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Mini-Workshop: Recent Developments in Representation Theory and Mathematical Physics

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Abstract. This mini-workshop was devoted to foster the interactions between mathematicians and mathematical physicists who are working on questions related to representation theory. This includes for example the representation theory of supergroups, vertex operator algebras and quantum groups. Another focus was on link and manifold invariants and TQFTs.

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Introduction by the Organizers

The aim of this mini-workshop was to bring together mathematical physicists and mathematicians with research interests close to specific areas of representation theory. The meeting provided a format for intense discussions and interactions. The mini-workshop was attended by 16 in-person participants and another 6 online participants.

In order to catalyze interactions among participants of diverse backgrounds, the first half of the workshop began with short, 20-minute talks in which all of the participants gave an informal overview of their work, highlighting important ideas, conjectures, theorems, and/or open questions related to mathematical physics and representation theory. These short talks inspired many further questions and interactions among the participants, which were then taken up in lively, hour-long open discussion sessions. In the second half of the workshop, several of the most interesting topics from the first few days were revisited in greater depth during hour-long research talks.

One of the main themes discussed was the representation theory of supergroups and superalgebras, as it appears in mathematics and mathematical physics. The talks of V. Serganova presented results on volumes of super-Grassmannians; while the talks of B. Williams discussed the appearance of superalgebras (especially exceptional superalgebras) in holomorphic twists of superconformal quantum field theories. These subjects turned out to be surprisingly related through the representation-theoretic techniques used to relate different Grassmannians and (respectively) analyze twists. This was brought to light during one of the open discussions.

Another main theme was the role of non-semisimple representation theory in topological quantum field theory (TQFT). This topic was introduced in N. Reshetikhin's overview talk on open problems in TQFT, and further expanded on from the perspective of vertex operator algebras (VOA's) in B. Feigin's overview talk. D. Reuter then discussed relations between semi-simplicity (or lack thereof) in TQFT and the strength of its associated topological invariants, while L. Woike presented new results on derived/dg extensions of TQFT in the non-semisimple setting. Both the work of D. Reuter and L. Woike was revisited in the second half of the workshop. L. Rozansky and M. Aganagic discussed the most recent results on categorification of the TQFT's related to quantum knot invariants, and their relation to supersymmetric gauge theory.

Several participants' talks and discussions connected further to the representation theory of VOA's. T. Creutzig presented new results on representation theory of logarithmic VOA's, while J. Teschner explained how some of these results (as well as ideas from N. Reshetikhin's previous overview) could be used to reinterpret and generalize the celebrated AGT (Alday-Gaiotto-Tachikawa) conjecture, relating conformal blocks of W-algebras with instanton partition functions and topological string theory. These connections formed the central focus of an open discussion. D. Gaiotto presented conjectures and open problems in introducing arithmetic fields other than the complex numbers (commonly used by physicists) in the correlation functions of VOA's and conformal field theories. The talks by E. Mukhin, G. Felder, and B. Vlaar focused on representation theory of quantum groups and quantum affine algebras, closely related to VOA's.

Finally, the talks of H. Jockers and A. Klemm presented some very recent results and advances in enumerative invariants of Calabi-Yau manifolds, related to representation theory of quantum DQ-modules, and their surprising arithmetic properties.

Mini-Workshop: Recent Developments in Representation Theory and Mathematical Physics

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Abstracts

Categories of modules of affine vertex algebras

Thomas Creutzig

There is a very long-term program on understanding categories of modules associated to affine vertex algebras at admissible levels. Here, I will summarize recent progress, highlighting difficulties and open questions.

