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Mini-Workshop: Regularization by Noise: Theoretical Foundations, Numerical Methods and Applications

Organized by
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Abstract. The regularizing effects of noisy perturbations of differential equations is a central subject of stochastic analysis. Recent breakthroughs initiated a new wave of interest, particularly concerning non-Markovian, infinite dimensional, and rough–stochastic/Young–stochastic hybrid systems. The mini–workshop aimed to build on these developments by bringing together young researchers in the field. Particular emphasis was given to the connection to numerical stochastic analysis, aiming to put the regularizing effects of the noise into quantitative numeric use.

Mathematics Subject Classification (2010): 60H50, 65C30, 60H10.

Introduction by the Organizers

The mini-workshop *Regularization by noise: theoretical foundations, numerical methods and applications*, organized by Oleg Butkovsky (Weierstrass Institute Berlin), Ana Djurdjevac (FU Berlin), and Máté Gerencsér (TU Wien) was held in February 2022. There were 21 participants from all over Europe (Austria, France, Germany, Netherlands, Sweden, United Kingdom) at various stages of their career (PhD students, postdocs, junior professors, full professors). Despite the challenges of the pandemic, we were happy to see a majority personal turnout: 14 participants attended the workshop in person, and 7 more online. The workshop brought together experts in various aspects of regularization by noise who shared their experience and knowledge with each other. This led to numerous new collaborations.

The scientific focus of the workshop was on different aspects of regularization by noise and its various connections: numerical analysis, stochastic differential equations (SDEs), partial differential equations (PDEs), rough paths, and more.

The most classical instance of regularization, studied since 1970s, is that the addition of a noise source into an ill-posed deterministic system might make it well-posed. Novel aspects of this phenomenon of restoring well-posedness and

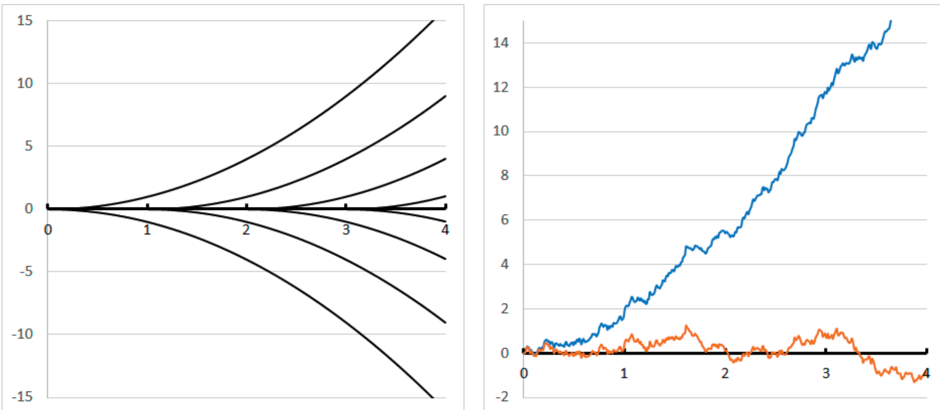

Figure 1. Regularization by noise. (a): Multiple solutions of ODE $dX_t = \text{sign}(X_t)\sqrt{|X_t|}dt$. (b): The unique solution of SDE $dX_t = \text{sign}(X_t)\sqrt{|X_t|}dt + dB_t$ (in blue) and white noise B (in brown).

more generally, of the averaging of stochastic processes, were discussed in the talks of Altman, Galeati, Kremp, and Menozzi.

By going to infinite dimensions, in other words, introducing a contiuous spatial variable, new challenges arise and the influence of the randomness in small or large space (in addition to time) gains relevance. Various such aspects were discussed in the talks on SPDEs by Gerolla, Lange, Rosati, and Tapia.

The regularizing effects of noise are also highly relevant when it comes to the question of approximations of SDEs/PDEs/SPDEs. By making the link to the classical and modern tools of regularization, several new and interesting results were presented in the context of numerical analysis by Dareiotis, Butkovsky, Eisenmann, Kruse, Lê, Ling, and Yaroslavtseva. Applications of these principles to mathematical finance, McKean-Vlasov SDEs, and machine learning were discussed by Bayer and Cox.

Apart from the talks, the other highlight of the workshop was the large number of many informal blackboard discussions among the participants, which constituted an important first step for future research and publications.

On behalf of all participants, the organizers would like to thank the staff and the director of the Mathematisches Forschungsinstitut Oberwolfach for their outstanding support and providing such a stimulating and inspiring atmosphere.

Mini-Workshop: Regularization by Noise: Theoretical Foundations, Numerical Methods and Applications

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Abstracts

RKHS regularization of singular local stochastic volatility McKean-Vlasov models

Christian Bayer

(joint work with Denis Belomestny, Oleg Butkovsky, John Schoenmakers)

In order to be useful in practice, models for financial assets (equity) are required to provide exact fits to liquid vanilla options, but have reasonable long-term dynamics. Dupire's *local volatility model* easily satisfies the first condition, but fails at the second. Conversely, some *stochastic volatility models* have reasonable long-term dynamics, but are not able to exactly fit all liquid vanilla option prices. It is, hence, tempting to use a combination of both approaches, called *local stochastic volatility model*. Given a stochastic instantaneous variance process v – chosen such as to give realistic dynamics of the asset price process S –, we are, therefore, looking for a corresponding local volatility function $\sigma = \sigma(t, x)$, such that the local stochastic volatility model

$$\mathrm{d}S_t = \sqrt{v_t}\sigma(t, S_t)S_t\mathrm{d}W_t \tag{1}$$

recovers given market prices of vanilla options. Formally, [1] realize that this is indeed the case if σ can be expressed in terms of Dupire's local volatility function σ_{Dup} by $\sigma_{\mathrm{Dup}}(t,x)^2 = \sigma(t,x)^2 E[v_t \mid S_t = x]$, $x > 0$, $t \geq 0$ – an easy consequence of Gyöngy's theorem, see [2]. However, the resulting stochastic differential equation of McKean-Vlasov type

$$\mathrm{d}S_t = \sigma_{\mathrm{Dup}}(t, S_t)S_t \frac{\sqrt{v_t}}{\sqrt{E[v_t \mid S_t]}}\mathrm{d}W_t \tag{2}$$

is quite difficult to both analyze and solve numerically. In a nutshell, the problem is that the function $(\mathcal{L}(S_t, v_t), x) \mapsto E[v_t \mid S_t = x]$ is not (Lipschitz) continuous in any usual sense, where $\mathcal{L}(S_t, v_t)$ denotes the joint distribution of (S_t, v_t). Indeed, existence and uniqueness of solutions to (2) is an open problem, see [3] for a literature review and an interesting partial result.

In this paper, we consider a *regularized* version of (2), which is motivated from a numerical point of view. A standard approach to numerical approximation of conditional expectations $E[v_t \mid S_t]$ consists in *global linear regression* – often considered more efficient than local regression as suggested by [1]. We regularize the regression estimate by including an l_2-penalty (*ridge regression*), but still observe exploding Lipschitz constants as the number of basis functions increases. In order to better control the approximation, we, hence, work in a framework provided by a *reproducing kernel Hilbert space* (RKHS) $\mathcal{H}$, see, for instance, [4]. Indeed, we use the approximation

$$E[v_t \mid S_t = \cdot] \approx \operatorname*{arg\,min}_{f \in \mathcal{H}} \left\{ E\left[(v_t - f(S_t))^2\right] + \lambda \|f\|_{\mathcal{H}}^2 \right\}, \tag{3}$$

with $\lambda > 0$ and $\|\cdot\|_{\mathcal{H}}$ denoting the norm in the RKHS.

Inserting the approximation (3) into the system (2) gives, for any $\lambda > 0$, a well-posed McKean-Vlasov system of SDEs, if v is itself a solution to an SDE. Moreover, we can also show convergence of the solution to the corresponding particle system to the solution of the regularized McKean-Vlasov system (*propagation of chaos*). It turns out that the particle system can be seen as an excellent numerical approximation to the original system (2). In particular, numerical evidence shows excellent fits to imposed market prices of vanilla options, corresponding to the actual calibration problem which motivates the whole problem.

