module over K° is a finite free K°-module E provided with a connection $\nabla : E \to \omega^1_{K^\circ} \otimes_{K^\circ} E$. For such a connection, there is an associated endomorphism δ of the $\mathbf{C}$-vector space $E/(K^{\circ\circ})$, called the residue of the connection ∇. The D-module E is said to be distinguished if the set $\mathrm{sp}(E)$ of eigenvalues δ lies in $\mathbf{Q}\cap[0,1)$. One then has $E = \bigoplus_{\lambda\in\mathrm{sp}(E)} E(\lambda)$ for D-submodules $E(\lambda)$ with $\mathrm{sp}(E(\lambda)) = \{\lambda\}$.

Fact 3.1. One can construct, for every restricted (resp. bounded) K-analytic space $\widehat{X}$ (resp. $\check{X}$) and every $n \geq 0$, a distinguished D-module over K°, $H^n_{\mathrm{dR}}(\widehat{X}/K^\circ)$ (resp. $H^n_{\mathrm{dR,c}}(\check{X}/K^\circ)$), functorial in $\widehat{X}$ (resp. $\check{X}$) and such that, if X is rig-smooth and compact, then $H^n_{\mathrm{dR}}(X/K^\circ) \otimes_{K^\circ} K \xrightarrow{\sim} H^n_{\mathrm{dR}}(X/K) = R^n\Gamma(X, \Omega^\cdot_{X/K})$.

Furthermore, consider the free K°-module of rank one $L = \mathrm{Hom}_{K^\circ}(K^{\circ\circ}, K^\circ)$. It is generated by the homomorphism $\ell_\varpi : K^{\circ\circ} \to K^\circ$ with $\ell_\varpi(\varpi) = 1$ for any generator ϖ of the ideal $K^{\circ\circ}$, and the K°-algebra $S = \mathrm{Sym}_{K^\circ}(L)[\frac{1}{\ell_\varpi}]$ does not depend on ϖ. The formal complex analytic space $\mathbb{K}^\star$ is the inductive limit of the analytifications $\mathbb{K}^\star_r = \mathrm{Spec}(S/(K^{\circ\circ})^r)^h$. It is provided with the log structure for which the canonical morphism $\mathbb{K}^\star \to \mathrm{Spf}(K^\circ)^h = \varinjlim \mathrm{Spec}(K^\circ/(K^{\circ\circ})^r)^h$ is strict. The element $z = \varpi\ell_\varpi \in M(\mathbb{K}^\star) \subset \mathcal{O}(\mathbb{K}^\star)$ does not depend on ϖ, and it defines a strict morphism $\mathbb{K}^\star \to \mathrm{Spf}(\widehat{\mathcal{K}}^\circ)^h$, where $\mathcal{K}$ is the fraction field of $\mathcal{O}_{\mathbb{C},0}$ and z is a fixed coordinate function on $\mathbb{C}$. One can show that $\omega^1_{K^\circ} \otimes_{K^\circ} \mathcal{O}_{\mathbb{K}^\star} \xrightarrow{\sim} \omega^1_{\mathbb{K}^\star/\widehat{\mathcal{K}}^\circ}$ and that, for a D-module E, the tensor product $\mathcal{E} = E \otimes_{K^\circ} \mathcal{O}_{\mathbb{K}^\star}$ is provided with a connection $\nabla : \mathcal{E} \to \omega^1_{\mathbb{K}^\star/\widehat{\mathcal{K}}^\circ} \otimes_{\mathcal{O}_{\mathbb{K}^\star}} \mathcal{E}$.

A distinguished de Rham structure over $\mathbb{K}^\star$ is a triple $(\mathcal{H}_{\mathbf{Z}}, E, \alpha)$ consisting of a local system of finitely generated abelian groups $\mathcal{H}_{\mathbf{Z}}$ on $\mathbb{K}^\star_1$, a distinguished D-module E, and an isomorphism of flat vector bundles on $\mathbb{K}^\star$

$$\alpha : \mathcal{H}_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathcal{O}_{\mathbb{K}^\star} \xrightarrow{\sim} E \otimes_{K^\circ} \mathcal{O}_{\mathbb{K}^\star} .$$

Then the residue of the connection on E gives rise to a homomorphism of local systems $N : \mathcal{H}_{\mathbf{Q}} \to \mathcal{H}_{\mathbf{Q}}(-1)$ and, for any generator ϖ of $K^{\circ\circ}$, α induces an isomorphism of K°-modules $\alpha^{(\varpi)} : \mathcal{H}_{\mathbf{Z},\hat{\varpi}} \otimes_{\mathbf{Z}} K^\circ \xrightarrow{\sim} E$ for the image $\hat{\varpi}$ of ϖ in $\mathbb{K}^\star_1$.

Fact 3.2. In Fact 3.1, there are distinguished de Rham structures $dRh^n(\widehat{X}/K^\circ) = (\mathcal{H}^n(\overline{\widehat{X}}, \mathbf{Z}), H^n_{\mathrm{dR}}(\widehat{X}/K^\circ), \alpha)$ (resp. $dRh^n_c(\check{X}/K^\circ) = (\mathcal{H}^n_c(\overline{\check{X}}, \mathbf{Z}), H^n_{\mathrm{dR,c}}(\check{X}/K^\circ), \alpha)$).

4. Hodge theory

Given a generator ϖ of $K^{\circ\circ}$, there is an associated ϖ-conjugation on K° that preserves ϖ and acts as the complex conjugation on $\mathbf{C} \subset K^\circ$. It induces a similar ϖ-conjugation on $V_{K^\circ} = V \otimes_{\mathbf{R}} K^\circ$ for any $\mathbf{R}$-vector space V. A ϖ-mixed Hodge structure over K° is a triple $(H_{\mathbf{Z}}, W, F)$ consisting of a finitely generated abelian group $H_{\mathbf{Z}}$, a finite increasing filtration W on $H_{\mathbf{Q}}$, and a finite decreasing filtration F by K°-submodules on H_{K°, which possess the usual properties of a mixed Hodge structure but on the K°-module H_{K° with ϖ-conjugation on it. A K°-submodule P of a D-module E over K° is said to be D-marked if the quotient E/P has no torsion and $P = \bigoplus_{\lambda\in\mathrm{sp}(E)}(P \cap E(\lambda))$.

A distinguished mixed Hodge structure over $\mathbb{K}^\star$ is a tuple

$$\underline{\mathcal{H}} = (\mathcal{H}_{\mathbf{Z}}, (\mathcal{H}_{\mathbf{Q}}, W), (E, F), \alpha) \ ,$$

where $(\mathcal{H}_{\mathbf{Z}}, E, \alpha)$ is a distinguished de Rham structure over $\mathbb{K}^\star$, W is a finite increasing filtration on $\mathcal{H}_{\mathbf{Q}}$ with $N(W_k) \subset W_{k-2}(-1)$, and F is a finite decreasing filtration on E by D-marked K°-submodules with $\nabla(F^k) \subset \omega^1_{K^\circ} \otimes_{K^\circ} F^{k-1}$ such that, for any generator ϖ of $K^{\circ\circ}$, the triple $\underline{\mathcal{H}}_\varpi = (\mathcal{H}_{\mathbf{Z},\hat{\varpi}}, W_{\hat{\varpi}}, F_\varpi)$ is a ϖ-mixed Hodge structure over K°, where F_ϖ is the preimage of F with respect to $\alpha^{(\varpi)}$.

Fact 4.1. For every bounded K-analytic space $\breve{X}$ without boundary, $dRh^n(\widehat{X}/K^\circ)$ extend to functorial distinguished mixed Hodge structures over $\mathbb{K}^\star$

$$\mathcal{H}dg^n(\breve{X}/K^\circ) = (\mathcal{H}^n(\overline{\breve{X}}, \mathbf{Z}), (\mathcal{H}^n(\overline{\breve{X}}, \mathbf{Q}), W), (H^n_{\mathrm{dR}}(\breve{X}/K^\circ), F), \alpha)$$

so that the following is true:

(1) if $h^{p,q} \neq 0$, then $p, q \geq 0$ and, if $\breve{X}$ is smooth, then $p, q \leq n$;
(2) there are canonical isomorphisms $\mathcal{H}^n(|\overline{X}|, \mathbf{Q}) \xrightarrow{\sim} W_0(\mathcal{H}^n(\overline{\breve{X}}, \mathbf{Q}))$;
(3) if X is proper smooth, then the Hodge filtration on $H^n_{\mathrm{dR}}(X/K^\circ)$ is induced by the stupid filtration on the de Rham complex $\Omega^\cdot_{X/K}$, and the Hodge-de Rham spectral sequence $E_1^{p,q} = H^q(X, \Omega^p_{X/K}) \Longrightarrow H^{p+q}_{\mathrm{dR}}(X/K)$ degenerates at E_1.

Fact 4.2. For every smooth bounded K-analytic space $\breve{X}$ of pure dimension n, $dRh^p_c(\breve{X}/K^\circ)$ extend to functorial distinguished mixed Hodge structures over $\mathbb{K}^\star$

$$\mathcal{H}dg^p_c(\breve{X}/K^\circ) = (\mathcal{H}^p_c(\overline{\breve{X}}, \mathbf{Z}), (\mathcal{H}^p_c(\overline{\breve{X}}, \mathbf{Q}), W), (H^p_{\mathrm{dR,c}}(\breve{X}/K^\circ), F), \alpha)$$

so that the following is true (Poincaré duality):

(1) there is a surjective trace mapping $\mathrm{Tr}_{\breve{X}} : \mathcal{H}dg^{2n}_c(\breve{X}/K^\circ) \to \mathbf{Z}_{\mathbb{K}^\star}(-n)$ and, if X is geometrically connected, it is an isomorphism;
(2) for any $0 \leq p \leq 2n$, there is a perfect pairing of distinguished mixed Hodge structures over $\mathbb{K}^\star$

$$\mathcal{H}dg^p_{\mathbf{Q}}(\breve{X}/K^\circ) \otimes \mathcal{H}dg^{2n-p}_{c,\mathbf{Q}}(\breve{X}/K^\circ) \ \to \ \mathcal{H}dg^{2n}_{c,\mathbf{Q}}(\breve{X}/K^\circ) \ \stackrel{\mathrm{Tr}}{\to} \ \mathbf{Q}_{\mathbb{K}^\star}(-n) \ .$$

A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry

Lucas Mann

We develop a p-adic 6-functor formalism in p-adic rigid-analytic geometry. The main motivation for this theory was to provide a new proof of the following Poincaré duality result:

Theorem. *Let C be an algebraically closed non-archimedean field over $\mathbb{Q}_p$ and let X be a proper smooth rigid-analytic variety of pure dimension d over C. Then for all $i \in \mathbb{Z}$ there is a natural perfect pairing*

$$H^i(X_{\mathrm{et}}, \mathbb{F}_p) \times H^{2d-i}(X_{\mathrm{et}}, \mathbb{F}_p) \to \mathbb{F}_p(-d).$$

To develop the 6-functor formalism we will not work directly with $\mathbb{F}_p$-sheaves, as these have a poor behavior under base-change. Instead, we make use of Scholze's primitive comparison theorem (saying that $H^i(X_{\mathrm{et}}, \mathbb{F}_p) \otimes \mathcal{O}_C/p$ and $H^i(X_{\mathrm{et}}, \mathcal{O}_X^+/p)$ are almost isomorphic), which allows us to instead work with certain sheaves of $\mathcal{O}_X^+/p$-modules. In this context it is also very convenient to allow more general spaces than rigid varieties and more general pseudouniformizers than p – more concretely, we will denote by $\mathrm{vStack}_\pi^\sharp$ the category of pairs (X, π) where X is a small v-stack in the sense of [1] together with a map $X \to \mathrm{Spd}\mathbb{Z}_p$ (e.g. this could be a rigid variety) and π is a pseudouniformizer on X (e.g. any pseudouniformizer in C if X lives over a field C). To every (X, π) in $\mathrm{vStack}_\pi^\sharp$ we can now associate a suitable category of "$\mathcal{O}_X^+/\pi$-modules":

Theorem. *There is a unique hypercomplete v-sheaf of ∞-categories*

$$(\mathrm{vStack}_\pi^\sharp)^{\mathrm{op}} \to \mathcal{C}at_\infty, \qquad (X, \pi) \mapsto \mathcal{D}^a_\square(\mathcal{O}_X^+/\pi),$$

such that for every affinoid perfectoid space $X = \mathrm{Spa}(A, A^+)$ which is weakly of perfectly finite type over some totally disconnected perfectoid space we have

$$\mathcal{D}^a_\square(\mathcal{O}_X^+/\pi) = \mathcal{D}^a_\square(A^+/\pi).$$

This somewhat technical result lies at the heart of our work. Let us explain some of the occuring terminology. We first refer the reader to [1] for the definition of totally disconnected perfectoid spaces. Moreover, a map $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ of perfectoid spaces is called *weakly of perfectly finite type* if there is a surjection $A\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}\rangle \to B$ of topological A-algebras. The ∞-category $\mathcal{D}^a_\square(A^+/\pi)$ is the derived ∞-category of solid almost A^+/π-modules which are introduced in [2] and can roughly be described as "complete topological A^+/π-modules modulo almost isomorphisms".

With the ∞-category $\mathcal{D}^a_\square(\mathcal{O}_X^+/\pi)$ at hand, we can now introduce the desired 6-functor formalism:

Definition. Let $f\colon Y \to X$ be a map in $\mathrm{vStack}_\pi^\sharp$.

(a) The *tensor product* $\otimes$ on $\mathcal{D}^a_\square(\mathcal{O}_X^+/\pi)$ is induced from the solid tensor product of modules.

(b) The *internal hom* $\underline{\mathrm{Hom}}$ on $\mathcal{D}^a_\square(\mathcal{O}_X^+/\pi)$ is defined as the right adjoint of the tensor product.

(c) The *pullback functor* $f^*\colon \mathcal{D}^a_\square(\mathcal{O}^+_X/\pi) \to \mathcal{D}^a_\square(\mathcal{O}^+_Y/\pi)$ is the "restriction map" of the sheaf $X \mapsto \mathcal{D}^a_\square(\mathcal{O}^+_X/\pi)$.
(d) The *pushforward functor* $f_*\colon \mathcal{D}^a_\square(\mathcal{O}^+_Y/\pi) \to \mathcal{D}^a_\square(\mathcal{O}^+_X/\pi)$ is the right adjoint of f^*.
(e) If f is "nice" (more precisely *bdcs*, which we will not define properly here) we can define the *shriek pushforward* $f_!\colon \mathcal{D}^a_\square(\mathcal{O}^+_Y/\pi) \to \mathcal{D}^a_\square(\mathcal{O}^+_X/\pi)$. If f is étale then $f_!$ is a left adjoint of f^* and if f is proper then $f_! = f_*$.
(f) If f is bdcs then $f^!\colon \mathcal{D}^a_\square(\mathcal{O}^+_X/\pi) \to \mathcal{D}^a_\square(\mathcal{O}^+_Y/\pi)$ is the right adjoint of $f_!$.

While the precise definition of bdcs maps is rather subtle and will not be discussed here, we remark that every map of rigid-analytic varieties over some base field C is bdcs. The above six functors define all the properties of a 6-functor formalism:

Theorem. *The functors $\otimes$, $\underline{\mathrm{Hom}}$, f^*, f_*, $f_!$ and $f^!$ satisfy the following properties:*

(i) Functoriality: $(f \circ g)_! = f_! \circ g_!$, etc.
(ii) Special Cases: If f is étale then $f^! = f^*$ and if f is proper then $f_! = f_*$.
(iii) Proper base-change: For every bdcs map $f\colon Y \to X$ and map $g\colon X' \to X$ in $\mathrm{vStack}^\sharp_\pi$ with base-change $f'\colon Y' \to X'$ and $g'\colon Y' \to Y$ there is a natural isomorphism $g^* f_! = f'_! g'^*$.
(iv) Projection Formula: For every map $f\colon Y \to X$ in $\mathrm{vStack}^\sharp_\pi$, every $\mathcal{M} \in \mathcal{D}^a_\square(\mathcal{O}^+_X/\pi)$ and every $\mathcal{N} \in \mathcal{D}^a_\square(\mathcal{O}^+_Y/\pi)$ there is a natural isomorphism $f_!(\mathcal{N} \otimes f^*\mathcal{M}) = f_!\mathcal{N} \otimes \mathcal{M}$.

With the full power of a 6-functor formalism at hand, we can now come back to the original motivation of proving Poincaré duality. We first get the following result:

Theorem. *Let $f\colon Y \to X$ be a smooth map of analytic adic spaces over $\mathbb{Q}_p$ and let $\pi \in \mathbb{Q}_p$ be a pseudouniformizer. Then for all $\mathcal{M} \in \mathcal{D}^a_\square(\mathcal{O}^+_X/\pi)$ the natural map $f^*\mathcal{M} \otimes f^!(\mathcal{O}^{+a}_X/\pi) \xrightarrow{\sim} f^!\mathcal{M}$ is an isomorphism and $f^!(\mathcal{O}^{+a}_X/\pi)$ is an invertible object of $\mathcal{D}^a_\square(\mathcal{O}^+_Y/\pi)$.*

By general properties of a 6-functor formalism, the proof of this smoothness result reduces to the case that X is a totally disconnected space and Y is the relative 1-dimensional torus over X. In this case it is an explicit computation, similar to a previous compuations by Faltings [3]. In order to deduce Poincaré duality it remains to see that $\mathcal{L}_f := f^!\mathcal{O}^{+a}_X/p$ satisfies $\mathcal{L}_f = \mathcal{O}^{+a}_Y/p[2d](d)$ in the case that f has pure dimension d. A priori we only know that $\mathcal{L}_f$ is invertible, which is a surprisingly subtle notion in the almost setting (in particular invertible objects need not be locally free in any sense). In order to get a better handle on this sheaf, we will show that it must automatically come from an invertible $\mathbb{F}_p$-sheaf. This follows from the following p-torsion Riemann-Hilbert correspondence.

Definition. Let $\mathrm{vStack}^\sharp_{\pi|p} \subset \mathrm{vStack}^\sharp_\pi$ denote the full subcategory spanned by those pairs (X, π) where $\pi|p$. Given any $X \in \mathrm{vStack}^\sharp_{\pi|p}$ we denote $\mathcal{D}^a_\square(\mathcal{O}^+_X/\pi)^\varphi$

the ∞-category of pairs $(\mathcal{M}, \varphi_M)$, where $\mathcal{M} \in \mathcal{D}^a_\square(\mathcal{O}^+_X/\pi)$ and $\varphi_M\colon \varphi^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$ is an isomorphism. Here φ denotes the Frobenius $x \mapsto x^p$ on $\mathcal{O}^+_X/\pi$.

One easily constructs an adjoint pair of functors

$$- \otimes_{\mathbb{F}_p} \mathcal{O}^{+a}_X/\pi\colon \mathcal{D}_{\mathrm{et}}(X, \mathbb{F}_p)^{\mathrm{oc}} \rightleftarrows \mathcal{D}^a_\square(\mathcal{O}^+_X/\pi)^\varphi : (-)^\varphi,$$

where $\mathcal{D}_{\mathrm{et}}(X, \mathbb{F}_p)^{\mathrm{oc}}$ denotes the ∞-category of overconvergent étale $\mathbb{F}_p$-sheaves on X.

Theorem. *For every $X \in \mathrm{vStack}^\sharp_{\pi|p}$, the functor $- \otimes_{\mathbb{F}_p} \mathcal{O}^{+a}_X/\pi\colon \mathcal{D}_{\mathrm{et}}(X, \mathbb{F}_p)^{\mathrm{oc}} \to \mathcal{D}^a_\square(\mathcal{O}^+_X/\pi)^\varphi$ is fully faithful and induces an equivalence of dualizable objects on both sides.*

One checks that all six functors extend to the φ-module setting and induce a 6-functor formalism on φ-modules. In particular, if $f\colon Y \to X$ is a smooth map as above and $\mathcal{L}_f := f^!\mathcal{O}^{+a}_X/p$ then $\mathcal{L}_f$ is an invertible φ-module and thus of the form $\mathcal{L}_f = \mathcal{L}_{f,0} \otimes_{\mathbb{F}_p} \mathcal{O}^{+a}_Y/p$ for some invertible $\mathbb{F}_p$-sheaf $\mathcal{L}_{f,0}$ on Y. One can now use a "deformation of the normal cone" argument to deduce:

Theorem. *Let $f\colon Y \to X$ be a smooth map of analytic adic space over $\mathbb{Q}_p$ and assume that f is of pure dimension d. Then $f^!\mathcal{O}^{+a}_X/p = \mathcal{O}^{+a}_Y/p[2d](d)$.*

This finishes the proof of Poincaré duality in rigid-analytic geometry. We expect many more applications of our 6-functor formalism, especially when applied to the geometric Langlands program. A natural question to ask is whether one can define similar ∞-categories "$\mathcal{D}^a_\square(\mathcal{O}^+_X)$" and "$\mathcal{D}_\square(\mathcal{O}_X)$", but this is currently unknown (one has to prove a version of the descent result at the beginning of this extended abstract for $\mathcal{D}^a_\square(A^+)$ and $\mathcal{D}_\square(A, A^+)$ for this to work). Another interesting question is whether the above Riemann-Hilbert correspondence can be extended to all étale $\mathbb{F}_p$-sheaves (not just the overconvergent ones) – in fact we conjecture that it can even be extended to v-sheaves.

