Mathematisches Forschungsinstitut Oberwolfach

Report No. 13/2022

DOI: 10.4171/OWR/2022/13

The Laguerre-Pólya Class and Combinatorics

Organized by
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Olga Holtz, Berkeley
Alan Sokal, London

13 March – 19 March 2022

Abstract. The talks at the workshop were focused on zero localization and zero finding of entire functions, with applications to analytic number theory and combinatorics. The discussions included specific areas such as stable and hyperbolic polynomials, the Laguerre-Pólya class of entire functions, Pólya frequency sequences, total positivity for sequences and functions, and zeros of generating functions arising in probability and combinatorics.

Mathematics Subject Classification (2010): Primary 26CXX, 30DXX; Secondary: 05CXX, 11MXX, 32AXX, 37FXX, 11Y65, 15A15, 15B05, 30B70, 30H15, 33C45, 42C05, 82B20, 82C20.

Introduction by the Organizers

The workshop brought together experts in several fields with interest in functions from the Laguerre-Pólya class. Problems and techniques discussed during the workshop came from complex analysis, algebraic and enumerative combinatorics, commutative algebra, continued fractions, and structured matrices and operators. The workshop extended and strengthened collaborative networks around the central topic of the Laguerre-Pólya class. A more detailed account of the workshop was written by Thu Hien Nguyen as included below.

The course of the workshop

Thu Hien Nguyen

The workshop *The Laguerre-Pólya Class and Combinatorics* in Oberwolfach Mathematical Research Institute, organized by Kathy Driver (University of Cape Town), Olga Holtz (University of California-Berkeley), and Alan Sokal (University College London) was attended with over 20 participants from different countries. Unfortunately, due to the Corona pandemic and the political conflict between Russia and Ukraine, namely Russia's invasion of Ukraine, many intended participants had no choice but to give talks online. Thus, only one of the organizers could be present physically (Olga Holtz), overall, 6 participants could attend in person, including Petter Brändén, James Eldred Pascoe, Thu Hien Nguyen, Dmitry Karp, and Jonathan Leake.

The talks were given in the format of a 50-minute lecture followed by a 10-minute questions and discussions session. On average, there were 2 morning and 2 afternoon talks with general discussion sessions in between which gave the extra time for participants to cover unanswered questions as well as to brainstorm new ideas for the open problems. The talks were recorded by the VCA Jonathan Leake.

Despite the hybrid meeting format, the workshop brought together experts in various fields with interest and experience involving entire functions from the Laguerre-Pólya class which extended and strengthened collaborative networks around this and related subjects. The talks covered a wide variety of topics which showed an interesting approach to the Laguerre-Pólya class from the perspective of analysis and combinatorics and the surprising interrelation between these areas. During the discussions, the participants found many interesting relations, in problems and in methods, between zero localization and zero finding of polynomials.

Among the techniques used to tackle questions of zero localization are geometric methods in complex analysis, algebraic methods from commutative algebra, the theory of continued fractions, and the theory of associated structured matrices and operators. The last approach, albeit classical, is currently gaining new attention and providing new insights into zero distribution of functions.

Total positivity is a property that is closely related to stability but also plays an important role in other domains of mathematics, mechanics, statistics, and operations research, viz., in problems involving convexity and moment spaces, approximation theory, in the study of vibrating coupled mechanical systems, and many others. Many talks and discussions were around total positivity, along with the associated topics of variation diminution and Pólya frequency functions.

The organizers as well as the participants would like to thank the MFO for making the workshop happen during the difficult times. In particular, we were very impressed by the excellent organization of the hybrid format, which allowed both in-person and remote participants to interact smoothly. Despite the exceptional situation, the participants spent a productive week in an inspiring atmosphere which led to some progress in a few problems. Everyone pointed out that they enjoyed the experience and felt very welcomed thanks to the staff of the Institute who made the stay comfortable and pleasant.

Workshop: The Laguerre-Pólya Class and Combinatorics

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Abstracts

Combinatorics and geometry of polynomials

Petter Brändén

In the past 20 years several connections between the geometry of polynomials and combinatorics have been made. I will talk about some of these connections, with a focus on stable polynomials, Lorentzian polynomials and M-convexity.

Total-positivity characterization of the Laguerre–Pólya class LP^+

Alexander Dyachenko
(joint work with Alan Sokal)

A nontrivial polynomial is called standard if its leading coefficient is positive. The Laguerre–Pólya class LP^+ comprises all entire functions which can be approximated by standard negative-real-rooted polynomials in the topology of uniform convergence. Laguerre [7] showed that $f \in LP^+$ if and only if it can be expressed in the form

$$f(t) = Ce^{\gamma t} t^m \prod_{i=1}^{\infty} (1 + \alpha_i t)$$

with $m \in \mathbb{N}$, $C, \gamma, \alpha_i \geq 0$ and $\sum_i \alpha_i < \infty$.

This talk is devoted to various characterizations of the class LP^+ via total positivity (i.e. nonnegativity of all minors) of certain related matrices. The basic result of this type is the Aissen–Edrei–Whitney–Schoenberg theorem [1, 3]:
An infinite Toeplitz matrix is totally positive if and only if the corresponding generating function has the form $\dfrac{f(t)}{g(-t)}$, *where both* $f(t)$ *and* $g(t)$ *are in* LP^+.
(An intriguing open question: may the complex analytic techniques of [3, 4] be replaced with something more straightforward? What about Okounkov's proof [8]?).

In particular, a power series $f(t) = \sum_{k \geq 0} a_k t^k$ converges to an LP^+ function precisely when the corresponding infinite Toeplitz matrix $T(f) = (a_{n-k})_{n,k=0}^{\infty}$ is totally positive and $\lim_{k \to \infty} \dfrac{a_{k-1}}{a_k} = \infty$. Our talk aims at other characterisations of this type.

Recall Grommer's theorem [6]: let $f(t)$ be an entire function, then $f \in LP^+$ if and only if the infinite Hankel matrix $(s_{n+k})_{n,k=0}^{\infty}$ built from the Taylor coefficients of $\dfrac{f'(-t)}{f(-t)} = \sum_{k \geq 0} s_k t^k$ is positive semidefinite. In this particular setting, positive semidefiniteness of the Hankel matrix equates to its total positivity, c.f. [5]. The related work [2] shows that $f \in LP^+$ is equivalent to total positivity of the Hurwitz-type matrix $H(f', f)$ consisting of the alternating rows of $T(f')$ and $T(f)$.