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An affine infinite-dimensional stochastic volatility model

Sonja Cox
(joint work with Sven Karbach, Asma Khedher)

We first showed the existence of a broad class of affine Markov processes on the cone of positive self-adjoint Hilbert-Schmidt operators. Such processes are well-suited as infinite-dimensional stochastic covariance models. The class of processes we consider is an infinite-dimensional analogue of the affine processes on the cone of positive semi-definite and symmetric matrices studied in [3].

As in the finite-dimensional case, the processes we construct allow for a drift depending affine linearly on the state, as well as jumps governed by a jump measure that depends affine linearly on the state. The fact that the cone of positive self-adjoint Hilbert-Schmidt operators has empty interior calls for a new approach to proving existence: instead of using standard localisation techniques, we employ the theory on generalized Feller semigroups introduced in [5] and further developed in Cuchiero and Teichmann [4]. For the precise formulation of our main result we refer to [1, Theorem 2.8].

We then proceed to introduce a flexible and tractable infinite-dimensional stochastic volatility model. More specifically, we consider a Hilbert space H-valued Ornstein–Uhlenbeck-type process Y, whos instantaneous covariance is given by above-mentioned pure-jump stochastic process taking values in the cone of positive self-adjoint Hilbert-Schmidt operators X:

$$dY_t = \mathcal{A}Y_t\,dt + X_t^{1/2}\,dW_t^Q, \quad t \geq 0, \quad Y_0 = y \in H\,, \tag{1}$$

where $\mathcal{A}\colon D(\mathcal{A}) \subseteq H \to H$ is a possibly unbounded operator with dense domain $D(\mathcal{A})$ and $(W_t^Q)_{t\geq 0}$ is a Q-Brownian motion independent of X, with Q a positive self-adjoint trace-class operator on H.

The tractability of our model lies in the fact that the two processes (X,Y) involved are *jointly affine*, i.e., we show that their characteristic function can be given explicitely in terms of the solutions to a set of generalized Riccati equations, see [2, Section 3]. The flexibility lies in the fact that we allow multiple modeling options for the instantaneous covariance process, including state-dependent jump intensity.

Infinite dimensional volatility models arise e.g. when considering the dynamics of forward rate functions in the Heath-Jarrow-Morton-Musiela modeling framework using the Filipović space. Several examples are discussed in [2, Section 4]

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Approximation of stochastic equations with irregular drifts

Konstantinos Dareiotis

(joint work with Oleg Butkovsky, Máté Gerencsér, Khoa Lê)

In this talk we are interested in the approximation of stochastic equations whose drifts that are not Lipschitz continuous. The well-posedness of the equations under consideration relies on the regularising properties of the driving noise, since their deterministic counterparts are not, in general, well-posed. Our aim will be to quantify those regularising properties of the noise at a numerical analytic level in order to derive rates of convergence of discrete approximations. We deal with different settings, including SDEs driven by Brownian motion and by fractional Brownian motion, and with stochastic PDEs.

We consider the equation

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t, \qquad X_0 = x \in \mathbb{R}^d, \qquad t \in [0,1], \tag{1}$$

where B is a standard Brownian motion, b is a vector field which is merely measurable and bounded, and σ is C^2 and non-degenerate. Our first result states that under these conditions, the Euler approximation X^n of X converges to X in $L_p(\Omega; C([0,1]))$, for any $p \geq 1$, with rate (almost) $1/2$. This rate is known to be optimal even for $b \equiv 0$. In order to obtain this rate in our irregular setting, we exploit the regularising properties of the noise. The main tool for capturing those properties is Lê's *stochastic sewing lemma* (SSL) and appropriate generalisations. By using the SSL we study functionals of the type

$$I(f,t,x) = \int_0^t f(X_s + x)\,ds,$$

which are closely related to the well-posedness/ stability/numerical approximation of (1). In case σ is constant, we show that the rate improves to $(1+\alpha)/2$, provided that b also belongs to the homogeneous Sobolev space $\dot{W}^{\alpha}_{q}$, with $\alpha \in (0,1)$ and $q \geq \max\{2,d\}$.

Further, we study the asymptotic distribution of the error $X - X^n$ (see [5] for the case $b \in C^1$). We show that if $b \in C^\alpha$ for some $\alpha > 0$, then the process $V^n = \sqrt{n}(X - X^n)$ converges weakly to a process V, which is uniquely determined by the equation

$$dV_t^\ell = V_t^j dL_t^j[X, b^\ell] + \partial_j \sigma^{\ell i}(X_t) V_t^j dB_r^i + \frac{1}{\sqrt{2}} (\partial_j \sigma^{\ell i} \sigma^{jk})(X_t) dW_t^{ki}, \tag{2}$$

where $L_t^j[X, b^\ell]$, formally given by

$$L_t^j[X, b^\ell] = \int_0^t \partial_j b^\ell(X_s)\,ds, \qquad i, \ell = 1, ..., d,$$

is a process which can be defined via the SSL, and with probability one $L_t^j[X, b^\ell] \in C^{(1+\alpha)/2}([0,1])$. Equation (2) is then understood as a hybrid Itô-Young equation. In (2), W is a (matrix-valued) Brownian motion independent of B.

One of the advantages of the SSL is that it does not rely on the Markovian nature of the underlying problem, in contrast to other techniques such as PDE transformations of Zvonkin-type. In this talk we use it further to study the approximation of SDEs driven by fractional noise: Let B^H be a fractional Brownian motion with Hurst parameter $H \in (0,1)$ and let $b \in C^\alpha$ for some $\alpha > 0 \vee (1 - 1/(2H))$. It is known by [3] that under this condition the equation

$$dX_t = b(X_t)\,dt + dB_t^H, \qquad X_0 = x \in \mathbb{R}^d, \qquad t \in [0,1], \tag{3}$$

is well-posed. We sketch in the talk that in this case, the Euler scheme converges with rate $(1/2 + \alpha H) \wedge 1$.

Further, we discuss equations in infinite dimensions. Let us consider the periodic problem

$$\partial_t u = \Delta u + f(u) + \xi, \qquad u(0,x) = u_0(x), \qquad (t,x) \in [0,1] \times \mathbb{T}, \tag{4}$$

where ξ is a space-time white noise on $[0,1] \times \mathbb{T}$. While the approximation of the solution has been extensively studied for $f \in C^1$, in the case $f \in C^\alpha$, with $\alpha < 1$, no quantitative result was available for the approximation of u. By using an infinite dimensional version of the SSL, we show that the fully discrete explicit finite difference scheme converges with rate $1/4$ in time and $1/2$ in space even for merely bounded and measurable f, a rate which is known to be optimal even for $f \equiv 0$.

A problem related to these results is the optimality of the rate $(1/2 + \alpha H)$ for (3). To the best of our knowledge, the only available work in this direction is [6], for the case $H = 1/2$. It follows from the results in [6] that there exists (discontinuous) $b \in W_2^{1/2}$ for which the corresponding rate of convergence cannot be better than $3/4$ ($= (1 + \alpha H)$ for $\alpha = H = 1/2$). Even in the case $H = 1/2$, it would be interesting to show that the rate $(1+\alpha)/2$ cannot be improved for some $b \in C^\alpha$.

The talk is based on results from [1, 2, 4] and some work in progress.

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Randomized operator splitting schemes for abstract evolution equations

Monika Eisenmann

Background. Abstract evolution equations are an important building block for modeling processes in physics, biology and social sciences. Equations of the form

$$u'(t) + A(t)u(t) = f(t), \quad u(0) = u_0, \tag{1}$$

include standard examples of $A(t)v$ like

$$-\nabla \cdot \big(\alpha(t)|\nabla v|^{p-2}\nabla v\big) \quad \text{and} \quad -\Delta\big(\alpha(t)|v|^{p-2}v\big) \tag{2}$$

for a function $v\colon \mathcal{D} \to \mathbf{R}$ on a domain $\mathcal{D} \subset \mathbf{R}^d$. Moreover, they are tightly connected to optimization problems appearing in a machine learning context, where we want to find w^* such that

$$w^* = \operatorname*{argmin}_w F(w) = \frac{1}{N}\sum_{i=1}^N f_i(w). \tag{3}$$

Standard examples for this are classification problems that we solve by minimizing the function F. One approach to finding the minimum w^* is to compute the steady state of the gradient flow

$$w'(t) + \nabla F(w(t)) = 0, \quad w(0) = w_0. \tag{4}$$

This equation is a special case of (1) with $A(t) \equiv \nabla F$.