References

[1] P. Scholze, *Étale cohomology of diamonds*, Preprint (2018).
[2] P. Scholze, D. Clausen, *Lectures on Condensed Mathematics* (2019).
[3] G. Faltings, *p-Adic Hodge Theory*, Journal of the American Mathematical Society **1** (1988), 255–299.

p-adic Jacquet-Langlands correspondence: duality, dimension, nonvanishing

David Hansen
(joint work with Lucas Mann)

Fix a prime p, a finite extension $F/\mathbf{Q}_p$, and an integer $n > 1$. Set $G = \mathrm{GL}_n(F)$, and let D/F be the division algebra of invariant $1/n$. If H is any p-adic Lie group, we write $D(H)$ for the derived category of the abelian category of smooth $\mathbf{F}_p$-representations of H.

By a classical theorem of Jacquet-Langlands and Rogawski, there is a natural bijection $\mathrm{Irr}_{L^2}(G) \xrightarrow{\sim} \mathrm{Irr}(D^\times)$ from the set of square-integrable irreducible smooth representations of G towards the set of irreducible smooth representations of $D^\times$. This bijection is characterized by a simple character identity, and is compatible with twisting and contragredients.

In light of the emerging p-adic Langlands program, it is very natural to hope for some kind of Jacquet-Langlands transfer with p-adic or mod-p coefficients. Here the story cannot be as simple as a bijection between irreducible representations; nevertheless, using p-adic geometry, Scholze defined a canonical Jacquet-Langlands *functor* [1]. More precisely, Scholze defines an exact covariant functor

$$\begin{aligned} \mathrm{Rep}^{\mathrm{sm}}_{\mathbf{F}_p}(G) &\to \mathrm{Sh}\left((\mathbf{P}^{n-1}_{\breve{F}}/D^\times)_{\text{ét}}, \mathbf{F}_p\right) \\ \pi &\mapsto \mathcal{F}_\pi \end{aligned}$$

using the geometry of the Gross-Hopkins period map. For any $\pi \in \mathrm{Rep}^{\mathrm{sm}}_{\mathbf{F}_p}(G)$, the module

$$\mathcal{J}^i(\pi) \overset{\mathrm{def}}{=} H^i_{\text{ét}}(\mathbf{P}^{n-1}_{\mathbf{C}_p}, \mathcal{F}_\pi)$$

is then naturally a smooth $D^\times$-representation. Scholze proved that if π is an *admissible* G-representation, then each $\mathcal{J}^i(\pi)$ is also an admissible $D^\times$-representation, and $\mathcal{J}^i(\pi) = 0$ unless $0 \le i \le 2n-2$.

We prefer to package the $\mathcal{J}^i$'s into a single operation on the level of derived categories. Since $\pi \mapsto \mathcal{F}_\pi$ is exact, this requires no extra effort: the functor $\mathcal{F}_\bullet$ extends naturally to a triangulated functor

$$\begin{aligned} D(G) &\to D\left((\mathbf{P}^{n-1}_{\breve{F}}/D^\times)_{\text{ét}}, \mathbf{F}_p\right) \\ V &\mapsto \mathcal{F}_V, \end{aligned}$$

and for any $V \in D(G)$ we simply define

$$\mathcal{J}(V) = R\Gamma(\mathbf{P}^{n-1}_{\mathbf{C}_p}, \mathcal{F}_V) \in D(D^\times).$$

Of course, if $\pi \in \mathrm{Rep}^{\mathrm{sm}}_{\mathbf{F}_p}(G) = D(G)^\heartsuit$ then $H^i(\mathcal{J}(\pi)) \cong \mathcal{J}^i(\pi)$. Scholze's results on admissibility translate into the fact that $\mathcal{J}$ restricts to a functor $D^b_{\mathrm{adm}}(G) \to D^b_{\mathrm{adm}}(D^\times)$, and one can then ask how this restricted functor interacts with duality.

In the context of mod-p representations of p-adic Lie groups, naive smooth duality is essentially useless. However, for any p-adic Lie group H, Kohlhaase has defined a (derived) duality functor $\mathcal{S}_H \circlearrowright D(H)$ with excellent properties [4]. Our main result is that Kohlhaase's duality interacts with the functor $\mathcal{J}$ in the simplest possible way.

Theorem. *For any $V \in D^b_{\mathrm{adm}}(G)$, there is a natural isomorphism*

$$(\mathcal{J} \circ \mathcal{S}_G)(V) \cong (\mathcal{S}_{D^\times} \circ \mathcal{J})(V)[2-2n](1-n)$$

in $D^b_{\mathrm{adm}}(D^\times)$.

This result has several applications. To state them, recall first that any admissible smooth $\mathbf{F}_p$-representation π of a p-adic Lie group H has a canonically associated *Gelfand-Kirillov dimension* $\dim_H \pi \in \mathbf{Z} \cap [0, \dim H]$. One immediate corollary of our result is that the functors $\mathcal{J}^i(-)$ cannot increase the Gelfand-Kirillov dimension by very much.

Theorem. *Let π be an admissible smooth $\mathrm{GL}_n(F)$-representation, and let $N_\pi \geq 0$ be the least integer such that $\mathcal{J}^i(\pi) = 0$ for all $i > N_\pi$. Then*

$$\dim_{D^\times} \mathcal{J}^i(\pi) \leq \dim_{\mathrm{GL}_n(F)}(\pi) + N_\pi$$

for all i. In particular, since $N_\pi \leq 2n-2$, we have

$$\dim_{D^\times} \mathcal{J}^i(\pi) \leq \dim_{\mathrm{GL}_n(F)}(\pi) + 2n - 2$$

for all π and all i.

Note that the possible increase of dimension by $2n-2$ is rather small compared to the maximal Gelfand-Kirillov dimension in this setting, which is $\dim G = \dim D^\times = n^2[F : \mathbf{Q}_p]$.

Now we specialize to the case $n = 2$, but with $F/\mathbf{Q}_p$ still arbitrary. Set $d = [F : \mathbf{Q}_p]$. Let $B \subset G = \mathrm{GL}_2(F)$ be the upper-triangular Borel subgroup, with $T = F^\times \times F^\times \subset B$ the diagonal maximal torus. Let $\omega : F^\times \to \mathbf{F}_p^\times$ be the character defined as the composition

$$F^\times \xrightarrow{\mathrm{Nm}_{F/\mathbf{Q}_p}} \mathbf{Q}_p^\times \xrightarrow{x \mapsto x|x|} \mathbf{Z}_p^\times \xrightarrow{\mathrm{red}} \mathbf{F}_p^\times.$$

If $\chi_1 \otimes \chi_2 : T \to \overline{\mathbf{F}_p}^\times$ is any smooth character, we can consider the smooth induction $\pi(\chi_1, \chi_2) := \mathrm{Ind}_B^G(\chi_1\omega \otimes \chi_2)$. This is irreducible iff $\chi_1\omega \neq \chi_2$. Moreover, $\mathcal{S}^i_G(\pi(\chi_1, \chi_2)) = 0$ for all $i \neq d$,[1] and $\pi(\chi_1, \chi_2)$ and $\pi(\chi_2, \chi_1)$ are swapped by the functor $\mathcal{S}^d_G(-) \otimes (\chi_1\chi_2\omega \circ \det)$.

Now let $\chi_1 \otimes \chi_2$ be a smooth character of T such that $\chi_1\chi_2^{-1} \neq \omega^{\pm 1}$. It is easy to see that $\mathcal{J}^0(-)$ applied to either $\pi(\chi_1, \chi_2)$ and $\pi(\chi_2, \chi_1)$ is zero. Much less obviously, by a result of Ludwig and Johansson-Ludwig [2, 3], it is known that $\mathcal{J}^2(-)$ applied to either is also zero. With this in mind, we set $\tau(\chi_1, \chi_2) = \mathcal{J}^1(\pi(\chi_1, \chi_2))$. The duality theorem then immediately implies the following dichotomy.

Theorem. *Let $\chi_1 \otimes \chi_2$ be a smooth character of T such that $\chi_1\chi_2^{-1} \neq \omega^{\pm 1}$. Then in the above notation, exactly one of the following two possibilities occurs.*

i. $\tau(\chi_1, \chi_2) = \tau(\chi_2, \chi_1) = 0$.

ii. $\tau(\chi_1, \chi_2)$ *and* $\tau(\chi_2, \chi_1)$ *are both nonzero, and are swapped by the functor* $\mathcal{S}^d_{D^\times}(-) \otimes (\chi_1\chi_2\omega \circ \mathrm{Nm})$. *Moreover,* $\mathcal{S}^i_{D^\times}(-)$ *applied to either is identically zero for any* $i \neq d$.

Finally, we specialize further to the case $F = \mathbf{Q}_p$ (with $p > 3$). In this situation, Paskunas has shown by global methods [5] that $\tau(\chi_1, \chi_2)$ and $\tau(\chi_2, \chi_1)$ cannot *both* be zero! Combining this with the dichotomy estabilished in part ii. of the previous theorem, we deduce the following.

Proposition. *Let $D/\mathbf{Q}_p$ be the quaternion division algebra with $p > 3$, and let $\chi_1 \otimes \chi_2 : \mathbf{Q}_p^\times \times \mathbf{Q}_p^\times \to \mathbf{F}_p^\times$ be any smooth character with $\chi_1\chi_2^{-1} \neq \omega^{\pm 1}$.*

[1] Here and elsewhere, for any $\pi \in \mathrm{Rep}^{\mathrm{sm}}_{\mathbf{F}_p}(H)$, we write $\mathcal{S}^i_H(\pi)$ for the ith cohomology of the complex $\mathcal{S}_H(\pi)$.

Then $\tau(\chi_1, \chi_2)$ is a nonzero $D^\times$-representation with Gelfand-Kirillov dimension one. Moreover, $\tau(\chi_1, \chi_2)$ admits no nonzero finite-dimensional H-stable quotient representation for any open compact subgroup $H \subset D^\times$.

This last claim amounts to the statement that $\tau(\chi_1, \chi_2)$ is "maximally non-semisimple" in some precise sense.

The proof of the duality theorem crucially uses Mann's p-adic six-functor formalism.

References

[1] P. Scholze, *On the p-adic cohomology of the Lubin-Tate tower*, With an appendix by Michael Rapoport. Ann. Sci. Éc. Norm. Sup. (4) 51 (2018), no. 4, 811–863.

[2] J. Ludwig, *A quotient of the Lubin-Tate tower*, Forum Math. Sigma 5 (2017), Paper No. e17, 41 pp.

[3] C. Johansson and J. Ludwig, *A quotient of the Lubin-Tate tower II*, Math. Ann. 380 (2021), no. 1-2, 43–89.

[4] J. Kohlhaase, *Smooth duality in natural characteristic*, Adv. Math. 317 (2017), 1–49

[5] V. Paskunas, *On some consequences of a theorem of J. Ludwig*, to appear in J. Inst. Math. Jussieu

Hirzebruch-Riemann-Roch for rigid analytic spaces

Dustin Clausen

(joint work with Peter Scholze)

Let F be a complete non-archimedean field of characteristic 0, let X be a smooth and proper rigid analytic space over F, and let $V \in \mathrm{Vect}(X)$ be a vector bundle on X. According to a theorem of Kiehl, the total cohomology

$$\oplus_{i\geq 0} H^i(X;V)$$

is a finite-dimensional F-vector space. The individual dimensions $\dim_F H^i(X;V)$ are important to compute, but it turns out that they depend on rather fine properties of the geometric input data (X, V). In particular, they are not invariant under deformations of (X, V).

The situation improves if we pass to the *Euler characteristic*

$$\chi(X, V) := \sum_{i\geq 0} (-1)^i \dim_F H^i(X;V).$$

This quantity is invariant under deformations, and it is therefore reasonable to ask for a simpler formula for it, where the terms themselves are invariant under deformation, and indeed computable. This is provided by the *Hirzebruch-Riemann-Roch (HRR) formula*, which says that

$$\chi(X, V) = \int_X \mathrm{ch}(V) \cdot \mathrm{Td}(T_X).$$

To make sense of the right-hand side requires first of all choosing a cohomology theory on smooth proper rigid analytic spaces over F. In principle everything I say should have an analog for whatever cohomology theory you like, but here let's use the simplest option: *Hodge cohomology.* This is defined as

$$\oplus_{i\geq 0} H^i(X;\Omega^i)$$

where Ω^i is the bundle of i-forms. It will be helpful to view this as the 0^{th} cohomology group of an object in the derived category of F-vector spaces, namely

$$\mathrm{Hdg}(X) := R\Gamma(X;\oplus_i \Omega^i[i]).$$

The usual wedge product on differential forms supplies a multiplication on $\mathrm{Hdg}(X)$.

Next we recall what the *Chern character* $\mathrm{ch}(V)\in H^0\,\mathrm{Hdg}(X)$ and *Todd class* $\mathrm{Td}(V)\in (H^0\,\mathrm{Hdg}(X))^*$ of a vector bundle $V\in \mathrm{Vect}(X)$ are. First, these associations are functorial in (X,V) with respect to pullback. Second, if

$$0\to V'\to V\to V''\to 0$$

is a short exact sequence of vector bundles, then the Chern character is addtitive and the Todd class is multiplicative:

$$\mathrm{ch}(V)=\mathrm{ch}(V')+\mathrm{ch}(V''),$$
$$\mathrm{Td}(V)=\mathrm{Td}(V')\cdot\mathrm{Td}(V'').$$

By the splitting principle for Hodge cohomology, this reduces the determination of these classes to the case of $V=\mathcal{L}$ a line bundle, where they are given by:

$$\mathrm{ch}(\mathcal{L})=e^{c_1\mathcal{L}},$$
$$\mathrm{Td}(\mathcal{L})=\frac{c_1\mathcal{L}}{1-e^{-c_1\mathcal{L}}}.$$

Here the first Chern class $c_1\mathcal{L}$ is a class in the summand $H^1(X;\Omega^1)$ of Hodge cohomology, gotten from the map $d\log:\mathbb{G}_m\to\Omega^1$ by applying H^1 via the isomorphism $\mathrm{Pic}(X)=H^1(X;\mathbb{G}_m)$. Since $\Omega^i=0$ for $i>\dim(X)$, this class $c_1\mathcal{L}$ is nilpotent, and one should read the right-hand side of the above formulas as a formal power series (with $\mathbb{Q}$ coefficients) in $c_1\mathcal{L}$. For example, Bernoulli numbers appear in the Todd class.

Finally we should define the map

$$\int_X : H^0\,\mathrm{Hdg}(X)\to F.$$

In fact, it comes from a map $\mathrm{Hdg}(X)\to F$, defined as follows: take the composition

$$\mathrm{Hdg}(X)\to H^d(X;\Omega^d)\to F,$$

where the first map is projection onto the top-dimensional summand and the second map is the trace map coming from Serre duality on X. For future reference, we note that this map $\mathrm{Hdg}(X)\to F$, composed with the multiplication $\mathrm{Hdg}(X)\otimes_F \mathrm{Hdg}(X)\to\mathrm{Hdg}(X)$, gives rise to a self-duality on $\mathrm{Hdg}(X)\in D(F)$, as a consequence of Serre duality.

Now all of the terms in the HRR formula have been defined. Before describing the method of proof, let me give some historical orientation. The HRR formula in the context of smooth projective varieties over $\mathbb{C}$ was proved by Hirzebruch.

Then it was proved for smooth projective varieties over an arbitrary field by Grothendieck. Then it was proved for arbitrary smooth proper complex analytic spaces by Atiyah-Singer. Then it was proved for smooth proper varieties over an arbitrary field by Fulton.

As far as I'm aware, it had not been proved for smooth proper rigid analytic varieties in general. The difficulty is that none of the above proofs can possibly work in this context. Both Hirzebruch and Grothendieck crucially relied on an embedding into projective space. Atiyah-Singer relied on the existence of an underlying real manifold and applied their index theorem. Fulton relied on Chow's lemma to reduce to the projective case.

On the other hand, we have been developing *condensed mathematics* with a special focus on analytic geometry, and we now have a substantial body of new techniques and, more importantly, new definitions in this context. We viewed the prospect of proving the HRR formula as a test of our formalism.

To describe our proof, let me start with a reminder about Grothendieck's generalization of HRR, since it is important for the proof to work with this generalization even if we're only interested in HRR itself. Grothendieck says we should focus on the association $V \mapsto \mathrm{ch}(V)$, and extend it to a homomorphism

$$\mathrm{ch} : K_0(\mathrm{Perf}(X)) \to H^0 \operatorname{Hdg}(X),$$

where on the left-hand side we have the Grothendieck group of perfect complexes on X.

Both $K_0(\mathrm{Perf}(-))$ and $H^0 \operatorname{Hdg}(-)$ have both a natural pullback and a natural pushforward functoriality for arbitrary maps $f : X \to Y$ of smooth proper rigid spaces. For $K_0 \operatorname{Perf}(-)$, these are simply induced by the corresponding pullback and pushforward on the categorical level of perfect complexes. For $H^0 \operatorname{Hdg}(X)$, or indeed for $\operatorname{Hdg}(X)$ itself, the pullback is the natural one, and the pushforward can be defined as the *dual* to the pullback, via the natural self-duality of $\operatorname{Hdg}(X) \in D(F)$ coming from Serre duality.

Now, the point is that while the homomorphism ch is functorial with respect to pullback, it is not functorial with respect to pushforward. The Grothendieck-Riemann-Roch (GRR) theorem says that pushforward functoriality is salvaged if we modify the pushforward on Hodge cohomology, by conjugating with the Todd class of the tangent bundle. GRR applied to the map $f : X \to *$ reduces to HRR.

Our method for proving the GRR theorem is to find a cohomology theory which is somehow "halfway between" $K(\mathrm{Perf}(-))$ and $\operatorname{Hdg}(X)$. More precisely, we factor ch as

$$K(\mathrm{Perf}(-)) \to HH_{D^\square(F)}(D^\square(X)) \to \operatorname{Hdg}(X).$$

The middle cohomology theory $HH_{D^\square(F)}(D^\square(X))$ is based on our theory of derived solid quasi-coherent sheaves, namely we take the Hochschild homology of $D^\square(X)$ relative to $D^\square(F)$.

Let us make precise what we mean by saying $HH_{D^\square(F)}(D^\square(-))$ lies halfway between $K(\mathrm{Perf}(-))$ and $\operatorname{Hdg}(X)$. First of all $HH_{D^\square(F)}(D^\square(-))$ also has pullback and pushforward functoriality, and just as for $K(\mathrm{Perf}(-))$ these are formally induced by pullback and pushfoward functorialities on categories. Moreover, the first map above commutes with *both* pullback and pushforward, again for essentially formal reasons. This is the sense in which the middle term is close to $K(\mathrm{Perf}(-))$: it has exactly the same functoriality.

On the other hand, the map $HH_{D^{\square}(F)}(D^{\square}(X)) \to \mathrm{Hdg}(X)$ is actually an isomorphism. This is a version of the *Hochschild-Konstant-Rosenberg* (HKR) theorem, and it gives the sense in which $HH_{D^{\square}(F)}(D^{\square}(X))$ is close to $\mathrm{Hdg}(X)$, although they don't have the same pushforward functoriality.

Given this, the GRR problem reduces to the following: we are given a cohomology theory with two different pushfoward functorialities, and we want to show that they are in fact the same. The cohomology theory is $\mathrm{Hdg}(X)$; the first pushfoward functoriality comes from transporting that on $HH_{D^{\square}(F)}(D^{\square}(X))$ via the HKR isomorphism, and the second pushfoward functoriality comes from conjugating the natural pushforward functoriality on $\mathrm{Hdg}(X)$ with the Todd class of the tangent bundle.

To handle this situation we prove an abstract GRR theorem which says that, under suitable axiomatics (notably including the presence of a Künneth formula), to check that two different pushforwards on the same cohomology theory agree, it is enough to check that they have the same Euler classes of line bundles. In our present situation, this equality of Euler classes, and hence the GRR theorem, is a trivial calculation which exactly reduces the definition of the Todd class.