Recently we obtained another fact of the same type. Let us call a real sequence $(a_n)_{n \geq 0}$ an LP^+ sequence if the corresponding power series $f(t) = \sum_{n \geq 0} a_n t^n$

converges to an LP^+ function. Let $\vartheta = t\frac{d}{dt}$, that is $(\vartheta f)(t) = tf'(t)$. Then, *given a power series* $f(t) = \sum_{n\geq 0} a_n t^n$, *the following conditions are equivalent:*

- (a) $(a_n)_{n\geq 0}$ *is an* LP^+ *sequence;*
- (b) $T(\alpha f + \vartheta f)$ *is totally positive for some set of* $\alpha \to +\infty$*;*
- (b') $((\alpha + n)a_n)_{n\geq 0}$ *is an* LP^+ *sequence for some set of* $\alpha \to +\infty$*;*
- (c) $T(\alpha f + \beta\vartheta f)$ *is totally positive for all* $\alpha, \beta \geq 0$*;*
- (c') $((\alpha + \beta n)a_n)_{n\geq 0}$ *is an* LP^+ *sequence for all* $\alpha, \beta \geq 0$*;*
- (d) $T\left(\prod_{i=1}^{N}(\alpha_i + \beta_i\vartheta)f\right)$ *is totally positive for all* N *and all* $\alpha_i, \beta_i \geq 0$*;*
- (d') $\left(\prod_{i=1}^{N}(\alpha_i + \beta_i n)a_n\right)_{n\geq 0}$ *is an* LP^+ *sequence all* N *and for all* $\alpha_i, \beta_i \geq 0$*;*
- (e) $T(g(\vartheta)f)$ *is totally positive for all* $g \in LP^+$*;*
- (e') $(g(n)a_n)_{n\geq 0}$ *is an* LP^+ *function for all* $g \in LP^+$*.*

Laguerre showed that $(a) \implies (e')$, and the implications

$$\begin{array}{ccccccc}
(e') & \Longrightarrow & (d') & \Longrightarrow & (c') & \Longrightarrow & (b') \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
(e) & \Longrightarrow & (d) & \Longrightarrow & (c) & \Longrightarrow & (b)
\end{array}$$

are obvious. So, our contribution here is the proof of $(b) \implies (a)$, which we conduct using methods of real analysis.

References

[1] M. Aissen, I. J. Schoenberg, A. M. Whitney, *On the generating functions of totally positive sequences I*, J. Analyse Math. **2** (1952), 93–103.

[2] A. Dyachenko, *Total Nonnegativity of Infinite Hurwitz Matrices of Entire and Meromorphic Functions*, Complex Anal. Oper. Theory **8**:5 (2014), 1097–1127.

[3] A. Edrei, *On the generating functions of totally positive sequences II,* J. Analyse Math. **2**(1952), 104–109.

[4] A. Edrei, *Proof of a conjecture of Schoenberg on the generating function of a totally positive sequence,* Canad. J. Math. **5** (1953), 86–94.

[5] F. Gantmakher, M. Krein, *Sur les matrices complètement non négatives et oscillatoires,* Compos. Math. **4** (1937), 445–476.

[6] J. Grommer, *Ganze transzendente Funktionen mit lauter reellen Nullstellen*, J. Reine Angew. Math. **144** (1914), 114–166.

[7] E. Laguerre, *Sur les fonctions du genre zéro et de genre un*, C. R. Acad. Sci. **95** (1882), 174–177.

[8] A. Yu. Okounkov, *On representations of the infinite symmetric group,* J. Math. Sci. **96**:5 (1999), 100–107.

The Laguerre-Pólya class via density criteria

Dimitar Dimitrov

We survey some recent progress towards long-standing open questions concerning the Laguerre-Pólya class. Among those are the results of M. Griffin, K. Ono, L. Rolen, and D. Zagier about hyperbolicity of the Jensen polynomials associated with the Riemann ξ-function and B. Rogers and T. Tao's proof of the de Bruijn-Newman conjecture. We discuss a new density criterion for an entire function that is represented via a Fourier transform to be in the Laguerre-Pólya class. This result may be considered the L^1 counterpart of the celebrated Nyman-Beurling L^2 density criterion for the Riemann hypothesis.

A spectral approach to polytope diameter

Nikhil Srivastava

A classic question in discrete geometry is: what is the maximum possible diameter of the skeleton of a polytope $P = x \in R^d : Ax \leq b$ in R^d defined by m constraints, as a function of m and d? We describe a new approach to this problem which uses the log-concavity of the volume of a polytope to bound the eigenvalues of a certain weighted adjacency matrix of its skeleton, yielding a good upper bound whenever A and b have integer entries of bounded height. Joint work with H. Narayanan and R. Shah.

On the de Bruijn-Newman constant: Kim and Lee's results on the zeros of Jensen polynomials

Haseo Ki

1. The de Bruijn-Newman constant (joint work with Jungseob Lee and Young-One Kim)

Define

$$\Xi_\lambda(t) = \int_0^\infty e^{\frac{\lambda}{4}(\log x)^2 + \frac{it}{2}\log x}\varphi(x)\frac{dx}{x} = \int_{-\infty}^\infty e^{\lambda u^2}\Phi(u)e^{itu}du.$$

Here

$$\psi(x) = \sum_{n=1}^\infty e^{-n^2\pi x}, \quad \varphi(x) = x^{5/4}\left(2x\psi''(x) + 3\psi'(x)\right), \quad \Phi(u) = 2\varphi\left(e^{2u}\right).$$

The Riemann Ξ-function :

$$\Xi(t) = \Xi_0(t) = \frac{s(s-1)}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s) \qquad \left(s = \frac{1}{2} + it\right).$$

Fact.

(1) The zeros of $\Xi_0(t)$ lie in $\{t : |\operatorname{Im} t| < 1/2\}$;

(2) The Riemann hypothesis is valid if and only if $\Xi_0(t)$ has real zeros only.

By the De Brujin's idea [1], we have the following.

Proposition A. *If $\lambda_1 \leq \lambda_2$, $\Delta \geq 0$, and the zeros of $\Xi_{\lambda_1}(t)$ lie in $\{t : |\operatorname{Im} t| \leq \Delta\}$, then those of $\Xi_{\lambda_2}(t)$ lie in $\{t : |\operatorname{Im} t| \leq \widetilde{\Delta}\}$, where*

$$\widetilde{\Delta} = \sqrt{\max\{\Delta^2 - 2(\lambda_2 - \lambda_1), 0\}}.$$

In particular, Ξ_λ has only real zeros when $\lambda \geq 1/8$.

In the wonderful paper [10], we have

Theorem [Newman]. Ξ_λ *has non-real zeros for some* $\lambda < 0$*. In particular, for a real constant* $\lambda^{(0)} \leq 1/8$, Ξ_λ *has only real zeros when* $\lambda^{(0)} \leq \lambda$ *but has non-real zeros when* $\lambda < \lambda^{(0)}$.