To find a suitable approximation of the solution of (1), we can begin to discretize with respect to the temporal variable t. For a step size h and the initial value $U^0 = u_0$, two simple, standard schemes to find an approximation U^n of $u(nh)$ ($n \in \mathbb{N}$) are given by

(5) $$U^n = \big(I - hA((n-1)h)\big)U^{n-1} + hf((n-1)h), \qquad \text{forward Euler,}$$

(6) $$U^n = \big(I + hA(nh)\big)^{-1}(U^{n-1} + hf(nh)), \qquad \text{backward Euler.}$$

In many applications, it is expensive to consider the entire operator $A(nh)$ in (5) and (6). However, after decomposing $A(nh)$ into sub-operators $A_i(nh)$, $i = 1, \dots, s$, such that $A(nh) = \sum_{i=1}^{s} A_i(nh)$, an operator splitting can be applied to rewrite the expensive equation into several small-scale equations.

Deterministic domain decomposition. For differential operators of the type stated in (2), a domain decomposition scheme is a powerful tool. The domain $\mathcal{D}$ can easily be decomposed into sub-domains. The idea is that $A_i(nh)$ only acts on one subset of the domain. It thus becomes possible to solve (6) only on the sub-domains. It remains to combine the solutions on the sub-domains to a solution on the entire domain. How to piece the solutions together is the key part of the abstract operator splitting and gives rise to several different temporal approximation schemes.

Stochastic optimizers. While the approach described above is purely deterministic, a very similar theoretical background can be found in optimization problems. For a large-scale problem, it becomes unreasonable to consider the entire sum in (3) to evaluate ∇F in every step of (5) with $A(t) \equiv \nabla F$. An alternative is given by a stochastic method. Instead of including every summand in (3), we choose a random subset B_ξ of $\{1, \dots, N\}$ and evaluate the corresponding summands. More precisely, we obtain a random approximation of F given by

$$f(w, \xi) = \frac{1}{|B_\xi|} \sum_{i \in B_\xi} f_i(w).$$

In every step of (5), we choose a new batch B_ξ and therefore a new stochastic approximation. In the optimization community, this method is known as the stochastic gradient descent method. The scheme can be interpreted as an operator splitting, where the decomposition happens in a randomized fashion. We save computational costs as we do not need to evaluate the entire operator in every step.

Combination of strategies and results. We apply a randomized operator splitting in combination with a domain decomposition. It is common to use a uniform distribution to decompose the sum in (3) if no additional information on the data set is known. For an evolution equation (1) with a differential operator of the form mentioned in (2), such an equation has a local structure. This means that the solution at a point x depends on the solution at the point $x + \varepsilon y$ for a small deviation εy but not on the solution far away. Thus, it is not advisable to choose a uniform distribution here. Instead, we choose a different probability distribution to select parts of the operator more efficiently by looking at the support of the solutions of previous time steps and the source term.

We then use a prediction of the support of the solution at the new time-step to give every sub-domain a certain probability of being chosen in our random approximation of the operator. If the predictor indicates that nothing or very little happens in a sub-domain, then we only choose this sub-domain with a small probability. On the other hand, if the predictor suggests that change is very likely to occur in one sub-domain, then we choose it with a high probability. To handle the fact that certain parts of the domain are unlikely to occur, we then take a time step considering such a sub-domain with a larger step size to compensate for the low probability. If it is almost certain that one sub-domain is chosen, we take the corresponding step with the almost ordinary length.

We provide error bounds in an abstract non-linear setting for a randomized operator splitting scheme where the error is measured in expectation. While the convergence rate in a purely deterministic framework is higher for smooth solutions, in the non-linear framework solutions often lack the needed higher-order regularity anyway. Moreover, the computational cost of time steps can be reduced in the randomized framework. We verify the theoretical results with a preliminary numerical example for a randomized splitting combined with a domain decomposition scheme for a simple test equation.

Scaling Limits of Additive functionals of rough processes without self-similarity

Henri Elad Altman
(joint work with Khoa Lê)

This talk addresses the long-time behaviour of additive functionals of a family of stochastic processes known as mixed fractional Brownian motions.

For a stochastic process $(X_t)_{t\geq 0}$ with values in $\mathbb{R}^d$, given a measurable function $f : \mathbb{R}^d \to \mathbb{R}$, the process $\int_0^t f(X_s)\,ds$, $t \geq 0$, is called an additive functional of $(X_t)_{t\geq 0}$. Under standard assumptions, for all $a \in \mathbb{R}^d$ we may also consider the additive functional associated with $f = \delta_a$, the Dirac measure at a: one obtains an object $L_t^X(a)$, called *local time* of the process X at the point a, and given formally as $\int_0^t \delta_a(X_s)\,ds$, see e.g. [4]. Jointly continuous local times exist for a 1-dimensional Brownian motion, and more generally for any fractional Brownian motion (fBM) in $\mathbb{R}^d$ with Hurst parameter H such that $Hd < 1$.

We first recall a known result for the case of standard Brownian motion, and more generally fractional Brownian motions, c.f. [6].

Theorem 1 (Darling-Kac Theorem). (1) *Let $(B_t)_{t\geq 0}$ be a standard Brownian motion in $\mathbb{R}$. Let $f \in L^1(\mathbb{R})$ and $t \geq 0$ fixed. Then, as $\lambda \to \infty$, $\frac{1}{\sqrt{\lambda}} \int_0^{\lambda t} f(B_s)\,ds$ converges in distribution to $(\int_{\mathbb{R}} f(x)\,dx)\,L_t^B(0)$.*

(2) *More generally, let $d \geq 1$ and $H \in (0,1)$ such that $Hd < 1$, and let $(B_t^H)_{t\geq 0}$ be a d-dimensional fBM of Hurst parameter H. Let $f \in L^1(\mathbb{R}^d)$ and $t \geq 0$ fixed. Then, as $\lambda \to \infty$, $\lambda^{Hd-1}\int_0^{\lambda t} f(B_s^H)\,ds$ converges in distribution to $(\int_{\mathbb{R}^d} f(x)\,dx)\,L_t^{B^H}(0)$.*

The above results rely heavily on the self-similarity of the underlying process. Thus the question arises: what can we say for a stochastic process that is not self-similar? We consider a toy model of stochastic process that is not self-similar.

Definition 1 (see [1]). *For $\alpha \in \mathbb{R} \setminus \{0\}$ and $H \in (0,1)$, we call mixed fractional Brownian motion (mixed fBM for short) a process of the form $X_t := B_t + \alpha B_t^H$, where B (resp. B^H) is a d-dimensional Brownian motion (resp. fractional Brownian motion), and B^H is independent from B.*

As soon as $H \neq 1/2$, a mixed fBM as defined above is not self-similar. The analysis of the long-time behaviour of its additive functionals therefore becomes highly challenging. We have the following, main result:

Theorem 2 (Elad Altman, Lê, 2022+). *For all $f \in L^1(\mathbb{R}^d)$ and any fixed $t \geq 0$, the following limits hold:*

(1) *when $H < \frac{1}{2}$ and $d = 1$, $\lambda^{-\frac{1}{2}}\int_0^{\lambda t} f(X_r)dr$ converges as $\lambda \to \infty$ in distribution to $\int_{\mathbb{R}} f(x)dx\, L_t^B(0)$,*
(2) *when $H > \frac{1}{2}$ and $d = 1$, $\lambda^{H-1}\int_0^{\lambda t} f(X_r)dr$ converges as $\lambda \to \infty$ in distribution to $\int_{\mathbb{R}} f(x)dx\, L_t^{B^H}(0)$,*
(3) *when $d \geq 2$ and $H < \frac{1}{d}$, then $\lambda^{Hd-1}\int_0^{\lambda t} f(X_r)dr$ converges as $\lambda \to \infty$ in probability to 0.*

One of the major ingredients in the proof of the above Theorem consists in identifying continuity properties of the local time process $L_t^X(a)$ with respect to the underlying process X, an agenda one does not seem to be able to carry out using standard techniques. In this perspective, a crucial tool to perform our analysis is the Stochastic Sewing Lemma of Khoa Lê [7], which we use both to construct the local times of mixed fBM, and study their continuity properties with respect to the underlying process. Our techniques do not rely in a crucial way on Gaussianity of the process X, although this feature makes computations easier. We believe our construction can be used to tackle more general processes, even beyond the Gaussian setting.