Removing normalization from the construction of integral models of Shimura varieties of abelian type, "motivically"

Yujie Xu

1. Introduction

Let (G, X) be a Shimura datum of Hodge type, i.e. it is equipped with an embedding $(G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), S^{\pm})$, where V is a $\mathbb{Q}$-vector space equipped with a symplectic pairing ψ. The embedding of Shimura data induces an embedding of Shimura varieties $\mathrm{Sh}_K(G, X) \hookrightarrow \mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$ for suitable choices of compact opens $K \subset G(\mathbb{A}_f)$ and $K' \subset \mathrm{GSp}(\mathbb{A}_f)$. For K' sufficiently small, the moduli interpretation of the Siegel modular variety $\mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$ naturally gives rise to an integral model $\mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$. We consider the integral model $\mathscr{S}_K(G, X)$ of $\mathrm{Sh}_K(G, X)$ with hyperspecial (resp. parahoric) level structure at p, as constructed in [6] (resp. [8]), which is initially defined as the normalization of the closure of $\mathrm{Sh}_K(G, X)$ inside $\mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$. In this report, we discuss the author's recent work [16], which shows that this construction can be simplified, in that the normalization step is redundant, and that $\mathscr{S}_K(G, X)$ is simply the closure of $\mathrm{Sh}_K(G, X)$ inside $\mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$.

The construction of smooth (resp. normal) integral models of Shimura varieties plays an important part in the Langlands program, which seeks to describe the zeta function of a Shimura variety in terms of automorphic L-functions [13]. For a more detailed historical exposition, see [6, 7]. On the other hand, special instances of results such as Theorems 1, 2 and 3, to be presented in section 2, have been useful in various other aspects of arithmetic geometry, e.g. in the construction of p-adic L-functions using Euler systems techniques, in the arithmetic intersection theory of special cycles on Shimura varieties and their integral models as in the Kudla-Rapoport program, arithmetic Gan-Gross-Prasad Program etc.

2. Main Results

Our main theorem is the following, which is independent of the choice of symplectic embeddings.

Theorem 1. [16] *For $K \subset G(\mathbb{A}_f)$ small enough and hyperspecial, there exists some $K' \subset \mathrm{GSp}(\mathbb{A}_f)$, such that we have a closed embedding ("the Hodge embedding")*

$$\mathscr{S}_K(G, X) \hookrightarrow \mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$$

More precisely, the normalization step $\mathscr{S}_K(G, X) \xrightarrow{\nu} \mathscr{S}_K^-(G, X)$ is redundant as the closure $\mathscr{S}_K^-(G, X)$ is already smooth.

We remark that experts have long known the finiteness of the Hodge morphism $\mathscr{S}_K(G, X) \to \mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$ for PEL type $\mathscr{S}_K(G, X)$. Indeed, the finiteness of such morphisms boils down to the finiteness of certain H^1_{fppf} (see [17] for details). Note that for a general abelian type integral model, since the normalization step really occurs in the construction of its Hodge type component, removing normalization from the construction of Hodge type models also removes normalization from the construction of any abelian type integral model.

We also state the following two analogues of our Theorem 1. Firstly, in the case of parahoric integral models constructed in [8], we impose mild technical assumptions from [19, 6.18]. We certainly expect this technical assumption from *loc.cit.* to be eventually unnecessary in our Theorem below.

Theorem 2. *Let $G_{\mathbb{Q}_p}$ be residually split, and K a parahoric level structure. There exists a closed embedding $\mathscr{S}_K(G, X) \hookrightarrow \mathscr{S}_{K'}(\mathrm{GSp}, S^{\pm})$ of integral models, for some suitable K'.*
More precisely, the normalization step in the construction of $\mathscr{S}_K(G, X)$ is redundant as the closure $\mathscr{S}_K^-(G, X)$ is already normal.

The second analogue concerns toroidal compactifications of integral models of Hodge type constructed in [14] (the PEL cases were constructed earlier in [10]). Combining our main theorem 1 with an analysis as in [11] on the boundary components of toroidal compactifications, one immediately obtains the following result.

Theorem 3. *Let (G, X) be a Shimura datum of Hodge type. For each $K \subset G(\mathbb{A}_f)$ sufficiently small*[1]*, there exist collections Σ and Σ' of cone decompositions, and*

[1] For parahoric level, assume the same condition as in Theorem 2.

$K' \subset \mathrm{GSp}(\mathbb{A}_f)$, such that we have a closed embedding of toroidal compactifications of integral models

$$\mathscr{S}_K^{\Sigma}(G,X) \hookrightarrow \mathscr{S}_{K'}^{\Sigma'}(\mathrm{GSp},S^{\pm})$$

extending the Hodge embedding of integral models.
In particular, the normalization step is redundant, and $\mathscr{S}_K^{\Sigma}(G,X)$ can be constructed by simply taking the closure of $\mathrm{Sh}_K(G,X)$ inside $\mathscr{S}_{K'}^{\Sigma'}(\mathrm{GSp},S^{\pm})$.

In particular, the Hodge morphism is a closed embedding in the PEL case[2], where we consider integral models constructed by Kottwitz [9] (resp. Rapoport-Zink [15]). Note, however, that the Hodge embedding result holds even for "exotic" PEL type integral models that are not covered by Kottwitz or Rapoport-Zink PEL models–indeed, one can simply consider a moduli space for abelian schemes equipped with an action by a non-maximal order $\mathcal{O}_B$, and the resulting integral model is not necessarily flat (and the level structure may be deeper than parahoric), but this exotic PEL model still embeds into a suitable Siegel integral model.

To see the result in this (broadly understood) PEL case, whether "exotic" or not, recall that the Hodge morphism is simply given by forgetting the $\mathcal{O}_B$-action on an abelian scheme $\mathcal{A}$, where B is a semisimple $\mathbb{Q}$-algebra attached to the PEL moduli problem. Let $T^{(p)}(\mathcal{A})$ be the prime-to-p Tate module. For a point on $\mathscr{S}_K(G,X)$, the corresponding level structure

$$\eta : V \otimes \mathbb{A}_f^p \xrightarrow{\sim} T^{(p)}(\mathcal{A}) \otimes \mathbb{A}_f^p$$

is compatible with the $\mathcal{O}_B$-actions on both sides. This shows that the $\mathcal{O}_B$-action on $\mathcal{A}$ is already determined by η, and hence the Hodge morphism is an embedding. Strictly speaking, this argument only applies when we let the level structure away from p go to zero, but it is not hard to deduce Theorem 1 from this (see [18] for details).

In the general Hodge type case, the mod p points of the integral model $\mathscr{S}_K(G,X)$ can be interpreted as abelian varieties equipped with certain "mod p Hodge cycles", which come from reduction mod p of Hodge cycles in characteristic zero. These mod p Hodge cycles are indeed motivated cycles in characteristic p in the sense of [1, 2, 3]. We denote the mod p Hodge cycle at a mod p point $x \in \mathscr{S}_K(G,X)$ by a tuple $(s_{\alpha,\ell,x}, s_{\alpha,\mathrm{cris},x})$, which is determined by either its ℓ-adic étale component or its cristalline component (see Proposition below). This is analogous to the case of Hodge cycles in characteristic 0, which are determined by either their étale components or their de Rham components [5].

More specifically, let $\mathscr{S}_K^-(G,X)$ be the closure of $\mathrm{Sh}_K(G,X)$ in $\mathscr{S}_{K'}(\mathrm{GSp},S^{\pm})$. By a criterion in [7] (resp. [19]), two mod p points $x,x' \in \mathscr{S}_K(G,X)(k)$ that have the same image in $\mathscr{S}_K^-(G,X)(k)$ are equal if and only if $s_{\alpha,\mathrm{cris},x} = s_{\alpha,\mathrm{cris},x'}$. Therefore, to show that the normalization morphism is an isomorphism, it reduces to proving the following statement on cohomological tensors:

Proposition. [16] $s_{\alpha,\ell,x} = s_{\alpha,\ell,x'} \Longrightarrow s_{\alpha,\mathrm{cris},x} = s_{\alpha,\mathrm{cris},x'}$.

[2]This terminology can cause ambiguity. The reader should note that we mean Kottwitz' (hyperspecial) or Rapoport-Zink's (parahoric) PEL moduli problem.

By a CM lifting result on $\mathscr{S}_K(G,X)$ due to [7] (resp. [19]), these cohomological tensors lift, up to G-isogenies, to Hodge cycles on CM abelian varieties. A key observation is that when two mod p points $x, x' \in \mathscr{S}_K(G,X)(k)$ map to the same image in $\mathscr{S}_K^-(G,X)(k)$, they can be CM-lifted using the same torus, whose cocharacter induces the filtration on the Dieudonné modules $\mathbb{D}(\mathcal{A}_x) = \mathbb{D}(\mathcal{A}_{x'})$ which then identifies the filtrations on the Dieudonné modules associated to CM-liftable mod p points, giving rise to an isogeny *in characteristic zero* between the two CM lifts. This observation allows us to match up the mod p cristalline tensors using the input from ℓ-adic étale tensors, precisely due to the rationality of Hodge cycles in characteristic zero and the existence of an isogeny lift in characteristic zero.

It is worth pointing out that, in the case where the aforementioned cohomological tensors are algebraic–for example, at points where the Hodge conjecture is true–the family of Hodge cycles (tensors) s_α that naturally lives over the Hodge type integral model $\mathscr{S}_K(G,X)$ becomes a flat family of algebraic cycles over $\mathscr{S}_K(G,X)$. In this case, $s_{\alpha,\ell,x} = s_{\alpha,\ell,x'}$ implies that the two algebraic cycles corresponding to the two ℓ-adic cycles are ℓ-adic cohomologically equivalent, hence numerically equivalent, and we only need to show that they are also cristalline-cohomologically equivalent. Recall that the Grothendieck Standard Conjecture D says that numerical equivalence and cohomological equivalence agree for algebraic cycles. The proof of the Proposition thus follows from a cristalline realisation of this Standard Conjecture D, for points on the integral model of Hodge type and their associated cristalline tensors, which are mod p Hodge cycles.

In general, without the algebraicity of s_α, the Proposition is essentially a weaker form of a conjecture due to Yves André on the rationality of motivated cycles in characteristic $p > 0$ [1, 3], which has implications for the Tannakian category constructed by Langlands-Rapoport [12].

References

[1] Y. André, *Pour une théorie inconditionnelle des motifs*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **83** (1996), no. 1, 5–49.

[2] Y. André, *Déformation et spécialisation de cycles motivés*, J. Inst. Math. Jussieu **5** (2006), no. 4, 563–603.

[3] Y. André, *Cycles de Tate et cycles motivés sur les variétés abéliennes en caractéristique $p > 0$*, J. Inst. Math. Jussieu **5** (2006), no. 4, 605–627.

[4] P. Deligne, *Travaux de Shimura*, Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389, 123–165. Lecture Notes in Math., Vol. 244, 1971.

[5] P. Deligne, *Hodge Cycles on Abelian Varieties*, Hodge Cycles, Motives, and Shimura Varieties, pp.9–100, Springer Berlin Heidelberg, 1982.

[6] M. Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), 967–1012.

[7] M. Kisin, mod *p points on Shimura varieties of abelian type*, J. Amer. Math. Soc. **30** (2017), no. 3, 819–914.

[8] M. Kisin and G. Pappas, *Integral models of Shimura varieties with parahoric level structure*, Publ. Math. Inst. Hautes Études Sci. **128** (2018), no. 4, 121–218.

[9] R. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), 373–444.

[10] K.W. Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, London Mathematical Society Monographs Series, vol.36, Princeton University Press, Princeton, NJ, 2013.
[11] K.W. Lan, *Closed immersions of toroidal compactifications of Shimura varieties*, to appear in Math. Res. Letters, 2019.
[12] R. P. Langlands and M. Rapoport, *Shimuravarietäten und Gerben*, J. Reine Angew. Math. **378** (1987), 113–220.
[13] R. P. Langlands, *Some contemporary problems with origins in the Jugendtraum*, Mathematical developments arising from Hilbert problems (Proc. Sympos. Pure Math., Vol. XXVIII, Northern Illinois Univ., De Kalb, Ill., 1974), 1976, pp. 401–418.
[14] K. Madapusi Pera, *Toroidal compactifications of integral models of Shimura varieties of Hodge type*, Ann. Sci. Éc. Norm. Supér. (4) **52** (2019), no. 2, 393–514.
[15] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, vol. 141, Princeton University Press, Princeton, NJ, 1996.
[16] Y. Xu, *Normalization in integral models of Shimura varieties of Hodge type*, arXiv: 2007.01275, 2021.
[17] Y. Xu, *Finiteness of fppf cohomology*, arXiv: 2110.04287, 2021.
[18] Y. Xu, *On the Hodge embedding for PEL type integral models of Shimura varieties*, arXiv: 2111.04209, 2021.
[19] R. Zhou, *Mod-p isogeny classes on Shimura varieties with parahoric level structure*, Duke Math. J. **169** (2020), no. 15, 2937–3031.

Moduli spaces in p-adic non-abelian Hodge theory

Ben Heuer

1. Introduction

We report on work in progress on p-adic analogues of the moduli theoretic aspects of the non-abelian Hodge correspondence:

For a smooth projective complex variety X and $x \in X(\mathbb{C})$, the non-abelian Hodge correspondence due to Corlette–Simpson is an equivalence of categories

$$\mathrm{Rep}^{\mathrm{f.d.}}_{\mathbb{C}}(\pi_1(X,x)) \xleftrightarrow{\sim} \left\{\begin{matrix}\text{semistable Higgs bundles on } X\\ \text{with vanishing Chern classes}\end{matrix}\right\}$$

which on isomorphism classes induces a homeomorphism

$$\mathcal{M}_{\mathrm{Betti}}(\mathbb{C}) \to \mathcal{M}_{\mathrm{Dol}}(\mathbb{C})$$

between the Betti and Dolbeault moduli space [6, §7].

If X is instead a connected proper smooth rigid space over a complete algebraically closed non-archimedean extension K of $\mathbb{Q}_p$, with fixed base point $x \in X(K)$, one is led to ask whether there is an analogous equivalence

$$\mathrm{Rep}^{\mathrm{f.d.}}_{K}(\pi_1^{\text{ét}}(X,x)) \xleftrightarrow{\sim} \left\{\begin{matrix}\text{Higgs bundles on } X\\ \text{satisfying...??}\end{matrix}\right\}$$

where the category on the left is now given by the continuous finite dimensional representations of the étale fundamental group. While an equivalence as above is not yet known in general, instances of such functors have been constructed in many special cases, starting with the works of Deninger–Werner [1], Faltings [2], and more recently Liu–Zhu [4]. We note that the equivalence is in general expected to be non-canonical, namely it will depend on choices of a $B_{\mathrm{dR}}^{+}/\xi^2$-lift of X (related to a splitting of the Hodge–Tate sequence of X) and an exponential on K.

Our goal is to show that there are also analogues of the homeomorphism of moduli spaces, and that this can help study the open question what condition has to be imposed on the "Higgs" side of the correspondence, as well as to give a more canonical statement. Our basic idea is to find a non-abelian version of Scholze's pro-étale approach to the Hodge–Tate spectral sequence [5, §3]: For this one first shows that for the projection of sites $\nu : X_{\text{proét}} \to X_{\text{ét}}$ there are isomorphisms

$$R^i\nu_*\mathcal{O} = \Omega_X^i \quad \text{for all } i \geq 0$$

(up to a Tate twist which we ignore), then one considers the Leray sequence of ν.

2. v-vector bundles on X

On the "Betti" side of the correspondence, our key object of study will be vector bundles on Scholze's pro-étale site $X_{\text{proét}}$, i.e. locally free modules with respect to the completed structure sheaf. This is because there is a fully faithful functor

$$\mathrm{Rep}_K^{\text{f.d.}}(\pi_1^{\text{ét}}(X,x)) \hookrightarrow \mathrm{VB}_v(X)$$

from the representations in question, which are "global" objects on X, to pro-étale vector bundles, which we can study locally on X. It is constructed in terms of the pro-finite-étale universal cover of X, which is a pro-étale $\pi_1^{\text{ét}}(X,x)$-torsor

$$\widetilde{X} \to X,$$

by interpreting any continuous representation $\pi_1^{\text{ét}}(X,x) \to \mathrm{GL}(V)$ on a finite dimensional K-vector space V as a descent datum for the vector bundle $V \otimes_K \mathcal{O}_{\widetilde{X}}$ on $\widetilde{X}$. This construction is essentially due to Faltings, as pro-étale vector bundles are equivalent to his notion of "generalised representations". Moreover, pro-étale vector bundles on X are the same as v-vector bundles on (the diamond associated to) X, which is the language we shall use in the following. But either category is in general strictly bigger than the full subcategory of analytic vector bundles.

Towards the p-adic non-abelian Hodge correspondence, the general expectation is now that this should be induced by an equivalence of categories

$$\Big\{v\text{-vector bundles on } X\Big\} \to \Big\{\text{Higgs bundles on } X\Big\}.$$

To get a moduli theoretic approach to this conjectural correspondence, we are thus led to consider for any $n \in \mathbb{N}$ the following moduli functors fibred in groupoids, defined on the category of affinoid perfectoid spaces over K:

$$\mathcal{B}\mathrm{un}_{v,n} : T \mapsto \{v\text{-vector bundles on } X \times T \text{ of rank } n\}$$
$$\mathcal{H}\mathrm{iggs}_n : T \mapsto \{\text{Higgs bundles on } X \times T \text{ of rank } n\}$$

Here the notion of Higgs bundles on the adic space $X \times T$ requires some explanation, because it requires a notion of Kähler differentials: To get a good local definition of the latter, we introduce the category of **smoothoid spaces**, which consist of analytic adic spaces Y that locally admit a smooth morphism of adic spaces to a perfectoid space over K. For any smoothoid adic space Y, we then set

$$\Omega^i_Y := R^i\nu_*\mathcal{O}_Y$$

where $\nu : Y_v \to Y_{\text{ét}}$ is the natural morphism of sites. On $Y = X \times T$ one can show that this agrees with the pullback of Ω^i_X, so gives a sensible notion of differentials.

Our first main result towards a p-adic Hodge correspondence is then:

Theorem 2.1. For any smoothoid space Y and $n \in \mathbb{N}$, there is a canonical and functorial isomorphism of sheaves of pointed sets on $Y_{\text{ét}}$

$$R^1\nu_*\mathrm{GL}_n(\mathcal{O}) \xrightarrow{\sim} (M_n(\mathcal{O}) \otimes \Omega^1_Y)^{\wedge=0}/\mathrm{GL}_n(\mathcal{O}).$$

Here the condition "$\wedge = 0$" on the right is the Higgs field condition, so we can think of the right hand side as the isomorphism classes of Higgs bundles on Y up to sheafification. We therefore regard the theorem as a "sheafified non-abelian Hodge correspondence". This is closely related to Faltings "local correspondence" (in a more general setting of smoothoids). The main difference of the above to Faltings' result is that it is completely canonical and functorial, at the expense of forgetting about morphisms, i.e. only relating isomorphism classes. We will now explain why this functoriality is enough to derive a much stronger, geometric incarnation of the correspondence, in two special cases: line bundles and curves.

Before, we note that the theorem is also interesting for understanding the difference between étale and pro-étale vector bundles on adic spaces. For example, it implies that the moduli functor $\mathcal{H}\mathrm{iggs}_n$ is a v-stack (for $\mathcal{B}\mathrm{un}_{v,n}$ this is clear).

3. The case of $\mathbb{G}_m$

In the case of $\mathrm{GL}_1 = \mathbb{G}_m$, Theorem 2.1 says that there is an isomorphism

$$R^1\nu_*\mathbb{G}_m = \Omega^1_Y.$$

Let us now for simplicity assume that X is projective, and switch from the moduli stacks $\mathcal{B}\mathrm{un}_{n,v}$ and $\mathcal{H}\mathrm{iggs}_n$ to the associated moduli functors $\mathbf{Bun}_{v,n}$ and $\mathbf{Higgs}_n$ obtained by passing to sets of isomorphism classes and v-sheafifying. Explicitly, for $n = 1$, this yields the v-Picard functor of X, namely the functor on perfectoid test objects

$$\mathbf{Pic}_v := \mathbf{Bun}_{v,1} : T \mapsto \mathrm{Pic}_v(X \times T)/\mathrm{Pic}_v(T).$$

We similarly define the étale Picard functor $\mathbf{Pic}_{\text{ét}}$ using étale line bundles instead. Then using Theorem 2.1, one can deduce the following (see [3] for more details):

Theorem 3.1. Both $\mathbf{Pic}_{\text{ét}}$ and $\mathbf{Pic}_v$ are represented by smooth rigid group varieties, and there is a short exact sequence in the étale topology

$$0 \to \mathbf{Pic}_{\text{ét}} \to \mathbf{Pic}_v \to H^0(X, \Omega^1_X) \otimes \mathbb{G}_a \to 0.$$

Choices of an exponential and a B^+_{dR}/ξ^2-lift of X lead to a splitting on K-points.

This shows that $\mathbf{Higgs}_1$ and $\mathbf{Bun}_{v,1}$ can be regarded as twists of each other over the base $H^0(X, \Omega^1) \otimes \mathbb{G}_a$. This is the perspective which we expect to generalise.