We observe that the Riemann hypothesis is true if and only if $\lambda^{(0)} \leq 0$.

Newman [10] came up with the following.

Conjecture [Newman] . $0 \leq \lambda^{(0)}$.

The de Bruijn-Newman constant. $\Lambda = 4\lambda^{(0)}$.

Recently, this conjecture was settled by B. Rodgers and T. Tao [14].

Theorem [Rodgers and Tao]. $0 \leq \Lambda$.

We recall that the Riemann hypothesis is valid if and only if $\Lambda \leq 0$.

From the Euler product for the Riemann zeta function $\zeta(s)$, we can prove that $\zeta(s)$ has no zeros in $\operatorname{Re} s > 1$ and so we have $\Lambda \leq 0.5$. We list the upper bounds of Λ.

[Euler product]. $\Lambda \leq 0.5$.

[Ki, Kim and Lee, 2009]. $\Lambda < 0.5$.

[Polymath, 2018]. $\Lambda < 0.22$.

[Platt and Trudgian, 2020]. $\Lambda < 0.2$.

See [6], [13] and [12] for these.

Fact.
(1) The zeros of $\Xi_0^{(m)}(t)$ lie in the strip $\{t : |\operatorname{Im} t| < 1/2\}$ for every $m \geq 0$;
(2) If the Riemann hypothesis were true, then $\Xi_0^{(m)}(t)$ would have only real zeros for every $m \geq 0$.

Due to Levinson's method [8], Conrey [3] justified

Theorem [Conrey]. The "proportion" of real zeros of $\Xi_0^{(m)}(t)$ tends to 1 as $m \to \infty$.

Define $\lambda^{(m)} = \inf \left\{ \lambda : \Xi_\lambda^{(m)}(t) \text{ has only real zeros} \right\}$ $(m = 0, 1, 2, \dots)$.

We have the following in [6].

Theorem 1. The sequence $\{\lambda^{(m)}\}$ is non-increasing, and its limit is ≤ 0.

Theorem 2. For every $\lambda > 0$ all but a finite number of zeros of $\Xi_\lambda(t)$ are real and simple.

Concerning Theorem 2, we recall

The quasi Riemann hypothesis. For some $0 < \epsilon < 1/2$, the zeros of the Riemann zeta function $\zeta(s)$ lie in $\epsilon < \operatorname{Re} s < 1 - \epsilon$.

For instant, this conjecture implies that $\pi(x)$, the number of primes less than x ($x > 1$) satisfies

$$\pi(x) = \int_2^x \frac{1}{\log t} dt + O_\epsilon\left(x^{1-\epsilon}\right) \qquad (x \to \infty)$$

for some $\epsilon > 0$. We might say that Theorem 2 is like the quasi Riemann hypothesis. However, Theorem 2 is irrelevant to the quasi Riemann hypothesis, because we cannot expect any number theoretical consequence from it.

Enumerate the real parts of the zeros of the Riemann Ξ-function in $\mathrm{Re}\, t > 0$ by

$$\gamma_1 \le \gamma_2 \le \gamma_3 \le \cdots .$$

From Montgomery's pair correlation conjecture [9], we expect

Conjecture.

$$\liminf_{n\to\infty} \left(\gamma_{n+1} - \gamma_n\right) \frac{\log \gamma_n}{2\pi} = 0;$$
$$\limsup_{n\to\infty} \left(\gamma_{n+1} - \gamma_n\right) \frac{\log \gamma_n}{2\pi} = \infty;$$

We recall that we have

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

where $N(T)$ is the number of zeros of $\zeta(s)$ in $0 < \mathrm{Im}\, s < T$. Thus, the average gap of zeros of the Riemann zeta function in $0 < \mathrm{Im}\, s < T$ is

$$\frac{2\pi}{\log T}.$$

Therefore, the conjecture above implies that consecutive zeros of the Riemann zeta function are arbitrarily close to the average gap and arbitrarily far from the average gap. This phenomenon appears only for genuine zeta functions.

On the other hand, enumerate the zeros of $\Xi_\lambda(t)$ in $\{t : \mathrm{Re}\, t > T_\lambda\}$ by

$$\gamma_{(\lambda,1)} < \gamma_{(\lambda,2)} < \gamma_{(\lambda,3)} < \cdots .$$

Set $N_\lambda(T) =$ the number of zeros of $\Xi_\lambda(t)$ in $\{t : 0 \le \mathrm{Re}\, t \le T\}$.

Theorem 3. If $\lambda > 0$, then

$$N_\lambda(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{\lambda}{4} \log \frac{T}{2\pi} + O(1) \qquad (T \to \infty).$$
$$\lim_{n\to\infty} \left(\gamma_{(\lambda,n+1)} - \gamma_{(\lambda,n)}\right) \frac{\log \gamma_{(\lambda,n)}}{2\pi} = 1;$$

This theorem says that zeros of $\Xi_\lambda(t)$ behave differently in relation to those of the Riemann zeta function or the Riemann ξ function. The error term for $N_\lambda(T)$ is bounded as $T \to \infty$ and then zeros of $\Xi_\lambda(t)$ behave regularly. However, the error term for $N(T)$ of the Riemann Ξ function possesses profound arithmetic information for the Riemann zeta function. This signs that $\Xi_\lambda(t)$ would be not appropriate in investigating the behavior of zeros of the Riemann zeta function.

We recall that for every $\lambda > 0$ all but a finite number of zeros of $\Xi_\lambda(t)$ are real and simple. Also, if we can prove that all zeros of $\Xi_\lambda(t)$ are real for any $\lambda > 0$, we get the Riemann hypothesis!

Can we resolve the Riemann hypothesis by the method in this talk, although we already observed a negative sign for that?

Let $\{a_n\}$ and $\{b_n\}$in $\mathbb{R}$ be such that $a_1 \neq 0$, $0 < b_1 < b_2 \le b_3 \le b_4 \le \dots$ and for a fixed $A > 0$

$$\sum_{n=1}^{\infty} |a_n| b_n^{-A} < \infty.$$

Put

$$L(s) = \sum_{n=1}^{\infty} a_n b_n^{-s} \quad \text{and} \quad \psi(x) = \sum_{n=1}^{\infty} a_n e^{-b_n x}.$$

For some $k, c_0, \dots, c_N \in \mathbb{R}$ $(c_N \neq 0)$, define

$$\varphi(x) = x^{k/2} \sum_{n=0}^{N} c_n x^n \psi^{(n)}(x).$$

Assume that for some $\delta \in \{0, 1\}$

$$(-1)^\delta \varphi(x) = \varphi\left(\frac{1}{x}\right) \qquad (\operatorname{Re} x > 0).$$

Put

$$h(s) = \int_0^\infty x^s \left(\sum_{n=0}^{N} c_n x^n \psi^{(n)}(x)\right) \frac{dx}{x}.$$

Then, for some polynomial $P(s)$,

$$h(s) = (-1)^\delta h(k - s); \qquad h(s) = P(s)\Gamma(s)L(s) \qquad (\operatorname{Re} s > A).$$

Define

$$H_\lambda(t) = i^\delta \int_0^\infty e^{\lambda(\log x)^2 + it \log x} \varphi(x) \frac{dx}{x}.$$

Essentially, we have the following [6].