Let $\mathfrak{g}$ be a Lie algebra or Lie superalgebra and $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ an invariant, supersymmetric bilinear form. Then one can associate to this data a vertex algebra called the universal affine vertex algebra of $\mathfrak{g}$ associated to B. If $B = k\kappa$ for κ the Killing form on $\mathfrak{g}$, then it is called the affine vertex algebra of $\mathfrak{g}$ at level k. Denote by $L_k(\mathfrak{g})$ its unique graded simple quotient. Known vertex algebras are related to affine vertex algebras via certain standard constructions, as e.g. cosets, extensions and cohomologies and so the affine vertex algebras form a very important class of these algebras. A module of the affine vertex algebra is automatically a smooth module of the underlying affine Lie algebra. From a representation theory point of view the important problem is the as complete as possible understanding of categories of modules of interest, e.g. one would like to

(1) Classify simple modules and find their projective covers and injective hulls; in particular show that projective objects are of finite length;
(2) Establish existence of a braided tensor structure on the category, enumerate fusion rules and establish rigidity

The difficulty of this problem depends on the choice of category of modules. Specialize to $\mathfrak{g}$ a simple Lie algebra or $\mathfrak{g} = \mathfrak{osp}_{1|2n}$ and k an admissible level. The smallest interesting category would be the category of ordinary modules, that is modules that have finite dimensional conformal weight spaces. This category is a semisimple [1, 2] braided tensor category [3] and in most cases also rigid [4, 5, 2]. The category of ordinary modules exhausts all possible modules only if k is a non-negative integer, otherwise there are modules whose weight spaces are not finite-dimensional and also conformal weight needs not be lower bounded, denote the category of such modules by $\mathcal{C}_k(\mathfrak{g})$. This leads to difficulties because classification techniques usually restrict to lower bounded modules [6]. Luckily every simple module is related to a lower bounded module by twisting by an automorphism, called spectral flow [7]. The main issues in the understanding of such module categories as abelian categories is the construction of possible projective modules and then to prove that they are indeed projective. The construction problem has been solved for the cases $\mathfrak{sl}_2, \mathfrak{osp}_{1|2}$ and partially $\mathfrak{sl}_3$, via embeddings of the vertex algebra in suitable larger structures [8, 9]. Moreover effective criteria for the vanishing of extensions allowed to find all projectives in the case of $\mathfrak{sl}_2$ [7]. An interesting observation is that the principal block then coincides with the principal block of the unrolled small quantum group of $\mathfrak{sl}_2$ at associated root of unity. This is conjecturally no coincidence, but has presently only been further exploited in the case of $\mathfrak{sl}_3$ and $k = -3/2$ and $u_i^H(\mathfrak{sl}_3)$ [10].

The existence of braided tensor category is completely open; but at least for $\mathfrak{g} = \mathfrak{sl}_2$ it is within reach now. The main issue is associativity which follows from analytic properties of correlation function. These analytic properties in turn hold if correlation functions are solutions to suitable differential equations and the existence of such differential equations follows from good finiteness conditions: in fact we need C_1-cofiniteness and finite length [11]. A priori this looks hopeless for affine vertex algebras as then C_1-cofinite modules are precisely the ordinary ones. Luckily there are dualities that can help. Dualities are close relations between naively unrelated vertex algebras. In the case of the affine vertex algebra of $\mathfrak{sl}_2$, this duality is often also called Kazama-Suzuki duality and the dual algebra is the $N = 2$ super conformal algebra. This duality is a special case of the ones studied in [12] and there are good functors between representation categories of the two algebras. Modules of the $N = 2$ superconformal algebra are lower bounded and have finite dimensional conformal weight spaces, i.e. they have a chance to be C_1-cofinite. If one can show that these modules (at the relevant central charges) are indeed C_1-cofinite then $\mathcal{C}_k(\mathfrak{sl}_2)$ would inherit braided tensor category structure via the good functors.

Rigidity and fusion rules are the next problems that lack a general technique. One such technique for fusion rules would be Verlinde's formula, that is conjectured for $\mathcal{C}_k(\mathfrak{sl}_2)$ in [13]. This conjecture is however completely open.