4. The Hitchin fibration

By a classical construction due to Hitchin, there is a natural morphism

$$\mathcal{H} : \mathbf{Higgs}_n \to \mathcal{B}_n := \bigoplus_{i=1}^{n} H^0(X, \mathrm{Sym}^i \Omega^1) \otimes_K \mathbb{G}_a$$

defined by sending a Higgs bundle (E, θ) to the characteristic polynomial of θ. As a consequence of Theorem 2.1, one can define an analogous morphism

$$\widetilde{\mathcal{H}} : \mathbf{Bun}_{v,n} \to \mathcal{B}_n$$

on the v-bundle side that we call the "twisted Hitchin fibration". By restricting to the locus of representations, we obtain a natural morphism of rigid spaces

$$\widetilde{\mathcal{H}} : \mathbf{Hom}(\pi_1^{\text{ét}}(X, x), \mathrm{GL}_n) \to \mathcal{B}_n.$$

that is both an analogue of the Hitchin fibration and of the Hodge–Tate morphism.

Assume now that X is a curve of genus $g \geq 2$. In this case, the Hitchin fibration $\mathcal{H}$ is well known to be a Picard-torsor generically on $\mathcal{B}_n$: More precisely, there is a relative "spectral curve"

$$f : \mathcal{X} \to \mathcal{B}_n$$

which has smooth connected fibres over a Zariski-dense open locus $\mathcal{B}_n^\circ$. Let us denote by $\mathcal{X}^\circ$, $\mathbf{Higgs}_n^\circ$, ... the respective fibres over $\mathcal{B}_n^\circ$, then $\mathcal{H}^\circ : \mathbf{Higgs}_n^\circ \to \mathcal{B}_n^\circ$ is a split torsor under $\mathbf{Pic}_{\mathcal{X}^\circ|\mathcal{B}_n^\circ,\text{ét}}$, the relative étale Picard functor of f, which agrees with the analytification of the algebraic Picard functor. Our main result is:

Theorem 4.1.

(1) Over the locus $\mathcal{B}_n^\circ$, the twisted Hitchin fibration

$$\widetilde{\mathcal{H}} : \mathbf{Bun}_{v,n}^\circ \to \mathcal{B}_n^\circ$$

is a non-split $\mathbf{Pic}_{\mathcal{X}^\circ|\mathcal{B}_n^\circ,\text{ét}}$-torsor.

(2) In particular, $\mathbf{Bun}_{v,n}^\circ$ is represented by a smooth rigid space.

(3) Any B_{dR}^+/ξ^2-lift $\mathbb{X}$ of X induces an étale local system $\mathbf{L}_\mathbb{X}$ on $\mathcal{B}_n^\circ$ and a canonical isomorphism of rigid spaces

$$\mathbf{Bun}_{v,n}^\circ \times_{\mathcal{B}_n^\circ} \mathbf{L}_\mathbb{X} \xrightarrow{\sim} \mathbf{Higgs}_n^\circ \times_{\mathcal{B}_n^\circ} \mathbf{L}_\mathbb{X}.$$

This is our desired geometric incarnation of the p-adic non-abelian Hodge correspondence in terms of an isomorphism of rigid analytic moduli spaces. While we do not know of any immediate analogue of this theorem in the complex setting, we remark that any choice of an exponential on K induces a splitting of the morphism $\mathbf{L}_\mathbb{X} \to \mathcal{B}_n^\circ$ on K-points, from which we obtain a homeomorphism

$$\mathbf{Bun}_{v,n}^\circ(K) \xrightarrow{\sim} \mathbf{Higgs}_n^\circ(K)$$

in close analogy to the complex case. In upcoming work, we will also investigate the behaviour of the locus of representations under this isomorphism.

References

[1] C. Deninger, A. Werner. *Vector bundles on p-adic curves and parallel transport*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 4, 553–597.
[2] G. Faltings. *A p-adic Simpson correspondence*, Adv. Math. **198** (2005), no. 2, 847–862.
[3] B. Heuer. *Diamantine Picard functors of rigid spaces*, arxiv:2103.16557, 2021.
[4] R. Liu, X. Zhu. *Rigidity and a Riemann-Hilbert correspondence for p-adic local systems*, Invent. Math. **207** (2017), no. 1, 291–343.
[5] P. Scholze. *Perfectoid spaces: a survey*, Current developments in mathematics 2012, 193–227, Int. Press, Somerville, MA, 2013
[6] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. II*, Inst. Hautes Études Sci. Publ. Math. **80** (1994), 5–79 (1995).

Six operations and holonomicity for $\widehat{\mathcal{D}}$-modules on rigid analytic spaces

Andreas Bode

1. Background

Let X be a smooth complex algebraic variety. The theory of $\mathcal{D}_X$-modules, modules over the sheaf of differential operators on X, is an important and powerful tool in geometric representation theory. Before turning to the p-adic analytic analogue, we quickly summarize this complex algebraic picture.

There are three finiteness conditions we can consider in studying $\mathcal{D}$-modules: firstly, we can ask for coherence over $\mathcal{D}_X$. Secondly, we can use the fact that $\mathcal{O}_X$ is contained in $\mathcal{D}_X$ and restrict to $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules. It can be shown that an $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module is the same as a vector bundle with an integrable connection, so we will usually speak of 'integrable connections' rather than $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules to simplify the terminology.

The third finiteness condition is in many ways the most remarkable: one can develop a dimension theory for coherent $\mathcal{D}_X$-modules (for example, by interpreting the homological grade as the codimension) and prove that for any non-zero coherent $\mathcal{D}_X$-module $\mathcal{M}$, we have $\dim(\mathcal{M}) \geq \dim X$ (Bernstein's inequality). We say that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is holonomic if it is of minimal dimension, i.e. if $\dim(\mathcal{M}) \leq \dim X$.

Any integrable connection is holonomic, so that we have the following inclusions:

$$\{\text{integrable connections}\} \subset \{\text{holonomic modules}\} \subset \{\text{coherent modules}\}.$$

The importance of the notion of holonomicity stems from the following result: we can define six functors on the derived category $\mathrm{D}^b(\mathcal{D}_X)$ – the tensor product $\otimes^{\mathbb{L}}_{\mathcal{O}_X}$, the duality functor $\mathbb{D}$, and for any morphism f between smooth varieties the direct image f_+, the inverse image f^+, and their 'shriek' versions $f_!$ and $f^!$.

Theorem 1. *Holonomicity is preserved by all six functors, i.e. each of the functors preserves the full subcategory* $\mathrm{D}^b_{\mathrm{hol}}(\mathcal{D})$.

Moreover, when restricted to $\mathrm{D}^b_{\mathrm{hol}}(\mathcal{D})$, the six operations obey the usual laws for composition, base change, projection and adjunction, turning this framework into (a version of) a six functor formalism. In fact, if we impose additional regularity conditions, the Riemann–Hilbert correspondence (almost) identifies these six functors with the 'classical' six functor formalism for constructible sheaves.

In the following sections, we discuss a p-adic analytic analogue of this theory by providing a derived categorical framework in which all six functors can be defined, and arrive at a suitable notion of holonomicity.

2. $\widehat{\mathcal{D}}$-modules on rigid analytic spaces

Now let K be a finite extension of $\mathbb{Q}_p$ with valuation ring R and uniformizer $\pi \in R$, and let X be a smooth rigid analytic K-variety. In [1], Ardakov–Wadsley introduced the sheaf $\widehat{\mathcal{D}}_X$, whose sections over affinoids are given by a Fréchet completion of the usual ring of differential operators, and studied the abelian category of coadmissible $\widehat{\mathcal{D}}_X$-modules, which is a direct analogue of the category of coherent $\mathcal{D}$-modules in the algebraic theory.

For a sensible derived picture, we need to find a category which is retaining the analytic flavour of the theory, so that e.g. the functor $\widehat{\otimes}_{\mathcal{O}_X}$ makes sense, while at the same time exhibiting good behaviour from a homological standpoint, so that a good theory of the derived category and derived functors exists. We propose for this the category of complete bornological $\widehat{\mathcal{D}}_X$-moduiles.

Definition. *A (convex) bornology on a K-vector space V is a collection $\mathcal{B}$ of 'bounded' subsets such that*

(i) $\{v\} \in \mathcal{B}$ *for any* $v \in V$.
(ii) $\mathcal{B}$ *is closed under finite unions.*
(iii) if $B \in \mathcal{B}$ *and* $B' \subset B$*, then* $B' \in \mathcal{B}$.
(iv) if $B \in \mathcal{B}$ *and* $\lambda \in K$*, then* $\lambda \cdot B \in \mathcal{B}$.
(v) if $B \in \mathcal{B}$*, then* $R \cdot B \in \mathcal{B}$.

The notion of bornological vector spaces has been in use in functional analysis for quite a long time, and recently Bambozzi, Ben-Bassat and Kremnitzer have studied bornologies in the context of nonarchimedean analytic geometry, see [3].

We say that a bornological vector space is complete if its bornology is generated by π-adically complete R-submodules. In particular, any complete bornological vector space can be written as a colimit of Banach spaces, and the completed projective tensor product of Banach spaces can be extended to give the category $\widehat{\mathcal{B}c}_K$ of complete bornological spaces a closed symmetric monoidal structure.

Moreover, $\widehat{\mathcal{B}c}_K$ is a quasi-abelian category in the sense of Schneiders [7], so that we can form its derived category. It has enough projectives, and its left heart (its 'abelian envelope') has enough injectives.

It is quite straightforward to verify that the Fréchet structure of sections of $\widehat{\mathcal{D}}_X$ can be used to give $\widehat{\mathcal{D}}_X$ the structure of a sheaf of complete bornological K-algebras. It follows that we can form the unbounded derived category $\mathrm{D}(\widehat{\mathcal{B}c}(\widehat{\mathcal{D}}_X))$

of complete bornological $\widehat{\mathcal{D}}_X$-modules, on which analogues of all six functors can be defined without any difficulty.

The following theorem ensures that Ardakov-Wadsley's work fits into this framework.

Theorem 2 ([4, Theorem 1.2]). *Let X be a smooth rigid analytic K-space. Then there is an exact fully faithful functor*

$$\{\text{coadmissible } \widehat{\mathcal{D}}_X-\text{modules}\} \to \widehat{\mathcal{B}}c(\widehat{\mathcal{D}}_X).$$

3. $\mathcal{C}$-complexes and holonomicity

The category $\mathrm{D}(\widehat{\mathcal{B}}c(\widehat{\mathcal{D}}_X))$ is way too large to afford good behaviour under the six functors, so we need to restrict to certain full subcategories playing the roles of $\mathrm{D}^b_{\mathrm{coh}}(\mathcal{D})$ and $\mathrm{D}^b_{\mathrm{hol}}(\mathcal{D})$.
We first transpose the definition of coadmissibility to the derived setting. Let X be a smooth affinoid. Then $\widehat{\mathcal{D}}(X) \cong \varprojlim \mathcal{D}_n(X)$ for certain Noetherian Banach algebras $\mathcal{D}_n(X)$, and a complete bornological $\widehat{\mathcal{D}}_X$-module is coadmissible if and only if it is obtained by glueing coherent $\mathcal{D}_n$-modules. Analogously, the category of $\mathcal{C}$-complexes is defined to be the full subcategory of $\mathrm{D}(\widehat{\mathcal{B}}c(\widehat{\mathcal{D}}_X))$ obtained by glueing $\mathrm{D}^b_{\mathrm{coh}}(\mathcal{D}_n)$ in a suitable sense. A bounded $\mathcal{C}$-complex turns out to be the same as a bounded complex with coadmissible cohomology, but we caution the reader that unbounded $\mathcal{C}$-complexes also exist.
The category of $\mathcal{C}$-complexes behaves similarly to $\mathrm{D}^b_{\mathrm{coh}}(\mathcal{D})$ with respect to the six functors.

Theorem 3 ([4, Theorem 1.3]). *Let $f : X \to Y$ be a morphism between smooth rigid analytic K-spaces.*

(i) If f is smooth, then $f^!$ preserves $\mathcal{C}$-complexes.
(ii) If f is projective, then f_+ preserves $\mathcal{C}$-complexes.
(iii) $\mathbb{D}$ preserves $\mathcal{C}$-complexes, and $\mathbb{D}^2(\mathcal{M}) \cong \mathcal{M}$ for any $\mathcal{C}$-complex $\mathcal{M}$.

Theorem 4 ([4, Theorem 7.12]). *Let X be a smooth rigid analytic space, and let $f : X \to \mathrm{Sp}\ K$ be the structure morphism. Then*

$$f_+\mathcal{O}_X \cong \mathrm{R}f_*(\Omega^\bullet_{X/K})[\dim X].$$

The theory of holonomicity turns out to be more subtle. While it is possible to develop a dimension theory for coadmissible $\widehat{\mathcal{D}}$-modules and prove an analogue of Bernstein's inequality, the resulting modules of minimal dimension (which we call 'weakly holonomic') can be quite pathological (see [2]). We follow Caro's strategy for arithmetic $F-\mathscr{D}$-modules ([5]) by introducing a notion of holonomicity which has many stability properties already built into the definition.

Definition. *We say that a $\mathcal{C}$-complex $\mathcal{M}$ on a smooth rigid analytic K-variety X is*

(i) 0-holonomic if for any smooth $f : X' \to X$ and any divisor $Z \subset X'$, $\mathrm{R}\underline{\Gamma}_Z(f^!\mathcal{M})$ is a $\mathcal{C}$-complex.

(ii) *n-holonomic if it is $(n-1)$-holonomic, and for any f, Z as above, $\mathbb{D}\mathrm{R}\underline{\Gamma}_Z(f^!\mathcal{M})$, $\mathbb{D}f^!\mathcal{M}$ are $(n-1)$-holonomic.*

(iii) ***holonomic*** *if it is n-holonomic for all $n \geq 0$.*

One of the main results of [2] is that any integrable connection is 0-holonomic, and by reducing to the test case of smooth divisors it is then easy to prove that the notion of holonomicity developed above is indeed reasonable.

Theorem 5. *Any integrable connection is holonomic.*

Using results on closed embeddings and several diagram chases, many stability properties of holonomicity now become purely formal. We summarize everything in a conjecture.

Conjecture.

(i) *Holonomicity is stable under $f^!$, f^+, $\mathbb{D}$, as well as $f_+ = f_!$ whenever f is projective.*

(ii) *Holonomicity is stable under tensor products $\widehat{\otimes}^{\mathbb{L}}_{\mathcal{O}_X}$.*

(iii) *Holonomicity allows for the usual composition, bsae change and projection rules.*

(iv) *Holonomic modules are algebraic (i.e. obtained from a coherent $\mathcal{D}$-module via completion) and weakly holonomic.*

We remark that the restriction for direct images is necessary, as we have seen above that $f_+\mathcal{O}_X$ need not be a $\mathcal{C}$-complex in general. One expects however that rather than requiring being projective, properness is enough.

Conjectures (i) and (iii) are fairly straightforward – we stress in particular that no bornological subtleties arise as long as we remain within the realm of $\mathcal{C}$-complexes, so that everything can be done quite formally. The tensor product of (ii) should be possible by mimicking Caro–Tsuzuki's work in [6] and proving that holonomic $\widehat{\mathcal{D}}_X$-modules are 'constructible' out of suitable integrable connections. This would also provide finite length results which then resolve (iv) by understanding the irreducible case.

References

[1] K. Ardakov, S. Wadsley, *$\widehat{\mathcal{D}}$-modules on rigid analytic spaces I*, J. Reine Angew. Math. **747** (2019), 221–276.

[2] K. Ardakov, A. Bode, S. Wadsley, *$\widehat{\mathcal{D}}$-modules on rigid analytic spaces III: weak holonomicity and operations*, Compos. Math. **157** (2021), no. 12, 2553–2584.

[3] F. Bambozzi, O. Ben-Bassat, K. Kremnitzer, *Stein domains in Banach algebraic geometry*, J. Funct. Anal. 274 (2018), no. 7, 1865–1927.

[4] A. Bode, *Six operations for $\widehat{\mathcal{D}}$-modules on rigid analytic spaces*, (2021), preprint.

[5] D. Caro, *$\mathscr{D}$-modules arithmétiques surholonomes*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 1, 141–192.

[6] D. Caro, N. Tsuzuki, *Overholonomicity of overconvergent F-isocrystals over smooth varieties*, Ann. of Math. (2) **176** (2012), no. 2, 747–813.

[7] J.-P. Schneiders, *Quasi-abelian categories and sheaves*, Mém. Soc. Math. Fr. (N.S.), no. 76, 1999.

Geometrization of the local Langlands correspondence ([1])

Laurent Fargues
(joint work with Peter Scholze)

1. Semi-simple parameters

Let E be a non-archimedean local field with residue field $\mathbb{F}_q$. Thus, either E is a finite degree extension of $\mathbb{Q}_p$ or $\mathbb{F}_q((\pi))$. Let G be a reductive group over E. Choose $\ell \neq p$ and let Λ be a $\mathbb{Z}_\ell$-algebra (with the assumption $\ell \gg 0$ with some explicit bound when ℓ is not invertible). We denote ${}^LG$ for the Langlands dual of G over Λ. Let π be a smooth representation of $G(E)$ with coefficients in Λ that is Schur irreducible i.e. $\mathrm{End}(\pi) = \Lambda$. We construct its *semi-simple Langlands parameter*

$$\varphi_\pi : W_E \longrightarrow {}^LG$$

that is a pseudo-representation, not a "real morphism" in general, but a usual semi-simple representation/parameter when $\Lambda = \overline{\mathbb{F}}_\ell$ or $\overline{\mathbb{Q}}_\ell$.

2. The stack Bun_G

To construct those Langlands parameters we need some geometry. For this let $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ be the category of $\overline{\mathbb{F}}_q$-perfectoid spaces equipped with the v-topology, an analog of the fpqc topology. We note $* = \mathrm{Spa}(\overline{\mathbb{F}}_q)$ for the final object of the v-topos (that is not representable by a perfectoid space). We introduce the moduli stack of G-bundles on the curve

$$\mathrm{Bun}_G \longrightarrow *.$$

This sends $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ to the groupoid of principal G-bundles on X_S where X_S is the "relative curve" attached to S, an E-adic space. More precisely, the collection of curves (the one studied by Fargues and Fontaine) $(X_{k(s),k(s)^+})_{s\in S}$ attached to each point $s \in S$ and the associated perfectoid residue field $k(s)$, can be "put in family" as an E-adic space X_S functorial in S. Here is the first result we obtain.

Theorem 1. *The moduli space* Bun_G *is an Artin v-stack (ℓ-cohomologically) smooth of dimension* 0. *More precisely, its diagonal is represented by a locally spatial diamond and there exists a locally spatial diamond U together with a (ℓ-coho.) smooth morphism $U \to \mathrm{Bun}_G$ such that $U \to *$ is (ℓ-coho.) smooth.*

The geometry of Bun_G is described in the following way. Let us note $\breve{E}$ for the completion of the maximal unramified extension of E with it's Frobenius σ. We denote $B(G) = G(\breve{E})/\sigma$-conjugation for the Kottwitz set of twisted conjugacy classes, $b \sim gbg^{-\sigma}$, that can be identified with the set of isomorphism classes of G-isocrystals. For each $b \in G(\breve{E})$ we can construct a principal G-bundle $\mathcal{E}_b$ on X_S. Let us recall the following.

Theorem 2 (Fargues-Fontaine, Fargues)**.** *If $S = \mathrm{Spa}(F, F^+)$ is a geometric point i.e. F is an algebraically closed perfectoid field, there is a bijection*

$$\begin{aligned} B(G) &\xrightarrow{\sim} H^1_{\acute{e}t}(X_{F,F^+}, G) \\ [b] &\longmapsto [\mathcal{E}_b]. \end{aligned}$$

Thus, we have

$$B(G) = |\mathrm{Bun}_G|.$$

We obtain the following further results concerning the geometry of Bun_G.

Theorem 3.

(1) The first Chern class of a principal G-bundle is a locally constant function with connected fibers. It induces a bijection

$$\pi_0(\mathrm{Bun}_G) \xrightarrow{\sim} \pi_1(G)_\Gamma$$

where $\pi_1(G)$ is Borovoi's fundamental group and $\Gamma = \mathrm{Gal}(\overline{E}|E)$.

(2) There is a "nice" Harder-Narasimhan stratification for which each stratum is locally closed and the semi-stable locus, Bun_G^{ss}, is open.

(3) Each connected component contains a unique semi-stable point and we have

$$\mathrm{Bun}_G^{ss} = \coprod_{[b]\ basic} \left[\, * / \underline{G_b(E)}\, \right]$$

where G_b for b basic is an inner form of G ($G_1 = G$ for example).

(4) More generally, for each $[b] \in B(G)$, the associated HN-stratum is

$$\left[\, * / \widetilde{G}_b \,\right]$$

where $\widetilde{G}_b$ is a group diamond such that $\widetilde{G}_b = \widetilde{G}_b^0 \rtimes \underline{G_b(E)}$ where G_b is an inner form of a Levi subgroup of G^* and $\widetilde{G}_b^0$ is a "unipotent diamond", a successive extension of positive Banach-Colmez spaces.