For $\lambda > 0$, all but finitely many zeros of $H_\lambda(t)$ are real and simple.

Unfortunately, in general, $H_0(t)$ violates the analogue of the Riemann hypothesis!

Let χ be a primitive Dirichlet character modulo $q(> 2)$. For the Dirichlet L-function $L(s, \chi)$, define

$$L(s) = L(s - 1/2, \chi) L(s + 1/2, \chi).$$

We can associate $H_\lambda(t)$ for $L(s)$. In [6],

Theorem 4. For any $\lambda > 0$, all but finitely many zeros of $H_\lambda(t)$ are real and simple.

Obviously, $H_0(t)$ violates the Riemann hypothesis (indeed, $H_0(t)$ has no real zeros at all), but still Theorem 4 is valid!! This theorem clearly tells us that by the methods in this talk, it should be very difficult in resolving the Riemann hypothesis.

2. Kim and Lee's results on the zeros of Jensen polynomials

Recently, Kim and Lee [7] got interesting results on the zeros of Jensen polynomials. It would be meaningful to introduce briefly their results in this talk.

We say that a real polynomial is *hyperbolic* if it has real zeros only. Set

$$\mathbb{S} = \{z \in \mathbb{C} : |\operatorname{Im} z| < 1/2\}.$$

Define

$$J^{d,n}(z) = \sum_{j=0}^{d} \binom{d}{j} \gamma(n+j) z^j \qquad \text{and} \qquad P^{d,n}(z) = \sum_{j=0}^{d} \binom{d}{j} \gamma(n+j) H_{d-j}(z),$$

where

$$\gamma(m) = \frac{(-1)^m m!}{(2m)!} \Xi^{(2m)}(0) \ \ (m = 0, 1, 2, \ldots); \ \ H_d(z) = d! \sum_{k=0}^{[d/2]} \frac{(-1)^k}{k!(d-2k)!2^k} z^{d-2k}.$$

Also, consider

$$\widetilde{J}^{d,n}(z) - \sum_{j=0}^{d} \binom{d}{j} \Xi(n+j) z^j \qquad \text{and} \qquad \widetilde{P}^{d,n}(z) = \sum_{j=0}^{d} \binom{d}{j} \Xi(n+j) H_{d-j}(z),$$

Due to a theorem of Pólya and Schur, each of the following four statements is equivalent to the Riemann hypothesis.

(1) All the polynomials $J^{d,n}$ are hyperbolic.
(2) All the polynomials $P^{d,n}$ are hyperbolic.
(3) All the polynomials $\widetilde{J}^{d,n}$ are hyperbolic.
(4) All the polynomials $\widetilde{P}^{d,n}$ are hyperbolic.

In 2019, Griffin et al. [5] proved that for every positive integer d $J^{d,n}$ is hyperbolic for all sufficiently large n. In [4], it is improved that there is a constant $c > 0$ such that $J^{d,n}$ is hyperbolic for $d \geq 1$ and $n \geq ce^{d/2}$. O'Sullivan [11] showed that for all sufficiently large d, $P^{d,n}$ is hyperbolic for $n/\log^2 n \geq d^{3/4}/2$. Chasse [2] proved that if $T \geq 1/2$, Ξ has real zeros only in the rectangle $\{z \in \mathbb{S} : |\operatorname{Re} z| \leq T\}$ and $d \leq T^2$, then $J^{d,n}$ is hyperbolic for every n. It is known in [12] that all zeros of Ξ in $z \in \mathbb{S}$ and $|\operatorname{Re} z| \leq 3 \cdot 10^{12}$ is real. Kim and Lee [7] improve all results for the polynomials as follows.

Theorem 5 [Kim-Lee]. *For every $c > 1$, there is a positive integer d_0 such that $J^{d,n}$, $P^{d,n}$, $\widetilde{J}^{d,n}$ and $\widetilde{P}^{d,n}$ are hyperbolic whenever $d \geq d_0$ and $n \geq d^{c/2}$.*

Theorem 6 [Kim-Lee]. *If $T \geq 1/2$, Ξ has only real zeros in the rectangle $\{z \in \mathbb{S} : |\operatorname{Re} z| \leq T\}$ and $d \leq 1 + 4T^2$, then $\widetilde{J}^{d,n}$ and $\widetilde{P}^{d,n}$ are hyperbolic for all n.*

In fact, Kim and Lee [7] justified a general theorem that implies Theorem 5.

Theorem 7 [Kim-Lee]. *Suppose that f is a transcendental real entire function, c is a positive constant, the order of f is strictly less than $\min\{c, 2\}$ and all zeros of f are in $\mathbb{S}$. Then, there is a positive integer d_0 such that $J(f^{(n)}; d)$ is hyperbolic whenever $d \geq d_0$ and $n \geq d^{c/2}$. If f is even, then there is a positive integer d_1 such that $J(f_0^{(n)}; d)$ is hyperbolic whenever $d \geq d_1$ and $n \geq d^{c/2}$, where $f(z) = f_0\left(z^2\right)$ and for an entire function g*

$$J(g; d) = \sum_{k=0}^{d} \binom{d}{k} f^{(k)}(0) z^k.$$

In the previous section, we considered $L(s) = L(s-1/2, \chi)L(s+1/2, \chi)$ and $H_\lambda(t)$ for $L(s)$. Set

$$\widetilde{\Xi}(z) = H_0(4z).$$

Then, all zeros of $\widetilde{\Xi}(z)$ are in $\mathbb{S}$. We observe that $\widetilde{\Xi}(z)$ has no real zeros at all, because all nontrivial zeros of $L(s, \chi)$ are in $0 < \operatorname{Re} s < 1$ and the critical line $\operatorname{Re} s = 1/2$ corresponds to the real line when we get $\widetilde{\Xi}(z)$ from $L(s)$. However, Theorem 7 implies that Theorem 5 for $\widetilde{\Xi}(z)$ is valid! Namely, Theorem 7 is too general to control zeros of genuine zeta functions. Thus, it is better not to expect that one can prove the Riemann hypothesis due to the methods in this section.

References

[1] N. G. de Bruijn, *The roots of trigonometric integrals*, Duke Math. J. **17** (1950), 197–226.

