Mathematisches Forschungsinstitut Oberwolfach

Report No. 7/2022

DOI: 10.4171/OWR/2022/7

Mini-Workshop: Interpolation, Approximation, and Algebra

Organized by
Carl de Boor, Madison
Tomas Sauer, Passau
Hal Schenck, Auburn
Tatyana Sorokina, Towson

13 February – 19 February 2022

Abstract. This report involves two concepts of geometric modeling: multivariate data interpolation by polynomials, and the study of generalized barycentric coordinates. These topics are connected to a wide range of of applications, from computer aided design (CAD) systems for designing airplanes and automobiles to animation in movies to problems in numerical analysis and partial differential equations. Traditionally these topics were studied mostly from an analytic standpoint, but recently advanced algebraic tools have come into the picture. The purpose of the mini-workshop was to bring together a diverse group of researchers with different areas of expertise, drawing from both the approximation theory and algebraic geometry communities, and to explore the connections between the two areas in greater detail.

Mathematics Subject Classification (2010): 13Pxx, 14Qxx, 41xx.

Introduction by the Organizers

Interpolation of data and approximation of functions is a fundamental theme in applied mathematics. Roughly speaking, given a dataset arising, for example, from some sampling process, the aim is to fit a function f to this data, and then use f to estimate or predict the behavior at points that were not sampled. Though this process is mostly well understood in one variable, there are still many open questions in two and more variables where the *geometry* of the sampling set plays a crucial role even in the "simplest" case of polynomial interpolation, cf. [23, 32]. In particular, already the construction of sampling points for a given space of polynomials becomes a nontrivial issue.

• One much studied set of interpolating functions arise when the underlying sample set consists of points which can be *geometrically categorized* (GC). Such sets are important because they give very simple and computationally inexpensive methods for interpolating the data. The Gasca-Maeztu conjecture proposes a specific structure for some particular sets of points in dimension two which satisfy the GC condition due to Chung and Yao [13]. Work of Sauer and Xu [33] shows that GC sets are very special algebraically: the vanishing ideal is generated by products of linear forms. We aim to understand and exploit this structure to attack the conjecture and understand potential generalizations to higher dimensions.

• The second theme of the proposal is closely tied to computer science and computer graphics. Generalized barycentric coordinates are used in numerical analysis and approximation theory when moving a dynamic shape in space. These coordinates were introduced by Wachspress in [35], and fundamental results on their structure were obtained by Warren [36], [37]. Recently Irving and Schenck [28] gave a complete dimension formula for the space of barycentric coordinates in the two dimensional case. One aim of the workshop is to understand what the proper extension of these results is in dimension three or more.

1. Approximation theory and interpolation

Though polynomial interpolation has a long and distinguished history, there nevertheless remain many intriguing open questions. One fundamental problem from approximation theory is the following: for a space of functions, a set of data sites is called *correct* for this space if it contains, for each choice of data values at the sites, exactly one function matching those data. Even when the space of functions consists of polynomials (possibly all) of degree $\leq k$, finding a correct set of data sites is very subtle; for more details and variants see the surveys [6, 23, 32]. The most prominent and well-studied configuration of correct interpolation sites is due to Chung and Yao [13] and can be described as follows.

Definition 1.1. *A set X of $\binom{n+2}{2}$ points in $\mathbb{R}^2$ is called a GC_n set if for each point $p_i \in X$, there exists a product*

$$Q_i = \prod_{k=1}^{n} l_k$$

of linear forms l_k such that $Q_i(p_j) = \delta_{ij}$.

In 1982, Gasca and Maeztu made the following

Conjecture 1.2. [[21]] *For a GC_n set, there is a line $V(l)$ with $l \mid Q_i$ for some Q_i, such that $V(l)$ contains $n+1$ points of X.*

The Gasca-Maeztu conjecture traces its origin back to the last century, in particular to the 1914 work of Berzolari [4] and subsequent work of Radon [31]. Roughly speaking, the idea is to solve the interpolation problem inductively: find a set T which has the desired property for degree $n-1$, then choose a set of n points from a line which does not intersect T, and adjoin them to T. This process not only yields a correct set of sites, but even a GC_n set in the sense of Definition 1.1. If the Gasca-Maeztu conjecture holds, then in fact any GC_n can be be constructed using the Berzolari-Radon technique.

As pointed out in [21], Conjecture 1.2 is easily seen to hold for $n = 1, 2$. For $n = 3, 4$, it was first proved by Busch in [8]. Additional proofs for $n = 4$, using different techniques, appear in [9] and [25]. Last year, Hakopian-Jetter-Zimmerman showed in [26] that the conjecture holds for $n = 5$. For more on the problem and related work, see [7, 10, 11, 22, 23].

Work of Sauer-Xu in [33] implicitly shows that the vanishing ideal I_X of a geometrically categorized set of $\binom{n+d}{d}$ points in $\mathbb{R}^d$ is generated by $\binom{n+d}{n+1}$ polynomials of degree $n + 1$, which are products of linear forms. Recent work [18] shows that by replacing the linear forms with variables leads to combinatorial structure, in the form of Stanley-Reisner ideals. This not only gives rise to deeper understanding of the Gasca-Maeztu conjecture, but also provides concepts for potential generalizations to an arbitrary number of variables.
Goal: Understand how to exploit the combinatorial properties to attack and to generalize the Gasca-Maeztu conjecture.

2. Generalized Barycentric Coordinates

The problem of determining the implicit equation of the image of a rational map $\phi : \mathbb{P}^m \dashrightarrow \mathbb{P}^n$ is of theoretical interest in algebraic geometry, and of practical importance in geometric modeling. There are essentially three methods which can be applied to the problem: Gröbner bases, resultants, and syzygies. Gröbner basis methods work via elimination, which tends to be computationally intensive. Thus, it is primarily the latter two techniques which are used in practice.

If ϕ is given by homogeneous polynomials $\{f_0, \ldots, f_n\}$, then the resultant method fails if the f_i simultaneously vanish at some point of $\mathbb{P}^m$. The common zeroes of the f_i are called the *base locus* of ϕ; maps with nonempty base locus arise frequently in practical applications where polynomial systems of equations have to be solved. So the use of syzygies to compute the implicit equation is of real practical importance.

In his work on finite elements, Wachspress (1975) was led to define generalized barycentric coordinates for a convex polygon P_d with d edges. The idea is as follows: to deform a planar shape, first place the shape inside a control polygon. Then move the vertices of the control polygon, and use barycentric coordinates to extend this motion to the entire shape. For $d \geq 4$ the barycentric coordinates are rational functions β_i depending on the vertices of P_d. Although the β_i are not unique, about 10 years ago, Warren showed that if one requires the β_i to be of minimal degree, then the resulting rational functions are unique.

The generalized barycentric coordinates define a rational map w_d on $\mathbb{P}^2$, whose value at a point $p \in P_d$ is the d-tuple of barycentric coordinates of p. The closure of the image of w_d defines the Wachspress surface W_d, first studied by Garcia–Puente and Sottile in their work on linear precision. Garcia-Sottile asked if it was possible to determine the implicit equations for the image. Irving and the PI answer the question, finding the defining equations for the ideal of W_d. The work involves beautiful connections to fatpoints (W_d is the blowup of $\mathbb{P}^2$ at a set of special points determined by P_d) and to combinatorics (the initial ideal of $I(W_d)$ is a Stanley-Reisner ideal). We also prove that the image W_d is a smooth surface, and that the ring $k[x_1, \ldots, x_d]/I(W_d)$ is very well behaved: it is arithmetically Cohen-Macaulay and has regularity two.

Goal: There is also much interest in three dimensions, where things are more subtle. Suitably merge algebraic and analytic techniques to attack the problem.

3. Aim of Workshop

The results discussed above suggest that the interaction between commutative algebra and approximation theory is unusually rich. The purpose of the mini-workshop is to bring together a diverse group of researchers with different areas of expertise with the aim of clarifying the connections, the possibilities and the limitations of the different approaches.

The first day will consist of surveys on the different concepts and tools, focused on familiarizing the participants with the background material. The second day more specialized talks will be given, and the remaining time will be mainly devoted to questions and discussions which could lead to new projects in the field.

Mini-Workshop: Interpolation, Approximation, and Algebra

Table of Contents

Abstracts

Wachspress coordinates

Kathlén Kohn

(joint work with Ragni Piene, Kristian Ranestad, Felix Rydell, Boris Shapiro, Rainer Sinn, Miruna-Stefana Sorea, Simon Telen)

Wachspress aimed to generalize barycentric coordinates on simplices to arbitrary polytopes and further to certain semialgebraic subsets of $\mathbf{R}^n$, that he called *polypols* [7, 9]. The central ingredient in his constructions is the *adjoint hypersurface*, which is the focus of this chapter. Wachspress' work investigated mainly planar polypols, which we will discuss in Section 2. In Section 1, we focus on polytopes of arbitrary dimension. We present several applications, including barycentric coordinates, of adjoint hypersurfaces in Section 3.

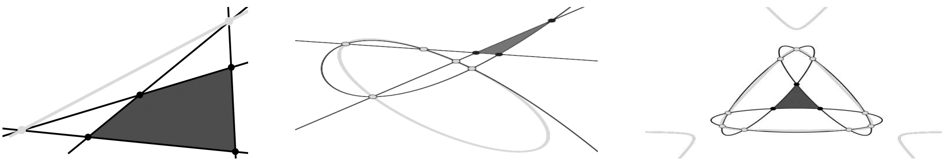


Figure 1. Three polypols (dark gray) and their adjoints (light gray). Boundary curves and vertices are black. Residual points are light gray.

1. Polytopes

Wachspress introduced the *adjoint curve* A_P of a polygon $P \subset \mathbb{P}^2$ as the minimal degree curve passing through the intersection points of pairs of lines containing non-adjacent edges of P [7]. The degree of the adjoint is the number of vertices minus three; see the first example in Figure 1. Warren generalized this to convex polytopes $P \subset \mathbb{R}^n$ [11]. For a fixed triangulation $\tau(P)$ of P that uses only the vertices $V(P)$ of P, he defined the *adjoint polynomial* as

$$\mathrm{adj}_{\tau(P)}(t) := \sum_{\sigma \in \tau(P)} \mathrm{vol}(\sigma) \prod_{v \in V(P) \setminus V(\sigma)} \ell_v(t),$$

where $t = (t_1, \ldots, t_n)$ and $\ell_v(t) = 1 - v_1 t_1 - v_2 t_2 - \ldots - v_n t_n$. He then showed that this definition is in fact independent of the chosen triangulation – so we write $\mathrm{adj}_P := \mathrm{adj}_{\tau(P)}$ – and that it is a generalization of Wachspress' construction in the sense that the zero locus of Warren's polynomial is Wachspress' adjoint curve of the polygon P^* dual to P, i.e., $Z(\mathrm{adj}_P) = A_{P^*}$.

We gave a geometric definition of the *adjoint hypersurface* using a vanishing condition à la Wachspress [5]. For a polytope $P \subset \mathbb{P}^n$, we write $\mathcal{H}_P$ for the hyperplane arrangement spanned by the facets of P, and $\mathcal{R}_P$ for the *residual* *arrangement* of linear spaces that are intersections of hyperplanes in $\mathcal{H}_P$ and do not contain any face of P; see Figure 2 for an example. If $\mathcal{H}_P$ is simple (i.e., through any point in $\mathbb{P}^n$ pass at most n of the hyperplanes), we show that there is a unique hypersurface A_P in $\mathbb{P}^n$ of minimal degree that passes through $\mathcal{R}_P$. We call A_P the *adjoint hypersurface.* Its degree is the number of facets of P minus n minus one. To mitigate the simpleness assumption on $\mathcal{H}_P$, we show that adjoint hypersurfaces are well-defined when taking limits; so one can obtain A_P of any polytope P by perturbing P such that $\mathcal{H}_P$ becomes simple. We also show that the adjoint A_P is in fact described by Warren's polynomial, i.e., $Z(\mathrm{adj}_P) = A_{P^*}$.

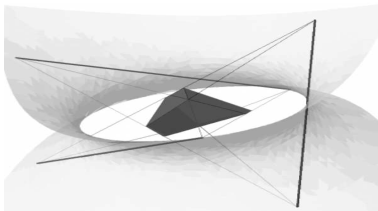


FIGURE 2. The residual arrangement of a perturbed cube consists of three skew lines. Its adjoint is the light gray quadric.

2. Rational planar polypols

A *planar polypol* P is given by its $k \geq 2$ irreducible boundary curves $C_1, C_2, \ldots, C_k \subset \mathbb{P}^2$ and *vertices* $v_{12} \in C_1 \cap C_2$, $v_{23} \in C_2 \cap C_3$, ..., $v_{k1} \in C_k \cap C_1$ such that v_{ij} is smooth on C_i and C_j, and the curves C_i and C_j intersect transversally at v_{ij}. The polypol is called *rational* if the curves $C_1, C_2, \ldots, C_k$ are rational. Writing $C := C_1 \cup C_2 \cup \ldots \cup C_k$, we define the *residual points* $\mathcal{R}_P$ of P as the scheme of singular points of C minus the vertices v_{ij}; see Figure 1 for examples. Wachspress argued that there is a unique curve A_P of minimal degree passing with appropriate multiplicities through the residual points $\mathcal{R}_P$ [9]; see [4] for a formal proof. This is called the *adjoint curve* of the polypol and its degree is $\sum_i \deg(C_i) - 3$.

3. Applications

3.1. **Barycentric Coordinates.** Wachspress used the adjoint curve to define barycentric coordinates on convex polygons and aimed to generalize this to *regular* rational planar polypols [7, 9]. To define regularity, we consider a polypol P together with a choice of (real) segments connecting $v_{i-1,i}$ with $v_{i,i+1}$ on C_i, called the *sides*, and a closed semi-algebraic set $P_{\geq 0}$ whose boundary is exactly the union of the sides. We say that P is *regular* if all points on the sides except the vertices are smooth on $C = C_1 \cup \ldots \cup C_k$ and the curve C does not intersect the interior of $P_{\geq 0}$. For instance, a polygon is regular if and only if it is convex.

Warren generalized Wachspress' coordinates from polygons to polytopes as follows [11]: For a convex polytope $P \subset \mathbb{R}^n$, we recall that the set of facets $\mathcal{F}(P)$

of P is in one-to-one correspondence with the vertex set $V(P^*)$ of the dual polytope. We write v_F for the vertex associated with the facet F. Vice versa, we write $F_v \in \mathcal{F}(P^*)$ for the facet corresponding to the vertex $v \in V(P)$. The *Wachspress coordinates* of P are

$$\forall u \in V(P): \quad \beta_u(t) := \frac{\mathrm{adj}_{\mathrm{F_u}}(\mathrm{t}) \cdot \prod_{F \in \mathcal{F}(P): u \notin F} \ell_{v_F}(t)}{\mathrm{adj}_{P^*}(t)}.$$

This is well-defined on the interior of the convex polytope P as the adjoint hypersurface A_P does not have zeroes there. Wachspress provided a similar construction for regular rational polypols in the plane, with the adjoint as the common denominator. However, it is not obvious that the adjoint the does not have zeroes in the interior of a regular polypol. This motivated Wachspress to formulate the following conjecture (in its original form for *polycons* (i.e., polypols with lines and conics as boundary curves) [8] and recently extended to polypols [4]):

Conjecture 1. *The adjoint curve A_P of a regular rational planar polypol P does not intersect its interior.*

The conjecture is widely open. The first non-solved case is polycons bounded by three conics. It has been recently attempted by Wachspress' grandson [10] and by us [4]. For convex polygons, a case already known by Wachspress, we showed that the adjoint curve is hyperbolic and gave an explicit description of its nested ovals. For convex 3D polytopes, the adjoint surface is not hyperbolic anymore.