3. The triangulated category $D_{lis}(\mathrm{Bun}_G, \Lambda)$

We define and study a triangulated category

$$D_{lis}(\mathrm{Bun}_G, \Lambda)$$

that is $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ when Λ is torsion and a subcategory of $D_{\text{proét}}(\mathrm{Bun}_G, \Lambda_\blacksquare)$ (solid complexes) in general. For each $[b] \in B(G)$ let us denote $i^b : [*/\widetilde{G}_b] \hookrightarrow \mathrm{Bun}_G$ for the inclusion of the associated HN stratum. There is then a functor

$$(i^b)^* : D_{lis}(\mathrm{Bun}_G, \Lambda) \longrightarrow D_{lis}([*/\widetilde{G}_b], \Lambda) \underbrace{=}_{\ell \neq p} D(G_b(E), \Lambda)$$

where $D(G_b(E), \Lambda)$ is the derived category of smooth representations of $G_b(E)$ with coefficients in Λ.

In particular, via $(i^1)^*$ and $(i^1)_!$, $D(G(E), \Lambda)$ is a direct factor in $D_{lis}(\mathrm{Bun}_G, \Lambda)$. Thus, we can see the "classical smooth representation theory of $G(E)$" as a "direct factor" of $D_{lis}(\mathrm{Bun}_G, \Lambda)$. This is a general principle in our work. *In some* *sense the good objects for the local Langlands program are not the usual smooth representations of* $G(E)$ *but rather objects of* $D_{lis}(\mathrm{Bun}_G, \Lambda)$. In fact more general notions from classical representation theory extend like the notion of admissible and finite type representations. In fact we have the following theorem. We take $\Lambda = \overline{\mathbb{Q}}_\ell$ to simplify.

Theorem 4. *For* $A \in D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$,

(1) *A is a compact object if and only if it has finite support and for each $[b] \in B(G)$, $(i^b)^*A \in D(G_b(E), \Lambda)$ is bounded with finite type cohomology.*
(2) *A is ULA (universally locally acyclic) if and only if for each $[b] \in B(G)$ and K a compact open subgroup of $G_b(E)$, $[(i^b)^*A]^K$ is a bounded complex of finite dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces.*
(3) *There is an involution $\mathbb{D}_{BZ} : D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega \xrightarrow{\sim} D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$ that extends the usual Bernstein-Zelevinsky involution on $D^b(G(E), \overline{\mathbb{Q}}_\ell)$.*

4. The moduli of Langlands parameters

The other object that shows up in this work, after Bun_G, is

$$\mathrm{LocSys}_{\widehat{G}} \longrightarrow \mathrm{Spec}(\mathbb{Z}_\ell),$$

the moduli of Langlands parameters. One has

$$\mathrm{LocSys}_{\widehat{G}} = \left[Z^1(W_E, \widehat{G})/\widehat{G}\right]$$

where $Z^1(W_E, \widehat{G})$ is an infinite disjoint union of finite type affine $\mathbb{Z}_\ell$-schemes. Moreover, $\mathrm{LocSys}_{\widehat{G}} \to \mathrm{Spec}(\mathbb{Z}_\ell)$ is locally complete intersection of dimension 0. We are mainly interested in $\mathrm{Coh}(\mathrm{LocSys}_{\widehat{G}})$, the bounded derived category of coherent sheaves with quasicompact support and its subcategory $\mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}})$ of perfect complexes.

5. The spectral action

Using an enhanced version of Beilinson-Drinfeld/Vincent Lafforgue formalism of factorization sheaves and a Geometric Satake correspondence for the B_{dR}-affine Grassmanian we obtain the following.

Theorem 5. *One can construct a monoidal action of* $\mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}})$ *on* $D_{lis}(\mathrm{Bun}_G, \mathbb{Z}_\ell)$.

This monoidal action allows us to construct the semi-simple Langlands parameters in families in the following way. We note $\mathfrak{Z}$ for the center of a category. It induces morphisms

$$\mathcal{O}(Z^1(W_E, \widehat{G}))^{\widehat{G}} = \underbrace{\mathfrak{Z}(\mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}}))}_{\text{spectral stable center}} \longrightarrow \underbrace{\mathfrak{Z}(D_{lis}(\mathrm{Bun}_G, \mathbb{Z}_\ell)}_{\text{geometric stable center}} \xrightarrow{(i^1)^*} \underbrace{\mathfrak{Z}(D(G(E), \mathbb{Z}_\ell))}_{\text{classical Bernstein center}} .$$

The composite of those two morphisms constructs the semi-simple local Langlands correspondence.

6. The geometrization conjecture

We finally state the categorical form of the geometrization conjecture. For this we suppose that G is quasi-split. Let U be the unipotent radical of a Borel subgroup and $\psi : U(E) \to \overline{\mathbb{Z}}_\ell^\times$ be a non-degenerate character. We note

$$\mathcal{W}_\psi = (i^1)_!\big(\text{c-ind}_{U(E)}^{G(E)}\ \psi\big),$$

the so-called Whittaker sheaf. We note $\mathcal{F} * A$ for the spectral action of $\mathcal{F}$ on A.

Conjecture. *The functor*

$$\begin{aligned}\mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}}/\overline{\mathbb{Z}}_\ell) &\longrightarrow D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)^\omega \\ \mathcal{F} &\longmapsto \mathcal{F} * \mathcal{W}_\psi\end{aligned}$$

extends to an equivalence

$$\mathrm{Coh}_{Nilp}(\mathrm{LocSys}_{\widehat{G}}/\overline{\mathbb{Z}}_\ell) \xrightarrow{\sim} D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$$

compatible with the spectral action.

In the preceding the index "$Nilp$" means Arinkin-Gaitsgory singular support condition. This means that the singular support of our complexes is contained in the nilpotent cone, the case when this is in the zero set inside the nilpotent cone being the subcategory $\mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}})$. In our situation this condition becomes automatic when we invert ℓ but we need it if we work integrally.

References

[1] Fargues, L. and Scholze, P., "Geometrization of the local Langlands correspondence", https://arxiv.org/abs/2102.13459

A comparison theorem for ordinary *p*-adic modular forms

Ana Caraiani
(joint work with Elena Mantovan, James Newton)

Let $G = \mathrm{GL}_2/\mathbb{Q}$, with the standard Borel subgroup $B = T \ltimes N \subset G$. Assume that $K = K^pK_p \subset G(\mathbb{A}_f)$ is a sufficiently small compact open subgroup. Let $Y_K/\mathbb{Q}$ be the modular curve of level K, with (minimal) compactification X_K. Let C be the completion of an algebraic closure of $\mathbb{Q}_p$. Faltings's Hodge–Tate decomposition [7] gives a short exact sequence

$$0 \to H^1(X_{K,C}, \omega^{-k}(-D))(k+1) \to H^1_{\mathrm{et},c}(Y_{K,C}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} C \to H^0(X_{K,C}, \Omega^1 \otimes \omega^k) \to 0,$$

where $k \geq 0$, $\mathcal{V}_k$ is the local system on Y_K corresponding to the representation $\mathrm{Sym}^k(\mathbb{Z}_p^2)$ of K_p, ω is the usual automorphic line bundle on X_K, and D is the divisor of the cusps. This short exact sequence, which relates the étale and coherent cohomology of modular curves, admits a Hecke and Galois equivariant splitting, and there are also versions for usual and interior cohomology. In this talk, I described how to obtain an integral version of the Hodge–Tate decomposition for families of ordinary p-adic modular forms, interpolating both the weight k and the level K_p.

Assume that $K_p = G(\mathbb{Z}_p)$ from now on. To interpolate the étale cohomology groups of Y_K with compact support, one considers the following *ordinary completed cohomology* group, which was originally introduced by Hida:

$$\widetilde{H}_c^{1,\mathrm{ord}} := \varprojlim_n \left(\varinjlim_m H^1_{\mathrm{et},c}(Y_{K_1(p^m),\overline{\mathbb{Q}}}, \mathbb{Z}/p^n)^{\mathrm{ord}} \right). \tag{1}$$

Here, we let $K_1(p^m) := \{k \in K \mid k \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{p^m}\}$, and the superscript ord denotes the ordinary part, which can be obtained by applying Hida's idempotent to the cohomology group at each finite level. This cohomology group is a p-adically admissible representation of $T(\mathbb{Z}_p)$ and can also be recovered as the (derived) *ordinary part* functor, as defined by Emerton [6], applied to the full completed cohomology of the modular curve with tame level K^p.

Later, it will be convenient to view (1) as a module over the Iwasawa algebra $\Lambda := \mathbb{Z}_p[\![T(\mathbb{Z}_p)]\!]$ - in some sense, we work on a "dual side". Explicitly, we set $\Lambda_n := \Lambda \otimes \mathbb{Z}/p^n$ and define a functor $\mathcal{M}_n$, taking smooth representations of $T(\mathbb{Z}_p)$ with $\mathbb{Z}/p^n$-coefficients to Λ_n-modules, given by $H \mapsto \mathcal{M}_n(H) := \mathrm{Hom}_{\Lambda_n}(H^\vee, \Lambda_n)$. We define

$$\widetilde{\mathcal{M}}_c^1 := \varprojlim_n \mathcal{M}_n \left(\widetilde{H}_c^{1,\mathrm{ord}}/p^n \right). \tag{2}$$

Using the fact that the ordinary part of cohomology with compact support is concentrated in degree 1, one can show that $\widetilde{\mathcal{M}}_c^1$ is a finite projective Λ-module.

To interpolate the coherent cohomology groups on either side of Faltings's decomposition, one considers Hida theory (for coherent cohomology in degree 0) and higher Hida theory (for coherent cohomology in degree 1), as recently defined by Boxer–Pilloni in the case of the modular curve [2]. For this, we consider the ordinary locus $\mathfrak{X}^{\mathrm{ord}}$ (at level K) as a formal scheme over Spf $\mathbb{Z}_p$; it is in fact affine. Over it, one can construct the *Igusa tower*, a pro-étale $T(\mathbb{Z}_p)$-torsor $\pi : \mathfrak{Ig} \to \mathfrak{X}^{\mathrm{ord}}$ obtained by trivialising the graded pieces of the slope filtration of the universal p-divisible group. We also have a diagram

$$\begin{array}{ccc} & \mathfrak{Ig} & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathfrak{Ig} & & \mathfrak{Ig} \end{array}, \tag{3}$$

where $p_1 = \widetilde{F}$ is (up to an isomorphism) the "canonical lift" of the relative Frobenius, and where p_2 is an isomorphism. We have the cohomological correspondence $U_p = \frac{1}{p}\mathrm{tr}_{\widetilde{F}} : p_2^* \mathcal{O}_{\mathfrak{Ig}} \to p_1^! \mathcal{O}_{\mathfrak{Ig}}$ and we set

$$H^0 := H^0(\mathfrak{Ig}, \mathcal{J})^{U_p-\mathrm{ord}}, \text{ with } \mathcal{J} \subset \mathcal{O}_{\mathfrak{Ig}} \text{ the ideal of the cusps.}$$

The space of global sections of $\mathfrak{Ig}$ is what Hida calls the space of p-adic modular forms, and we are restricting to the subspace of cusp forms that are also ordinary at p. On the dual side, we have the universal character $\kappa : T(\mathbb{Z}_p) \to \Lambda^\times$, which induces the sheaf $\omega^\kappa := \pi_*(\mathcal{J} \otimes_{\mathbb{Z}_p} \Lambda)^{T(\mathbb{Z}_p)}$ on the ordinary locus $\mathfrak{X}^{\text{ord}}$. We obtain a finite projective Λ-module

$$M^0 := H^0(\mathfrak{X}^{\text{ord}}, \omega^\kappa)^{U_p-\text{ord}} \simeq \varprojlim_n \mathcal{M}_n(H^0/p^n).$$

The module M^0 interpolates the spaces of ordinary p-adic cusp forms of varying weights (which are recovered by specialising the universal character), and its finiteness over Λ is not immediate: it follows from classicality in weight $k \geq 3$. Projectivity is easier to establish: it follows from the affineness of $\mathfrak{X}^{\text{ord}}$.

In [2], Boxer and Pilloni define a second Λ-adic family using higher coherent cohomology. One can reverse the roles of the two projections in diagram 3 and obtain the cohomological correspondence $\widetilde{F} : p_1^*\mathcal{O}_{\mathfrak{I}\mathfrak{g}} \to p_2^!\mathcal{O}_{\mathfrak{I}\mathfrak{g}}$. We consider

$$H_c^1 := H_c^1(\mathfrak{I}\mathfrak{g}, \mathcal{J})^{\widetilde{F}-\text{ord}},$$

using Hartshorne's definition of coherent cohomology with compact support, and, dually,

$$M_c^1 := H_c^1(\mathfrak{X}^{\text{ord}}, \omega^\kappa)^{\widetilde{F}-\text{ord}} \simeq \varprojlim_n \mathcal{M}_n(H_c^1/p^n).$$

Using affineness and classicality in weight $k \leq -1$, Boxer and Pilloni show that M_c^1 is a finite projective Λ-module.

Theorem 1. *Set $\Lambda_{\mathcal{O}_C} := \Lambda \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C$. There is a Hecke-equivariant short exact sequence of finite projective $\Lambda_{\mathcal{O}_C}$-modules*

$$0 \to M_c^1 \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C \xrightarrow{(1)} \widetilde{M}_c^1 \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C \xrightarrow{(2)} M^0 \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C \to 0,$$

where the map (1) is equivariant with respect to the map that sends $t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right) \in T(\mathbb{Z}_p)$ to $t^{w_0} = \left(\begin{smallmatrix} t_2 & 0 \\ 0 & t_1 \end{smallmatrix}\right)$ and the map (2) is equivariant with respect to the map that sends $t \in T(\mathbb{Z}_p)$ to $t_1^2\kappa(t)$.

Remark.

(1) Theorem 1 was originally proved by Ohta in [8] and reproved by Cais [4], but, in our set-up, the connection to higher Hida theory is more direct.

(2) After inverting p, one can consider the finite slope case, which is more general than the ordinary, slope $= 0$ case. In this case, one can describe finite slope p-adic modular forms defined using modular symbols in terms of Coleman (map (2), due to [1] and [5]) and higher Coleman theory (map (1), more recently defined and studied by Rodríguez Camargo [9]).

To establish Theorem 1, we first construct a sequence of morphisms

$$H_c^1 \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C \to \widetilde{H}_c^{1,\text{ord}} \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C \to H^0 \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C. \tag{4}$$

For this, we apply a version of Scholze's primitive comparison (almost) isomorphism $\widetilde{H}_c^{1,\text{ord}} \hat{\otimes}_{\mathbb{Z}_p} \mathcal{O}_C \overset{a}{\simeq} H^1_{\text{qproet}}(\mathcal{X}_{N(\mathbb{Z}_p)}, \widehat{\mathcal{I}}^+)$, where $\mathcal{X}_{N(\mathbb{Z}_p)} = \varprojlim_{N(\mathbb{Z}_p) \subset K_p} \mathcal{X}_{K^pK_p}$ as diamonds. We then use the Hodge–Tate period map $\pi : \mathcal{X}_{N(\mathbb{Z}_p)} \to |\mathbb{P}^1/N(\mathbb{Z}_p)|$ at this intermediate level and pull back the Bruhat decomposition from $\mathbb{P}^1/N(\mathbb{Z}_p)$. We obtain a decomposition of $\mathcal{X}_{N(\mathbb{Z}_p)}$ into a closed, anti-canonical subset, and an open neighbourhood of the canonical locus. The morphisms in (4) arise from the corresponding long exact sequence for cohomology in a closed subset. The dynamic of the Hecke operator U_p simplifies this long exact sequence when taking ordinary parts, and a comparison result in the style of Bhatt–Morrow–Scholze [3] allows us to recover the coherent cohomology of the Igusa tower. Finally, to prove that the dual sequence is (almost) exact, one uses the classicality results in weight $k \geq 3$ for M^0 and in weight $k \leq -1$ for M^1_c.

References

[1] F. Andreatta, A. Iovita and G. Stevens, *Overconvergent modular sheaves and modular forms for* GL_2/F. Israel J. Math. **201** (2014), no. 1, 299–359.

[2] G. Boxer and V. Pillloni, *Higher Hida and Coleman theories on the modular curve*, arXiv:2002.06845.

[3] B. Bhatt, M. Morrow and P. Scholze, *Integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci **128**, 219–397.

[4] B. Cais, *The geometry of Hida families II: Λ-adic (φ, Γ)-modules and Λ-adic Hodge theory*, Compos. Math. **154** (2018), no. 4. 719–760.

[5] P. Chojecki, D. Hansen and C. Johansson, *Overconvergent modular forms and perfectoid Shimura curves*, Doc. Math **22** (2017), 191–262.

[6] M. Emerton, *Ordinary parts of admissible representations of p-adic reductive groups II. Derived functors*, Astérisque **331** (2010), 403–459.

[7] G. Faltings, *Hodge–Tate structures and modular forms*, Math. Ann. **278** (1987), No. 1-4, 133–149.

[8] M. Ohta, *On the p-adic Eichler–Shimura isomorphism for Λ-adic cusp forms*, J. Reine Angew. Math. **463** (1995) 49–98.

[9] J.E. Rodríguez Camargo, *Eichler–Shimura maps for the modular curve*, arXiv:2102.13099.

Huber's GOS-formula and dimensions of representations of quaternion algebras

Johannes Anschütz

(joint work with Arthur-César Le Bras)

Let p be a prime and let E be a local field with residue field $\mathbb{F}_q$ of characteristic p. Let $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ be the category of perfectoid spaces over $\overline{\mathbb{F}}_q$. To any $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ let X_S be the relative Fargues–Fontaine curve associated with S and E. We let Bun be the small v-stack on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ sending S to the groupoid of vector bundles on X_S.

Assume now that S is a small v-stack and $\mathcal{E} \in \mathrm{Bun}(S)$ fiberwise of positive slopes. Then we define the "positive Banach–Colmez space" $\mathrm{BC}(\mathcal{E})$ associated with $\mathcal{E}$ as the small v-sheaf over S by

$$(T \to S) \mapsto H^0(X_T, \mathcal{E}_T),$$

and the "negative Banach–Colmez space" $\mathrm{BC}(\mathcal{E}^\vee)$ via

$$(T \to S) \mapsto H^1(X_T, \mathcal{E}^\vee_T).$$

There exists a natural pairing

$$\alpha\colon \mathrm{BC}(\mathcal{E}) \times \mathrm{BC}(\mathcal{E}^\vee) \to [S/\underline{E}]$$

because the stack $[S/\underline{E}]$ indentifies with the stack of extensions of $\mathcal{O}$ by $\mathcal{O}$. Given a non-trivial character $\psi\colon \underline{E} \to \Lambda$, where $\Lambda = \overline{\mathbb{F}}_\ell$ with $\ell \neq p$, the pullback $\alpha^*\mathcal{L}_\psi$ of the associated local system $\mathcal{L}_\psi$ on $[S/\underline{E}]$, serves as the kernel for a "Fourier transform"

$$\mathcal{F}_\psi\colon D_{ét}(\mathrm{BC}(\mathcal{E}),\Lambda) \to D_{ét}(\mathrm{BC}(\mathcal{E}^\vee),\Lambda).$$

Theorem 1 (A.-Le Bras)**.** *The functor $\mathcal{F}_\psi$ is an equivalence, commuting with Verdier duality.*

The statement holds more generally for a class of "flat coherent sheaves" on the Fargues–Fontaine curve (still of fiberwise positive slopes), and then generalizes Ramero's Fourier transform for the adic affine line $\mathbb{A}^{1,\mathrm{ad}}_{\mathbb{C}_p}$.

