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MFO-RIMS Tandem Workshop: Nonlocality in Analysis, Probability and Statistics

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Abstract. The central theme of the workshop were nonlocal operators which appear in various branches of mathematics (PDEs, fractional calculus, stochastic processes, statistics). Although the basic concepts are similar, both language and methods differ depending on one's own community. The aim of the workshop was to bring together leading researchers from these disciplines, in order to alert the different communities about the problems, methods, and progress achieved separately, and to bridge the gap caused by different background and different mathematical terminology.

Mathematics Subject Classification (2020): 31xx, 35xx, 46xx, 47xx, 60xx, 62xx.

Introduction by the Organizers

The *MFO-RIMS Tandem Workshop: Nonlocality in Analysis, Probability and Statistics* (20/March/2022–26/March/2022) was the second joint workshop of the MFO (Mathematisches Forschungsinstitut Oberwolfach) and RIMS (Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan), organized by K. Bogdan (Wrocław), A. Kohatsu-Higa (Kyoto), X. Ros-Oton (Barcelona) and R.L. Schilling (Dresden). It took place within the newly created Tandem format: two separate events at MFO and RIMS share a common Zoom session. Due to the restrictive Covid policy of Japan, the RIMS part was fully online while the MFO part took place as a hybrid workshop. In total we had 41 regular participants and 7 students (2 Leibniz fellows, 5 further PhD students) with a broad geographic representation from all over the world. At Oberwolfach there were 19 invited researchers (of those: 7 online), 4 organizers (of those: 2 online), 2 Leibniz fellows and 5 PhD-students (online), RIMS hosted 18 invited researchers, 1 organizer and 2 students – all online.

On the technical side, there was a live joint session, broadcast via Zoom and taking place simultaneously in the morning at MFO and in the late afternoon at RIMS, respectively (time difference being 8 hrs). All other activities, the afternoon session at MFO, and the morning/early afternoon events at RIMS were both broadcast and recorded, so that all participants could follow these activities either live or asynchronously. The slides of all talks were available on an internet repository (cloud). There were 18 joint talks, 9 talks at RIMS and 10 talks at MFO, 37 talks in total. All talks were of about 40 min duration, with ample time for discussion and further joint activities. Some of the joint talks were surveys; all survey speakers were offered a further opportunity to talk in case they wanted to report on recent progress.

The participants had various mathematical backgrounds, spanning the theory of partial differential equations, fractional calculus, stochastic analysis, the theory of stochastic processes and mathematical statistics, but they had one common interest: nonlocal operators. In order to address the different mathematical backgrounds and to overcome problems in terminology, some of the participants were asked to contribute survey talks so that the talks containing the latest results in the respective fields would be more accessible to all participants. These presentations took place on Monday and Tuesday in the joint live session.

The topics discussed in the workshop can be summarized as follows:

- *Boundary value problems* for non-local operators: If L is a non-local operator on a domain $D \subset \mathbb{R}^n$ the equation $Lu = f$ on D has to be supplemented by a "boundary" condition $Bu = g$ on $\mathbb{R}^n \setminus D$ (rather than the familiar $Bu = g$ on ∂D for partial differential operators). Already the formulation and the choice of the operator L – e.g. fractional Laplacian vs. *regional* fractional Laplacian – is a problem, and the concrete identification of the boundary operator is still open. While the Dirichlet problem ($u = 0$ in $\mathbb{R}^n \setminus D$) is relatively straightforward, already the non-local counterpart for the Neumann problem allows for several choices, and general Wentzell-type conditions are only understood for very few operators L. Apart from the statement of the problem, the regularity theory for the solution is still in its infancy. In the workshop these problems were addressed and interesting new approaches for a regularity theory for a wider class of non-local operators were discussed. On the probabilistic side, stochastic representations for the solutions and the behaviour of the associated processes were discussed.
- *Stochastic analysis:* The key connection between probability (Markov processes) and analysis is the Kolmogorov equation $\partial_t u(t,x) = L_x u(t,x)$ with initial datum $u(0,\cdot) = \delta_0$. For jump processes, L is a non-local operator of Courrège–von Waldenfels form, e.g. a fractional Laplace operator or a pseudo-differential operator with negative definite symbol. Important questions are the existence and regularity of fundamental solutions. The

probabilistic counterpart is the existence and properties of well-behaved (Feller) processes with jumps. Several approaches to the construction and uniqueness were discussed, including methods from harmonic analysis and singular integral operators, quadratic form methods, the parametrix construction and the martingale problem.

- *Statistics of Stochastic Processes* and other applications: A central theme was the calibration of stochastic differential equations (SDE) driven by jump processes. In particular questions relating to the model selection and the estimation of the parameters were addressed. A key tool are transition density estimates for the (perturbed) SDEs and their non-local infinitesimal generators. Some of the talks in this direction discussed rigorous numerical simulations of the Markov processes solving the SDE, as well as novel statistical tools for estimating parameters of Markovian models, and theoretical optimality properties such as the local asymptotic normality property.

The interaction between the PDE community and probability community turned out to be very lively. Some of topics that stirred much activity and exchange of knowledge were the Liouville property and the unique continuation property for harmonic functions of nonlocal operators, and evolutionary phenomena related to minimization of the nonlocal perimeter.

MFO-RIMS Tandem Workshop: Nonlocality in Analysis, Probability and Statistics

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Abstracts

Higher-order fractional Laplacians, maximum principles, and related issues

Nicola Abatangelo
(joint work with Sven Jarohs, Alberto Saldaña)

The higher-order fractional Laplacian $(-\Delta)^s$, $s > 1$, is a nonlocal operator of (in most cases, non-integer) order $2s$, obtained by abstractly computing a positive power of the Laplace operator. The term *higher-order* is intended to stress the fact that the power s is greater than 1, opposed to the family of operators obtained by considering

$$(1)\quad (-\Delta)^s u(x) := c_{n,s} \int_{R^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}}\, dy,$$

$$\text{where } c_{n,s} = \frac{2^{2s-1}\Gamma(n/2+s)}{\pi^{n/2}|\Gamma(-s)|} \text{ and } s \in (0,1).$$

Mind that the limitation $s < 1$ in (1) is due to the singularity of the kernel $|x - y|^{-n-2s}$ and that, therefore, the same formula does not carry over to $s > 1$. One possibility to extend the definition of the operator [3] is to consider

$$(2)\quad (-\Delta)^s u(x) := c_{n,m,s} \int_{\mathbb{R}^n} \frac{\displaystyle\sum_{j=-m}^{m} (-1)^j \binom{2m}{m-j} u(x+jy)}{|y|^{n+2s}}\, dy,$$

$$\text{for } m \in \mathbb{N} \text{ and } s \in (0,m).$$

The constant $c_{n,m,s}$ is a necessary normalization in order to have a Fourier symbol equal to $|\xi|^{2s}$, which gives that (2) is actually independent of the parameter $m \in \mathbb{N}$ as a byproduct.

It is possible to naturally associate a bilinear form to (2). This can be introduced in the following or other equivalent different ways

$$\mathcal{E}_s(u,v) = \int_{\mathbb{R}^n} |\xi|^{2s}\, \mathcal{F}u(\xi)\, \overline{\mathcal{F}v(\xi)}\, d\xi.$$

With this, if $u, v \in H^s(\mathbb{R}^n)$ with $uv \equiv 0$ in $\mathbb{R}^n$, then [5, 11]

$$\mathcal{E}_s(u,v) = \frac{2^{2s-1}\Gamma(n/2+s)}{\pi^{n/2}\Gamma(-s)} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{u(x)\, v(y)}{|x-y|^{n+2s}}\, dx\, dy.$$

As a consequence,

$$\text{if} \quad u, v \geq 0,\ uv \equiv 0 \text{ in } \mathbb{R}^n, \qquad \text{then} \qquad \mathcal{E}_s(u,v) \begin{cases} \leq 0 & \text{if } \lfloor s \rfloor \in 2\mathbb{N}, \\ \geq 0 & \text{if } \lfloor s \rfloor \in 2\mathbb{N} - 1, \end{cases}$$

where the inequalities are strict if both u and v are non-trivial, and

(3) if $u, v \geq 0$, $uv \equiv 0$ in $\mathbb{R}^n$,

$$\text{then} \quad \mathcal{E}_s(u+v, u+v) \begin{cases} \leq \mathcal{E}_s(u-v, u-v) & \text{if } \lfloor s \rfloor \in 2\mathbb{N}, \\ \geq \mathcal{E}_s(u-v, u-v) & \text{if } \lfloor s \rfloor \in 2\mathbb{N}-1. \end{cases}$$

The last obtained inequality is roughly saying that, in some particular situations and for $\lfloor s \rfloor \in 2\mathbb{N}-1$, it could be energetically more convenient to consider a positive-negative oscillating function rather than one with fixed sign. Indeed, manipulations on (3) show that the weak maximum principle fails on disconnected domains for $\lfloor s \rfloor \in 2\mathbb{N}-1$ and, in particular, for $s \in (1, 2)$. Another almost direct consequence is the sign of the nonlocal Poisson kernel for the ball B_1, which amounts to be

$$P_s(x, y) \begin{cases} \geq 0 & \text{if } \lfloor s \rfloor \in 2\mathbb{N}, \\ \leq 0 & \text{if } \lfloor s \rfloor \in 2\mathbb{N}-1. \end{cases} \qquad \text{for } x \in B_1,\ y \in \mathbb{R}^n \setminus B_1.$$

A more precise information to this regard is given by its explicit expression [6]

$$P_s(x, y) = (-1)^{\lfloor s \rfloor} \frac{\gamma(n, s - \lfloor s \rfloor)}{|x-y|^n} \left(\frac{1-|x|^2}{|y|^2-1} \right)^s \qquad \text{for } x \in B_1,\ y \in \mathbb{R}^n \setminus \overline{B_1},$$

which is entailed by the Boggio's formula for the Green function [4, 9, 10]

$$G_s(x, y) = k_{n,s} |x-y|^{2s-n} \int_0^{\rho(x,y)} \frac{v^{s-1}}{(v+1)^{n/2}} \, dv \qquad \text{for } x, y \in \mathbb{R}^n,\ x \neq y,$$

where

$$\rho(x, y) = \frac{(1-|x|^2)_+ (1-|y|^2)_+}{|x-y|^2}, \qquad k_{n,s} = \frac{1}{n|B_1|} \frac{2^{1-2s}}{\Gamma(s)^2}.$$

Counterexamples to the validity of the weak maximum principle for $(-\Delta)^s$ in the range $s \in (2, 3)$ can be built in terms of polynomial-like functions on ellipsoidal domains [7]: these require the ability of performing explicit computations of fractional Laplacians at least on a good set of examples, and this constitutes a non-trivial technical challenge.

Given the general absence of weak maximum principles, and the oscillatory behaviour of energy minimizing functions, it is also interesting to study the shape of the first eigenfunction. It turns out that, for example, on the domain defined as the union of two disjoint balls (recall that this is the prototypical domain not complying with a positivity preserving property, as recalled above) the first eigenvalue is simple and the first eigenfunction is of fixed sign for $\lfloor s \rfloor \in 2\mathbb{N}$, whereas it is positive on one ball and negative on the other for $\lfloor s \rfloor \in 2\mathbb{N}-1$. A number of shape optimization questions now arise, especially considering that rearrangement inequalities are no longer available.

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Fractional-Order Operators on Nonsmooth Domains

Helmut Abels

(joint work with Gerd Grubb)

We consider the following nonlocal elliptic boundary value problem:

$$Pu = f \qquad \text{in } \Omega, \tag{1}$$

$$u = 0 \qquad \text{in } \mathbb{R}^n \setminus \Omega, \tag{2}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain and P is a suitable (nonlocal) elliptic operator of order $2a$, $a \in (0,1)$, as e.g.

$$Pu(x) = (-\Delta)^a u(x) = \mathcal{F}^{-1}[|\xi|^{2a}\hat{u}(\xi)] = c_{a,n} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2a}}\, dy$$

for suitable $u\colon \mathbb{R}^n \to \mathbb{C}$, where $\mathcal{F}$ denotes the Fourier transformation and $c_{a,n}$ is a suitable constant.

We note that usual statements on elliptic regularity in general fail for nonlocal elliptic boundary value problems of fractional order as in (1)-(2)! – In particular, u is not smooth in general even if $f \in C^\infty(\overline{\Omega})$ and $\partial\Omega \in C^\infty$. Some selected known results on regularity for (1)-(2) are:

- Ros-Oton and Serra [6, 7] showed: If $\partial\Omega \in C^{1,1}$, then
$$f \in C^s(\overline{\Omega}) \quad \Rightarrow \quad u/d^a \in C^{s+a}(\overline{\Omega})$$
for sufficiently small $s > 0$, where $d(x) = \operatorname{dist}(x, \partial\Omega)$.
- Grubb [4, 5] proved: If $\partial\Omega \in C^\infty$ and $s > 0$, then
$$f \in C^s(\overline{\Omega}) \quad \Rightarrow \quad u/d^a \in C^{s+a}(\overline{\Omega})$$
- Abatangelo and Ros-Oton [1] obtained: If $\partial\Omega \in C^{1+s+a}$, then
$$f \in C^s(\overline{\Omega}) \quad \Rightarrow \quad u/d^a \in C^{s+a}(\overline{\Omega})$$

It is the goal of our work to extend the results in [4, 5] to nonsmooth domains and operators and the results in [1] to a wider class of operators. For the following we denote by
$$H^s_q(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : (1-\Delta)^{s/2}u \in L^q(\mathbb{R}^n)\}$$
the standard L^p-Bessel potential space on $\mathbb{R}^n$ and
$$H^s_q(\Omega) = \{u \in \mathcal{D}'(\Omega) : u = v|_\Omega \text{for some } v \in H^s_q(\mathbb{R}^n)\},$$
$$\dot{H}^s_q(\Omega) = \{u \in H^s_q(\Omega) : \operatorname{supp} u \subseteq \overline{\Omega}\},$$
where $1 < q < \infty$ and $s \in \mathbb{R}$. Moreover, let $C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, $\tau > 0$, be the set of all continuous $p\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}\colon (x,\xi) \mapsto p(x,\xi)$ that are smooth with respect to the second variable $\xi \in \mathbb{R}^n$, in the Hölder space $C^\tau(\mathbb{R}^n)$ with respect to the first variable $x \in \mathbb{R}^n$ and satisfy
$$\|\partial^\alpha_\xi p(.,\xi)\|_{C^\tau(\mathbb{R}^n)} \le C_\alpha \langle\xi\rangle^{m-|\alpha|}$$
for all $\alpha \in \mathbb{N}^n_0$, $\xi \in \mathbb{R}^n$, where C_α is independent of ξ. Here $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$. $p \in C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ are symbols of nonsmooth pseudodifferential operators, which are defined by
$$p(x,D_x)u(x) \equiv (\mathrm{OP}(p)u))(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, d\xi$$
e.g. for $u \in \mathcal{S}(\mathbb{R}^n)$. Our main result is:

Theorem 1 (A. & Grubb [3]). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with $C^{1+\tau}$-boundary, $a \in (0,1)$, $\tau > 2a$, $1 < q < \infty$, and $p \in C^\tau S^{2a}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ be an even and elliptic symbol, $P = \mathrm{OP}(p)$. Let $u \in \dot{H}^a_q(\overline{\Omega})$ be a solution of*
$$Pu = f \qquad \text{in } \Omega$$
for some $f \in H^s_q(\Omega)$ with $s \in [0, \tau - 2a)$ and $s + a - \frac{1}{q} \notin \mathbb{Z}$. Then
$$u = d^a v + w \qquad \text{for some } v \in H^{s+a}_q(\Omega), w \in \dot{H}^{s+2a}_q(\overline{\Omega}).$$

The idea of the proof is:By localization and scaling we reduce the statement to the case that
$$\Omega = \mathbb{R}^n_\gamma = \{x \in \mathbb{R}^n : x_n > \gamma(x_1,\ldots,x_{n-1})\}$$
for some sufficiently small $\gamma \in C^{1+\tau}(\mathbb{R}^{n-1})$. One proves the result in the latter case by perturbing the case $\Omega = \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0,\infty)$ (with smooth boundary) and "freezing of coefficients", i.e., one considers $(x,\xi) \mapsto p(x_0,\xi)$ for a fixed $x_0 \in$

$\mathbb{R}^n$. In order to perform the pertubation result it is important to use the precise information on the regularity of solution in the case $\Omega = \mathbb{R}^n_+$: If $p \in C^\tau S^{2a}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ is independent of x, even, elliptic and $p(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$, then

$$r_+ \operatorname{OP}(p) \colon H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_+) \to H^s_q(\mathbb{R}^n_+)$$

is invertible for every $1 < q < \infty$ and $s > 0$, where

$$H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_+) := \operatorname{OP}((\langle \xi' \rangle + i\xi_n)^{-a}) e_+ H_q^{s+a}(\mathbb{R}^n_+)$$

is the so-called transmission spaces, $\langle \xi' \rangle = (1 + |\xi'|^2)^{\frac{1}{2}}$, $\xi' = (\xi_1, \ldots, \xi_{n-1})$, and $e_+ f$ is the extension by zero of $f \colon \mathbb{R}^n_+ \to \mathbb{C}$ to $\mathbb{R}^n$ and $r_+ g = g|_{\mathbb{R}^n_+}$. To understand the definition better we note that:

(1) For every $f \in C^\infty_{(0)}(\overline{\mathbb{R}^n_+})$

$$\operatorname{supp} \mathcal{F}^{-1}[(\langle \xi' \rangle + i\xi_n)^{-a} \widehat{e_+ f}(\xi)] \subseteq \overline{\mathbb{R}^n_+}$$

since $\xi_n \mapsto (\langle \xi' \rangle + i\xi_n)^{-a} \widehat{e_+ f}(\xi)$ has an analytic extension to $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ (Paley-Wiener Theorem).

(2) $x \mapsto \mathcal{F}^{-1}[(\langle \xi' \rangle + i\xi_n)^{-a} \widehat{e_+ f}(\xi)]$ is smooth in $\mathbb{R}^n_+$, but has a singularity at $x_n = 0$. Larger a give higher regularity close to $x_n = 0$.

(3) If $u \in H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_+)$ and $s + a > \frac{1}{q}$ and $s + a - \frac{1}{q} \notin \mathbb{N}$, then

$$u(x) = x_n^a v(x) + w(x),$$

where $v \in H_q^{s+a}(\mathbb{R}^n_+)$ and $w \in \dot{H}_q^{s+2a}(\mathbb{R}^n_+)$ due to Grubb [5].

In order to prove the regularity result for $\Omega = \mathbb{R}^n_\gamma = \{x \in \mathbb{R}^n : x_n > \gamma(x_1, ..., x_{n-1})\}$ let $F_\gamma \colon \mathbb{R}^n \to \mathbb{R}^n$ be a $C^{1+\tau}$-diffeomorphism such that $F_\gamma(\mathbb{R}^n_\gamma) = \mathbb{R}^n_+$ and

$$(P_\gamma u)(x) = (P(u \circ F_\gamma^{-1}))(F_\gamma(x))$$

for some $\gamma \in C^{1+\tau}(\mathbb{R}^{n-1})$. Then for any $0 \leq s < \tau - 2a$

$$r_+ P_\gamma \colon H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_+) \to H^s_q(\mathbb{R}^n_+)$$

is bounded and invertible if $\|\gamma\|_{C^{1+\tau}}$ is sufficiently small since

$$r_+ \operatorname{OP}(p) \colon H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_+) \to H^s_q(\mathbb{R}^n_+)$$

is invertible. To prove the boundedness of $r_+ P_\gamma \colon H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_+) \to H^s_q(\mathbb{R}^n_+)$ one uses that

$$r_+ P_\gamma \colon H_q^{a(s+2a)}(\overline{\mathbb{R}}^n_+) \to H^s_q(\mathbb{R}^n_+)$$

is bounded. To show the latte one uses that

$$P_\gamma u(x) = \frac{1}{(2\pi)^n} \operatorname{Os} - \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} q_\gamma(x, y, \xi) u(y) \, dy \, d\xi,$$

where $q_\gamma \in C^\tau S^{2a}_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$ is even. Moreover, for any $\ell \in \mathbb{N}$ with $\ell < \tau$

$$q_\gamma(x, y, \xi) = \sum_{|\alpha| \leq \ell} p_\alpha(x, \xi) + \sum_{|\alpha| = \ell} D^\alpha_\xi r_\alpha(x, y, \xi),$$

where $p_\alpha(x,\xi) = \frac{1}{\alpha!}\partial_y^\alpha D_\xi^\alpha q_\gamma(x,y,\xi)|_{y=x} \in C^{\tau-|\alpha|}S^{m-|\alpha|}_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$ and $D_\xi^\alpha r_\alpha \in C^{\tau-\ell}S^{m-\ell}_{1,0}(\mathbb{R}^{2n}\times\mathbb{R}^n)$ for all $|\alpha|=\ell$, and one shows for $|s|<\tau$

$$p_\alpha(x,D_x)\colon H_q^{a(s-|\alpha|+2a)}(\overline{\mathbb{R}}^n_+) \to H_q^s(\mathbb{R}^n_+)$$

with the aid of nonsmooth Green operators using that $p_\alpha(x,\xi)(i\xi_n+\langle\xi'\rangle)^{-a}$ satisfies the transmission condition, cf. A. [2].

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On the fractional Yamabe problem: nonlocal ODE method

Weiwei Ao

(joint work with Hardy Chan, Azahara DelaTorre, Marco A. Fontelos, María del Mar González, Juncheng Wei)

Let $\gamma\in(0,1)$. We introduce some new method to deal with nonlocal ODE in the study of fractional singular Yamabe problem:

$$(-\Delta)^\gamma u = cu^{\frac{n+2\gamma}{n-2\gamma}} \text{ in } \mathbb{R}^n,\ u\to\infty \text{ as } x\to\Sigma$$

for $c>0$, and Σ is a k dimensional submanifold for $k<\frac{n-2\gamma}{2}$ arising from the study of conformal geometry. The result of this talk basically come from the references [1, 2, 3, 4].

We use gluing method to construct singular solution, i.e., we find solution of the form $u(x)=u_*(x)+\phi$ around a good approximate solution u_* with perturbation ϕ. Then the equation for ϕ becomes

$$L\phi = E + N(\phi),$$

with

$$L\phi = (-\Delta)^\gamma\phi - pu_*^{p-1}\phi,\ E=(-\Delta)^\gamma u_* - u_*^p,$$

and the higher order term:

$$N(\phi) = (u_*+\phi)^p - u_*^p - pu_*^{p-1}\phi.$$

Basically there are two key problems in the construction:

- Existence of the building blocks u_*;
- Mapping property of the building blocks: the linearized operator $L\phi$.

We are led to consider radially symmetric solutions of the fractional Laplacian equation

$$(-\Delta)^\gamma u = Au^p \quad \text{in } \mathbb{R}^n \setminus \{0\}, \tag{1}$$

with an isolated singularity at the origin and the corresponding linearized operator. Here $p \in (\frac{n}{n-2\gamma}, \frac{n+2\gamma}{n-2\gamma}]$, and the constant A is chosen so that $u_0(r) = r^{-\frac{2\gamma}{p-1}}$ is a singular solution to the equation. Note that this is the exact growth rate around the origin of any other solution with non-removable singularity.

In this talk we take the analytical point of view and study several nonlocal ODE that are related to problem (1). A nonlocal equation such as (1) for radially symmetric solutions $u = u(r)$, $r = |x|$, requires different techniques than regular ODE. For instance, existence and uniqueness theorems are not available in general, so one cannot reduce it to the study of a phase portrait. Moreover, the asymptotic behavior as $r \to 0$ or $r \to \infty$ is not clear either.

We developed some nonlocal ODE method to study this nonlocal ODE. The main underlying idea, is to write problem (1) as an infinite dimensional ODE system. Each equation in the system is a standard second order ODE, the nonlocality appears in the coupling of the right hand sides. The advantage of this formulation comes from the fact that, even though we started with a nonlocal ODE, we can still use a number of the standard results, such as the indicial roots for the system and a Wrońskian-type quantity which is useful in the uniqueness proof.

More precisely, we first consider existence theorems for (1), both in the critical and subcritical case. We show that the change of variable

$$r = e^{-t}, \quad u(r) = r^{-\frac{2\gamma}{p-1}} v(-\log r) \tag{2}$$

transforms (1) into the nonlocal equation of the form

$$\int_{\mathbb{R}} \tilde{\mathcal{K}}(t-t')[v(t) - v(t')]\, dt' + Av(t) = Av(t)^p, \quad v = v(t), \quad t \in \mathbb{R}, \tag{3}$$

for some singular kernel satisfying $\tilde{\mathcal{K}}(t) \sim |t|^{-1-2\gamma}$ as $|t| \to 0$. The advantage of (3) over the original (1) is that in the new variables the problem becomes *autonomous* in some sense. We use some idea from conformal geometry. We give an interpretation of the change of variable (2) in terms of the conformal fractional Laplacian on the cylinder. In the local case, the corresponds to a second order ODE for which we can get the classification of solutions by analyzing the Hamiltonian level, i.e., the solutions is either homoclinic or periodic. But in the nonlocal case, there is no Hamiltonian, but anyhow, we use variational method to prove the existence of the periodic solution. But up to today, the classification result for nonlocal ODE (3) is still unknown and it is an open question.

For p subcritical, $p \in (\frac{n}{n-2\gamma}, \frac{n+2\gamma}{n-2\gamma})$, we study the existence of radial fast decaying solution u_* and the mapping property of the linearized problem around u_*.

For the existence of solutions for the nonlocal ODE, we use method from PDE to deal with it. We basically combine variational method, bifurcation method and blow up analysis to show the existence of radial solution. For the linear problem, the resulting equation may be written as

$$(-\Delta)^\gamma \phi - pAu_*^{p-1}\phi = 0, \quad \phi = \phi(r).$$

Defining the radially symmetric potential $\mathcal{V}_*(r) := pAr^{2\gamma}u_*^{p-1}$, this equation is equivalent to

$$L_*\phi := (-\Delta)^\gamma \phi - \frac{\mathcal{V}_*(r)}{r^{2\gamma}}\phi = 0, \quad \phi = \phi(r). \tag{4}$$

Note that $\mathcal{V}_*(r) \to \kappa$ as $r \to 0$ for some positive constant. Therefore to understand operators with critical Hardy potentials such as L_* we consider first the constant coefficient operator

$$L_\kappa := (-\Delta)^\gamma - \frac{\kappa}{r^{2\gamma}}.$$

The fractional Hardy inequality asserts the non-negativity of such operator up to $\kappa = \Lambda_{n,\gamma}$.

Using conformal change of variables, we can write Green's function for the constant coefficient operator L_κ in suitable weighted spaces. We calculate the indicial roots of the problem to characterize invertibility. This is done by writing a variation of constants formula to produce solutions to $L_\kappa\phi = h$ from elements in the kernel $L_\kappa\phi = 0$. Such ϕ is governed by the indicial roots of the equation. However, in contrast to the local case where a second order ODE only has two indicial roots, for the nonlocal operator, we find an infinite number of them and, moreover, the solution is not just a combination of two linearly independent solutions of the homogeneous problem, but an infinite sum.

We obtain, as a consequence, a Frobenius type theorem which yields a precise asymptotic expansion for solutions to (4) in terms of the asymptotics of the potential as $r \to 0$. We can use what we know about L_κ in to obtain information about L_*. In particular, we find the indicial roots of L_* both as $r \to 0$ and as $r \to \infty$.

We introduce a new Wrońskian quantity for a nonlocal ODE such as (4), which allows to compare any two solutions and plays the role of the usual Wrońskian $W = w_1'w_2 - w_1w_2'$ for the second order linear ODEs. Using similar techniques as for the non-linear second order ODE, we give full account of non-degeneracy of equation (1) for the particular solution u_*.

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Liouville theorems for non-local operators

David Berger
(joint work with René L. Schilling)

The talk is based on the work [2]. In this paper we anaylzed the Liouville theorem for generators of Lévy processes: Let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded function and let $L = (L_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ with symbol ψ. We say that L has the Liouville property, if $\psi(D)f = 0$ weakly implies that f is constant a.e. This question was first solved in [1], which presented the result in terms of the characteristic triplet of the Lévy process. We proved in two completely different ways an equivalent theorem, which describes the condition found in [1] in terms of the zero-set of the symbol.

Theorem 1. *Let $\psi(D)$ be the generator of a Lévy process with characteristic exponent ψ. The operator $\psi(D)$ has the Liouville property if, and only if, the zero-set of the characteristic exponent satisfies $\{\psi = 0\} = \{0\}$.*

The first proof uses a method based on ideas in distributional theories, which works in the case of C^∞-symbols, symbols with Lévy processes with every polynomial moment. Our second proof uses a result of convolution equation and fixed points from Choquet and Deny [3].

Theorem 2. *Let μ be a probability measure on $\mathbb{R}^d$ and h a bounded and continuous function. One has $h = h * \mu$ if, and only if, every point of the support of μ is a period of h.*

One can now show that the condition $\psi(D)f = 0$ is equivalent to the fact that $\mathbb{E}f(X_t + x) = f(x)$ for every $x \in \mathbb{R}^d$, which can then be described as a convolution equation. By applying the above theorem and by using some results on probability measures on lattices, one obtains our main theorem. We can apply this result on Subordinators to show that for a Lévy process L the Liouville property holds if, and only if, the Liouville property for the subordinated process $L_S = (L_{S_t})_{t\geq 0}$ holds for every subordinator S. By using a further result of Choquet and Deny [3] one can show an extension of the Liouville property for functions growing at most like a submultiplicative function:

Theorem 3. *Let $\psi(D)$ be the generator of a Lévy process with characteristic exponent ψ. Assume that there exists a locally bounded, submultiplicative function $g : \mathbb{R}^d \to [1, \infty)$ satisfying $\int_{|y|\geq 1} g(y)\nu(dy) < \infty$. The following assertions are equivalent:*

(1) every measurable, positive and g-bounded function $0 \leq f \leq g$ such that $\psi(D)f = 0$ weakly is constant.
(2) $\{\xi \in \mathbb{R}^d \mid \psi(\xi) = 0\} = \{\eta \in \mathbb{R}^d \mid \psi(-i\eta) = 0\} = \{0\}$.

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Blowing-up solutions for a nonlocal mean-field equation in a union of intervals

Matteo Cozzi

(joint work with Antonio J. Fernández)

In [6], DelaTorre, Hyder, Martinazzi, and Sire considered the Dirichlet problem

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u = \lambda \dfrac{e^u}{\int_{-1}^{1} e^u \, dx} & \text{in } (-1,1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1,1), \end{cases}$$

for a nonlocal mean field equation driven by the half-Laplacian

$$(-\Delta)^{\frac{1}{2}} u(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \frac{2u(x) - u(x+z) - u(x-z)}{z^2} \, dz.$$

They showed that this problem has a solution u_λ if and only if $\lambda \in (0, 2\pi)$. These solutions are even, positive in $(-1,1)$, decreasing in $[0,1]$, and, as a family, they blow-up at the origin as $\lambda \nearrow 2\pi$, meaning that $\lim_{\lambda \nearrow 2\pi} u_\lambda(0) = 0$. It also follows from their analysis that

$$\lim_{\lambda \nearrow 2\pi} \frac{1}{\lambda} \int_{-1}^{1} e^{u_\lambda} \, dx = +\infty. \tag{1}$$

Due to the nonlocality of the half-Laplacian, it makes sense to study this problem even in more general (disconnected) subsets of the real line, such as the union $I = \bigcup_{k=1}^{d} I_k$ of $d \geq 2$ open intervals I_k with pairwise disjoint closures. If one is merely interested in detecting families of blowing-up solutions, setting $\varepsilon := \lambda \left(\int_I e^u \, dx \right)^{-1}$and recalling (1) this problem can be equivalent recast as the Liouville type equation

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u = \varepsilon \kappa(x) e^u & \text{in } I, \\ u = 0 & \text{in } \mathbb{R} \setminus I, \end{cases} \tag{2}$$

in the regime $\varepsilon \searrow 0$, for $\kappa \equiv 1$.

Problem (2) has an interesting geometric interpretation. Understanding the half-Laplacian as a Dirichlet-to-Neumann operator, (2) is equivalent to the local mixed boundary value problem

$$(3)\qquad \begin{cases} -\Delta U = 0 & \text{in } \mathbb{R}^2_+ := \mathbb{R}\times(0,+\infty),\\ \partial_\nu U = \varepsilon\kappa(x)e^U & \text{in } I\subset\partial\mathbb{R}^2_+,\\ U=0 & \text{in } \partial\mathbb{R}^2_+\setminus I,\end{cases}$$

for the harmonic extension U of u to the half-plane $\mathbb{R}^2_+$. This is in turn equivalent to finding a metric $g = e^{2U}g_{\mathrm{eu}}$ in $\mathbb{R}^2_+$, conformally equivalent to the Euclidean one g_{eu}, with respect to which $\mathbb{R}^2_+$ has Gaussian curvature identically equal to 0, the intervals I have geodesic curvature equal to $\varepsilon\kappa$, and such that the two metrics induced by g and g_{eu} on the boundary of $\mathbb{R}^2_+$ coincide outside of I. See [2, 4, 8, 9] for related geometric problems.

Problem (2) can also be regarded as the one-dimensional analogue of

$$(4)\qquad \begin{cases} -\Delta u = \varepsilon^2\kappa(x)e^u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain with smooth boundary. Notice that the Laplacian in the plane shares many similarities with the one-dimensional half-Laplacian, such as fundamental solutions and criticality of the Sobolev embedding. Blowing-up families of solutions of (4) have been largely investigated, most prominently in [1, 5, 7]. In particular, in [5] del Pino, Kowalczyk, and Musso have shown that, given any non-simply connected domain Ω, any $m\in\mathbb{N}$, and any function $\kappa\in C^2(\overline{\Omega})$ with $\inf_\Omega\kappa>0$, there is a family of solutions $\{u_\varepsilon\}$ to (4) which blows-up at m points of Ω as $\varepsilon\searrow 0$ and satisfies

$$\lim_{\varepsilon\searrow 0}\varepsilon^2\int_\Omega \kappa(x)e^{u_\varepsilon(x)}\,dx = 8m\pi.$$

In [3] we establish a somewhat analogous result for the nonlocal problem (2). Note that in 1D we lack the topological richness already present in the plane. In a sense, the non-simply connectedness is replaced by the disconnectedness of our union of intervals I. The precise statement of our result reads as follows.