There remain many interesting open questions:

(1) One may study fluctuation results, by asking what is the correct scaling in the first two statements of Theorem 2 above, when $\int f(x)\,dx = 0$. The case of a fBM was studied in [6].
(2) Alternatively we may wonder what happens if f is not integrable. Note that such results are known in the case of a standard Brownian motion [8] and in the case of stable processes [3]. It would be interesting to obtain an extension of such results to the non-self-similar setting.

(3) Correct scaling when $d \geq 2$, $H < \frac{1}{2}$ and $dH < 1$? Namely, the 3rd scaling limit result in Theorem 2 is sub-optimal, and it seems very interesting, albeit quite challenging, to find the correct scaling at which the additive functionals exhibit a non-trivial limit. Note that, in $d \geq 2$, the standard Brownian motion does not possess local times. It is plausible that the correct scaling may be logarithmic, in a way reminiscent of the case of a 2-dimensional Brownian motion, see [5] and [2].
(4) Scaling limits for additive functionals of more complicated processes (e.g. solutions to stochastic partial differential equations)? This seems to be a very open question, but the Stochastic Sewing Lemma appears as a promising tool to tackle it.

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Some recent advances on SDEs with fractional noise

Lucio Galeati
(joint work with Máté Gerencsér)

It is by now well established, after the pioneering work of Catellier and Gubinelli [1], that fractional noise can have a strong regularizing effect on ODEs. Specifically, denoting by W^H a fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$, SDEs of the form

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \mathrm{d}W^H_t, \quad X|_{t=0} = x_0 \in \mathbb{R}^d \tag{1}$$

are wellposed whenever $b \in C^\alpha_x$ with $\alpha > 1 - 1/(2H)$ (consider autonomous b for the moment). More recently, the same result has been extended in [5] the cover the whole regime $H \in (0, +\infty) \setminus \mathbb{N}$; observe that any choice of $H < \infty$ allows for values $\alpha < 1$ (i.e. non-Lipschitz drift b), so that explicit counterexamples to uniqueness for the ODE would be available in the absence of W^H.

The above results give rise to several natural questions:

A) Is the regime $\alpha > 1/(2H)$ (possibly $\alpha \geq 1/(2H)$) optimal?
B) Even when wellposedness is already known, can one obtain finer information on the solutions, e.g. construct the associated stochastic flow?
C) Is the stochastic transport equation associated to the SDE also wellposed?

One of the main challenges in addressing the above questions lies in the fact that, for $H \in (0,1) \setminus \{1/2\}$, W^H is not a Markov process nor a semimartingale; therefore, all standard stochastic calculus tools used to solve the SDE (Itô formula, Zvonkin transform, martingale problem, PDE methods, etc.) completely break down and new strategies must be developed in order to overcome this diffuculty.

It is worth observing that a simple scaling argument already suggests the expected optimal regularity needed to solve the SDE. Recall that W^H is self-similar with parameter H, in the sense that $\tilde{W}^H_t := \lambda^{-H} W_{\lambda t}$ is again distributed as W^H; applying the same scaling to (1), we see that $\tilde{X}_t := \lambda^{-H} X_{\lambda t}$ solves the SDE associated to $\tilde{W}^H$ and new drift

$$\tilde{b}(t,x) = \lambda^{1-H} b(\lambda t, \lambda^H x).$$

Considering $b \in L^q_t C^\alpha_x$ and looking at how it behaves under rescaling, the suggested *subcritical regime* (i.e. where $\|\tilde{b}\|_{L^q C^\alpha} \to 0$ as $\lambda \to 0$) corresponds to

$$\alpha > 1 - \frac{1}{q' H} \tag{2}$$

where q' denotes to the conjugate exponent of q. Our main result from [3] roughly speaking states that, for the choice $q = 2$, we can solve the SDE (1) in the full subcritical regime.

Theorem 1. *Let $H \in (0,+\infty) \setminus \mathbb{N}$, W^H fBm of parameter H, $b \in L^2_t C^\alpha_x$ with $\alpha > 1 - \frac{1}{2H}$, $\alpha \in \mathbb{R}$. Then:*

1) *Strong existence, pathwise uniqueness and path-by-path uniqueness hold for the SDE (1), for any $x_0 \in \mathbb{R}^d$.*
2) *The SDE admits a stochastic flow of diffeomorphisms.*
3) *Strong stability estimates are available. Namely, let X^i be solutions to (1) associated to (x^i_0, b^i), $i = 1,2$, and same W^H; then for any $p \in [1,\infty)$ there exists a constant C, depending on $\alpha, H, T, \|b^i\|_{L^q C^\alpha}$, such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X^1_t - X^2_t|^p\Big]^{\frac{1}{p}} \leq C(|x^1_0 - x^2_0| + \|b^1 - b^2\|_{L^2 C^{\alpha-1}}) \tag{3}$$

The proof relies on a combination of the strategy developed in [5] and the use of an appropriate variant of the stochastic sewing lemma (SSL) developed in [6]; specifically, we need a version of the SSL accomodating the presence of controls, shifts and conditional norms. For the concept of path-by-path uniqueness we refer to [2]; in the case $\alpha < 0$ (distributional drift), the SDE can be given pathwise meaning by means of nonlinear Young integration as in [1]. Finally, the stability estimate (3) is an improvement of the one presented in [4] and can be used as therein to solve distribution dependent SDEs driven by W^H (this result will also appear in [3]).

Theorem 1 provides further insight on questions A) and B) presented above; we are currently working on C) and we believe we can give it a positive answer for b satisfying the same assumptions as in Theorem 1. At the same time, although we have further weakened the time regularity on b, our result does not improve the spatial regularity threshold $\alpha > 1 - 1/(2H)$. Thus the fundamental question of whether one can lower the value of α, by assuming $b \in L^q_t C^\alpha_x$ for higher values of q, in accordance to formula (2), remains open.

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Fluctuations of the stochastic heat equation in dimensions three and higher.

Luca Gerolla

(joint work with Xue-Mei Li, Martin Hairer)

Recent results concerning the stochastic heat equation (SHE) and KPZ equation in dimension $d \geq 3$ showed that their large scales fluctuations converge to a Gaussian field, given by the solution of the Edwards-Wilkinson equation

$$\partial_t \mathcal{U} = \tfrac{1}{2}\Delta \mathcal{U} + \beta\nu \dot{W}.$$

Here β denotes a small coupling constant, ν the effective variance and $\dot{W}$ space-time white noise. We start by introducing the models and the weak/strong disorder regimes dictated by the value of β [7]. We then review currently available results [6, 4, 2, 3, 1, 5] on the equations, in particular the effective variance and the role of the compactly supported (integrable) spatial covariance in the noise considered. Hence we motivate ongoing work on the non-linear SHE

$$\partial_t u = \tfrac{1}{2}\Delta u + \beta\sigma(u)\xi,$$

where ξ is white in time Gaussian noise with non-integrable spatial covariance R, displaying power law decay $R(x) \sim |x|^{-\kappa}$ at infinity, with $\kappa \in (2, d)$. In these settings, we expect analogue Gaussian large scales fluctuations. However, compared to current results, we observe different effective variance and order of fluctuations, with a Edwards-Wilkinson limit which is now driven by noise that has Riesz kernel $|x|^{-\kappa}$ correlations in space. We outline the work in progress results and the main techniques employed.

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Weak rough-path-type solutions for singular Lévy SDEs

Helena Katharina Kremp

(joint work with Nicolas Perkowski)

The talk is based on [5] and the recent works [6, 7]. We study the weak well-posedness of multidimensional Lévy-driven stochastic differential equations of the form,

$$dX_t = V(t, X_t)dt + dL_t, \qquad X_0 = x \in \mathbb{R}^d, \tag{1}$$

where L is a non-degenerate, symmetric, α-stable Lévy process for $\alpha \in (1, 2]$, and $V \in C([0,T], (\mathscr{C}^\beta)^d)$ is a Besov distribution in the space variable for $\beta < 0$.

Since the works by Delarue, Diel [4] and Cannizzaro, Chouk [2] (in the Brownian noise setting), and our previous work [5], the existence and uniqueness of solutions to the martingale problem associated to (1) is known, when the Besov regularity of the drift satisfies $\beta > (2 - 2\alpha)/3$ (rough regime). In the socalled Young regime, where $\beta > (1 - \alpha)/2$, weak existence and uniqueness for the SDE with stable noise was already established in [3] and [1] (with strong existence in dimension $d = 1$).