The rest of the talk focused on the case that $\mathcal{E} = \mathcal{O}(1)$ for $S = \mathrm{Spd}(\overline{\mathbb{F}}_q)$. As[1]

$$\mathrm{BC}(\mathcal{O}(1)) \cong \mathrm{Spd}(\overline{\mathbb{F}}_q((t)))$$

the fibers of the Fourier transform are computed by compactly supported cohomology of sheaves on the open, rigid-analytic unit disc $\mathbb{D}_C$ for some non-archimedean, algebraically closed field $C/\overline{\mathbb{F}}_q$. Theorem 1 implies that the stalks of the Fourier transform of an object $A \in D_{ét}(\mathrm{BC}(\mathcal{O}(1)),\Lambda)$ with perfect stalks, are again perfect. The example of $\mathrm{BC}(\mathcal{O}(1))$ is particularly interesting as

$$(\mathrm{BC}(\mathcal{O}(1)) \setminus \{0\})/\underline{E^\times} \cong \mathrm{Div}^1_{\overline{\mathbb{F}}_q} := \mathrm{Spd}(\breve{E})/\varphi^{\mathbb{Z}}$$

and Λ-local systems $\mathbb{L}$ on $\mathrm{Div}^1_{\overline{\mathbb{F}}_q}$ are equivalent to Λ-representations of the Weil group W_E of E. Pulling back $\mathbb{L}$ to $\mathrm{BC}(\mathcal{O}(1)) \setminus \{0\}$, then !-extending to $\mathrm{BC}(\mathcal{O}(1))$ and doing the Fourier transform yields an object $\tilde{B} \in D_{ét}(\mathrm{BC}(\mathcal{O}(-1),\Lambda)$, which naturally descents to an object $B \in D_{ét}(\mathrm{BC}(\mathcal{O}(-1))/\underline{E^\times},\Lambda)$. Now,

$$\mathrm{BC}(\mathcal{O}(-1)) \setminus \{0\}/\underline{E^\times} \cong \mathrm{BC}(\mathcal{O}(1/2)) \setminus \{0\}/\underline{D^\times}$$

for the divison quaternion algebra D over E. If $\mathbb{L}$ is of dimension 2 and irreducible, then Fargues' conjecture and results of Frenkel/Gaitsgory/Vilonen on the geometric Langlands correspondence for GL_n suggest that the restriction of B to the punctured BC-space descents further to $[\mathrm{Spd}(\overline{\mathbb{F}}_q)/\underline{D^\times}]$ and corresponds there to the Jacquet-Langlands correspondent π_σ of the local Langlands correspondent of the Weil representation σ associated with $\mathbb{L}$.

Theorem 2 (A.-Le Bras)**.** *Assume $\mathbb{L}$ is irreducible of rank 2. If B_x is the stalk of B for a geometric point $x\colon \mathrm{Spd}(C,\mathcal{O}_C) \to \mathrm{BC}(\mathcal{O}(1/2)) \setminus \{0\}/\underline{D^\times}$, then $B_x[1]$ is concentrated in degree 0 and has dimension $\dim_\Lambda(\pi_\sigma)$.*

If the Swan conductor $\mathrm{sw}(\sigma)$ of σ cannot be lowered by twisting with a character, then more concretely

$$\dim_\Lambda(\pi_\sigma) = \begin{cases} 2q^{\mathrm{sw}(\sigma/2} & \text{if } \mathrm{sw}(\sigma) \text{ is even,} \\ (q+1)q^{\mathrm{sw}(\sigma)/2-1/2} & \text{if } \mathrm{sw}(\sigma) \text{ is odd.} \end{cases}$$

[1] Here $\overline{\mathbb{F}}_q((t))$ should better be replaced by the field of norms for the completed maximal abelian extension of E.

The proof of this theorem applies Huber's Grothendieck-Ogg-Shafarevich formula to the punctured open rigid-analytic unit disc $\mathbb{D}_C^*$ to reduce the question to a determination of Swan conductors near the origin and the boundary, and then uses Ramero's result on convexity of the discriminant function of a local system to calculate these Swan conductors. To get an expression in terms of the Swan conductor $\mathrm{sw}(\sigma)$ (and not just in terms of the Swan conductor of $\sigma_{|\mathrm{Gal}(\overline{E}/\mathrm{E}^{\mathrm{ab}})}$ via the field of norms) we then use results of Fontaine–Wintenberger to analyze how the Swan conductor changes under the field of norms construction.

References

[1] Roland Huber. Swan representations associated with rigid analytic curves. 2001.

[2] Laurent Fargues and Peter Scholze. Geometrization of the local langlands correspondence. *arXiv preprint arXiv:2102.13459*, 2021.

[3] Johannes Anschütz and Arthur-César Le Bras. A fourier transform for banach-colmez spaces. *arXiv preprint arXiv:2111.11116*, 2021.

Mod-p Poincaré Duality in p-adic Analytic Geometry

Bogdan Zavyalov

We discuss the proof of the mod-p Poincaré Duality for p-adic rigid-analytic varieties. For the rest of the report, we fix a non-archimedean field K of mixed characteristic $(0, p)$, and denote its completed algebraic closure by $C = \widehat{\overline{K}}$.

Theorem 1. *Let X be a smooth and proper rigid-analytic variety of pure dimension d. Then there is a trace map $t_X \colon \mathrm{H}^{2d}(X_C, \mathbf{F}_p(d)) \to \mathbf{F}_p$ that induces a perfect pairing*

$$\mathrm{H}^i(X_C, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathrm{H}^{2d-i}(X_C, \mathbf{F}_p(d)) \xrightarrow{\cup} \mathrm{H}^{2d}(X_C, \mathbf{F}_p) \xrightarrow{t_X} \mathbf{F}_p$$

for every $i \geq 0$.

Remark 2. The trace map t_X was already constructed by Berkovich in [1], it was also shown that t_X is always surjective. However, it seems difficult to establish Theorem 1 by the methods of that paper.

A version with $\mathbf{Q}_p$-coefficients and K discretely valued was established in [5].

The essential idea of the proof is to reduce Theorem 1 to almost duality for $\mathcal{O}^+/p$-coefficients. More precisely, using the trace map defined by Berkovich, we can assume that $K = C$ is algebraically closed. Then one can use the primitive comparison theorem

$$\mathrm{H}^i(X, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathcal{O}_C/p \simeq^a \mathrm{H}^i(X, \mathcal{O}_X^+/p)$$

to reduce the question to showing that the complex $\mathrm{R}\Gamma(X, \mathcal{O}_X^+/p(d))[2d]$ is *almost* dual to $\mathrm{R}\Gamma(X, \mathcal{O}_X^+/p)$. We show this by choosing a formal $\mathcal{O}_C$-model $\mathcal{X}$ with the mod-p fiber $\mathcal{X}_0$ and writing

$$\mathrm{R}\Gamma(X, \mathcal{O}_X^+/p) \simeq \mathrm{R}\Gamma(\mathcal{X}_0, \mathrm{R}\nu_* \mathcal{O}_X^+/p)$$

where $\nu_* \colon (X_{\mathrm{proet}}, \mathcal{O}_X^+/p) \to (\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0})$ is the natural morphism of ringed sites.

This approach gives us the opportunity to study $\mathrm{R}\Gamma(X, \mathcal{O}_X^+/p)$ in two steps: first understand the "p-adic nearby cycles" $\mathrm{R}\nu_*\mathcal{O}_X^+/p$, and then understand the effect of $\mathrm{R}\Gamma(X, -)$. It turns out that $\mathrm{R}\nu_*\left(\mathcal{O}_X^+/p\right)$ is of "coherent flavour", so one can apply coherent methods (such as Grothendieck Duality) to study the effect of $\mathrm{R}\Gamma(\mathcal{X}_0, -)$ on $\mathrm{R}\nu_*\left(\mathcal{O}_X^+/p\right)$.

In order to make sense of this "coherent flavour", we need to introduce almost coherent sheaves:

Definition. *Let X be a finitely presented $\mathcal{O}_C$-scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is almost coherent if $\mathfrak{m}_C \otimes \mathcal{F}$ is quasi-coherent and there is an open affine covering $X = \bigcup_I U_i$ such that $\mathcal{F}(U_i)$ is almost finitely presented over $\mathcal{O}_X(U_i)$ (in the sense of* [4]*).*

Theorem 3.

(1) If $X = \mathrm{Spec} A$, the global section functor

$$\Gamma(X, -)\colon \mathbf{Mod}_X^{qc,acoh} \to \mathbf{Mod}_A^{acoh}$$

induces an equivalence between quasi-coherent, almost coheret modules and almost coherent A-modules.

(2) (Quasi-coherent) almost coherent $\mathcal{O}_X$-modules form a Weak Serre subcategory of all $\mathcal{O}_X$-modules.

(3) If $f\colon X \to Y$ a proper morphism of finitely presented $\mathcal{O}_C$-schemes, the derived pushforward functor $\mathrm{R}f_*$ carries $\mathbf{D}^+_{qc,acoh}(X)$ to $\mathbf{D}^+_{qc,acoh}(Y)$.

(4) If X is a proper, finitely presented $\mathcal{O}_C$-scheme with a formal p-adic completion $\widehat{X}$. Then the pullback functor $\mathrm{L}c^*$ induces an equivalence

$$\mathrm{L}c^*\colon \mathbf{D}_{qc,acoh}(X) \to \mathbf{D}_{qc,acoh}(\widehat{X})^1.$$

(5) If $f\colon X \to Y$ is a morphism of separated, finitely presented $\mathcal{O}_C$-schemes. Then the twisted inverse image functor $f^!\colon \mathbf{D}^+_{qc}(Y) \to \mathbf{D}^+_{qc}(X)$ induces a functor

$$f^!\colon \mathbf{D}^+_{qc,acoh}(Y) \to \mathbf{D}^+_{qc,acoh}(X).$$

Using Theorem 3, we can show that the "p-adic nearby cycles" are almost coherent:

Theorem 4. *Let X be a rigid-analytic space of dimension d with an admissible formal $\mathcal{O}_C$-model $\mathcal{X}$. Then $\mathfrak{m}_C \otimes \mathrm{R}\nu_*\mathcal{O}^+/p \in \mathbf{D}^{[0,d]}_{qc,acoh}(\mathcal{X}_0)$.*

Remark 5. Somewhat surprisingly, the hardest part of Theorem 4 is to show that $\mathrm{R}\nu_*\mathcal{O}^+/p$ is almost concentrated in degrees $[0, d]$. This step uses the stronger version of the almost purity theorem from [3]. This bound will be crucial in the proof later.

Using Theorem 3(5) (applied to the structure morphism $f_0\colon \mathcal{X}_0 \to \mathrm{Spec}\,(\mathcal{O}_C/p)$), the adjunction $(\mathrm{R}f_{0,*}, f_0^!)$, and Theorem 4 one easily sees that almost duality for $\mathrm{R}\Gamma(X, \mathcal{O}_X^+/p)$ follows from the following theorem:

---

[1] With the appropriate definition of almost coherent modules on a formal scheme $\widehat{X}$.

Theorem 6. *Let X be a smooth rigid-analytic space of pure dimension d, and $\mathcal{X}$ its formal $\mathcal{O}_C$-model. Then there is an almost morphism*

$$\mathrm{Tr}_{F,\mathcal{X}_0} : \mathrm{R}\nu_* \mathcal{O}_X^+/p \to \omega^\bullet_{\mathcal{X}_0}(-d)[-2d]$$

(called Faltings' trace map) that induces an almost perfect pairing

$$\mathrm{R}\nu_* \mathcal{O}_X^+/p \otimes^L_{\mathcal{O}_{\mathcal{X}_0}} \mathrm{R}\nu_* \mathcal{O}_X^+/p \xrightarrow{\cup} \mathrm{R}\nu_* \mathcal{O}_X^+/p \xrightarrow{\mathrm{Tr}_{F,\mathcal{X}_0}} \omega^\bullet_{\mathcal{X}_0}(-d)[-2d].$$

The proof of Theorem 6 has two essentially independent steps: the first one is to construct Faltings' trace map, and the second is to show that it induces an almost perfect pairing. For the first step, one uses degree bound from Theorem 4 to argue that it suffices to construct an almost morphism

$$\mathrm{R}^d \nu_* \mathcal{O}_X^+/p \to \omega_{\mathcal{X}_0}(-d)$$

where $\omega_{\mathcal{X}_0} := \mathcal{H}^{-d}(\omega^\bullet_{\mathcal{X}_0})$ is the dualizing *sheaf* of $\mathcal{X}_0$. Then one shows that it suffices to construct Faltings' trace map integrally, i.e. an almost morphism $\mathrm{R}^d \nu_* \widehat{\mathcal{O}}_X^+ \to \omega_{\mathcal{X}}(-d)$. Up to some $(\zeta_p - 1)$-torsion issues, the desired morphism was constructed in [2] on the smooth locus $\mathcal{X}^{\mathrm{sm}}$, and in [6] on the generic fiber. Then one uses the reduced fiber theorem, and an "extend from codimension-2" type argument to construct the desired morphism on the whole formal scheme $\mathcal{X}$.

To show that $\mathrm{Tr}_{F,\mathcal{X}_0}$ induces an almost perfect pairing, we do an explicit computation with continuous group cohomology on polystable formal models, and then reduce to the case of polystable models via the following Meta-Theorem.

Meta-Theorem 7. Suppose we want to prove some claim $\mathbf{P}(X, \mathcal{X})$ for *all* pairs of a smooth rigid-analytic space X and its formal model $\mathcal{X}$. Then it suffices to :

(1) show that $\mathbf{P}(X, \mathcal{X})$ is étale local on $\mathcal{X}$;
(2) descends through rig-isomorphisms $\mathcal{X}' \to \mathcal{X}$;
(3) descends through "nice" finite group quotients $\mathcal{X}' \to \mathcal{X}$, i.e. $\mathcal{X} = \mathcal{X}'/G$ for an $\mathcal{O}_C$-action of a finite group G on X such that the action is free on the adic generic fiber;
(4) show that $\mathbf{P}(X, \mathcal{X})$ holds for $\mathcal{X}$ that is a finite product of standard semi-stable formal schemes $\mathrm{Spf}\, \mathcal{O}_C\langle T_0, \ldots, T_n\rangle/(T_0 \ldots T_m - \pi)$ for some integers $m \le n$ and $\pi \in \mathcal{O}_C \setminus \{0\}$.

References

[1] V. Berkovich, "Étale cohomology for non-Archimedean analytic spaces". In *Inst. Hautes Études Sci. Publ. Math.* 78 (1993), 5–161 (1994).
[2] B. Bhatt, M. Morrow, and P. Scholze "Integral p-adic Hodge theory". In *Publ. Math. Inst. Hautes Études Sci.* 128 (2018), pp. 219–397.
[3] B. Bhatt and P. Scholze. "Prisms and prismatic cohomology" URL: https://arxiv.org/abs/1905.08229
[4] O. Gabber and L. Ramero. *Almost Ring Theory*. Lecture Notes in Mathematics. Springer, 2003, pp. viii+318
[5] K.-W. Lan, R. Liu, and X. Zhu. "De Rham Comparisoan and Poincaré Duality For Rigid Varieties", URL: https://arxiv.org/pdf/1912.13030.pdf
[6] P. Scholze. "p-adic Hodge theory for rigid-analytic varieites". In *Forum Math. Pi*, 2013

Specialization for the pro-étale fundamental group

Piotr Achinger
(joint work with Marcin Lara, Alex Youcis)

1. The étale fundamental group

The story starts with the following problem, addressed by Grothendieck in [6, Exposé X]: *In a family of algebraic varieties, how does one compare the (étale) fundamental groups of the fibers?* More precisely (and preparing the statements slightly to suit are needs), let K be a field which is complete with respect to a rank one valuation, with $\mathcal{O} \subseteq K$ its valuation ring and $k = \mathcal{O}/\mathfrak{m}$ its residue field, and let X be a proper $\mathcal{O}$-scheme with nonempty connected fibers. A geometric point $\overline{x}_K \in X(\overline{K})$ comes from a unique element $\overline{x}_{\mathcal{O}} \in X(\overline{\mathcal{O}})$ ($\overline{\mathcal{O}}$ being the valuation subring of $\overline{K}$), which then induces a point $\overline{x}_k \in X(\overline{k})$ where $\overline{k}$ is the residue field of $\overline{\mathcal{O}}$ and also an algebraic closure of k. Then, there is a continuous *specialization homomorphism*

$$\mathrm{sp}\colon \pi_1^{\text{ét}}(X_K, \overline{x}_K) \longrightarrow \pi_1^{\text{ét}}(X_k, \overline{x}_k).$$

If X is flat over $\mathcal{O}$ and is either normal or its special fiber is reduced, then the map sp is surjective. The latter statement is aptly called the ‘semicontinuity theorem’ by Grothendieck. He also shows a ‘continuity theorem,’ which addresses the question whether sp is an isomorphism if $K = \overline{K}$ and X is in addition smooth over $\mathcal{O}$ (answer: it is after prime-to-p completion).

2. The pro-étale fundamental group

In situations of geometric interest (e.g. semistable curves) one often has to deal with X as above where the special fiber X_k is not normal, and which (therefore) might admit natural covering spaces which are not seen by its étale fundamental group. For a concrete example (to be referenced several times below), let us consider a cubic $X \subseteq \mathbf{P}^2_{\mathcal{O}}$ such that X_K is smooth while X_k is isomorphic to $\mathbf{P}^1_k$ with 0 and ∞ identified. Let Y be an infinite chain of $Y_n = \mathbf{P}^1_k$'s indexed by $n \in \mathbf{Z}$, where we identify 0 in Y_n with ∞ in Y_{n+1}. There is a natural étale map $Y \to \mathbf{P}^1_k = X_k$ which looks very much like a covering space, and in fact is a $\mathbf{Z}$-torsor and satisfies the valuative criterion for properness. That said, it does not fit into the framework of the étale fundamental group because it is not of finite type.

In order to treat such coverings, one may use the *pro-étale fundamental group* $\pi_1^{\text{proet}}(X, \overline{x})$ of a connected locally topologically Noetherian scheme X introduced by Bhatt and Scholze [4]. Its definition is based on the category Cov_X of *geometric coverings* of X, which are étale maps $Y \to X$ satisfying the valuative criterion of properness. The base point $\overline{x}$ induces a fiber functor $F_{\overline{x}}\colon \mathrm{Cov}_X \to \mathrm{Sets}$, and one defines $\pi_1^{\text{proet}}(X, \overline{x})$ as the automorphism group of this functor, endowed with the topology coming from the compact-open topologies on the infinite symmetric groups $\mathrm{Aut}(F_{\overline{x}}(Y))$ for all objects Y of Cov_X. The theorem here is that the pair

$(\mathrm{Cov}_X, F_{\overline{x}})$ is a *tame infinite Galois category*, which basically means that the fiber functor induces an equivalence

$$F_{\overline{x}}\colon \mathrm{Cov}_X \xrightarrow{\sim} \pi_1^{\mathrm{proet}}(X,\overline{x})\text{-sets}$$

where the target is the category of discrete sets with a continuous action of $\pi_1^{\mathrm{proet}}(X,\overline{x})$. The profinite completion of $\pi_1^{\mathrm{proet}}(X,\overline{x})$ is $\pi_1^{\mathrm{\acute{e}t}}(X,\overline{x})$.

The chain of $\mathbf{P}^1$'s mentioned above is thus an example of a geometric covering, and in fact (if $k = \overline{k}$) it is the "universal covering" of the nodal curve X_k, so that $\pi_1^{\mathrm{proet}}(X_k) \simeq \mathbf{Z}$. In fact, non-normality of X_k is crucial here: if X is a geometrically unibranch locally Noetherian scheme, then geometric coverings are disjoint unions of finite étale coverings, and hence $\pi_1^{\mathrm{proet}}(X_k) \simeq \pi_1^{\mathrm{\acute{e}t}}(X_k)$, so this group is profinite. We also note that, unlike in our motivating example, general geometric coverings are not disjoint unions of finite étale coverings étale locally on the base, already for X being the union of two copies of $\mathbf{G}_m$ glued together at a point.Related to this is the fact that $\pi_1^{\mathrm{proet}}(X)$ is in general not a pro-discrete group, but what is called a Noohi topological group.

Coming back to Grothendieck's specialization map, one might ask naively if there is a similarly defined continuous specialization homomorphism

$$\pi_1^{\mathrm{proet}}(X_K,\overline{x}_K) \xrightarrow{?} \pi_1^{\mathrm{proet}}(X_k,\overline{x}_k),$$

compatible with the version for $\pi_1^{\mathrm{\acute{e}t}}$. Unfortunately, the answer is no, already in the case of our degenerating cubic! Indeed, the target is the discrete group $\mathbf{Z}$ (if k is algebraically closed) and the source is profinite (as X_K is normal). If the map is continuous, it has to have compact image, and hence would have to be trivial. But, the specialization map for $\pi_1^{\mathrm{\acute{e}t}}$ is nontrivial in this case, contradiction.

3. The de Jong fundamental group

The proposed way of clarifying this issue employs formal and rigid geometry. In fact, formal geometry is present already in the construction of Grothendieck's specialization map. If we denote by FEt_X the category of finite étale coverings of a (formal) scheme X, then in the situation of §1 the crucial point is that the inclusion $i\colon X_k \to X$ induces an equivalence

$$i^*\,\mathrm{FEt}_X \xrightarrow{\sim} \mathrm{FEt}_{X_k}\,.$$

Composing its inverse with the restriction $\mathrm{FEt}_X \simeq \mathrm{FEt}_{X_K}$, we obtain a functor $\mathrm{FEt}_{X_k} \to \mathrm{FEt}_{X_K}$ which then induces a map on fundamental groups in the reverse direction. However, in order to prove that i^* is an equivalence, one introduces the formal scheme $\widehat{X}$, the formal completion of X with respect to the topology on $\mathcal{O}$. Then, the topological invariance of the étale site easily gives $\mathrm{FEt}_{\widehat{X}} \simeq \mathrm{FEt}_{X_k}$, while Grothendieck's existence theorem (which applies since X is proper) gives an equivalence $\mathrm{FEt}_X \simeq \mathrm{FEt}_{\widehat{X}}$. In the example of the $\mathbf{Z}$-cover of the nodal cubic X_k, we still obtain a $\mathbf{Z}$-cover of the formal scheme $\widehat{X}$, but since it is infinite the existence theorem does not apply. However, we can still take its *rigid-analytic* generic fiber, obtaining an étale space over $\widehat{X}_{\mathrm{rig}}$, which in fact is the famous Tate uniformization of the rigid elliptic curve X_K^{an}!