3.2. **Moments of Uniform Distributions.** We consider the uniform probability distribution μ_P on a convex polytope $P \subset \mathbb{R}^n$ and its moments $m_{\mathcal{I}}(P) = \int_{\mathbb{R}^n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \, d\mu_P$ for $\mathcal{I} = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$. The adjoint appears in the following normalized generating function over all moments [6]:

$$\sum_{\mathcal{I} \in \mathbb{Z}^n_{\geq 0}} c_{\mathcal{I}} m_{\mathcal{I}}(P) t^{\mathcal{I}} = \frac{\mathrm{adj}_P(t)}{\mathrm{vol}(P) \prod_{v \in V(P)} \ell_v(t)}, \text{ where } c_{\mathcal{I}} := \binom{i_1 + i_2 + \cdots + i_n + n}{i_1, i_2, \ldots, i_n, n}.$$

3.3. **Segre Classes of Monomial Schemes.** We consider a smooth variety V with smooth hypersurfaces $X_1, \ldots, X_n$ that meet with normal crossings in V. For $\mathcal{I} = (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$, we write $X^{\mathcal{I}}$ for the hypersurface obtained by taking X_{i_j} with multiplicity i_j. Any finite subset $\mathcal{A} \subset \mathbb{Z}^n_{\geq 0}$ defines a *monomial subscheme* $S_{\mathcal{A}} := \bigcap_{\mathcal{I} \in \mathcal{A}} X^{\mathcal{I}}$ and a *Newton region* $N_{\mathcal{A}} := \mathbb{R}^n_{\geq 0} \setminus \mathrm{convHull}\left(\bigcup_{\mathcal{I} \in \mathcal{A}} (\mathbb{R}^n_{\geq 0} + \mathcal{I})\right)$. Aluffi [1, 2] showed that the Segre class of $S_{\mathcal{A}}$ in the Chow ring of V is

$$\frac{n! \, X_1 \cdots X_n \, \mathrm{adj}_{N_{\mathcal{A}}}(-X)}{\prod_{v \in V(N_{\mathcal{A}})} \ell_v(-X)}.$$

The Newton region may have vertices at infinity in the direction of the standard basis vectors $e_1, \ldots, e_n$. For such a vertex v_i in the direction of e_i, the linear form becomes $\ell_{v_i}(t) := -t_i$.

3.4. **Scattering Amplitudes.** Arkani-Hamed, Bai, and Lam [3] introduced positive geometries in their studies of scattering amplitudes in particle physics. We consider a projective, complex, irreducible variety X of dimension n. Let $X_{\geq 0}$ be a non-empty closed semi-algebraic subset of the real part of X such that its Euclidean interior $X_{>0}$ is an open oriented n-dimensional manifold whose closure equals $X_{\geq 0}$. We consider the Euclidean boundary $\partial X_{\geq 0} := X_{\geq 0} \setminus X_{>0}$ and its Zariski closure ∂X in X. We write $C_1, \ldots, C_k$ for the irreducible components of ∂X, and denote by $C_{i,\geq 0}$ the Euclidean closure of the interior of $C_i \cap X_{\geq 0}$.

The pair $(X, X_{\geq 0})$ is a *positive geometry* if there is a unique non-zero rational n-form $\Omega(X, X_{\geq 0})$, called its *canonical form*, satisfying: 1) if $n = 0$, then $X = X_{\geq 0}$ is a point and $\Omega(X, X_{\geq 0}) = \pm 1$, and 2) if $n > 0$, every boundary component $(C_i, C_{i,\geq 0})$ is a positive geometry whose canonical form is the residue of $\Omega(X, X_{\geq 0})$ along C_i, and $\Omega(X, X_{\geq 0})$ is holomorphic on $X \setminus (C_1 \cup \ldots \cup C_k)$.

One-dimensional positive geometries $(X, X_{\geq 0})$ are finite disjoint unions of closed segments on rational curves X such that each open segment in $X_{>0}$ is smooth. The canonical form of such a closed segment can be identified with the canonical form of the interval $[a, b] \subset \mathbb{P}^1$, which is $\frac{b-a}{(t-a)(b-t)}dt$.

We show that planar positive geometries $(X, X_{\geq 0})$, i.e., with $X = \mathbb{P}^2$, are generalized rational planar polypols [4]. Moreover, we show that the canonical form of a rational polypol $P_{\geq 0}$ with boundary curves $C_1, \ldots, C_k$ is

$$\Omega(\mathbb{P}^2, P_{\geq 0}) = \eta \frac{\alpha_P}{f_1 \cdots f_k} dx \wedge dy$$

where α_P and f_i are defining equations of A_P resp. C_i, and η is a normalizing constant. The analogous formula for polytopes has been shown in unpublished lecture notes by Gaetz.

References

[1] Paolo Aluffi, *Segre classes of monomial schemes*, Electron. Res. Announc. Math. Sci. **20** (2013), 55–70.

[2] Paolo Aluffi, *Segre classes as integrals over polytopes*, J. Eur. Math. Soc. **18** (2016), 2849–2863.

[3] Nima Arkani-Hamed, Yuntao Bai, Thomas Lam, *Positive geometries and canonical forms*, Journal of High Energy Physics **2017.11** (2017), 1–124.

[4] Kathlén Kohn, Ragni Piene, Kristian Ranestad, Felix Rydell, Boris Shapiro, Rainer Sinn, Miruna-Stefana Sorea, Simon Telen, *Adjoints and Canonical Forms of Polypols*, arXiv:2108.11747 (2021).

[5] Kathlén Kohn, Kristian Ranestad, *Projective geometry of Wachspress coordinates*, Foundations of Computational Mathematics (2019), 1–39.

[6] Kathlén Kohn, Boris Shapiro, Bernd Sturmfels, *Moment Varieties of Measures on Polytopes*, Annali della Scuola Normale Superiore di Pisa **21** (2020), 739–770.

[7] Eugene L. Wachspress, *A rational finite element basis*, Academic Press, 1975.

[8] Eugene L. Wachspress, *The case of the vanishing denominator*, Mathematical Modelling, 1980(1), 395–399.

[9] Eugene L. Wachspress, *Rational Bases and Generalized Barycentrics*, Springer, 2016.

[10] Jacob Wachspress, *On the denominator of Wachspress basis functions for polycons of order six*, Princeton University Senior Theses (2020).

[11] Joe Warren, *Barycentric coordinates for convex polytopes*, Advances in Computational Mathematics **6** (1996), 97–108.

Apozyan's counterexample to the GM_d conjecture

Carl de Boor

The GM_d conjecture [3] is a ready generalization of the GM conjecture [6]. It concerns polynomial interpolation at the points of a finite set $X \subset \mathbb{R}^d$, i.e., the inversion of the restriction map

$$|_X : \Pi_n(\mathbb{R}^d) \to \mathbb{R}^X : p \mapsto p|_X := (p(\mathbf{x}) : \mathbf{x} \in X).$$

If it is invertible, X is called n-**correct**. In that case, the inverse is necessarily of the form

$$(|_X)^{-1} : \mathbb{R}^X \to \Pi_n(\mathbb{R}^d) : \mathbf{v} \mapsto \sum_{\mathbf{x}\in X} \ell_{\mathbf{x}} v_{\mathbf{x}},$$

with the $\ell_{\mathbf{x}}$ the **Lagrange polynomials** of the process, i.e., $\ell_{\mathbf{x}}|_X = (\delta_{\mathbf{x},\mathbf{y}} : \mathbf{y} \in X)$. An n-correct X is said [5] to satisfy the **Geometric Characterization of degree** n, in notation

$$X \in GC_n,$$

if for each $\mathbf{x} \in X$ there exist n hyperplanes in $\mathbb{R}^d$ whose union contains all of X except $\mathbf{x}$. Associating with each such hyperplane h some $0 \neq p_h \in \Pi_1$ that vanishes on h, this says, in effect, that all $\ell_{\mathbf{x}}$ have only linear factors. The GM_d **conjecture** is the claim that for any GC_n set X in $\mathbb{R}^d$, there exist $d+1$ **maximal hyperplanes**, i.e., hyperplanes h whose intersection with X is an n-correct set (for $\Pi_n(h)$). The GM conjecture [6] is the more modest claim that every GC_n set $X \subset \mathbb{R}^2$ has at least one maximal straight line h. It reflects the hope that interpolation on any GC_n set could be done à la Newton, by inductive application of the Berzolari [2], Radon [7] recipe which obtains from an $(n-1)$-correct set $X \subset \mathbb{R}^2$ the n-correct set $X' := X \cup L$, with L an arbitrary $(n+1)$-set on an arbitrary straight line h that does not intersect X. The stronger GM_d conjecture was based on the fact [4] that the GM conjecture implies that every GC_n set in $\mathbb{R}^2$ has at least 3 maximal straight lines.

Note that, for an n-correct set in $\mathbb{R}^d$ and any k-dimensional flat F, $\#(X \cap F)$ cannot exceed $\dim \Pi_n(F) = \dim \Pi_n(\mathbb{R}^k)$ since $|_X$, being onto, must map $\Pi_n(F)$ onto $\mathbb{R}^{X\cap F}$, hence F is called **maximal** for X if $\#(X \cap F) = \dim \Pi_n(F)$, in which case $X \cap F$ is n-correct for $\Pi_n(F)$.

The statement of the GM_d conjecture in [3] is followed immediately by a counterexample in $\mathbb{R}^3$, of a GC_2 set with only 3 maximal hyperplanes, obtained from a standard GC_2 set by moving one edge point. Apozyan's counterexample in $\mathbb{R}^6$ is obtained by 7 such moves from cleverly chosen 7 edges in that standard GC_2.

A standard GC_2 set in $\mathbb{R}^d$ is obtained by picking $d+1$ hyperplanes $h_0, h_1, \ldots, h_d$ in general position. Get the GC_1 set

$$A := \{\mathbf{a}_i : i = 0{:}d\} \quad \text{via} \quad \cap_{j\neq i}\, h_j =: \{\mathbf{a}_i\},$$

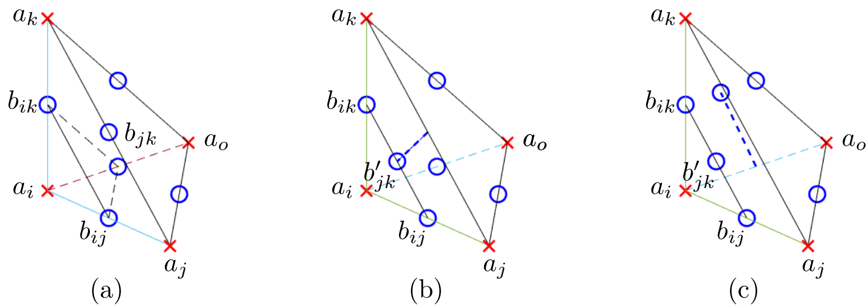


FIGURE 1. (a) A 3-dimensional GC_2 set with 4 maximal hyperplanes changed into (b) one with only 3 maximal hyperplanes by the move of one edge point, and (c) one obtained by another such move which is still 2-correct but not anymore GC_2.

hence h_i is the affine hull or flat $\flat\{\mathbf{a}_j : j \neq i\}$ containing all of $A \backslash \{\mathbf{a}_i\}$ but not $\mathbf{a}_i$, and augment A with the set

$$B := \{\mathbf{b}_{ij} : j \neq i\} \quad \text{with} \quad \mathbf{b}_{ij} \in \flat\{\mathbf{a}_i, \mathbf{a}_j\} \backslash \{\mathbf{a}_i, \mathbf{a}_j\}.$$

$X := A \cup B$ is readily seen to be GC_2: For $\mathbf{a}_i$, h_i is one of the hyperplanes, while $k_i := \flat\{\mathbf{b}_{ij} : j \neq i\}$, being spanned by the remaining d points yet to be covered, is at most a hyperplane, hence cannot contain $\mathbf{a}_i$ since that would imply that it contain all of A, contradicting that A is GC_1. For any $\mathbf{b}_{ij}$, $X \backslash \{\mathbf{b}_{ij}\} \subset h_i \cup h_j$.

Each h_i, $i = 0{:}d$, is a maximal hyperplane for X since $\#(X \backslash h_i) = d + 1 = \dim \Pi_n(\mathbb{R}^d) - \dim \Pi_n(h_i)$, hence X satisfies the GM_d conjecture. The counterexample for $d = 3$ in [3] is obtained by choosing a triple $A_i = \{\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k\}$ and replacing $\mathbf{b}_{jk}$ by some $\mathbf{b}'_{jk}$ in $\flat\{\mathbf{b}_{ij}, \mathbf{b}_{ik}\} \backslash \{\mathbf{b}_{ij}, \mathbf{b}_{ik}\}$, thus depriving h_i of its maximality while all other h_j are still maximal.

The resulting X' is still GC_2 as one sees by verifying that $|_{X'}$ is still invertible, and then noticing that all points in X' but $\mathbf{a}_i$ are off a maximal hyperplane, hence its Lagrange polynomial must have that hyperplane as a linear factor, leaving another linear factor, while for $\mathbf{a}_i$, all that happened was that a point on h_i moved into k_i, hence its Lagrange polynomial is unchanged.

In contrast to what the author claimed in his talk, it is not possible in the case $d = 3$ to carry out a second such move, e.g., for the triple $A_k = \{\mathbf{a}_k, \mathbf{a}_i, \mathbf{a}_o\}$ (see Figure (c)). While the resulting X'' is still 2-correct, it fails to be GC_2 since such a move removes, for the edge point $\mathbf{b}_{ik}$ from the pair $\{\mathbf{a}_i, \mathbf{a}_k\}$ common to A_i and A_k, some point of X formerly in $h_i \cup h_k$, making it impossible for $\mathbf{b}_{ik}$'s Lagrange polynomial to have linear factors. This belated realization, caused by a question from Jesús Carnicer during a second talk in which the author attempted to give a counterexample already in $\mathbb{R}^4$, terminated that attempt.

Probably because of such an example, Apozyan [1] obtained his counterexample for $d = 6$ by choosing, for each $i = 0{:}d$, a 3-set $A_i = \{\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k\}$ in such a way that no two have more than one point in common. Such a choice is not possible for $d = 3$. However, for $3 < d \le 6$, a $(d+1)$ set can have $2(d-3)$ such 3-sets, hence Apozyan's proof also proves that such $\mathbb{R}^d$ contains GC_2 sets with just $d+1-2(d-3) = 6-d$ maximal hyperplanes.

Here is Apozyan's choice for $d = 6$:

$$\begin{array}{ccccccc}
x & x & x & . & . & . & . \\
. & x & . & x & . & x & . \\
. & . & x & . & x & x & . \\
x & . & . & x & x & . & . \\
. & x & . & . & x & . & x \\
x & . & . & . & . & x & x \\
. & . & x & x & . & . & x
\end{array}$$

Apozyan obtains his counterexample X' from X by replacing, for each $A_i = \{\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k\}$, the edge point $\mathbf{b}_{jk}$ in the manner already described, thus depriving each h_i of its maximality while not increasing the points in any other h_j.

The resulting X' is still a GC_2 set: For each $\mathbf{a}_i$, h_i and k_i still work. For each moved $\mathbf{b}'_{jk}$ and for each unmoved $\mathbf{b}_{jk}$ whose $\{\mathbf{a}_j, \mathbf{a}_k\}$ lies in no A_i, h_j and h_k still work since $h_j \cup h_k$ contains all other points of X'.

That leaves those $\mathbf{b}_{ij}$ with $\{\mathbf{a}_i, \mathbf{a}_j\}$ in A_i or A_j, say in A_i wlog. Then $\mathbf{b}_{ij}$ is off h_j, hence can use h_j for one hyperplane but still has to take care of $\mathbf{b}_{jr}$ for all $r \ne i, k$, and of $\mathbf{b}'_{jk}$ and $\mathbf{a}_j$, a total of d points hence certainly contained in some hyperplane k'_j. But such k'_j therefore also contains $\mathbf{a}_r$ for all $r \ne i, k$, and that prevents it from containing $\mathbf{b}_{ij}$ since then it would have to contain A, contradicting that A is a GC_1 set.

Finally, if perchance the seven moves have generated another maximal hyperplane H then $\#(X \backslash H) = d+1$ while, with $\mu := \#(A \cap H)$, $\#(X \backslash H) \ge (d+1-\mu)(1+\mu) > d+1$ for $0 < \mu < d$, leaving only the possibilities $\mu = 0, d$. $\mu = d$ corresponds to one of the h_i while $\mu = 0$ implies that $X' \cap H = B'$, and its maximality is easily destroyed by moving some $\mathbf{b}_{jk}$ off H along $\flat\{\mathbf{a}_j, \mathbf{a}_k\}$ without destroying the GC_2 property.

References

[1] A. Apozyan, *A six dimensional counterexample for the GM_d conjecture*, Jaen J. Approximation, **3** (2011), 161–172.

[2] L. Berzolari, *Sulla determinazione d'una curva o d'una superficie algebrica e su alcune questioni di postulazione*, Rendiconti del R. Instituto Lombardo, Ser. (2) **47** (1914), 556–564.

[3] C. de Boor, *Multivariate polynomial interpolation: conjectures concerning GC-sets*, Numer.Algor. **45** (2007), 113–125.

[4] J. M. Carnicer, M. Gasca, *On Chung and Yao's geometric characterization for bivariate polynomial interpolation.* In: *Tom Lyche, Marie-Laurence Mazure, and Larry L. Schumaker (eds.) Curve and Surface Design: Saint-Malo* (2002), (11–30). *Nashboro Press, Brentwood TN* (2003).