The key ingredient in the rough case is to solve the associated backward Kolmogorov equation

$$\mathcal{G}^V u = (\partial_t - \mathcal{L}_\nu^\alpha + V \cdot \nabla)u = f, \quad u(T, \cdot) = u^T \tag{2}$$

for $f \in L^\infty$, $u^T \in \mathscr{C}^3$, with the paracontrolled ansatz. Here, $-\mathcal{L}_\nu^\alpha$ denotes the generalized fractional Laplacian, that is the generator of the stable process (ν being the spherical component of the jump measure). The idea of the paracontrolled ansatz, is to treat u as a perturbation of the linearized equation, $\partial_t w = \mathcal{L}_\nu^\alpha w - V$, and to leverage this to gain some regularity. This works as long as the product $V \cdot \nabla u$ is of lower order than the linear operator $\mathcal{L}_\nu^\alpha$, i.e. if $\alpha > 1$ and $\beta > 1-\alpha$ (subcritical regime). For that, in the rough case, we assume that the drift V can be enhanced, i.e. that the resonant product $\mathscr{V}_2 := J^T(\nabla V) \odot V \in C([0,T], (\mathscr{C}^{\alpha+2\beta-1})^d)$ exists, where $J^T(v)(t) := \int_t^T P_{r-t} v_r dr$ and (P_t) being the $(-\mathcal{L}_\nu^\alpha)$-semigroup (notation: $\mathscr{V} = (\mathscr{V}_1, \mathscr{V}_2) = (V, \mathscr{V}_2) \in \mathscr{X}^\beta$).

Motivated by the equivalence of probabilistic weak solutions to SDEs with L^∞-drift and solutions to the martingale problem, we define a (non-canonical) weak solution concept for singular Lévy diffusions, proving moreover equivalence to the martingale solution in both the Young, as well as in the rough regime. In the Young regime, a canonical weak solution concept is well-posed (cf. also in [1]). Here, the canonical weak solution is a tuple of stochastic processes (X, L) on some probability space, such that L is a symmetric, α-stable Lévy process and X is given by $X = x + Z + L$, where Z satisfies, for all $0 \leqslant s < t \leqslant T$,

$$(3) \qquad \|Z_{s,t}\|_{L^2(\mathbb{P})} \lesssim |t-s|^{(\alpha+\beta)/\alpha}, \quad \|\mathbb{E}_s[Z_{s,t}]\|_{L^\infty(\mathbb{P})} \lesssim |t-s|^{(\alpha+\beta)/\alpha}$$

and is such that it exists a sequence (V^n) of smooth V^n with $V^n \to V$ in $C([0,T], (\mathscr{C}^\beta)^d)$ and with

$$(4) \qquad Z_t = \lim_{n\to\infty} \int_0^t V^n(r, X_r) dr =: \lim_{n\to\infty} Z_t^n$$

in $L^2(\mathbb{P})$. Notice that $(\alpha+\beta)/\alpha > 1/2$, such that Z is a zero quadratic variation process and X is a Dirichlet process. Constructing a counterexample, we prove in [6] that, contrary to the Young case, in the rough regime, the canonical weak solution is in general non-unique in law. This is due to the fact, that we can construct sequences (V^n), (W^n) with $J^T(\nabla V) \odot V^n \to \eta$ and $J^T(\nabla V) \odot W^n \to \eta + C$ for a constant $C > 0$.

By imposing a rough-path-type assumption for a weak solution X, we obtain a well-posed solution concept. That is, we furthermore require for a weak solution X, that the iterated integrals

$$(5) \qquad \mathbb{Z}_{s,t}^V := \lim_{n\to\infty} \int_s^t (J^T(\nabla V)(r, X_r) - J^T(\nabla V)(s, X_s)) dZ_r^n$$

are well-defined in $L^2(\mathbb{P})$ and satisfy the Hölder-type bounds, for all $0 \leqslant s < t \leqslant T$,

$$(6) \qquad \|\mathbb{Z}_{s,t}^V\|_{L^2(\mathbb{P})} \lesssim |t-s|^{(\alpha+\beta)/\alpha}, \quad \|\mathbb{E}_s[\mathbb{Z}_{s,t}^V]\|_{L^\infty(\mathbb{P})} \lesssim |t-s|^{(2(\alpha+\beta)-1)/\alpha},$$

where $2(\alpha+\beta) - 1 > \alpha + \beta$. That is, we define:

Definition 1 (Weak solution).
We call $(X, L, \mathbb{Z}^V)$ a weak solution, if X is given by $X = x + Z + L$ and if it exists a sequence of smooth V^n, $n \in \mathbb{N}$, with $V^n \to V$ in $\mathscr{X}^\beta$ and (4) and (5) holding true. Furthermore Z satisfies (3) and $\mathbb{Z}^V$ satisfies (6).

The main theorem of [6] is then the following.

Theorem 1.
Let $V \in \mathscr{X}^\beta$ for β in the rough regime and $x \in \mathbb{R}^d$. Then $(X, L, \mathbb{Z}^V)$ is a weak solution if and only if X solves the $(\mathcal{G}^V, x)$-martingale problem (cf. [5, Definition 4.1]*).*

For a weak solution $(X, L, \mathbb{Z}^V)$, an application of the stochastic sewing lemma by Khoa Lê [8], yields existence and stability of the rough stochastic integral with germ $\Xi_{s,t} = \nabla u(s, X_s)[Z_{s,t} + \mathbb{Z}^V_{s,t}]$ in $L^2(\mathbb{P})$ for u solving (2). This enables to prove, that a weak solution is a martingale solution, in particular the weak solution is unique in law. To prove, that for a martingale solution X, the iterated integrals $\mathbb{Z}^V$ satisfy the bounds (6), we solve the backward Kolmogorov equation with *singular paracontrolled* terminal conditions (theory contained in [7]) and utilize bounds on those.

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Finite p-variation of solutions to (stochastic) evolution equations and applications in numerical analysis

Raphael Kruse

(joint work with Johanna Weinberger, Rico Weiske)

In this contribution to the MFO mini-workshop 2207c we discuss the temporal regularity of stochastic evolution equations in terms of the finite p-variation norm. In addition, we illustrate how this norm can be used in numerical analysis to derive error estimates for discretization methods. The content of the talk and this extended abstract is based on joint work [4, 5] with Johanna Weinberger and Rico Weiske (both MLU Halle-Wittenberg).

First, we briefly recall the analytical framework for stochastic evolution equations from [1]: For $T \in (0, \infty)$ let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbf{P})$ be a filtered probability space satisfying the usual conditions. Further, let H, U be separable Hilbert spaces.

Then, we consider the mild solution $X\colon [0,T]\times\Omega\to H$ to a stochastic evolution equation of the form

$$\mathrm{d}X(t) = AX(t)\,\mathrm{d}t + G(X(t))\,\mathrm{d}W(t), \quad t\in(0,T], \quad X(0)=X_0, \tag{1}$$

where $X_0\in L^2(\Omega,\mathcal{F}_0,\mathbf{P};H)$ denotes the initial condition, $(W(t))_{t\in[0,T]}$ is a U-valued Wiener process with covariance operator $Q\colon U\to U$ satisfying $\mathrm{Tr}(Q)<\infty$, and the Lipschitz continuous mapping $G\colon H\to\mathcal{L}_2(U_0,H)$ takes values in the space of Hilbert–Schmidt operators.

In addition, we assume that the operator $A\colon \mathrm{dom}(A)\subset H\to H$ is densely defined, linear, self-adjoint, and negative definite with a compact inverse. This implies that A is the infinitesimal generator of an analytic semigroup $(E(t))_{t\in[0,\infty)}$ on H. Under these assumption (1) admits a unique mild solution of the form

$$X(t) = E(t)X_0 + \int_0^t E(t-s)G(X(s))\,\mathrm{d}W(s), \quad t\in[0,T]. \tag{2}$$

For all details regarding existence and uniqueness of the solution X we refer to [1].

In numerical analysis of stochastic evolution equations one often depends on estimates of the temporal regularity of the exact solution X to derive the optimal order of convergence. For instance, the order of convergence of the backward Euler method typically coincides with the exponent $\gamma\in(0,1]$ of Hölder continuity of the exact solution as discussed in, e.g., [3, 6].