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Skew Howe duality and dynamical Weyl group

Giovanni Felder
(joint work with Rea Dalipi and Tommaso Botta)

The symmetric groups play two different roles in the representation theory of the general linear group $GL_N(\mathbb{C})$ of linear automorphisms of $V = \mathbb{C}^N$. On one side the symmetric group S_N is the Weyl group of $GL_N(\mathbb{C})$ and is identified with the subgroup of permutation matrices; it therefore acts on all representations of $GL_N(\mathbb{C})$. On the other hand the group S_M acts on the tensor product $V^{\otimes M}$ of M copies of the vector representation V by permutations of the factors. The Schur–Weyl duality is the statement that the images of the actions of S_M and $GL_N(\mathbb{C})$ in $\mathrm{End}_{\mathbb{C}}(V^{\otimes M})$ are commutants of each other.

An explanation and extension of the relation between this two roles of the symmetric group is provided by the (GL_N, GL_M) Howe duality [7]. Let $V = \mathbb{C}^N$ and $W = C^M$. Then we have commuting GL_N and GL_M actions on the exterior algebra $\bigwedge(V \otimes W)$ with a multiplicity-free decomposition in irreducible $GL_N \times GL_M$ subrepresentations

$$\bigwedge(V \otimes W) = \oplus_\lambda V^N_\lambda \otimes V^M_{\lambda^t} \tag{1}$$

Here we denote by V^n_λ the irreducible representation of GL_n with highest weight λ, which we view as a Young diagram. The sum is over the Young diagrams λ fitting in an $N \times M$ box and λ^t denotes the transposed diagram. A similar statement holds for the symmetric algebra, but here we focus on the skew symmetric version.

By writing $V \otimes W$ as $V \oplus \cdots \oplus V$, or as $W \oplus \cdots \oplus W$, in the ordering given by the standard bases, we get isomorphisms

$$\bigwedge V \otimes \cdots \otimes \bigwedge V \leftarrow \bigwedge(V \otimes W) \rightarrow \bigwedge W \otimes \cdots \otimes \bigwedge W \tag{2}$$

On the left we have M factors and the subspace $\bigwedge^{k_1} V \otimes \cdots \otimes \bigwedge^{k_M} V$ is the weight subspace of weight $(k_1, \ldots, k_M)$ for the GL_M-action. In particular $V^{\otimes M}$ appears as weight subspace of weight $(1, \ldots, 1)$, preserved by $S_M \subset GL_M$, which acts by permutations of factors. This reproduces the Schur–Weyl duality.

This story has an analogue for the Drinfeld–Jimbo quantum universal enveloping algebra $U_q\mathfrak{gl}_N$, see e.g. [3]. Recall that $U_q\mathfrak{gl}_N$ is a family of Hopf algebras reducing to the universal enveloping algebra of $\mathfrak{gl}_N$ at $q = 1$. It has a presentation by generators and relations deforming the Chevalley–Serre presentation in the $q = 1$ case. The exterior powers of the vector representation generalize straightforwardly to $U_q\mathfrak{gl}_N$. The point is that the relations such as $[e_i, f_i] = (q^{h_i} - q^{-h_i})/(q - q^{-1})$ between Chevalley generators reduces to the classical relation $[e_i, f_i] = h_i$ in representations where the coroots h_i act semisimply with eigenvalues in $\{1, 0, -1\}$.

The quantum Schur–Weyl duality is due to Jimbo [6] and quantum versions of Howe duality are known. In the skew case they appear in [1, 2]. In the following formulation we exploit the fact that tensor products of representations are defined for bialgebras via the coproduct Δ. But we also use the opposite coproduct $\Delta' = \sigma \circ \Delta$, defined by composition with the flip $\sigma(a \otimes b) = b \otimes a$.