[5] K. C. Chung, T. H. Yao, *On lattices admitting unique Lagrange interpolations*, Siam J. Numer.Anal. **14** (1977), 735–743.

[6] Gasca, M., Maeztu, J. I., On Lagrange and Hermite interpolation in $\mathbb{R}^k$. Numer.Math. **39** (1982), 1–14.
[7] J. Radon, *Zur mechanischen Kubatur*, Monatshefte der Math. Physik **52**(4) (1948), 286–300.

Berzolari, Radon, H-bases, syzygies and maximal lines

Tomas Sauer
(joint work with Jesus Carnicer, Carmen Godés)

While polynomial interpolation in one variable is essentially a matter of counting, geometry matters in several variables. In general, the *Lagrange interpolation problem*

$$f(x) = y_x, \qquad x \in X \subset \mathbb{R}^s, \tag{1}$$

is not *n-correct*, i.e., cannot be solved by $f \in \Pi_n$, the space of all polynomials of total degree $\le n$, even if $\#X = \deg n$. Elementary linear algebra shows that (1) is n-correct if and only if $\#X = \dim \Pi_n$ and there exist *fundamental polynomials* or *Lagrange polynomials* $\ell_x \in \Pi_n$, $x \in X$, such that

$$\ell_x(x') = \delta_{x,x'}, \qquad x, x' \in X.$$

While the set of n-correct is open and dense among all sets of proper cardinality, i.e., interpolation fails only in very rare occasions, explicit constructions for n-correct sets are rare. A particular case of such a construction, attributed to Berzolari [1] and Radon [8] in the bivariate case, works as follows: using an n-correct set Y of points and a line T such that $Y \cap T = \emptyset$, one picks $n+2$ points on the line that extend Y to an $(n+1)$-correct set; the idea can easily be generated to an arbitrary number of variables replacing "line" by "hyperplane".

A different approach was presented by Chung and Yao [4] who introduced the concept of the *geometric characterization*: for each point $x \in X$ there exist n hyperplanes $H_{x,1}, \dots, H_{x,n}$ such that x is contained in none of them, but any other point in $X \setminus \{x\}$ lies on at least one of them. It is not hard to see that this condition in equivalent to having fundamental polynomials that can be factorized into linear polynomials. Gasca and Maeztu [6] conjectured that any set X in $\mathbb{R}^2$ that satisfies the geometric characterization is indeed generated by a Berzolari-Radon construction. Moreover, this is equivalent to the existence of a *maximal line*, i.e., a line that contains $n+1$ points of X. So far the conjecture is proven up to $n = 5$, see [7].

The problem can be transformed by means of algebraic geometry. To that end, one considers H-bases of the ideal $I(X) = \{f \in \Pi : f(X) = 0\}$, which is generated by $n+2$ polynomials of degree $n+1$ that can be arranged into a vector $h = (h_0, \dots, h_{n+1}) \in \Pi_{n+1}^{n+2}$. These polynomials induce a linear *syzygy matrix* $\Sigma \in \Pi_1^{(n+1)\times(n+2)}$ which satisfied $\Sigma h = 0$ and whose minors of size $(n+1) \times (n+1)$ yield the H-basis up to a nonzero constant. The syzygy matrices for two different H-bases are related by

$$\Sigma' = A \Sigma B, \qquad A \in \mathbb{R}^{(n+1)\times(n+1)},\ B \in \mathbb{R}^{(n+2)\times(n+2)},$$

where A and B are nonsingular. The following result, originally from [5] was proved in the above context in [2].

Theorem. *An n-correct set X contains a maximal line if and only if there exists an H-basis for $I(X)$ with associated syzygy matrix Σ such that one column of Σ contains only one nonzero element. Moreover, the variety associated to this nonzero affine polynomial is the maximal line.*

The talk describes the background of this story and some further implications and reformulations from [3].

References

[1] L. Berzolari, *Sulla determinazione di una curva o di una superficie algebrica e su algune questioni di postulazione*, Lomb. Ist. Rend. **47** (1914), 556–564.

[2] J. Carnicer and T. Sauer, *Observations on interpolation by total degree polynomials in two variables*, Constr. Approx., **47** (2018), 373–389.

[3] J. Carnicer, C. Godés, and T. Sauer, *Radon's construction and matrix relations generating syzygies*, Monatsh. Math., **192** (2020), 311–332.

[4] K. C. Chung and T. H. Yao, *On lattices admitting unique Lagrange interpolation*, SIAM J. Num. Anal. **14** (1977), 735–743.

[5] N. Fieldsteel and H. Schenck, *Polynomial interpolation in higher dimension: From simplicial complexes to GC sets*, SIAM J. Numer. Anal. **55** (2017), 131–143.

[6] M. Gasca and J. I. Maeztu, *On Lagrange and Hermite interpolation in $\mathbb{R}^k$*, Numer. Math. **39** (1982), 1–14.

[7] H. Hakopian, K. Jetter, and G. Zimmermann, *The Gasca-Maeztu conjecture for $n = 5$*, Numer. Math. **127** (2014), 685–713.

[8] J. Radon, *Zur mechanischen Kubatur*, Monatsh. Math. **52** (1948), 286–300.

Sparse Interpolation in Terms of Multivariate Chebyshev Polynomials

Evelyne Hubert, Michael Singer

The goal of *sparse interpolation* is the exact recovery of a function as a short linear combination of elements in a specific set of functions, usually of infinite cardinality, from a limited number of evaluations, or other functional values. The function to recover is sometimes refered to as a *blackbox*: it can be evaluated, but its expression is unknown. We consider the case of a multivariate function $f(x_1, \ldots, x_n)$ that is a sum of generalized Chebyshev polynomials and present an algorithm to retrieve the summands. We assume we know the number of summands, or an upper bound for this number, and the values of the function at a finite set of well chosen points.

Beside their strong impact in analysis, Chebyshev polynomials arise in the representation theory of simple Lie algebras. In particular, the Chebyshev polynomials of the first kind may be identified with orbit sums of weights of the Lie algebra sl_2 and the Chebyshev polynomials of the second kind may be identified with characters of this Lie algebra. Both types of polynomials are invariant under the action of the symmetric group $\{1, -1\}$, the associated Weyl group, on the exponents of the monomials. In presentations of the theory of Lie algebras [5, Ch.5,§3], this identification is often discussed in the context of the associated root systems, and we will take this approach. In particular, we define the *generalized Chebyshev polynomials* associated to a root system, as similarly done in [11, 16, 18, 21, 22, 27]. Several authors have already exploited the connection between Chebyshev polynomials and the theory of Lie algebras or root systems [7, 23] and successfully used this in the context of quadrature problems [15, 17, 19, 22] or differential equations [27].

A forebear of our algorithm is Prony's method to retrieve a univariate function as a linear combination of exponential functions from its values at equally spaced points [26]. The method was further developed in a numerical context [24]. In exact computation, mostly over finite fields, some of the algorithms for the sparse interpolation of multivariate polynomial functions in terms of monomials bear similarities to Prony's method and have connections with linear codes [3, 1]. General frameworks for sparse interpolation were proposed in terms of sums of characters of Abelian groups and sums of eigenfunctions of linear operators [8, 10]. The algorithm in [13] for the recovery of a linear combination of univariate Chebyshev polynomials does not fit in these frameworks though. Yet, as observed in [2], a simple change of variables turns Chebyshev polynomials into Laurent polynomials with a simple symmetry in the exponents. This symmetry is most naturally explained in the context of root systems and Weyl groups and leads to a multivariate generalization.

Previous algorithms [2, 9, 13, 25] for sparse interpolation in terms of Chebyshev polynomials of one variable depend heavily on the relations for the products, an identification property, and the commutation of composition. We show in this paper how analogous results hold for generalized Chebyshev polynomials of several variables and stem from the underlying root system. As already known, expressing the multiplication of generalized Chebyshev polynomials in terms of other generalized Chebyshev polynomials is presided over by the Weyl group. As a first original result we show how to select n points in $\mathbb{Q}^n$ so that each n-variable generalized Chebyshev polynomial is determined by its values at these n points. A second original observation permits to generalize the commutation property in that we identify points where commutation is available.

To provide a full algorithm, we revisit sparse interpolation in an intrinsically multivariate approach that allows one to preserve and exploit symmetry. For the interpolation of sparse sums of Laurent monomials the algorithm presented has strong ties with a multivariate Prony method [12, 20, 28]. It associates to each sum of r monomials $f(x) = \sum_\alpha a_\alpha x^\alpha$, where $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and a_α in a field $\mathbb{K}$, a linear form $\Omega : \mathbb{K}[x, x^{-1}] \to \mathbb{K}$ given by $\Omega(p) = \sum_\alpha a_\alpha p(\zeta_\alpha)$ where $\zeta_\alpha = (\xi^{\alpha_1}, \dots, \xi^{\alpha_n})$ for suitable ξ. This linear form allows us to define a Hankel operator from $\mathbb{K}[x, x^{-1}]$ to its dual whose kernel is an ideal I having precisely the ζ_α as its zeroes. The ζ_α can be recovered as eigenvalues of multiplication maps on $\mathbb{K}[x, x^{-1}]/I$. The matrices of these multiplication maps can actually be calculated directly in terms of the matrices of a Hankel operator, without explicitly calculating I. One can then find the ζ_α and the a_α using only linear algebra and evaluation of the original polynomial $f(x)$ at well-chosen points. The calculation of the $(\alpha_1, \ldots, \alpha_n)$ is then reduced to the calculation of logarithms.

The usual Hankel or mixed Hankel-Toepliz matrices that appeared in the literature on sparse interpolation [3, 13] are actually the matrices of the Hankel operator mentioned above in the different univariate polynomial bases considered. The recovery of the support of a linear form with this type of technique also appears in optimization, tensor decomposition and cubature [4, 6, 14]. We present new developments to take advantage of the invariance or semi-invariance of the linear form. This allows us to reduce the size of the matrices involved by a factor equal to the order of the Weyl group.

For sparse interpolation in terms of Chebyshev polynomials, one again recasts this problem in terms of a linear form on a Laurent polynomial ring. We define an action of the Weyl group on this ring and note that the linear form is invariant or semi-invariant according to whether we consider generalized Chebyshev polynomials of the first or second kind. Evaluations, at specific points, of the function to interpolate provide the knowledge of the linear form on a linear basis of the invariant subring or semi-invariant module. In the case of interpolation of sparse sums of Laurent monomials the seemingly trivial yet important fact that $(\xi^\beta)^\alpha = (\xi^\alpha)^\beta$ is crucial to the algorithm. In the multivariate Chebyshev case we identify a family of evaluation points that provides a similar commutation property in the Chebyshev polynomials.

Since the linear form is invariant, or semi-invariant, the support consists of points grouped into orbits of the action of the Weyl group. Using tools developed in analogy to the Hankel formulation above, we show how to recover the values of the fundamental invariants on each of these orbits and, from these, the values of the Chebyshev polynomials that appear in the sparse sum. Furthermore, we show how to recover each Chebyshev polynomial from its values at n carefully selected points.

The relative cost of our algorithms depends on the linear algebra operations used in recovering the support of the linear form and the number of evaluations needed. Recovering the support of a linear form on the Laurent polynomial ring is solved with linear algebra after introducing the appropriate Hankel operators. Symmetry reduces the size of matrices, as expected, by a factor equal to the order of the group. Concerning evaluations of the function to recover, we need evaluations to determine certain submatrices of maximum rank used in the linear algebra component of the algorithms. To bound the number of evaluations needed, we rely on the interpolation property of sets of polynomials indexed by the hyperbolic cross, a result generalizing the case of monomials in [28]. We show that our method uses fewer evaluations than when one expands the Chebyshev polynomials into Laurent polynomials and determines the total support of a r-sparse sum of Chebyshev polynomials now considered as a sum of r times the order of the group Laurent monomials. One stricking result is that for multivariate Chebyshev polynomials associated with $\mathcal{A}_2$ the number of evaluations needed to recover the support of a sum of r such polynomials is the same as the number of evaluations to recover the support of a sum of r Laurent monomials.

The related article will appear in the journal *Foundations of Computational Mathematics*.

References

[1] A. Arnold. *Sparse Polynomial Interpolation and Testing.* PhD thesis, University of Waterloo, 3 2016.

[2] A. Arnold and E. Kaltofen. Error-correcting sparse interpolation in the Chebyshev basis. In *Proceedings of the 2015 ACM on International Symposium on Symbolic and Algebraic Computation*, ISSAC '15, pages 21–28, New York, NY, USA, 2015. ACM.

[3] M. Ben-Or and P. Tiwari. A deterministic algorithm for sparse multivariate polynomial interpolation. In *Proceedings of the Twentieth Annual ACM Symposium on Theory of Computing*, STOC '88, pages 301–309, New York, NY, USA, 1988. ACM.

[4] A. Bernardi and D. Taufer. Waring, tangential and cactus decompositions. *J. Math. Pures Appl. (9)*, 143:1–30, 2020.

[5] N. Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines.* Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968.

[6] M. Collowald and E. Hubert. Algorithms for computing cubatures based on moment theory. *Studies in Applied Mathematics*, 141(4):501–546, 2018.

[7] J. Dieudonné. *Special functions and linear representations of Lie groups*, volume 42 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, R.I., 1980. Expository lectures from the CBMS Regional Conference held at East Carolina University, Greenville, North Carolina, March 5–9, 1979.

[8] A. Dress and J. Grabmeier. The interpolation problem for k-sparse polynomials and character sums. *Adv. in Appl. Math.*, 12(1):57–75, 1991.

[9] M. Giesbrecht, G. Labahn, and W. Lee. Symbolic-numeric sparse polynomial interpolation in chebyshev basis and trigonometric interpolation. In *CASC 2004*, 2004.

[10] D. Grigoriev, M. Karpinski, and M. Singer. The interpolation problem for k-sparse sums of eigenfunctions of operators. *Adv. in Appl. Math.*, 12(1):76–81, 1991.

[11] M. Hoffman and W. Withers. Generalized Chebyshev polynomials associated with affine Weyl groups. *Trans. Amer. Math. Soc.*, 308(1):91–104, 1988.

[12] S. Kunis, T. Peter, T. Römer, and U. von der Ohe. A multivariate generalization of Prony's method. *Linear Algebra Appl.*, 490:31–47, 2016.

[13] Y. Lakshman and D. Saunders. Sparse polynomial interpolation in nonstandard bases. *SIAM J. Comput.*, 24(2):387–397, 1995.

[14] M. Laurent. Sums of squares, moment matrices and optimization over polynomials. In *Emerging applications of algebraic geometry*, volume 149 of *IMA Vol. Math. Appl.*, pages 157–270. Springer, New York, 2009.

[15] H. Li and Y. Xu. Discrete Fourier analysis on fundamental domain and simplex of A_d lattice in d-variables. *J. Fourier Anal. Appl.*, 16(3):383–433, 2010.

[16] V. D. Lyakhovsky and Ph. V. Uvarov. Multivariate Chebyshev polynomials. *J. Phys. A*, 46(12):125201, 22, 2013.

[17] R. Moody, L. Motlochová, and J. Patera. Gaussian cubature arising from hybrid characters of simple Lie groups. *J. Fourier Anal. Appl.*, 20(6):1257–1290, 2014.

[18] R. Moody and J. Patera. Computation of character decompositions of class functions on compact semisimple Lie groups. *Math. Comp.*, 48(178):799–827, 1987.

[19] R. Moody and J. Patera. Cubature formulae for orthogonal polynomials in terms of elements of finite order of compact simple Lie groups. *Adv. in Appl. Math.*, 47(3):509–535, 2011.