Theorem 1. *For any integer $m\in[1,d]$ and ε sufficiently small, there exists a solution u_ε of problem* (2) *satisfying*

$$\lim_{\varepsilon\searrow 0}\varepsilon\int_I \kappa(x)e^{u_\varepsilon(x)}\,dx = 2m\pi.$$

Moreover, there exist m distinct points $\xi_1,\dots,\xi_m\in I$ such that, given any infinitesimal sequence $\{\varepsilon_n\}$ and any $\delta>0$, the following holds true up to a subsequence:

- *$\{u_{\varepsilon_n}\}$ is uniformly bounded in $I\setminus\bigcup_{j=1}^m(\xi_j-\delta,\xi_j+\delta)$;*
- *$\sup_{(\xi_j-\delta,\xi_j+\delta)} u_{\varepsilon_n}\to+\infty$ as $n\to+\infty$, for all $j=1,\dots,m$.*

We stress that κ is a general function of class $C^2(\overline{I})$ satisfying $\inf_I\kappa>0$.

Our proof of Theorem 1 is perturbative and based on a Lyapunov-Schmidt reduction close in spirit to that pioneered in [5]. This approach naturally produces a rather precise description of the blow-up behavior of the family of solutions $\{u_\varepsilon\}$ nearby the points ξ_j's. Indeed, focusing for simplicity on the case $\kappa \equiv 1$, the solution $u = u_\varepsilon$ has the form $u = \mathscr{U} + \psi$, with

$$\mathscr{U}(x) = \sum_{j=1}^{m} \left(\log\left(\frac{2\mu_j}{\mu_j^2 \varepsilon^2 + (x - \xi_j)^2} \right) + H_j(x) \right),$$

for some appropriately chosen parameters $\mu_1, \ldots, \mu_m \in (0, +\infty)$, suitable corrector functions H_j's, and a remainder term ψ having small-in-ε L^∞ norm. The proof proceeds by linearizing the equation in (2) around $\mathscr{U}$ and showing that the corresponding problem for ψ admits a solution which is small in ε. It turns out that this is possible if the vector $(\xi_1, \ldots, \xi_m)$ is a critical point of a functional involving the Green function of the half-Laplacian in I and the trace of its regular part, often called Robin function. We are able to find such a critical point (actually, a minimizer) if $m \leq d$ and each ξ_j lies in a different interval I_k.

Our analysis leaves a few questions open.

- Is it possible to construct blowing-up solutions at $m > d$ points? Or even at $m \leq d$ points, but with two or more points lying in the same interval? We have a partial non-negative answer in the case of $d = 2$ intervals and a sufficiently large number m of blow-up points—see [3].
- Are there blowing-up solutions for the generalization of the geometric problem (3) in which one prescribes a non-trivial Gaussian curvature in $\mathbb{R}^2_+$? This amounts to introducing an exponential nonlinearity in the first equation in (3) and, as a result, it would require techniques which are not (solely) based on the analysis of the trace problem for the half-Laplacian.

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Nonlocal minimal surfaces

Serena Dipierro
(joint work with Ovidiu Savin, Enrico Valdinoci)

The theory of nonlocal minimal surfaces deals with an energy functional accounting for "pointwise interactions" between a given set and its complement. These contributions from "far-away" can have a significant influence on the local structure of these new objects and produce unexpected phenomena. In particular, differently from classical minimal surfaces, the nonlocal minimal surfaces have the strong tendency to "stick at the boundary", as discovered in [3].

More specifically, following Caffarelli-Roquejoffre-Savin, given $s \in (0,1)$ and a smooth, bounded, open set $\Omega \subset \mathbf{R}^n$, the s-perimeter $\mathrm{Per}_s(E;\Omega)$ of a (measurable) set $E \subseteq \mathbf{R}^n$ in Ω is defined as

$$L(E\cap\Omega,(\mathcal{C}E)\cap\Omega)+L(E\cap\Omega,(\mathcal{C}E)\cap(\mathcal{C}\Omega))+L(E\cap(\mathcal{C}\Omega),(\mathcal{C}E)\cap\Omega),$$

where

$$L(A,B):=\int_A\int_B \frac{1}{|x-y|^{n+s}}\,dx\,dy$$

and $\mathcal{C}E:=\mathbf{R}^n\setminus E$.

Minimizers of Per_s are called nonlocal minimal sets and they exhibit several boundary stickiness phenomena (see [3]). For instance: if $K_\delta := \big(B_{1+\delta}\setminus B_1\big)\cap\{x_n<0\}$ and E_δ is the nonlocal minimal set among all the sets E such that $E\setminus B_1 = K_\delta$, then $E_\delta = K_\delta$, provided that $\delta>0$ is sufficiently small.

Also, given a sufficiently large $M>1$, the nonlocal minimal set E_M in $(-1,1)\times\mathbf{R}$ with datum outside $(-1,1)\times\mathbf{R}$ given by the jump $\big((-\infty,-1]\times(-\infty,-M)\big)\cup\big([1,+\infty)\times(-\infty,M)\big)$ satisfies, for some constant C,

$$[-1,1)\times[CM^{\frac{1+s}{2+s}},\,M]\subseteq E_M^c \qquad\text{and}\qquad (-1,1]\times[-M,\,-CM^{\frac{1+s}{2+s}}]\subseteq E_M.$$

Interestingly, halfspaces are nonlocal minimal sets, but arbitrarily small perturbations of their data are sufficient to produce stickiness.

After [3] (among many others) three foundational questions remained open: How regular are the nonlocal minimal surfaces coming from inside the domain? Is the nonlocal mean curvature equation satisfied up to the boundary? How typical is the stickiness phenomenon?

These questions have been addressed in [4] for nonlocal minimal sets in the plane with graphical structure. In this framework, we have that at the boundary "continuity implies differentiability", namely: if a nonlocal minimal graph u in $(0,1)\times\mathbf{R}$

is discontinuous at 0 with respect to the external datum, then its derivatives blows up at 0; if instead u is continuous at 0 then it is automatically of class at least $C^{1,\frac{1+s}{2}}$.

We note that this dichotomy is a purely nonlinear, or geometric, effect, since the boundary behavior of linear equation is of Hölder type.

Furthermore, as a curve, the nonlocal minimal graph turns out to be always $C^{1,\frac{1+s}{2}}$: namely, it is either the graph of a $C^{1,\frac{1+s}{2}}$-function (when it is continuous at the boundary), or it is discontinuous and sticks vertically detaching in a $C^{1,\frac{1+s}{2}}$ fashion (but then the inverse function is a $C^{1,\frac{1+s}{2}}$ function).

Since the nonlocal mean curvature can be understood as a "$C^{1,s}$ operator", the fact that $\frac{1+s}{2} > s$ allows us to "pass the equation to the limit" and obtain that the nonlocal mean curvature equation is satisfied up to the boundary.

This in turn implies that the stickiness phenomenon is "generic": namely, either a nonlocal minimal graph in the plane is boundary discontinuous, or there is an arbitrarily small perturbation of its exterior datum which produces a boundary discontinuous nonlocal minimal graph. The proof of this genericity result leverages the fact that the nonlocal mean curvature equation holds true up to the boundary, since one can take any configuration, add an arbitrarily small bump and use the unperturbed configuration as a barrier.

These results entail a "butterfly effect", since an arbitrarily small perturbation produces a boundary discontinuity and an infinite derivative at the boundary (that is, an arbitrarily small perturbation suddenly changes the boundary derivative from zero to infinity).

Higher dimensional situations are addressed in [5]. For further examples of stickiness phenomena, see [1, 2].

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Stable cones in the fractional Alt–Caffarelli problem

Xavier Fernández-Real
(joint work with X. Ros-Oton)

Consider the energy functional

$$\mathcal{J}(u) = [u]^2_{H^1(B_1)} + \big|\{u>0\}\cap B_1\big| = \int_{B_1} \left(|\nabla u|^2 + \chi_{\{u>0\}}\right) \tag{1}$$

where χ_A is the characteristic function of the set A.

The study of the critical points and minimizers of (1) is known as the (classical) one-phase free boundary problem (or Bernoulli free boundary problem), which is a typical model for flame propagation and jet flows. From a mathematical point of view, it was originally studied by Alt and Caffarelli in [3], and since then multiple contributions have been made.

In this talk, we deal with the fractional analogue of (1), in which the Dirichlet energy in the functional is replaced by the H^s fractional semi-norm of order $s \in (0,1)$,

$$\mathcal{J}_s(u) = [u]^2_{H^s(B_1)} + \big|\{u>0\}\cap B_1\big|, \tag{2}$$

(see (4) below) which corresponds to the case in which turbulence or long-range interactions are present, and appears in particular in flame propagation; see [5, 8] and references therein.

This problem was first studied by Caffarelli, Roquejoffre, and Sire in [5], where they obtained the optimal C^s regularity for minimizers, the free boundary condition on $\partial\{u>0\}$, and showed that Lipschitz free boundaries are C^1 in dimension $n=2$. More recently, further regularity results for the free boundary have been obtained in [7, 2, 6, 10, 9, 8, 11] among others. These results imply that free boundaries are regular outside a certain set of singular points Σ, with $\dim_{\mathcal{H}}(\Sigma) \le n - k_s^*$ and $k_s^* \ge 3$. The value of k_s^* is the lowest dimension in which there are stable/minimal cones.

The non-local energy functional. Let us consider the energy functional,

$$\mathcal{J}_\Lambda(v,\mathbb{R}^n) = [v]^2_{H^s(\mathbb{R}^n)} + \Lambda^2\big|\{v>0\}\big|, \tag{3}$$

depending on the parameter $\Lambda \in \mathbb{R}$, with the fractional semi-norm

$$[v]^2_{H^s(\mathbb{R}^n)} = \frac{c_{n,s}}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(v(x)-v(y))^2}{|x-y|^{n+2s}}\,dx\,dy, \quad \text{where} \quad c_{n,s} = \frac{s2^{2s}\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{n/2}\Gamma(1-s)} \tag{4}$$

is the constant appearing in the fractional Laplacian,

$$(-\Delta)^s u(x) = c_{n,s} PV \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy.$$

Obtaining local minimizers to $\mathcal{J}_\Lambda$ is the *fractional one-phase* free boundary problem. When $s=\frac12$ this is equivalent to the thin one-phase free boundary problem. It is a free boundary problem because, a priori, the zero-level set of the minimizer is unknown, and its boundary is called the "free boundary". After understanding the optimal regularity of minimizers, the study of the free boundary constitutes the main topic of research for this type of problem.

Let u be a local minimizer (or critical point) to (3) in a ball B. Let $\Omega = \{u > 0\}$, and let us suppose Ω is smooth enough. Let

$$d(x) = \mathrm{dist}(x, \partial\Omega).$$

Then, by standard variational arguments we have that $(-\Delta)^s u = 0$ in $\Omega \cap B$. Moreover, we have that u solves the following problem involving a condition on the fractional derivative on $\partial\Omega$,

$$\begin{cases} (-\Delta)^s u &= 0 \quad \text{in } \Omega \cap B \\ u &= 0 \quad \text{in } \Omega^c \cap B \\ \Gamma(1+s)\dfrac{u}{d^s} &= \Lambda \quad \text{on } \partial\Omega \cap B. \end{cases} \tag{5}$$

This is the first variation of the energy functional.

The stability condition. Our main goal is to obtain the second variation of the energy functional. Namely, we will find the stability condition for (3).

In order to state the result, we need the following definition:

Definition 1. Let Ω be a $C^{1,\alpha}$ domain outside the origin, and let $G_{\Omega,s}(x,y)$ be the Green function of the operator $(-\Delta)^s$ for the domain Ω. Then, we define the kernel $\mathcal{K}_{\Omega,s} : \partial\Omega \times \partial\Omega \to \mathbb{R}$ as

$$\mathcal{K}_{\Omega,s}(x,y) = \lim_{\substack{\Omega \ni \bar{x} \to x \\ \Omega \ni \bar{y} \to y}} \frac{G_{\Omega,s}(\bar{x},\bar{y})}{d^s(\bar{x}) d^s(\bar{y})}. \tag{6}$$

By well-known boundary regularity estimates for the fractional Laplacian ([14, 15]), (6) is well-defined as soon as the boundary is $C^{1,\alpha}$.

Furthermore, we also define the following curvature-type term

$$\mathcal{H}_{\Omega,s}(x) := \int_{\partial\Omega} |\nu(x) - \nu(y)|^2 \mathcal{K}_{\Omega,s}(x,y) d\sigma(y)$$

for $x \in \partial\Omega$, and where $\nu : \partial\Omega \to \mathbb{S}^{n-1}$ denotes the unit inward normal vector on $\partial\Omega$, and σ denotes the area measure on $\partial\Omega$.

We can now state the second-variation condition for the energy functional (3). In the local case, this result was obtained in [4, 13].

Theorem 2. *Let $s \in (0,1)$ and let $u \in C^s(\mathbb{R}^n)$ be a global s-homogeneous stable solution to* (5)*. Assume that $\Omega := \{u > 0\}$ is a $C^{2,\alpha}$ domain outside the origin.*

Let $\mathcal{K}_{\Omega,s}$ and $\mathcal{H}_{\Omega,s}$ be given by Definition 1. Then, we have

$$\int_{\partial\Omega}\int_{\partial\Omega} \big(f(x) - f(y)\big)^2 \mathcal{K}_{\Omega,s}(x,y)\, d\sigma(x)\, d\sigma(y) \geq \int_{\partial\Omega} H_{\Omega,s} f^2\, d\sigma(x) \tag{7}$$

for all $f \in C_c^\infty(\partial\Omega \setminus \{0\})$.

Furthermore, $\mathcal{K}_{\Omega,s}$ is $(-n)$-homogeneous and

$$\mathcal{K}_{\Omega,s}(x,y) \asymp \frac{1}{|x-y|^n} \quad \textit{for all } \; x, y \in \partial\Omega, \tag{8}$$

while $\mathcal{H}_{\Omega,s}$ *is* (-1)*-homogeneous, and*

$$\mathcal{H}_{\Omega,s}(x) \asymp \frac{1}{|x|} \quad \text{for all } \; x \in \partial\Omega,$$

if Ω *is not a half space.*

Here, we have denoted $g_1(x) \asymp g_2(x)$ if $C^{-1}g_2(x) \leq g_1(x) \leq Cg_2(x)$ for some positive constant C independent of x.

The result stated here is for s-homogeneous solutions since we are mainly interested in blow-ups at free boundary points. A more general result is stated in the body of the paper.

The stability condition (7) has an equivalent formulation in terms of large solutions for the fractional Laplacian (which were introduced and studied in [1, 12]). More precisely, (7) turns out to be equivalent to

$$\int_{\partial\Omega} f\, T_{\Omega,s} f \, d\sigma \geq \kappa_s \int_{\partial\Omega} U_1 f^2 \, d\sigma \tag{9}$$

for all $f \in C_c^\infty(\partial\Omega \setminus \{0\})$, where

$$U_1 := -\frac{1}{\Lambda}\,\partial_\nu \left(\frac{u}{d^s}\right), \qquad T_{\Omega,s} f := -\partial_\nu \left(\frac{F}{d^{s-1}}\right),$$

and F is the unique solution of

$$\begin{cases} (-\Delta)^s F = 0 & \text{in } \Omega \\ F = 0 & \text{in } \Omega^c \\ \dfrac{F}{d^{s-1}} = f & \text{on } \partial\Omega \end{cases}$$

satisfying $F \to 0$ for $|x| \to \infty$. (Notice that F blows-up on the free boundary $\partial\Omega$.)

Such equivalence is not trivial, and actually U_1 is related, but not equal, to $\mathcal{H}_{\Omega,s}$.

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A heat equation approach to some boundary value problems in analysis and geometry

Nicola Garofalo

In this talk I report on some parts of the following joint works with Giulio Tralli [6], [7] and [8]. Consider the following parabolic version of the Caffarelli-Silvestre extension problem [2]. Little known to most people, this problem was first introduced by Frank Jones in 1968 [11]. He also constructed the Poisson kernel and solved the case $s = 1/2$: given $a \in (-1, 1)$ and a function $u \in C_0^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ find a function $U \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}_t \times \mathbb{R}^+_y)$ such that

$$\begin{cases} \partial_{yy}U + \frac{a}{y}\partial_y U + \Delta_x U - \partial_t U = 0 \\ U(x,t,0) = u(x,t). \end{cases} \tag{1}$$

The solution $U((x,t),y)$ to (1) possesses the fundamental property

$$(\partial_t - \Delta_x)^s u(x,t) = -2^{-a}\frac{\Gamma(\frac{1-a}{2})}{\Gamma(\frac{1+a}{2})}\lim_{y\to 0^+} y^a \frac{\partial U}{\partial y}((x,t),y),$$

where $s = \frac{1-a}{2} \in (0,1)$. It turns out that the conformal invariances of $(-\Delta)^s$ are hidden in the heat kernel

$$\mathfrak{q}^{(s)}(x,y,t) = (4\pi t)^{-\left(\frac{n}{2}+(1-s)\right)} e^{-\frac{|x|^2+y^2}{4t}}. \tag{2}$$

of the extension PDE in (1)

$$\partial_{yy}U + \frac{1-2s}{y}\partial_y U + \Delta_x U - \partial_t U = 0. \tag{3}$$

An important (beautiful) property enclosed in (2) is that

$$\int_0^\infty \mathfrak{q}^{(\pm s)}(x,y,t)dt = \frac{\Gamma(\frac{n\mp 2s}{2})}{\pi^{\frac{n}{2}\mp s}}(|x|^2+y^2)^{-\frac{n\mp 2s}{2}}, \tag{4}$$

which gives the Talenti-Aubin-Lieb extremals for the Sobolev embedding, and their intertwined functions obtained by changing s into $-s$.

The starting point of my talk is a parabolic extension problem, inspired to (1), and yet completely different from the Caffarelli-Silvestre one. Such problem arises in conformal geometry and complex hyperbolic scattering. The relevant geometric framework is that of Lie groups of Heisenberg type, a special class of the so-called stratified, nilpotent Lie groups, aka Carnot groups. These ambients model physical systems with constrained dynamics, in which motion is only possible in a prescribed set of directions in the tangent space (sub-Riemannian [3], versus Riemannian geometry). Every Carnot group is endowed with an important second order partial differential operator called horizontal Laplacian

$$\mathscr{L} = \sum_{j=1}^{m} X_j^2.$$

The idea goes back to the visionary address of Eli Stein [14]. The operator $\mathscr{L}$ fails to be elliptic at every point of the ambient space $\mathbb{G}$, unless the group is Abelian, in which case $\mathbb{G} = \mathbb{R}^m$ and $\mathscr{L} = \Delta$, the standard Laplacian. I will focus on the simplest genuinely non-Abelian setting: Carnot groups of step two. This means that the Lie algebra of the group admits a representation $\mathfrak{g} = \mathfrak{g}_1 \oplus g_2$, where $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$ and $[\mathfrak{g}_1, \mathfrak{g}_2] = \{0\}$. The most important of these Lie groups is the ubiquitous $2n+1$-dimensional Heisenberg group $\mathbb{H}^n$, first introduced by H. Weyl in his group representation theory approach to quantum mechanics [15, 16]. In the Heisenberg group $\mathbb{H}^n$, with horizontal Laplacian $\mathscr{L}$, I consider the following parabolic extension problem: given a number $s \in (0,1)$, and a function $u \in C_0^\infty(\mathbb{H}^n \times \mathbb{R})$, find $U \in C^\infty(\mathbb{H}^n \times \mathbb{R} \times \mathbb{R}_y^+)$ such that

$$\begin{cases} \partial_{yy} U + \frac{1-2s}{y} \partial_y U + \frac{y^2}{4} \partial_{\sigma\sigma} U + \mathscr{L} U - \partial_t U = 0, \\ U(g,t,0) = u(g,t), \qquad (g,t) \in \mathbb{H}^n \times \mathbb{R}. \end{cases} \tag{5}$$

I stress that, without the term $\frac{y^2}{4}\partial_{\sigma\sigma} U$, the problem (5) would simply be similar to the extension problem (1), but for the fractional powers of the heat operator $(\partial_t - \mathscr{L})^s$ on $\mathbb{H}^n$! In such case the relevant PDE would decouple, which means that its heat kernel would simply be the product of the heat kernels of $\partial_{yy} + \frac{1-2s}{y}\partial_y - \partial_t$ and $\mathscr{L} - \partial_t$, i.e. $\mathfrak{q}^{(s)}(z,\sigma), y, t) = (4\pi t)^{1-s} e^{-\frac{y^2}{4t}} p(z,\sigma,t)$, where p is the Gaveau-Hulanicki [9, 10] heat kernel on $\mathbb{H}^n$. The additional term $\frac{y^2}{4}\partial_{\sigma\sigma} U$ makes the problem (5) completely different, and much harder, than (1). One critical obstruction is that the operator $\mathscr{L}$ involves differentiation in the variable σ. One has in fact, in the real coordinates $(z,\sigma) \in \mathbb{H}^n$, $\mathscr{L} = \Delta_z + \frac{|z|^2}{4}\partial_{\sigma\sigma} + \partial_\sigma(\sum_{i=1}^n x_i \partial_{y_i} - y_i \partial_{x_i})$. Concerning the extension problem (5) the following is one of our main results.

Theorem 1. *Let $\mathbb{G}$ be a group of Heisenberg type* [5]*, with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, and let $m = \dim \mathfrak{g}_1$, $k = \dim \mathfrak{g}_2$. For every $0 < s < 1$ the heat kernel with pole in the origin of the operator in* (5) *is given by*

(6)

$$\mathfrak{q}_{(s)}((z,\sigma),t,y) = \frac{2^k}{(4\pi t)^{\frac{m}{2}+k+1-s}} \int_{\mathbb{R}^k} e^{-\frac{i}{t}\langle \sigma,\lambda\rangle} \left(\frac{|\lambda|}{\sinh|\lambda|}\right)^{\frac{m}{2}+1-s} e^{-\frac{|z|^2+y^2}{4t}\frac{|\lambda|}{\tanh|\lambda|}} d\lambda.$$

It is notable that, in the limiting cases when $s \nearrow 1$ and $y \to 0^+$, we recover from (6) the heat kernel of $\partial_t - \mathscr{L}$ of Gaveau and Hulanicki! In our works mentioned in the opening we use the heat kernels (6), and its intertwined counterpart obtained by changing s into $-s$, to develop in groups of Heisenberg type an extensive analysis of the fractional powers $\mathscr{L}_s$ of the so-called *conformal horizontal Laplacian.* We have also constructed the inverses of the pseudodifferential operators $\mathscr{L}_{(\pm s)}$, and computed their fundamental solutions $\mathfrak{e}_{(s)}$ and $\mathfrak{e}_{(-s)}$.

Additional literature:

- the operators $\mathscr{L}_s$ were first introduced in [1] via the spectral formula

$$\mathscr{L}_s = 2^s|T|^s \frac{\Gamma(-\frac{1}{2}\mathscr{L}|T|^{-1} + \frac{1+s}{2})}{\Gamma(-\frac{1}{2}\mathscr{L}|T|^{-1} + \frac{1-s}{2})}, \qquad 0 < s < 1, \tag{7}$$

  where $T = \partial_\sigma$ is the differentiation in the vertical direction.
- in $\mathbb{H}^n$ the time-independent version of (5) was first introduced in [4]. Using scattering theory the authors proved that its solution U satisfies the following fundamental weighted Dirichlet-to-Neumann mapping

$$-\frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)} \lim_{y\to 0^+} y^{1-2s}\frac{\partial U}{\partial y}((z,\sigma),y) = \mathscr{L}_s u(z,\sigma). \tag{8}$$

- in their papers [12] and [13] the authors considered the problem (5), and proved that the representation (7) of $\mathscr{L}_s$ is equivalent to the following formula:

$$\mathscr{L}_s u(g) = -\frac{s}{\Gamma(1-s)} \int_0^\infty \frac{1}{t^{1+s}} \left[P_{(-s),t}u(g) - u(g)\right] dt. \tag{9}$$

For an explanation of the operators $P_{(-s),t}$, and their intertwined $P_{(+s),t}$, see the slides of my talk. It is important to keep in mind that these operators do not generate semigroups! Therefore, the standard theory does not apply.

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The Dirichlet problem for fractional-order operators; transmission spaces; a local Dirichlet boundary value

Gerd Grubb

1. The homogeneous Dirichlet problem

The talk deals with *the fractional Laplacian* $(-\Delta)^a$, $0 < a < 1$, and suitable pseudodifferential operators (ps.d.o.s) P of order $2a$. Recall that the Fourier transform $\mathcal{F}u = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx$ serves to define the ps.d.o. P with symbol $p(x,\xi)$ by:

$$Pu(x) = \mathcal{F}^{-1}_{\xi\to x}\big(p(x,\xi)\hat{u}(\xi)\big) = \operatorname{Op}(p)u.$$

Then $(-\Delta)^a u = \operatorname{Op}(|\xi|^{2a})u$. $(-\Delta)^a$ can also be described in real terms as a singular integral operator, the convolution with $c_{n,a}|y|^{-n-2a} = \mathcal{F}^{-1}|\xi|^{2a}$.

Let $\Omega \subset \mathbb{R}^n$ be either $\mathbb{R}^n_+$ or a bounded open subset with $C^{1+\tau}$-boundary, some $\tau \in \mathbb{R}_+$. By r^+ we denote restriction from $\mathbb{R}^n$ to Ω, and by e^+ extension by 0 from $\overline{\Omega}$ to $\mathbb{R}^n$. Together with the Sobolev space $H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}((1+|\xi|)^s\hat{u}) \in L_2(\mathbb{R}^n)\}$, $s \in \mathbb{R}$, we use the *restricted* space $\overline{H}^s(\Omega) = r^+H^s(\mathbb{R}^n)$ and the *supported* space $\dot{H}^s(\overline{\Omega}) = \{u \in H^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}$; there are similar spaces H^s_q for L_q, $1 < q < \infty$.

Let $P = (-\Delta)^a$ on $\mathbb{R}^n$, $0 < a < 1$, or a pseudodifferential generalization (see later). The homogeneous Dirichlet problem for P on Ω is:

$$r^+Pu = f \text{ on } \Omega, \quad u = 0 \text{ on } \mathbb{R}^n \setminus \overline{\Omega}. \tag{1}$$

Assume Ω bounded. Since P is of order $2a$, r^+P maps $\dot{H}^a(\overline{\Omega})$ to $\overline{H}^{-a}(\Omega)$. Applying the Lax-Milgram lemma to the sequilinear form $Q(u,v) = \int_\Omega Pu\,\bar{v}\,dx$, we find that the realization P_D with domain

$$D(P_D) = \{u \in \dot{H}^a(\overline{\Omega}) \mid r^+Pu \in L_2(\Omega)\},$$

is bijective from $D(P_D)$ onto $L_2(\Omega)$. We now ask about the regularity of solutions.

It follows from Vishik and Eskin [3] that $D(P_D) = \dot H^{2a}(\overline\Omega)$ if $a < \frac12$, and $D(P_D) \subset \dot H^{a+\frac12-\varepsilon}(\overline\Omega)$ if $a \ge \frac12$. More recent results:

$$f \in L_\infty(\Omega) \implies u \in d^a \overline{C}^t(\Omega), \text{ for } t \text{ up to } a\ , \tag{2}$$

$$f \in \overline{H}_q^s(\Omega) \iff u \in H_q^{a(s+2a)}(\overline\Omega), \text{ when } s \ge 0, \tag{3}$$

$$f \in C^\infty(\overline\Omega) \iff u \in e^+ d^a C^\infty(\overline\Omega) \equiv \mathcal{E}_a(\overline\Omega), \tag{4}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$ near $\partial\Omega$ (extended > 0 to Ω). In (3), $H_q^{a(s+2a)}(\overline\Omega)$ is the socalled *a-transmission space*, it satisfies

$$H_q^{a(s+2a)}(\overline\Omega) \subset \dot H_q^{s+2a}(\overline\Omega) + d^a e^+ \overline{H}_q^{s+a}(\Omega), \text{ when } s + a > \tfrac1q.$$

(2) is shown in Ros-Oton and Serra [6] for $C^{1,1}$-domains, (3), (4) in G. [4] for C^∞-domains, (3) in Abels-G. [2] for $C^{1+\tau}$-domains, $\tau > 2a$. The theory initiated in an unpublished lecture note by Hörmander 1966. There are estimates for Lipschitz domains e.g. by Borthagaray and Nochetto.

2. Transmission spaces

A simple example. Let $P = (1-\Delta)^a$, $0 < a < 1$, with symbol $p(\xi) = (1+|\xi|^2)^a$. The Lax-Milgram method applies straightforwardly to $P = (1-\Delta)^a$, showing unique solvability of the homogeneous Dirichlet problem on $\mathbb{R}^n_+$. Now

$$(1+|\xi|^2)^a = (\langle\xi'\rangle^2 + \xi_n^2)^a = (\langle\xi'\rangle - i\xi_n)^a(\langle\xi'\rangle + i\xi_n)^a.$$

Define for general $t \in \mathbb{R}$ the order-reducing operators:

$$\Xi^t_\pm = \operatorname{Op}((\langle\xi'\rangle \pm i\xi_n)^t),$$

then P has the factorization $(1-\Delta)^a = \Xi^a_-\,\Xi^a_+$, with inverse $(1-\Delta)^{-a} = \Xi^{-a}_+\,\Xi^{-a}_-$. Relative to $\mathbb{R}^n_+$, the operators $\Xi^t_\pm$ map as follows for all $s \in \mathbb{R}$:

$$\Xi^t_+ \colon \dot H^s(\overline{\mathbb{R}}^n_+) \overset{\sim}{\to} \dot H^{s-t}(\overline{\mathbb{R}}^n_+), \quad r^+\Xi^t_- e^+ \colon \overline{H}^s(\mathbb{R}^n_+) \overset{\sim}{\to} \overline{H}^{s-t}(\mathbb{R}^n_+),$$

with inverses $(\Xi^t_+)^{-1} = \Xi^{-t}_+$, $(r^+\Xi^t_- e^+)^{-1} = r^+\Xi^{-t}_- e^+$. The map $r^+P \colon \dot H^a(\overline{\mathbb{R}}^n_+) \overset{\sim}{\to} \overline{H}^{-a}(\mathbb{R}^n_+)$ can be factored as $r^+Pu = (r^+\Xi^a_- e^+)\,\Xi^a_+ u$. Hence the solution operator to the Dirichlet problem for $(1-\Delta)^a$ over $\mathbb{R}^n_+$ maps with bijective factors:

$$\overline{H}^{-a}(\mathbb{R}^n_+) \overset{r^+\Xi^{-a}_- e^+}{\longrightarrow} L_2(\mathbb{R}^n_+) \overset{\Xi^{-a}_+}{\longrightarrow} \dot H^a(\overline{\mathbb{R}}^n_+).$$

When we lift these bijections to higher-order Sobolev spaces, the range space at the right end will be an a-transmission space:

$$\overline{H}^s(\mathbb{R}^n_+) \overset{r^+\Xi^{-a}_- e^+}{\longrightarrow} \overline{H}^{s+a}(\mathbb{R}^n_+) \overset{\Xi^{-a}_+}{\longrightarrow} \Xi^{-a}_+ e^+ \overline{H}^{s+a}(\mathbb{R}^n_+) \equiv H^{a(s+2a)}(\overline{\mathbb{R}}^n_+).$$

More generally, we define for $1 < q < \infty$, $H_q^{a(t)}(\overline{\mathbb{R}}^n_+) = \Xi^{-a}_+ e^+ \overline{H}_q^{t-a}(\mathbb{R}^n_+)$. The definition extends to bounded $C^{1+\tau}$-domains Ω by use of local coordinates, when $a - \frac{1}{q'} < t < 1 + \tau$.

The general ps.d.o.s P are taken with the symbol being C^τ in x, *classical* (expanded in homogeneous term $p \sim \sum_{j\in\mathbb{N}_0} p_j$, p_j of degree $2a - j$), *strongly* *elliptic:* $\operatorname{Re} p_0(x,\xi) \geq c > 0$, and *even:* $p_j(x,-\xi) = (-1)^j p_j(x,\xi)$, all j, $|\xi| \geq 1$. The idea is that for such P on $\mathbb{R}^n_+$ one writes $P = \Xi^a_- Q \Xi^a_+$ and applies the Boutet de Monvel calculus to Q.

Theorem 1 [2] *For P and Ω with $\tau > 2a$, $0 \leq s < \tau - 2a$, the solutions of the homogeneous Dirichlet problem* (1) *satisfy* (3)*.*

3. A nonzero Dirichlet boundary value

For motivation, consider the C^∞-results in (4). P maps forwardly [4]:

$$r^+P\colon \mathcal{E}_{a+k}(\overline{\Omega}) \to C^\infty(\overline{\Omega}) \text{ for all integer } k \geq -1.$$

Observe the Taylor expansions at the boundary, in local coordinates where Ω is replaced by $\mathbb{R}^n_+ = \{x = (x', x_n) \mid x_n > 0\}$ so that $d(x) = x_n$:

- In $\mathcal{E}_0$: $u(x) \sim v_0(x') + v_1(x')x_n + v_2(x')x_n^2 + \dots$.
- In $\mathcal{E}_1$: $u(x) \sim v_0(x')x_n + v_1(x')x_n^2 + v_2(x')x_n^3 + \dots$.
- In $\mathcal{E}_a$: $u(x) \sim v_0(x')x_n^a + v_1(x')x_n^{a+1} + v_2(x')x_n^{a+2} + \dots$.
- In $\mathcal{E}_{a-1}$: $u(x) \sim v_0(x')x_n^{a-1} + v_1(x')x_n^a + v_2(x')x_n^{a+1} + \dots$.