Theorem *Let q be a non-zero complex number. The action of $U_q\mathfrak{gl}_N$ and of $U_q\mathfrak{gl}_M$ on $\bigwedge(V \otimes W)$ induced from the action of $(U_q\mathfrak{gl}_N, \Delta)$ on $(\bigwedge V)^{\otimes M}$ and of $(U_q\mathfrak{gl}_M, \Delta'))$ on $(\bigwedge W)^{\otimes N}$ via (2) commute. If q is not a root of unity then $\bigwedge(V \otimes W)$ decomposes into the direct sum (1) of irreducible highest weight representations of $U_q\mathfrak{gl}_N \otimes U_q\mathfrak{gl}_M$.*

The universal enveloping algebra of the loop algebra $L\mathfrak{gl}_N = \mathfrak{gl}_N \otimes \mathbb{C}[t, t^{-1}]$ of $\mathfrak{gl}_N$ admits a quantum Hopf algebra deformation $U_qL\mathfrak{gl}_N$. It contains $U_q\mathfrak{gl}_N$ as a Hopf subalgebra. For $z \in \mathbb{C}^\times$ we have an evaluation algebra homomorphism $\mathrm{ev}_z : U_qL\mathfrak{gl}_N \to U_qL\mathfrak{gl}_N$, deforming the evaluation map $L\mathfrak{gl}_N \to \mathfrak{gl}_N$, $a \otimes f(t) \mapsto f(z)a$. This allows us to define an evaluation representation $M(z)$ of $U_qL\mathfrak{gl}_N$ for any representation M of $U_q\mathfrak{gl}_N$ and nonzero complex number z. In particular we have exterior powers $\bigwedge^k V(z)$ with evaluation point z. As in the classical case the tensor products of such representations with generic evaluation points are irreducible and we have isomorphisms

$$\check{R}_{k,k'}(z_1/z_2)\colon \bigwedge^k V(z_1) \otimes \bigwedge^{k'} V(z_2) \to \bigwedge^{k'} V(z_2) \otimes \bigwedge^k V(z_1)$$

unique if we impose that $v_k \otimes v_{k'} \mapsto (-1)^{kk'} v_{k'} \otimes v_k$ for tensor products of highest weight vectors. These braiding matrices $\check{R}_{k,k'}(z)$ are rational functions of the spectral parameter z. They (or more precisely the R-matrices $P \circ \check{R}$ obtained by the composition with the permutation of factors) obey the Yang–Baxter equation expressing the equality of the two isomorphisms obtained by two ways of composing braiding matrices to go from z_1, z_2, z_3 to z_3, z_2, z_1. Moreover we have the inversion relation $\check{R}_{k',k}(z^{-1})\check{R}_{k,k'}(z) = \mathrm{id}$.

The R-matrices for exterior powers of vector representations were computed by Date and Okado [4]. We offer an alternative "fermionic" formula via Howe duality, which gives a meaning to the residues at the poles as a function of the spectral parameter. Our observation is that $\check{R}_{k,k'}(z)$ commutes in particular with the subalgebra $U_q\mathfrak{gl}_N$ and can be expressed as the action of some element of $U_q\mathfrak{gl}_2$, mapping the weight subspace of weight (k, k') to the weight subspace of weight (k', k). Up to a sign this is an element of the subalgebra $U_q\mathfrak{sl}_2$ depending only on $k - k'$. Let us denote by $E, F, K^{\pm 1}$ the standard generators of $U_s\mathfrak{sl}_2$ and use the divided power notation $E^{(k)} = E^k/[k]_q!$, $F^{(k)} = F^k/[k]_q!$, with $[k]_q! = \prod_{j=1}^k (q^j - q^{-j})/(q - q^{-1})$.