[2] M. Chasse, *Laguerre multiplier sequences and sector properties of entire functions*, Complex Var. Elliptic Equ. **58** (2013), 875–885.

[3] B, Conrey, *Zeros of derivatives of Riemann's ξ-function on the critical line*, J. Number Theory **16** (1983), 49–74.

[4] M. Griffin, K. Ono, L. Rolen, J. Thorner, Z. Tripp and I. Wagner, *Jensen Polynomials for the Riemann Xi Function Michael Griffin*, `arXiv:1910.01227`.

[5] M. Griffin, K. Ono, L. Rolen, Larry and D. Zagier, *Jensen polynomials for the Riemann zeta function and other sequences*, Proc. Natl. Acad. Sci. USA **116** (2019), 11103–11110.

[6] H. Ki, Y.-O. Kim and J. Lee, *On the de Bruijn-Newman constant*, Advances in Mathematics **222** (2009), 281–306

[7] Y.-O. Kim and J. Lee, *A note on the zeros of Jensen polynomials*, to appear in J. Korean Math. Soc.

[8] N. Levinson, *More than one third of zeros of Riemann's zeta-function are on $\sigma = 1/2$*, Advances in Mathematics **13** (1974), 383–436

[9] H. L. Montgomery, *The pair correlation of zeros of the zeta function*, Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), pp. 181–193. Amer. Math. Soc., Providence, R.I., 1973.

[10] C. M. Newman, *Fourier transforms with only real zeros*, Proc. Amer. Math. Soc. **61** (1976), 245–251.

[11] C. O'Sullivan, *Zeros of Jensen polynomials and asymptotics for the Riemann xi function*, Res. Math. Sci. **8** (2021), Paper No. 46, 27 pp.

[12] D. Platt and T. Trudgian, *The Riemann hypothesis is true up to* $3 \cdot 10^{12}$, Bulletin of the London Mathematical Society **53** (2021), 792–797.

[13] D.H.J. Polymath, *Effective approximation of heat flow evolution of the Riemann ξ function, and a new upper bound for the de Bruijn-Newman constant*, Research in the Mathematical Sciences **6**, Article number: 31 (2019).
[14] B. Rodgers and T. Tao, *The De Bruijn-Newman Constant is non-negative*, Forum of Mathematics, Pi **8** (2020), e6.

A generalisation of the Laguerre and Newton inequalities

Mikhail Tyaglov
(joint work with M.J. Atia, O. Katkova, and A. Vishnyakova)

Given a real polynomial $p(z)$ with only real zeroes, we estimate the number of non-real zeroes of the differential polynomial

$$F_\kappa[p](z) = p(z)p''(z) - \kappa[p'(z)]^2,$$

where κ is a real number.

A counterexample to a conjecture by B. Shapiro on the number of real zeroes of the polynomial $F_{\frac{n-1}{n}}[p](z)$ in the case when the real polynomial $p(z)$ of degree n has non-real zeroes is constructed.

We will discuss similar results for functions in the Laguerre-Pólya class and other generalisations of the Hawaii conjecture. Some ideas for an alternative proof of the Hawaii conjecture will also be presented.

The talk is based on a joint works with M.J. Atia, O. Katkova, and A. Vishnyakova.

Relaxations of the Laguerre-Pólya class and trace inequalities

James Eldred Pascoe

We classify functions $f : (a, b) \to \mathbb{R}$ which satisfy the inequality

$$tr(f(A) + f(C)) \geq tr(f(B) + f(D))$$

when $A \leq B \leq C$ are self-adjoint matrices, $D = A + C - B$, the so-called trace minmax functions. (Here $A \leq B$ if $B - A$ is positive semidefinite, and f is evaluated via the functional calculus.) A function is trace minmax if and only if its derivative analytically continues to a self map of the upper half plane. The negative exponential of a trace minmax function satisfies the inequality

$$\det g(A) \det g(C) \leq \det g(B) \det g(D)$$

for A, B, C, D as above. We call such functions determinant isoperimetric. We show that determinant isoperimetric functions are in the "radical" of the the Laguerre-Pólya class. We derive an integral representation for such functions which is essentially a continuous version of the Hadamard factorization for functions in the Laguerre-Pólya class. We apply our results to give some equivalent formulations of the Riemann hypothesis. Moreover, natural relaxations arising from the theory of matrix monotonicity in conjunction with certain pair-correlation type conjectures on the zero distribution give that if the Riemann hypothesis fails, then any zeros witnessing the failure need to be far away from the critical line in some sense.

Some analytic properties of the partial theta function

Vladimir Kostov

The partial theta function is defined as the sum of the double series

$$\theta(q,x) := \sum_{j=1}^{\infty} q^{j(j+1)/2} x^j ,$$

which for each fixed value of the parameter q, $|q| < 1$, is an entire function in x. When q is real (i.e. either $q \in (0,1)$ or $\theta \in (-1,0)$), the complex conjugate pairs of zeros of θ remain within a fixed bounded domain. Its spectral values (i.e. values of q for which θ has a multiple zero) are sequences $\{q_j\}$ tending to 1 or -1 respectively. For each spectral value, θ has exactly one multiple zero which is of multiplicity 2. The spectral values and the corresponding double zeros have asymptotic expansions in j. When q is complex, the double zeros of θ belong to a fixed bounded domain, one and the same for all values of $q \in \mathbb{D}_1$.

References

[1] V.P. Kostov and B.Z. Shapiro, *Hardy-Petrovitch-Hutchinson's problem and partial theta function*, Duke Math. J. **162**, No. 5 (2013), 825–861.

[2] V.P. Kostov, *About a partial theta function*, Comptes Rendus Acad. Sci. Bulgare **66**, No 5 (2013), 629–634.

[3] V.P. Kostov, *On the zeros of a partial theta function*, Bull. Sci. Math. **137**, No. 8 (2013), 1018–1030.

[4] V.P. Kostov, *On the spectrum of a partial theta function*, Proc. Royal Soc. Edinb. A **144**, No. 5 (2014), 925–933.

[5] V.P. Kostov, *Asymptotics of the spectrum of partial theta function*, Revista Mat. Complut. **27**, No. 2 (2014), 677–684. DOI: 10.1007/s13163-013-0133-3.

[6] V.P. Kostov, *A property of a partial theta function*, Comptes Rendus Acad. Sci. Bulgare **67**, No.10 (2014), 1319–1326.

[7] V.P. Kostov, *On a partial theta function and its spectrum*, Proc. Royal Soc. Edinb. A **146**, No. 3 (2016), 609–623.

[8] V.P. Kostov, *Asymptotic expansions of zeros of a partial theta function*, Comptes Rendus Acad. Sci. Bulgare **68**, No. 4 (2015), 419–426.