Denoting $u|_{\partial\Omega} = \gamma_0 u$, we observe that for all $a > 0$,

$$\mathcal{E}_a \text{ is the subset of } \mathcal{E}_{a-1} \text{ where } \gamma_0(u/d^{a-1}) = 0.$$

Let $f \in C^\infty(\overline{\Omega})$, $\varphi \in C^\infty(\partial\Omega)$. Compare boundary value problems for Δ and $(-\Delta)^a$. *Old fact:* The **nonhomogeneous Dirichlet problem for Δ:**

$$\Delta u = f \text{ on } \Omega, \quad \gamma_0 u = \varphi \text{ on } \partial\Omega,$$

is uniquely solvable in $C^\infty(\overline{\Omega}) \simeq \mathcal{E}_0(\overline{\Omega})$. In particular, the **homogeneous Dirichlet problem for Δ** (with $\varphi = 0$) is uniquely solvable in $\mathcal{E}_1(\overline{\Omega})$.

New result from [4]*:* The **homogeneous Dirichlet problem** (1) **for** $(-\Delta)^a$ is uniquely solvable in $\mathcal{E}_a(\overline{\Omega})$. Here $\mathcal{E}_a(\overline{\Omega})$ has a role like $\mathcal{E}_1(\overline{\Omega})$ for Δ. It is then natural to define a **nonhomogeneous Dirichlet problem for** $(-\Delta)^a$ by going out to the larger space $\mathcal{E}_{a-1}(\overline{\Omega})$. The problem

$$(-\Delta)^a u = f \text{ on } \Omega, \quad \gamma_0(u/d^{a-1}) = \varphi \text{ on } \partial\Omega, \quad \operatorname{supp} u \subset \overline{\Omega}, \tag{5}$$

is uniquely solvable in $\mathcal{E}_{a-1}(\overline{\Omega})$ [4]. The proof is done by subtracting a function $w \in \mathcal{E}_{a-1}$ with $\gamma_0(w/d^{a-1}) = \varphi$.

Abatangelo [1] approached the problem from a different angle, starting with a Green's function $G_\Omega(x,y)$ for the homogeneous Dirichlet problem for $(-\Delta)^a$ and developing integral representation formulas imitating the formulas known for Δ. This led to a boundary operator $u \mapsto Eu$, known to be proportional to $\gamma_0(u/d^{a-1})$ (cf. [5]). It enters also e.g. in recent works of Chan, Gomez-Castro and Vazquez; Fernandez-Real and Ros-Oton. The solutions are called "blow-up solutions", since they behave like d^{a-1} near $\partial\Omega$. They are in $L_q(\Omega)$ for $1 < q < (1-a)^{-1}$.

More generally, the roles of $\mathcal{E}_a$ and $\mathcal{E}_{a-1}$ are taken over by the a- and $(a-1)$-transmission spaces, such as $H_q^{(a-1)(t)}(\overline{\mathbb{R}}^n_+) = \Xi_+^{-a+1} e^+ \overline{H}_q^{t-a+1}(\mathbb{R}^n_+)$.

For $C^{1+\tau}$-domains, the boundary map $u \mapsto \gamma_0(u/d^{a-1})$ is continuous and surjective from $H_q^{(a-1)(t)}(\overline{\Omega})$ to $B_q^{t-a+\frac{1}{q'}}(\partial\Omega)$ when $a - \frac{1}{q'} < t < \tau + a - 1$.

Theorem 2 [5] *When $f \in \overline{H}_q^s(\Omega)$, $\varphi \in B_q^{s+a+1/q'}(\partial\Omega)$, $0 \leq s < \tau - 2a - 1$, the nonhomogeneous Dirichlet problem* (5) *for P is uniquely solvable with a solution $u \in H_q^{(a-1)(s+2a)}(\overline{\Omega})$ (Fredholm solvable if* 0 *is a Dirichlet eigenvalue).*

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Morrey smoothness spaces: A new approach

Dorothee D. Haroske
(joint work with Hans Triebel)

In the recent years so-called Morrey smoothness spaces attracted a lot of interest. They can (also) be understood as generalisations of the classical spaces of Besov and Triebel-Lizorkin type,

$$A^s_{p,q}(\mathbb{R}^n) \quad \text{with} \quad A \in \{B, F\}, \quad s \in \mathbb{R} \quad \text{and} \quad 0 < p, q \leq \infty, \tag{1}$$

where the parameters satisfy $s \in \mathbb{R}$ (smoothness), $0 < p \leq \infty$ (integrability) and $0 < q \leq \infty$ (summability). They have been extended in several directions, most notably into two types of Morrey smoothness spaces,

$$\mathcal{N}^s_{u,p,q}(\mathbb{R}^n) \quad \text{and} \quad \mathcal{E}^s_{u,p,q}(\mathbb{R}^n) \qquad \text{with} \quad p \leq u < \infty, \tag{2}$$

and

$$A^{s,\tau}_{p,q}(\mathbb{R}^n), \quad A \in \{B, F\}, \qquad \text{with} \quad 0 \leq \tau < \infty. \tag{3}$$

When $p = u$ in (2), and $\tau = 0$, $p < \infty$ in (3), one re-obtains the corresponding spaces in (1). In our opinion, among the various approaches to Morrey type spaces, these two scales (2), (3) enjoy special attention, also in view of possible applications to PDEs. This was already the case for the 'basic' Morrey spaces extending L_p, cf. [8]. Later, one of the milestones in this direction, was the famous paper [4] where they used spaces of type (2) to study Navier-Stokes equations; we also refer in this context to the papers [7, 1, 13, 5, 6], as well as to the monographs [9, 10, 14, 11, 12].

Our intention is to reorganise these two prominent types of Morrey smoothness spaces. The classical parameters (s,p,q) in (1), but also in (2) and (3), are untouchable. Now we add the so–called slope parameter ϱ, preferably (but not exclusively) with $-n \le \varrho < 0$, that is, we replace u in (2) and τ in (3) by the common parameter ϱ (appropriately chosen). The corresponding spaces

$$\varrho\text{-}A^s_{p,q}(\mathbb{R}^n) = \big\{ \Lambda^\varrho A^s_{p,q}(\mathbb{R}^n), \Lambda_\varrho A^s_{p,q}(\mathbb{R}^n) : \ A \in \{B,F\}\big\}, \tag{4}$$

cover all spaces in (2) and (3), in particular, with

$$\Lambda^{-n} A^s_{p,q}(\mathbb{R}^n) = \Lambda_{-n} A^s_{p,q}(\mathbb{R}^n) = A^s_{p,q}(\mathbb{R}^n). \tag{5}$$

We call ϱ the slope parameter because $|\varrho|$ quite often takes over the rôle of the slope n, and $\min(|\varrho|,1)$ replaces 1 in slopes of (broken) lines in the typical $\big(\frac{1}{p}, s\big)$–diagram characterising distinguished properties of the spaces $A^s_{p,q}(\mathbb{R}^n)$, where any space of type $A^s_{p,q}$ is indicated by its smoothness parameters s and the integrability p, neglecting the fine index q for the moment.

Let us illustrate this approach by a few examples, like embeddings in $L_\infty(\mathbb{R}^n)$ and in $L_1^{\mathrm{loc}}(\mathbb{R}^n)$ (regular distributions) or traces on hyper–planes. Moreover, one may ask to which spaces the δ–distribution or the characteristic function χ_Q of the unit cube $Q=(0,1)^n$, $n\in\mathbb{N}$, belong. Some final answers in the setting of Morrey smoothness spaces have been obtained recently. But the related conditions for the above questions are often not very appealing, producing, for instance, curved lines in the well–known $\big(\frac{1}{p}, s\big)$–diagram; we refer to [2] for such examples. This suggested to search for a re–parametrisation of the spaces in (2), (3) such that the outcome produces natural and transparent conditions for distinguished properties of these spaces. It turns out that the previous, separately obtained results, based on independent arguments, can thus not only be understood in a better way, detached from the (sometimes quite involved) technical requirements. But one might also observe more intrinsic reasons for common phenomena. This will also constitute some basis for unified results and, occasionally, lead to appropriate conjectures. In particular, the sharp embedding

$$A^s_{p,q}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \qquad \text{if} \quad s > \frac{n}{p}, \quad 0<p<\infty, \tag{6}$$

as far as the breaking line is concerned, has now the sharp counterpart

$$\varrho\text{-}A^s_{p,q}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \quad \text{if} \quad s > \frac{|\varrho|}{p}, \quad 0<p<\infty. \tag{7}$$

The sharp inclusion

$$A^s_{p,q}(\mathbb{R}^n) \subset L_1^{\mathrm{loc}}(\mathbb{R}^n) \qquad \text{if} \quad s > \sigma^n_p, \quad 0<p<\infty, \tag{8}$$

as far as the breaking line is concerned, has now the sharp counterpart

$$\varrho\text{-}A^s_{p,q}(\mathbb{R}^n) \subset L_1^{\mathrm{loc}}(\mathbb{R}^n) \quad \text{if} \quad s > \sigma^{|\varrho|}_p, \quad 0<p<\infty, \tag{9}$$

where we use the notation

$$\sigma^t_p = t\Big(\max\Big(\frac{1}{p}, 1\Big) - 1\Big), \qquad t\ge 0, \quad 0<p<\infty. \tag{10}$$

For the characteristic function χ_Q of the cube $Q = (0,1)^n$ the sharp assertion

$$\chi_Q \in A^s_{p,q}(\mathbb{R}^n) \qquad \text{if} \quad s < \frac{1}{p}, \quad 0 < p < \infty, \tag{11}$$

as far the breaking line is concerned, has now the sharp counterpart

$$\chi_Q \in \varrho\text{-}A^s_{p,q}(\mathbb{R}^n) \qquad \text{if} \quad s < \frac{1}{p}\min(|\varrho|, 1), \quad 0 < p < \infty. \tag{12}$$

The generalisation of the slope n in (6), (8) by $|\varrho|$ in (7), (9) obeys the so–called *Slope–n–Rule*, whereas the replacement of 1 in (11) by $\min(|\varrho|, 1)$ in (12) is a typical example of the so–called *Slope–1–Rule*.

Our aim in the paper [3] is two–fold: we reformulate some assertions already available in the literature (many of them quite recently), and we establish on this basis new properties, a few of them became visible only in the context of the offered new approach.

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Pseudodifferential Operators with Negative Definite Symbols. The Balayage Dirichlet Problem and the geometry of Transition Functions

Niels Jacob

In this survey we first recall the notion of a pseudo-differential operator with a negative definite symbol, that is a pseudo-differential operator $q(x, D)$ with symbol $q(x, \xi)$ which is with respect to the covariable ξ a continuous negative definite function; i.e. $\xi \mapsto q(x, \xi)$ admits a Lévy-Khinchin representation. We then discuss for certain of such operators the balayage Dirichlet problem: $q(x, D)u = 0$ in a bounded open set $G \subset \mathbb{R}^n$ and $u|_{G^c} = f$. The results are contained in [2] and we discuss them in relation to problems related to the fractional Laplacian, an operator of much interest since the work of Caffarelli and Sylvestre.
The second and larger part of this survey discusses a programme initiated in [3] by V. Knopova, S. Landwehr, R. Schilling and NJ in order to obtain a geometric understanding of transition densities of Lévy or Lévy-type processes. The first basic observation in [3] is that if ψ is a continuous negative definite function, hence $-\psi(D)$ generates a Feller semigroup $(T_t^{\psi})_{t\geq 0}$, then the condition $\psi(\xi) = 0$ if and only if $\xi = 0$ implies that $d_{\psi}(\xi, \eta) := \sqrt{|\psi(\xi - \eta)|}$ is a metric on $\mathbb{R}^n$ which generates the Euclidean topology if and only if $\liminf_{|\xi|\to\infty} \psi(\xi) > 0$. In this case we have

$$(1) \qquad \|T_t^{\psi}\|_{L^1-L^\infty} = p_t^{\psi}(0) = (2\pi)^{-n} t\mathcal{L}(V_{\psi}(\sqrt{r}))(t)$$

where $\mathcal{L}$ is the Laplace transform, $\lambda^{(n)}$ the Lebesgue measure in $\mathbb{R}^n$ and $V_{\psi}(\rho) = \lambda^{(n)}(B^{d_{\psi}}(0, \rho))$. Moreover, if the metric measure space $(\mathbb{R}^n, d_{\psi}, \lambda^{(n)})$ has the doubling property, we have $p_t^{\psi}(0) \asymp V_{\psi}(\frac{1}{\sqrt{t}})$. Note that (1) implies that $p_t^{\psi}(0)$ has in general not a power-type decay and that therefore Nash-type inequalities are not the most suitable tool to handle the transition density p_t^{ψ} of a Lévy or Lévy-type process. We suggest in [2] to assume for p_t^{ψ} the structure

$$(2) \qquad p_t^{\psi}(x - y) = p_t^{\psi}(0)e^{-\delta_{\psi,t}^2(x,y)}$$

where $\delta_{\psi,t}$ is a further metric on $\mathbb{R}^n$. For many examples this can be verified however there is still no general result. We now consider following [1] two explicit examples given by the negative definite symbols with at least C_b–coefficients a_k and b_k, respectively, as:

$$(3) \qquad L(x, \xi) := \sum_{k=1}^{n} a_k(x)|\xi_k|,\ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n,\ 0 < \nu_0 \leq a_k(x) \leq \nu_1,$$

and

$$(4) \qquad \Lambda(x, \xi) := \left(\sum_{k=1}^{n}(b_k(x)\xi_k^2)\right)^{\frac{1}{2}},\ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n,\ 0 < \mu_0 \leq b_k(x) \leq \mu_1,$$

and we freeze the coefficients at $x_0 \in \mathbb{R}^n$. For $p_t^{L(x_0,\cdot)}$ and $p_t^{\Lambda(x_0,\cdot)}$ we obtain now

$$p_t^{L(x_0,\cdot)}(x-y) = \frac{1}{t^n}\prod_{k=1}^{n}\frac{1}{\pi a_k(x_0)}e^{-\sum_{k=1}^{n}\ln\left(\frac{|x_k-y_k|^2+a_k^2(x_0)t^2}{t^2}\right)} \tag{5}$$

and

$$p_t^{\Lambda(x_0,\cdot)}(x-y) = \frac{1}{t^n}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n-1}{2}}}\frac{1}{\left(\prod_{k=1}^{n}b_k(x_0)\right)^{1/2}}e^{-\frac{n+1}{2}\ln\left(\frac{1}{t^2}\left(\sum_{k=1}^{n}\frac{|x_k-y_k|^2}{b_k(x_0)}+t^2\right)\right)}. \tag{6}$$

The two sets of corresponding metrics are

$$d^{L(x_0,\cdot)}(\xi,\eta) := \left(\sum_{k=1}^{n}a_k(x_0)|\xi_k-\eta_k|\right)^{\frac{1}{2}},$$

$$\delta_{L(x_0,\cdot),t}(x,y) := \left(\sum_{k=1}^{n}\ln\left(\frac{|x_k-y_k|^2+a_k^2(x_0)t^2}{t^2}\right)\right)^{1/2}$$

and

$$d^{\Lambda(x_0,\cdot)}(\xi,\eta) := \left(\sum_{k=1}^{n}b_k(x_0)(\xi_k-\eta_k)^2\right)^{\frac{1}{2}},$$

$$\delta_{\Lambda(x_0,\cdot),t} := \sqrt{\frac{n+1}{2}}\left(\ln\left(\frac{1}{t^2}\left(\sum_{k=1}^{n}\frac{|x_k-y_k|^2}{b_k(x_0)}+t^2\right)\right)\right)^{\frac{1}{2}}.$$

We observe that the metrics $d_{L(x_0,\cdot)}$ and $d_{\Lambda(x_0,\cdot)}$, hence $p_t^{L(x_0,\cdot)}(0)$ and $p_t^{\Lambda(x_0,\cdot)}(0)$, are comparable, even for different points x_0 and x_1, however the metrics $\delta_{L(x_0,\cdot)}$ and $\delta_{\Lambda(x_0,\cdot)}$ are not comparable as $p_t^{L(x_0,\cdot)}(x-y)$ and $p_t^{\Lambda(x_0,\cdot)}(x-y)$ are not comparable, although the symbols $L(x_0,\xi)$ and $\Lambda(x_0,\xi)$ are comparable. Both examples are discussed further to obtain (under further conditions) estimates for $(T_t)_{t\geq 0}$, the semigroup generated by $-L(x,D)$ $(-\Lambda(x,D))$ in terms of $(S_t)_{t\geq 0}$, the semigroup generated by $-L(x_0,D)$ $(-\Lambda(x_0,D))$. For example we may find for $t\to 0$

$$(T_t\varphi)(x) \geq -\kappa_0(\varphi)t + \gamma_0 V_{L(x_0,\cdot)}\left(\frac{1}{\sqrt{t}}\right)\inf_{y\in G_1} e^{-\delta^2_{L(x_0,\cdot),t}(x,y)}\lambda^{(n)}(G_1)$$

and

$$(T_t\varphi)(x) \leq \kappa_0(\varphi)(t) + \gamma_2 V_{L(x_0,\cdot)}\left(\frac{1}{\sqrt{t}}\right)\sup_{y\in G_2} e^{-\delta^2_{L(x_0,\cdot),t}(x,y)}\lambda^{(n)}(G_2),$$

where $G_1 \subset G_2$ are bounded open sets, $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\mathcal{X}_{G_1} \leq \varphi \leq \mathcal{X}_{G_2}$ where $\mathcal{X}_G$ denotes the characteristic function of G, and $\kappa_0(\varphi)$ is a constant depending on φ, whereas γ_0 and γ_1 are further constants. These two bounds should be sharpened by taking on the right hand side of the first estimate the maximum with 0, and in the second estimate the minimum with the kernel of T_t on the diagonal at the point x.

These results make it evident that we are still far away of a full understanding of the transition densities of Lévy and Lévy-type processes and new, for example geometric ideas are needed.

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As a text for the general background we refer to Springer Lecture Notes in Mathematics, vol. 2099 (Lévy Matters vol. III) by R. Schilling and co-authors.

A Sobolev inequality and a Faber-Krahn inequality for the regional fractional Laplacian

Tianling Jin
(joint work with Rupert L. Frank, Dennis Kriventsov, Jingang Xiong)

Let $n \geq 1$, $\sigma \in (0,1)$ and $\Omega \subset \mathbb{R}^n$ be an open set. There are two natural fractional Sobolev norms which may be defined for $u \in C_c^\infty(\Omega)$:

$$I_{n,\sigma,\mathbb{R}^n}[u] := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}}\, \mathrm{d}x\, \mathrm{d}y$$

and

$$I_{n,\sigma,\Omega}[u] := \iint_{\Omega \times \Omega} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\sigma}}\, \mathrm{d}x\, \mathrm{d}y.$$

Depending on the choices of n, σ and Ω, these two norms may or may not be equivalent. Even when they are equivalent, there are still subtle differences in how they depend on the domain Ω.

One significant difference is the behavior of their corresponding best Sobolev constants:

$$S_{n,\sigma}(\Omega) := \inf\left\{ I_{n,\sigma,\Omega}[u] : u \in C_c^\infty(\Omega), \int_\Omega |u|^{\frac{2n}{n-2\sigma}}\, \mathrm{d}x = 1 \right\}$$

and

$$\widetilde{S}_{n,\sigma}(\Omega) := \inf\left\{ I_{n,\sigma,\mathbb{R}^n}[u] : u \in C_c^\infty(\Omega), \int_\Omega |u|^{\frac{2n}{n-2\sigma}}\, \mathrm{d}x = 1 \right\}.$$

Using the dilation or translation invariance of $\widetilde{S}_{n,\sigma}(\mathbb{R}^n)$, it is not difficult to see that

$$\widetilde{S}_{n,\sigma}(\Omega) = \widetilde{S}_{n,\sigma}(\mathbb{R}^n) = S_{n,\sigma}(\mathbb{R}^n).$$

Moreover, a result of Lieb classifies all minimizers for $\widetilde{S}_{n,\sigma}(\mathbb{R}^n)$ and shows that they do not vanish anywhere on $\mathbb{R}^n$. Therefore, the infimum $\widetilde{S}_{n,\sigma}(\Omega)$ is not attained unless $\Omega = \mathbb{R}^n$.

However, in [1], we discovered that the minimization problem for $S_{n,\sigma}(\Omega)$ behaves differently from $\widetilde{S}_{n,\sigma}(\Omega)$, due to a Brézis-Nirenberg effect.

Theorem 1 ([1]). Suppose $n \geq 4\sigma$. Then

- If the complement Ω^c has an interior point, then
$$S_{n,\sigma}(\Omega) < S_{n,\sigma}(\mathbb{R}^n).$$
- If $\sigma \neq 1/2$, then $S_{n,\sigma}(\mathbb{R}^n_+)$ is achieved (this half space case was also proved by Musina-Nazarov independently around the same time).
- If $\sigma > 1/2$, Ω is a bounded domain such that $B_1^+ \subset \Omega \subset \mathbb{R}^n_+$, then
$$S_{n,\sigma}(\Omega) < S_{n,\sigma}(\mathbb{R}^n_+).$$
Moreover, if $\partial\Omega$ is smooth then $S_{n,\sigma}(\Omega)$ is achieved.

In probability, $I_{n,\sigma,\Omega}$ is called the Dirichlet form of the censored 2σ-stable process in Ω. Its generator,

$$(-\Delta)^\sigma_\Omega u := 2 \lim_{\varepsilon \to 0} \int_{\{y\in\Omega:\ |y-x|\geq\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{n+2\sigma}}\, \mathrm{d}y, \tag{1}$$

is usually called the regional fractional Laplacian operator. Next, we would like to study the Faber-Krahn inequality for the regional fractional Laplacian operator.

Let us recall the Faber-Krahn inequality for the classical Laplacian: Let $\lambda_1(\Omega)$ be the first Dirichlet eigenvalue of Δ on the bounded open set Ω. Then

$$\lambda_1(\Omega) \geq \lambda_1(B),$$

where B is a ball having the same measure as Ω. Moreover, if the above inequality holds, then Ω must be a ball.

Let $\mathring{H}^\sigma(\Omega)$ be the completion of $C^1_c(\Omega)$ with respect to $I_{n,\sigma,\Omega}$. Consider the first eigenvalue

$$\lambda_{1,\sigma}(\Omega) = \min\{I_{n,\sigma,\Omega}[u] : u \in \mathring{H}^\sigma(\Omega), \|u\|_{L^2(\Omega)} = 1\},$$

which solves

$$\begin{cases} (-\Delta)^\sigma_\Omega u = \lambda_1 u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

We are interested in

$$\inf\{\lambda_{1,\sigma}(\Omega) :\ \Omega \subset \mathbb{R}^n \text{ a bounded open set such that } |\Omega| = |B_1|\}.$$

This shape optimization problem is equivalent to (up to scaling):

$$\inf\left\{ \frac{I_{n,\sigma,\{u>0\}}[u]}{\|u\|^2_{L^2(\mathbb{R}^n)}} + |\{u>0\}| : u \in \mathring{H}^\sigma(\mathbb{R}^n),\quad u \not\equiv 0,\ u \geq 0 \text{ in } \mathbb{R}^n \right\}.$$

One difficulty is that radially symmetric rearrangement does not reduce the $I_{n,\sigma,\Omega}[u]$ norm.

Theorem 2 ([2]). Let $n \geq 2$ and $\sigma \in (\frac{1}{2}, 1)$. There exists $u_0 \in \mathring{H}^\sigma(\mathbb{R}^n)$, $u_0 \not\equiv 0$, $u_0 \geq 0$ in $\mathbb{R}^n$ such that the set $\{u_0 > 0\}$ is bounded, and

$$\inf\left\{\frac{I_{n,\sigma,\{u>0\}}[u]}{\|u\|^2_{L^2(\mathbb{R}^n)}}+|\{u>0\}| : u\in \mathring{H}^\sigma(\mathbb{R}^n),\quad u\not\equiv 0,\ u\geq 0 \text{ in } \mathbb{R}^n\right\}$$

is achieved by u_0.

Open Question: Is there a minimizer that is continuous in $\mathbb{R}^n$?

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Energy methods for nonlocal operators

Moritz Kassmann

(joint work with Guy Foghem)

Short summary: Within the framework of Hilbert spaces, we solve nonlocal problems in bounded domains with prescribed conditions on the complement of the domain. Our main focus is on the inhomogeneous Neumann problem in a rather general setting. We also study the transition from complement value problems to local boundary value problems. Several results are new even for the fractional Laplace operator. The setting also covers relevant models in the framework of peridynamics. The talk is based on [FG20].

Over the last years, there have been several studies of nonlocal Neumann problems of the following type: Given a bounded open set $\Omega\subset\mathbb{R}^d$, one is interested in well-posedness for

$$(N) \qquad Lu = f \text{ in } \Omega, \qquad \mathcal{N}u = g \text{ on } \mathbb{R}^d\setminus\Omega\,,$$

where L is an integro-differential operator and $\mathcal{N}$ is a related integral operator, which plays the role of some kind of normal derivative on $\mathbb{R}^d\setminus\Omega$. Here, the main goal is to prove well-posedness results for (N) in a general setting. We assume:

$$Lu(x) = \text{pv.}\int_{\mathbb{R}^d}\big(u(x)-u(y)\big)k(x,y)\mathrm{d}y \qquad (x\in\mathbb{R}^d)\,,$$

$$\mathcal{N}u(y) = \int_\Omega (u(y)-u(x))k(x,y)\mathrm{d}x \qquad (y\in\Omega^c)\,.$$

Here, $k:\mathbb{R}^d\times\mathbb{R}^d\setminus\text{diag}\to[0,\infty)$ is measurable and satisfies

$$(\text{E}) \qquad \Lambda^{-1}\nu(y-x)\leq k(x,y)\leq\Lambda\nu(y-x)\quad (x,y\in\mathbb{R}^d)\,,$$

where $\nu:\mathbb{R}^d\setminus\{0\}\to[0,\infty)$ is the density of a symmetric Lévy measure,

$$(\text{L}) \qquad \nu(h)=\nu(-h) \text{ for all } h\neq 0 \quad\text{and}\quad \int_{\mathbb{R}^d}\big(1\wedge|h|^2\big)\nu(h)\mathrm{d}h<\infty\,.$$

In short, in [FG20] we provide a general framework that includes integrable and singular kernels at the same time. Extending previous results, e.g. from [DROV17], [MPL19], [DTZ22], we study the inhomogeneous problem for natural choices of data g. Note that some of the results are new even for the fractional Laplace Operator.

Let us explain condition (E). Note that, in the case $k(x,y) = \nu(y-x)$ with ν as above, the operator L is translation invariant and generates a symmetric Lévy process. The density ν defines the "order" of the operator L, which becomes apparent in the case of $\nu(h) = C_{d,\alpha}|h|^{-d-\alpha}$ for $h \neq 0$ where $\alpha \in (0,2)$ is fixed and $C_{d,\alpha}$ is an appropriate constant. The resulting operator is the so called fractional Laplace operator $(-\Delta)^{\alpha/2}$. For details about the fractional Laplace operator $(-\Delta)^{\alpha/2}$ and the constant $C_{d,\alpha}$ we refer to [NPV12, FG20]. It is worth mentioning that the nonlocal operator $\mathcal{N}$ was initially introduced by [DROV17]. Another type of such an operator appeared earlier in the literature, see for instance [DGLZ12].

Let us quickly review the classical Neumann problem. Let $\Omega \subset \mathbb{R}^d$ be a bounded open subset whose boundary $\partial\Omega$ is sufficiently regular. Given $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ measurable, the classical inhomogeneous Neumann problem consists in finding a function $u : \Omega \to \mathbb{R}$ satisfying

$$-\Delta u = f \quad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial u}{\partial n} = g \text{ on } \partial\Omega. \tag{1}$$

Here $\frac{\partial u}{\partial n}$ denotes the outward normal derivative of u on $\partial\Omega$. It is interesting to note that the Neumann boundary problem has received considerably less attention in the literature when compared with the Dirichlet boundary problems.

Following [FKV15, SV14] we introduce a bilinear form $\mathcal{E}$ by

$$\mathcal{E}(u,v) = \frac{1}{2} \int\int_{(\Omega^c\times\Omega^c)^c} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\, \nu(x-y)\mathrm{d}x\,\mathrm{d}y \tag{2}$$

for all smooth functions with compact support. As in the local case, a main tool in the study of Neumann problems, is a Gauss-Green type formula for $u,v \in C_c^\infty(\mathbb{R}^d)$:

$$\int_\Omega Lu(x)v(x)\mathrm{d}x = \mathcal{E}(u,v) - \int_{\Omega^c} \mathcal{N}u(y)v(y)\mathrm{d}y. \tag{3}$$

Relation (3) motivates us to introduce an energy space $V_\nu(\Omega|\mathbb{R}^d)$ as the vector space of all measurable functions $u : \mathbb{R}^d \to \mathbb{R}$ such that the restriction $u|_\Omega$ belongs to $L^2(\Omega)$ and $\mathcal{E}(u,u)$ is finite. The energy space $V_\nu(\Omega|\mathbb{R}^d)$ can be seen as a nonlocal analog of $H^1(\Omega)$.

Let us summarize the main results:

(1) The first step is to define a base space $L^2(\mathbb{R}^d;\widetilde{\nu})$, in which we can define the complement value problems. We define $\widetilde{\nu}$ and two altervative options $\overline{\nu},\nu^*$. We study embedding results of corresponding function spaces.
(2) The next step is to introduce $T_\nu(\Omega^c)$ as the trace space of $V_\nu(\Omega|\mathbb{R}^d)$. We provide equivalent norms on the trace space and a density result. We show that the trace spaces introduced in [DK20] and [BGPR20] coincide.

(3) An important tool in the proof of well-posedness results is the compact embedding $V_\nu(\Omega|\mathbb{R}^d) \hookrightarrow L^2(\Omega)$, which is a core result.
(4) We focus on the Neumann problem but we also discuss a more general Robin-type complement value problem.
(5) We propose a Dirichlet form for the study of reflected jump processes, which is different from the one in [Von21].
(6) The setup allows to define a fully nonlocal Dirichlet-to-Neumann map with the help of the nonlocal Neumann-type derivative $\mathcal{N}$. For $\Omega \subset \mathbb{R}^d$, the Dirichlet data are given on Ω^c and mapped to $\mathcal{N}u$ on Ω^c, where u satisfies the nonlocal equation in Ω. Thus, this map is a nonlocal analog of the well-known Dirichlet-to-Neumann operator given in [CS07].
(7) The analogy between the classical Neumann problem and problem (N) leads to a convergence result when considering a sequence of complement value problems for the fractional Laplace operator $(-\Delta)^{\frac{\alpha_n}{2}}$ where $\alpha_n \to 2$. We establish the convergence of the corresponding sequence of solutions u_{α_n}.

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Sharp heat kernel estimates for Markov process with singular jump

Kyung-Youn Kim
(joint work with Lidan Wang)

Let $Z = (Z^1, \dots, Z^d)$ be d-dimensional Lévy processes, where Z^i's are independent 1-dimensional Lévy processes with jump kernel $\nu^1(u,w) = |u-w|^{-1}\phi(|u-w|)^{-1}$ for $u, w \in \mathbb{R}$. Here ϕ is an increasing function satisfying that for $\underline{\alpha}, \overline{\alpha} \in (0,2)$, there exists a positive constant $c > 1$ such that

$$(\mathbf{WS}) \qquad c^{-1}\left(\frac{R}{r}\right)^{\underline{\alpha}} \le \frac{\phi(R)}{\phi(r)} \le c\left(\frac{R}{r}\right)^{\overline{\alpha}} \qquad \text{for } 0 < r < R.$$

For $x, y \in \mathbb{R}^d$, define

$$\nu(x,y) := \begin{cases} \nu^1(x^i, y^i) & \text{if } x^i \neq y^i \text{ for some } i \text{ and } x^j = y^j \text{ for all } j \neq i, \\ 0 & \text{if } x^i \neq y^i \text{ for more than one index } i. \end{cases}$$

Since Z^i's are independent, the transition density $p^Z(t,x,y)$ of Z has the following estimates: there exists a positive constant $c_1 > 1$ such that for any $t > 0, x, y \in \mathbb{R}^d$,

$$c_1^{-1}\prod_{i=1}^d \big([\phi^{-1}(t)]^{-1} \wedge t\nu^1(|x^i - y^i|)\big) \le p^Z(t,x,y) \le c_1 \prod_{i=1}^d \big([\phi^{-1}(t)]^{-1} \wedge t\nu^1(|x^i - y^i|)\big).$$

The following conjecture is formulated in [7]:

Conjecture: *Let L_t be a Lévy process (a non-degenerate α-stable process) in $\mathbb{R}^d$ with Lévy measure μ. Let M_t be a symmetric Markov process whose Dirichlet form has a symmetric jump intensity $j(x,dy)$ that is comparable to the one of L_t, i.e., $j(x,dy) \asymp \mu(x-dy)$. Then the heat kernel of M_t is comparable to the one of L_t.*

Denote $f \asymp g$ if f/g is comparable to some positive constants in the domains of f and g. Let $J(x,y)$ be a symmetric measurable function comparable to $\nu(x,y)$, and define a symmetric bilinear form $\mathcal{E}(\cdot,\cdot)$ on $L^2(\mathbb{R}^d)$ that

$$\mathcal{E}(u,v) := \int_{\mathbb{R}^d} \Big(\sum_{i=1}^d \int_{\mathbb{R}} \big(u(x+e^i h) - u(x)\big)\big(v(x+e^i h) - v(x)\big) J(x, x+e^i h) dh \Big) \mathrm{d}x\,;$$
$$\mathcal{D} := \{u \in L^2(\mathbb{R}^d) |\ \mathcal{E}(u,u) < \infty\}.$$

We prove the existence of a conservative Feller process $X = (X^1, \dots, X^d)$ associated to the (non-isotropic) regular Dirichlet form $(\mathcal{E}, \mathcal{D})$. Furthermore, X has a jointly continuous transition density function $p(t,x,y)$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$, which enjoys the following estimates: there exists a positive constant $c_2 > 1$ such that for any $t > 0, x, y \in \mathbb{R}^d$,

$$c_2^{-1}[\phi^{-1}(t)]^{-d} \prod_{i=1}^d \left(1 \wedge \frac{t\phi^{-1}(t)}{|x^i - y^i|\phi(|x^i - y^i|)}\right)$$
$$\le p(t,x,y) \le c_2[\phi^{-1}(t)]^{-d} \prod_{i=1}^d \left(1 \wedge \frac{t\phi^{-1}(t)}{|x^i - y^i|\phi(|x^i - y^i|)}\right).$$

The above gives the answer the conjecture for the general Lévy process Z since $p(t,x,y) \asymp p^Z(t,x,y)$. Also we prove that bounded harmonic functions of X is Hölder continuous.