Theorem *The braiding matrix $R_{k,k'}(z)$ is given by $(-1)^{\min(k,k')}$ times the action of the element $A_{k-k'}(z) \in U_q\mathfrak{sl}_2$ defined as*

$$A_m(z) = \sum_{j \geq 0} (-q)^j \frac{1 - q^{|m|}z}{1 - q^{2j+|m|}z} \begin{cases} E^{(j)}F^{(j+m)}, & \text{if } m \geq 0, \\ E^{(j+|m|)}F^{(j)}, & \text{if } m \leq 0. \end{cases}$$

The infinite sum in the definition of $A_m(z)$ reduces to a finite sum in any finite dimensional representation. The Yang–Baxter equation and the inversion relations translate via Howe duality into properties of $A_m(z)$. Let the symmetric group S_M act on $\mathbb{C}^M$ and on the weight lattice $P = \mathbb{Z}^M/\mathbb{Z}(1, \ldots, 1)$ of $\mathfrak{sl}_M$ by permutations of the coordinates and denote by s_i the transposition $(i, i+1)$. For $\mu \in P$ let $A_{i,\mu}(z_1, \ldots, z_M)$ denote the image of $A_{\mu(h_i)}(z_i/z_{i+1})$ by the embedding of $U_q\mathfrak{sl}_2$ into $U_q\mathfrak{sl}_M$ corresponding to the ith simple root. For any finite dimensional representation U of $U_q\mathfrak{sl}_M$ with weight spaces $U[\mu]$ let $A_{i,U}(z)$ be the $\operatorname{End} U$-valued function of $z \in (\mathbb{C}^\times)^M$ so that $A_{i,U}(z)|_{U[\mu]} = A_{i,\mu}(z)\colon U[\mu] \to U[s_i\mu]$.

Corollary *Let U be any finite dimensional representation of $U_q\mathfrak{sl}_M$. Then $(s_i f)(z) = A_{i,U}(z) f(s_i z)$ defines a representation of S_M on U-valued rational functions of $z = (z_1, \ldots, z_M)$.*

It turns out that this representation is closely related to the dynamical action of the braid group constructed in [9, 5] from intertwining operators: the action of generators of the braid group differ from our action by multiplication with a rational endomorphism-valued function. The more general result is that the dynamical action of the Weyl group of [5] defined on zero-weight spaces of integrable modules of $U_q\mathfrak{g}$ for semisimple $\mathfrak{g}$ extends to an action on the whole modules.

Finally, let us mention that our formula for the braiding matrix, being universal in N, is well-defined in the limit $N \to \infty$, and gives explicit expressions for R-matrices on Fock spaces for $U_q\mathfrak{gl}_\infty$ and quantum toroidal algebras along the lines of the paper [8] on the rational Yangian case, which was an inspiration for the present work. The duality between braiding matrices and dynamical Weyl group action appears to be an instance of 3D mirror symmetry.

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Quantum K-Theory Rings

Hans Jockers
(joint work with Urmi Ninad, Peter Mayr, Alexander Tabler)

Let X be a smooth projective variety and let $K(X)$ be (the torsion-free part of) the topological K-theory ring of X. We want to study a quantum deformation of $K(X)$ referred to as the quantum K-theory ring $K_Q(X)$. While the additive group structure of the rings $K(X)$ and $K_Q(X)$ are identical, the multiplicative structure of $K(X)$ — given by the tensor product of K-theory elements — is deformed in $K_Q(X)$ as

$$E_\alpha \otimes_Q E_\beta = \sum_\gamma C_{\alpha\beta}^\gamma(Q) E_\gamma = E_\alpha \otimes E_\beta + O(Q) \ .$$

Here the set $\{E_\alpha\}$ generates both $K(X)$ and $K_Q(X)$ as a group, and $C_{\alpha\beta}^\gamma(Q)$ are the deformed structure constants of $K_Q(X)$ that are formal series in the Novikov variables $Q = (Q_1, \ldots, Q_r)$ dual to $H^2(X, \mathbb{Z})/\text{Torsion}$. The constant terms $C_{\alpha\beta}^\gamma(Q)\Big|_{Q=0}$ reproduce the structure constants of $K(X)$, while the corrections $O(Q)$ are encoded in the K-theoretic genus 0 Gromov–Witten invariants [1, 2, 3]