[9] V.P. Kostov, *On the double zeros of a partial theta function*, Bull. Sci. Math. **140**, No. 4 (2016), 98–111.

[10] V.P. Kostov, *Stabilization of the asymptotic expansions of the zeros of a partial theta function*, Comptes Rendus Acad. Sci. Bulgare **68**, No. 10 (2015), 1217-1222.

[11] V.P. Kostov, *On the multiple zeros of a partial theta function*, Funct. Anal. Appl. (Russian version **50**, No. 2 (2016), 84–88, English version **50**, No. 2 (2016), 153–156).

[12] V.P. Kostov, *The closest to* 0 *spectral number of the partial theta function*, Comptes Rendus Acad. Sci. Bulgare **69**, No. 9 (2016), 1105–1112.

[13] V.P. Kostov, *Uniform bounds on locations of zeros of partial theta function*, Acta Universitatis Matthiae Belii Ser. Math. (2016), 16–20.

[14] V.P. Kostov, *A separation in modulus property of the zeros of a partial theta function*, Analysis Mathematica **44**, No. 4 (2018), 501–519.

[15] V.P. Kostov, *A domain containing all zeros of the partial theta function*, Publicationes Mathematicae Debrecen **93**, No. 1-2 (2018), 189–203.
[16] V.P. Kostov, *On the complex conjugate zeros of the partial theta function*, Funct. Anal. Appl. (English version **53**, No.2 (2019), 149–152, Russian version **53**, No. 2 (2019), 87-91).
[17] V.P. Kostov, *On the zero set of the partial theta function*, Serdica Math. J. **45** (2019), 225–258.
[18] V.P. Kostov, *Partial theta function and separation in modulus property of its zeros*, Vietnam Journal of Mathematics **48**, No.1 (2020), 145-157.
[19] S.O. Warnaar, *Partial theta functions. I. Beyond the lost notebook.* `arxiv:math/0105002v1[math.QA]`.
[20] A.D. Sokal, *The leading root of the partial theta function*, Adv. Math. **229** (2012), 2603-2621. `arXiv:1106.1003[math.CO]`, `http://dx.doi.org/10.1016/j.aim.2012.01.012`.
[21] A.D. Sokal, `https://webspace.maths.qmul.ac.uk/p.j.cameron/csgnotes/sokal/`.
[22] A.D. Sokal, `https://physics.nyu.edu/faculty/sokal/Queen_Mary_Jul_9_13_talk.pdf`.

On necessary and sufficient conditions for entire functions to belong to the Laguerre-Pólya class in terms of their Taylor coefficients

Thu Hien Nguyen
(joint work with Anna Vishnyakova)

We discuss new conditions for the entire functions with positive Taylor coefficients to belong to the Laguerre–Pólya class. For an entire function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ let us define the second quotients of Taylor coefficients as $q_n(f) := \frac{a_{n-1}^2}{a_{n-2}a_n}, n \geq 2$.

The *partial theta-function*, $g_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{a^{k^2}}$, $a > 1$, was studied by many authors. In [Katkova, Lobova, Vishnyakova 2003] it is proved that there exists a constant $q_\infty, q_\infty \approx 3.23363666$, such that the partial theta-function (and all its odd Taylor sections) belongs to the Laguerre-Pólya class if and only if $a^2 \geq q_\infty$.

We obtained new necessary and sufficient conditions on the Taylor coefficients of entire functions to belong to the Laguerre-Pólya class. The following theorem illustrates a sufficient condition for the case when $q_n(f)$ are decreasing in n.

Theorem ([Nguyen, Vishnyakova 2018]). *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ for all k, be an entire function. Suppose that $q_n(f)$ are decreasing in n, i.e. $q_2(f) \geq q_3(f) \geq q_4(f) \geq \ldots$, and $\lim_{n\to\infty} q_n(f) = b \geq q_\infty$. Then all the zeros of f are real and negative, in other words $f \in \mathcal{L} - \mathcal{P}$.*

It is easy to see that, if only the estimation of $q_n(f)$ from below is given and the assumption of monotonicity is omitted, then the Hutchinson's constant 4 in $q_n(f) \geq 4$ is the smallest possible to conclude that $f \in \mathcal{L} - \mathcal{P}$.

The following result provides a necessary condition for an entire function with positive coefficients and with the increasing second quotients of Taylor coefficients to belong to the Laguerre-Pólya class.

Theorem ([Nguyen, Vishnyakova 2019]). *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ for all k, be an entire function such that the quotients $q_n(f)$ are increasing in n. If f belongs to the Laguerre-Pólya class, then $\lim_{n\to\infty} q_n(f) \geq q_\infty$.*

We also present other new related necessary and sufficient conditions.

References

[1] O. Katkova, T. Lobova and A. Vishnyakova, *On power series having sections with only real zeros*, Comput. Methods Funct. Theory **3**, No 2 (2003), 425–441.

[2] T.H. Nguyen and A. Vishnyakova, *On the entire functions from the Laguerre-Pólya class having the decreasing second quotients of Taylor coefficients*, Journal of Mathematical Analysis and Applications **465**, No. 1 (2018), 348–359.

[3] T.H. Nguyen and A. Vishnyakova, *On a necessary condition for an entire function with the increasing second quotients of Taylor coefficients to belong to the Laguerre-Pólya class*, Journal of Mathematical Analysis and Applications, **480**, No. 2 (2019).

Determinantal representations and the principal minor map

Cynthia Vinzant

The principal minor map takes an n by n matrix to the vector of its 2^n principal minors. When the matrix is Hermitian, the associated multiaffine generating polynomial is determinantal and real stable. In this talk, I will describe a characterization of the image of Hermitian matrices under this map based on a connection with determinantal representations and factorizations of “Rayleigh differences”. This is based on joint work with Abeer Al Ahmadieh.

Pólya frequency sequences: from Laguerre and Fekete to Jacobi and Trudi

Apoorva Khare

We discuss the class of Toeplitz bi-infinite “totally nonnegative matrices” (those with all minors nonnegative). These matrices are called Polya frequency sequences, and they are connected to the Laguerre-Polya class, with a rich history. We present several examples: two of them are connected to the Jacobi-Trudi identities, while a third is connected to an 1883 result of Laguerre, shown in 1912 by Fekete in correspondence with Polya. We also present a novel characterization of Polya frequency sequences of order k, via the non-negativity of only the $k \times k$ minors.