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Elliptic (non-local) PDEs with singular data - the method of orthogonal projection

Tomasz Klimsiak

(joint work with A. Rozkosz)

We are concerned with the following Dirichlet type problem (see [2, 3, 4])

$$-Au = f(\cdot,u) + \mu \quad \text{in } D, \quad u = \gamma \quad \text{on } \partial_h D. \tag{1}$$

Here

- D is an open subset of E and $\partial_h D$ is the harmonic boundary of D;
- E is a locally compact separable metric space;
- m is a Radon measure on E with full support;
- $(-A, \mathfrak{D}(A))$ is a self-adjoint operator on $L^2(E;m)$ that generates a strongly continuous Markov semigroup $(T_t)_{t\geq 0}$:

$$0 \leq g \leq 1 \implies 0 \leq T_t g \leq 1;$$

- $f : E \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that is non-increasing with respect to the second variable;
- μ is a Borel measure on E and $\gamma : D^c \to \mathbb{R}$ is a Borel measurable function.

We deal with two basic problems: definition of a solution to (1), and the existence and uniqueness problem for (1). As to the problem of right definition of a solution to (1), we propose the method of orthogonal projection used for the first time in context of the Dirichlet problem by S. Zaremba, O.M. Nikodym and H. Weyl (see [5, 6, 7]). The basic tool in the method is a Dirichlet form $(\mathcal{E}, \mathfrak{D}(\mathcal{E}))$, associated with $(-A, \mathfrak{D}(A))$, given by the following formula

$$\mathfrak{D}(\mathcal{E}) := \mathfrak{D}(\sqrt{-A}), \quad \mathcal{E}(u,v) := (\sqrt{-A}u, \sqrt{-A}v)_{L^2(E;m)}, \quad u, v \in \mathfrak{D}(\mathcal{E}).$$

We assume that $(\mathcal{E}, \mathfrak{D}(\mathcal{E}))$ is regular:

(A1) $\mathfrak{D}(\mathcal{E}) \cap C_c(E)$ is dense in $C_c(E)$ and in $\mathfrak{D}(\mathcal{E})$,

and that $(\mathcal{E}, \mathfrak{D}(\mathcal{E}))$ is transient:

(A2) there exists a strictly positive function g on E and a constant $c > 0$ satisfying

$$\int_E ug\, dm \le c\sqrt{\mathcal{E}(u,u)}, \quad u \in \mathfrak{D}(\mathcal{E}).$$

The last condition implies that $(\mathcal{E}, \mathfrak{D}(\mathcal{E}))$ may be completed to a Hilbert space. We denote this completion by $\mathfrak{D}_e(\mathcal{E})$. It is well known that $\mathfrak{D}_e(\mathcal{E}) \subset L^1(E; g \cdot m)$.

Potential theory

Consider a Choquet capacity $Cap : 2^E \to \mathbb{R}^+ \cup \{\infty\}$, which for open $U \subset E$ admits the form

$$Cap(U) := \inf\{\mathcal{E}(u,u) : u \in \mathfrak{D}(\mathcal{E}),\, u \ge \mathbf{1}_U \text{ a.e.}\}.$$

We say that a relation holds q.e. if it holds outside a set $N \subset E$ that satisfies $Cap(N) = 0$. A set $V \subset E$ is called *quasi-open* if for any $\varepsilon > 0$ there exists an open set G_ε containing V with $Cap(G_\varepsilon \setminus V) < \varepsilon$. Let $\mathcal{O}_q$ denote the family of all quasi-open subsets of E. A function $u : E \to \mathbb{R} \cup \{\pm\infty\}$ is called quasi-continuous if u is finite q.e. and $u^{-1}(I) \in \mathcal{O}_q$ for any open set $I \subset \mathbb{R}$. Any function $u \in \mathfrak{D}_e(\mathcal{E})$ has a quasi-continuous m-version (we always consider such versions). A function $u : E \to \mathbb{R} \cup \{\pm\infty\}$ is called nearly Borel if it equals a Borel function q.e.

Orthogonal projection

We let $F := \mathfrak{D}_e(\mathcal{E})$ and for any quasi-open $V \subset E$ we let $F(V) := \mathfrak{D}_e(\mathcal{E}_{|V})$, which is a closed linear subspace of F - here $\mathcal{E}_{|V}$ is the restriction of the form $\mathcal{E}$ to V. Consequently,

$$F = F(V) \oplus F(V)^{\perp}.$$

We let

$$\Pi_V : F \to F(V)$$

denote the orthogonal projection onto $F(V)$, and $H_V := Id_F - \Pi_V$. By [1] there exists a family of *sub-stochastic kernels* $\{P_V(x,dy),\, x \in E,\, V \in \mathcal{O}_q\}$ on E such that

$$H_V(\gamma)(x) = \int_{V^c} \gamma(y)\, P_V(x,dy), \quad q.e. \tag{2}$$

for any $\gamma \in F$ and quasi-open $V \subset E$. We introduce the following notions (harmonic boundary ∂_h and harmonic closure cl_h): for any $V \in \mathcal{O}_q$

$$\mathcal{S}(V) := \bigcup_{x \in V} \mathrm{supp}[P_V(x,dy)], \quad \partial_h V := \mathcal{S}(V) \setminus V, \;\; cl_h V := V \cup \partial_h V.$$

Two crucial spaces connected with problem (1)

Definition 1. Let $W \in \mathcal{O}_q$. We say that a family $\mathcal{Q} \subset \mathcal{O}_q$ is *W-total* if

- (i) $\bigcup_{V \in \mathcal{Q}} V = W$ q.e.,
- (ii) $V_1, V_2 \in \mathcal{O}_q$, $V_1 \subset V_2$, $V_2 \in \mathcal{Q}$ implies $V_1 \in \mathcal{Q}$,
- (iii) if $V_1, V_2 \in \mathcal{Q}$, then $V_1 \cup V_2 \in \mathcal{Q}$.

We let

$$F_{q.loc}(W) := \{u \in \mathcal{B}^n(E) : \exists W\text{-total family } \mathcal{Q} \subset \mathcal{O}_q : \Pi_V(u) \in F(V),\, V \in \mathcal{Q}\}.$$

For any $u \in F_{q.loc}(W)$ we denote by $\mathcal{Q}_W[u]$ the class of all W-total families $\mathcal{Q}$ such that $\Pi_V(u) \in F(V)$, $V \in \mathcal{Q}$.

For given strictly positive function $\rho : E \to \mathbb{R}$, satisfying $\|\rho\|_{L^1} = 1$, and nearly Borel function $u : E \to \mathbb{R}$, we denote

$$\|u\|_{\mathbb{D}^1_\rho} := \sup_{V \in \mathcal{O}_q} \|H_V(|u|)\|_{L^1_\rho}.$$

We let $\mathbb{D}^1_\rho$ denote the space of all nearly Borel functions $u : E \to \mathbb{R}$ such that

$$\|(|u| - n)^+\|_{\mathbb{D}^1_\rho} \to 0, \quad n \to \infty,$$

and we let $\mathbb{D}^1 := \bigcup \mathbb{D}^1_\rho$, where the sum is over all ρ as in the foregoing.

Smooth measures

Definition 2. A Borel measure μ is called smooth if

- (i) $\mu \ll Cap$,
- (ii) there exists a strictly positive quasi-continuous function $u : E \to \mathbb{R}$ such that $\int_E u\, d|\mu| < \infty$.

It is well known that for any smooth measure μ there exists a total family $\mathcal{Q}$ such that $\mathbf{1}_V \cdot \mu \in F^*$, $V \in \mathcal{Q}$. We let $\mathcal{R}_D[\mu]$ denote the class of all D-total families $\mathcal{Q}$ such that $\mathbf{1}_V \cdot \mu \in F^*$, $V \in \mathcal{Q}$. Observe that for any positive smooth measure μ, we may define $R^D\mu :=\uparrow \lim_{n\to\infty} R^D(\mathbf{1}_{V_n} \cdot \mu)$, where R^D is the potential operator of the form $\mathcal{E}_{|D}$, i.e. $R^D : [F(D)]^* \to F(D)$,

$$\mathcal{E}_{|D}(R^D\eta, v) = \langle \eta, v \rangle_{[F(D)]^*, F(D)}, \quad v \in F(D).$$

Here (V_n) is an increasing sequence of $V_n \in \mathcal{Q}$ ($\in \mathcal{R}_D[\mu]$) such that $\bigcup_{n\geq 1} V_n = D$ q.e.

Main results

Definition 3. A quasi-continuous function $u : E \to \mathbb{R}$ is a solution to (1) if

(i) $u \in F_{q.loc}(D) \cap \mathbb{D}^1$,
(ii) $R^D|f(\cdot,u)| < \infty$ m-a.e.,
(iii) for some $\mathcal{Q} \in \mathcal{Q}_D[u] \cap \mathcal{R}_D[f(\cdot,u)] \cap \mathcal{R}_D[\mu]$ and any $V \in \mathcal{Q}$,

$$\begin{aligned}\mathcal{E}(\Pi_V(u), \eta - \Pi_V(u)) \geq \langle \mathbf{1}_V f(\cdot,u), \eta - \Pi_V(u)\rangle_{F^*,F} \\ + \langle \mathbf{1}_V \cdot \mu, \eta - \Pi_V(u)\rangle_{F^*,F}, \quad \eta \in F(V),\end{aligned}$$

(iv) $u = \gamma$ q.e. on $\partial_h D$.

Theorem 4. *There exists at most one solution to* (1).

Theorem 5. *Assume that*

(1) μ is a smooth measure such that $R^D|\mu| < \infty$ m-a.e.,
(2) $\gamma \in \mathbb{D}^1$ is quasi-continuous,
(3) for any $y \in \mathbb{R}$, the function $x \mapsto f(x,y)$ belongs to $L^1(E;m)$,
(4) $R^D|f(\cdot,0)| < \infty$ m-a.e.

Then there exists a solution to (1).

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Generalized couplings for stochastic systems with delays and non-localities

Oleksii Kulyk

We develop a new technique which makes it possible to study stochastic differential equations whose coefficients are assumed to be only *Hölder continuous*, and which does not rely on analytical results from the PDE theory. The analytic approach to the study of diffusion processes dates back to Kolmogorov, and nowadays is a common tool for the analysis of SDEs with low regularity of coefficients; e.g. [15]. For stochastic systems of more complicated structure, e.g. those described by stochastic equations with delay, this approach is not realistic because of the necessity to study PDEs in (infinite-dimensional) functional spaces. For such systems, the Itô-Lévy stochastic approach is typically used which requires (one-sided local) Lipschitz continuity of the coefficients; e.g. [13] or [14]. It appears that the range of application of the standard stochastic analysis tools can be substantially extended, including delay equations with low regularity of the coefficients and stochastic differential equations with jumps.

Our approach is based on the concept of *generalized coupling*, which extends the classical notion of *coupling* in the following way. By definition, a coupling is a probability measure on a product space with prescribed marginal distributions. For a *generalized* coupling the marginals satisfy instead milder deviation bounds from the prescribed distributions. The class of generalized couplings is much wider than of classical couplings, and it is typically much easier to construct for a given system a generalized coupling with desired properties than a true one. This makes generalized couplings quite an efficient tool in the ergodic theory of Markov processes, see the recent paper [4] where they were used as a key ingredient in the construction of contracting/nonexpanding distance-like functions for complicated SPDE models.

In [4], generalized couplings were first constructed using stochastic control arguments, and then used for the construction of true couplings; in this last step the change of the marginal laws caused by the control terms was in a sense reimbursed. We call this type of argument a *Control-and-Reimburse (C-n-R)* strategy. The same general idea — to apply a stochastic control in order to improve the system, and then to take into account the impact of the control — is scattered in the literature; e.g. it is used in [9, Section 5.2] in a construction of of contracting/nonexpaning distance-like function $d(x, y)$ for delay equations, in [8] in an approach to the study of weak ergodicity of SPDEs, in [1] in the proof of ergodicity in total variation for degenerate diffusions, and in [3] in the proof of ergodicity in total variation for solutions to Lévy driven SDEs. Related ideas were used to establish the Harnack inequality for SDEs and SFDEs [16, 7].

In [12], in the framework of stochastic delay equations, this general idea was further developed in the following two directions. First, it was shown that the C-n-R trategy is well applicable under just Hölder continuity assumptions on the coefficients (actually, one-sided Hölder continuity for the drift). This makes it possible to establish ergodic rates for delay equations with non-Lipschitz coefficients.

Moreover, essentially the same generalized coupling construction allows one to prove well-posedness of the system, i.e. that the weak solution to the equation is uniquely defined and the corresponding *segment process* is a time-homogeneous Markov process with the Feller property. Second, stabilization rates for *sensitivities* for the model(that is, for the derivatives of the semigroup rather than for the semigroup itself) were obtained. The natural and commonly adopted way to get such rates in a finite-dimensional setting is based on the Bismut-Elworthy-Li type formulae ([2], [6]) which give integral representations of sensitivities based on the integration-by-parts formulae. Such a *regularization* effect in an infinite-dimensional setting becomes much more structure demanding, since the random noise (which is the source of the integration-by-parts formula) needs to be non-degenerate in the entire space; for one result of such type and a detailed discussion see [5], where reaction-diffusion equations with a cylindrical noise are considered. In the delay case the noise is finite-dimensional and thus is strongly degenerate; hence the Bismut-Elworthy-Li type formula for the (Fréchet) derivatives of the semigroup is hardly available. Nevertheless, the C-n-R strategy allows one to derive a family of representation formulae for these derivatives, which can be understood as 'poor man's Bismut-Elworthy-Li type formulae', see (2.21) and (6.27) in [12]. These formulae are not completely free from gradient terms like ∇f, but the weights in the corresponding integral expressions can be forced to decay exponentially fast at an arbitrarily large rate. Using these representation formulae one can manage to establish stabilization rates for sensitivities (derivatives) of arbitrary order; note that the (full) regularization effect now has no reason to appear, and thus for these results certain smoothness of the coefficients is to be assumed.

The C-n-R strategy can also provide a considerable help in a study of general Lévy driven stochastic equations differential equations. Analytically, the differential Kolmogorov equations for such stochastic models involve non-local integro-differential operators, and the heat kernels of the associated Markov processes are identified as the fundamental solutions to corresponding Cauchy problems. However, a close study of the matter reveals that, for complicated Lévy-driven models, the heat kernel properties may become substantially complicated and the associated analytical methods may require structural limitations on the underlying Lévy structure, e.g. [10], [11]. The stochastic calculus approach, based on the generalized couplings, is free from any such limitations and allows one both to prove well-posedness and to establish ergodic rates for general Lévy-driven stochastic equations with non-Lipschitz coefficients.

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Estimation and selection of ergodic Lévy driven SDE: an overview of some recent developments

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(joint work with Shoichi Eguchi)

Setup and objective. Suppose that we observe an equally spaced high-frequency sample $X^{(n)} = (X_{t_j^n})_{j=0}^n$ for $t_j^n = t_j = jh$, where $X = (X_t)_{t\in\mathbb{R}_+}$ is a solution to the d-dimensional stochastic differential equation (SDE):

$$dX_t = A(X_t)dt + C(X_{t-})dZ_t, \tag{1}$$

where $A : \mathbb{R}^d \to \mathbb{R}^d$ and $C : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^r$, and $Z = (Z_t)_{t\in\mathbb{R}p}$ is a non-Gaussian r-dimensional Lévy process independent of the initial variable X_0. We suppose that $E[Z_1] = 0$ and $\mathrm{cov}[Z_1] = I_r$ (the r-dimensional identity matrix), and that $E[|Z_1|^q] < \infty$ for every $q > 0$. The sampling stepsize $h = h_n > 0$ is a known real number such that $T_n := nh \to \infty$ and $nh^2 \to 0$ as $n \to \infty$. We wish to infer the coefficients A and C from $X^{(n)}$, without specifying the distribution $\mathcal{L}(Z)$.

Suppose that we are given the following *statistical (candidate) models* for the *data generating process* (1):

- $c_1(x,\gamma_1), \dots, c_{M_1}(x,\gamma_{M_1})$ for the scale $C(x)$;
- $a_1(x,\alpha_1), \dots, a_{M_2}(x,\alpha_{M_2})$ for the drift $A(x)$.

Then, each candidate model $\mathcal{M}_{m_1,m_2}$ is described by

$$dX_t = a_{m_2}(X_t, \alpha_{m_2})dt + c_{m_1}(X_{t-}, \gamma_{m_1})dZ_t, \tag{2}$$

where $\gamma_{m_1} \in \Theta_{\gamma_{m_1}} \subset \mathbb{R}^{p_{\gamma_{m_1}}}$ ($m_1 = 1, \dots, M_1$) and $\alpha_{m_2} \in \Theta_{\alpha_{m_2}} \subset \mathbb{R}^{p_{\alpha_{m_2}}}$ ($m_2 = 1, \dots, M_2$) are finite-dimensional unknown parameters. We assume that each model $\mathcal{M}_{m_1,m_2}$ is correctly specified: $c_{m_1}(\cdot, \gamma_{m_1,0}) = C(\cdot)$ and $a_{m_2}(\cdot, \alpha_{m_2,0}) = A(\cdot)$ for some $\gamma_{m_1,0} \in \Theta_{\gamma_{m_1}}$ and $\alpha_{m_2,0} \in \Theta_{\alpha_{m_2}}$.

With this setup, we are interested which coefficient is *relatively the best* one among the candidates $\{\mathcal{M}_{m_1,m_2}\}_{m_1,m_2}$. We have shown how Akaike's AIC and Schwarz's BIC, the classic twin jewels for relative model assessment, can apply.

Two-stage Gaussian quasi-likelihood function. For notational brevity, removing the model indices we look at a single model

$$dX_t = a(X_t, \alpha)dt + c(X_{t-}, \gamma)dZ_t, \tag{3}$$

where $\gamma = (\gamma_k) \in \Theta_\gamma \subset \mathbb{R}^{p_\gamma}$ and $\alpha = (\alpha_l) \in \Theta_\alpha \subset \mathbb{R}^{p_\alpha}$, both parameter spaces being bounded convex domains. The Euler approximation for (3) under P_θ (image measure of X assciated with $\theta := (\alpha, \gamma)$) is given by

$$X_{t_j} \approx X_{t_{j-1}} + a_{j-1}(\alpha)h + c_{j-1}(\gamma)\Delta_j Z. \tag{4}$$

Taking the small-time (fake) Gaussian approximation ($S(x,\gamma) := c(x,\gamma)^{\otimes 2}$)

$$\mathcal{L}(X_{t_j} | X_{t_{j-1}} = x) \approx N_d\left(x + a(x,\alpha)h, hS(x,\gamma)\right)$$

into account, we introduce the joint *Gaussian quasi-likelihood (GQLF,* [4]*)*

$$\mathbb{H}_n(\theta) = \mathbb{H}_n(\theta; X^{(n)}) := \sum_{j=1}^n \log \phi_d\left(X_{t_j};\, X_{t_{j-1}} + a_{j-1}(\alpha)h,\, hS_{j-1}(\gamma)\right) \tag{5}$$

with $\phi_d(\cdot; \mu, \Sigma)$ denoting the d-dimensional $N_d(\mu, \Sigma)$-density. We can write $\mathbb{H}_n(\theta) = \mathbb{H}_{1,n}(\gamma) + \mathbb{H}_{2,n}(\theta)$ where, with the multilinear-form notation,

$$\mathbb{H}_{1,n}(\gamma) := \sum_{j=1}^n \log \phi_d\left(X_{t_j};\, X_{t_{j-1}},\, hS_{j-1}(\gamma)\right),$$

$$\mathbb{H}_{2,n}(\theta) := \sum_{j=1}^n \left(S_{j-1}^{-1}(\gamma)\left[\Delta_j X,\, a_{j-1}(\alpha)\right] - \frac{h}{2} S_{j-1}^{-1}(\gamma)\left[a_{j-1}^{\otimes 2}(\alpha)\right] \right).$$

The joint GQLF $\mathbb{H}_n(\theta)$ has two distinct "resolutions", which occurs since the last term $c_{j-1}(\gamma)\Delta_j Z$ in the right-hand side of (4) is stochastically dominant compared with the second one $a_{j-1}(\alpha)h$. As in [5], it can be seen under suitable conditions that both $n^{-1}\mathbb{H}_{1,n}(\gamma)$ and $T_n^{-1}\mathbb{H}_{2,n}(\alpha, \gamma)$ have non-trivial limits (of the ergodic theorem) for each θ, the former limits depending on γ only.

Estimation. We suggested the following *two-stage* estimation strategy:

- First, we estimate γ by $\hat{\gamma}_n \in \mathrm{argmax}_\gamma \mathbb{H}_{1,n}(\gamma)$;
- Next, for $\mathbb{H}_{2,n}(\alpha) := \mathbb{H}_{2,n}(\alpha, \hat{\gamma}_n)$, we estimate α by $\hat{\alpha}_n \in \mathrm{argmax}_\alpha \mathbb{H}_{2,n}(\alpha)$.

Our regularity conditions include sufficient smoothness of the coefficients and the uniform non-degeneracy of the scale matrix $S(x, \gamma)$, the exponential ergodicity and certain moment boundedness, and the parameter-identifiability condition. Then, we can deduce the following convergence of the scaled *Gaussian quasi-MLE (GQMLE)* $\hat{u}_n := \sqrt{T_n}(\hat{\theta}_n - \theta_0)$: for some explicitly given $V(\theta_0)$, we have

$$\lim_n E\left[f(\hat{u}_n)\right] = \int f(u)\phi(u; 0, V(\theta_0))du \tag{6}$$

for any continuous function $f : \mathbb{R}^p \to \mathbb{R}$ of at most polynomial growth; in particular, we have $E(\hat{u}_n) \to 0$ and $E(\hat{u}_n^{\otimes 2}) \to V(\theta_0)$. The mighty mode of convergence (6) of the GQMLE is important in many situations in theoretical statistics. Indeed, it is the basis for developing the information criteria given below.

Selection. We consider both AIC- and BIC-type statistics through the GQLFs $\mathbb{H}_{1,n}$ and $\mathbb{H}_{2,n}$ in this order. The two ICs are based on the different philosophies:

- AIC for *prediction* purpose: AIC is designed to pick up the best predictive model for an independent data set with the same distribution;
- BIC for *model-description* purpose: BIC is based on the marginal likelihood in the Bayesian setup with a prior distribution of θ and designed to pick the most simple true (descriptive) model.

They may select different models, and neither one is universally better than the other; we refer to [1] for a historical and systematic account.

Recall our model setup (2). In [3], it turned out that *stepwise* procedures do efficiently work for both AIC and BIC. The specific formulae of the proposed information criteria are given as follows; we keep omitting the model indices (m_1, m_2).

For the AIC-type, first we select a scale-coefficient model as a minimizer of

$$\mathrm{GQAIC}_{1,n} := -2\,\mathbb{H}_{1,n}(\hat{\gamma}_n) + \frac{2}{h}\mathrm{trace}\left(\hat{\Gamma}_{\gamma,n}^{-1}\hat{W}_{\gamma,n}\right) \tag{7}$$

over the candidates $c_1(x, \gamma_1), \ldots, c_{M_1}(x, \gamma_{M_1})$, and then, building on the selected (and estimated) scale coefficient, we select a drift-coefficient model by minimizing

$$\mathrm{GQAIC}_{2,n} := -2\,\mathbb{H}_{2,n}(\hat{\alpha}_n) + 2p_\alpha. \tag{8}$$

over the candidates $a_1(x, \alpha_1), \ldots, a_{M_2}(x, \alpha_{M_2})$; the second stage GQAIC is of the standard form in the correctly specified setting, see [1].

For the BIC-type, we follow the same way, with replacing (7) and (8) by

$$\mathrm{GQBIC}_{1,n} = -2\,\mathbb{H}_{1,n}(\hat{\gamma}_n) + \frac{p_\gamma}{h}\log T_n \tag{9}$$

and

$$\mathrm{GQBIC}_{2,n} = -2\,\mathbb{H}_{2,n}(\hat{\alpha}_n) + p_\alpha \log T_n, \tag{10}$$

respectively; (10) is the same as in the diffusion case [2], while (9) is essentially different. All the details can be found in [3].

Remark. Notably, our studies revealed some non-standard features.

- Although the model is regular (smooth in the parameters), concerned with the selection of the scale coefficients we must partly employ the non-standard forms for both AIC and BIC statistics.
- Especially for the BIC-type methodology, it turned out that, to execute consistent model selection, we cannot follow the classic route where we show stochastic expansion of the marginal (quasi-)likelihood; instead, approximating the "heated up" free energy seems to be the right way.

These annoying features are essentially due to the mixed-rates structure in the sense of [6] of the joint GQLF (5), which does not emerge for the case of diffusions where Z is a standard Wiener process. The features can be sidestepped through the stepwise procedure introduced above.

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On the optimality of refraction–reflection strategies for Lévy processes

Kei Noba

1. Introduction

This talk is based on [3], where we deal with optimal dividend problem with capital injections for Lévy processes. We assume that the assets of an enterprise behave as a Lévy process $X = \{X_t : t \geq 0\}$. This enterprise pays dividends to shareholders when it can afford the assets. On the other hand, when the enterprise does not have much in the way of assets, it receives capital injections from shareholders to avoid bankruptcy. In this setting, the enterprise wants to know the strategy to pay larger dividends and receive smaller capital injections. The optimal dividend problem when X is a spectrally negative Lévy process was studied by [1]. Since then, a number of researchers have studied the problem when X is a spectrally negative process. On the other hand, a method for solving the optimal dividend problem dealing with general Lévy processes, which may have both positive and negative jumps, has been developed in [2].

In [3] we deal in particular with the case where the stochastic process for the cumulative amount of dividends is absolutely continuous with respect to the Lebesgue measure and its density is suppressed by $\alpha > 0$. The optimal dividend problem with capital injections under such condition was solved by [4] when X is a spectrally positive Lévy process and by [5] when X is a spectrally negative Lévy process. In [3], I extended the main results of [4] and [5] to general Lévy processes. Here we summarise the setting of the problem and the main result.

2. Preliminaries

2.1. Lévy processes. Let $X = \{X_t : t \geq 0\}$ be a real-valued Lévy process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For all $x \in \mathbb{R}$, we denote $\mathbb{P}_x$ the law of X when it starts from x. We write Ψ for the characteristic exponent of X, which satisfies

$$e^{-t\Psi(\lambda)} = \mathbb{E}_0\left[e^{i\lambda X_t}\right], \quad \lambda \in \mathbb{R}, \quad t \geq 0.$$

The characteristic exponent Ψ has the form

$$\Psi(\lambda) = -i\gamma\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}\backslash\{0\}} (1 - e^{i\lambda x} + i\lambda x 1_{|x|<1})\Pi(dx), \quad \lambda \in \mathbb{R},$$

where $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and Π is a Lévy measure on $\mathbb{R}\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}\backslash\{0\}} (1 \wedge x^2)\Pi(dx).$$

When X has bounded variation paths, we write

$$\delta = \gamma - \int_{(-1,1)\backslash\{0\}} x\Pi(dx).$$

We write $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ for the natural filtration generated by X.

2.2. Refracted and refracted–reflected Lévy processes. We think about the following two cases:

Case 1 X has unbounded variation paths or has bounded variation paths with $\delta \notin [0, \alpha]$.

Case 2 X has bounded variation paths and $\delta \in [0, \alpha]$.

We define the function h^b with $b \in \mathbb{R}$ as

$$h^b(y) = \begin{cases} \alpha 1_{(b,\infty)}(y) & \text{in Case 1,} \\ \alpha 1_{(b,\infty)}(y) + \delta 1_{\{b\}}(y) & \text{in Case 2,} \end{cases} \qquad y \in \mathbb{R}.$$

Let $Y^b = \{Y^b_t : t \geq 0\}$ be the strong solution of a following stochastic differential equation:

$$Y^b_t = X_t - \int_0^t h^b(Y^b_s)ds, \quad t \geq 0. \tag{1}$$

The process Y^b is called a refracted Lévy process at b.

The refracted–reflected Lévy process $Z^b = \{Z^b_t : t \geq 0\}$ at $b \geq 0$ is defined by reflecting Y^b at 0, see [3, Section 2.3] for details.

2.3. **The setting of the problem.** We fix the cost per unit injected capital $\beta > 1$ and the discount rate $q > 0$. Let $\mathcal{A}$ be the set of a process $\pi = (L^\pi, R^\pi) = \{(L_t^\pi, R_t^\pi) : t \geq 0\}$ satisfying the following conditions.

(1) There exists process $l^\pi = \{l_t^\pi : t \geq 0\}$ that is progressively measurable with respect to $\mathbb{F}$ and satisfies

$$l_t^\pi \in [0, \alpha], \quad L_t^\pi = \int_0^t l_s^\pi ds, \quad t \geq 0.$$

(2) The process R^π is $\mathbb{F}$-adapted, non-decreasing, right-continuous,

$$X_t - L_t^\pi + R_t^\pi \geq 0, \quad t \geq 0,$$

and

$$\mathbb{E}_x \left[\int_{[0,\infty)} e^{-qt} dR_t^\pi \right] < \infty, \qquad x \in \mathbb{R}.$$

A process π which belongs to $\mathcal{A}$ is called a strategy. The expected net present value of the total dividends and capital injections when we use the strategy π is defined as

$$v_\pi(x) = \mathbb{E}_x \left[\int_{[0,\infty)} e^{-qt} dL_t^\pi - \beta \int_{[0,\infty)} e^{-qt} dR_t^\pi \right].$$

Our purpose is to find a strategy $\pi^* \in \mathcal{A}$ which satisfies

$$v_{\pi^*}(x) = \sup_{\pi \in \mathcal{A}} v_\pi(x), \quad x \in \mathbb{R}.$$

Such strategy π^* is called an optimal strategy.

2.4. **Refraction–reflection strategies.** Let π^b with $b \geq 0$ be the strategy which satisfies the following:

$$l_t^{\pi^b} = h^b(Z_t^b), \qquad R_t^{\pi^b} = - \inf_{s \in [0,t]} \left(\left(X_s - L_s^{\pi^b} \right) \wedge 0 \right), \qquad t \geq 0.$$

The strategy π^b is called the reflection–reflection strategy at b.

3. Main result

Then, we have the following main result.

Theorem 1. *We assume the following:*

(1) The Lévy measure Π *satisfies*

$$\int_{(-\infty,-1)} |x| \Pi(dx) < \infty.$$

(2) The stochastic differential equation (1) *has an unique strong solution.*

(3) When X *has bounded variation paths, the function* $x \mapsto \mathbb{E}_x \left[e^{-q\kappa_0^b} \right]$ *with* $b \geq 0$ *and* $\kappa_0^b = \inf\{t > 0 : Y_t^b < 0\}$ *has a locally bounded density* ν_b' *on* $(0, \infty)$ *with respect to the Lebesgue measure. In addition, the density* ν_b' *is continuous a.e. with respect to the Lebesgue measure.*

Then, the refraction–reflection strategy π^{b^*} at b^* is an optimal strategy, where

$$b^* = \inf\left\{b \geq 0 : \beta\mathbb{E}_b\left[e^{-q\kappa_0^b}\right] < 1\right\}.$$

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Local asymptotic normality for discretely observed ergodic jump-diffusion processes

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(joint work with Yuma Uehara)

Let X_t^α be a parametrized m-dimensional stochastic process satisfying

$$X_t^\alpha = x_0 + \int_0^t a(X_s^\alpha, \theta)ds + \int_0^t b(X_s^\alpha, \sigma)dW_s + \sum_{j=1}^{N_t} Y_i, \tag{1}$$

where a and b are $\mathbb{R}^m$- and $\mathbb{R}^m \otimes \mathbb{R}^m$-valued continuous functions, respectively, N_t is a Poisson process with intensity $\lambda(\theta)$, W_t is an m-dimensional standard Brownian motion, $(Y_i)_{i=1}^\infty$ is a sequence of independent, identically distributed random variable with a density function F_θ, and $\alpha = (\sigma, \theta) \in \mathbb{R}^d$ is a parameter to be estimated. Let $\alpha_0 = (\sigma_0, \theta_0)$ be the true value of the parameter, and let $X_t = X_t^{\alpha_0}$. We observe $(X_{t_k})_{k=0}^n$, where $t_k = kh_n$, $h_n \to 0$ and $nh_n \to \infty$ as $n \to \infty$. A quasi-maximum-likelihood estimator $(\hat{\sigma}_n, \hat{\theta}_n)$ for the parameter (σ_0, θ_0) is studied in Shimizu and Yoshida [4]. They constructed the estimator by using thresholding techniques that detect jumps, and showed the asymptotic normality of the estimator:

$$(\sqrt{n}(\hat{\sigma}_n - \sigma_0), \sqrt{nh_n}(\hat{\theta}_n - \theta_0)) \overset{d}{\to} N(0, \Gamma^{-1}) \tag{2}$$

for some positive definite matrix Γ under suitable conditions.

We consider the optimality of estimators. To investigate this, we consider the local asymptotic normality (LAN) of the statistical model. Let $\{P_{\alpha,n}\}_{\alpha,n}$ be a family of probability measures on measurable space $(\mathcal{X}_n, \mathcal{A}_n)$ and let α be a d-dimensional parameter. Then, $\{P_{\alpha,n}\}_{\alpha,n}$ satisfies the LAN property at $\alpha = \alpha_0$ if there exist a positive definite matrix Γ, a random vector $\mathcal{N}$, and a positive definite matrix ϵ_n such that $\mathcal{N} \sim N(0, I_d)$ and

$$\log \frac{dP_{\alpha_0+\epsilon_n u,n}}{dP_{\alpha_0,n}} \to u^\top \sqrt{\Gamma}\mathcal{N} - \frac{1}{2}u^\top \Gamma u$$

in $P_{\alpha_0,n}$-probability for any $u \in \mathbb{R}^d$.