[20] B. Mourrain. Polynomial–exponential decomposition from moments. *Foundations of Computational Mathematics*, 18(6):1435–1492, Dec 2018.
[21] H. Munthe-Kaas. On group Fourier analysis and symmetry preserving discretizations of PDEs. *J. Phys. A*, 39(19):5563–5584, 2006.
[22] H. Munthe-Kaas, M. Nome, and B. Ryland. Through the kaleidoscope: symmetries, groups and Chebyshev-approximations from a computational point of view. In *Foundations of computational mathematics, Budapest 2011*, volume 403 of *London Math. Soc. Lecture Note Ser.*, pages 188–229. Cambridge Univ. Press, Cambridge, 2013.
[23] M. Nesterenko, J. Patera, and A. Tereszkiewicz. Orthogonal polynomials of compact simple Lie groups. *Int. J. Math. Math. Sci.*, 2011.
[24] V. Pereyra and G. Shcerer, editors. *Exponential Data Fitting and its Applications*. Bentham e-books, `http://www.benthamscience.com/ebooks/9781608050482`, 2010.
[25] D. Potts and M. Tasche. Sparse polynomial interpolation in Chebyshev bases. *Linear Algebra Appl.*, 441:61–87, 2014.
[26] C. (Baron de Prony) Riche. Essai expérimental et analytique sur les lois de la dilatabilité des fluides élastique et sur celles de la force expansive de la vapeur de l'eau et de la vapeur de l'alkool, à différentes températures. *J. de l'École Polytechnique*, 1:24–76, 1795.
[27] B. Ryland and H. Munthe-Kaas. On multivariate Chebyshev polynomials and spectral approximations on triangles. In *Spectral and high order methods for partial differential equations*, volume 76 of *Lect. Notes Comput. Sci. Eng.*, pages 19–41. Springer, Heidelberg, 2011.
[28] T. Sauer. Prony's method in several variables: symbolic solutions by universal interpolation. *J. Symbolic Comput.*, 84:95–112, 2018.

Analysis of the geometry of edges with the Taylorlet transform

Thomas Fink

In this talk, we consider an extension of the continuous shearlet transform which additionally uses higher order shears. This extension, called the Taylorlet transform, allows for a detection of the position, the orientation, the curvature, and other higher order geometric information of singularities. Employing the novel vanishing moment conditions of higher order, $\int_{\mathbb{R}} g(\pm t^k)t^m dt = 0$ for $k, m \in \mathbb{N}$, $k \geq 1$, on the analyzing function $g \in \mathcal{S}(\mathbb{R})$, we can show that the Taylorlet transform exhibits different decay rates for decreasing scales depending on the choice of the higher order shearing variables. This enables a faster detection of the geometric information of singularities in terms of the decay rate with respect to the dilation parameter. Subsequently, we consider an extension of the 3D continuous shearlet transform which uses a different scaling matrix and second order shears. This extension, called the 3D Taylorlet transform, allows for a detection of the curvature tensor of a smooth singular surface in addition to its orientation and position. To this end, the 3D Taylorlet is equipped with special vanishing moment and non-vanishing moment properties. They provide the means to distinguish between hyperbolic and non-hyperbolic points in the singular surface of the analyzed function (after application of dilation, translation, first and second order shears), thereby enabling the 3D Taylorlet transform to resolve the curvature tensor through its decay at fine scales.

A plethora of different methods exists for the task of edge detection. Multiscale approaches play a special role as they offer a good noise robustness and are motivated from a continuous setting where the term of an edge finds a more general mathematical analogue in the singular support. The detection of this feature, i.e. the distinction between regular and singular points by the decay rate of continuous multi-scale methods such as the continuous wavelet transform has been thoroughly discussed in the literature, e.g. for the continuous wavelet transform in [1]. In addition to the identification of singularities, the continuous curvelet and shearlet transform as well offer a detection of directional information, i.e. a resolution of the wavefront set [2, 3]. When the continuous shearlet transform is additionally endowed with second order shears

$$S_s : \mathbb{R}^2 \to \mathbb{R}^2, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + s_1 x_1 + s_2 x_2^2 \\ x_2 \end{pmatrix},$$

the resulting bendlet transform is capable of extracting the curvature of an edge [4].

In this talk, we discuss the Taylorlet transform [6], which utilizes higher order shears like the bendlet transform, and allows for an extraction of the position, orientation, curvature and higher order geometric information of edges. It extends the bendlet transform by conditions that ensure a high decay rate for a more robust detection of the desired features. The approach is based on a modeling of a singular support as graph of a singularity function $q \in C^\infty(\mathbb{R})$, i.e. $\operatorname{sing\,supp}(f) = \{x \in \mathbb{R}^2 : \; x_1 = q(x_2)\}$. The geometrical data accessible by the Taylorlet transform consists of the Taylor coefficients of q. In this perspective, the 2D continuous wavelet and shearlet transform essentially identify the 0^{th} rsp. the 0^{th} and 1^{st} Taylor coefficients of the singularity function.

For these detection properties, vanishing moment conditions for the respective analyzing function are essential. They are responsible for the ability of the continuous wavelet transform to detect singularities of high regularity [1, Thm 3] and ensure the decay rate of the continuous shearlet transform for decreasing scales [5, Thm 3.1]. It can be shown that analogously an analyzing Taylorlet $\tau(x) = g(x_1) \cdot h(x_2)$ satisfying vanishing moment conditions of the type

$$\int_{\mathbb{R}} g(x_1^k) x_1^m dx_1 = 0 \in \mathbb{R} \text{ for all } k \in \{1, \ldots, n\},$$

are of similar importance for the extraction of higher order geometric information.

Moreover, we present an extension of the Taylorlet transform and its detection results in 3D. As was shown in [7], the continuous 3D shearlet transform is capable of resolving the wavefront set of the characteristic function of an open set with a piecewise smooth boundary, thereby enabling the extraction of directional information of the analyzed function. When proving a similar result for the detection of also the curvature of the boundary using the 3D Taylorlet transform, a distinction between hyperbolic and non-hyperbolic points of the boundary becomes necessary. In this regard, certain vanishing and non-vanishing moment properties force a different decay rate of the 3D Taylorlet transform in the case of hyperbolic points and non-hyperbolic points for small scales. These case distinctions finally allow us to devise a 3-step detection algorithm for the curvature tensor of the boundary.

References

[1] Stephane Mallat and Wen Liang Hwang. Singularity detection and processing with wavelets. *IEEE transactions on information theory*, 38(2):617–643, 1992.

[2] Emmanuel J Candes and David L Donoho. Continuous curvelet transform: I. resolution of the wavefront set. *Applied and Computational Harmonic Analysis*, 19(2):162–197, 2005.

[3] Gitta Kutyniok and Demetrio Labate. Resolution of the wavefront set using continuous shearlets. *Transactions of the American Mathematical Society*, 361(5):2719–2754, 2009.

[4] Christian Lessig, Philipp Petersen, and Martin Schäfer. Bendlets: A second-order shearlet transform with bent elements. *arXiv preprint arXiv:1607.05520*, 2016.

[5] Philipp Grohs. Continuous shearlet frames and resolution of the wavefront set. *Monatshefte für Mathematik*, 164(4):393–426, 2011.

[6] Fink, Thomas. Higher order analysis of the geometry of singularities using the Taylorlet transform. *Advances in Computational Mathematics*, 45(2), pages 675-705, 2019.

[7] Guo, Kanghui, and Demetrio Labate. Characterization of piecewise-smooth surfaces using the 3D continuous shearlet transform. Journal of Fourier Analysis and Applications 18(3), pages 488-516, 2012.

GC sets with few maximal lines

Jesús Carnicer

(joint work with Carmen Godés)

The *Lagrange interpolation problem* by polynomials in the space $P_n(\mathbb{R}^2)$ of bivariate polynomials of total degree not greater than n on a planar set $X \subset \mathbb{R}^2$ consists of finding a polynomial $p \in P_n(\mathbb{R}^2)$ for a given $f \in \mathbb{R}^X$ such that

$$p(x) = f(x), \quad x \in X.$$

If, for a set X, the interpolation polynomial exists and it is unique, we say that the set X is P_n*-correct.*

The *Lagrange polynomials* of a P_n-correct set are the polynomials $\ell_{x,X} \in P_n(\mathbb{R}^2)$ can be defined by

$$\ell_{x,X}(x) = 1, \quad \ell_{x,X}(y) = 0, \quad \forall y \in X \setminus \{x\}.$$

Using the Lagrange polynomials, we can express the solution of the interpolation problem by the Lagrange formula

$$p(t) = \sum_{x \in X} f(x)\ell_{x,X}(t), \quad t \in \mathbb{R}^2.$$

If a set X is P_n-correct, no line of the plane L contains more than $n+1$ nodes, $\#(L \cap X) \leq n+1$. A line with exactly $n+1$ nodes is called a *maximal line* (see [1]). Chung and Yao in 1977 [10], obtained a *geometric characterization* of sets of nodes such that the Lagrange polynomials are products of first degree polynomials.

Definition 1 (GC_n set)**.** A set $X \subset \mathbb{R}^2$, $\#X = (n+2)(n+1)/2$ satisfies the *geometric characterization* of degree n, (X is a GC_n *set* for short), if for each $x \in X$ there exist a set of n lines $\Gamma_{x,X}$ such that

$$x \notin \bigcup_{L \in \Gamma_{x,X}} L, \quad X \setminus \{x\} \subseteq \bigcup_{L \in \Gamma_{x,X}} L.$$

If X is a GC_n set, then it is P_n-correct. Gasca and Maeztu [11] stated the following conjecture.

Conjecture 1 (Gasca-Maeztu conjecture)**.** *Each planar GC_n set has a maximal line.*

Different authors have contributed to the conjecture showing that it holds for all GC_n sets up to degree $n \leq 5$ (see [2, 4, 12]).

Theorem 1. *If X is a GC_n set with $n \leq 5$, there exists a maximal line.*

A hope for solving the conjecture consists of understanding the structure of all known GC sets. Some GC sets have some particular structure, leading to interesting algebraic relations in the corresponding syzygy matrices.

Definition 2. A P_n-correct set $X \subset \mathbb{R}^2$ with $n \geq 1$ has *defect* d if the number of maximal lines is $n + 2 - d$.

The defect of a GC_n set is an integer between 0 and $n+2$. In [3] it was shown that the Gasca-Maeztu conjecture implies that each GC_n sets contains at least 3 maximal lines.

Theorem 2. *Assume that the Gasca-Maeztu conjecture holds for all GC sets up to degree ν, that is, each GC_k set with $k \leq \nu$ has at least a maximal line. If X is a GC_n set with $n \leq \nu$, there exists at least three maximal lines.*

So, if the Gasca-Maeztu conjecture holds, the defect of any GC_n set is an integer between 0 and $n-1$. The GC_n sets with defect $n-1$ have a very special structure

Definition 3. A set $X \subset \mathbb{R}^2$ is a *generalized principal lattice of degree n* if there exist 3 families of lines each containing $n+1$ lines

$$(L_i^r(X))_{i \in \{0,1,\ldots,n\}}, \quad r = 0, 1, 2,$$

such that the $3n+3$ lines are distinct,

$$L_i^0(X) \cap L_j^1(X) \cap L_k^2(X) \cap X \neq \emptyset \Leftrightarrow i + j + k = n$$

and

$$X = \{x_{ijk} \mid x_{ijk} := L_i^0(X) \cap L_j^1(X) \cap L_k^2(X),\ 0 \leq i, j, k \leq n,\ i + j + k = n\}.$$

If the Gasca-Maeztu conjecture holds, it can be shown that each GC_n set with defect $n-1$ is a generalized principal lattice [5].

Theorem 3. *Let $X \subset \mathbb{R}^2$ be a generalized principal lattice of degree $n \geq 1$. Then X is a GC_n set with defect $n-1$ whose maximal lines are $L_0^0(X), L_0^1(X), L_0^2(X)$. If the Gasca-Maeztu conjecture holds for all degrees up to ν and $1 \leq n \leq \nu + 3$, then X is a generalized principal lattice of degree n if and only if it is a GC_n set with defect $n-1$.*

In the talk, we review the characterizations of GC_n sets according to their defect, assuming that the Gasca-Maeztu conjecture holds [5, 6, 7, 8]. A description of the GC_n sets with defect 0, 1, 2 and 3 is discussed. We sketch part of the proof of the following result in [9]

Theorem 4. *If X is a GC_n set with $n \geq 6$, then the defect of X cannot be 4.*

As a consequence, if the Gasca-Maeztu conjecture holds, only GC_n sets with defect $0, 1, 2, 3$ and $n - 1$ may exist.

References

[1] C. de Boor, *Multivariate polynomial interpolation: conjectures concerning GC-sets*, Numer. Algor. **45** (2007), 113–125.

[2] J. R. Busch, *A note on lagrange interpolation in* $\mathbb{R}^2$, Rev. Union Matem. Argent. **36** (1990), 33–38.

[3] Carnicer, J. M., and M. Gasca, On Chung and Yao's geometric characterization for bivariate polynomial interpolation, in *Curve and Surface Design: Saint-Malo 2002*, T. Lyche, M.-L. Mazure, and L. L. Schumaker (eds.), Nashboro Press, Brentwood, 2003, 21–30.

[4] J. M. Carnicer and M. Gasca, *A conjecture on multivariate polynomial interpolation*, Rev. R. Acad. Cien. Serie A. Mat. **95** (2001), 145–153.

[5] J. M. Carnicer and C. Godés, Geometric characterization and generalized principal lattices, *J. Approx. Theory* **143** (2006) 2–14.

[6] J. M. Carnicer and C. Godés, *Geometric characterization of configurations with defect three*, Curve and Surface Fitting: Avignon 2006 (A. Cohen, J. L. Merrien, and L. L. Schumaker, eds.), Nashboro Press, Brentwood, 2007, pp. 61–70.

[7] J. M. Carnicer and C. Godés, Generalized principal lattices and cubic pencils, *Numer. Algorithms* **44** (2007), 133–145.

[8] J. M. Carnicer and C. Godés, Classification of sets satisfying the geometric characterization. Numer. Algorithms **50** (2009), 145–154.

[9] J. M. Carnicer and C. Godés, Configurations of nodes with defects greater than three. J. Comput. Appl. Math. **233** (2010), 1640–1648.

[10] K. C. Chung and T. H. Yao, *On lattices admitting unique Lagrange interpolation*, SIAM J. Num. Anal. **14** (1977), 735–743.

[11] M. Gasca and J. I. Maeztu, *On Lagrange and Hermite interpolation in* $\mathbb{R}^k$, Numer. Math. **39** (1982), 1–14.

[12] H. Hakopian, K. Jetter, and G. Zimmermann, *The Gasca-Maeztu conjecture for* $n = 5$, Numer. Math. **127** (2014), 685–713.

On n-independent, n-correct, and GC_n sets

Hakop Hakopian

This talk considers some of the recent results on n-independent, n-correct, and GC_n sets, concerning the Gasca-Maeztu conjecture.

Along with the maximal line we consider also line passing through exactly n-nodes, as well as the subset of the nodes in an n-correct set that use a given line. Next, the space of curves of given degree passing through the points of an n-independent set of certain cardinality is studied and a characterization for an extremal case is provided.

Denote the space of all bivariate polynomials of total degree not exceeding n by Π_n. We have that

$$N := N_n := \dim \Pi_n = (1/2)(n+1)(n+2).$$

Consider a planar set $\mathcal{X}$ of s distinct nodes. The problem of finding a polynomial $p \in \Pi_n$, which satisfies the conditions $p(A) = c_A \ \forall A \in \mathcal{X}$, is called *interpolation problem.*

A set $\mathcal{X}$ is called *n-correct* if the interpolation problem is unisolvent for arbitrary data $\{c_A : A \in \mathcal{X}\}$. For n-correct sets we have that $|\mathcal{X}| = N$.

A *fundamental polynomial* for a node $A \in \mathcal{X}$ is, by definition, a polynomial p which vanishes on the set $\mathcal{X}$ but A and $p(A) = 1$.

A set $\mathcal{X}$ is called *n-independent* if each node has a fundamental polynomial from Π_n. Otherwise, it is called *n-dependent.*

A set of nodes $\mathcal{X}_s$ is called *essentially n-dependent*, if no node has an n-fundamental polynomial.

For n-independent sets we have that $|\mathcal{X}| \le N$.

In view of the Lagrange formula one gets readily that a set $\mathcal{X}$ is n-independent if and only if the interpolation problem is solvable in Π_n for arbitrary data.

On the other hand each n-independent set $\mathcal{X}$ with $|\mathcal{X}| < N$ can be enlarged to an n-correct set [5]. Thus n-independent sets can be characterized also as subsets of n-correct sets.

We call a line k-node if it passes through exactly k nodes of $\mathcal{X}$.

At most $n+1$ nodes can be collinear in any n-independent set and an $(n+1)$-node line is called a *maximal line* (C. de Boor, 2007).

We say that a node A in an n-correct set $\mathcal{X}$ *uses* a line ℓ if ℓ divides the fundamental polynomial of A. Denote the subset of nodes of $\mathcal{X}$ that use the line ℓ by $\mathcal{X}^\ell$.

An n-correct set $\mathcal{X}$ is called a *GC_n set* if the fundamental polynomial of each node is a product of n linear factors [3]. Thus in a GC_n set each node uses exactly n lines.

The Gasca-Maeztu (or briefly GM) conjecture (1982) states that every GC_n set has a maximal line [4]. Until now the conjecture has been proved only for the cases $n \le 5$ ([4], [1], [6]).