Motion of zeros of polynomial solutions of the one-dimensional heat equation: A first-order Calogero-Moser system

Alan Sokal

I study the motion of zeros of polynomial solutions $\varphi(x,t) = \prod_{i=1}^{n}[x - x_i(t)]$ of the one-dimensional heat equation $\dfrac{\partial \varphi}{\partial t} = \kappa \dfrac{\partial^2 \varphi}{\partial x^2}$; they satisfy the first-order Calogero–Moser system

$$\frac{dx_i}{dt} = \sum_{j \neq i} \frac{-2\kappa}{x_i - x_j}$$

I am interested in the behavior at *complex* time t (usually with real initial conditions $x_1^\circ, \ldots, x_n^\circ$). My goals are to

(a) Determine the complex times t at which collisions can or cannot occur; and

(b) Control the location of $x_1(t), \ldots, x_n(t)$ in the complex plane.

I have no nontrivial theorems, but many interesting conjectures.

References

[1] D.V. Choodnovsky and G.V. Choodnovsky, *Pole expansions of nonlinear partial differential equations*, Nuovo Cimento B **40** (1977), 339–353.

[2] F. Calogero, *Motion of poles and zeros of special solutions of nonlinear and linear partial differential equations and related "solvable" many-body problems*, Nuovo Cimento B **43** (1978), 177–241.

Total positivity of combinatorial matrices

Yi Wang

(joint work with Xi Chen)

A finite or infinite matrix is called *totally positive* if all its minors are nonnegative. Such matrices have a wide variety of applications across pure and applied mathematics. In this talk, we present several sufficient conditions for the total positivity of combinatorial matrices, including the Riordan arrays, the Krylov matrices, and Aigner's recursive matrices.

Riordan arrays play an important unifying role in enumerative combinatorics. A *proper Riordan array*, denoted by $(d(t), h(t))$, is an infinite lower triangular matrix whose generating function of the kth column is $d(t)h^k(t)$, where $d(0) \neq 0, h(0) = 0$ and $h'(0) \neq 0$. An infinite nonnegative sequence $(a_n)_{n\geq 0}$ is called a *Pólya frequency sequence*, if its Toeplitz matrix $[a_{i-j}]_{i,j\geq}$ is totally positive. We show that if the coefficients of $d(t)$ and $h(t)$ both form Pólya frequency sequences, then the Riordan array $(d(t), h(t))$ is totally positive [1].

In general, a Riordan array can be viewed as a Krylov matrix. Given $A \in \mathbb{C}^{n\times n}$ and $v \in \mathbb{C}^{n\times 1}$, where n is finite or infinite. The *Krylov matrix* $K(v, A)$ of A generated by v is an $n \times n$ matrix

$$K(v, A) = [v, Av, A^2v, \ldots, A^{n-1}v].$$

A *row Krylov matrix* $\widehat{K}(u, B)$ means the transpose of a Krylov matrix, that is, $\widehat{K}(u, B) = [K(u^T, B^T)]^T$. Krylov matrices play a fundamental role in Krylov subspace method, which are counted among the "Top 10 Algorithms of the 20th century" and can be used to develop iterative methods for finding eigenvalues of large sparse matrices or solving large systems of linear equations. We [2] show that if the matrix $[v, A]$ is totally positive, the so is the Krylov matrix $K(v, A)$; if the matrix $\left[\begin{array}{c} u \\ B \end{array}\right]$ is totally positive, then so is the row Krylov matrix $\widehat{K}(u, B)$.

As applications, we obtain another two criteria for the total positivity of Riordan arrays. The first one is in [3]. Let $d(t) = \sum_{n\geq 0} d_n t^n$ and $h(t) = \sum_{n\geq 0} h_n t^n$, if the matrix

$$\left[\begin{array}{ccccc} d_0 & h_0 & & & \\ d_1 & h_1 & h_0 & & \\ d_2 & h_2 & h_1 & h_0 & \\ \vdots & \vdots & & & \ddots \end{array}\right]$$

is totally positive, then so is the Riordan array $(d(t), h(t))$. A Riordan array $R = [r_{n,k}]_{n,k\geq 0}$ can also be characterized by its A-sequence $(a_n)_{n\geq 0}$ and Z-sequence $(z_n)_{n\geq 0}$ such that

$$r_{n+1,0} = \sum_{j\geq 0} z_j r_{n,j}, \quad r_{n+1,k+1} = \sum_{j\geq 0} a_j r_{n,k+j}$$

for $n, k \geq 0$. The second criterion is presented by menas of A- and Z-sequences [4]: if the *production matrix*

$$P(R) = \left[\begin{array}{ccccc} z_0 & a_0 & & & \\ z_1 & a_1 & a_0 & & \\ z_2 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & & & \ddots \end{array}\right]$$

is totally positive, then so is the Riordan array R.

Another application is the total positivity of Aigner's recursive matrices $A = [a_{n,k}]_{n,k\geq 0}$, which is an infinite lower triangular matrix defined by

$$a_{0,0} = 1, \quad a_{n+1,k} = a_{n,k-1} + s_k a_{n,k} + t_{k+1} a_{n,k+1},$$

where $(s_k)_{k\geq 0}$ and $(t_k)_{k\geq 1}$ are nonnegative sequences. We [5] show that if

$$J = \left[\begin{array}{cccc} s_0 & 1 & & \\ t_1 & s_1 & 1 & \\ & t_2 & s_2 & \ddots \\ & & \ddots & \ddots \end{array}\right]$$

is totally positive, then so is Aigner's recursive matrix A. We also present several sufficient conditions for the total positivity of the tridiagonal matrix J.

References

[1] X. Chen and Y. Wang, *Notes on the total positivity of Riordan arrays*, Linear Algebra Appl. **569** (2019), 156–161.

[2] X. Chen, J. Mao, L. Mu and Y. Wang, *Total positivity of Krylov matrices*, in preparation.

[3] J. Mao, L. Mu and Y. Wang, *Yet another criterion for the total positivity of Riordan arrays*, Linear Algebra Appl. **634** (2022), 106–111.

[4] X. Chen, H. Liang and Y. Wang, *Total positivity of Riordan arrays*, European J. Combin. **46** (2015), 68–74.

[5] X. Chen, H. Liang and Y. Wang, *Total positivity of recursive matrices*, Linear Algebra Appl. **471** (2015), 383–393.

Electrostatic partners and zeros of orthogonal and multiple orthogonal polynomials

Andrei Martínez-Finkelshtein

(joint work with R. Orive and J. Sanchez-Lara)

The well-known electrostatic interpretation of the zeros of Hermite, Laguerre or Jacobi polynomials, which goes back to the 1885 work of Stieltjes, is one of the most popular models in the theory of orthogonal polynomials. It was picked up and extended to several contexts, such as orthogonal and quasi-orthogonal polynomials on the real line and the unit circle, for classical and semiclassical weights. Our first goal is to generalize the known electrostatic interpretations.