Under the LAN property, any regular estimator $\{V_n\}$ satisfies

$$\liminf_{n\to\infty} E_{\alpha_0,n}[l(|\epsilon_n^{-1}(V_n-\alpha_0)|)] \geq E[l(|\Gamma^{-1/2}\mathcal{N}|)] \tag{3}$$

for any increasing function $l : [0,\infty) \to \mathbb{R}$ with $l(0) = 0$. An estimator which attains the lower bound of the above inequality is called asymptotically efficient.

When we try to show the LAN property of jump-diffusion processes, we need to specify the limit of $\log(dP_{\alpha_0+\epsilon_n u,n}/dP_{\alpha_0,n})$. It is difficult to deal with the transition density ratio for two different jump-diffusion processes. In the proof of the LAN property for diffusion processes in Gobet [1], the Aronson estimate

$$C_1G_1(x,y) \leq p_k(x,y) \leq C_2G_2(x,y)$$

is used for transition density p_k of the diffusion process to control transition density ratios, where G_1, G_2 are Gaussian density functions and C_1, C_2 are positive constants. However, it is difficult to obtain an Aronson-type estimate for jump-diffusion processes.

To avoid this problem, we employ the scheme with the L^2 regularity condition in Jeganathan [2]. Jeganathan [2] studied sufficient conditions for the local asymptotic mixed normality (LAMN, which is an extension of LAN) including the following L^2 regularity condition. Let $p_{k,\alpha}(x_{k-1},x_k)$ be the transition density function of an Markov process observations. We assume that $\alpha \mapsto p_{k,\alpha}$ is a C^2 function and the zero points of $p_{k,\alpha}$ do not depend on the parameter α. Moreover, we assume

$$\sum_{k=1}^n E_{\alpha_0}\left[\int \left(\sqrt{p_{k,\alpha_u}} - \sqrt{p_{k,\alpha_0}} - \frac{u^\top \epsilon_n \partial_\theta p_{k,\alpha_0}}{2\sqrt{p_{k,\alpha_0}}}\right)^2 (x_{k-1},x_k)dx_k\right] \to 0, \tag{4}$$

where ϵ_n is a scaling matrix, and $\alpha_u = \alpha_0 + \epsilon_n u$. This condition is called the 'L^2 regularity condition'. The integrand in the left-hand side of the above equation can be rewritten as follows;

$$\int_0^1 (1-s)\int \left\{u^\top \epsilon_n\left(\frac{\partial_\alpha^2 p_{k,\alpha_{su}}}{2p_{k,\alpha_{su}}} - \frac{\partial_\alpha p_{k,\alpha_{su}}\partial_\alpha p_{k,\alpha_{su}}^\top}{4(p_{k,\alpha_{su}})^2}\right)\epsilon_n u\right\}^2 p_{k,\alpha_{su}}dx_k ds.$$

In the last expression, only the transition density at α_{su} appears, and hence we do not need Aronson-type estimates for transition density functions. Jeganathan [2] showed the LAMN property for Markov processes under the L^2 regularity condition and some conditions for $\partial_\alpha^l \log p_{k,\alpha_0}$ ($l \in \{1,2\}$). This original scheme by [2] cannot be applied for jump-diffusion processes because the expectation appears in (4) is unbounded for jump-diffusion due to their fat-tailed behaviors. Therefore, we extend the scheme using a conditional expectation instead of the expectation so that it can be applied to jump-diffusion processes.

Another problem to show the LAN property is that the transition probability for no jump is quite different from that for the presence of jumps. This fact makes it difficult to identify the asymptotic behavior of the likelihood function. To deal with this problem, we approximate the original likelihood function by using a thresholding likelihood function that detects the existence of jumps.

As a consequence of these techniques, we obtain the LAN property for jump-diffusion processes; let $\{P_{\alpha,n}\}_{\alpha,n}$ be the family of probability measures generated by observations $(X^{\alpha}_{t_k})_{k=0}^{n}$. Then, $\{P_{\alpha,n}\}_{\alpha,n}$ satisfies LAN at $\alpha = \alpha_0$ with Γ the same as the one in Shimizu and Yoshida [4], and

$$\epsilon_n = \begin{pmatrix} n^{-1/2} I_{d_1} & 0 \\ 0 & (nh_n)^{-1/2} I_{d_2} \end{pmatrix}, \tag{5}$$

where d_1 and d_2 are the dimensions of the parameter σ and θ, respectively. Moreover, the quasi-maximum-likelihood estimator $(\hat{\sigma}_n, \hat{\theta}_n)$ in [4] is shown to be asymptotically efficient for any bounded, continuous loss function l.

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Nonlocal nonlinear Douglas identity

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(joint work with Krzysztof Bogdan, Tomasz Grzywny, Artur Rutkowski)

1. Introduction. Classical Hardy-Stein and Douglas formulas

Let $D = \{z \in \mathbb{R}^2 : |z| < 1\}$, $\partial D = S^1 \sim [0, 2\pi)$ be the unit disc and its boundary, and let $g : S^1 \to \mathbb{R}$ be a measurable function. Let

$$u(z) = \int_{S^1} g(\theta) P_D(z,\theta) \mathrm{d}\theta \text{ if the integral is defined , where } P_D(z,\theta) = \frac{1}{2\pi} \frac{1-|z|^2}{|z-\theta|^2}.$$

This extension of g from the circle to the unit disc is a harmonic function – for g regular enough one has $\Delta u = 0$. The following formula holds (J. Douglas [4]):

$$\int_D |\nabla u(z)|^2 \mathrm{d}z = \iint_{S^1 \times S^1} (g(\theta) - g(\eta))^2 \frac{1}{8\pi} \frac{1}{\sin^2((\theta-\eta)/2)} \mathrm{d}\theta \mathrm{d}\eta. \tag{1}$$

We also have (Hardy-Stein identity):

$$\frac{1}{2\pi}\int\limits_{S^1} |g(\theta)|^2 \mathrm{d}\theta = |u(0)|^2 + 2\int\limits_{D} G_D(0,z)|\nabla u(z)|^2 \mathrm{d}z,$$

where $G_D(\cdot,\cdot)$ is the classical Green function of the unit disc.

We intend to give similar formulas in the nonlocal setting – the Laplace operator Δ will be replaced by jump unimodal Lévy operators defined as follows. Let $\nu\colon [0,\infty) \to (0,\infty]$ be nonincreasing and $d \in \{1,2,\ldots\}$. Denote $\nu(z) = \nu(|z|)$, $z \in \mathbb{R}^d$ and $\nu(x,y) = \nu(|x-y|)$, $x,y \in \mathbb{R}^d$. Assume $\int_{\mathbb{R}^d} \nu(z)\mathrm{d}z = \infty$ and $\int_{\mathbb{R}^d} \left(|z|^2 \wedge 1\right) \nu(z)\mathrm{d}z < \infty$. For $x \in \mathbb{R}^d$ and $u : \mathbb{R}^d \to \mathbb{R}$ let

$$Lu(x) \quad = \quad \lim_{\epsilon\to 0^+} \int_{|x-y|>\epsilon} (u(y)-u(x))\nu(x,y)\,\mathrm{d}y.$$

The limit exists e.g.for $u \in C_c^2(\mathbb{R}^d)$. An example of such an operator is the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$, $\alpha \in (0,2)$. In this case, $\nu(z) = C_{d,\alpha}|z|^{-d-\alpha}$, $z \in \mathbb{R}^d$.

2. Nonlocal Sobolev-type spaces

For $x \in \mathbb{R}$ and $\kappa \in \mathbb{R}$ we write $x^{<\kappa>} = |x|^\kappa \operatorname{sgn}(x)$ and

$$F_p(a,b) = |b|^p - |a|^p - pa^{<p-1>}(b-a), \quad p>1, \quad a,b \in \mathbb{R}.$$

It is nonnegative as the second-order Taylor remainder of a convex function $|t|^p$, This is an example of a *Bregman divergence*. For $p=2$ we just have $F_2(a,b) = (a-b)^2$. In general, $F_p(a,b) \approx (a-b)^2(|a|+|b|)^{p-2} \approx (a-b)(a^{\langle p-1\rangle} - b^{\langle p-1\rangle})$.

Let $D \subset \mathbb{R}^d$ be open, nonempty and Lipschitz. Moreover, we assume that the Lévy kernel ν satisfies: (1) $\nu'' \in C(0,\infty)$, $|\nu'(r)|, |\nu''(r)| \leq c\nu(r)$ for $r>1$; (2) for certain $\beta \in (0,2)$ it holds $\nu(\lambda r) \leq c\lambda^{-d-\beta}\nu(r)$, for $0<\lambda$, $r \leq 1$, and $\nu(r) \leq c\nu(r+1)$, for $r \geq 1$; (3) for certain $\alpha \in (0,2)$ $\nu(\lambda r) \quad \geq \quad c\lambda^{-d-\alpha}\nu(r)$, for $0<\lambda$, $r\leq 1$. With these assumptions, we define by $G_D(\cdot,\cdot)$ the Green function of D relative to the operator L, and the nonlocal Poisson kernel – by $P_D(x,z) := \int_D G_D(x,y)\nu(y,z)\mathrm{d}y$. Further, we let

$$\gamma_D(w,z) = \int_D\int_D \nu(w,x)G_D(x,y)\nu(y,z)\,\mathrm{d}x\mathrm{d}y = \int_D \nu(w,x)P_D(x,z)\mathrm{d}x$$

be the *interaction kernel* of D.

For $p>1$ let we consider the expressions

$$\mathcal{E}_D^{(p)}[u] = \tfrac{1}{2}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} F_p(u(x),u(y))\nu(x,y)\mathrm{d}x\mathrm{d}y,$$

$$\mathcal{H}_D^{(p)}[g] \quad = \quad \tfrac{1}{p}\iint\limits_{D^c\times D^c} F_p(g(w),g(z))\gamma_D(w,z)\,\mathrm{d}w\mathrm{d}z.$$

and the spaces

$$\mathcal{V}_D^p := \{u|\ \mathcal{E}_D^{(p)}[u] < \infty\}, \qquad \mathcal{X}_D^p := \{g|\ \mathcal{H}_D^{(p)}[g] < \infty\}.$$

For $p = 2$ these expressions are quadratic forms, and the resulting spaces were first considered in [5, 7], where they were used for solving the nonlocal Dirichlet problem on D (see below).

3. The results

3.1. The Douglas identity.

Theorem 1 (Douglas identity, for $p = 2$ see [2], for general $p > 1$ see [3]). Let $p > 1$. Assume that the Lévy measure ν and the set $D \subset \mathbb{R}^d$ are as above. Then:

(i) Let $g\colon D^c \to \mathbb{R}$ be such that $\mathcal{H}_D^{(p)}[g] < \infty$. Then $P_D[g]$ is well-defined and satisfies

$$\mathcal{H}_D^{(p)}[g] = \mathcal{E}_D^{(p)}[P_D[g]].$$

(ii) Furthermore, if $u\colon \mathbb{R}^d \to \mathbb{R}$ satisfies $\mathcal{E}_D^{(p)}[u] < \infty$, then $\mathcal{H}_D^{(p)}[u|_{D^c}] < \infty$.

In particular, for $p = 2$ the identity reads

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} (u(x)-u(y))^2\nu(x,y)\mathrm{d}x\mathrm{d}y = \iint\limits_{D^c\times D^c} (g(w)-g(z))^2\gamma_D(w,z)\mathrm{d}w\mathrm{d}z$$

which is easily seen to be the nonlocal counterpart of the Douglas identity (1).

In this case, the Sobolev space $\mathcal{V}_D^{(2)}$ is suitable for solving the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{on } D, \\ u = g & \text{on } D^c, \end{cases}$$

and the space $\mathcal{X}_D^{(2)}$ is the optimal space for the exterior boundary condition g. See [5, 6].

In general, we have the following trace and extension theorem (for $p > 1$).

Corollary 1. Let $\operatorname{Ext} g = P_D[g]$ be the Poisson extension, and $\operatorname{Tr} u = u|_{D^c}$ – the restriction to D^c. Then $\operatorname{Ext}\colon \mathcal{X}_D^p \to \mathcal{V}_D^p$, $\operatorname{Tr}\colon \mathcal{V}_D^p \to \mathcal{X}_D^p$, and $\operatorname{Tr}\operatorname{Ext}$ is the identity operator on $\mathcal{X}_D^p$.

3.2. Hardy-Stein identity.

Main tool for obtaining the Douglas identity is the following.

Theorem 2 (Hardy-Stein identity [3]; for $\Delta^{\alpha/2}$ see [1]). If $u = P_D[g]$ and $x \in D$, then

$$\int_{D^c} |g(z)|^p\, P_D(x,z)\mathrm{d}z = |u(x)|^p + \int_D G_D(x,y)\int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y.$$

Moreover, we make use of the following useful relation:

Lemma 1. Let X be a random variable with $\mathbb{E}|X| < \infty$. Then,

$$\mathbb{E}F_p(\mathbb{E}X, X) = \mathbb{E}|X|^p - |\mathbb{E}X|^p \geq 0,$$

and

$$\mathbb{E}F_p(a, X) = F_p(a, \mathbb{E}X) + \mathbb{E}F_p(\mathbb{E}X, X), \quad a \in \mathbb{R}.$$

We refer to [2], [3] for further discussion, examples, and an extended list of references.

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Poisson process and sharp constants in L^p and Schauder estimates for a class of degenerate Kolmogorov operators

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(joint work with Lorenzo Marino, Stéphane Menozzi)

Let us start with the two-dimensional Kolmogorov example (cf. [8]). Namely, we are interested in the following Cauchy problem:

$$\begin{cases} \partial_t u(t,x,y) \,=\, \partial^2_{xx} u(t,x,y) + x\partial_y u(t,x,y) + f(t,x,y), \text{ on } [0,T]\times\mathbb{R}^2; \\ u(0,x,y) \,=\, 0, \text{ on } \mathbb{R}^2; \end{cases} \tag{1}$$

where (x,y) in $\mathbb{R}^2$; $T>0$ is fixed and $f\in C_0^\infty((0,T)\times\mathbb{R}^2)$. This is a degenerate parabolic Cauchy problem which has been also considered by Hörmander in [6].

It is not difficult to prove existence and uniqueness of a classical bounded solution u. Moreover, according to [2] there exists a constant $C_p>0$ independent of u and f such that

$$\|\partial^2_{xx} u\|_{L^p((0,T)\times\mathbb{R}^2)} \le C_p\ \|f\|_{L^p((0,T)\times\mathbb{R}^2)} = C_p\|\partial_t u - L^{\mathrm{Kol}} u\|_{L^p((0,T)\times\mathbb{R}^2)}, \tag{2}$$

where $p\in(1,+\infty)$ (we are considering L^p-spaces with respect to the Lebesgue measure; we set $L^{\mathrm{Kol}}u = \partial^2_{xx}u + x\partial_y u$). For related L^p-estimates, also called Sobolev estimates, concerning kinetic equations we refer to [1], [4], [7] and the references therein.

In [12] we deal with the more general Cauchy problem:

$$\begin{cases} \partial_t w(t,x,y) = \partial^2_{xx} w(t,x,y) + x\partial_y w(t,x,y) + s(t)\partial^2_{yy} w(t,x,y) + f(t,x,y), \\ w(0,x,y) \,=\, 0, \text{ on } \mathbb{R}^2. \end{cases} \tag{3}$$

Here $s(t)$ is a continuous and non-negative function defined on $[0,T]$. In [12] we prove in particular that the unique bounded solution w verifies

$$\|\partial^2_{xx} w\|_{L^p((0,T)\times\mathbb{R}^2)} \le C_p\ \|f\|_{L^p((0,T)\times\mathbb{R}^2)} \tag{4}$$

with the same constant C_p appearing in (2) (hence, C_p is independent of $s(t)$).

Remark 1. *In* [12] *to show the previous stability result for* (3) *we use a probabilistic approach based on the Poisson process. This approach has been introduced in* [9]*. There, it was established in particular that L^p-estimates like* (2) *for solutions to the heat equation are valid with constants that are independent of the dimension.*

The previous stability result for (3) *has two parts:*
(i) it shows that there exists $M_p > 0$ such that the solution w to the perturbed equation verifies:

$$\|\partial^2_{xx} w\|_{L^p((0,T)\times\mathbb{R}^2)} \le M_p \, \|f\|_{L^p((0,T)\times\mathbb{R}^2)};$$

(ii) it shows that actually $M_p = C_p$.

We do not know analytic methods to get even (i). Thus it remains a challenging open problem to have a purely analytic proof of our regularity results.

Remark 2. *In* [12] *we also consider related Schauder estimates and L^p-estimates of different type. In particular, starting from anisotropic Schauder estimates for solutions u to* (1)*:*

$$\sup_{0\le t\le T} \|u(t,\cdot)\|_{C^{2+\beta}_{b,d}} \le C_\beta \sup_{0\le t\le T} \|f(t,\cdot)\|_{C^{\beta}_{b,d}}, \tag{5}$$

for $\beta \in (0,1)$ (d is a distance on $\mathbb{R}^2$ related to L^{Kol}; cf. [5], [10], [11] *and the references therein) we can derive*

$$\sup_{0\le t\le T} \|w(t,\cdot)\|_{C^{2+\beta}_{b,d}} \le C_\beta \sup_{0\le t\le T} \|f(t,\cdot)\|_{C^{\beta}_{b,d}}$$

for the solution w to (3) *with the same constant C_β as before.*

In the sequel we will only discuss the main result of [12] concerning L^p-estimates like (4).

We consider $\mathbb{R}^N = \mathbb{R}^{d_0} \times \mathbb{R}^{d_1}$, d_0, d_1 are non-negative integers such that $d_0 + d_1 = N$, $d_0 \ge 1$. We introduce a non-negative symmetric matrix B in $\mathbb{R}^N \otimes \mathbb{R}^N$ given by

$$B = \begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix},$$

where B_0 is a symmetric, positive definite matrix in $\mathbb{R}^{d_0} \otimes \mathbb{R}^{d_0}$ such that

$$\nu \textstyle\sum_{i=1}^{d_0} \xi_i^2 \le \sum_{i,j=1}^{d_0} (B_0)_{ij}\xi_i\xi_j \le \frac{1}{\nu} \sum_{i=1}^{d_0} \xi_i^2,$$

for all $\xi \in \mathbb{R}^{d_0}$, for some $\nu > 0$. We define a possibly degenerate *Ornstein-Uhlenbeck operator* which generalizes L^{Kol} in (1):

$$L^{\mathrm{ou}} f(z) = \mathrm{Tr}(BD^2 f(z)) + \langle Az, Df(z)\rangle, \quad z = (x,y) \in \mathbb{R}^{d_0+d_1} = \mathbb{R}^N, \tag{6}$$

for A in $\mathbb{R}^N \otimes \mathbb{R}^N$, where $\langle\cdot,\cdot\rangle$ denote the usual inner product in $\mathbb{R}^N$.

We assume *the Kalman condition:*

[K] There exists a non-negative integer k, such that the vectors

$$\{e_1, \dots, e_{d_0}, Ae_1, \dots, Ae_{d_0}, \dots, A^k e_1, \dots, A^k e_{d_0}\} \quad \text{generate } \mathbb{R}^N, \tag{7}$$

where $\{e_i\}_{i\in\{1,\cdots,d_0\}}$ are the first d_0 vectors of the canonical basis for $\mathbb{R}^N$.

Assumption **[K]** is equivalent to the Hörmander condition on the commutators (see [6] and [5]) ensuring the hypoellipticity of $\partial_t - L^{\text{ou}}$. Note that L^{Kol} verifies **[K]** with $d_0 = 1$, $N = 2$ and $A := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

First one can prove existence and uniqueness of bounded regular solutions to

$$\begin{cases} \partial_t u(t,z) \;=\; L^{\text{ou}} u(t,z) + f(t,z), & \text{on } (0,T)\times\mathbb{R}^N; \\ u(0,z) \;=\; 0, & \text{on } \mathbb{R}^N, \end{cases} \tag{8}$$

when f belongs to $B_b\left(0,T; C_0^\infty(\mathbb{R}^N)\right)$ which contains $C_0^\infty((0,T)\times\mathbb{R}^N)$ (see [12] for the precise definition of such space). Equation (8) is understood in an integral form.

By [3] for any fixed p in $(1,+\infty)$, there exists $C_p = C_p(\nu, A, d_0, d_1, T)$ such that

$$\|D_x^2 u\|_{L^p((0,T)\times\mathbb{R}^N)} \;\le\; C_p \|\partial_t u - L^{\text{ou}} u\|_{L^p((0,T)\times\mathbb{R}^N)} \;=\; C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N)}; \tag{9}$$

here $D_x^2 u(t,z)$ is the Hessian matrix in $\mathbb{R}^{d_0}\otimes\mathbb{R}^{d_0}$ with respect to the variable x.

Fix a continuous map: $t \mapsto S(t) \in \mathbb{R}^N \otimes \mathbb{R}^N$ such that $S(t)$ is a *symmetric* and *non-negative definite*, $t \in [0,T]$; consider the following perturbation of L^{ou} :

$$\begin{aligned} L_t^{\text{ou},S} f(z) \;&:=\; \text{Tr}(BD^2 f(z)) + \text{Tr}(S(t)D^2 f(z)) + \langle Az, Df(z)\rangle \\ &=\; L^{\text{ou}} f(z) + \text{Tr}(S(t)D^2 f(z)), \end{aligned} \tag{10}$$

$z = (x,y)$ is in $\mathbb{R}^{d_0+d_1} = \mathbb{R}^N$. In the next result we denote by u_S the regular bounded solution of the Cauchy problem

$$\begin{cases} \partial_t u_S(t,z) \;=\; L_t^{\text{ou},S} u_S(t,z) + f(t,z), & \text{on } (0,T)\times\mathbb{R}^N; \\ u_S(0,z) \;=\; 0, & \text{on } \mathbb{R}^N, \end{cases} \tag{11}$$

Theorem 1 ([12])**.** Let us consider (11) with $f \in B_b\left(0,T; C_0^\infty(\mathbb{R}^N)\right)$. Then, the solution u_S verifies, with the same constant C_p as in (9),

$$\|D_x^2 u_S\|_{L^p((0,T)\times\mathbb{R}^N)} \le C_p \,\|\partial_t u_S - L_t^{\text{ou},S} u_S\|_{L^p((0,T)\times\mathbb{R}^N)} = C_p\,\|f\|_{L^p((0,T)\times\mathbb{R}^N)}.$$

Let us mention that our stability results can be useful to investigate the well-posedness of related martingale problems. We plan to consider this topic in the future. On this respect see also [13] and the references therein.

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On tree-based methods for (partial) differential equations

Nicolas Privault
(joint work with Guillaume Penent and Jiang Yu Nguwi)

Stochastic branching mechanisms have been used to represent the solutions of partial differential equations in [15], [7], [10], [8], and recently extended in [6] to the treatment of polynomial nonlinearities in first order gradient terms. This talk reviews an extension of such tree-based methods to functional nonlinearities with gradients of arbitrary orders.

Consider the ODE

$$u'(t) = f(u(t)), \qquad u(0) = u_0 \in \mathbb{R}^d, \quad t \in \mathbb{R}_+, \tag{1}$$

whose solution can be expanded as

$$u(t) = u_0 + tf(u_0) + \frac{t^2}{2} f'f(u_0) + \frac{t^3}{6} f'f'f(u_0) + \frac{t^3}{6} f''[f,f](u_0) + \cdots$$

which rewrites as the sum

$$u(t) = u_0 + \sum_{\mathcal{T}} \frac{t^{r(\mathcal{T})}}{\sigma(r(\mathcal{T}))\gamma(r(\mathcal{T}))} F(\mathcal{T})$$

over the family of Butcher trees $\mathcal{T}$, see [1], [2], Chapters 4-6 of [4], and [9], based on early work of [3]. In order to solve (1), we may also write

$$u(s) = u_0 + \int_0^s u'(r)dr = u_0 + \int_0^s f(u(r))dr,$$

and more generally we can expand the derivative $f^{(l)}(u(r))$ as

$$f^{(l)}(u(r)) = f^{(l)}(u_0) + \int_0^r f(u(r))f^{(l+1)}(u(v))dv, \qquad l \geq 1.$$

We note that the above family of equations can be rewritten as

$$c(u)(t) = c(u)(0) + \sum_{Z \in \mathcal{M}(c)} \int_0^t \prod_{z \in Z} z(u)(s)ds \tag{2}$$

where c runs through a set $\mathcal{C} := \{\mathrm{Id},\ f^{(l)},\ l \geq 0\}$, of functions called *codes* and $\mathcal{M}(c)$ is defined by letting $\mathcal{M}(\mathrm{Id}) := \{f\}$ and $\mathcal{M}(g) := \{(f, g')\}$ for g a smooth function on $\mathbb{R}_+ \times \mathbb{R}$, see [12].

Next, consider a nonlinear PDE of the form

$$\begin{cases} \partial_t u(t,x) + \frac{1}{2}\Delta u(t,x) + f(u(t,x)) = 0 \\ u(T,x) = \phi(x), \qquad (t,x) \in [0,T] \times \mathbb{R}. \end{cases} \tag{3}$$

Letting $v(t,x) := g(u(t,x))$, we now have

$$\partial_t v(t,x) + \frac{1}{2}\Delta v(t,x) = g'(u(t,x))\left(\partial_t u(t,x) + \frac{1}{2}\Delta u(t,x)\right) + \frac{1}{2}(\partial_x u(t,x))^2 g''(u(t,x))$$
$$= -f(u(t,x))g'(u(t,x)) + \frac{1}{2}(\partial_x u(t,x))^2 g''(u(t,x)),$$

which shows that the functions u, $\partial_x u$, $af^{(k)} \circ u$ satisfy the integral equations

$$\begin{cases} u(t,x) = \int_{-\infty}^{\infty} \varphi(T-t, y-x)\phi(y)dy + \int_t^T \int_{-\infty}^{\infty} \varphi(s-t, y-x) f(u(s,y))dyds \\ af^{(k)}(u(t,x)) = \int_{-\infty}^{\infty} \varphi(T-t, y-x) af^{(k)}(\phi(y))dy \\ + \int_t^T \int_{-\infty}^{\infty} \varphi(s-t, y-x) \\ \qquad \times \left(af(u(s,y))f^{(k+1)}(u(s,y)) - \frac{a}{2}(\partial_x u(s,y))^2 f^{(k+2)}(u(s,y))\right) dyds \\ \partial_x u(t,x) = \int_{-\infty}^{\infty} \varphi(T-t, y-x)\partial_x \phi(y)dy \\ \quad + \int_t^T \int_{-\infty}^{\infty} \varphi(s-t, y-x) f'(u(s,y))\partial_x u(x,y)dyds, \end{cases}$$

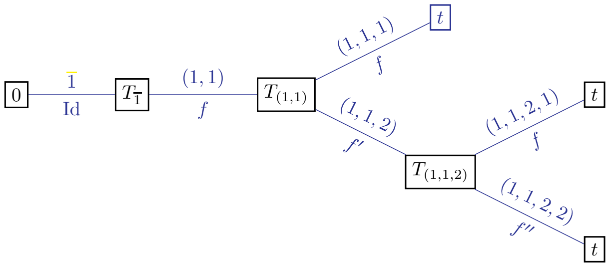
$a \neq 0$, $k \in \mathbb{N}$. We note that the above set of equations admits a formulation identical to (2) provided that we use the codes

$$\mathcal{C} := \{\mathrm{Id},\ \partial_x,\ af^{(k)},\ a \neq 0,\ k \in \mathbb{N}\}$$

and the mechanism defined as

$$\mathcal{M}(\mathrm{Id}) := \{f^*\}, \quad \mathcal{M}(g^*) := \left\{(f^*, (g')^*), \left(\partial_x, \partial_x, -\frac{1}{2}(g'')^*\right)\right\},$$

and $\mathcal{M}(\partial_x) := \{((f')^*, \partial_x)\}$.



We consider a random coding tree $\mathcal{T}_{t,x,c}$ illustrated by the above sample, started at (t,x) with a code $c \in \mathcal{C}$ and partitioned as $\mathcal{K}^{\partial} \cup \mathcal{K}^{\circ}$, where $\mathcal{K}^{\circ}$ denotes the set of leaves. In the next result, we use the random functional

$$\mathcal{H}(\mathcal{T}_{t,x,c}) := \prod_{\overline{k} \in \mathcal{K}^{\circ}} \frac{1}{q_{c_{\overline{k}}} \rho(\tau_{\overline{k}})} \prod_{\overline{k} \in \mathcal{K}^{\partial}} \frac{c_{\overline{k}}(u)\big(T, X^{\overline{k}}_{T_{\overline{k}}}\big)}{\overline{F}(T - T_{\overline{k}-})}.$$

of the random coding tree $\mathcal{T}_{t,x,c}$, in which branching at a node $\overline{k}$ occurs at the random time $T_{\overline{k}}$, the interjump time $\tau_{\overline{k}} = T_{\overline{k}} - T_{\overline{k}-}$ has tail CDF $\overline{F}$ and PDF ρ, and $\big(X^{\overline{k}}_t\big)_{t \geq T_{\overline{k}-}}$ is an independent Brownian motion started at time $T_{\overline{k}-}$.

Theorem 1. *([13]) Assume that the integral solution of the system* (2) *is unique and that there exists a constant $K > 0$ such that:*

$$|f^{(k)} \circ \phi|_{\infty} \leq K, \quad k \geq 0, \quad |\phi|_{\infty} \leq K, \quad |\phi'|_{\infty} \leq K.$$

Then, there exists $T > 0$ such that the solution of (3) *admits the probabilistic representation*

$$u(t,x) = \mathbb{E}\left[\mathcal{H}(\mathcal{T}_{t,x,\mathrm{Id}})\right], \qquad (t,x) \in [0,T] \times \mathbb{R}.$$

The above method also extends to fully nonlinear PDEs of the form

$$\begin{cases} \partial_t u(t,x) + \dfrac{1}{2}\Delta u(t,x) + f\big(u(t,x), \nabla u(t,x), \ldots, \nabla^n u(t,x)\big) = 0, \\ u(T,x) = \phi(x), \qquad (t,x) = (t, x_1, \ldots, x_d) \in [0,T] \times \mathbb{R}^d, \end{cases}$$

$d \geq 1$, see [13], [11]. As an example, we consider a cosine nonlinearity with a gradient of order four, for which our method appears more accurate than the deep Galerkin method [14]. Related comparisons can be found in [11] with respect to the deep BSDE method [5].

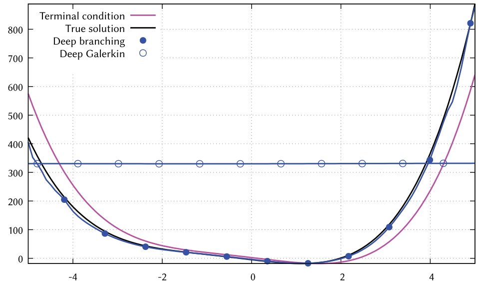


FIGURE 1. Comparison graphs in dimension $d = 5$.

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Probabilistic representations for the solutions of nonlinear PDEs with fractional Laplacians

Nicolas Privault
(joint work with Guillaume Penent)

This talk presents tree-based probabilistic algorithms for the existence of solutions of nonlinear fractional PDEs with their numerical implementation.

1. Parabolic case

Given $\eta : (0,\infty) \to [0,\infty)$ a Bernstein function, consider the (nonlocal) semilinear PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \eta(-\Delta/2)u(t,x) + f\left(t,x,u(t,x),\dfrac{\partial u}{\partial x_1}(t,x),\ldots,\dfrac{\partial u}{\partial x_m}(t,x)\right) = 0, \\ u(T,x) = \phi(x), \qquad x = (x_1,\ldots,x_d) \in \mathbb{R}^d, \end{cases}$$

where $f(t,x,y,z_1,\ldots,z_m)$ is a polynomial nonlinearity given by

$$f(t,x,y,z_1,\ldots,z_m) = \sum_{l=(l_0,\ldots,l_m)\in\mathcal{L}_m} c_l(t,x) y^{l_0} z_1^{l_1}\cdots z_m^{l_m},$$

and $\mathcal{L}_m \subset \mathbb{N}^{m+1}$ is finite. By choosing $\eta(\lambda) := (2\lambda)^{\alpha/2}$, this setting includes the case of the standard fractional Laplacian Δ_α. We assume that the coefficients $c_l(t,x)$ are uniformly bounded and that the terminal condition ϕ is Lipschitz and bounded on $\mathbb{R}^d$.

Theorem 1. *([8]) Suppose that $\int_{\lambda_0}^\infty \frac{1}{\sqrt{\lambda}\eta(\lambda)} d\lambda < \infty$ for some $\lambda_0 > 0$. Then, there exists a small enough $T > 0$ such that the PDE*

$$u(t,x) = \int_{\mathbb{R}^d} \varphi(T-t,y-x)\phi(y)dy$$
$$+ \sum_{l=(l_0,\ldots,l_m)\in\mathcal{L}_m} \int_t^T \int_{\mathbb{R}^d} \varphi(s-t,y-x) c_l(s,y) u^{l_0}(s,y) \prod_{j=1}^m \left(\frac{\partial u}{\partial y_j}(s,y)\right)^{l_j} dyds,$$

admits an integral solution on $[0,T]$.

To prove the above result, for each $i = 0,1,\ldots,d$ we construct a sufficiently integrable functional $\mathcal{H}_\phi(\mathcal{T}_{t,x,i})$ of a random tree $\mathcal{T}_{t,x,i}$ driven by a subordinated Lévy process $(Z_t)_{t\in\mathbb{R}_+} := (B_{S_t})_{t\in\mathbb{R}_+}$, where $(B_t)_{t\in\mathbb{R}_+}$ is a multidimensional Brownian motion, such that we have the representations

$$u(t,x) := \mathbb{E}\big[\mathcal{H}_\phi(\mathcal{T}_{t,x,0})\big], \quad (t,x) \in [0,T]\times\mathbb{R}^d,$$

and

$$\frac{\partial u}{\partial x_i}(t,x) := \mathbb{E}\big[\mathcal{H}_\phi(\mathcal{T}_{t,x,i})\big], \quad (t,x) \in [0,T]\times\mathbb{R}^d, \quad i = 1,\ldots,d.$$

Dealing with gradient terms requires to perform an integration by parts, which is made possible using the Gaussian density of B_t in the subordination $Z_t := B_{S_t}$, as done in [7] in the case of stable processes with $\eta(\lambda) := (2\lambda)^{\alpha/2}$. Related local and global-in-time existence results have been obtained by deterministic arguments under more technical conditions in e.g. [5], [6]

Corollary 2. *([8]) Taking $\eta(\lambda) := (2\lambda)^{\alpha/2}$ with $\alpha \in (1,2)$, under the above assumptions there exists a small enough $T > 0$ such that the PDE*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - (-\Delta)^{\alpha/2} u(t,x) + f\left(t, x, u(t,x), \dfrac{\partial u}{\partial x_1}(t,x), \ldots, \dfrac{\partial u}{\partial x_m}(t,x)\right) = 0, \\ u(T,x) = \phi(x), \qquad x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \end{cases}$$

with α-fractional Laplacian admits an integral solution on $[0,T]$.