A *plane algebraic curve* is the zero set of some bivariate polynomial of degree ≥ 1. To simplify notation, we shall use the same letter, say p, to denote the polynomial p and the curve given by the equation $p(x, y) = 0$.

Set $d(n, k) := N_n - N_{n-k} = (1/2)k(2n + 3 - k)$.

Let q be an algebraic curve of degree $k \le n$ with no multiple components and $\mathcal{X}$ be an n-correct set. Then q contains not more than $d(n, k)$ nodes of $\mathcal{X}$. A curve of degree $k \le n$ passing through exactly $d(n, k)$ points of an n-correct set $\mathcal{X}$ is called a *maximal curve* [13].

It is worth mentioning that any n-independent node set in the curve q of cardinality less than $d(n, k)$ can be enlarged to a maximal n-independent set of cardinality $d(n, k)$ [9].

Results. Let us start with two results on GC_n sets.

Theorem 1 ([11])**.** Assume that GM conjecture holds. Let $\mathcal{X}$ be a GC_n set, and ℓ be a k-node line, $k \ge 2$. Then, we have that $\mathcal{X}^\ell = \emptyset$, or

$\mathcal{X}^\ell$ is an GC_{s-2} subset of $\mathcal{X}$, hence $|\mathcal{X}^\ell| = \binom{s}{2}$, where $s \le k$.

Moreover, if λ is a maximal line then $|\lambda \cap \mathcal{X}^\ell| = 0$ or $|\lambda \cap \mathcal{X}^\ell| = s - 1$.

Let us mention that J.M. Carnicer and M. Gasca proved earlier that a k-node line ℓ can be used by at most $\binom{k}{2}$ nodes and, in addition, there are no k collinear nodes that use ℓ [2].

Theorem 2 ([10])**.** Assume that GM conjecture holds. Let $\mathcal{X}$ be a GC_n set with $n \geq 4$. Then there are at most three n-node lines. Moreover, any two n-node lines intersect at a node of $\mathcal{X}$.

Then let us bring two recent results on n-independent sets.

Theorem 3. Assume that $\mathcal{X}$ is an n-independent set of $d(n, k-s)+s$ nodes with $s+1 \leq k \leq n-1$, where $s = 1, 2, 3$. Then at most $N_s - s$ linearly independent curves of degree $\leq k$ may pass through all the nodes of $\mathcal{X}$. Moreover, there are such $N_s - s$ curves for the set $\mathcal{X}$ if and only if all the nodes of $\mathcal{X}$ but s lie in a (maximal) curve of degree $k - s$.

The cases $s = 1, 2, 3$, are proved in [9], [7], [8], respectively.

Theorem 4. Assume that $\mathcal{X}$ is an n-independent set of $d(n, k-s)+s+1$ nodes with $s+1 \leq k \leq n-s$, where $s = 2, 3$. Then at most $N_s - s - 1$ linearly independent curves of degree $\leq k$ may pass through all the nodes of $\mathcal{X}$. Moreover, there are such $N_s - s - 1$ curves for the set $\mathcal{X}$ if and only if either all the nodes of $\mathcal{X}$ lie in a curve of degree $k - s + 1$ or all the nodes of $\mathcal{X}$ but $s + 1$ lie in a (maximal) curve of degree $k - s$.

The case $s = 2$ is proved in [8].
Next let us bring a result on n-correct sets.

Corollary. Let $\mathcal{X}$ be an n-correct set of nodes and ℓ be an $(s+2)$-node line, where $s = 1, 2, 3$. Then ℓ can be used at most by N_s nodes from $\mathcal{X}$. Moreover, if ℓ is used by at least $N_{s-1} + 1$ nodes from $\mathcal{X}$ then it is used by exactly N_s nodes from $\mathcal{X}$. Furthermore, if it is used by N_s nodes, then these nodes form an s-correct set.

The cases $s = 1, 2, 3$ are proved in [9], [7], [8], respectively.
The results presented above are useful tools in the study of the Gasca-Maeztu hypothesis for $n = 6$.
One may conjecture that Theorems 3, 4, and Corollary hold also for the values $s \geq 4$. Finally, we present the following

Theorem 5 ([12])**.** A set $\mathcal{X}$ with $\#\mathcal{X} = mn$, $m \leq n$, is the set of intersection points of two algebraic curves of degrees m and n respectively, if and only if the following two conditions hold:
(i) The set $\mathcal{X}$ is essentially $(m + n - 3)$-dependent;
(ii) The set $\mathcal{X}$ contains an $(m - 1)$-correct set.

It is worth mentioning that the necessity of the conditions (i) and (ii) follow from the Cayley-Bacharach and Noether theorems, respectively.

References

[1] J. R. Busch, *A note on Lagrange interpolation in* $\mathbb{R}^2$, Rev. Un. Mat. Argentina **36** (1990), 33–38.

[2] J. M. Carnicer and M. Gasca, *On Chung and Yao's geometric characterization for bivariate polynomial interpolation*, in Curve and Surface Design, eds. T. Lyche, M.-L. Mazure, and L. L. Schumaker (Nashboro Press, Saint Malo 2002, Brentwood, 2003) 21–30.

[3] K. C. Chung and T. H. Yao, *On lattices admitting unique Lagrange interpolations*, SIAM J. Numer. Anal. **14** (1977), 735–743.

[4] M. Gasca, and J. I. Maeztu, *On Lagrange and Hermite interpolation in* $\mathbb{R}^n$, Numer. Math. **39** (1982), 1–14.

[5] H. Hakopian, K. Jetter, and G. Zimmermann, *Vandermonde matrices for intersection points of curves,* Jáen J. Approx. **1** (2009), 67–81.

[6] H. Hakopian, K. Jetter, and G. Zimmermann, *The Gasca-Maeztu conjecture for* $n = 5$, Numer. Math. **127** (2014), 685–713.

[7] H. Hakopian and H. Kloyan, *On the dimension of spaces of algebraic curves passing through* n*-independent nodes*, Proceedings of YSU. Physical and Mathematical Sciences no. 2 (2019), 3–13.

[8] H. Hakopian, H. Kloyan, and D. Voskanyan, *On plane algebraic curves passing through* n*-independent nodes,* Journal of Contemporary Mathematical Analysis **56** (2021), 280–294.

[9] H. Hakopian and S. Toroyan, *On the Uniqueness of algebraic curves passing through* n*-independent nodes*, New York J. Math. **22** (2016), 441–452.

[10] H. Hakopian and N. Vardanyan, *On basic properties of* GC_n *sets,* Journal of Knot Theory and Its Ramifications **29** (2020), 1-26.

[11] H. Hakopian, V. Vardanyan, *On a correction of a property of* GC_n *sets*, Adv. Comput. Math., **45** (2019), 311–325.

[12] H. Hakopian and D. Voskanyan, *On the intersection points of two plane algebraic curves,* Journal of Contemporary Mathematical Analysis **54** (2019), 90–97.

[13] L. Rafayelyan, *Poised nodes set constructions on algebraic curves*, East J. Approx., **17** (2011), 285–298.

Higher order interpolation through the singular locus of a reflection arrangement

Alexandra Seceleanu

We discuss *higher order interpolation* from an algebraic-geometric view point.

Higher order interpolation. Classically, the interpolation problem fixes a finite set of points $X = \{p_1, \ldots, p_n\}$ together with desired values v_i, one for each point, and asks for the equation of a polynomian function F that satisfies $F(p_i) = v_i$. In this talk the values to be interpolated will always be $v_i = 0$ and an interpolating function, that is, a polynomial F that satisfies $F(p_i) = 0$ is the equation of a *hypersurface* that contains X.

Higher order interpolation also fixes a finite set of points $X = \{p_1, \ldots, p_n\}$ and requires a function $F \in k[x_0, \ldots, x_d]$ that vanishes to order m at each of the points. This is a polynomial function which satisfies

$$(1) \quad \frac{\partial F}{\partial x_{i_1} \cdots \partial x_{i_\ell}}(p_i) = 0 \text{ for all } 0 \leq \ell \leq m-1, 1 \leq i_1 \leq \ldots \leq i_\ell \leq d, 1 \leq i \leq n.$$

To summarize, we now require F as well as all its derivatives of order up to $m-1$ to vanish at X. If F is assumed homogeneous, for equation (1) to hold it suffices that all derivatives of order $m-1$ vanish.

Let $I_{p_i} = \{f \in k[x_1, \ldots, x_d] : f(p_i) = 0\}$ be the defining ideal of p_i. Then the solutions to the classical interpolation problem form the ideal

$$I_X = I_{p_1} \cap I_{p_2} \cap \cdots \cap I_{p_n}$$

and the polynomials which vanish to order m at X form the ideal

$$I_X^{(m)} = I_{p_1}^m \cap I_{p_2}^m \cap \cdots \cap I_{p_n}^m.$$

The latter is called a *fat point ideal*.

Example 1. Consider the points $X = \{[1:0:0], [0:1:0], [0:0:1]\}$. Then

$$\begin{aligned} I_X &= \langle y,z\rangle \cap \langle x,z\rangle \cap \langle x,y\rangle &&= \langle xy, xz, yz\rangle \\ I_X^{(3)} &= \langle y,z\rangle^3 \cap \langle x,z\rangle^3 \cap \langle x,y\rangle^3 &&= \langle x^2y^2z, x^2yz^2, xy^2z, x^3y^3, x^3z^3, y^3z^3\rangle. \end{aligned}$$

Problems 2. Two fundamental open problems in higher order interpolation are:

(1) Given X, find the smallest degree of a polynomial in $I_X^{(m)}$.
(2) Given X, δ, find the dimension of the space of degree δ polynomials in $I_X^{(m)}$.

If the points in X are *general* there are important conjectural answers to both of the above problems. Nagata's conjecture [7] predicts that the smallest degree of a hypersurface vanishing to order m at n points in d-dimensional space is at least $m\sqrt[d]{n}$. For *general* points in the plane the Segre-Harbourne-Gimigliano-Hirshowitz (SHGH) conjecture predicts that the answer to the second problem is obtained by counting the conditions (number of derivatives) imposed by the higher vanishing at each point, with some geometrically determined exceptions.

The containment problem. Instead of asking enumerative questions regarding the solutions to the higher order interpolation problem, one can ask the following structural question:

Problem 3. How can polynomials vanishing to order m at X be expressed in terms of products of polynomials vanishing at X? Equivalently, which pairs of natural numbers r, m give rise to containments $I_X^{(m)} \subseteq I_X^r$?

Clearly, if $f_1, \ldots, f_m$ vanish at X then $f_1 \cdots, f_m$ vanishes to order m at X. In other words there is an ideal containment $I_X^m \subseteq I_X^{(m)}$. But not all polynomials vanishing at to order m at X are such products. In Example 1 we find that $x^2y^2z \in I_X^{(3)} \setminus I_X^3$. However, it turns out that $x^2y^2z \in I_X^2$ as $x^2y^2z = (xy)^2z$. In fact, in Example 1 there is a containment $I_X^{(3)} \subseteq I_X^2$; all polynomials which vanish to order 3 at X can be expressed in terms of products of two polynomials in I_X.

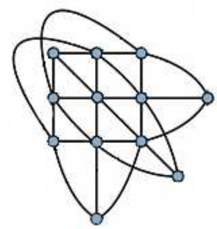
Two prominent answers to the containment problem are:

Theorem 4 (Ein-Lazarsfeld-Smith [5], Hochster-Huneke [6])**.** If $X \subseteq \mathbb{P}^d$ then $I_X^{(dn)} \subseteq I_X^n, \ \forall n \in \mathbb{N}$.

Theorem 5 (Bocci-Harbourne [2])**.** If X is a *generic* set of points in $\mathbb{P}^2$ then $I_X^{(2n-1)} \subseteq I_X^n,\ \forall n \in \mathbb{N}$.

In view of the difference between Theorems 4 and 5 it became important to understand the sets of points $X \subseteq \mathbb{P}^d$ for which the containment $I_X^{(dn-1)} \subseteq I_X^n$ does *not* hold for some $n \in \mathbb{N}$. Such sets are termed *containment-tight* in [3]. The first open case is $d = 2, n = 3$, i.e., sets of points in the plane for which $I_X^{(3)} \not\subseteq I_X^2$.

Example 6 (Dumnicki-Szemberg-Tutaj-Gasińska [4])**.** Let X be the set of 12 points of intersection of the 9 lines pictured below (note that some of the lines are depicted as curves) and let $F = (x^3 - y^3)(y^3 - z^3)(z^3 - x^3)$ be the product of the 9 linear forms defining these lines. Then we have $F \in I_X^{(3)} \setminus I_X^2$.



The containment problem for singular loci of reflection arrangements. Further examples of point configurations X in the plane which satisfy $I_X^{(3)} \not\subseteq I_X^2$ are given in [1]: the Klein configuration consists of the 49 points of intersection of 21 lines (21 of the points are quadruple intersections and 28 are triple) and the Wiman configuration consists of the 201 points of intersection of 45 lines (36 points are quintuple, 45 quadruple, and 120 triple). The important feature of these configurations as well as that of Example 6 above is that all singular points of the respective line arrangements are at least triple intersections. This cannot occur in the real plane (i.e. any real arrangement of lines that do not all meet at a point has at least one point where only 2 lines meet) by the dual Sylvester-Gallai theorem.

With hindsight, all the examples discussed here so far are seen to be reflection arrangements. A *reflection group* is a finite group generated by reflections, that is, linear transformations which fix a hyperplane pointwise. A *reflection arrangement* is the collection of fixed hyperplanes for all reflections in a given reflection group. Irreducible complex reflection groups have been classified by Shephard–Todd in an infinite family denoted $G(m, p, n)$ and 34 exceptional cases. Above we have discussed the points of intersection of the reflection arrangements $G(1, 1, 4) = S_3$ (the symmetric group on 3 elements) in Example 1, $G(3, 3, 3)$ in Example 6, and the exceptional groups G_{24}, G_{27} which have the Klein and the Wiman configurations as their respective singular points.

Finally, we are able to classify the complex reflection arrangements in terms of their behavior with respect to the containment problem.

Theorem 7 (Drabkin-Seceleanu [3])**.** If I is an ideal defining the singular locus of an irreducible reflection arrangement $\mathcal{A}$ in $\mathbb{P}^d$ then $I^{(3)} \subseteq I^2$ unless $\mathcal{A}$ is one of

(1) G_{24} (Klein), G_{27} (Wiman) for $d = 2$
(2) $G_{29}(d = 3), G_{33}(d = 4), G_{34}(d = 5)$,
(3) $G(m, m, d + 1)$ with $m \geq 3, d \geq 2$.

Moreover $I^{(2n-1)} \subseteq I^n$ $\forall n \geq 3$ for all examples with $d = 2$.

An key ingredient of the above result is the determination of syzygies for the respective ideals. In particular, the containment $I^{(3)} \subseteq I^2$ corresponds to the cases when I has a linear (degree one) syzygy, whereas the non containment corresponds to the absence of linear syzygies.

References

[1] T. Bauer, S. DiRocco, B. Harbourne, J. Huizenga, A. Seceleanu, T. Szemberg, *Negative curves on symmetric blowups of the projective plane, resurgences, and Waldschmidt constants*, Int. Math. Res. Not. IMRN 2019 (2018), no. 24, 7459–7514.

[2] C. Bocci, B. Harbourne, *Comparing powers and symbolic powers of ideals*, J. Algebraic Geom. 19 (2010), no. 3, 399–417.

[3] B. Drabkin, A. Seceleanu, *Singular loci of reflection arrangements and the containment problem*, Math. Z. 299 (2021), no. 1–2, 867–895.

[4] M. Dumnicki, T. Szemberg, H. Tutaj-Gasińska, *Counterexamples to the $I^{(3)} \subseteq I^2$ containment*, Journal of Algebra 393 (2013), 24–29.

[5] L. Ein, R. Lazarsfeld, K. Smith, *Uniform bounds and symbolic powers on smooth varieties*, Inventiones Math 144 (2) (2001), 241–252.

[6] M. Hochster, C. Huneke, *Comparison of symbolic and ordinary powers of ideals*, Invent. Math. 147 (2002), no. 2, 349–369.

[7] M. Nagata, *On the 14-th problem of Hilbert*, Amer. J. Math. 81 (1959), 766–772.

On Polynomial Interpolation on Arbitrary Varieties

Boris Shekhtman

This talk is based on joint work with Tom Mckinley and Brian Tuesink [2].

The interpolation problem had mostly been studied as interpolation on points. The singular exception is the study of interpolation on “flats” done by Carl de Boor, Nira Dyn and Amos Ron in [1].