$$\begin{aligned}&\left\langle E_1 L_1^{k_1}, \ldots, E_n L_n^{k_n} \right\rangle_{0,n,d}^X \\ &\qquad = \chi_{\overline{M}_{0,n}(X,d)} \left(\mathcal{O}^{\text{vir}} \otimes (\text{ev}_1^*(E_1) \otimes L_1^{k_1}) \otimes \ldots \otimes (\text{ev}_n^*(E_n) \otimes L_n^{k_n}) \right) \in \mathbb{Z} \ ,\end{aligned}$$

which are given by the holomorphic Euler characteristics $\chi_{\overline{M}_{0,n}(X,d)}$ over the moduli space of stable maps $\overline{M}_{0,n}(X,d)$ of genus 0 with n-marked points of class $d \in H_2(X, \mathbb{Z})$. Here $\mathcal{O}^{\text{vir}}$ is the virtual structure sheaf of $\overline{M}_{0,n}(X,d)$ [3], $\text{ev}_\ell : \overline{M}_{0,n}(X,d) \to X$ denotes the evaluation map and L_ℓ is the universal cotangent line at the ℓ-th marked point. There is a further generalization of these invariants and the associated quantum K-theory rings $K_Q(X)$ by including a non-trivial level structure [4].

In physics, the quantum K-theory rings $K_Q(X)$ and their generalizations with level structure arise from Wilson line algebras of 3d $\mathcal{N} = 2$ gauge theories [5, 6] with the projective variety X as their target spaces [7] and the level structure arising from 3d Chern-Simons terms [8].

In the following we consider for X the complex Grassmannians $\text{Gr}(M, N)$ (with $1 \leq M < N$). From the gauge theory perspective we determine that for the canonical choice of Chern–Simons terms the quantum K-theory ring $K_Q(X)$ for $X = \text{Gr}(M, N)$ is given by [8, 9]

$$\mathbb{Z}[X_1(\delta_a), \ldots, X_M(\delta_a), Q]/J_{M,N} \ , \quad J_{M,N} = \left\langle\!\left\langle \mathcal{O}_\lambda(\delta_a) - (-1)^M Q \middle| \lambda \in \Lambda \right\rangle\!\right\rangle \ ,$$

where $X_\ell(\delta_a)$ are the elementary symmetric polynomials and $\mathcal{O}_\lambda(\delta_a)$ are the Grothendieck polynomials in the variables δ_a, $a = 1, \ldots, M$, labeled by partitions λ.

Λ is the set of hook partitions

$$\Lambda = \left\{ (N-M+1, \underbrace{1,\ldots,1}_{k}) \middle| k = 0,1,\ldots,M-1 \right\} .$$

This universal expression for the quantum K-theory ring $K_Q(X)$ is in agreement with the findings of refs. [10, 11].

Analogously, the quantum K-theory rings for $\mathrm{Gr}(M,N)$ with level structure are determined from 3d $\mathcal{N}=2$ gauge theories with non-trivial choices of Chern–Simons terms. In particular, for specific choices the quantum K-theory ring with level structure becomes isomorphic to the quantum cohomology ring $H^*_Q(\mathrm{Gr}(M,N))$ [12, 13, 14], demonstrating the connection to the Verlinde algebra of refs. [15, 16, 8].

Generalizing the obtained results to more general smooth Fano varieties X — as performed on the level of quantum cohomology in ref. [14] — offers an immediate future research direction.

This talk is based on refs. [8, 9] in collaboration with Urmi Ninad, Peter Mayr, and Alexander Tabler.