Therefore we begin to suspect that the "correct" analog of the specialization map with target $\pi_1^{\mathrm{proet}}(X_k)$ should have as a source some kind of π_1 of the rigid-analytic generic fiber. Fortunately, such a theory has been developed by de Jong [5] (in the context of Berkovich spaces). Let X be a rigid-analytic space over K (or an adic space locally of finite type over $\mathrm{Spa}(K)$). A *de Jong covering* is an étale map $Y \to X$ such that X admits an open covering by partially proper open (i.e. wide open) subsets $X_i \subseteq X$ with the property that $Y \times_X X_i$ is the disjoint union of finite étale coverings of X_i for all i. De Jong (essentially) proved that the category $\mathrm{Cov}_X^{\mathrm{dJ}}$ of de Jong coverings of X, together with a fiber functor induced by a geometric point $\overline{x}$, is a tame infinite Galois category if X is connected. We therefore obtain a Noohi topological group $\pi_1^{\mathrm{dJ}}(X, \overline{x})$ which we call the *de Jong fundamental group*. Variants of de Jong's fundamental group were recently introduced in [2, 3].

4. The results

We are now able to state our main result:

Theorem 1 ([1]). *Let X be a formal scheme locally of finite type over $\mathcal{O}$.*

- *(a) If $Y \to X$ is an étale map of formal schemes whose special fiber $Y_k \to X_k$ is a geometric covering, then its rigid generic fiber $Y_{\mathrm{rig}} \to X_{\mathrm{rig}}$ is a de Jong covering.*
- *(b) Consequently, if X_k and X_{rig} are both connected and non-empty, and if $\overline{x}_{\mathrm{rig}}$ is a geometric point of X_{rig} specializing to a geometric point $\overline{x}_k$ of X_k, we obtain a continuous specialization homomorphism*

$$\mathrm{sp}\colon \pi_1^{\mathrm{dJ}}(X_{\mathrm{rig}}, \overline{x}_{\mathrm{rig}}) \longrightarrow \pi_1^{proet}(X_k, \overline{x}_k).$$

- *(c) The map* sp *has dense image if X is moreover η-normal (i.e. if X is flat over $\mathcal{O}$ and $\mathcal{O}_X$ is integrally closed in $\mathcal{O}_X \otimes_{\mathcal{O}} K$).*

Part (a) (of which (b) is a formal consequence) is at least a bit surprising. Indeed, as we have noted before, a geometric covering $Y_k \to X_k$ does not have to split into the disjoint union of finite étale coverings étale locally on X_k. Therefore there is a priori no reason why $Y_{\mathrm{rig}} \to X_{\mathrm{rig}}$ should split even locally on X_{rig}, not to mention the "wide open" condition in the definition of a de Jong covering. The key idea in the proof is that the tubes (preimages under the specialization map) of the irreducible components Z_i of X_k form a wide open cover X_i of X_{rig}. Therefore, if these components are normal (or geometrically unibranch), then $Y_k \to X_k$ is the disjoint union of finite étale coverings, and then the same holds for the restriction of $Y_{\mathrm{rig}} \to X_{\mathrm{rig}}$ to X_i. (An important role in this argument is played by fact that X_i is the rigid generic fiber of the non-adic formal scheme obtained by completing X along Z_i.)

In order to make use of the previous paragraph, one has to deal with non-normal irreducible components of X_k. To this end, we need to perform some admissible blowups of X. Unfortunately, we do not know if there exists an admissible blowup

$X' \to X$ such that the irreducible components of X'_k are normal, even if X itself is normal. However, what saves us is the key following technical auxiliary result: *There exists an admissible blowup $X' \to X$ such that for every irreducible component Z' of X'_k with image Z in X_k, the map $Z' \to Z$ factors through the normalization of Z.*

Part (c) of the theorem is an analog of Grothendieck's semicontinuity theorem. As the groups in question are no longer compact, it is difficult to find a criterion for surjectivity (for profinite groups dense image means surjective). For dense image, the criterion is that if $Y \to X$ is étale and $Y_k \to X_k$ is a connected geometric covering, then Y_{rig} is connected. Using idempotents, it is trivial to see that this is the case if Y is η-normal. So to prove (c), one needs to show that if $Y \to X$ is étale and X is η-normal, then so is Y. Proving this is a bit technical, especially in the non-noetherian case, and relies on a version of Serre's normality criterion. Namely, X is η-normal if and only if it is flat over $\mathcal{O}$, its local rings at the generic points of X_k are valuation rings, and for every (equiv. for some) pseudouniformizer $\pi \in \mathcal{O}$, the scheme $X \otimes \mathcal{O}/\pi$ has no embedded points (a function which vanishes on a dense open has to be zero).

References

[1] Piotr Achinger, Marcin Lara, and Alex Youcis, *Specialization for the pro-étale fundamental group*, arXiv preprint arXiv:/2107.06761, 2021.
[2] Piotr Achinger, Marcin Lara, and Alex Youcis, *Geometric arcs and fundamental groups of rigid spaces*, arXiv preprint arXiv:/2105.05184, 2021.
[3] Piotr Achinger, Marcin Lara, and Alex Youcis, *Variants of the de Jong fundamental group*, arXiv preprint arXiv:/2203.11750, 2021.
[4] Bhargav Bhatt and Peter Scholze, *The pro-étale topology for schemes*, Astérisque (2015), no. 369, 99–201. MR 3379634
[5] Aise Johan de Jong, *Étale fundamental groups of non-Archimedean analytic spaces*, Compositio Math. **97** (1995), no. 1-2, 89–118. Special issue in honour of Frans Oort. MR 1355119
[6] Alexander Grothendieck (ed.), *Revêtements étales et groupe fondamental*, Lecture Notes in Mathematics, Vol. 224, Springer-Verlag, Berlin-New York, 1971, Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud. MR 0354651

The residually wild locus for an étale morphism of adic spaces

Katharina Hübner
(joint work with Michael Temkin)

Let (k, k^+) be a nonarchimedean field (whose valuation is allowed to be trivial). We consider an étale morphism $f : X \to Y$ of smooth adic spaces over k. For a point $x \in X$ mapping to $y \in Y$ the residue field extension $k(x)/k(y)$ is a finite separable extension of valued fields. The ramification behaviour of the extensions $k(x)/k(y)$ for varying x encodes information about an extension of f to a morphism of formal models $\mathcal{X} \to \mathcal{Y}$. In most situations nice formal models are not available without assuming resolution of singularities and singular models are difficult to handle. By examining the residue field extensions $k(x)/k(y)$ instead of working with a model, we can avoid using resolution of singularities in many cases.

Let us assume that the residue characteristic of k^+ is $p > 0$. If the morphism $f : X \to Y$ is residually tame (i.e. all residue field extensions $k(x)/k(y)$ for $x \in X$ are tame), the locus R_f of all $x \in X$ where $k(x)/k(y)$ ramifies is a skeleton of f. In case of wild ramification, however, the residual ramification locus R_f is expected to be much bigger. For curves over an algebraically closed field, the residually wild ramification locus has been described by Adina Cohen, Michael Temkin, and Dmitri Trushin in [1]. It is a radial subset with respect to an appropriate skeleton, i.e. can be thought of as a tubular neighbourhood of the skeleton.

The goal of this project is to understand the residual ramification behaviour of $f : X \to Y$ for higher dimensional spaces over fields that are not necessarily algebraically closed. We expect that the residually wild ramification locus is either empty or large in an appropriate sense. A first step in this direction is the following theorem. Before we state it let us recall that a divisorial point of X is a point such that the residue field extension $\widetilde{k(x)}/\tilde{k}$ (of the extension of valuation rings $k(x)^+/k^+$) has transcendence degree $\dim_x X$ (or $\dim_x X - 1$ if $|k^\times| = \{1\}$). These are the points that correspond to irreducible components of the special fiber of some model $\mathcal{X}/k^+$ of X.

Theorem 1. *Let $f : X \to Y$ be an étale morphism of smooth adic spaces over k. Then the residually wild locus R_f is either empty or contains a divisorial point.*

The theorem tells us that we can read off wild ramification at divisorial points and do not need to deal with more complicated valuations. It is proved by first reducing to the case of a finite étale morphism of degree p of curves. If the base change of f to the algebraic closure of k stays ramified, we can use the description of R_f for curves over algebraically closed fields. However, it can also happen that this base change is unramified. For this case we prove a version of local uniformization over perfectoid fields that are not necessarily algebraically closed. We then show that we can find a perfectoid field k' over k such that the base change of f to k' stays ramified. What remains is a direct computation on discs and annuli.

As an application of the theorem we answer the following question by Hélène Esnault: Let Y be a smooth scheme over a field k of positive characteristic. Let $X \to Y$ be a finite étale galois morphism that is curve tame (i.e. the base change to any curve is tame, see [2]). Then does f extend to a map $\bar{f} : \bar{X} \to \bar{Y}$ of normal compactifications wich is numerically tame (i.e., for every point $x \in \bar{X}$ the inertia group I_x has order prime to p)? A positive answer is given in [2] under the assumption of resolution of singularities. Using the above theorem we are able to prove the statement without this assumption.

References

[1] M. Temkin, *Morphisms of Berkovich curves and the different function*, Advances in Mathematics **303** (2016), 800–858.

[2] M. Kerz, A. Schmidt, *On different notions of tameness in arithmetic geometry*, Mathematische Annalen **346** (2010), 641–668.

Semistable Higgs bundles in p-adic non-abelian Hodge theory

Matti Würthen

We reported on work in progress on a fundamental open problem in the p-adic Simpson correspondence.

Let X be a connected smooth projective variety over $\mathbb{C}_p$. The p-adic Simpson correspondence aims to relate Higgs-bundles on X to vector bundles on the pro-étale site, i.e. locally free sheaves over the completed structure sheaf $\hat{\mathcal{O}}_X$ on $X_{pro\acute{e}t}$. The pioneering work was done by Faltings [4], which has been extended by Abbes-Gros and Tsuji [1] and recently by Wang [13] in the general context of rigid analytic spaces, building on work of Liu-Zhu [11].

Let $LF(\hat{\mathcal{O}}_X)$ be the category of finite locally free sheaves over the completed structure sheaf $\hat{\mathcal{O}}_X$ on the pro-étale site $X_{pro\acute{e}t}$. Denote by $\pi_1^{\acute{e}t}(X) = \pi_1^{\acute{e}t}(X, x)$ the étale fundamental group of X with respect to some chosen base point x. By the so-called primitive comparison theorem of Faltings-Scholze, $LF(\hat{\mathcal{O}}_X)$ contains the category $Rep^{cont}_{\pi_1^{\acute{e}t}(X)}(\mathbb{C}_p)$ of continous representations of the étale fundamental group on finite dimensional $\mathbb{C}_p$-vector spaces as a full subcategory. Naturally, one may ask which Higgs-bundles lie in the essential image of $Rep^{cont}_{\pi_1^{\acute{e}t}(X)}(\mathbb{C}_p)$ under the p-adic Simpson correspondence. In analogy to the complex situation, one may hope that the essential image consists of the semistable Higgs-bundles with vanishing Chern classes (here meant to be l-adic Chern classes).

An attempt to relate $\pi_1^{\acute{e}t}(X)$-representations to vector bundles (with vanishing Higgs-field) was first undertaken by Deninger and Werner (see [2] and [3]). In particular, they obtained the following: Let E be a vector bundle which admits an integral model $\mathcal{E}$ over an integral model $\mathcal{X}$ of X, such that the special fiber $\mathcal{E} \otimes \mathcal{O}_{\mathbb{C}_p}/\mathfrak{m}$ is a numerically flat vector bundle. Then there is a $\pi_1^{\acute{e}t}(X)$-representation associated to E. The work of Deninger-Werner is compatible with the p-adic Simpson correspondence, which was first shown by Xu for curves in [15]. The work has also been generalized to arbitrary rigid spaces in [14]. Recent work of Xu [16] on curves also indicates that - using so called twisted pullbacks - one might even hope to be able to reduce the problem for general Higgs-fields to the problem of finding models with numerically flat reduction.

Unfortunately, finding models which have numerically flat reduction seems to be very difficult in general. For plane curves, examples have been constructed in [5].

In general, stronger results have been obtained in the case of Higgs line bundles [6] and for Higgs bundles on abeloid varieties [7].

For smooth algebraic varieties in characteristic p, Ogus and Vologodsky [12] have introduced a correspondence between Higgs bundles (more precisely so-called PD-Higgs bundles) and bundles with flat connections (whose p-curvature extends to an F-PD-Higgs field). Using this work, Lan-Sheng-Zuo [10] have introduced the notion of strongly semistable Higgs bundles (henceforth called LSZ-strongly semistable). In the case of bundles with vanishing Chern classes on a projective variety over a finite field, these are Higgs bundles which initiate a so-called preperiodic Higgs-de-Rham flow. It was shown by Lan-Sheng-Zuo [10][Theorem 6.5] and independently by Langer that on a projective smooth variety over a finite field any semistable nilpotent Higgs bundle with vanishing Chern classes of small rank $< p$ is LSZ-strongly semistable. Note that in particular every semistable vector bundle of rank $< p$ with vanishing Higgs field is thus LSZ-strongly semistable (even though it may not be strongly semistable (i.e. numerically flat) in the classical sense as a vector bundle).

The goal of our work is to prove that Higgs bundles are associated to $\pi_1^{ét}(X)$-representations in the good reduction case, if they admit an integral model whose special fiber is LSZ-strongly semistable with vanishing Chern classes. In this way we can in some cases simultaneously extend the approach of Deninger-Werner to Higgs-bundles, while avoiding the problem of finding models with numerically flat reduction.

More precisely, we are able to prove the following.

Theorem. *Let k be a finite field and let $\mathcal{X}$ be a smooth projective scheme over $W(k)$ and denote by K the fraction field of $W(k)$. Let further $(\mathcal{E}, \theta)$ be a Higgs bundle on $\mathcal{X}$, such that the reduction $(E, \theta)_k$ is nilpotent of level $< p$ and LSZ-strongly semistable with vanishing Chern classes. Then $(\mathcal{E}, \theta)_{\hat{\bar{K}}}$ corresponds to a $\pi_1^{ét}(X)$-representation via the p-adic Simpson correspondence.*

Assume that (E, θ) is a semistable Higgs-bundle on the generic fiber X of $\mathcal{X}$. By a general version of Langton's theorem (by Langer [8]), it is always possible to find an integral model of (E, θ) on $\mathcal{X}$, whose reduction to the special fiber $\mathcal{X}_k$ is semistable (in fact, local freeness of the integral model is non-obvious, but assured in our situation by [9][Theorem 2.2]). As mentioned above, by results of Lan-Sheng-Zuo/Langer, any nilpotent semistable Higgs bundle of rank $< p$ with vanishing Chern classes on $\mathcal{X}_k$ is strongly semistable. This leads to the following.

Corollary. *Let k be a finite field and let $\mathcal{X}$ be a smooth projective scheme over $W(k)$ and denote by K the fraction field of $W(k)$. Let further (E, θ) be a semistable Higgs bundle of rank $< p$ with vanishing Chern classes on X, such that (E, θ) admits an integral model, whose Higgs-field is nilpotent modulo p. Then $(E, \theta)_{\hat{\bar{K}}}$ corresponds to a $\pi_1^{ét}(X)$-representation.*

References

[1] A. Abbes, M. Gros, and T. Tsuji, *The p-adic Simpson correspondence*, Ann. of Math. Stud. **193** (2016), Princeton University Press.

[2] C. Deninger and A. Werner, *Vector bundles on p-adic curves and parallel transport.* Ann. Sci. Éc. Norm. Supér., Ser. 4 **38(4)** (2005), 553–597.

[3] C. Deninger and A. Werner, *Parallel transport for vector bundles on p-adic varieties*, J. Algebraic Geom. **29** (2020): 1–52, 2020.

[4] G. Faltings. *A p-adic Simpson correspondence*, Adv. Math. **198(2)** (2005), 847 – 862.

[5] M. A. Hahn and A. Werner, *Strongly semistable reduction of syzygy bundles on plane curves*, arXiv:1907.02466.
[6] B. Heuer, *Line bundles on rigid spaces in the v-topology*, arXiv:2012.07918.
[7] B. Heuer, L. Mann and A. Werner, *The p-adic Corlette-Simpson correspondence for abeloids*, arXiv:2107.09403.
[8] A. Langer, *Semistable modules over Lie algebroids in positive characteristic*, Doc. Math. **19** (2014), 509–540.
[9] A. Langer, *Nearby cycles and semipositivity in positive characteristic*, arXiv:1902.05745.
[10] G. Lan, M. Sheng and K. Zuo, *Semistable Higgs bundles, periodic Higgs bundles and representations of algebraic fundamental groups*, J. Eur. Math. Soc. (JEMS) **21** (2019), 3053–3112.
[11] R. Liu and X. Zhu, *Rigidity and a Riemann-Hilbert correspondence for p-adic local systems*, Invent. Math. **207(1)** (2017), 291–343.
[12] A. Ogus and V. Vologodsky, *Nonabelian Hodge theory in characteristic p*, Publ. Math. Inst. Hautes Études Sci. **106** (2007), 1–138.
[13] Y. Wang, *A p-adic Simpson correspondence for rigid analytic varieties*, arXiv:2101.00263.
[14] M. Würthen, *Vector bundles with numerically flat reduction on rigid analytic varieties and p-adic local systems*, to appear in Int. Math. Res. Not. IMRN (2022).
[15] D. Xu, *Transport parallèle et correspondance de Simpson p-adique*, Forum Math. Sigma **5:e13** (2017).
[16] D. Xu, *Parallel transport for Higgs bundles over p-adic curves*, arXiv:2201.06697.

δ-Forms on Lubin–Tate space

Andreas Mihatsch

Let $\mathcal{X}/\mathbb{Z}$ be a regular, proper, flat and generically smooth scheme. In Gillet–Soulé's formulation of Arakelov theory, arithmetic cycles on $\mathcal{X}$ are defined as pairs $(\mathcal{Z}, g)$, where $\mathcal{Z}$ is a cycle on $\mathcal{X}$ in the algebraic sense and g a Green current on $\mathcal{X}(\mathbb{C})$ for $\mathcal{Z}(\mathbb{C})$. The intersection product of two such pairs is defined scheme-theoretically in the $\mathcal{Z}$-component and analytically ($\star$-product) in the g-component.

A folklore expectation is that the employed complex-analytic theory of forms and currents has an analog on non-archimedean analytic spaces. Presumably, this will alleviate the need for integral models in the above definition. Namely, instead of $\mathcal{X}$ and $\mathcal{Z}$, one could then consider smooth, proper varieties $X/\mathbb{Q}$ and tuples $(Z, (g_p)_p)$, where Z is a cycle on X and g_p a Green current on the p-adic analytification X_p^{an} for Z_p^{an}.

A candidate definition for a suitable theory of differential forms has been put forth by Chambert-Loir–Ducros [1]. Let, from now on, (k, v) denote a non-archimedean, additively valued field and X/k a non-archimedean space in the sense of Berkovich. Any tuple of invertible functions $f = (f_1, \ldots, f_n) \in \mathcal{O}_X(U)^\times$ defines a continuous map

$$v(f) = \big(v(f_1), \ldots, v(f_n)\big) \colon U \to \mathbb{R}^n.$$

Given a (p, q)-form α on $\mathbb{R}^n$ in the sense of Lagerberg [5], Chambert-Loir–Ducros define a pullback (p, q)-form $v(f)^*(\alpha)$. Sheafifying this construction gives rise to *smooth differential forms* on X. These behave a lot like their complex analytic counterparts. We mention that Gubler–Jell–Rabinoff [3] have recently expanded this formalism by considering pullbacks $\psi^*(\alpha)$ for more general $\psi \colon U \to \mathbb{R}^n$.

The subtlety of Chambert-Loir–Ducros' definition lies with the fact that the sheafification is performed on the topological space of X, while constructions involving formal models are usually G-local. Thus arises the need to also incorporate *piecewise smooth* functions and forms into the theory. To this end, Gubler–Künnemann [4] have introduced their so-called δ-forms. Roughly speaking, these enrich smooth forms by pullbacks $v(f)^*(T)$ of tropical cycles T on $\mathbb{R}^n$. The differential calculus of δ-forms is then developed in terms of tropical intersection theory on $\mathbb{R}^n$.