Here is the general question addressed:

Problem 1. Let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be subvarieties of $\mathbb{C}^d$ and let $p_1, \ldots, p_n$ be polynomials in $\mathbb{C}[\mathbf{x}] := \mathbb{C}[x_1, \ldots, x_d]$ that I will refer to as “data”. When does there exist a polynomial $f \in \mathbb{C}[x_1, \ldots, x_d]$ such that the restriction f onto $\mathcal{V}_j$ coincides with p_j on $\mathcal{V}_j$?

To the best of my knowledge this is the first effort to study interpolation on general varieties, at least in the field of approximation theory. The best news is that the proofs are extremely simple and rely on nothing more than The Hilbert Nullstellensatz.

Of course the answers depend on varies and the data. Here is a simple example:

Theorem 2. *Given a collection of pairwise non-intersecting varieties $\mathcal{V}_1, \ldots, \mathcal{V}_n$ in $\mathbb{C}^d$ and arbitrary polynomials $p_1, \ldots, p_n$ in $\mathbb{C}[\mathbf{x}]$ there always exists a polynomial $f \in \mathbb{C}[\mathbf{x}]$ such that $f \mid \mathcal{V}_k = p_k \mid \mathcal{V}_k$ for all $k = 1, \ldots, n$.*

This is the direct generalization of the classical case when the varieties are points.

Of course if the varieties have a non-empty intersection the data has to coincide on the intersections. From now on we will let $\mathcal{V}(J)$ to denote the variety associated with an ideal J

$$\mathcal{V}(J) := \left\{\mathbf{x} \in \mathbb{k}^d : f(\mathbf{x}) = 0 \text{ for all } f \in J\right\},$$

and with every variety $\mathcal{V} \subset \mathbb{k}^d$ we associate a radical ideal

$$J(\mathcal{V}) = \left\{f \in \mathbb{k}[\mathbf{x}] : f(\mathbf{x}) = 0 \text{ for all } \mathbf{x} \in \mathbb{k}^d\right\}.$$

Lemma 3. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two varieties in $\mathbb{C}^d$. Then the following are equivalent:

a) For every pair of polynomials $p_1, p_2 \in \mathbb{C}[\mathbf{x}]$ such that

$$p_1 \mid (\mathcal{V}_1 \cap \mathcal{V}_2) = p_2 \mid (\mathcal{V}_1 \cap \mathcal{V}_2) \tag{1}$$

there exists a polynomial $f \in \mathbb{C}[\mathbf{x}]$ such that

$$f \mid \mathcal{V}_k = p_k \mid \mathcal{V}_k \text{ for } k = 1, 2. \tag{2}$$

b) The ideal $J(\mathcal{V}_1) + J(\mathcal{V}_2)$ is radical.

More generally

Theorem 4. *Let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be a collection of varieties in $\mathbb{C}^d$. The following are equivalent:*

a) *For every collection of polynomials $p_1, \ldots, p_n \in \mathbb{C}[\mathbf{x}]$ such that*

$$p_k \mid (\mathcal{V}_k \cap \mathcal{V}_j) = p_j \mid (\mathcal{V}_k \cap \mathcal{V}_j) \text{ for all } k, j = 1, \ldots, n \tag{3}$$

there exists a polynomial f such that $f \mid \mathcal{V}_k = p_k \mid \mathcal{V}_k$ for $k = 1, \ldots n$.

b) *For every $m < n$ the ideal $J\left(\bigcup_{j=1}^{m} \mathcal{V}_j\right) + J(\mathcal{V}_m)$ is radical.*

In the general case we where able to obtain only a sufficient conditions (which is also necessary for two varieties):

Theorem 5. *Let $J_i = J(\mathcal{V}_i)$, $i = 1, \ldots, n$ be given ideal and $p_1, \ldots, p_n$ be the polynomial data on the varieties $\mathcal{V}_i$. Suppose that for any i, j*

$$(p_i - p_j) \in J_i + \bigcap_{k \neq i} J_k.$$

Then the interpolation problem has a solution.

References

[1] Carl de Boor, Nira Dyn and Amos Ron, *Polynomial Interpolation to Data on Flats in $\mathbb{R}^d$*, Journal of Approximation Theory **105** (2000), 313–343.

[2] Tom McKinley, Boris Shekhtman and Brian Tuesink, *On Polynomial Interpolation on Arbitrary Varieties* (submitted).

A Conjecture on Bernstein Basis Polynomials

Larry L. Schumaker

I made the following conjecture in 2003. It has not yet been completely proved mathematically. However, there has been some progress to report on.

Conjecture 1. *Given d and a triangle $T := \langle v_1, v_2, v_3 \rangle$, let Γ be an arbitrary subset of the set $D_{d,\triangle}$ of domain points associated with the triangle. Then the matrix*

$$M_\Gamma := \left[B^d_\eta(\xi)\right]_{\xi,\eta\in\Gamma}$$

is nonsingular, and so for any real numbers $\{z_\xi\}_{\xi\in\Gamma}$, there is a unique $p := \sum_{\eta\in\Gamma} c_\eta B^d_\eta$ such that $p(\xi) = z_\xi$ for all $\xi \in \Gamma$.

It has been shown see [2] that the conjecture holds for $\Gamma := D_{d,\triangle} \setminus \{\xi_{d00}, \xi_{0d0}, \xi_{00d}\}$, and for $\Gamma := D_{d,\triangle} \setminus \{\xi_{i,j,k}\}$ such that $i \geq m_1,\ j \geq m_2,\ k \geq m_3\}$ for some $m_1, m_2, m_3 \geq 0$ with $m := m_1 + m_2 + m_3 < d$.

It was also verified numerically for all $d \leq 16$ in [1].

References

[1] Jaklic, G., Kunduc, T.: On positivity of principal minors of bivariate Bézier collocation matrix; Appl. Math. Comput.; 227; 2014; 320–328;

[2] Lai, M. J., Schumaker, L. L.: *Spline Functions on Triangulations*, Cambridge University Press (Cambridge), 2007.

Adjoints, a short introduction with a view to interpolation and Calabi-Yau varieties

Kristian Ranestad

Let

$$F \in \mathbb{C}[x_0, x_1, x_2]_d$$

be a homogeneous form of degree d, and assume that

$$C = Z(F) \subset \mathbb{P}^2_{\mathbb{C}}$$

throughout has only ordinary double points as singularities. Then C has at most $\binom{d}{2}$ double points, with equality if and only if C is the union of d lines with no three through any point. If, furthermore, F is irreducible, then C has at most $\binom{d-1}{2}$ double points, and they impose independent conditions on forms of degree $d-3$.

Any curve $D \subset \mathbb{P}^2_{\mathbb{C}}$ of degree $d-3$ that passes through all the double points of C is called an *adjoint* to C. We immediately deduce:

Proposition 1 (Π_{d-3}-correct)**.** If C is irreducible with the maximal number $\binom{d-1}{2}$ of double points, then the set of double points is Π_{d-3}-correct. In particular C has no adjoint.

Remark 2. The general Π_{d-3}-correct set of points ($d > 5$) is not the set of double points of an irreducible curve of degree d. ♢

Consider the example of plane sextics ($d = 6$). Let $C = Z(F)$ be an irreducible sextic curve with 10 double points. Enumerate the double points, $n_1, ..., n_{10}$. Then there is a unique cubic curve $E_i = Z(G_i)$ through all the double points except n_i, moreover, any curve $C_s = Z(F + sG_i^2)$ is a sextic with nine double points. The curves C_s form a *Halphen pencil*, the nine double points form a *Halphen set of points*. Nine points form a *Halphen set*, if there is a pencil of curves of degree $3n, n > 1$ with multiplicity at least n at each of the double points. Halphen sets, with a given $n > 1$, form a 17-dimensional set in the 18-dimensional set of all 9-tuples of points.

Let $C = Z(F)$ be an irreducible curve with exactly $\binom{d-1}{2} - 1$ double points. Then there is a unique curve A_C of degree $d-3$ through all the double points. Furthermore, C and A_C have no common points outside the nodes, since by Bezout, the number of intersection points are accounted for at the double points:

$$C \cdot A_C = d(d-3) = 2(\binom{d-1}{2} - 1)$$

In this case C is an elliptic curve, a "Calabi-Yau curve".

More generally, consider a hypersurface $X \subset \mathbb{P}^n_{\mathbb{C}}$ of degree d with ordinary singularities of multiplicity k in codimension $k-1$. An *adjoint* to X is a hypersurface Y of degree $d - n - 1$ that passes through all the singularities. A hypersurface X is *Calabi-Yau* if it has a unique adjoint Y, and this adjoint intersects X only along the singular locus of X. (It is customary to have a more restrictive definition of Calabi-Yau varieties, but for our purposes this suffices.) The first examples of Calabi-Yau varieties are nonsingular hypersurfaces $X \subset \mathbb{P}^n_{\mathbb{C}}$ of degree $n + 1$.

Adjoints have played a key role in the classification of isomorphism classes of nonsingular varieties (cf [1]): Let

$$X \subset \mathbb{P}^N_{\mathbb{C}}, \qquad N \geq n$$

be a nonsingular variety of dimension $n - 1$, and let

$$X \to \bar{X} \subset \mathbb{P}^n_{\mathbb{C}}$$

be a general projection. A first classification of X is then given by the number of adjoints to $\bar{X}$: (no adjoints, a unique adjoint, many adjoints). This was historically the initial approach to classification.

Calabi-Yau varieties have received interest from physicists, and are conjectured to form a finite set of families when $n = 4$. So it is tempting to use the theory of adjoints to find possible new families of Calabi-Yau varieties.

Consider a relaxation of the "irreducibility condition" on the hypersurface X, in order to find "reducible Calabi-Yau varieties". Let $X \subset \mathbb{P}^n_{\mathbb{C}}$ be any hypersurface of degree d with ordinary singularities of multiplicity k in codimension $k - 1$. Fix an algebraic subset

$$R \subset \mathrm{sing}(X)$$

We define an *R-adjoint* to X to be a hypersurface Y of degree $d-n-1$ that passes through R (non-standard notation).

A hypersurface X with a unique R-adjoint for an algebraic subset $R \subset \operatorname{sing}(X)$, is potentially a Calabi-Yau variety if the remaining singularities

$$\operatorname{sing}(X) \setminus R$$

are smoothable. The adjoints of simple polytopes or polypols (cf. [2],[3]) are "R-adjoints".

Reducible plane curves with unique R-adjoints are central objects in the study [3] of polypols after Wachspress. In [2] we find all simple polytopes in $\mathbb{P}^3_{\mathbb{C}}$ with a unique adjoint ("R-adjoint"), such that the polytope can be deformed, keeping the singular locus R, to a surface that is nonsingular outside R . The desingularization of the deformed surface is then a minimal K3-surface (a Calabi-Yau surface), a nonminimal K3-surface or a proper elliptic surface.

References

[1] M. Beltrametti, A.J. Sommese: The adjunction theory of complex manifolds, Expositions in Mathematics **16** Walter De Gruyter, Berlin, (1995).

[2] Kathlén Kohn, Kristian Ranestad: Projective geometry of Wachspress coordinates, Foundations of Computational Mathematics (2019), 1-39.

[3] Kathlén Kohn, Ragni Piene, Kristian Ranestad, Felix Rydell, Boris Shapiro, Rainer Sinn, Miruna-Stefana Sorea, Simon Telen: Adjoints and Canonical Forms of Polypols, arXiv:2108.11747 (2021).

Polynomial interpolation and cubature rules

Yuan Xu

Let w be a nonnegative regular weight function on a domain $\Omega \subset \mathbb{R}^d$. A cubature rule of degree m for the integral $\int_\Omega f(x)w(x)dx$ is a finite linear combination of function evaluations

$$\mathsf{C}(f) := \sum_{k=1}^{N} \lambda_k f(x_k), \qquad x_k \in \mathbb{R}^d, \quad \lambda_k \in \mathbb{R},$$

that approximates the integral. We say it is of degree m if

$$\int_\Omega f(x)w(x)dx = \mathsf{C}(f), \qquad \text{for all } f \in \Pi_m^d,$$

where Π_m^d denotes the space of polynomials of degree at most m in d variables, and the identity fails to hold for at least one polynomial of degree $m+1$. If $\mathsf{C}(f)$ is of degree m, then its number of nodes must satisfy

$$N \geq \dim \Pi_{n-1}^d = \binom{n+d-1}{d}, \qquad m = 2n-1 \text{ or } m = 2n-2.$$

We call a cubature rule Gaussian if $N = \binom{n+d-1}{d}$. If $d = 1$, then a Gauss quadrature rule of degree $2n-1$ always exists and its nodes are zeros of the orthogonal polynomial of degree n with respect to w; moreover, it is the integral of the polynomial that interpolates f on its nodes. For $d > 1$, it is known if a Gauss cubature rule of degree $2n-1$ exists, then its nodes must be common zeros of all orthogonal polynomials of degree n with respect to w. For $m = 2n-2$, the nodes need to be common zeros of a basis of quasi-orthogonal polynomials of degree n. Whenever a Gauss cubature rule exists, its set of nodes admits unique polynomial interpolation of degree $n-1$. See, for example, [1, Chapter 3].

Gauss cubature rules exist rarely and they do not exist, for example, if w and Ω are symmetric with respect to the origin (called centrally symmetric). There are two families of weight functions for which Gauss cubature rules of degree $2n-1$ exist; both are derived with the help of symmetric functions and on domains that are bounded by curves. The existence of the Gauss cubature rules of degree $2n-2$ can be characterized by the solution of a non-linear system of equations, writing in terms of the coefficient matrices of the three-term relations of orthogonal polynomials. Apart from the two families of weight functions, the product Chebyshev weight function of the second kind

$$w(x_1, x_2) = \frac{1}{\sqrt{1-x_1^2}\sqrt{1-x_2^2}}, \qquad (x_1, x_2) \in [-1,1]^2$$

is an exceptional case, for which the Gauss cubature rules of degree $2n-2$ exist. An open question of considerable interest is if such a rule exists for the product Chebyshev weight function in three variables. At the moment, the existence is known only for $n = 2$ and $n = 3$.

In the centrally symmetric setting, the number of nodes of a cubature rule of degree $2n-1$ satisfies an improved lower bound (called Möller's bound)

$$N \geq \dim \Pi_{n-1}^2 + \left\lfloor \frac{n}{2} \right\rfloor.$$

The bound is attained if and only if the nodes of the cubature rule are common zeros of $\lceil \frac{n+1}{2} \rceil$ orthogonal polynomials of degree n. Moreover, the nodes of such a cubature admit a unique interpolation in a subspace $\Pi_n^d \subset U$, where U is the set of polynomials that vanish on the nodes. Cubature rules that attain Möller's lower bound hold for two families of weight functions, $w_{\alpha,\beta,-\frac{1}{2}}$ and $w_{\alpha,\beta,\frac{1}{2}}$,

$$w_{\alpha,\beta,\pm\frac{1}{2}}(x,y) = |x+y|^{2\alpha+1}|x-y|^{2\beta+1}(1-x^2)^{\pm\frac{1}{2}}(1-y^2)^{\pm\frac{1}{2}}, \quad \alpha, \beta > -\tfrac{1}{2}$$

on $[-1,1]^2$, which include the product Chebyshev weight function of the first kind and the second kind as special cases. See [2] for a survey on cubature rules on the square.

The Moller's lower bound is not sharp for the Chebyshev weight function on the unit disk. Moreover, it is likely not sharp for most weight functions. We can formulate the problem as that of minimal cubature rules. For a given integral or a weight function, a cubature rule is called *minimal* if it has the smallest number of nodes among all cubature rules of the same degree. Those cubature rules that are Gaussian or attain Möller's lower bound are minimal. The existence of minimal cubature rule is a tautology. More generally, we are looking for cubature rules with fewer nodes. For $d = 3$, we do not know any centrally symmetric weigh function that admits a cubature rule with $\dim \Pi_{n-1}^3 + O(n^2)$ nodes. The problem of constructing cubature rules with fewer nodes is challenging. There are few available tools. To illustrate the difficulty, let us formulate the problem in the language of idea and variety.

A polynomial P is called m-orthogonal if it satisfies

$$\int_\Omega P(x)Q(x)w(x)dx = 0, \qquad \forall Q \,\text{such that} \,\,\, \deg P + \deg Q \leq m.$$

The existence of a cubature rule of degree $2n-1$ for w is equivalent to the existence of a polynomial ideal $\mathcal{I}$ generated by m-orthogonal polynomials so that the variety V is *real*, zero-dimentional, and its cardinality $|V|$ is equal to the codimension of the idea; that is, $\mathrm{codim}\mathcal{I} := \dim \Pi^d/\mathcal{I} = |V|$. The problem is thus in the realm of algebraic geometry and, more restrictively, of *real* algebraic geometry.