For a given polynomial P with simple zeros, and a given semiclassical weight w, we present a construction that yields a linear second-order differential equation (ODE), and in consequence, an electrostatic model for zeros of P. The coefficients of this ODE are written in terms of a dual polynomial that we call the electrostatic partner of P. This construction is absolutely general and can be carried out for any polynomial with simple zeros and any semiclassical weight on the complex plane. An additional assumption of quasi-orthogonality of P with respect to w allows us to give more precise bounds on the degree of the electrostatic partner. In the case of orthogonal and quasi-orthogonal polynomials, we recover some of the known results and generalize others.

For the Hermite–Padé or multiple orthogonal polynomials of type II, this approach yields a system of linear second-order differential equations, from which we derive an electrostatic interpretation of their zeros in terms of a vector equilibrium (something that was unknown). More detailed results are obtained in the special cases of Angelesco, Nikishin, and generalized Nikishin systems. We also discuss the discrete-to-continuous transition of these models in the asymptotic regime, as the number of zeros tends to infinity, into the known vector equilibrium problems. If time permits, we will discuss how the system of obtained second-order ODEs yields a third-order differential equation for these polynomials, well described in the literature, as well as present several illustrative examples.

This is a joint work with R. Orive (Universidad de La Laguna, Canary Islands, Spain) and J. Sanchez-Lara (Granada University, Spain).

Results and conjectures on log-concavity and zeros of hypergeometric and basic hypergeometric functions

Dmitry Karp

In the talk we will discuss log-concavity for generic power series with coefficients involving gamma and q-gamma functions with respect to the variable contained in their arguments. The motivating example of such series are hypergeometric and basic hypergeometric series for which we can prove more than for the generic case. We show how log-concavity with respect to the simultaneous shift of all parameters implies Laguerre inequalities and relate it to the questions of belongingness of the generalized hypergeometric function to the Laguerre-Pólya class. We further show the connection to certian classes of polynomials constructed from arbitrary real sequences in terms of the rising factorials of the argument. We prove a coefficient-wise positivity statement for one type of such polynomials and propose several conjectures about their stability and zeros under condition that the initial sequence is a Pólya frequency sequence of certain order.

References

[1] D. Karp and S.M. Sitnik, *Log-convexity and log-concavity of hypergeometric-like functions*, Journal of Mathematical Analysis and Applications **364** (2010), 384–394. DOI:10.1016/j.jmaa.2009.10.057.

[2] D. Karp, *Positivity of Toeplitz determinants formed by rising factorial series and properties of related polynomials*, Zapiski Nuachnykh Seminarov POMI **404** (2012), 184–199. Springer version: Journal of Mathematical Sciences **193**, No. 1 (2013), 106-114. DOI:10.1007/s10958-013-1438-y.

[3] S.I. Kalmykov and D.B. Karp, *Log-concavity for series in reciprocal gamma functions and applications*, Integral Transforms and Special Functions **24**, No. 11 (2013), 859–872.

[4] S.I. Kalmykov and D.B. Karp, *Log-convexity and log-concavity for series in gamma ratios and applications*, Journal of Mathematical Analysis and Applications **406** (2013), 400–418.

[5] S.I. Kalmykov and D.B. Karp, *Log-concavity and Turán type inequalities for the generalized hypergeometric function*, Analysis Mathematica **43**, No. 4 (2017), 567-580. DOI:10.1007/s10476-017-0503-z.

[6] S.I. Kalmykov and D.B. Karp, *Inequalities for series in q-shifted factorials and q-gamma functions*, Journal of Mathematical Analysis and Applications **460** (2018), 332-351. DOI:10.1016/j.jmaa.2017.11.041.

[7] S.I. Kalmykov and D.B. Karp, *Inequalities for some basic hypergeometric functions*, Issues of Analysis (Problemy Analiza) **8(26)**, No. 1 (2019), 47–64. DOI:10.15393/j3.art.2019.5210.

Some recent developments on positivity of combinatorial polynomials and matrices

Bao-Xuan Zhu

Many important problems have the close relations with certain positivity. There has been an extensive literature in recent years on positivity in combinatorics. In this talk, we will introduce some recent results concerning positivity (i.e., Pólya frequency, q-log-convexity, strong q-log-convexity, Stieltjes moment property and q-Stieltjes moment property) of combinatorial polynomials and matrices from real-rooted polynomials, total positivity and continued fractions.

References

[1] L.L. Liu and Y. Wang, *On the log-convexity of combinatorial sequences*, Adv. in Appl. Math. **39** (2007), 453–476.

[2] M. Pétréolle, A.D. Sokal and B.-X. Zhu, *Lattice paths and branched continued fractions: An infinite sequence of generalizations of the Stieltjes-Rogers and Thron-Rogers polynomials, with coefficientwise Hankel-total positivity*, to appear in Mem. Amer. Math. Soc. `arXiv:1807.03271`.

[3] A.D. Sokal, *Continued fractions and Hankel-total positivity*, 15th International Symposium on Orthogonal Polynomials, Special Functions and Applications, Hagenberg, Austria, 23 July 2019.

[4] A.D. Sokal, *Coefficientwise total positivity (via continued fractions) for some Hankel matrices of combinatorial polynomials*, in preparation (2018).

[5] Y. Wang and B.-X. Zhu, *Log-convex and Stieltjes moment sequences*, Adv. in Appl. Math. **81** (2016), 115–127.

[6] B.-X. Zhu, *Log-convexity and strong q-log-convexity for some triangular arrays*, Adv. in Appl. Math. **50** (2013), 595–606.

[7] B.-X. Zhu, *Stieltjes moment properties and continued fractions from combinatorial triangles*, preprint in Adv. in. Appl. Math. **130** (2021), Article 102232, 33pp.

[8] B.-X. Zhu, *Total positivity from the exponential Riordan arrays*, SIAM J. Discrete Math. **35** (2021), 2971-3003. `arXiv:2006.14485`.

Total nonnegativity of matrices and zero localization of entire functions

Olga Holtz

I will review a number of old and new results connecting total nonnegativity of special matrices with zero localization of polynomials and entire functions.

Lorentzian polynomials on cones and the Heron-Rota-Welsh conjecture

Jonathan Leake

About 5 years ago, the Heron-Rota-Welsh conjecture (log-concavity of the coefficients of the characteristic polynomial of a matroid) was proven by Adiprasito, Huh, and Katz via the exciting development of a new combinatorial Hodge theory for matroids. In recent work with Petter Brändén, we have given a new short "polynomial proof" of the Heron-Rota-Welsh conjecture. Our proof uses an extension of the theory of Lorentzian polynomials to convex cones, which generalizes real stable and hyperbolic polynomials. In this talk, I will briefly discuss the basics of Lorentzian (aka completely log-concave) polynomials, and then I will give an overview of our new proof of the Heron-Rota-Welsh conjecture.

Reporter: Thu Hien Nguyen

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