2. Elliptic case

We consider the class of semilinear elliptic PDEs on the open ball $B(0,R)$ of radius $R > 0$ in $\mathbb{R}^d$, of the form

$$\begin{cases} \Delta_\alpha u(x) + f\big(x, u(x), \nabla u(x)\big) = 0, \qquad x \in B(0,R), \\ u(x) = \phi(x), \qquad x \in \mathbb{R}^d \backslash B(0,R), \end{cases} \tag{1}$$

where $\phi : \mathbb{R}^d \to \mathbb{R}$ is a bounded Lipschitz function on $\mathbb{R}^d \setminus B(0,R)$, Δ_α denotes the fractional Laplacian with parameter $\alpha \in (0,2)$, and $f(x,y,z)$ is a polynomial nonlinearity term. Next, we provide existence results and probabilistic representations for the solution of (1) under boundedness and smoothness conditions on polynomial coefficients. Our approach allows us to take into account gradient nonlinearities, which has not been done by deterministic finite difference methods, see e.g. [4].

Theorem 3. *([9], [10]) Let $d \geq 2$ and $\alpha \in (1,2)$, assume that the boundary condition ϕ belongs to $H^\alpha(\mathbb{R}^d)$ and is bounded on $\mathbb{R}^d \setminus B(0,R)$. Under the above assumptions, the semilinear elliptic PDE*

$$\begin{cases} \Delta_\alpha u(x) + f\big(x, u(x), \nabla u(x)\big) = 0, \qquad x \in B(0,R), \\ u(x) = \phi(x), \qquad x \in \mathbb{R}^d \backslash B(0,R), \end{cases}$$

admits a viscosity solution in $\mathcal{C}^1(B(0,R)) \cap \mathcal{C}^0(\overline{B}(0,R))$ provided that R and $|c_l|_\infty$, $l \in \mathcal{L}_m$, are sufficiently small.

Existence of solutions are obtained through a probabilistic representation of the form $u(x) := \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,0})]$, $x \in B(0,R)$, where $\mathcal{H}_\phi(\mathcal{T}_{x,0})$ is a functional of a random branching tree $\mathcal{T}_{x,0}$. For each $i = 0, 1, \ldots, d$ we construct a sufficiently integrable functional $\mathcal{H}_\phi(\mathcal{T}_{x,i})$ of a random tree $\mathcal{T}_{x,i}$ such that we have the representation

$$u(x) = \mathbb{E}\big[\mathcal{H}_\phi(\mathcal{T}_{x,0})\big], \qquad x \in \mathbb{R}^d. \tag{2}$$

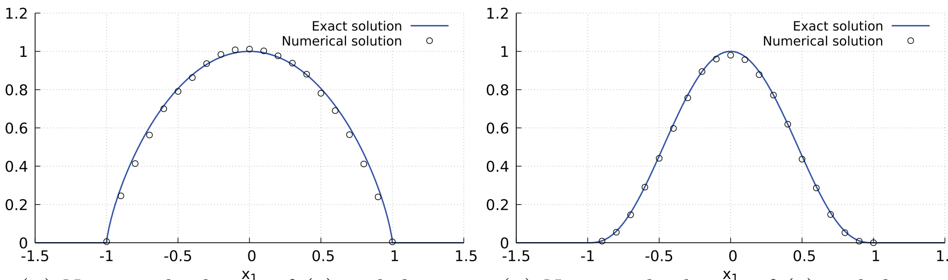
The main difficulty in the proof is to show the uniform integrability required on $\mathcal{H}(\mathcal{T}_{x,i})$ for $\mathbb{E}[\mathcal{H}(\mathcal{T}_{x,i})]$ to be continuous in $x \in \mathbb{R}^d$ is satisfied for $\alpha \in (1,2)$, as required in the framework of viscosity solutions. For this, we extend arguments of

[1] from the standard Laplacian Δ and Brownian motion to the fractional Laplacian $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ and its associated stable process, and use bounds on the fractional Green and Poisson kernel and stable process hitting times from [3], [2].

As an example, consider the elliptic PDE with nonlinear gradient term

(3) $\Delta_s u(x) + \Psi_{k,\alpha}(x) + (2k+\alpha)^2|x|^4(1-|x|^2)^{2k+\alpha} + ((1-|x|^2)x \cdot \nabla u(x))^2 = 0,$

$x \in B(0,1)$, with $u(x) = 0$ for $x \in \mathbb{R}^d \setminus B(0,R)$, and explicit solution $u(x) = \Phi_{k,\alpha}(x) = (1-|x|^2)_+^{k+\alpha/2}$, $x \in \mathbb{R}^d$. Numerical estimates of (2) by the Monte Carlo method are presented in the figure below.



(A) Numerical solution of (3) with $k = 0$. (B) Numerical solution of (3) with $k = 2$.

FIGURE 1. Numerical solutions with $d = 10$ and $\alpha = 1.75$.

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On (global) unique continuation properties of the fractional discrete Laplacian

Luz Roncal

(joint work with Aingeru Fernández-Bertolin, Angkana Rüland)

Qualitative and quantitative unique continuation properties (UCP) hold for large classes of local and nonlocal equations and have been thoroughly studied. However, for their discrete counterparts only weaker versions of these persist in general. It is the main objective of this talk to discuss UCP, both for local and nonlocal equations on the lattice $(h\mathbb{Z})^d$, with correction terms in the discretization error and describe consequences for associated inverse problems.

For the fractional Laplacian on $\mathbb{R}^d$ a remarkable *global* UCP, which is not available for local elliptic operators such as the Laplacian, holds:

Theorem 1 (Global UCP, [3])**.** Let $s \in (0,1)$ and let $\Omega \subset \mathbb{R}^d$ with $d \geq 1$ be open. Let $u \in H^r(\mathbb{R}^d)$ for some $r \in \mathbb{R}$ and assume that $u = 0 = (-\Delta)^s u$ in Ω. Then $u \equiv 0$ in $\mathbb{R}^d$.

We illustrate that for the fractional *discrete* Laplacian, the global UCP fails on the discrete lattice $(h\mathbb{Z})^d$ with $h \in \mathbb{R}_+$. Given a function $u : (h\mathbb{Z})^d \to \mathbb{R}$, we define the discrete Laplacian as

$$(-\Delta_{\mathrm{d}})u(hj) := \frac{1}{h^2}\sum_{i=1}^{d}\big(u(h(j+e_i)) - 2u(hj) + u(h(j-e_i))\big), \quad hj \in (h\mathbb{Z})^d.$$

Based on this definition, it is possible to define the *fractional discrete Laplacian* $(-\Delta_{\mathrm{d}})^s$ by means of its heat semigroup representation, or its Fourier symbol, or the associated semi-discrete Caffarelli–Silvestre extension.

Let us write $u_j := u(hj)$. We have that, on the lattice, the direct counterpart of Theorem 1 fails:

Theorem 2. Let $X \subset (h\mathbb{Z})^d$ be a finite set of cardinality $M \in \mathbb{N}$. Then there exists a non-zero function $u \in \ell_s$ such that $u_j = 0 = (-\Delta_{\mathrm{d}})^s u_j$ for $j \in X$.

Here, for $0 \leq s \leq 1$, ℓ_s is the function space allowing the definition of $(-\Delta_{\mathrm{d}})^s$ under minimal decay conditions.

While the strongest version of the global UCP of Theorem 1 fails for discrete operators, we highlight that a weaker (still qualitative) counterpart of it persists in the form of global UCP from the exterior:

Theorem 3. Let $d \geq 1$, $h \in \mathbb{R}_+$ and $u \in H^r((h\mathbb{Z})^d)$ for some $r \in \mathbb{R}$. Let $s \in (0,1)$ and assume that for some $R \in \mathbb{R}_+$, $R \geq h$, $u = 0 = (-\Delta_{\mathrm{d}})^s u$ in $(h\mathbb{Z})^d \setminus B_R$. Then $u \equiv 0$ in $(h\mathbb{Z})^d$.

Here for $R > 0$, we have set $B_R = B_R(0) \cap (h\mathbb{Z})^d$; the function space $H^r((h\mathbb{Z})^d)$ is the discrete Sobolev space.

While the main focus of the talk is the investigation of the degree of the failure of the global UCP for the fractional discrete Laplacian, we also briefly consider the weak UCP for fractional discrete Schrödinger equations in slab domains and we show that in this situation the weak UCP from (thin) slab domains fails.

A class of quantitative results which is strongly related to the UCP of the fractional Laplacian consists of boundary-bulk inequalities for elliptic equations. Indeed, the relation between boundary-bulk UCP and unique continuation estimates for the fractional Laplacian follows from the characterization of the fractional Laplacian by means of the Caffarelli–Silvestre extension [1]. In the setting of the fractional continuous Laplacian for $s = 1/2$, it is well-known [4] that a boundary-bulk inequality holds.

We here present the analogous question on the lattice: Consider a function $\tilde{u}$ solving the equation

$$(1) \qquad \begin{aligned} (\partial_t^2 + \Delta_{\mathrm{d}})\tilde{u} &= V\tilde{u} \text{ in } (h\mathbb{Z})^d \times \mathbb{R}_+, \\ \tilde{u} &= u \text{ on } (h\mathbb{Z})^d \times \{0\}, \end{aligned}$$

where $V : (h\mathbb{Z})^d \times \mathbb{R}_+ \to \mathbb{R}$ is a bounded potential.

We prove the following theorem, which illustrates that the boundary-bulk unique continuation estimates only "barely" fail with correction terms that decay exponentially in the lattice size.

Theorem 4. Let $u \in H^1((h\mathbb{Z})^d)$ and $\tilde{u} : (h\mathbb{Z})^d \times \mathbb{R}_+ \to \mathbb{R}$ be a solution to (1). Then, there exist $h_0 > 0$, $C > 1$ (depending on $\|V\|_{L^\infty((h\mathbb{Z})^d \times \mathbb{R}_+)}$ and d) and $r_0, \alpha \in (0,1)$ (depending only on d) such that for all $h \in (0, h_0)$ it holds that

$$\begin{aligned} \|\tilde{u}\|_{L^2(B_{r_0}^+)} \le C \max\{\|\tilde{u}\|_{L^2(B_1^+)}, \|u\|_{H^1(B_1')} + \|\partial_t \tilde{u}\|_{L^2(B_1')}\}^{1-\alpha} \\ \times (\|u\|_{H^1(B_1')} + \|\partial_t \tilde{u}\|_{L^2(B_1')})^\alpha + Ce^{-Ch^{-1}} \|\tilde{u}\|_{L^2(B_1^+)}. \end{aligned}$$

Here, we use the notation $B_r^+ := \{(x,t) \in \mathbb{R}_+^{d+1} : |(x,t)| < r\}$ and $B_r' := \{(x,0) \in \mathbb{R}^d \times \{0\} : \; |x| < r\}$.

As an application of the boundary-bulk inequality from above, we prove that the global UCP from Theorem 1 persists in a certain sense if one is "sufficiently close" to the continuum setting and if global information on the data is present. To this end, we consider the inverse problem of recovering a function $f \in C_c^\infty(W)$ from partial measurements of its half-Laplacian $(-\Delta_{\mathrm{d}})^{\frac{1}{2}} f|_\Omega$ on the open domain Ω which we assume to be disjoint from the open set W.

We can use Theorem 4 to infer a stability estimate for this inverse problem:

Theorem 5. Let $W, \Omega \subset (h\mathbb{Z})^d$ be non-empty, open sets with $\overline{W} \cap \overline{\Omega} = \emptyset$ and $f \in C_c^\infty(W)$. Let h_0 be the value from Theorem 4. Then there exists $\nu > 0$ such that if

$$\begin{aligned} 0 < h_0 \le 10^{-1} \Big| \log\Big(\frac{\|(-\Delta_{\mathrm{d}})^{\frac{1}{2}} f\|_{L^2(\Omega)}}{\|f\|_{H^1(W)}} \Big) \Big|^{-1+\nu} \Big| \\ \times \log\Big(-C \log\Big(\frac{\|(-\Delta_{\mathrm{d}})^{\frac{1}{2}} f\|_{L^2(\Omega)}}{\|f\|_{H^1(W)}} \Big) \Big) \Big|^{-1}, \end{aligned}$$

the following estimate holds, for $h \in (0, h_0)$:

$$\|f\|_{L^2(W)} \leq C\|f\|_{H^1(W)}\Big|\log\Big(\frac{\|(-\Delta_{\rm d})^{\frac{1}{2}}f\|_{L^2(\Omega)}}{\|f\|_{H^1(W)}}\Big)\Big|^{-\nu}$$
$$+ C\exp\Big(-(Ch)^{-1}\Big|\log\Big(\frac{\|(-\Delta_{\rm d})^{\frac{1}{2}}f\|_{L^2(\Omega)}}{\|f\|_{H^1(W)}}\Big)\Big|^{-1+\nu}\Big)\|f\|_{H^1(W)}.$$

We remark that in particular, for $h_0 \in (0, 1)$ sufficiently small (depending on the size of the measurement data $\|(-\Delta_{\rm d})^{\frac{1}{2}}f\|_{L^2(\Omega)}$ and the oscillation of f measured in terms of $\|f\|_{H^1(W)}$), we obtain a stability estimate for the discrete inverse problem of recovering $f \in C_c^\infty(W)$ from the data $\|(-\Delta_{\rm d})^{\frac{1}{2}}f\|_{L^2(\Omega)}$ under a priori oscillation control for f. We hope that this eventually also allows one to obtain similar stability estimates for nonlinear discrete inverse problems such as the discrete fractional Calderón problem. This question is left as a problem for future research.

The results we report in this talk are contained in the preprint [2].

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On the Fractional Calderón Problem – Uniqueness, Stability and Single Measurement Recovery

Angkana Rüland

(joint work with T. Ghosh, M. Salo, G. Uhlmann)

In this talk I present uniqueness, stability and single measurement recovery results for a nonlocal inverse problem, the *fractional Calderón problem.* The fractional Calderón problem is a nonlocal variant of the celebrated Calderón problem [1, 7] and had been introduced in [2]. In its study one seeks to recover an unknown potential q (in a suitable function space) on a domain $\Omega \subset \mathbb{R}^n$ from the knowledge of the generalized Dirichlet-to-Neumann map Λ_q of the associated fractional Schrödinger equation. More precisely, let $u : \mathbb{R}^n \to \mathbb{R}$ be a solution to

$$\begin{aligned} (-\Delta)^s u + qu = 0 \text{ in } \Omega, \\ u = f \text{ on } \Omega_e := \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned}$$

Then, the generalized Dirichlet-to-Neumann map Λ_q takes the form

$$\Lambda_q : \tilde{H}^s(\Omega_e) \to H^{-s}(\Omega_e),\ f \mapsto (-\Delta)^s u|_{\Omega_e}.$$

In studying this inverse problem for the fractional Calderón problem, a number of questions arise:

- **Injectivity**: Is it true that if q_1, q_2 are in suitable function spaces and if $\Lambda_{q_1} = \Lambda_{q_2}$, that then $q_1 = q_2$?
- **Stability**: Is it true that under appropriate a priori regularity assumptions on the potentials q_1, q_2 a stability estimate holds, thus quantifying injectivity?
- **Reconstruction**: It it possible to algorithmically recover the unknown potential q from the knowledge of Λ_q?

Presenting some of the results from the articles [2, 5, 4, 3, 6], in this talk I provide answers to these questions, explaining the following main points:

- **Injectivity** holds in critical function spaces which are given as certain multiplier spaces. Moreover, injectivity holds already in the (infinite measurement) *partial data* setting in which measurements for the generalized Dirichlet-to-Neumann operator suffice in certain, possibly very small, possibly disjoint open subset $W_1, W_2 \subset \Omega_e$, see [5].
- **Stability** holds under suitable a priori information for the (infinite measurement) partial data problem. Contrary to the uniqueness properties, in this context nonlocality does not substantially improve the stability properties of the inverse problem – as the classical Calderón problem also the fractional Calderón problem remains highly unstable with only logarithmic moduli of continuity, see [5]. In particular, the logarithmic modulus of stability is optimal and cannot be improved, see [4].
- **Single measurement uniqueness and reconstruction** is possible. Instead of the knowledge of the full, infinite dimensional (partial data) generalized Dirichlet-to-Neumann operator, only a single pair of data $(f, \Lambda_q(f))$ with $f \neq 0$ suffices to uniquely and algorithmically recover the potential q, if q is sufficiently regular, see [3].
- A **single measurement stability result** proving that also the single measurement problem is logarithmically stable (under suitable a priori assumption on the potential and data) quantifies the single measurement uniqueness result, see [6].

These results strongly rely on the nonlocality of the fractional Calderón problem and many of these results are not known in the setting of the classical Calderón problem. A key ingredient is a qualitative, quantitative and algorithmic duality between rigidity and flexibility in the form of global unique continuation results and Runge approximation properties.

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Nonlinear problems with partially symmetric solutions

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(joint work with Giuseppe Devillanova, Giovanni Molica Bisci)

Several important problems arising in many research fields such as physics and differential geometry lead to consider semilinear variational elliptic equations defined on unbounded domains of the Euclidean space and a great deal of work has been devoted to their study. From the mathematical point of view, probably the main interest relies on the fact that often the tools of nonlinear functional analysis, based on compactness arguments, cannot be used, at least in a straightforward way, and some new techniques have to be developed.

The seminal paper [4] by P.-L. Lions has inspired a (nowadays usual) way to overcome the lack of compactness by exploiting symmetry. This approach is fruitful in the study of variational elliptic problems in presence of a suitable continuous action of a topological group on the Sobolev space where the solutions are being sought.

Along this direction, in the paper [3] we exploit a group theoretical scheme, raised in the study of problems which are invariant with respect to the action of orthogonal subgroups, to show the existence of multiple solutions distinguished by their different symmetry properties. We emphasize that a wide class of nonlinear problems of this kind can be handled by constructing suitable subspaces, of "partially symmetric" functions, of the ambient Sobolev space, and by applying an appropriate version of the so-called Principle of Symmetric Criticality proved in the seminal paper [6] by Palais.

In [3] we are interested in getting existence and multiplicity results of weak solutions to the following problem

$$(P_\lambda) \qquad \begin{cases} -\Delta u = \lambda \alpha(x,y) f(u) & \text{in } \omega \times \mathbb{R}^{d-m} \\ u = 0 & \text{on } \partial\omega \times \mathbb{R}^{d-m}, \end{cases}$$

where λ is a positive parameter and $\omega \times \mathbb{R}^{d-m}$ is an unbounded strip of $\mathbb{R}^d$, being ω an open bounded subset of $\mathbb{R}^m$ with smooth boundary $\partial\omega$ and $d, m \in \mathbb{N}$, $d \geq m+2$. Moreover, we assume that $\alpha : \omega \times \mathbb{R}^{d-m} \to \mathbb{R}$ verifies the following integrability, symmetry and sign conditions

$$(\alpha_1) \qquad \alpha \in L^1(\omega \times \mathbb{R}^{d-m}) \cap L^\infty(\omega \times \mathbb{R}^{d-m})$$

$$(\alpha_2) \qquad \alpha(x,y) = \alpha(x,|y|) \text{ a.e. } (x,y) \in \omega \times \mathbb{R}^{d-m}$$

$$(\alpha_3) \qquad \alpha \geq 0 \text{ a.e. in } \omega \times \mathbb{R}^{d-m} \quad \text{and there exist } r > 0 \text{ and } \alpha_0 > 0 \text{ such that } \operatorname{essinf}_{\omega \times B(0,r)} \alpha \geq \alpha_0 \,,$$

where $B(0,r)$ is the ball in $\mathbb{R}^{d-m}$ centered at 0 with radius r, while on $f : \mathbb{R} \to \mathbb{R}$ we require the next hypotheses

$$(f_1) \qquad f \text{ is continuous in } \mathbb{R}$$

$$(f_2) \qquad f(t) = o(|t|) \text{ as } |t| \to 0$$

$$(f_3) \qquad \text{there exists } \sigma > 2 \text{ such that } 0 < \sigma F(t) \leq t f(t) \text{ for any } t \in \mathbb{R} \setminus \{0\},$$

where F is the following antiderivative of the function f

$$(1) \qquad F(t) = \int_0^t f(\tau)\, d\tau \,, \quad t \in \mathbb{R} \,,$$

$$(f_4) \qquad \sup_{t \in \mathbb{R}\setminus\{0\}} \frac{|f(t)|}{|t| + |t|^{q-1}} < +\infty \text{ for some } q \in (2, 2^*),$$

where 2^* is the critical Sobolev exponent given by $2^* := 2d/(d-2)$. Assumption (f_3) is the well-known Ambrosetti-Rabinowitz condition, which is a superlinear assumption on the term f, namely a superquadratic one on its antiderivative F at infinity.

In [3] we study Problem (P_λ) also under sublinear conditions at infinity on the nonlinearity f. More precisely, we also consider the case in which, instead of (f_3), the function f satisfies the following hypotheses

$$(f_5) \qquad f(t) = o(|t|) \text{ as } |t| \to +\infty$$

$$(f_6) \qquad \text{there exists } t_0 \in \mathbb{R}^+ \text{ such that } F(t_0) > 0 \text{ and } F(t) \geq 0 \text{ on } [0,\, t_0]\,,$$

where F is given in (1).

Problem (P_λ) has a clear variational structure, indeed its solutions can be found as critical points of the following energy functional defined by setting for all $u \in H^1_0(\omega \times \mathbb{R}^{d-m})$

$$\mathcal{I}_\lambda(u) := \frac{1}{2} \int_{\omega \times \mathbb{R}^{d-m}} |\nabla u(x,y)|^2 \, dx\, dy - \lambda \int_{\omega \times \mathbb{R}^{d-m}} \alpha(x,y) F(u(x,y))\, dx\, dy,$$

where F is given in (1).

Since the problem is set on the strip-like domain $\omega \times \mathbb{R}^{d-m}$, there is no compactness property which can be used with $\mathcal{I}_\lambda$ on the whole space. Hence, in order to find a weak solution to Problem (P_λ), we need to construct a suitable subspace of $H^1_0(\omega \times \mathbb{R}^{d-m})$ which allows us, from one side, to recover compactness and to get, by an application of the Mountain Pass Theorem, a constrained critical point for the energy functional $\mathcal{I}_\lambda$ and, from the other side, to apply the Principle of Symmetric Criticality got by Palais in [6] to show that the restriction to that subspace does not play any role.

Finally, when $d = m+4$ or $d \geq m+6$ and the nonlinearity f is odd, by exploiting a flower-shape geometric structure in the Sobolev space $H^1_0(\omega \times \mathbb{R}^{d-m})$, we get a multiplicity result for Problem (P_λ), using again variational and topological arguments.

In the superlinear framework our result reads as follows:

Theorem 1. (*Superlinear setting*). *Let $\omega \times \mathbb{R}^{d-m}$ be an unbounded strip of $\mathbb{R}^d$, with ω open bounded subset of $\mathbb{R}^m$ with smooth boundary $\partial\omega$, $d, m \in \mathbb{N}$, $d \geq m+2$, and let λ be a positive parameter. Let α satisfy conditions (α_1), (α_2) and (α_3) and let f satisfy assumptions (f_1), (f_2), (f_3) and (f_4).*

Then,

- (i) *Existence: for any $\lambda > 0$ there exists a nontrivial weak solution u_λ of Problem (P_λ) in $H^1_0(\omega \times \mathbb{R}^{d-m})$ with cylindrical symmetry;*
- (ii) *Multiplicity: if, in addition, $d = m+4$ or $d \geq m+6$ and f is odd, then for any $\lambda > 0$ Problem (P_λ) admits $s_{d,m}$ sequences of nontrivial weak solutions, with different symmetries, where $s_{d,m}$ is defined as follows*

$$s_{d,m} = (-1)^{d-m} + \left\lfloor \frac{d-m-3}{2} \right\rfloor + 1\,. \tag{2}$$

In the sublinear setting our main result for Problem (P_λ) is stated here below.

Theorem 2. (*Sublinear setting*). *Let $\omega \times \mathbb{R}^{d-m}$ be an unbounded strip of $\mathbb{R}^d$, with ω open bounded subset of $\mathbb{R}^m$ with smooth boundary $\partial\omega$, $d, m \in \mathbb{N}$, $d \geq m+2$, and let λ be a positive parameter. Let α satisfy conditions (α_1), (α_2) and (α_3) and let f satisfy (f_1), (f_2), (f_5) and (f_6).*

Then,

- (i) *there exists $\bar{\lambda} > 0$ such that for any $\lambda < \bar{\lambda}$ there are no nontrivial weak solutions for Problem (P_λ);*
- (ii) *there exists $\lambda^*_E > 0$ such that for any $\lambda > \lambda^*_E$ there exist at least two nontrivial weak solutions of Problem (P_λ) in $H^1_0(\omega \times \mathbb{R}^{d-m})$ with cylindrical symmetry;*
- (iii) *if, in addition, $d = m+4$ or $d \geq m+6$ and f is odd, then there exists $\lambda^*_M > 0$ such that for any $\lambda > \lambda^*_M$ Problem (P_λ) admits $s_{d,m}$ pairs of nontrivial weak solutions, with different symmetries ($s_{d,m}$ is defined by (2)).*

The main theorems of [3] may be seen as an extension of existence and multiplicity results, already appeared in the literature, for nonlinear problems set in the entire space $\mathbb{R}^d$, as for instance, the ones obtained in the papers [1, 2] due to Bartsch and Willem (see also [5]).

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Fredholm properties of boundary value problems for the one-dimensional regional fractional Laplacian

Eugene Shargorodsky
(joint work with Tony Hill)

1. Notation

Let r_+ denote the operator of restriction to $\mathbb{R}_+ = (0,\infty)$ of functions defined on $\mathbb{R}$, and let e_+ be the operator of extension to $\mathbb{R}$ by 0 of functions defined on $\mathbb{R}_+$.

Let F and F^{-1} denote the direct and the inverse Fourier transforms.

Bessel-potential spaces:

$$H^s_p(\mathbb{R}) := \left\{ f \in S'(\mathbb{R}) |\ \|f|\ H^s_p(\mathbb{R})\| = \left\| F^{-1}(1+\xi^2)^{s/2} Ff |\ L_p(\mathbb{R}) \right\| < \infty \right\},$$
$$H^s_p(\overline{\mathbb{R}_+}) := r_+ H^s_p(\mathbb{R}), \quad s \in \mathbb{R},\ 1 < p < \infty.$$

If $s > 1 + 1/p$, then the following subspace is well defined

$$H^s_{p,0}(\overline{\mathbb{R}_+}) := \left\{ u \in H^s_p(\overline{\mathbb{R}_+}) :\ u'(0) = 0 \right\}.$$

We do not require $u(0) = 0$ here.

2. Auxiliary results

Let B denote the beta function:

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \operatorname{Re} x > 0,\ \operatorname{Re} y > 0.$$

The equation

$$\frac{\sin \pi\alpha}{\pi} B(\tau - 2\alpha + 1 + i\xi, 2\alpha) = \frac{\sin\left[\pi(1-\tau+\alpha-i\xi)\right]}{\sin\left[\pi(1-\tau+2\alpha-i\xi)\right]} \tag{1}$$

does not have solutions with $0 < \alpha < 1/2$, $0 < \tau < 1$ or with $0 < \alpha < 1$, $1 < \tau < 2$, $\xi \neq 0$.

If $\xi = 0$, then (1) is equivalent to

$$\Gamma(2\alpha-\tau)\Gamma(\tau+1)\sin\pi(\alpha-\tau) = \Gamma(2\alpha)\sin\pi\alpha. \tag{2}$$

If $0 < \alpha < 1/2$, equation (2) has no solutions $\tau \in (0,1)$.
If $0 < \alpha < 1$, equation (2) has a unique solution $\tau \in (1,2)$ of the form $\tau = 1 + \alpha_c$, where $0 < \alpha_c < \alpha$.

3. Main results

Let

$$\begin{aligned} A &:= (D^2+1)^\alpha = F^{-1}(\xi^2+1)^\alpha F, \\ \mathbf{1}_G(x) &:= \begin{cases} 1 & \text{if } x \in G, \\ 0 & \text{if } x \in \mathbb{R}\setminus G, \end{cases} \qquad G \subset \mathbb{R}, \\ \mathcal{A}u &:= r_+\, A\, e_+ u + u\, r_+ A(\mathbf{1}_{\mathbb{R}_-}). \end{aligned}$$

Theorem 1. *Let $0 < \alpha < \frac{1}{2}$, $1 < p < \infty$, and $1/p < s < 1 + 1/p$. Then the operator $\mathcal{A} : H_p^s(\overline{\mathbb{R}_+}) \to H_p^{s-2\alpha}(\overline{\mathbb{R}_+})$ is bounded and invertible.*

Theorem 2. *Let $0 < \alpha < 1$ and $1 < p < \infty$.*
If $1 + 1/p < s < 1 + 1/p + \alpha_c$, then the operator $\mathcal{A} : H_{p,0}^s(\overline{\mathbb{R}_+}) \to H_p^{s-2\alpha}(\overline{\mathbb{R}_+})$ is bounded and invertible.
If $1 + 1/p + \alpha_c < s < 2 + 1/p$, then $\mathcal{A}$ has a trivial kernel and is Fredholm with index equal to -1.

4. Ingredients of the proof

Lemma 1. *Let $0 < \alpha < 1$ and $u \in C_0^\infty(\mathbb{R})$. Additionally, let $u'(0) = 0$ in the case $\frac{1}{2} \le \alpha < 1$. Then*

$$x^{-2\alpha}(u(x) - u(0)) = \int_0^\infty K_{2\alpha}\left(\frac{x}{y}\right)\left(C_{0+}^{2\alpha}u\right)(y)\,\frac{dy}{y}\,,$$

where

$$K_{2\alpha}(t) = \frac{\mathbf{1}_{[1,\infty)}(t)}{\Gamma(2\alpha)t^{2\alpha}(t-1)^{1-2\alpha}}\,,$$

and C_{0+}^{β} denotes the Caputo fractional derivative of order $\beta > 0$:

$$\left(C_{0+}^{\beta}f\right)(x) := \frac{1}{\Gamma(1-\{\beta\})}\int_0^x \frac{f^{([\beta]+1)}(y)}{(x-y)^{\{\beta\}}}\,dy, \quad x > 0,$$

and $\{\beta\} = \beta - [\beta]$ is the fractional part of β.

Using this lemma, one can reduce the proof of theorems 1 and 2 to the study of the operator

$$W(a_1) + c\,M^0(b)W(a_2) + T : L_p(\mathbb{R}_+) \to L_p(\mathbb{R}_+), \tag{3}$$

where T is compact, $c \in C_0^\infty(\mathbb{R})$, $W(a) := r_+ \, a(D) \, e_+$ is a Wiener-Hopf operator, $M^0(b) := \mathcal{M}_p^{-1} b \mathcal{M}_p$ is a Mellin operator, and $\mathcal{M}_p^{\pm 1}$ are the direct and the inverse Mellin tranforms:

$$(\mathcal{M}_p u)(\eta) := \int_0^\infty x^{1/p-1-i\eta} u(x) \, dx, \quad \eta \in \mathbb{R},$$
$$\left(\mathcal{M}_p^{-1} v\right)(x) := \frac{1}{2\pi} \int_{-\infty}^\infty x^{-1/p+i\eta} v(\eta) \, d\eta, \quad x \in \mathbb{R}_+,$$

If $0 < \alpha < \frac{1}{2}$, $1/p < s < 1 + 1/p$, then

$$a_1(\xi) = (\xi^2+1)^\alpha (\xi - i)^{s-2\alpha-1} (\xi + i)^{1-s},$$
$$a_2(\xi) = (-i\xi)^{2\alpha} (\xi - i)^{s-2\alpha-1} (\xi + i)^{1-s},$$
$$b(\xi) = B(s - 2\alpha + 1 - 1/p + i\xi, 2\alpha)/\Gamma(2\alpha),$$
$$c(0) = -\frac{\Gamma(2\alpha) \sin \pi\alpha}{\pi}.$$

If $0 < \alpha < 1$, $1 + 1/p < s < 2 + 1/p$, then

$$a_1(\xi) = (\xi^2+1)^\alpha (\xi - i)^{s-2\alpha-2} (\xi + i)^{2-s},$$
$$a_2(\xi) = (-i\xi)^{2\alpha} (\xi - i)^{s-2\alpha-2} (\xi + i)^{2-s},$$

and b, c are as above.
Fredholm properties, including the index, of operator (3) can be derived from Duduchava's theory ([1], [2]). This operator is Fredholm if and only if equation (1) with $\tau = s - 1/p$ does not have solutions $\xi \in \mathbb{R}$.
If $p = 2$, then $(\mathcal{A}u, u) > 0$ for $u \neq 0$. Hence $\mathcal{A}$ has a trivial kernel: $\mathrm{Ker}\mathcal{A} = \{0\}$. A general result on Fredholm operators then implies that the kernel is trivial for any $p \neq 2$.

The proofs of the above results can be found in Tony Hill's PhD thesis [3].