References

[1] C. Dunkl and Y. Xu, *Orthogonal polynomials of several variables*, 2nd Edition Encyclopedia of Mathematics and its Applications, vol. **155**, Cambridge Univ. Press, 2014.
[2] Y. Xu, Optimal points for cubature rules and polynomial interpolation on the square. *Contemporary Computational Mathematics - a celebration of the 80th birthday of Ian Sloan,* p. 1287–1305, Springer, Cham, 2018.

On the dimension of trivariate spline spaces

Nelly Villamizar
(joint work with Michael DiPasquale)

A C^r-*spline* is a piecewise polynomial function on a partition Δ of some domain in $\mathbb{R}^n$ which is continuously differentiable to order r, for some integer $r \geq 0$. If the polynomial pieces have degree at most d, then the set of splines on Δ is a vector space which we denote by $\mathcal{S}_d^r(\Delta)$. Splines play an important role in many areas such as finite elements, computer-aided design, and data fitting [12]. In these applications it is important to construct a basis for $\mathcal{S}_d^r(\Delta)$, and a basic task is simply to compute the dimension of the space.

A formula for $\dim \mathcal{S}_d^1(\Delta)$, where Δ is a planar triangulation, was first proposed by Strang [17] and proved for $d \geq 2$ by Billera for *generic* vertex positions [6]. Subsequently the problem of computing the dimension of planar splines on triangulations has received considerable attention using a wide variety of techniques, see for example [12, 15, 18] and the references therein. An important feature of planar splines is that the formula which gives $\dim \mathcal{S}_d^r(\Delta)$ for $d \geq 3r+2$ is a lower bound for *any* degree $d \geq 0$ [16]. The computation of $\dim \mathcal{S}_d^r(\Delta)$ for planar Δ when $r+1 \leq d \leq 3r$ remains an open problem.

The literature on computing the dimension of splines on tetrahedral partitions is much less conclusive. The dimension has been computed if $r = 0$ (see [4] or [7]), and also if $r = 1$, $d \geq 8$, and Δ is generic [5]. For $r > 1$ bounds on $\dim \mathcal{S}_d^r(\Delta)$ have been computed in [1, 13, 3, 14]. A major difficulty is that for computing $\dim \mathcal{S}_d^r(\Delta)$ exactly for $d \gg 0$, we must be able to compute the dimension of the space of *homogeneous* splines $\mathcal{H}_d^r(\Delta_\gamma)$ exactly in all degrees, where γ is a vertex of Δ and Δ_γ is the *star* of γ (that is, Δ_γ consists of all tetrahedra having γ as a vertex). The computation of such spline spaces has only been made for $r \leq 1$; for $r = 1$, the partition Δ is required to be generic [5]. Vertex stars Δ_γ are a natural generalization of planar triangulations, and understanding splines on vertex stars is a crucial step to analyzing trivariate splines.

In [2], Alfeld, Neamtu, and Schumaker derive formulas for $\dim \mathcal{H}_d^r(\Delta_\gamma)$ for degree $d \geq 3r + 2$. A crucial difference from the planar case is that, for splines on tetrahedral complexes, these dimension formulas are not necessarily a lower bound on $\dim \mathcal{H}_d^r(\Delta_\gamma)$. On one hand, it is straightforward to show that this is the case for all $d \geq 0$ if γ is a boundary vertex, but if the vertex is completely surrounded by tetrahedra – we call these *closed* vertex stars, it is quite delicate to determine the degrees d for which the formula proved in [2] is a lower bound on $\dim \mathcal{H}_d^r(\Delta)$.

In our main result in [11], we show that the formula in [2] gives a lower bound on $\dim \mathcal{H}_d^r(\Delta_\gamma)$ for a closed vertex star Δ_γ with at least six boundary vertices when $d \geq (3r + 2)/2$ and the vertex coordinates of Δ_γ are general enough. If Δ_γ has only four (respectively five) boundary vertices, then we show that the formula in [2] is a lower bound when $d > 2r$ (respectively, $d > (5r + 2)/3$). Our proof uses *ideals of fat points* in $\mathbb{P}^2$ [9], and the so-called *Waldschmidt constant* of the set of points dual to the interior faces of the vertex star [8]. We furthermore observe that arguments of Whiteley [18] and Alfeld, Schumaker, and Whiteley [5] imply that the only splines of degree at most $(3r + 1)/2$ on a generic closed vertex star are global polynomials. This has a satisfying completeness: the formulas of [2] may not be a lower bound on $\dim \mathcal{H}_d^r(\Delta_\gamma)$ for closed vertex stars in small degrees; however for most vertex positions there will only be trivial splines in these small degrees anyway!

In [10], we apply our results on vertex stars from [11] to establish a formula $\mathrm{LB}(\Delta, d, r)$ which is a lower bound on the dimension of the spline space on most tetrahedral partitions of interest (any triangulation of a compact three-manifold with boundary) in large enough degree. We illustrate in several examples that for generic Δ, our formula begins to be a lower bound in degrees close to the *initial degree* of $\mathcal{S}^r(\Delta)$; by the initial degree of $\mathcal{S}^r(\Delta)$ we mean the smallest degree d in which $\mathcal{S}_d^r(\Delta)$ admits a spline which is not globally polynomial. It is worth noting that none of the lower bounds in the literature [13, 3, 14] give the exact dimension of the generic spline space (even in large degree) on these examples if $r \geq 2$.

References

[1] P. Alfeld. Upper and lower bounds on the dimension of multivariate spline spaces. *SIAM J. Numer. Anal.*, 33(2):571–588, 1996.

[2] P. Alfeld, M. Neamtu, and L. Schumaker. Dimension and local bases of homogeneous spline spaces. *SIAM J. Math. Anal.*, 27(5):1482–1501, 1996.

[3] P. Alfeld and L. Schumaker. Bounds on the dimensions of trivariate spline spaces. *Adv. Comput. Math.*, 29(4):315–335, 2008.

[4] P. Alfeld, L. Schumaker, and M. Sirvent. On dimension and existence of local bases for multivariate spline spaces. *J. Approx. Theory*, 70(2):243–264, 1992.

[5] P. Alfeld, L. Schumaker, and W. Whiteley. The generic dimension of the space of C^1 splines of degree $d \geq 8$ on tetrahedral decompositions. *SIAM J. Numer. Anal.*, 30(3):889–920, 1993.

[6] L. Billera. Homology of smooth splines: generic triangulations and a conjecture of Strang. *Trans. Amer. Math. Soc.*, 310(1):325–340, 1988.
[7] L. Billera. The algebra of continuous piecewise polynomials. *Adv. Math.*, 76(2):170–183, 1989.
[8] C. Bocci and B. Harbourne. Comparing powers and symbolic powers of ideals. *J. Algebraic Geom.*, 19(3):399–417, 2010.
[9] S. Cooper, B. Harbourne, and Z. Teitler. Combinatorial bounds on Hilbert functions of fat points in projective space. *J. Pure Appl. Algebra*, 215(9):2165–2179, 2011.
[10] M. DiPasquale and N. Villamizar. A lower bound for the dimension of tetrahedral splines in large degree. *arXiv e-prints*, page arXiv:2007.12274, July 2020.
[11] M. DiPasquale and N. Villamizar. A lower bound for splines on tetrahedral vertex stars. *SIAM Journal on Applied Algebra and Geometry*, 5(2):250–277, 2021.
[12] M. Lai and L. Schumaker. *Spline functions on triangulations*, volume 110 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2007.
[13] W. Lau. A lower bound for the dimension of trivariate spline spaces. *Constr. Approx.*, 23(1):23–31, 2006.
[14] B. Mourrain and N. Villamizar. Bounds on the dimension of trivariate spline spaces: a homological approach. *Math. Comput. Sci.*, 8(2):157–174, 2014.
[15] H. Schenck. Algebraic methods in approximation theory. *Comput. Aided Geom. Design*, 45:14–31, 2016.
[16] L. Schumaker. On the dimension of spaces of piecewise polynomials in two variables. In *Multivariate approximation theory (Proc. Conf., Math. Res. Inst., Oberwolfach, 1979)*, volume 51 of *Internat. Ser. Numer. Math.*, pages 396–412. Birkhäuser, Basel-Boston, Mass., 1979.
[17] G. Strang. Piecewise polynomials and the finite element method. *Bull. Amer. Math. Soc.*, 79:1128–1137, 1973.
[18] W. Whiteley. The combinatorics of bivariate splines. In *Applied geometry and discrete mathematics*, volume 4 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 587–608. Amer. Math. Soc., Providence, RI, 1991.

GBCs Part 2

Michael Floater

This talk was given as a response to the earlier talk of Scott Schaefer on generalized barycentric coordinates.

In this talk I tried to provide more details of Wachspress's rational elements [5], or 'coordinates' with respect to convex polygons, and how the theory developed through papers by Warren, Meyer, Schaefer, Ju, Desbrun, and others [4, 6], and in particular how we now have an explicit formula for the adjoint, the denominator in these elements, which immediately shows that it is positive inside (and on the boundary) of the polygon. Both Wachspress' linear elements and higher order elements can be used to solve PDEs over polygonal meshes, with the same convergence rates as when using piecewsie polynomials of triangular meshes [3].

I also reviewed the derivation of mean value coordinates [1]. These coordinates are no longer rational, and require computing square roots. However, they have two advantages. One is that for a vertex x of a triangulation, the coordinates can be used to express x as a convex combination of its neighbouring vertices, since in this case the coordinates are still positive. This was a motivation for using them to compute 'harmonic-like' mappings of surface triangular meshes into the plane, for example to construct parameterizations for surface fitting, or to make texture maps in computer graphics. The other advantage is that the coordinates extend in a simple way to points in arbitrary polygons, although they are not in general positive. This property means that they can be applied to morphing and deformation of images. The 3D version of the coordinates can be used to model deformations of surfaces.

I also mentioned some other properties of certain kinds of GBCs with respect to convex polygons, such as the monotonicity property [2].

References

[1] M. S. Floater, *Generalized barycentric coordinates and applications*, Acta Numerica **24** (2015), 161–214.

[2] M. S. Floater, *On the monotonicity of generalized barycentric coordinates on convex polygons*, Comp. Aided Geom. Design **42** (2016), 34–39.

[3] M. S. Floater and M.-J. Lai, *Polygonal spline spaces and the numerical solution of the Poisson equation*, SIAM J. Numer. Anal. **54** (2016), 797–824.

[4] T. Ju, S. Schaefer, J. Warren, and M. Desbrun, *A geometric construction of coordinates for convex polyhedra using polar duals*, in 'Symposium on Geometry Processing 2005', 2005, pp. 181–186.

[5] E. Wachspress, *A rational finite element basis*, Academic Press. 1975.

[6] J. Warren, S. Schaefer, A. N. Hirani, and M. Desbrun, *Barycentric coordinates for convex sets*, Adv. in Comp. Math. **27** (2007), 319–338.

Harmonic splines

Tatyana Sorokina

The motivation for introducing and studying harmonic splines arises from the following idea. Suppose we would like to use finite elements and Galerkin method to solve the following boundary value problem numerically:

$$\begin{aligned} -\Delta u &= 0, \quad \text{in } \ \Omega, \\ u &= f, \quad \text{on } \ \partial\Omega, \end{aligned}$$

where Ω is a bounded polytopal domain in $\mathbb{R}^n$. Typically, the full polynomial space P_d is used to build a finite element, i.e., the corresponding spline space includes all piecewise polynomials of degree $\leq d$. It seems natural to try to use piecewise harmonic polynomials only to build the finite element space. This would drastically reduce the number of unknowns in the Galerkin scheme, and lead to computational savings if the harmonic element assumes the same order of approximation.

We first note, that harmonic splines can be at most continuous, i.e., a piecewise harmonic polynomial of global smoothness $r \geq 1$ is a polynomial. The proof for the bivariate case provided in [1] easily generalizes to n variables. Thus, we can only hope to construct C^0 harmonic finite elements. Let $\widetilde{\Omega}$ be a simplicial partition of Ω. The following spline space is called a space of harmonic splines of degree d on $\widetilde{\Omega}$:

$$H_k^n(\widetilde{\Omega}) := \{s \in C(\Omega) : s|_T =: p_T \in P_d, \text{ and } \Delta p_T = 0, \text{ for all } T \in \widetilde{\Omega}\}.$$

In [4], [5], [6], and [7], we construct several families of bivariate harmonic finite elements and prove that they have optimal order of convergence. Our work shows that such constructions heavily depend on the geometry of the underlying partition. Of particular interest are interpolating sets of harmonic polynomials.

Bernstein-Bézier techniques, see [2], can be adopted to working with splines in $H_k^n(\widetilde{\Omega})$. We begin by writing $p_T \in P_d$ in the Bernstein-Bézier (BB) form relative a simplex T in $\widetilde{\Omega}$ as follows:

$$p_T = \sum_{i_1+\cdots+i_{n+1}=d} c_{i_1\ldots i_{n+1}} B^d_{i_1\ldots i_{n+1}} = \sum_{|I|=d} c_I B^d_I. \tag{1}$$

We denote $I := (i_1, \ldots, i_{n+1})$, and $|I| = \sum_{j=1}^{n+1} i_j$. The question is: what are the conditions on the BB-coefficients c_I that would guarantee that the Laplacian Δp_T vanishes? It turns out we need to involve some geometry. Let x_i^j is the j-th-coordinate of vertex number i in $\mathbb{R}^n$, $1 \le j \le n$. We use square brackets for the convex hull. Let $T := [v_1, \ldots, v_{n+1}]$ be a non-degenerate simplex in $\mathbb{R}^n$ with the vertices $v_i := (x_i^1, \ldots, x_i^n)$, and the associated matrix

$$M(T) := \begin{bmatrix} x_1^1 & x_2^1 & \ldots & x_{n+1}^1 \\ x_1^2 & x_2^2 & \ldots & x_{n+1}^2 \\ \ldots & \ldots & \ldots & \ldots \\ x_1^n & x_2^1 & \ldots & x_{n+1}^n \\ 1 & 1 & \ldots & 1 \end{bmatrix}$$

with the cofactors $M_j^k(T)$ of x_j^k, $k = 1, \ldots, n$, $j = 1, \ldots, n+1$, where the k-th row and the j-th column are deleted, and the minor is multiplied by $(-1)^{k+j}$. Let F_j be the facet of T opposite v_j. For each facet F_j we form the following vectors

$$h_j = \left(M_j^1, \ldots, M_j^n\right)/\det M, \text{ for any } 1 \le j \le n+1. \tag{2}$$

In [3], we show that each h_j is orthogonal to F_j, its magnitude is the reciprocal of the distance from v_j to the hyperplane containing F_j, and if h_j is attached to the centroid of F_j it points inward T. For the Laplacian of p_T in its BB-form (1), we obtain

$$\Delta p_T = \sum_{i=1}^n D_{ii} p_T = \sum_{|I|=d-2} \overset{\triangle}{c}_I B_I^{d-2},$$

where

$$\overset{\triangle}{c}_I = d(d-1) \sum_{j=1}^{n+1} \sum_{k=1}^{n+1} c_{I(j,k)} h_j \cdot h_k, \quad |I| = d-2,$$

with $I(j,k) := (i_1, \ldots, i_{j-1}, i_j + 1, i_{j+1}, \ldots, i_{k-1}, i_k + 1, i_{k+1}, \ldots, i_{n+1})$, and h_j is as in (2). Consequently, we obtain our next result.

Theorem 1. *A continuous polynomial spline is piecewise harmonic if and only if each polynomial piece satisfies the following conditions on its BB-coefficients:*

$$\sum_{j=1}^{n+1} \sum_{k=1}^{n+1} c_{I(j,k)} h_j \cdot h_k = 0, \quad \textit{for all} \quad |I| = d-2.$$

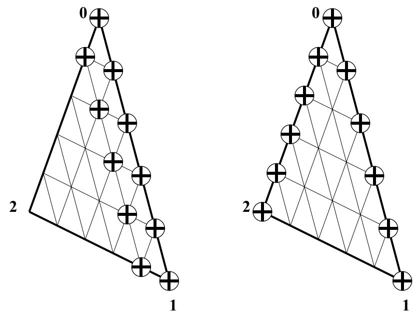


FIGURE 1. Two sets of MDS for $H_5^2(T)$; for the MDS on the right the angle at v_0 cannot be equal to π/k, for any $k = 2, 3, 4, 5$.