During my talk, I presented a new definition of δ-forms and an application to intersection theory on formal schemes. These were taken from [7] and [8]. Assume for this that X is of pure dimension d and without boundary. To any tuple $f = (f_1, \ldots, f_n) \in \mathcal{O}_X(U)^\times$ as above, Ducros [2] has associated the skeleton

$$\Sigma(U, f) = \left\{ x \in U \mid \text{tr.deg}\left(\tilde{k}\left(\tilde{f}_x\right) / \tilde{k} \right) = d \right\}.$$

Here, $\tilde{k}$ denotes the graded residue field of k and $\tilde{k}\left(\tilde{f}_x\right)$ the graded extension generated by the graded residue classes of the germ of f in x. By results of [1], [2] and [3], it is naturally a weighted piecewise linear space of pure dimension d. Furthermore, the distinguished piecewise linear functions $v(f_i)$, $i = 1, \ldots, n$, satisfy a balance condition which makes it into a *tropical space*. It is then possible to define pullbacks $v(f)^*(\alpha)|_{\Sigma(U,f)}$ as polyhedral currents on $\Sigma(U, f)$, meaning as linear combinations of polytopes on $\Sigma(U, f)$ with differential form coefficients.

Definition ([8, Def. 4.2]). A δ-form on X is the datum of polyhedral currents $(\omega_\Sigma)_{\Sigma = \Sigma(V,g)}$, for varying open subsets $V \subseteq X$ and tuples of invertible functions g on V, such that every point has a neighborhood U as well as f and α as before with $\omega_\Sigma = v(f)^*(\alpha)|_\Sigma$ for every $\Sigma \subseteq U$.

Example. Most importantly, every piecewise smooth function $\phi\colon X \to \mathbb{R}$ is a δ-form. This means that ϕ is continuous and G-locally smooth in the sense of Chambert-Loir–Ducros. If, for example, $\mathcal{X}/O_k$ is a formal model and $\mathcal{L}$ a line bundle on $\mathcal{X}$, then its generic fiber L/X is equipped with a piecewise smooth metric $|\cdot|$. There is a differential calculus for δ-forms that allows to define the curvature $c_1(L, |\cdot|)$, which is a δ-form of degree $(1, 1)$. The corresponding Monge–Ampère measure $c_1(L, |\cdot|)^d$ is explicitly computed as a certain polyhedral current on a suitable skeleton Σ.

My results on intersection theory are motivated by the following complex-analytic example. Consider the function $\phi = \log |z_1^2 + \ldots + z_r^2|$ on $\mathbb{C}^r \setminus \{0\}$. Then $\omega = (-1)^r \phi (\partial\bar{\partial}\phi)^{r-1}$ is a Green current for $\{0\}$, meaning here that $\int \omega \wedge (\partial\bar{\partial}\rho) = \rho(0)$ for every compactly supported smooth function ρ.

Let $Z = V(f_1, \ldots, f_r) \subseteq X$ be the closed subspace defined by a regular sequence $f_1, \ldots, f_r \in \mathcal{O}_X$. The non-archimedean analog of the above function is

$$\phi := \min\{v(f_1), \ldots, v(f_r)\}\colon X \setminus Z \to \mathbb{R}.$$

It is piecewise smooth and, in particular, a δ-form. Put $\omega = (-1)^{r-1}\phi(d'd''\phi)^{r-1}$.

Theorem ([8, Thm. 5.5]). The δ-form ω is a Green current for Z. More precisely, for every compactly supported δ-form η of degree $(d-r, d-r)$,

$$\int_{X\setminus Z} \omega \wedge (d'd''\eta) = -\int_Z \eta.$$

In some cases, this formalism can reproduce intersection numbers from formal schemes. Assume for this that k is discretely valued with uniformizer π and $v(\pi) = 1$. Let $\mathcal{X}/\mathrm{Spf}O_k$ be a local, flat formal scheme with generic fiber X that is formally of finite type and separated. Let $\mathcal{Z} = V(f_0, \ldots, f_d) \subseteq \mathcal{X}$ be an artinian closed formal subscheme that is defined by a regular sequence. Define $\phi = \min\{v(f_0), \ldots, v(f_d)\}$ and ω as before. Then [8, Thm. 5.9] states

$$(*) \qquad \mathrm{len}_{O_k}(\mathcal{O}_{\mathcal{Z}}) = \int_X \omega.$$

Originally, my search for such a result was motivated by an intersection problem in Lubin–Tate formal deformation spaces. Consider for this an unramified quadratic extension E/F of p-adic local fields and some $h \geq 1$. Then there is a closed immersion $\mathcal{N} \to \mathcal{M}$ of the Lubin–Tate space for height h and field E into that for height $2h$ and field F, namely the map forgetting the O_E-action. The dimension of $\mathcal{N}$ is h, exactly half the dimension of $\mathcal{M}$. The units $O_D^\times$ of the ring of integers in a division algebra of Hasse invariant $1/2h$ act on $\mathcal{M}$ and one may define an intersection number for $\gamma \in O_D^\times$,

$$\mathrm{Int}(\gamma) := \mathrm{len}_{O_{\breve{F}}} \mathcal{O}_{\mathcal{N}\cap\gamma\mathcal{N}} \in \mathbb{Z}_{\geq 1} \cup \{\infty\}.$$

It is finite for generic γ, which is the case of interest. Since both $\mathcal{M}$ and $\mathcal{N}$ are regular and local, $\mathcal{N}$ is defined by a regular sequence and $(*)$ may be applied. It is then possible to evaluate the integral in $(*)$ explicitly, cf. [8], and to recover the following formula for $\mathrm{Int}(\gamma)$, which is due to Q. Li.

Theorem (Q. Li [6]). There is a constant c that only depends on F and h such that for all generic γ,

$$\mathrm{Int}(\gamma) = c \int_{GL_{2h}(O_F)} |\mathrm{Res}(P_\gamma, P_g)|^{-1}\, dg.$$

Here, $\mathrm{Res}(P_\gamma, P_g)$ denotes the resultant of certain polynomials one attaches to γ and g. I refer to [6] and [8] for further details.

References

[1] A. Chambert-Loir, A. Ducros, *Formes différentielles réelles et courants sur les espaces de Berkovich*, ArXiv Preprint (2012), arXiv:1204.6277.

[2] A. Ducros, *Image réciproque du squelette par un morphisme entre espaces de Berkovich de même dimension*, Bull. Soc. Math. France **131** No. 4 (2003), 483–506.

[3] W. Gubler, P. Jell, J. Rabinoff, *Forms on Berkovich spaces based on harmonic tropicalizations*, ArXiv Preprint (2021), arXiv:2111.05741v1.

[4] W. Gubler, K. Künnemann, *A tropical approach to nonarchimedean Arakelov geometry*, Algebra Number Theory **11** No. 1 (2017), 77–180.

[5] A. Lagerberg, *Super currents and tropical geometry*, Math. Z. **270** (2011), 1011–1050.

[6] Q. Li, *An intersection number formula for CM cycles in Lubin-Tate towers*, ArXiv Preprint (2018), arXiv:1803.07553.
[7] A. Mihatsch, *On Tropical Intersection Theory*, ArXiv Preprint (2021), arXiv:2107.12067.
[8] A. Mihatsch, *δ-Forms on Lubin–Tate space*, ArXiv Preprint (2021), arXiv:2112.10018.

Duality for the proétale p-adic cohomology of analytic spaces

Pierre Colmez
(joint work with Sally Gilles, Wiesława Nizioł)

Let K be a finite extension of $\mathbf{Q}_p$ and let C be the completion of $\overline{K}$.

The geometric proétale cohomology of an open annulus. Let Y_K be an open annulus defined over K and set $Y_C := Y_K \times C$. Then, Kummer theory or syntomic computions give, for $k \in \mathbf{Z}$,

$$H^i_{\text{proét}}(Y_C, \mathbf{Q}_p(k)) \simeq \begin{cases} \mathbf{Q}_p(k) & \text{if } i = 0, \\ \mathbf{Q}_p(k-1) \oplus (\mathcal{O}(Y_C)/C)(k-1) & \text{if } i = 1, \\ 0 & \text{if } i \geq 2 \end{cases}$$

To define the cohomology with compact support, let us first define the boundary ∂Y_C of Y_C as $\varprojlim_X Y_C \setminus X$, where X runs over quasi-compact subspaces of Y_C (hence $\mathcal{O}(Y_C) = \varinjlim_X \mathcal{O}(Y_C \setminus X)$). This is the disjoint union of two "ghost circles". Then, define $\mathrm{R}\Gamma_{\text{proét},c}(Y_C, \mathbf{Q}_p(k))$ as the fiber of the map $\mathrm{R}\Gamma_{\text{proét}}(Y_C, \mathbf{Q}_p(k)) \to \mathrm{R}\Gamma_{\text{proét}}(\partial Y_C, \mathbf{Q}_p(k))$. The usual long exact sequence gives isomorphisms

$$H^1_{\text{proét},c}(Y_C, \mathbf{Q}_p(k)) \simeq H^{i-1}_{\text{proét}}(\partial Y_C, \mathbf{Q}_p(k))/H^{i-1}_{\text{proét}}(Y_C, \mathbf{Q}_p(k))$$

from which one gets:

$$H^i_{\text{proét},c}(Y_C, \mathbf{Q}_p(k)) \simeq \begin{cases} \mathbf{Q}_p(k) & \text{if } i = 1, \\ \mathbf{Q}_p(k-1) \oplus \frac{\mathcal{O}(\partial Y_C)/C^2}{\mathcal{O}(Y_C)/C}(k-1) & \text{if } i = 2, \\ 0 & \text{if } i \neq 1, 2 \end{cases}$$

which does not look too good for a duality between cohomology and cohomology with compact support. Nevertheless, $(\mathcal{O}(\partial Y_C)/C^2)/(\mathcal{O}(Y_C)/C)$ is naturally the C-dual of $\mathcal{O}(Y_C)/C$ by Serre duality (but not its $\mathbf{Q}_p$-dual since C is infinite dimensional over $\mathbf{Q}_p$).

The arithmetic case. One can use Hochschild-Serre spectral sequence to compute the cohomology of Y_K from the cohomology of Y_C, and Tate's results that $H^i(G_K, C(j)) = K$ if $j = 0$ and $i = 0, 1$ and $H^i(G_K, C(j)) = 0$ if $j \neq 0$ or $i \neq 0, 1$ (here $G_K = \mathrm{Gal}(\overline{K}/K)$). A case by case inspection, using Poitou-Tate duality between $H^i(G_K, \mathbf{Q}_p(j))$ and $H^{2-i}(G_K, \mathbf{Q}_p(1-j))$ shows that $H^4_{\text{proét},c}(Y_K, \mathbf{Q}_p(2)) = \mathbf{Q}_p$ and $H^{4-i}_{\text{proét},c}(Y_K, \mathbf{Q}_p(2-k))$ is the $\mathbf{Q}_p$-dual of $H^i_{\text{proét}}(Y_K, \mathbf{Q}_p(k))$. This leads to the following conjecture:

Conjecture. *Let Y_K be a partially proper analytic space* (or a dagger space) *defined over K, smooth, connected, of dimension d. Then*:

(i) $H^{2d+2}_{\text{proét},c}(Y_K, \mathbf{Q}_p(d+1)) = \mathbf{Q}_p$.

(ii) $H^i_{\text{proét}}(Y_K, \mathbf{Q}_p(k)) \times H^{2d+2-i}_{\text{proét},c}(Y_K, \mathbf{Q}_p(d+1-k)) \to \mathbf{Q}_p$ *is a duality of topological* $\mathbf{Q}_p$*-vector spaces.*

Remark.

(i) The conjecture is true if Y_K is proper thanks to the recently proved Poincaré duality for proper analytic spaces over C (proofs by L. Mann and by B. Zavyalov).

(ii) The conjecture is perhaps overoptimistic for topological reasons in the general case (one should probably phrase it using condensed mathematics of Clausen and Scholze), but we have the following result.

Theorem. *The conjecture is true in dimension* 1.

The geometric case. For a proper curve,

$$H^i_{\text{proét}}(Y_C, \mathbf{Q}_p(1)) \times H^{2-i}_{\text{proét},c}(Y_C, \mathbf{Q}_p(1)) \to \mathbf{Q}_p(1)$$

is a perfect duality. In the case of the open annulus appearing above there seems to be a mixing of $\mathbf{Q}_p$-duality and C-duality as well as a shift of cohomological degrees.

Now, $\mathbf{Q}_p$ and C are C-points of BC's (which are sums of copies of $\mathbf{G}_a$ "up to finite dimensional $\mathbf{Q}_p$-vector spaces"), namely $\mathbf{Q}_p$ is the C-points of $\mathbf{Q}_p$ and C is the C points of $\mathbf{G}_a$. In this category (or even in the bigger category of VS's in which finite dimensionality is dropped), one has

$$\mathrm{Hom}_{\mathrm{VS}}(\mathbf{Q}_p, \mathbf{Q}_p(1)) = \mathbf{Q}_p(1) \qquad \mathrm{Ext}^1_{\mathrm{VS}}(\mathbf{Q}_p, \mathbf{Q}_p(1)) = 0$$

$$\mathrm{Hom}_{\mathrm{VS}}(\mathbf{G}_a, \mathbf{Q}_p(1)) = 0 \qquad \mathrm{Ext}^1_{\mathrm{VS}}(\mathbf{G}_a, \mathbf{Q}_p(1)) = C$$

This leads to the following conjecture:

Conjecture. *Let* Y *be a partially proper analytic space* (or a dagger space) *defined over* C*, smooth, connected, of dimension* d*. Then*:

$$\mathrm{R}\Gamma_{\text{proét}}(Y, \mathbf{Q}_p(j)) \simeq \mathrm{RHom}_{\mathrm{VS}}(\mathrm{R}\Gamma_{\text{proét},c}(Y, \mathbf{Q}_p(d+1-j)), \mathbf{Q}_p(1))[+2d]$$

Theorem. *The conjecture is true in dimension* 1.

Remark.

(i) If Y is proper, all spaces are finite dimensional over $\mathbf{Q}_p$ and the statement is true (with usual duality) by the above mentionned results of Mann and of Zavyalov.

(ii) In order to make sense of the conjecture, one has to "geometrize" p-adic proétale cohomology to turn it into VS's. This is done in [2] where a "geometrized" comparison theorem with syntomic cohomology is also established.

(iii) Using the point of view on BC's developped in Le Bras's thesis, one can express everything that comes out of syntomic cohomology in terms of coherent cohomology of complexes of sheaves on the Fargues-Fontaine curve

X_{FF}. This suggests an ad hoc definition for $\mathrm{RHom}_{\mathrm{VS}}$ which, actually, was proved to be correct recently by Anschütz and Le Bras [1].

(iv) Using these tools, the proof for curves is rather straightforward, and probably extends, at least locally, to higher dimension.

References

[1] J. Anschütz, A.-C. Le Bras, *A Fourier Transform for Banach-Colmez spaces*, arXiv:2111.11116 [math.AG] (2021).

[2] P. Colmez, W. Nizioł, *On the cohomology of p-adic analytic spaces, I: The basic comparison theorem*, arXiv:2104.13448 [math.NT] (2021).

Reporter: Yujie Xu

Participants

Prof. Dr. Dan Abramovich
Department of Mathematics
Brown University
Box 1917
151 Thayer Street
Providence RI 02912
UNITED STATES

Piotr Achinger
Institute of Mathematics of the
Polish Academy of Sciences
P.O. Box 21
ul. Sniadeckich 8
00-656 Warszawa
POLAND

Prof. Dr. Yves André
Institut de Mathématiques de Jussieu
Paris Rive Gauche (IMJ-PRG)
4 place Jussieu
P.O. Box 247
75252 Paris Cedex 05
FRANCE

Dr. Johannes Anschütz
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Konstantin Ardakov
Mathematical Institute
Oxford University
Andrew Wiles Building
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Prof. Dr. Vladimir Berkovich
Department of Mathematics
The Weizmann Institute of Science
Rehovot 7630031
ISRAEL

Prof. Dr. Alessandra Bertapelle
Dipartimento di Matematica
"Tullio-Levi Civita"
Università di Padova
Via Trieste, 63
35121 Padova
ITALY

Prof. Dr. Amnon Besser
Department of Mathematics
Ben-Gurion University of the Negev
P.O.Box 653
84105 Beer-Sheva
ISRAEL

Prof. Dr. Bhargav Bhatt
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor MI, 48109-1043
UNITED STATES

Dr. Andreas Bode
Mathematics and Statistics
University of Calgary
2500 University Drive NW
T2N 1N4 Calgary
CANADA

Prof. Dr. Ana Caraiani
Imperial College London
Department of Mathematics
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Dr. Charlotte Chan
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Dr. Dustin Clausen
University of Copenhagen
Universitetsparken 5
København 2100
DENMARK

Prof. Dr. Pierre Colmez
Institut de Mathématiques de Jussieu
CNRS, Case 247, Theorie des Nombres
Université de Paris VI
4, place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Ehud de Shalit
Institute of Mathematics
The Hebrew University
Givat-Ram
Jerusalem 91904
ISRAEL

Dr. Gabriel Dospinescu
Département de Mathématiques
École Normale Supérieure de Lyon
46, Allée d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Antoine Ducros
Institut de Mathématiques de Jussieu
Sorbonne-Université ; Case 247
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Dr. h.c.mult. Hélène Esnault
Fachbereich Mathematik und Informatik
Freie Universität Berlin
Arnimallee 3
14195 Berlin
GERMANY

Prof. Dr. Gerd Faltings
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Laurent Fargues
Institut de Mathématiques de Jussieu
Université de Paris VI
CNRS, Case 247
4, place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Ulrich Görtz
Fakultät für Mathematik
Universität Duisburg-Essen
45117 Essen
GERMANY

Dr. Michael Groechenig
Department of Mathematics
University of Toronto
40 St. George Street
Toronto ON M5S 2E4
CANADA

Prof. Dr. Walter Gubler
Fakultät für Mathematik
Universität Regensburg
Universitätsstraße 31
93040 Regensburg
GERMANY

Prof. Dr. David Hansen
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Eugen Hellmann
Mathematisches Institut
Universität Münster
Einsteinstraße 62
48149 Münster
GERMANY

Dr. Ben Heuer
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Annette Huber-Klawitter
Mathematisches Institut
Universität Freiburg
Ernst-Zermelo-Straße 1
79104 Freiburg i. Br.
GERMANY

Dr. Katharina Hübner
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Prof. Dr. Mattias Jonsson
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Dr. Arthur-César Le Bras
C.N.R.S.
Université Paris XIII
99, avenue Jean-Baptiste Clement
93430 Villetaneuse Cedex
FRANCE

Prof. Dr. Ruochuan Liu
School of Mathematical Sciences/Peking University
No. 5 Yiheyuan Road, Haidian District
Beijing 100 871
CHINA

Prof. Dr. Francois Loeser
Institut de Mathématiques de Jussieu
Sorbonne Université
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Lucas Mann
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Tomoki Mihara
Institute of Mathematics
University of Tsukuba
1-1-1
Tennodai Tsukuba-Shi, Ibaraki 305-8571
JAPAN

Dr. Andreas Mihatsch
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Johannes Nicaise
Department of Mathematics
Imperial College London
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Wieslawa Niziol
CNRS, IMJ-PRG, Sorbonne Universite
69364 Paris
FRANCE

Prof. Dr. Jérôme Poineau
L M N O
Université de Caen Normandie
P.O. Box BP 5186
14032 Caen Cedex
FRANCE

Prof. Dr. Michael Rapoport
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Peter Schneider
Mathematisches Institut
Universität Münster
Einsteinstraße 62
48149 Münster
GERMANY

Prof. Dr. Peter Scholze
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Michael Temkin
Einstein Institute for Mathematics
The Hebrew University of Jerusalem
Givat Ram
Jerusalem 91904 01
ISRAEL

Prof. Dr. Ilya Tyomkin
Department of Mathematics
Ben-Gurion University of the Negev
P.O. Box 653
Beer-Sheva 8410501
ISRAEL

Prof. Dr. Torsten Wedhorn
Fachbereich Mathematik
Technische Universität Darmstadt
Schlossgartenstraße 7
64289 Darmstadt
GERMANY

Prof. Dr. Jared Weinstein
Department of Mathematics & Statistics
Boston University
111 Cummington Mall
Boston, MA 02215
UNITED STATES

Prof. Dr. Annette Werner
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Straße 8
60325 Frankfurt am Main
GERMANY

Dr. Matti Würthen
Institut für Mathematik
Goethe-Universität Frankfurt
Postfach 111932
60054 Frankfurt am Main
GERMANY

Prof. Dr. Chenyang Xu
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Yujie Xu
Department of Mathematics
Harvard University
Science Center, One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Tony Yue YU
Laboratoire de Mathématiques d'Orsay
Université Paris-Sud, Paris-Saclay
91400 Orsay Cedex
FRANCE

Bogdan Zavyalov
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Mingjia Zhang
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Shouwu Zhang
Department of Mathematics
Princeton University
Princeton NJ 08544
UNITED STATES