In [4] we illustrate that, in some situations, it is beneficial to measure the temporal regularity of X in terms of finite p-variation instead of the more common notion of Hölder continuity. To be more precise, for $p\in[1,\infty)$ we say that X is of finite p-variation with respect to the $L^2(\Omega;H)$ if

$$|X|_{p-\mathrm{var},L^2(\Omega;H)} := \sup_{\pi\subset[0,T]}\Big(\sum_{t_i\in\pi}\|X(t_i)-X(t_{i-1})\|^p_{L^2(\Omega;H)}\Big)^{\frac{1}{p}}<\infty,$$

where the supremum is taken over the set of all finite partitions $\pi=\{0=t_0<t_1<\ldots<t_N=T\}$ of the interval $[0,T]$.

It is straight-forward to show that if X is Hölder continuous with exponent $\gamma\in(0,1]$ then it also of finite p-variation for every $p\in(\frac{1}{\gamma},\infty)$. More importantly, the Garsia–Rodemich–Rumsey result, see [2, App. A], yields embeddings of more general (fractional) Sobolev and Besov spaces into the space of all functions with finite p-variation.

The main purpose of [4] is then to show that, under certain conditions, the mild solution X in (2) is not (or only with a very small exponent) γ-Hölder continuous but, at the same time, it is of finite p-variation for some $p<\frac{1}{\gamma}$. In fact, this situation already occurs when the initial condition X_0 is of low regularity measured in terms of fractional powers of the infinitesimal generator A. To be more precise, if $X_0\in L^2(\Omega,\mathcal{F}_0,\mathbf{P};\mathrm{dom}((-A)^\gamma))$ for some small $\gamma\in(0,1]$ then the mapping $[0,T]\ni t\mapsto E(t)X_0\in L^2(\Omega,\mathcal{F}_0,\mathbf{P};H)$ is γ-Hölder continuous with respect to the norm in $L^2(\Omega,\mathcal{F}_0,\mathbf{P};H)$. However, it is shown in [4] that the mapping $t\mapsto E(t)X_0$ is of finite 1-variation regardless of the value of $\gamma\in(0,1]$.

By applying techniques for the numerical analysis of rough differential equations from [7] it is then shown in [4, 5] that the temporal order of convergence of the backward Euler method and the BDF2-Maruyama method only depends on the value $\frac{1}{p}$ provided the exact solution is of finite p-variation. Together, these two results yield the optimal order of convergence also for stochastic evolution with, for instance, non-smooth initial values.

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On regularization by noise of an averaged version of the Navier-Stokes equations

Theresa Lange

(joint work with Martina Hofmanová)

Consider the three-dimensional deterministic Navier-Stokes equations describing the time evolution of an incompressible, viscous fluid on $\mathbb{R}^3$ by

$$\begin{aligned} \partial_t u + (\nabla \cdot u)u + \nabla p &= \Delta u, \\ \nabla \cdot u &= 0, \\ u(0,\cdot) &= u_0 \end{aligned} \tag{1}$$

with velocity field $u : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$ and pressure field $p : [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}$. In particular, projected onto the divergence-free vector fields these read

$$\begin{aligned} \partial_t u &= \Delta u + B(u,u), \\ u(0,\cdot) &= \Pi u_0 \end{aligned} \tag{2}$$

with Π the Leray projection, and B the symmetric Euler bilinear operator satisfying the cancellation property

$$\langle B(u,u), u \rangle = 0 \tag{3}$$

in $L^2(\mathbb{R}^3)$. Well-posedness in the sense of existence of global-in-time smooth solutions to (1) is a largely open problem: though in the regime of small initial data or hyperdissipative variants there exist positive results exploiting symmetry and cancellation property of B, these, however, do not extend to the general case. In fact, approaches that merely use these two properties will be doomed to fail: in [1], the author constructs a variant C of B satisfying symmetry and (3) such that the corresponding system of the form (2) experiences a blow-up in finite time.
On the other hand, the field of regularization by noise investigates whether by augmenting an ill-posed deterministic system by a stochastic perturbation improves its regularity. Of large interest in the context of (1) is a recently analysed Stratonovich noise of transport type employed in [2] and [3]: on $\mathbb{T}^3$ consider

$$\sum_{k\in\mathbb{Z}_0^3}\sum_{i=1}^{2}\theta_k\Pi((\sigma_{k,i}\cdot\nabla)\cdot)\circ \mathrm{d}W_t^{k,i} \tag{4}$$

with a specific choice of divergence-free vector fields $\sigma_{k,i}$, and $W^{k,i}$ a family of complex Brownian motions. Via a scaling limit argument, the authors of [2] were able to show that there exists a choice of parameters θ_k such that there exists a unique strong (in the probabilistic sense) solution to the perturbed vorticity equation of (1), and extended their result to hold for more general models as exploited in [3].
This talk aims at bringing together these two concepts: let C be a bilinear operator of the form as constructed in [1] on $\mathbb{T}^3$ and let $\xi := \nabla\times u$ denote the vorticity field of a solution u to (2) with nonlinearity C. Then the time evolution of ξ is of the form

$$\partial_t\xi = \Delta\xi + F(\xi)$$

and by means of the analysis in [2] and [3] we are able to show that the above regularization result also applies to

$$\mathrm{d}\xi = (\Delta\xi + F(\xi))\mathrm{d}t + \sqrt{\mathcal{C}}\sum_{k\in\mathbb{Z}_0^3}\sum_{i=1}^{2}\theta_k\Pi((\sigma_{k,i}\cdot\nabla)\xi)\circ \mathrm{d}W_t^{k,i} \tag{5}$$

and similarly for higher order derivatives of ξ. This illustrates a great strength of the transport noise (4) rendering its potential regularization skills in the context of (1). Note, however, that whether regularization in the above sense also holds for (5) on the level of the velocity u is yet unresolved. Furthermore it remains to be shown whether the concrete construction in [1] can be regularized by (4). On the other hand, an interesting task is to identify a noise intrinsically related to the model in [1] which delays the blow-up. Noise of type (4) might provide a good starting point in this direction, which is part of our future research.

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Quadrature estimates by stochastic sewing

Khoa Lê

(joint work with Konstantinos Dareiotis, Máté Gerencsér, Chengcheng Ling)

The talk is based on recent joint works with K. Dareiotis, M. Gerencsér ([6]) and C. Ling ([12]). Consider the stochastic differential equation (SDE)

$$dX = b(t, X)dt + \sigma(t, X)dB \tag{1}$$

driven by a multidimensional Brownian motion B. The diffusion coefficient σ is uniformly elliptic and regular while the drift vector field b is irregular, discontinuous and possibly singular. Two particular classes of drifts are Borel bounded functions and integrable functions in $L^q([0,1]; L^p(\mathbb{R}^d))$ with $\frac{d}{p} + \frac{2}{q} < 1$.

The corresponding equidistant tamed Euler–Maruyama scheme is given by

$$dX^n_t = b^n(t, X^n_{k_n(t)})dt + \sigma(t, X^n_{k_n(t)})dB_t \tag{2}$$

where $k_n(t) = j/n$ whenever $j/n \le t < (j+1)/n$. Here, b^n is an approximating drift coefficient, converging to b in some suitable topology. When b is a bounded measurable function, the Euler–Maruyama scheme is well-defined and one can simply take $b^n = b$. When b is merely integrable (and hence may have singularities), the simulation of the usual Euler–Maruyama scheme can enter a neighborhood of a singularity which can potentially make the scheme unstable and uncontrollable. In this situation, a strategy is to replace b by an approximation b^n which stabilizes the scheme, resulting in the tamed Euler–Maruyama scheme (2). The terminology is borrowed from [7] where a specific example of (2) is introduced for SDE's with regular but super-linear drifts. Thus "tamed" is understood in a generalized sense since (2) allows for any generic approximation b^n of b.

It is well-known that the strong convergence rate of Euler–Maruyama scheme is closely related to the rate of the weighted quadrature error

$$\left\| \int_0^1 (f(r, X^n_r) - f(r, X^n_{k_n(r)}))g(r, X^n_r)dr \right\|_{L_m(\Omega)} \tag{3}$$

where $m \ge 2$, f models b^n, and g is a weight arising from Zvonkin's transformation ([15, 14, 10]), typically g has one more (weak) regularity than f. This connection can be traced back at least to [9, 8] and recently in [6, 4, 13] with some combination with the Zvonkin's transformation.