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Calabi-Yau threefolds and modular forms

Albrecht Klemm

We consider the fourteen families W of Calabi-Yau threefolds with one complex structure parameter and Picard-Fuchs equation of hypergeometric type, like the mirror of the quintic in $\mathbb{P}^4$. Mirror symmetry identifies the masses of even–dimensional D–branes of the mirror Calabi-Yau M with four periods of the holomorphic $(3,0)$-form over a symplectic basis of $H_3(W,\mathbb{Z})$. It was discovered by Chad Schoen that the singular fiber at the conifold of the quintic gives rise to a Hecke eigenform of weight four under $\Gamma_0(25)$, whose Hecke eigenvalues are determined by the Hasse-Weil zeta function which can be obtained by counting points of that fiber over finite fields. Similar features are known for the thirteen other cases. In two cases we further find special regular points, so called rank two attractor points, where the Hasse-Weil zeta function gives rise to modular forms of weight four and two. We numerically identify entries of the period matrix at these special fibers as periods and quasi periods of the associated modular forms. In one case we prove this by constructing a correspondence between the conifold fiber and a Kuga-Sato variety. We also comment on simpler applications to local Calabi-Yau threefolds. This presentation is based on the preprint [1].

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Algebras behind the deformed $\mathcal{W}$-algebras

Evgeny Mukhin
(joint work with B. Feigin, M. Jimbo)

The generating currents of deformed $\mathcal{W}$-algebras are realized as sums of vertex operators and satisfy elliptic commutation relations. Introducing an additional Heisenberg current, one can modify these currents in a natural way so that the commutation relations between currents (and all summands) become rational. Then one obtains an algebra generated by the Heisenberg and modified current.

In the case of deformed $\mathcal{W}$-algebra of type A_n this algebra coincides with the Borel subalgebra of quantum toroidal algebra $\mathcal{E}_1$ of type $\mathfrak{gl}_1$ with level depending on n.

The deformed $\mathcal{W}$-algebras of types B_n, C_n, and D_n all produce the algebra $\mathcal{K}_1$ with various levels.

Let $q_1q_2q_3 = 1$. Let $g(z,w) = (z - q_1w)(z - q_2w)(z - q_3w)$.

The algebra $\mathcal{K}_1$ has generators

$$E(z) = \sum_{n\in\mathbb{Z}} E_n z^{-n}\,, \quad K^{\pm}(z) = \exp\Big(\sum_{\pm r>0} H_r z^{-r}\Big)\,,$$

and a central element C, satisfying the defining relations:

$$\begin{aligned}
&g(z,w)E(z)E(w)+g(w,z)E(w)E(z)\\
&\qquad=\frac{1}{g(1,1)}\Big(g(z,w)\delta\big(C^2\frac{z}{w}\big)K(z)+g(w,z)\delta\big(C^2\frac{w}{z}\big)K(w)\Big)\,,\\
&K^{\pm}(z)K^{\pm}(w)=K^{\pm}(w)K^{\pm}(z)\,,\\
&g(z,w)g(z,C^2w)\,K^+(z)K^-(w)=g(w,z)g(C^2w,z)\,K^-(w)K^+(z)\,,\\
&g(z,w)\,K^{\pm}(z)E(w)=g(w,z)E(w)\,K^{\pm}(z)\,,\\
&\operatorname*{Sym}_{z_1,z_2,z_3}\frac{z_2}{z_3}[E(z_1),[E(z_2),E(z_3)]]\\
&\quad=\operatorname*{Sym}_{z_1,z_2,z_3}\big(1-\frac{z_1^2}{z_2z_3}\big)\frac{z_1(z_1+z_2)(z_1+z_3)}{g(z_2,z_1)g(z_3,z_1)}g(z_2,z_3)\delta\Big(C^2\frac{z_2}{z_3}\Big)E(z_1)K(z_2)\,,
\end{aligned}$$

where $K(z)=K^-(z)K^+(C^2z)$, and $\delta(z)=\sum_{i\in\mathbb{Z}}z^i$.

The algebra $\mathcal{K}_1$ is a comodule over $\mathcal{E}_1$. Namely, there exists an algebra homomorphism $\Delta:\mathcal{K}_1\to\mathcal{E}_1\tilde{\otimes}\mathcal{K}_1$, where $\tilde{\otimes}$ is an appropriately completed tensor product. The formulas are reminiscent of quantum symmetric pairs and i-quantum groups.

The algebra $\mathcal{K}_1$ possesses a commutative family of integrals of motion.