Theorem 1 provides a tool to identify minimal determining sets for H_d^n. We next find a useful minimal determining set for harmonic polynomials, and by doing so, provide a proof for the dimension of this space, see Figure 1 (left) for an illustration.

Lemma 2. Given $T = [v_1, \ldots, v_{n+1}]$, the set of $\binom{d+n-1}{n-1} + \binom{d+n-2}{n-1}$ domain points

$$\{\xi_{I,0}\}_{|I|=d} \cup \{\xi_{I,1}\}_{|I|=d-1},$$

is a minimal determining set for harmonic polynomials of degree $\leq d$.

Our next result provides another MDS for bivariate harmonic polynomials, see Figure 1 (right) for an illustration.

Theorem 3. *All domain points on two edges of a triangle form an MDS for the space of harmonic polynomials of degree $\leq d$ if and only if the angle between the two edges is not equal to π/k for any $k = 1, \ldots, d$.*

As Theorem 3 suggests, special care is needed when choosing interpolating sets on the boundary of T, see Example 1.

Example 1. Consider bivariate quadratic harmonic polynomials defined over the standard triangle T with vertices $(0, 0), (1, 0), (0, 1)$. The dimension of this space is five. However, no five points located on the x and y axis form an interpolating set, since the harmonic polynomial $p(x, y) = xy$ does not vanish there. In particular, the five domain points on the edges of T aligned with the axes do not form an MDS for $H_2^2(T)$.

References

[1] P. Alfeld and T. Sorokina, *Linear differential operators on bivariate spline spaces and spline vector fields*, BIT Numerical Mathematics **56(1)** (2006), 15–32.

[2] M. Lai and L. Schumaker, *Spline functions on triangulations*, Cambridge University Press, Cambridge, 2007.

[3] T. Sorokina, *Bernstein-Bézier techniques for divergence of polynomial spline vector fields in* $\mathbb{R}^n$, Advances in Computational Mathematics **44** (2018), 227–244.

[4] T. Sorokina, and S. Zhang, *3D interpolated Galerkin finite elements for the Poisson equation*, Springer Proceedings in Mathematics and Statistics, to appear.

[5] T. Sorokina, and S. Zhang, *An inequality involving a triangle and an interior point and its application*, Mathematical Inequalities and applications, **23** (2020), 713–717.

[6] T. Sorokina, and S. Zhang, *Conforming harmonic finite elements on the Hsieh-Cough-Tocher split of a triangle*, IJNAM, **17** (2020), 54–67.

[7] T. Sorokina, and S. Zhang, *Conforming and nonconforming harmonic finite elements*, Applicable Analysis, **99** (2020), 569–584.

Problems presented during problem sessions

Dimensions of Spline spaces. (Schenck)
Let Δ be a simplicial subdivision (triangulation) of a simply connected bounded domain in the plane $\mathbb{R}^2$. For positive integers r, m, define $S^r_m(\Delta)$ to be the real vector space of piecewise polynomial functions of degree m on Δ that have r continuous derivatives. That is, on each triangle in Δ the function is a polynomial of degree at most m. This is a spline space and its elements are called splines.

The open question is to determine the dimension of $S^r_m(\Delta)$ in terms of the combinatorics of Δ, the local geometry of Δ, and the global geometry of Δ. This is open even in the case of $S^1_3(\Delta)$.

There is much to elaborate on in this problem, as it has a long history. A connection to algebra is found in the smoothness condition for a given spline. The boundary between two adjacent simplices is a line segment. Let ℓ be the affine form defining that segment and f, g the polynomials in the spline on the two simplices. The condition of r-smoothness is that ℓ^{r+1} divides the difference $f - g$, equivalently, $f - g \in \langle \ell^{r+1} \rangle$, the ideal generated by ℓ^{r+1}.

Interpolation spaces. (Shekhtman)
Let $\mathbb{K}$ be a field of characteristic zero. A subspace V of $\mathbb{K}[x, y]$ interpolates functions on a set $X \subset \mathbb{K}^2$ if the restriction map from V to $\mathbb{K}^X$ is surjective. It is known that no three-dimensional subspace of $\mathbb{K}[x, y]$ interpolates all subsets X consisting of three points, but at least one of

$$\mathbb{K}\{1, x, x^2\}\,,\ \mathbb{K}\{1, x, y\}\,, \mathbb{K}\{1, y, y^2\}\,,$$

works for any set of three points.

What is the corresponding minimal number of four-dimensional subspaces of $\mathbb{K}[x, y]$ with the property that at least one of them will interpolate functions for any set of four points?

It is not hard to see that an upper bound is provided by the spans of monomials corresponding to the five partitions of 4,

$$\mathbb{K}\{1, x, x^2, x^3\}\,,\ \mathbb{K}\{1, x, x^2, y\}\,,\ \mathbb{K}\{1, x, y, xy\}\,,\ \mathbb{K}\{1, x, y, y^2\}\,,\ \mathbb{K}\{1, y, y^2, y^3\}\,.$$

Also, at least three are needed. The bet is that four is the correct number.

Mean-value coordinates. (Sottile, but inspired by Schafer's talk)
Given a polygon with n sides in the plane, there are mean-value barycentric coordinates β_v, one for each vertex v of the polygon. These define a map from the polygon to the affine plane in $\mathbb{R}^n$ where the sum of the coordinates is 1, or to the projective space $\mathbb{P}^{n-1}$. What are the homogeneous equations for the Zariski closure of the image? These are the algebraic relations among the mean-value coordinates.

To see that is makes sense, one should note that the mean-value coordinates are algebraic functions, even though some definitions use transcendental expressions such as $\tan\frac{\theta}{2}$.

This is inspired by the Ph.D. work of Corey Irving who studied the same questions for Wachspress coordinates (see later).

Three dimensional Wachspress varieties. (Sottile, Schenck)
Given a polytope P in $\mathbb{R}^3$, its Wachspress coordinates define a rational map from $\mathbb{R}^3$ to $\mathbb{P}^P$, the projective space whose basis is indexed by the vertices of P. The Zariski closure of the image is a Wachspress variety, which is 3-dimensional.

Some questions: What are the equations of the Wachspress variety? Describe the Wachspress variety. The degree is known. What about in higher dimensions?

Schenck, Smith, and Sottile looked at this a little in 2013, but did not come to a conclusive result.

Moduli spaces of Wachspress varieties. (Sottile)
Consider Wachspress surfaces, given by polygons in the plane. All surfaces for n-gons lie in $\mathbb{P}^{n-1}$, and have the same Hilbert polynomial (this is a result of Irving and Schenck). The Wachspress surfaces move (deform) as the polygon moves, and there should be natural limiting varieties (schemes) as the polygons degenerate. Describe a reasonable space of Wachspress surfaces of $n-$gons: what are the limiting objects (flat limits) and can a reasonable and meaningful compactification be found?

Irving and Sottile worked out some examples of limits a decade ago, but nothing conclusive was found.

Adjoints of heptagons and beyond. (Kohn)
A general heptagon in the plane has 14 degrees of freedom. Its adjoint has degree four and hence also 14 degrees of freedom. There is a rational map from the space $(\mathbb{P}^2)^7/D_7$ of heptagons to quartics. Experimentally, it is dominant and has degree $864 = 2^5 \cdot 3^3$. What is this map?

There are other mutations possible. Given n points in $\mathbb{P}^2$, a choice of a cyclic order on them gives a directed n-gon, and the reverse of that order is the same undirected n-gon. There are $(n-1)!/2$ different n-gons for this choice of n points. How are their adjoint curves related?

Aluffi's adjoint of a toric variety. (Schenck)
Toric varieties have well-understood Weil divisors (which are torus orbits). It would be good to understand Aluffi's formula in this case, as presented in Kohn's talk.

References

[1] A. Apozyan, A six dimensional counterexample for the GM_d conjecture, *Jaen J. Approximation*, 3 (2011), 161–172.
[2] A. Apozyan, Conjectures on GC_n sets, *J. Contemp. Math. Anal.*, 46 (2011), 69–76.
[3] A. Apozyan, G. Avagyan, G. Ktryan, On the Gasca-Maeztu conjecture in $\mathbb{R}^3$, *East J. Approximation*, 16 (2010), 25–33.
[4] L. Berzolari, Sulla determinazione di una curva o di una superficie algebrica e su alcune questioni di postulazione, *Lomb. Ist. Rend.*, 47 (1914), 556-564.
[5] A. Björner, I. Peeva, J. Sidman, Subspace arrangements defined by products of linear forms, *J. London Math Soc.*, 71 (2005), 273-288.
[6] C. de Boor, Multivariate polynomial interpolation: Conjectures concerning GC-sets, *Numer. Algorithms*, 45 (2007), 113-125.
[7] C. de Boor, A. Ron, On multivariate polynomial interpolation, *Const. Approx.*, 6, (1990), 287-302.
[8] J. Busch, Lagrange interpolation on $\mathbb{R}^2$, *Rev. Union. Math. Argen.*, 36 (1990), 33-38.
[9] J. Carnicer, M. Gasca, A conjecture on multivariate polynomial interpolation, *Rev. R. Acad. Cien. Serie A. Mat.* 95 (2001), 145-153.
[10] J. Carnicer, M. Gasca, T. Sauer, Interpolation lattices in several variables, *Numer. Math.*, 102 (2006), 559-581.
[11] J. Carnicer, C. Godes, Classification of sets satisfying the geometric characterization, *Numer. Algorithms*, 50 (2009), 145-154.
[12] J. Carnicer and T. Sauer. Observations on interpolation by total degree polynomials in two variables. *Submitted for publication*, (2016). arXiv:1610.01850.
[13] K. Chung, T. Yao, On lattices admitting unique Lagrange interpolation, *SIAM J. Num. Anal.*, 14 (1977), 735–743.
[14] J. Eagon, V. Reiner, Resolutions of Stanley-Reisner rings and Alexander duality, *J. Pure Appl. Algebra*, 130 (1998), 265–275.
[15] D. Eisenbud, *Commutative Algebra with a view towards Algebraic Geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, Berlin-Heidelberg-New York, 1995.
[16] D. Eisenbud, *The geometry of syzygies*, Graduate Texts in Mathematics, vol. 229, Springer-Verlag, Berlin-Heidelberg-New York, 2005.
[17] S. Eliahou, M. Kervaire, Minimal resolutions of monomial ideals, *J. Algebra*, 129 (1990), 1–25.
[18] N. Fieldsteel, H. Schenck, Polynomial Interpolation in higher dimension: from simplicial complexes to GC sets, *SIAM Journal on Numerical Analysis*, 55 (2017), 131–143.
[19] G. Fløystad, J. Vatne, Bi-Cohen-Macaulay simplicial complexes and their associated coherent sheaves, *Comm. Algebra*, 33 (2005), 3121–3136.
[20] L. Garcıa-Puente, F. Sottile, *Linear precision for parametric patches*, Advances in Computational Mathematics, **33** (2010), 191–214.
[21] M. Gasca, J. Maeztu, On Lagrange and Hermite interpolation in $\mathbb{R}^k$, *Numer. Math.*, 39 (1982), 1-14.
[22] M. Gasca, T. Sauer, On the history of multivariate polynomial interpolation, *J. Comput. Appl. Math.*, 122 (2000), 23-35.

[23] M. Gasca, T. Sauer, Multivariate polynomial interpolation, *Advances in Comp. Math.*, 12 (2000), 377-410.

[24] E. Gorla, Lifting the determinantal property. *Algebra, geometry and their interactions, Contemp. Math* 448, (2007), 69-89.

[25] H. Hakopian, K. Jetter, G. Zimmermann, A new proof of the Gasca-Maeztu conjecture for $n = 4$, *Journal of Approximation Theory*, 159, (2009), 224-242.

[26] H. Hakopian, K. Jetter, G. Zimmermann, The Gasca-Maeztu conjecture for $n = 5$, *Numer. Math.*, 127, (2014), 685-713.

[27] H. Hakopian, L. Refayelyan, On a generalization of the Gasca-Maeztu conjecture, *New York J. Mathematics*, 21, (2015), 351-367.

[28] C. Irving, H. Schenck, Geometry of Wachspress surfaces, *Algebra and Number Theory*, 8, (2014), 369-396.

[29] A. Möbius, *Der barycentrische Calcul*, Verlag von Johann Ambrosius Barth, Leipzig, 1827.

[30] I. Peeva, Graded Syzygies, Springer, Berlin-Heidelberg-New York, 2011.

[31] J. Radon, Zur mechanischen Kubatur, *Monatsh. Math.* 52 (1948) 286-300.

[32] T. Sauer. Polynomial interpolation in several variables: Lattices, differences, and ideals. In M. Buhmann, W. Hausmann, K. Jetter, W. Schaback, and J. Stöckler (eds.), *Multivariate Approximation and Interpolation*, pages 189–228. Elsevier, 2006.

[33] T. Sauer, Y. Xu, On multivariate Lagrange interpolation, *Math. Comp.* 64 (1995) 1147-1170.

[34] H. Schenck, Computational Algebraic Geometry, Cambridge University Press, (2003).

[35] E. Wachspress, *A rational finite element basis*, Academic Press, 1975.

[36] J. Warren, *Barycentric coordinates for convex polytopes*, Adv. Comp. Math **6**, (1996) 97–108.

[37] J. Warren, *On the uniqueness of barycentric coordinates*, in "Topics in algebraic geometry and geometric modeling: Workshop on Algebraic Geometry and Geometric Modeling", AMS Contemporary Mathematics **334**, (2003) 93–99.

[38] G. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, Berlin-Heidelberg-New York, 1995.

Reporter: Thomas Fink

Participants

Prof. Dr. Jesus Carnicer
Depto. de Matematica Aplicada
Universidad de Zaragoza
Facultad de Ciencias
Pedro Cerbuna, 12
50009 Zaragoza
SPAIN

Prof. emer. Dr. Carl de Boor
Depts Math, Computer Science,
University of Wisconsin-Madison
81 Caprice Lane
P.O. Box 1076
Eastsound WA 98245
UNITED STATES

Dr. Thomas Fink
Fakultät f. Mathematik u. Informatik
Universität Passau
94032 Passau
GERMANY

Dr. Michael S. Floater
Department of Mathematics, University
of Oslo
Moltke Moes vei 35
P.O. Box 1053
0851 Oslo
NORWAY

Prof. Dr. Luis D. Garcia Puente
Department of Mathematics and
Computer Science,
Colorado College
14 East Cache la Poudre St.
Colorado Springs CO, 80903
UNITED STATES

Prof. Dr. Hakop A. Hakopian
Department of Informatics and
Applied Mathematics
Yerevan State University
A. Manoogian Str. 1
Yerevan 0025
ARMENIA

Prof. Dr. Evelyne Hubert
INRIA Méditerranée
2004 Route des Lucioles
06902 Sophia-Antipolis Cedex
FRANCE

Prof. Dr. Kathlén Kohn
Department of Mathematics
KTH
Lindstedtsvägen 25
10044 Stockholm
SWEDEN

Prof. Dr. Kristian Ranestad
Department of Mathematics
University of Oslo
Blindern
P.O. Box 1053
0316 Oslo
NORWAY

Prof. Dr. Tomas Sauer
Fakultät f. Mathematik u. Informatik
Universität Passau
94030 Passau
GERMANY

Prof. Dr. Scott Schafer
3112 Texas A&M University
Department of Computer Science &
Engineering
Texas A & M University
College Station, TX 77843-3112
UNITED STATES

Prof. Dr. Henry K. Schenck
Department of Mathematics
Auburn University
Auburn, AL 36849
UNITED STATES

Prof. Dr. Larry L. Schumaker
Department of Mathematics
Vanderbilt University
1326 Stevenson Center
Nashville, TN 37240
UNITED STATES

Prof. Dr. Alexandra Seceleanu
203 Avery Hall
Department of Mathematics
University of Nebraska, Lincoln
Lincoln NE 68588-0130
UNITED STATES

Prof. Dr. Boris Shekhtman
Department of Mathematics & Statistics
University of South Florida
Tampa, FL 33620-5700
UNITED STATES

Prof. Dr. Tatyana Sorokina
Department of Mathematics
Towson University
8000 York Road
Towson, MD 21252-001
UNITED STATES

Prof. Dr. Frank Sottile
Department of Mathematics
Texas A & M University
College Station, TX 77843-3368
UNITED STATES

Dr. Nelly Y. Villamizar
Department of Mathematics
Computational Foundry
Swansea University
Bay Campus
Fabian Way
Swansea SA1 8EN
UNITED KINGDOM

Prof. Dr. Yuan Xu
Department of Mathematics
University of Oregon
Eugene, OR 97403-1222
UNITED STATES