When b is Lipschitz continuous, the optimal rates of the Euler–Maruyama scheme and the corresponding quadrature error are the same and equal 1/2 [9]. Optimal or almost optimal rate (with an ε-loss or with a logarithmic factor) for *irregular* drift b is surprisingly difficult to obtain, albeit with recent successes started from [5] and extended in [6, 12, 2]. These works are inspired by recent progress from regularization-by-noise problems. While [5] uses direct moment computations, the other works employ stochastic sewing lemma from [11], an instrumental result which combines rough path and martingale techniques. To get the idea, we put

$$\mathcal{A}_t = \int_0^t (f(r,B_r) - f(r,B_{k_n(r)}))g(r,B_r)dr, \quad A_{s,t} = \mathbb{E}(\mathcal{A}_t - \mathcal{A}_s|\mathcal{F}_s),$$
$$J_{s,t} = \mathcal{A}_t - \mathcal{A}_s - A_{s,t}.$$

Using statistical properties of X^n, one can show that

$$\|J_{s,t}\|_{L_m(\Omega)} \leq Cw(s,t)^{\frac{1}{2}+\varepsilon}, \tag{4}$$

$$\|J_{s,t} - J_{s,u} - J_{u,t}\|_{L_m(\Omega)} \leq C(1/n)^{\frac{1}{2}} w(s,t)^{\frac{1}{2}}, \tag{5}$$

where C is some constant and w is a continuous control (i.e. $w = w(s,t)$ is continuous and satisfies $w(s,u) + w(u,t) \leq w(s,t)$ whenever $s \leq u \leq t$). Applying a version of the stochastic sewing lemma, one can find a constant $N = N(\varepsilon, m)$ such that

$$\|J_{s,t}\|_{L_m(\Omega)} \leq N[(1/n)^{\frac{1}{2}} \log n] w(s,t)^{\frac{1}{2}}. \tag{6}$$

This estimate immediately yields a rate for (3). Note that this rate differs from the optimal rate 1/2 by a logarithmic factor, and whether it can be improved remains open. The implication from (4) and (5) to (6) displays an intimate connection between statistical properties of X^n with moment estimates for additive functionals through the stochastic sewing argument. With some additional effort, the described method can be employed to obtain the same rate for the quantity

$$\left\| \sup_{t\in[0,1]} \left| \int_0^t (f(r,X_r^n) - f(r,X_{k_n(r)}^n))g(r,X_r^n)dr \right| \right\|_{L_m(\Omega)}.$$

For the details, the reader is referred to [6, 12]. Many open questions, for which this method could be implemented, remain to be explored. To name a few, the strong convergence rate for Euler–Maruyama schemes for SDE's driven by fractional Brownian motion ([2]) and the strong convergence rate for numerical schemes for stochastic partial differential equations with irregular drifts ([1, 3]).

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Regularization by noise: from a numerical (Wong-Zakai approximation) viewpoint

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(joint work with Sebastian Riedel, Michael Scheutzow)

Starting from a well-known example [1, P2, P8] which explains well on 'regularization by noise' effect concerning the existence and uniqueness of the solution to a stochastic differential equation (SDE for short) of which the coefficients usually are not Lipschitz continuous, we aim to show SDEs with multiplicative Stratonovich-type noise of the form

$$dX_t = b(X_t)\,\mathrm{d}t + \sigma(X_t) \circ \mathrm{d}W_t, \quad X_0 = x_0 \in \mathbb{R}^d, \quad t \geq 0,$$

with a possibly singular drift $b \in L^p(\mathbb{R}^d)$, $p > d$ and $p \geq 2$, such SDEs can be approximated by random ordinary differential equations by smoothing the noise and the singular drift at the same time. The main idea behind is the stability results with respect to the drift term. Then based on the classical Wong-Zakai theorem combining with the stability results obtained before, in the end we can show the Wong-Zakai theorem for singular SDEs. We further prove a support theorem for this class of SDEs in a rather simple way using the Girsanov theorem. Usually the support theorem is derived from Wong-Zakai approximation, which is not easy to get in most of the cases. However since the noise is non-degenerate, we can obtain the support theorem via Girsanov theorem.

After the talk, there are some interesting questions from the audiences. One of them is whether the method can be generalized to more singular settings, e.g. distributional valued drifts. It is an inspiring point and worthy to continue working in this direction.

This talk is based on the joint work [2] with S. Riedel (Leibniz Universität Hannover) and M. Scheutzow (TU Berlin).

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Stable SDEs with singular drift (the linear and McKean-Vlasov case)

Stéphane Menozzi

(joint work with Paul-Éric Chaudru de Raynal, Jean-Francois Jabir)

We investigate the well-posedness of $\mathbb{R}^d$-valued equations of the form

$$X_s = \xi + \int_0^s b(r, X_r)dr + \mathcal{W}_s \tag{L}$$

and

$$X_s = \xi + \int_0^s \Big\{ \int_{\mathbb{R}^d} b(r, X_r - y)\mu_r(dy) \Big\} dr + \mathcal{W}_s, \tag{NL}$$

where $\mathcal{W}$ is a d-dimensional stable process of order $\alpha \in (1, 2]$ defined on some probability space and for (NL), μ_r stands for the law of X_r. The law of the initial condition ξ (independent of the noise) can be any probability law on $\mathbb{R}^d$.

Importantly we will focus on the case where b is singular. To keep things simple, let us restrict to the case where $b \in L^\infty((0, T], \mathbb{B}^\beta_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^d)) =: E, \beta \in (-1, 0)$, where $\mathbb{B}^\beta_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ stands for the usual Besov space (see [9]). Put it differently, in the time homogeneous setting, the drift b can be seen as the generalized derivative of a Hölder continuous function, i.e. $b(t, \cdot) = b(\cdot) = DB(\cdot)$, $B \in \mathbb{B}^{1+\beta}_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}^d) = C^{1+\beta}(\mathbb{R}^d, \mathbb{R}^d)$.

Many questions arise. What notion of solution can we consider to solve (L) and (NL) and how can we specify a dynamics? This last difficulty is particularly clear in the case (L) since the equation is only *formal*. Indeed the drift is not a priori well-defined and we need to specify what we mean by (L). Before giving a precise description for the dynamics, the first step usually consists in investigating the martingale problem associated with the *formal* generator of (L), precisely

$$L_t\varphi(x) = \langle b(x), \nabla\varphi(x)\rangle + L^\alpha\varphi(x), \tag{G}$$

where L^α stands for the generator of the driving stable process. Again, it is difficult to give a pointwise meaning to $L_t\varphi(x)$ for distributional drifts b. This leads to consider slightly differently the martingale problem approach and in seeking probability measures on $C(\mathbb{R}^+, \mathbb{R}^d)$ or $D(\mathbb{R}^+, \mathbb{R}^d)$ (depending on α) s.t. for the corresponding canonical process, $u(t, X_t) - u(0, x) - \int_0^t f(s, X_s)ds$ is a martingale, for functions f in a rich enough function class to characterize the law, where u is a mild solution of

$$\text{(PDE)}\qquad \begin{cases}\partial_t u(t,x) + L_t u(t,x) = f(t,x),\ (t,x) \in [0,T) \times \mathbb{R}^d,\\ u(T,x) = 0,\ x \in \mathbb{R}^d,\end{cases}$$

for some fixed $T > 0$ small enough. Namely, one looks for a function u solving the integral equation

$$\text{(M)}\qquad u(t,x) = -\int_t^T P^\alpha_{s-t} f(s,x)ds + \int_t^T P^\alpha_{s-t}\langle b, \nabla u\rangle(s,x)ds,$$

where P^α stands for the semi-group generated by L^α generator of the stable driving process in the above SDEs. Observe that from the above implicit formulation, it seems clear that to address the martingale problem the gradient of u in (M) needs to be controlled. This is usually done through a Schauder type approach which says that for $b \in \mathbb{B}^\beta_{\infty,\infty}$ one can expect u to benefit from the related parabolic boostrap, namely u *should* belong to $\mathbb{B}^{\beta+\alpha}_{\infty,\infty}$ in its space variable, uniformly in time. Anyhow, developing a Schauder type theory for (M) requires that the second integral in the right hand side of (M). To this end, one can heuristically say that, provided u benefits from the previously mentioned bootstrap, from the Bony rule for paraproducts, this will be the case provided

$$\text{(Y)}\qquad (\alpha + \beta - 1) + \beta > 0 \iff \beta > \frac{1-\alpha}{2},$$

where $\alpha + \beta - 1$ is the expected spatial smoothness of $\nabla u(s,\cdot)$. The condition in (Y) corresponds to the so-called *Young regime* and appeared in several works (see [6], [10] for the Brownian case or [1], [8] in the strictly stable setting). In the indicated articles, the authors also considered various notions of solutions: *virtual solutions* in [6] replacing namely the drift by the increment of the equation (PDE) along the process with the drift as source, or viewing the drift in (L) as the limit of smooth approximations of the drift (the limit drift being a Dirichlet process), [10], [1].