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Non-local operators with critical perturbations

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(joint work with Jakub Minecki)

An operator $\mathfrak{L}$ is said to be *Lévy-type* if it acts on every smooth compactly supported function f according to the following formula

$$\mathfrak{L}f(x) = c(x)f(x) + b(x)\cdot\nabla f(x) + \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$
$$+ \int_{\mathbb{R}^d}\Big(f(x+z) - f(x) - \mathbf{1}_{|z|<1}\left\langle z, \nabla f(x)\right\rangle\Big)N(x,dz)\,.$$

Here $c(x)$, $b(x)$, $a_{ij}(x)$ and $N(x,dz)$ are called coefficients, and need to satisfy certain natural conditions. The coefficients model the infinitesimal behaviour of a particle at the point x. For instance, the vector $b(x)$ defines the drift, while $N(x,B)$ is the intensity of jumps from x to the set $x+B\subset\mathbb{R}^d$.

The case of constant coefficients, i.e., when $c(x)\equiv c$, $b(x)\equiv b$, $a_{ij}(x)\equiv a_{ij}$ and $N(x,dz)\equiv N(dz)$, leads to convolution semigroups of operators or Lévy processes, or Lévy flights, which are prevalent in probability, PDEs, physics, finance and statistics [1], [3]. In this case there is a well established one-to-one correspondence between the process, the semigroup and the operator $\mathfrak{L}$ [10].

It is extremely important to understand operators with x-dependent coefficients. Due to the Courrège-Waldenfels theorem, the infinitesimal generator of a Feller semigroup with a sufficiently rich domain is a Lévy-type operator; see [2, Theorem 2.21], [5, Theorem 4.5.21]. However, it is a highly non-trivial task to construct an operator semigroup for a given Lévy-type operator with rough (non-constant) coefficients. To resolve the problem many authors follow a scheme known as *the parametrix method*, whose primary role is to provide a candidate for the integral kernel of the semigroup in question. Each usage of the method brings about different technical difficulties to overcome that depend on the class of coefficients under consideration, but most applications follow a characteristic pattern, e.g., the decomposition of the candidate kernel into *a zero order approximation* and *a remainder*.

In the paper [9] we provide a general functional analytic framework for the parametrix method, namely, for the construction of the family of operators $\{P_t\colon t\in(0,1]\}$ for a given choice of *the zero order approximation* operator P_t^0 and *the error term* operator Q_t^0. The aforementioned remainder corresponds then to an operator given by P_t^0 and multiple compositions of Q_t^0. We single out natural hypotheses on P_t^0 and Q_t^0 that lead to the construction and basic properties of P_t. We furthermore point out key hypotheses on *the approximate solution* $P_{t,\varepsilon}$ that validate the semigroup property, non-negativity, etc., of P_t. We also discuss in a general context integral kernels associated with the constructed operators. Thus, the success of the parametrix method boils down to verifying the proposed hypotheses. It should be noted that they may only hold if P_t^0 is well chosen. In the literature there is enough evidence showing that the flexibility of the choice of the zero order approximation is crucial. Our results indeed provide such flexibility.

As an application of the general developed framework we consider the classical choice of P_t^0 by *freezing coefficients at the endpoint*. In doing so we focus on the case when the measure $N(x,dz)$ is non-symmetric in dz. In particular, a non-zero *internal drift*

$$\zeta_r^x := \int_{\mathbb{R}^d} z\left(\mathbf{1}_{|z|<r} - \mathbf{1}_{|z|<1}\right) N(x,dz)\,, \qquad r>0,$$

may be induced by non-symmetric jumps. For example, for $c\equiv 0$, $b\equiv 0$, $a_{ij}\equiv 0$, we get

$$\begin{aligned}\mathfrak{L}f(x) = &\int_{\mathbb{R}^d} \Big(f(x+z)-f(x)-\mathbf{1}_{|z|<r}\left\langle z,\nabla f(x)\right\rangle\Big)N(x,dz)\\ &+\left(\int_{\mathbb{R}^d} z\left(\mathbf{1}_{|z|<r}-\mathbf{1}_{|z|<1}\right)N(x,dz)\right)\cdot\nabla f(x)\,.\end{aligned}$$

The above can be interpreted as a decomposition of the operator into *the leading non-local part* and *the internal drift part*, and we wish to control the latter. As observed in examples, the internal drift ζ_r^x is more difficult to handle than the *external drift* $b(x)$. To be more specific, we study the operator

(1)
$$\mathcal{L}f(x) = b(x)\cdot\nabla f(x) + \int_{\mathbb{R}^d}\Big(f(x+z)-f(x)-\mathbf{1}_{|z|<1}\left\langle z,\nabla f(x)\right\rangle\Big)\,\kappa(x,z)J(z)dz\,.$$

Under certain assumptions on b, J, κ we prove the uniqueness and existence of a weak fundamental solution to the equation $\partial_t = \mathcal{L}$. We analyse the semigroup associated with the solution and discuss properties of its generator, which we identify as the closure of the operator $\mathcal{L}$ in (1) acting on the space of smooth compactly supported functions. Pointwise estimates of the fundamental solution are established under additional conditions.

Our main focus is the case when the mapping $z\mapsto \kappa(x,z)J(z)$ is non-symmetric. A surprising fact is that despite extensive study of non-local operators and rapidly growing literature of the subject, the following fundamental example has not yet been covered. Let $b=0$ and $J(z)=|z|^{-d-1}$ in (1), i.e.,

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d}\Big(f(x+z)-f(x)-\mathbf{1}_{|z|<1}\left\langle z,\nabla f(x)\right\rangle\Big)\frac{\kappa(x,z)}{|z|^{d+1}}dz\,, \tag{2}$$

and suppose only that $c_\kappa^{-1}\le\kappa(x,z)\le c_\kappa$ and $|\kappa(x,z)-\kappa(y,z)|\le c_\kappa|x-y|^{\varepsilon_\kappa}$ for some $c_\kappa>0$, $\varepsilon_\kappa\in(0,1]$ and all $x,y,z\in\mathbb{R}^d$. We note that the results available in [7], [11], [6], [4], require further assumptions on the coefficient κ to treat (2), which exclude natural examples. Our results remove those restrictions and we construct and estimate the semigroup $(P_t)_{t>0}$. Of course we also cover other interesting operators. We emphasize that the general functional analytic framework should apply to other zero order approximations P_t^0 [8], but even for the choice of P_t^0 resulting from freezing coefficients at the endpoint we obtain new results for the non-symmetric case.

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Numerical schemes for radial Dunkl processes

Dai Taguchi

(joint work with Hoang-Long Ngo)

Let R be a (reduced) root system in $\mathbb{R}^d$, that is, R is a finite set of nonzero vectors in $\mathbb{R}^d$ such that (R1) $R \cap \{c\alpha \ ; \ c \in \mathbb{R}\} = \{\alpha, -\alpha\}$, for any $\alpha \in R$; (R2) $\sigma_\alpha(R) = R$ for any $\alpha \in R$. Here σ_α is the orthogonal reflection with respect to $\alpha \in \mathbb{R}^d \setminus \{0\}$ defined by

$$\sigma_\alpha x = x - \frac{2\langle \alpha, x\rangle}{|\alpha|^2}\alpha = \left(I_d - \frac{2}{|\alpha|^2}\alpha\alpha^\top\right)x, \ x \in \mathbb{R}^d.$$

For a total ordering $>$ of $\mathbb{R}^d$, a positive subsystem of the root system R is denoted by R_+. A sub-group $W = W(R)$ of $O(d)$ is called the Weyl group generated by a root system R, if it is generated by the reflections $\{\sigma_\alpha \ ; \ \alpha \in R\}$, that is, $W = \langle \sigma_\alpha \mid \alpha \in R\rangle$.

The Dunkl operator T_i on $\mathbb{R}^d$ associated with W are introduced by Dunkl [5] and are differential-difference operators given by

$$T_i f(x) := \frac{\partial f(x)}{\partial x_i} + \sum_{\alpha \in R_+} k\alpha_i \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}.$$

Dunkl operators have been widely studied in both mathematics and physics. For example, there operators play a crucial role to the study special functions associated with root systems and the Hamiltonian operators of some Calogero-Moser-Sutherland quantum mechanical systems. Moreover, Rösler [8] studied Dunkl heat equation $(\Delta_k - \partial_t)u,\ u(\cdot,0) = f \in C_b(\mathbb{R}^d;\mathbb{R})$ where the Dunkl Laplacian defined by $\Delta_k f(x) := \sum_{i=1}^d T_i^2$ and has the following explicit form

$$\Delta_k f(x) = \Delta f(x) + 2\sum_{\alpha\in R_+} k\left\{\frac{\langle \nabla f(x),\alpha\rangle}{\langle \alpha, x\rangle} + \frac{f(\sigma_\alpha x) - f(x)}{\langle \alpha, x\rangle^2}\right\}.$$

Rösler and Voit [9] introduced Dunkl processes Y, which are càdlàg Markov processes with infinitesimal generator $\Delta_k/2$ and are martingales with the scaling property. On the other hand, a radial Dunkl process $X = (X(t))_{t\geq 0}$ is a continuous Markov process with infinitesimal generator $L_k^W/2$ defined by

$$\frac{L_k^W f(x)}{2} := \frac{\Delta f(x)}{2} + \sum_{\alpha\in R_+} k\frac{\langle \nabla f(x),\alpha\rangle}{\langle \alpha, x\rangle},$$

and is a W-radial part of the Dunkl process Y, that is, for the canonical projection $\pi : \mathbb{R}^d \to \mathbb{R}^d/W$, we have $X = \pi(Y)$, as identifying the space $\mathbb{R}^d/W$ to the (fundamental) Weyl chamber $\mathbb{W} := \{x \in \mathbb{R}^d\ ;\ \langle \alpha, x\rangle > 0,\ \alpha \in R_+\}$ of the root system R. Demini [3] proved that a radial Dunkl process X satisfies the following $\mathbb{W}$-valued stochastic differential equation (SDE)

$$\text{(1)}\qquad \mathrm{d}X(t) = \mathrm{d}B(t) + \sum_{\alpha\in R_+} \frac{k}{\langle \alpha, X(t)\rangle}\alpha\,\mathrm{d}t,\ X(0) = x(0) \in \mathbb{W},$$

where $B = (B(t))_{t\geq 0}$ is a d-dimensional standard Brownian motion (see also [10] for the radial Heckman–Opdam process). For example, if $R := \{\pm 1\}$ then X is a Bessel process, and if a type A_{d-1} root system, that is, $R := \{e_i - e_j \in \mathbb{R}^d\ ;\ i \neq j\} \subset \{x \in \mathbb{R}^d; \sum_{i=1}^d x_i = 0\}$, then X is a Dyson's Brownian motion.

In this talk, we consider the numerical approximation for a class of radial Dunkl processes corresponding to arbitrary (reduced) root systems. Inspired by [1, 4, 6, 7], we introduce a backward and truncated Euler-Maruyama scheme, which can be implemented on a computer, and study its rate of convergence in L^p-norm. The key idea of the proof is to use the change of measure based on Girsanov theorem for radial Dunkl processes, which was proved in [2] for general radial Dunkl processes, and in [11] for the Bessel case.

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Functional analytic techniques for Markov processes

Kazuaki Taira

This survey talk is devoted to the functional analytic approach to the problem of construction of Markov processes in probability theory. It is well known that, by virtue of the Hille–Yosida theory of semigroups, the problem of construction of Markov processes can be reduced to the study of boundary value problems for elliptic integro-differential operators of second order. In this talk we introduce a mathematical crossroads of functional analysis (macroscopic approach), partial differential equations (mesoscopic approach), and probability (microscopic approach) via the mathematics needed for the hard parts of Markov processes. This work brings these three fields of analysis together and provides a profound stochastic insight (microscopic approach) into the study of elliptic boundary value problems.

Let Ω be a bounded domain in Euclidean space $\mathbf{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega$. Its closure $\overline{\Omega} = \Omega \cup \partial\Omega$ is an n-dimensional, compact smooth manifold with boundary. Table 1 below gives a bird's-eye view of strong Markov processes, Feller semigroups and elliptic Ventcel' (Wentzell) boundary value problems for Waldenfels operators, and how these relate to each other (see [1]):

In this talk we consider the following problem:

"Conversely, construct a Feller semigroup $\{T_t\}_{t\geq 0}$ on the closure $\overline{\Omega}$ with prescribed analytic data (W, L)."

Several recent developments in the theory of partial differential equations have made possible further progress in the study of elliptic boundary value problems and hence of the problem of construction of Markov processes. This talk focuses on the relationship between Markov processes and elliptic boundary value problems with emphasis on the study of maximum principles. The approach here is distinguished by the extensive use of the theory of partial differential equations.

The content of this survey talk is summarized in Table 2 below (see [2], [3]):

Probability (Microscopic approach)	**Functional Analysis** (Macroscopic approach)	**Elliptic Boundary Value Problems** (Mesoscopic approach)
Strong Markov process $\boxed{\mathcal{X} = (x_t, \mathcal{F}, \mathcal{F}_t, P_x)}$	**Feller semigroup** $\boxed{T_t f(x) = \int_\Omega p_t(x, dy)\, f(y)}$ (Kakutani)	**Infinitesimal generator** $\boxed{T_t = e^{t\mathfrak{A}}}$ (Hille–Yosida)
Markov transition function $\boxed{P_x\{x_t \in E\} = p_t(x, E)}$ (Dynkin)	**Green kernel** $\boxed{G_\alpha(x, y) = \int_0^\infty e^{-\alpha t}\, p_t(x, dy)\, dt}$ (Riesz–Markov)	**Resolvent** $\boxed{(\alpha I - \mathfrak{A})^{-1}}$ (Hille–Yosida–Ray)
Chapman–Kolmogorov equation $p_{t+s}(x, dz) = \int_\Omega p_t(x, dy)\, p_s(y, dz)$	**Semigroup property** $T_{t+s} = T_t \cdot T_s$	**Waldenfels operator** $\boxed{W = A + S}$
Absorption, reflection, viscosity phenomena, two jump phenomena, diffusion along the boundary **(Six phenomena)**	**Function spaces** $C_0(\overline{\Omega})$ [Dirichlet case] $C(\overline{\Omega})$ [Other cases]	**Ventcel' (Wentzell) boundary condition** $\boxed{L}$

Table 1.

Diffusion operator A	**Lévy operator** S	**Ventcel' condition** $L = \Gamma - \delta(x')\, W$	**using the theory of**	**proved by**
Elliptic smooth case	$\boxed{S \equiv 0}$	$\Gamma = \mu(x') \frac{\partial}{\partial \mathbf{n}} + \gamma(x')$ $\boxed{\mu(x') > 0}$ on $\partial\Omega$	Classical potential theory	Sato–Ueno (1965)
Elliptic smooth case	general case (compact perturbations)	$\Gamma = \mu(x') \frac{\partial}{\partial \mathbf{n}} + Q + T$ Q (elliptic operator) T (Ventcel'–Lévy) $\boxed{\mu(x') > 0}$ on $\partial\Omega$	Classical potential theory	Bony et al. (1968)
Elliptic smooth case	$\boxed{S \equiv 0}$	$\Gamma = \mu(x') \frac{\partial}{\partial \mathbf{n}} + Q$ Q (degenerate case) $\boxed{\mu(x') + \delta(x') > 0}$ on $\partial\Omega$ (transversal case)	Pseudo-differential operators (**hypoelliptic case**)	Taira (1979) Taira (1988) Taira (2022)
Elliptic smooth case	general case (transmission property)	$\Gamma = \mu(x') \frac{\partial}{\partial \mathbf{n}} + Q + T$ Q (degenerate case) T (transmission property) $\boxed{\mu(x') + \delta(x') > 0}$ on $\partial\Omega$ (transversal case)	Pseudo-differential operators (**Boutet de Monvel calculus**)	Cancelier (1986) Taira (1992) Taira (2014) Taira (2020)
Elliptic discon-tinuous (**VMO**) case	general case (compact perturbations)	$\Gamma = \mu(x') \frac{\partial}{\partial \mathbf{n}} + Q + T$ Q (first order case) T (first order case) $\boxed{\mu(x') > 0}$ on $\partial\Omega$	Singular integral operators (Calderón and Zygmund)	My second talk (2022)

Table 2.

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Singular integrals and Feller semigroups with jump phenomena

Kazuaki Taira

This research talk is devoted to the real analysis methods for the problem of construction of Markov processes with boundary conditions in probability theory. Analytically, a Markovian particle in a domain of Euclidean space is governed by an integro-differential operator W, called Waldenfels operator, in the interior Ω of the domain, and it obeys a boundary condition L, called Ventcel' (Wentzell) boundary condition, on the boundary $\partial\Omega$ of the domain. Probabilistically, a Markovian particle moves both by continuous paths and by jumps in the state space and it obeys the Ventcel' boundary condition which consists of six terms corresponding to a diffusion along the boundary, an absorption phenomenon, a reflection phenomenon, a sticking (or viscosity) phenomenon and a jump phenomenon on the boundary and an inward jump phenomenon from the boundary.

In particular, second order elliptic differential operators are called diffusion operators which describe analytically strong Markov processes with continuous paths in the state space such as Brownian motion. We remark that second order elliptic differential operators with *discontinuous* coefficients enter naturally in connection with the problem of construction of Markov processes in probability theory. Since second order elliptic differential operators are pseudo-differential operators only if the coefficients are infinitely differentiable, we can not make use of the theory of pseudo-differential operators as in the book [2].

In this talk we consider the following problem:

"Conversely, construct a Feller semigroup $\{T_t\}_{t\geq 0}$ on the closure $\overline{\Omega} = \Omega \cup \partial\Omega$ with prescribed analytic data (W, L)."

Our approach here is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in the Calderón and Zygmund theory of singular integral operators with non-smooth (i.e., non-infinitely differentiable) kernels. It should be emphasized that singular integral operators with non-smooth kernels provide a powerful tool to deal with smoothness of solutions of partial differential equations, with minimal assumptions of regularity on the coefficients. The Calderón–Zygmund theory continues to be one of the most influential works in modern history of analysis, and is a very refined mathematical tool whose full power is yet to be exploited. Several recent developments in the theory of singular integrals have made possible further progress in the study of elliptic boundary value problems with discontinuous coefficients and hence in the study of Markov processes ([5]). The approach here is distinguished by the extensive use of function spaces such as the BMO (bounded mean oscillation) space due to John and Nirenberg and the VMO (vanishing mean oscillation) space due to Sarason, respectively. The presentation of these new results is the main purpose of this talk.

Main results of this research talk are summarized in Table 3 below.

Boundary conditions (various phenomena)	**Notation**	**Resolvents**	**Feller Semigroups**
Dirichlet case (absorption)	γ_0	$G^0_\alpha = (\alpha I - \mathfrak{W}_D)^{-1}$	$e^{t\mathfrak{W}_D}$
Oblique derivative case (absorption, reflection, drift)	Λ	$G^\lambda_\alpha = (\alpha I - \mathfrak{W}_\Lambda)^{-1}$	$e^{t\mathfrak{W}_\Lambda}$
Ventcel' case (absorption, reflection, drift, inward jump)	$\Gamma = \Lambda + \gamma_0 T$	$G^\gamma_\alpha = (\alpha I - \mathfrak{W}_\Gamma)^{-1}$	$e^{t\mathfrak{W}_\Gamma}$
General case (absorption, reflection, drift, inward jump, viscosity)	$L = \Lambda - \delta W$	$G_\alpha = (\alpha I - \mathfrak{W})^{-1}$	$e^{t\mathfrak{W}}$

TABLE 3.

More precisely, we prove four generation theorems for Feller semigroups with Dirichlet boundary condition, oblique derivative boundary condition and first order Ventcel' boundary condition for second order, uniformly elliptic differential operators with VMO coefficients, which extend earlier theorems due to Bony–Courrège–Priouret [1] to the VMO case. In other words, we construct Feller semigroups associated with the absorption, reflection, drift and sticking phenomena at the boundary and the inward jump phenomenon from the boundary.

Our proof is essentially based on various maximum principles for second order elliptic Waldenfels operators with discontinuous coefficients in the framework of L^p Sobolev spaces (see [3], [4]).

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Jump-type stochastic differential equations on manifolds

Atsushi Takeuchi

Let $T > 0$ be a constant, and (M, g) a connected, compact and smooth Riemannian manifold of dimension d with the Levi-Civita connection. Denote by $O(M)$ the bundle of orthnormal frames on M, and let $\pi : O(M) \to M$ be the canonical projection such that $\pi(r) = x$ for $r = (x, e) \in O(M)$. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider the $O(M)$-valued process $\{R_t\,;\, 0 \le t \le T\}$ determined by the following stochastic differential equation of Stratonovich type:

$$dR_t = \sum_{i=1}^{d} H_i(R_r) \circ dW_t^i, \quad R_0 = r \in O(M), \tag{1}$$

where $\{(W_t^1, \ldots, W_t^d)\,;\, 0 \le t \le T\}$ is a d-dimensional Brownian motion, and $H_1, \ldots, H_d$ are the canonical horizontal vector fields on $O(M)$. It is well known that the process R is strong Markovian with infinitesimal generator

$$\mathcal{L} := \frac{1}{2} \sum_{i=1}^{d} H_i H_i,$$

which is called the horizontal Laplacian of Bochner. Since the process W is rotationally invariant, the projected process $X := \pi(R)$ of the $O(M)$-valued process R determined by the equation (1) is also strong Markovian with the infinitesimal generator $\Delta_M/2$, where Δ_M is the Laplace-Beltrami operator on M. The process X as defined above is just the Brownian motion on M. Such procedure in which the M-valued process is constructed as the projected one in $O(M)$ is often called the Eells-Elworthy-Malliavin approach. See [4] on the detailed explanations.

It seems very natural to consider whether jump processes on M can be constructed by the procedure stated above or not. Hunt in [5] constructed Lévy processes on Lie groups from the viewpoint of functional analysis, and Applebaum and Kunita in [2] discussed stochastic flows of diffeomorphisms on Lie groups as solutions to stochastic differential equations driven by Lévy proceeese. One of the goals in this talk is to construct jump processes on M via the Eells-Elworthy-Malliavin approach. Before doing it, we shall prepare some notations. Let $\nu(dz)$ be the Lévy measure on $\mathbb{R}_0^d := \mathbb{R}^d \backslash \{0\}$, and $J(ds, dz)$ the Poisson random measure over $[0, T] \times \mathbb{R}_0^d$ with intensity $ds\, \nu(dz)$. For simplicity of notations, write

$$\bar{J}(ds, dz) = \mathbb{I}_{(|z| \le 1)} \left\{ J(ds, dz) - ds\, \nu(dz) \right\} + \mathbb{I}_{(|z|>1)}\, J(ds, dz).$$

For each $z \in \mathbb{R}_0^d$ and $r \in O(M)$, let $\{\Xi_\sigma^z(r)\,;\, 0 \le \sigma \le 1\}$ be the solution to the ordinary differential equation:

$$\frac{d}{d\sigma}\Xi_\sigma^z(r) = \sum_{i=1}^d H_i\big(\Xi_\sigma^z(r)\big)\, z_i, \quad \Xi_0^z(r) = r. \tag{2}$$

Now, let us consider the $O(M)$-valued process $\{\tilde{R}_t\,;\, 0 \le t \le T\}$ determined by the Marcus-type stochastic differential equation with jumps of the form:

$$d\tilde{R}_t = \sum_{i=1}^d H_i(\tilde{R}_t) \circ dW_t^i + \int_{\mathbb{R}_0^d} \big\{\Xi_1^z(\tilde{R}_{t-}) - \tilde{R}_{t-}\big\}\bar{J}(dt, dz), \quad \tilde{R}_0 = r \in O(M). \tag{3}$$

Then, the process $\tilde{R}$ is strong Markovian with infinitesimal generator

$$\mathcal{J}F(r) = \mathcal{L}F(r) + \int_{\mathbb{R}_0^d} \Big\{F\big(\Xi_1^z(r)\big) - F(r) - \sum_{i=1}^d H_i F(r)\, z_i\, \mathbb{I}_{(|z|\le 1)}\Big\}\, \nu(dz)$$

for $F \in C^\infty(O(M))$. But, the Markov property of the M-valued process $\tilde{X} := \pi(\tilde{R})$ is not so clear, because the process $\tilde{X}$ depends on the choice of frames. One of the sufficient conditions is introduced in this talk, which is the revisited result obtained by Applebaum and Estrade in [1]. See also [6].

The second interest in this talk is to study the commutativity on the procedures of the subordination and the projection. Let $0 < \alpha < 1$ be a constant. When our situation is in the Euclidean space, it is quite well known that the subordinated process W_τ of the Brownian motion W by the α-stable subordinator τ is just the 2α-stable process. The Lévy-Khintchine representation on the characteristic functions plays a crucial role. Now, let us proceed our position into the manifold M. Since the Brownian motion X on M can be constructed via the Eells-Elworthy-Malliavin approach stated above, the subordinated process X_τ of the Brownian motion X by the stable subordinator τ is well defined. On the other hand, we have already introduced the M-valued process $\tilde{X}$ via the projection of the $O(M)$-valued process $\tilde{R}$ determined by the equation (3). Then, we wonder what about the relationship between the M-valued processes X_τ and $\tilde{X}$ proposed in [3]. In this talk, the Wasserstein distance of the processes X_τ and $\tilde{X}$ implies that their probability laws are not always equivalent. The Wasserstein distance on the processes determined by stochastic differential equations of jump type is studied in [7], and the topics stated above is mentioned as one of the applications.

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On symmetric stable-type processes having singular Lévy densities

Toshihiro Uemura
(joint work with Masayoshi Takeda)

We are concerned with symmetric stable-type processes with singular Lévy densities:

$$(*) \qquad \mathcal{E}(u,v) = \iint_{x\neq y} \big(u(x)-u(y)\big)\big(u(x)-u(y)\big)k(x,y)dxdy,$$

where $k(x,y)$ is a Lévy density given by

$$k(x,y) = c_{d,\alpha}\frac{(|x|^{-\beta}\vee 1)(|y|^{-\beta}\vee 1)}{|x-y|^{d+\alpha}}$$

for $0<\alpha<2$ and $\beta\geq 0$.

When $\beta=0$, the (bilnear) functional $\mathcal{E}$ is the Dirichlet form of a translation invariant symmetric α-stable process on $\mathbb{R}^d$ and its domain is a fractional Sobolev space $W^{\alpha/2,2}(\mathbb{R}^d)$. The Lévy density $k(x,y)$ has sigularity at 0 not just on the diagonal $\{(x,x) : x\in\mathbb{R}^d\}$ when $\beta>0$. One of our motivations to consider the form $(*)$ is to know how the (singular) parameter β effects the paths behavior of the Markov process associated with the form $(*)$.

In the talk, we first introduce a class of test functions of the infinitesimal generators of the symmetric Dirichlet form $\mathcal{E}$ as a "core" and then show some estimates of the Markov process, which is called "a symmetric stable-type process", associated with the form.

We first show that the symmetric form defined by $(*)$ produces regular symmetric Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathbb{R}^d)$ under the conditions $0<\alpha<2$ and $0\leq 2\beta<d$. The Dirichlet form is conservative under the same conditions.

Then, introducing a class of test functions denoted by $C^\infty_\#(\mathbb{R}^d)$, the set of all smooth functions f defined on $\mathbb{R}^d$ having compact support such that f is constant on a neighborhood of the origin, we also show that the domain of the L^2-infinitesimal generator $(\mathcal{L},\mathcal{D}(\mathcal{L}))$ of $(\mathcal{E},\mathcal{F})$ contains the test functions $C^\infty_\#(\mathbb{R}^d)$ and that the form $\mathcal{L}f$ for $f\in C^\infty_\#(\mathbb{R}^d)$ can be expressed as follows:

$$\begin{aligned}\mathcal{L}f(x) = &\int_{h\neq 0}\Big(f(x+h)-f(x)-h\cdot\nabla f(x)\chi(h)\Big)k(x,x+h)dh \\ &+\frac{1}{2}\int_{h\neq 0} h\cdot\nabla f(x)\chi(h)\big(k(x,x+h)-k(x,x-h)\big)dh, \quad x\in\mathbb{R}^d,\end{aligned}$$

where χ is a *centering function*, which is a measurable bounded function defined on $\mathbb{R}^d$ so that $\chi(h) = \chi(-h)$ for $h \in \mathbb{R}^d$ and $\lim_{h\to 0}(\chi(h) - 1)/|h| = 0$.

Using the precise expression of the infinitesimal generator for the test function and considering the Fukushima decomposition (or the usual semimartingale decomposition), we will estimate the exit time of balls.

To this end, recall that, for a rotational invariant symmetric α-stable process with $\alpha < d$ (i.e., the process is transient), the following estimate are well-known: there exist positive constants $c > 0$ such that for any $r > 0$,

$$\mathbb{E}_z[\tau_r] = c(r^2 - |z|^2)^{\alpha/2}, \ |z| < r$$

(*e.g.*, [1, 3]). Here $\tau_r = \inf\{t > 0 : |X_t| > r\}$ is the exit time ball at 0 with radius $r > 0$. This implies that

$$\mathbb{E}_z[\tau_r] \approx r^\alpha \quad \text{for} \quad |z| < r/2.$$

Let us return to our process. We will obtain a portion of the above well-known result as follows: There exist a positive constant $c > 0$ such that for any $0 < r < 1$,

$$\mathbb{E}_z[\tau_r] \leq c r^{\alpha+2\beta} \quad \text{for} \quad |z| < r/2.$$

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Nonlocal capillarity theory

Enrico Valdinoci

(joint work with Alessandra De Luca, Serena Dipierro, Francesco Maggi)

We describe recent results motivated by a nonlocal capillarity theory introduced in [4]. The notion of capillarity, as related to the formation of droplets, is a classical topic of investigation and, in relation with surface tension, the phenomena described by this theory are topical in virtually all the branches of science (such as physics, material sciences, biology, chemistry, etc.). Surface tension is itself a very complex phenomenon, which can be seen as the macroscopic byproduct of complicated microscopic features due to molecule attraction, cohesion and adhesion.

For this, the classical capillarity theory aims at describing surface tension as a local average of intermolecular forces which in principle possess long-range contributions and, to account for the formation of droplets in a container Ω (supposed to be open and smooth), it prescribes the minimization of a "perimeter-like" energy functional

$$\mathcal{E}(E) := |(\partial E) \cap \Omega| + \sigma\, |(\partial E) \cap (\partial\Omega)|,$$

under a volume constraint $|E| = m$.

In this setting, E is the droplet, of a given volume m, and the parameter σ, which is called "relative adhesion coefficient" takes into account, roughly speaking, the different interfacial tensions between the droplet and the air (say γ_{DA}), the droplet and the container (say γ_{DC}) and the container and the air (say γ_{CA}; this would produce that $\sigma = \frac{\gamma_{DC}-\gamma_{CA}}{\gamma_{DA}}$).

When $\sigma = 1$, we have that $\mathcal{E}(E) = |\partial E|$ and the minimum of the functional, for small enough volumes, is a ball (placed wherever in Ω): this is a "completely non-wetting" situation and the container does not meet the droplet (or so it does only at a single point). Instead, when $\sigma = -1$, we have that $\mathcal{E}(E) = |(\partial E) \cap \Omega| - |(\partial E) \cap (\partial\Omega)|$ and the minimum of $\mathcal{E}$, for large enough volumes, is a the complement of a ball (placed wherever in Ω: this case is "perfectly wetting" and the container is in full contact with the droplet, possibly except at one single point).

Interesting cases are those which produce a droplet with some contact with the boundary of the container, and these correspond to the range $\sigma \in (-1, 1)$. The contact angle is indeed a quantity of physical interest, measured by a "contact angle goniometer" using high resolution cameras.

The first theoretical determination of the contact angle is due to Thomas Young (polymath who made notable contributions to the fields of vision, light, solid mechanics, energy, physiology, language, musical harmony, and Egyptology): remarkably, Young's Law determines the contact angle ϑ in terms of the relative adhesion coefficient σ according to the elegant formula $\cos\vartheta = -\sigma$. In particular, when $\sigma > 0$, we have that $\vartheta \in \left(\frac{\pi}{2}, \pi\right)$ and the material is "hydrophobic". Instead, when $\sigma < 0$, we have that $\vartheta \in \left(0, \frac{\pi}{2}\right)$ and the material is "hydrophilic".

The geometry of the droplet in the classical capillarity theory is also quite well-understood, since volume preserving perturbations of the functional $\mathcal{E}$ which do not touch the container show that the mean curvature of the droplet is necessarily constant: thus, a prototypical example of small classical droplets is that of spherical regions meeting the container at an angle ϑ in accordance with Young's Law.

In [4], we introduced a nonlocal capillarity theory in which the energy functional fully accounts for remote particle interplay in view of an integral kernel K. The prototypical case for K is that of an interaction kernel which is invariant under translations and rotations and takes the form $K_s(x) := \frac{1}{|x|^{n+s}}$. Point-set interactions (after Caffarelli-Roquejoffre-Savin), for given disjoint sets X, $Y \subset \mathbb{R}^n$ can be described by the multiple integral $\mathcal{I}(X,Y) := \iint_{X\times Y} K(x-y)\,dx\,dy$ and the corresponding capillarity functional can be thus taken of the form

$$\mathcal{E}_s(E) := \mathcal{I}(E, \Omega \setminus E) + \sigma\, \mathcal{I}(E, \mathbb{R}^n \setminus \Omega).$$

The first term mimics the molecular interactions of the droplet with the gas, the second with the external environment.

It is customary to consider the "regular part Reg_E of ∂E", namely, roughly speaking, the points of $\overline{(\partial E) \cap \Omega}$ that are locally of class $C^{1,\alpha}$ with $\alpha \in (s,1)$ and the nonlocal mean curvature of ∂E at $x \in \mathrm{Reg}_E$ with respect to the kernel K, defined as

$$\mathcal{H}^K_{\partial E}(x) := \int_{\mathbb{R}^n} \Big(\chi_{\mathbb{R}^n \setminus E}(y) - \chi_E(y) \Big)\, K(x-y)\, dy.$$

This integral converges in the principal value sense as soon as E is $C^{1,\alpha}$ near x with $\alpha \in (s,1)$. In this setting (see [4]), we have that if E is a critical set for the nonlocal capillarity functional for the kernel K_s and $x \in \mathrm{Reg}_E$, then

$$\mathcal{H}^{K_s}_{\partial E}(x) + (\sigma - 1) \int_{\mathbb{R}^n \setminus \Omega} K_s(x-y)\, dy = \text{const}.$$

Differently from the classical case, the relative adhesion coefficient σ appears in the above equation (but this dependence vanishes as $s \nearrow 1$).