Under some additional structure condition on the drift (e.g. provided it can be enhanced into a suitable rough path-structure) it is possible to go below the Young threshold reaching $\beta > \frac{2-2\alpha}{3}$ (see [5] in the Brownian setting and [7] for $\alpha \in (1,2])$.

Importantly, following the approach introduced in [5], enhancing as in that reference the martingale problem in order to have at hand a driving noise to reconstruct the drift as a suitable Young integral, we managed in [2] to specify the dynamics for (L) which actually writes:

$$\text{(LD)}\qquad X_t = \xi + \int_0^t \mathcal{F}(s, X_s, ds) + \mathcal{W}_s,\ \mathcal{F}(s,x,v-s) := \int_s^v dr P^\alpha_{s-r} b(s,x).$$

Let us observe that the above description is indeed well suited for possible associated numerical approximation schemes.

For the non-linear McKean-Vlasov dynamics (NL) we established weak, for $\beta > 1-\alpha$, and strong well-posedness, for $\beta > 2 - \frac{3}{2}\alpha$, for any initial condition ξ

independent of the driving noise. This was performed in [4] in which we carry out a specific *a priori estimates strategy* seeking for solutions to the related Fokker-Planck equations with densities in $\mathbb{B}^{-\beta}_{1,1}$ (which can be put in spatial duality with the drift). This space is natural for the stability analysis and has lower spatial regularity than the expected parabolic gain. The main idea is to proceed again through a (non-linear) Duhamel type expansion of type (M) and to use a so-called *dequadrification strategy* to get rid of the underlying quadratic dependence on the density for the first order term. The previous threshold clearly improves the one in (Y). Interestingly, it quantifies how a convolution with a singular interaction kernel can help to decrease the Young threshold. Importantly, this thresholds is the *natural* one for the driving noise from a scaling viewpoint, see e.g. the discussion in [3] in the Brownian case. Also, in this latter case, the dynamics are usual ones, the non-linear drift is a function.

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Spectral gap for projective processes of hyperviscous SPDEs

Tommaso Rosati

(joint work with Martin Hairer)

The study of Lyapunov exponents of stochastic PDEs has been mostly restricted to order preserving systems, for example in relation to KPZ or Burgers' equations [6, 3, 5]. For order preserving systems it is generally well understood that one force one solution principles (synchronization) hold: in particular Lyapunov exponents associated to the Jacobian of solutions of order preserving systems are expected to be strictly negative [4]. This picture can change dramatically if one considers more complex models. Recent results regarding the Navier–Stokes equations show that finite dimensional approximations or toy models for these equations exhibit chaos or non–uniqueness of invariant measures [1, 2].

Extending these results to infinite dimensions is very challenging, and even basic questions concerning Lyapunov exponents are not well understood. The main purpose of this work is to introduce tools through which to establish first results beyond the order preserving setting. To do so we consider a class of linear hyperviscous SPDEs

$$\partial_t u = -(-\Delta)^a u + u\xi \ ,$$

for a smooth non-degenerate noise ξ. If $a > 1$ this equation does not satisfy a maximum principle and hence is not order preserving. We then prove that there exists a unique $\lambda \in \mathbb{R}$ such that for all $u_0 \neq 0$

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \log \|u_t\|_{L^2} \ .$$

In addition we provide Furstenberg–Khasminskii type formulas for the Lyapunov exponent.

The main contribution of our work is a new approach to the analysis of the projective component $\pi_t = u_t / \|u_t\|_{L^2}$ on the infinite–dimensional sphere: through classical arguments, many results concerning Lyapunov exponents, including the ones just mentioned, follow from proving a spectral gap for this process. In addition, such a spectral gap is useful in establishing further properties such as fluctuations and large deviations principles for $\frac{1}{t} \log \|u_t\|_{L^2}$ around λ.

In order to prove the spectral gap we introduce a novel Lyapunov functional for π_t, which keeps track of the midpoint (in frequency space) of the process

$$M(\pi) = \min \left\{ M \in \mathbb{N} : \sum_{|k| > M} |\hat{\pi}|^2(k) \leq \sum_{|k| \leq M} |\hat{\pi}|^2(k) \right\} ,$$

as well as its high frequency regularity, controlled by norms of the kind:

$$\|\pi\|_{\gamma, M} = \left(\sum_{|k| > M} (|k| - M)^{2\gamma} |\hat{\pi}|^2(k) \right)^{\frac{1}{2}} ,$$

where M is the midpoint. The main challenge to overcome is that in the deterministic setting the equation can be trapped in high frequency states. In the stochastic case this is not anymore possible, as long as the noise satisfies some weak non-degeneracy assumptions (a finite number of Fourier modes are required to be non-trivial). Under this assumption, we prove that a discretized version of the midpoint follows roughly the dynamic of a discrete Ornstein–Uhlenbeck process: this is the essential building block for the construction of our Lyapunov functional, which is eventually roughly of the form

$$\mathfrak{F}(\pi_t) \simeq \exp\left\{\kappa M(\pi_t) + \|\pi_t\|^2_{\gamma,M}\right\} ,$$

for some $\gamma, \kappa > 0$.

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Transport and continuity equations with (very) rough noise

Nikolas Tapia

(joint work with Carlo Bellingeri, Ana Djurdjevac, Peter K. Friz)

We prove existence and uniqueness for rough flows. Using this results, we show well-posedness on a path-by-path sense of transport and continuity equations driven by general weakly geometric rough path. The talk is based on the joint work with Carlo Bellingeri, Ana Djurdjevac and Peter K. Friz [1].

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On strong approximation of SDEs with a discontinuous drift coefficient

Larisa Yaroslavtseva

(joint work with Thomas Mueller-Gronbach)

Consider a scalar autonomous stochastic differential equation (SDE)

$$\begin{aligned} dX_t &= \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \quad t \in [0,1], \\ X_0 &= x_0 \end{aligned}$$

with deterministic initial value $x_0 \in \mathbb{R}$, drift coefficient $\mu\colon \mathbb{R} \to \mathbb{R}$, diffusion coefficient $\sigma\colon \mathbb{R} \to \mathbb{R}$ and 1-dimensional standard Brownian motion W. In this talk we study L_p-approximation of X_1 by means of methods that use finitely many evaluations of the driving Brownian motion W in the case when the drift coefficient μ may have discontinuity points.

SDEs with a discontinuous drift coefficient arise e.g. in mathematical finance, insurance and stochastic control problems. In the past decade, an intensive study of strong approximation of such SDEs has begun. In many of the obtained results it is assumed that the drift coefficient μ has finitely many discontinuity points and is piecewise Lipschitz continuous and the diffusion coefficient σ is Lipschitz continuous and non-degenerate at the discontinuity points of μ. In this talk we present our contribution to the analysis of strong approximation of such SDEs.

We first discuss the performance of the classical Euler-Maruyama scheme. We show that under the above assumptions the Euler-Maruyama scheme achieves an L_p-error rate of at least $1/2$ for all $p \in [1, \infty)$ as in the case of SDEs with Lipschitz continuous coefficients.

We then discuss higher order methods. We present a Milstein-type scheme that achieves an L_p-error rate $3/4$ in terms of the number of evaluations of the driving Brownian motion W if, additionally to the assumptions stated above, both the drift and the diffusion coefficients are piecewise differentiable with Lipschitz continuous derivatives. We furthermore show that the L_p-error rate $3/4$ can not be improved in general under these assumptions by no numerical method based on evaluations of the driving Brownian motion W at fixed time points and, finally, we present a numerical method based on sequential evaluations of W, which achieves an L_p-error rate of at least 1 in terms of the average number of evaluations of W.

The talk is based on the articles [1, 2, 3, 4].

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