To determine the nonlocal contact angle, one can perform a blow-up at a regular boundary point (say, the origin) and obtain (see [4]) the following Nonlocal Young's Law: if H and V denote the half-spaces such that the blow-up of Ω approaches H and the blow-up of E approaches V, then, the angle ϑ between H and V satisfies, for every $v \in (\partial V) \cap H$, the identity

$$\mathcal{H}^{K_s}_{\partial(H \cap V)}(v) + (\sigma - 1) \int_{\mathbb{R}^n \setminus H} K_s(v-y)\, dy = 0.$$

This equation uniquely identifies the angle $\vartheta = \vartheta(s, \sigma)$ between H and V.

Other results in [4, 3] address the interior regularity of the minimizers and the behavior at the boundary. A detailed asymptotic expansion of the nonlocal contact angle has been obtained in [2].

In [1], we addressed the case of more general kernels which are not necessarily invariant under rotations. This setting produces several new features, such as the influence of different scales in the dilation arguments and the lack of cancellations in singular integrals. In a nutshell, one can consider two kernels K_1 and K_2 such that $\frac{\chi_{B_\varrho}(x)\, K_{s_j}(x)}{\lambda} \le K_j(x) \le \lambda K_{s_j}(x)$ for some $\varrho > 0$ and $\lambda \ge 1$, look at the interactions $\mathcal{I}_j(X,Y) := \iint_{X \times Y} K_j(x-y)\, dx\, dy$ and at the energy functional

$$\mathcal{E}(E) := \mathcal{I}_1(E, \Omega \setminus E) + \sigma\, \mathcal{I}_2(E, \mathbb{R}^n \setminus \Omega).$$

In this setting, if E is a critical set for the anisotropic nonlocal capillarity functional and $x \in \mathrm{Reg}_E$, then

$$\mathcal{H}^{K_1}_{\partial E}(x) - \int_{\mathbb{R}^n \setminus \Omega} K_1(x-y)\, dy + \sigma \int_{\mathbb{R}^n \setminus \Omega} K_2(x-y)\, dy = \text{const}.$$

To understand the contact angle in the anisotropic case, one considers the blow-up limit of the kernels, namely assuming that $K_j^*(x) := \lim_{r\searrow 0} r^{n+s_j} K_j(x) = \frac{a_j(\vec{x})}{|x|^{n+s_j}}$, for some positive, continuous, even function a_j, where $\vec{x} := \frac{x}{|x|}$.

The determination of the contact angle ϑ in this case heavily depends on the different homogeneity powers s_1 and s_2 of the kernels (and notice that when $\sigma = 0$ formally we have s_2 free for us to choose!). More specifically, if $s_1 < s_2$, then the kernel K_2 "dominates at small scales", hence it becomes determinant for the determination of the contact angle (the kernel K_1 becoming "ineffective" and the system only seeing the interaction of the droplet with the external environment): thus, when $s_1 < s_2$, if $\sigma < 0$ we have that $\vartheta = 0$ and if $\sigma > 0$ we have that $\vartheta = \pi$.

Instead, when $s_1 > s_2$ (or $s_1 \leq s_2$ and $\sigma = 0$) the kernel K_1 "dominates at small scales" (the kernel K_2 becoming "ineffective", and the environment playing only a marginal role): in this case, $\vartheta \in (0, \pi)$ and, for every $v \in (\partial V) \cap H$,

$$\mathcal{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{\mathbb{R}^n\setminus H} K_1^*(v-y)\, dy = 0.$$

The more interesting case is thus when $s_1 = s_2$, since the two kernels have a perfect scaling balance and one expects that both play a role in the determination of the contact angle.

To obtain nontrivial contact angles, one also assumes the following bound on the relative adhesion coefficient: $|\sigma|\, K_2(x) \leq (1-\epsilon_0)\, K_1(x)$ for all $x \in B_{\epsilon_0} \setminus \{0\}$, for some $\epsilon_0 \in (0,1)$. This condition can be seen as the natural counterpart of the hypothesis that $\sigma \in (-1,1)$ in the classical capillarity theory. In this framework, when $s_1 = s_2$, it holds that $\vartheta \in (0,\pi)$ and, for every $v \in (\partial V) \cap H$,

$$\mathcal{H}^{K_1^*}_{\partial(H\cap V)}(v) - \int_{\mathbb{R}^n\setminus H} K_1^*(v-y)\, dy + \sigma \int_{\mathbb{R}^n\setminus H} K_2^*(v-y)\, dy = 0.$$

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Exponential ergodicity for damping Hamiltonian dynamics with state-dependent and non-local collisions

Jian Wang
(joint work with Jianhai Bao)

Piecewise deterministic Markov processes (PDMPs for short) constitute a very natural class of non-diffusive stochastic processes, where the mathematical framework was built by Mark H. A. Davis in [3]. Roughly speaking, the PDMP is a process which jumps at some random time and moves continuously between two adjacent random times; see [4] for more details. According to [3, Section 3], the probability law of a PDMP with the state space E is determined by the following three ingredients: (i) a vector field Ξ, generating a deterministic flow; (ii) a jump rate function $J : E \to [0, \infty)$, giving the law of the random times between jumps; (iii) a jump measure $Q : E \times E \to (0, \infty)$ (i.e., for each fixed $A \in \mathcal{B}(E)$, $E \ni x \mapsto Q(x, A)$ is a measurable function, and, for each fixed $x \in E$, $\mathcal{B}(E) \ni A \mapsto Q(x, A)$ is a probability measure), giving the transition probability kernel of its jumps.

Here, we consider a special class of PDMPs $(X_t, V_t)_{t\geq 0}$ on the state space $\mathbb{R}^{2d} := \mathbb{R}^d \times \mathbb{R}^d$ and associated with the following infinitesimal generator

$$
\begin{aligned}
\big(\mathcal{L}f\big)(x,v) &= \big(\langle \nabla_x f(x,v), v\rangle - \langle \nabla_v f(x,v), \gamma v + \nabla U(x)\rangle\big) \\
&\quad + J(x,v)\int_{\mathbb{R}^d} \big(f(x,u) - f(x,v)\big)\varphi(u)\,du \qquad (1)\\
&=: (\mathcal{L}_{1,\gamma} f)(x,v) + (\mathcal{L}_2 f)(x,v), \qquad f \in C_b^1(\mathbb{R}^{2d}),
\end{aligned}
$$

where $\gamma > 0$, $U : \mathbb{R}^d \to \mathbb{R}$ is smooth, $J : \mathbb{R}^{2d} \to (0, \infty)$, and $\varphi(\cdot)$, which is radial (i.e., $\varphi(x) = \varphi(|x|)$ for all $x \in \mathbb{R}^d$), is a probability density function on $\mathbb{R}^d$. In (1), $C_b^1(\mathbb{R}^{2d})$ means the collection of bounded real-valued functions $f(x, v)$ on $\mathbb{R}^{2d}$, which are differentiable in x and v, respectively, and $\nabla_x f(x, v)$ and $\nabla_v f(x, v)$ denote the first order gradients of $f(x, v)$ with respect to the variable x and the variable v, respectively.

Now, we make some detailed expositions on the quantities involved in (1). More precisely, $(v, -\gamma v - \nabla U(x))$ is the vector field generating the damping Hamiltonian flow, where γ means the friction intensity that ensures a damped-driven Hamiltonian and $-\gamma v$ stands for the damping force; $J : \mathbb{R}^{2d} \to (0, \infty)$ is the jump rate; $\varphi(u)\,du$ represents the jump measure. In terminology, $\mathcal{L}_{1,\gamma}$ is called the Liouville operator associated with the damping Hamiltonian flow generated by the vector field $(x, -\gamma v - \nabla U(x))$, and $\mathcal{L}_2$ is the so-called non-local collision operator. In particular, if $\varphi(u)$ is the density function of the standard normal distribution and $J(x, v) = \lambda$ for all $x, v \in \mathbb{R}^d$, $\mathcal{L}_2$ is called the complete momentum randomization operator; see, for example, [2]. It is worthy to emphasize that, in statistical physics, the damping Hamiltonian system has been applied widely to model many vibration phenomena (e.g., the generalized Duffing oscillator); see e.g. [5, 6].

The purpose of this talk is to study the exponential ergodicity of the PDMP $(X_t, V_t)_{t\geq 0}$ whose generator $\mathcal{L}$ is given by (1). Before we state our main result, we first present the assumptions. First of all, we assume that

($\mathbf{H_0}$) *For any $\beta \in \mathbb{R}$, there exists a constant $K_{\beta,U} > 0$ such that for all $x, x' \in \mathbb{R}^d$,*

$$|\beta(x - x') + \nabla U(x') - \nabla U(x)| \le K_{\beta,U}|x - x'|.$$

In particular, ∇U is Lipschitz continuous under ($\mathbf{H_0}$).

For the jump rate J and the probability density φ of the jump measure, we assume that

($\mathbf{A_1}$) *$J : \mathbb{R}^{2d} \to (0, \infty)$ is uniformly bounded between two positive constants, i.e., there exist constants $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 \le J(x, v) \le \lambda_2$ for all $(x, v) \in \mathbb{R}^{2d}$. Moreover, J is globally Lipschitz continuous on $\mathbb{R}^{2d}$, i.e., there exists a constant $\lambda_J > 0$ such that for all $(x, v), (x', v') \in \mathbb{R}^{2d}$,*

$$|J(x, v) - J(x', v')| \le \lambda_J\big(|x - x'| + |v - v'|\big).$$

($\mathbf{A_2}$) *For any $\alpha, \kappa > 0$, there exist $c_*(\alpha, \kappa), c^*(\alpha, \kappa) > 0$ such that for all $z \in \mathbb{R}^d$,*

$$c_*(\alpha, \kappa) \le A_{\alpha,\kappa}(z) := \int_{\mathbb{R}^d} \psi_{\alpha(z)_\kappa}(u)\, du \quad \text{and} \quad 1 - A_{\alpha,\kappa}(z) \le c^*(\alpha, \kappa)|z|,$$

where for all $\xi, u \in \mathbb{R}^d$,

$$\psi_\xi(u) := \varphi(u) \wedge \varphi(u + \xi),$$

and, for the threshold $\kappa > 0$, the truncation counterpart of $z \in \mathbb{R}^d$ is defined by

$$(z)_\kappa = \frac{(\kappa \wedge |z|)z}{|z|} 1_{\{z \neq \mathbf{0}\}} + \mathbf{0}1_{\{z = \mathbf{0}\}}.$$

Since $A_{\alpha,\kappa}(0) = \int_{\mathbb{R}^d} \psi_0(u)\, du = \int_{\mathbb{R}^d} \varphi(u)\, du = 1$, in some sense ($\mathbf{A_2}$) indicates the non-degenerate property and the continuity of the probability density φ.

Besides all the assumptions above, we further need the following Lyapunov condition:

($\mathbf{B_1}$) *There exist a C^1-function $\mathcal{W} : \mathbb{R}^{2d} \to [1, \infty)$ and constants $c_0, C_0 > 0$ such that*

$$\lim_{|x|+|v| \to \infty} \mathcal{W}(x, v) = \infty$$

and for all $(x, v) \in \mathbb{R}^{2d}$,

$$(\mathcal{L}\mathcal{W})(x, v) \le -c_0\mathcal{W}(x, v) + C_0.$$

($\mathbf{B_2}$) *There exists a constant $c^{**} > 0$ such that for all $x, \xi \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} \mathcal{W}(x, u)\varphi(u)\, du \le c^{**} \inf_{v \in \mathbb{R}^d} \mathcal{W}(x, v), \; \int_{\mathbb{R}^d} \mathcal{W}(x, u)\, \Psi_\xi(u)\, du \le c^{**} \inf_{v \in \mathbb{R}^d} \mathcal{W}(x, v)|\xi|,$$

where for all $\xi, u \in \mathbb{R}^d$,

$$\Psi_\xi(u) := \varphi(u) - \psi_\xi(u).$$

Let $\mathcal{P}(\mathbb{R}^{2d})$ be the set of probability measures on $\mathbb{R}^{2d}$. For $\mu,\nu \in \mathcal{P}(\mathbb{R}^{2d})$, define the quasi-Wasserstein distance between μ and ν induced by a distance-like function $\Phi : \mathbb{R}^{2d} \times \mathbb{R}^{2d} \to [0,\infty)$ as below

$$W_\Phi(\mu,\nu) = \inf_{\Pi\in\mathcal{C}(\mu,\nu)} \int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} \Phi(x,y)\,\Pi(dx,dy),$$

where $\mathcal{C}(\mu,\nu)$ stands for the collection of all couplings of μ and ν. In particular, W_Φ goes back to the classical Wasserstein distance when Φ is a metric function. Note that $W_\Phi(\mu,\nu) = 0$ if and only if $\mu = \nu$, since Φ is a distance-like function. Moreover, the space

$$\mathcal{P}_\Phi(\mathbb{R}^{2d}) := \Big\{\mu \in \mathcal{P}(\mathbb{R}^{2d}) : \int_{\mathbb{R}^{2d}} \Phi(x,\mathbf{0})\,\mu(dx) < \infty\Big\}$$

is complete under W_Φ, i.e., each W_Φ-Cauchy sequence in $\mathcal{P}_\Phi(\mathbb{R}^{2d})$ converges with respect to W_Φ.

For each $t \geq 0$, let $P_t\big((x,v),\cdot\big)$ be the transition probability kernel of the Markov process $(X_t,V_t)_{t\geq 0}$ with initial value $(X_0,V_0) = (x,v)$ associated with the generator $\mathcal{L}$. Furthermore, we shall write μP_t to mean the distribution of (X_t,V_t) with initial distribution $\mu \in \mathcal{P}(\mathbb{R}^{2d})$.

Theorem 1. *Assume that* $(\mathbf{H_0})$, $(\mathbf{A_1})$, $(\mathbf{A_2})$, $(\mathbf{B_1})$ *and* $(\mathbf{B_2})$ *hold, and that the following inequality*

$$\beta \geq 4K_{\beta,U}$$

is solvable in the interval $(0,\gamma^2/4]$, *where* γ *was given in* (1) *and* $K_{\beta,U}$ *was given in* $(\mathbf{H_0})$. *Then, the PDMP* $(X_t,V_t)_{t\geq 0}$ *corresponding to the operator* $\mathcal{L}$ *in* (1) *is exponentially ergodic in the sense that there exist a unique invariant probability measure* $\mu \in \mathcal{P}_\Phi(\mathbb{R}^{2d})$ *and a constant* $\lambda^* > 0$ *such that for any* $\nu \in \mathcal{P}_\Phi(\mathbb{R}^{2d})$ *and* $t \geq 0$,

$$W_\Phi\big(\nu P_t,\mu\big) \leq C(\mu,\nu)\mathrm{e}^{-\lambda^* t},$$

where for all $(x,v),(x',v') \in \mathbb{R}^{2d}$,

$$\Phi\big((x,v),(x',v')\big) := \big((|x-x'|+|v-v'|)\wedge 1\big)\big(\mathcal{W}(x,v)+\mathcal{W}(x',v')\big)$$

and $C(\mu,\nu)$ *is a positive function depending on* μ *and* ν *(indepedent of* t*).*

To illustrate the effectiveness of Theorem 1, we consider the following example.

Example 2. Assume that Assumption $(\mathbf{A_1})$ holds. Let $U(x) = \theta|x|^2$ with

$$\frac{\gamma^2}{8} \geq \theta > \frac{(\lambda_1+\gamma)^2(\lambda_2-\lambda_1)^2}{4(2\lambda_1\lambda_2-\lambda_1^2+4\lambda_2\gamma+3\gamma^2)},$$

and $\varphi(x) = \varphi_1(x) := c_{d,\beta_1}(1+|x|)^{-d-\beta_1}$ with $\beta_1 > 0$ or $\varphi(x) = \varphi_2(x) := c_{d,\beta_2}\exp(-|x|^{\beta_2})$ with $\beta_2 > 0$. Then, the conclusion of Theorem 1 holds with $\mathcal{W}(x,v) = (1+|x|^2+|v|^2)$ and the previously defined φ_2 or φ_1 when $\beta_1 > 2$, and with $\mathcal{W}(x,v) = (1+|x|^2+|v|^2)^{(\beta_1-\varepsilon)/2}$ for any $\varepsilon \in (0,\beta_1)$ and the foregoing φ_1 when $\beta_1 \in (0,2]$.

The approach to the main result above is also motivated partly by our previous work [1] on exponential ergodicity of stochastic Hamiltonian systems with Lévy noises. However, in contrast to [1], the non-local collision operator in the present setting is not only highly degenerate but also state-dependent, so much more delicate work are to be implemented. In particular, we shall adopt a combination of the refined basic coupling and the refined reflection coupling (rather than the refined basic coupling exploited merely in [1]) in order to include more general probability measures (e.g., (sup-)Gaussian or (sub-)Gaussian probability measures and probability measures with heavy tails). So, in a certain sense, Theorem 1 is a continuation of the corresponding main result in [1] on exponential ergodicity of stochastic Hamiltonian systems with Lévy noises. Furthermore, we emphasize that the process under investigation in this paper has some essentially different properties from stochastic Hamiltonian systems with Lévy noises under consideration in [1]. For example, under some regular conditions the process associated with stochastic Hamiltonian systems with Lévy noises can possess the strong Feller property. Nonetheless, since the non-local collision operator $\mathcal{L}_2$ in (1) is a bounded operator on $B_b(\mathbb{R}^{2d})$ under Assumption $(\mathbf{A_1})$, the PDMP $(X_t, V_t)_{t\geq 0}$ corresponding to the operator $\mathcal{L}$ in (1) can never enjoy the strong Feller property.

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Regularity estimates for nonlocal operators related to nonsymmetric forms

Marvin Weidner
(joint work with Moritz Kassmann)

The aim of this talk is to prove regularity properties for nonlocal operators related to nonsymmetric bilinear forms. Such operators are determined by jumping kernels $K : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$ which might be nonsymmetric. The corresponding operators

$$-L^K u(x) = 2\,\mathrm{p.v.} \int_{\mathbb{R}^d} (u(x) - u(y))K(x, y)\mathrm{d}y$$

are associated with nonsymmetric bilinear forms

$$\mathcal{E}^K(u,v) = 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x)-u(y))v(x)K(x,y)\mathrm{d}y\mathrm{d}x$$
$$=: \mathcal{E}^{K_s}(u,v) + \mathcal{E}^{K_a}(u,v),$$

where

$$\mathcal{E}^{K_s}(u,v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)K_s(x,y)\mathrm{d}y\mathrm{d}x,$$
$$\mathcal{E}^{K_a}(u,v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\big(u(x)-u(y)\big)\big(v(x)+v(y)\big)K_a(x,y)\mathrm{d}y\mathrm{d}x,$$

and K_s, K_a are the symmetric respectively the antisymmetric part of K, i.e.,

$$K_s(x,y) = \frac{K(x,y)+K(y,x)}{2}, \qquad K_a(x,y) = \frac{K(x,y)-K(y,x)}{2}.$$

In the last 15 years, a lot of research has been devoted to the symmetric case, i.e., when $K_a = 0$. Regularity properties such as Hölder regularity, local boundedness or the validity of weak and full Harnack inequalities for weak solutions to nonlocal elliptic equations have been investigated via energy form approaches e.g., in [3], [5], [4], [6]. Weak parabolic Harnack inequalities and Hölder regularity estimates have been derived in the symmetric case in [2] for weak solutions to

$$\text{(PDE)} \qquad \partial_t u - Lu = f \text{ in } I_R(t_0)\times B_{2R} \subset \mathbb{R}^{d+1},$$

where $B_{2R} \subset \Omega$ is some ball, $I_R(t_0) := (t_0 - R^\alpha, t_0 + R^\alpha) \subset \mathbb{R}$, $t_0 \in \mathbb{R}$, $\Omega \subset \mathbb{R}^d$ is a fixed open set and $f \in L^\infty(I_R(t_0)\times B_{2R})$. They developed an energy form approach based on a nonlocal adaptation of Moser iteration under the following two assumptions on the jumping kernel K_s. Let $\alpha \in (0,2)$:

Assumption ($\mathcal{E}_\asymp$)**.** There is $\Lambda \geq 1$ such that for every ball $B_R \subset \Omega$, $0 < R < 1$:

$$(\mathcal{E}_\asymp) \qquad \Lambda^{-1}[f]^2_{H^{\alpha/2}(B_R)} \leq \mathcal{E}^{K_s}_{B_R}(f,f) \leq \Lambda[f]^2_{H^{\alpha/2}(B_R)},\ \forall f \in L^2(B_R).$$

Assumption (tail-est)**.** There is $\Lambda \geq 1$ such that for every $\rho > 0$:

$$\text{(tail-est)} \qquad \sup_{x\in\Omega}\int_{\mathbb{R}^d\setminus B_\rho(x)} K_s(x,y)\mathrm{d}y \leq \Lambda\rho^{-\alpha}.$$

($\mathcal{E}_\asymp$) can be regarded as a coercivity assumption, guaranteeing the validity of certain functional inequalities, like a fractional Sobolev-type embedding and a Poincaré inequality. (tail-est) is an integrated version of a pointwise upper bound.

In this talk, we extend some of the aforementioned approaches to the nonsymmetric case. The novelty caused by the absence of symmetry lies in the existence of the second summand $\mathcal{E}^{K_a}$ which is of different shape compared to $\mathcal{E}^{K_s}$. In order to control $\mathcal{E}^{K_a}$, we need to impose suitable conditions on the jumping kernel K.

Assumption (K1). Let $J : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ be symmetric and $\theta \in [\frac{d}{\alpha}, \infty]$.

- K satisfies (K1$_{loc}$) if there is $C > 0$ s.t. for every ball $B_{2r} \subset \Omega$ with $r \leq 1$:

$$\left\|\int_{B_{2r}} \frac{|K_a(\cdot,y)|^2}{J(\cdot,y)}\mathrm{d}y\right\|_{L^\theta(B_{2r})} \leq C, \quad \mathcal{E}^J_{B_{2r}}(v,v) \leq C\mathcal{E}^{K_s}_{B_{2r}}(v,v),\ \forall v \in L^2(B_{2r}).$$

- K satisfies (K1$_{glob}$) there is $C > 0$ s.t. for every ball $B_{2r} \subset \Omega$ with $r \leq 1$:

$$\left\| \int_{\mathbb{R}^d} \frac{|K_a(\cdot, y)|^2}{J(\cdot, y)} \mathrm{d}y \right\|_{L^\theta(\mathbb{R}^d)} \leq C, \quad \mathcal{E}^J_{B_{2r}}(v,v) \leq C\mathcal{E}^{K_s}_{B_{2r}}(v,v), \ \forall v \in L^2(B_{2r}).$$

Assumption (K2). There exist $C > 0$, $D < 1$ and a symmetric $j : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty]$ s.t. for every ball $B_{2r} \subset \Omega$ with $r \leq 1$:

$$K(x,y) \geq (1-D)j(x,y), \ \forall x,y \in B_{2r}, \quad \mathcal{E}^{K_s}_{B_{2r}}(v,v) \leq C\mathcal{E}^{j}_{B_{2r}}(v,v), \ \forall v \in L^2(B_{2r}).$$

Assumption (K1$_{glob}$) can be interpreted as a sector condition for the nonsymmetric form $\mathcal{E}^K$. Indeed, together with ($\mathcal{E}_{\asymp}$), we have that $(\mathcal{E}, V^{K_s}(\mathbb{R}^d|\mathbb{R}^d))$ is a regular lower bounded semi-Dirichlet form in the sense of Oshima, where $V^{K_s}(\mathbb{R}^d|\mathbb{R}^d) = \{v \in L^2(\mathbb{R}^d) : \mathcal{E}^{K_s}(v,v) < \infty\}$. The range of $\theta \in [\frac{d}{\alpha}, \infty]$ ensures that the antisymmetric part $\mathcal{E}^{K_a}$ does not have supercritical scaling.
(K1$_{loc}$) is a localized version of (K1$_{glob}$), which is sufficient in many cases because we are interested in interior regularity estimates. (K2) is an additional local coercivity assumption on $K - |K_a|$.

Example. We have in mind the following two examples of jumping kernels:

(i) Let $0 < \lambda \leq \Lambda < \infty$, $V : \mathbb{R}^d \to \mathbb{R}^d$ s.t. $|V(x) - V(y)| \leq \lambda$ for every $x, y \in \mathbb{R}^d$ and $j : \mathbb{R}^d \times \mathbb{R}^d \to [\lambda, \Lambda]$ symmetric. Define

$$K_1(x,y) = j(x,y)|x-y|^{-d-\alpha} + (V(x) - V(y))|x-y|^{-d-\alpha}.$$

One can prove that K_1 satisfies (K1$_{glob}$) if $V \in C^{0,\gamma}(\mathbb{R}^d)$ for some $\gamma > \frac{\alpha}{2}$. The associated operator L^{K_1} is a nonlocal analog to

$$-\operatorname{div}(a_{i,j}(x)\nabla f(x)) + 2d(x)\nabla f(x).$$

(ii) Let $C \subset \mathbb{R}^d$ be a single cone and $D \subset \mathbb{R}^d$ be a double cone such that $C \cap D = \emptyset$. Let $0 < \beta < \alpha/2 < \alpha < 2$. Consider

$$K_2(x,y) = |x-y|^{-d-\alpha}\mathbb{1}_D(x-y) + |x-y|^{-d-\beta}\mathbb{1}_C(x-y).$$

The aforementioned examples illustrate that the class of admissible operators with respect to ($\mathcal{E}_{\asymp}$), (tail-est), (K1) and (K2) can be seen as symmetric nonlocal operators with a lower order nonlocal drift term. Now we state our main result:

Theorem 1 (see [1]). *Assume* (K1$_{loc}$), (K2), (tail-est), *and* ($\mathcal{E}_{\asymp}$) *for some* $\alpha \in (0,2)$, $\theta \in [\frac{d}{\alpha}, \infty]$. *Then the following hold:*

(i) (weak Harnack inequality): There is $c > 0$ s.t. for every $0 < R \leq 1$, and every nonnegative, weak supersolution u to (PDE) *in $I_R(t_0) \times B_{2R}$:*

$$\inf_{\left(t_0+R^\alpha-(\frac{R}{2})^\alpha, t_0+R^\alpha\right) \times B_{\frac{R}{2}}} u \geq cR^{-d-\alpha} \int_{\left(t_0-R^\alpha, t_0-R^\alpha+(\frac{R}{2})^\alpha\right) \times B_{\frac{R}{2}}} u - cR^\alpha \|f\|_{L^\infty}.$$

(ii) (Hölder estimate): There are $c > 0$ and $\gamma \in (0,1)$ s.t. for every $0 < R \leq 1$ and every weak solution u to (PDE) *in $I_R(t_0) \times B_{2R}$ with $f \equiv 0$:*

$$|u(t,x) - u(s,y)| \le c\|u\|_{L^\infty(I_R(t_0)\times\mathbb{R}^d)} \left(\frac{|x-y| + |t-s|^{1/\alpha}}{R}\right)^\gamma$$

for almost every $(t,x),(s,y) \in I_{R/2}(t_0) \times B_R$.

Analogous results are established for weak (super)-solutions u to

$$(\widehat{\text{PDE}}) \qquad \partial_t u - \widehat{L}u = f \text{ in } I_R(t_0) \times B_{2R} \subset \mathbb{R}^{d+1},$$

where $\widehat{L}$ is the dual operator of L, given by $(-\widehat{L}u, v) = \widehat{\mathcal{E}}(u,v) = \mathcal{E}(v,u)$.

Theorem 2 (see [1])**.** *Assume* (K1$_{glob}$), (K2) (tail-est), *and* ($\mathcal{E}_{\asymp}$) *for some* $\alpha \in (0,2)$, $\theta \in (\frac{d}{\alpha}, \infty]$. *Then (i) and (ii) of Theorem 1 hold true for any nonnegative, weak supersolution* u *to* ($\widehat{\text{PDE}}$), *respectively any weak solution* u *to* ($\widehat{\text{PDE}}$) *in* $I_R(t_0) \times B_{2R}$ *with* $f \equiv 0$.

The aforementioned results can be seen as a nonlocal extension of the De Giorgi-Nash-Moser-theory for second order divergence form operators with a drift:

$$\mathcal{L}u = \partial_i(a_{i,j}\partial_j u) + b_i\partial_i u, \qquad \text{resp. } \widehat{\mathcal{L}}u = \partial_i(a_{i,j}\partial_j u - b_i u),$$

which were developed e.g., by Stampacchia (1965), Aronson-Serrin (1967).

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Local time penalizations with various clocks for Lévy processes

Kouji Yano

1. Conditionings and local time penalisations with various clocks

The *conditioning of a process* X_t *to avoid a set* C *with a clock* τ is the limit

$$(1) \qquad \lim_{\tau\to\infty} \mathbb{E}_x[F_t \mid T_C > \tau],$$

$T_C = \inf\{t > 0 : X_t \in C\}$ is the first hitting time of C, F_t is an arbitrary bounded $\mathcal{F}_t^X$-measurable functional (test functional), and the limit is taken via τ called a *clock*, which is a net of random times. The limit (1) can be interpreted as the procedure of preventing the process X_t from hitting the set C for all time, whatever the clock τ is. However, the limit may vary according to the choice of the clock.

Knight [4] studied two conditionings of a one-dimensional Brownian motion to stay in an interval $(-a, a)$; one is with the constant time clock:

$$\lim_{s\to\infty} \mathbb{E}_x[F_t \mid T_{(-a,a)^c} > s]; \tag{2}$$

the other one is with the inverse local time clock:

$$\lim_{u\to\infty} \mathbb{E}_x[F_t \mid T_{(-a,a)^c} > \eta_u] \tag{3}$$

with $\eta_u = \inf\{t > 0 : L_t > u\}$ being the inverse of the local time L_t of zero for the Brownian motion. He determined the limit processes of the two conditionings, which turned out to be different.

For a process X_t which admits the local time L_t of zero, the *local time penalization of X_t with a clock* τ is the limit

$$\lim_{\tau\to\infty} \frac{\mathbb{E}_x[F_t f(L_\tau)]}{\mathbb{E}_x[f(L_\tau)]} \tag{4}$$

for a given positive function f. The conditioning of X_t to avoid zero (or avoid the singleton $\{0\}$) can be regarded as a special case of local time penalizations if we take $f = 1_{\{0\}}$.

Yano–Yano [12] studied conditionings of one-dimensional diffusions to avoid zero with various clocks, where the limit processes may vary according to the choice of the clock. Profeta–Yano–Yano [6] generalized the conditioning results to local time penalizations.

2. Two kinds of conditionings for Lévy processes

For a one-dimensional Lévy process X_t, we may consider two kinds of conditionings, the conditioning to stay positive and that to avoid zero, namely,

$$\lim_{\tau\to\infty} \mathbb{E}_x[F_t \mid \tau_0^- > \tau] \quad \text{and} \quad \lim_{\tau\to\infty} \mathbb{E}_x[F_t \mid T_0 > \tau], \tag{5}$$

respectively, where $\tau_0^- = \inf\{t > 0 : X_t < 0\}$ stands for the first passage time of level zero and $T_0 = \inf\{t > 0 : X_t = 0\}$ for the first hitting time of point zero. Note that the two kinds of conditionings coincide when X_t is a Brownian motion starting from a positive point, and the limit process is the three-dimensional Bessel process:

$$\lim_{\tau\to\infty} \mathbb{E}_x[F_t \mid \tau_0^- > \tau] = \lim_{\tau\to\infty} \mathbb{E}_x[F_t \mid T_0 > \tau] = \mathbb{E}_x\left[F_t \cdot \frac{X_{t\wedge T_0}}{x}\right]. \tag{6}$$

The two conditionings may differ when X_t is a Lévy process; see Yano [10] for comparison between the two kinds of conditionings.

The conditioning of one-dimensional Lévy processes to stay positive with the exponential clock has been studied by Chaumont [1], Chaumont–Doney [2] and

Doney [3]. They showed, under some regularity conditions, that

$$\lim_{q\downarrow 0}\mathbb{E}_x[F_t \mid \tau_0^- > \boldsymbol{e}/q] = \mathbb{E}_x\left[F_t\cdot\frac{h^{\uparrow}(X_{t\wedge\tau_0^-})}{h^{\uparrow}(x)}\right], \tag{7}$$

where $h^{\uparrow}$ stands for the invariant function with respect to the stopped process $X_{t\wedge\tau_0^-}$ which was introduced by Silverstein [7]. Yano–Yano–Yor [14] studied the infimum (supremum) penalisation of one-dimensional Lévy processes with the constant clock, which may be regarded as a generalization of the conditionings to stay positive.

The conditioning of one-dimensional Lévy processes to avoid zero has been studied by Yano–Yano–Yor [13] in the symmetric case with the constant clock and generalized by Pantí [5] with the exponential clock. Under some regularity conditions, it holds that

$$\lim_{q\downarrow 0}\mathbb{E}_x[F_t \mid T_0 > \boldsymbol{e}/q] = \mathbb{E}_x\left[F_t\cdot\frac{h^{\times}(X_{t\wedge T_0})}{h^{\times}(x)}\right], \tag{8}$$

where $h^{\times}$ stands for an invariant function with respect to the stopped process $X_{t\wedge T_0}$; see also Tsukada [9] and Yano [11].

Recently, Takeda–Yano [8] studied local time penalizations with various clocks. In particular, they considered the two-point hitting time clock $\tau = T_a \wedge T_{-b}$ with $(a,b) \xrightarrow{\gamma} \infty$ in the sense that $a,b\to\infty$ and $\frac{a-b}{a+b}\to\gamma\in[-1,1]$. Under some regularity conditions, it holds that

$$\lim_{(a,b)\xrightarrow{\gamma}\infty}\mathbb{E}_x[F_t \mid T_a\wedge T_{-b} > \boldsymbol{e}/q] = \mathbb{E}_x\left[F_t\cdot\frac{h^{(\gamma)}(X_{t\wedge T_0})}{h^{(\gamma)}(x)}\right], \tag{9}$$

where $h^{(\gamma)}(x) = h^{\times}(x) + \frac{\gamma}{m^2}x$ with $m^2 = \mathbb{E}_0(X_1)^2$ is an invariant function with respect to the stopped process $X_{t\wedge T_0}$.

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