Let $\mu=C^2q_2^{-1}$. Then $\{I_n\}_{n=1}^{\infty}$ commute,

$$I_n=\int\cdots\int\boldsymbol{E}(z_1)\cdots\boldsymbol{E}(z_n)\cdot\prod_{j<k}\omega_2(z_k/z_j)\prod_{j=1}^{n}\frac{dz_j}{2\pi i z_j}\,,$$

where the integral is taken on the unit circle $|z_j|=1$, $j=1,\dots,n$ in the region $|q_1|,|q_3|>1$ and extended by analytic continuation everywhere else.

Here we used the dressed current $\boldsymbol{E}(z)=E(z)\prod_{s=0}^{\infty}(K^+(\mu^{-s}z))^{-1}$, and the kernel

$$\omega_2(z)=\frac{\Theta_\mu(z)\Theta_\mu(q_2^{-1}z)}{\Theta_\mu(q_1z)\Theta_\mu(q_3z)}\,,$$

where $\Theta_\mu(z)$ is the Jacobi theta function given by $\Theta_\mu(z)=(z,\mu z^{-1},\mu;\mu)_\infty$.

Unlike the quantum toroidal algebras, where the commutative family of integrals of motion stems out from the transfer matrices, the origin of this commutative family is unclear.

The algebra obtained from the deformed $\mathcal{W}$-algebra of type G_2 is described in [1]. The properties of this algebra are not known. The algebras related to other types have not been computed yet.

In [2] we give the higher rank $\mathcal{K}_n$ generalization of the algebra $\mathcal{K}_1$ and show that it has similar properties. In particular, $\mathcal{K}_n$ is a comodule over quantum toroidal algebra of type $\mathfrak{gl}_n$ and has a family of integrals of motion.

It is expected that similar $\mathcal{K}$-type algebras exists for types D_n, E_6, E_7, E_8.

Nothing is known about possible higher rank generalizations of the algebras for other types.

The main question is: what is the nature of these algebras? Are these examples a part of one construction?

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Unification of WRT and CGP invariants of 3-manifolds via BPS q-series

Pavel Putrov
(joint work with Francesco Costantino, Sergei Gukov)

Categorification of quantum invariants of 3-manifolds is an important problem in low-dimensional topology. One of the reasons is that homological invariants of 3-manifolds that behave functorially with respect to 4-dimensional bordisms in particular provide numerical invariants of smooth 4-manifolds. A famous (and essentially so far the only known non-trivial) example is Heegaard/Monopole Floer Homology of 3-manifolds that categorifies numerical invariants of 3-manifolds associated to $\mathfrak{gl}(1|1)$ Lie superalgebra, or equivalently Reidemeister-Turaev torsion. The link counterparts of such invariants is Knot Floer Homology which categorifies Alexander-Conway polynomial. The corresponding numerical invariants of 4-manifolds are celebrated Seiberg-Witten invariants.

One natural question is to find a suitable 3-manifold analog of Khovanov homology which is known to categorify Jones polynomial, a quantum knot invariant associated to Lie algebra $\mathfrak{sl}(2)$. The corresponding numerical 3-manifold invariant is known as Witten-Reshetikhin-Turaev (WRT) invariant [9, 7]. Unlike Jones polynomial, WRT invariant is only defined when q, the deformation parameter in the quantum group $U_q(\mathfrak{sl}_2)$, is a root of unity and is not a Laurent polynomial (or series) in q.

However from physics one expects existence of a family of certain q-series $\hat{Z}_{\mathfrak{s}}(Y)$ associated to a 3-manifold Y with a spin^c structure $\mathfrak{s}$ and which are closely related to the WRT invariant $\mathrm{WRT}_r(Y)$ [4, 3] defined for the r-th primitive root of unity[1]$q = e^{\frac{2\pi i}{r}}$ for some positive integer $r \geq 2$ (and for concreteness we will assume that $r = 2 \mod 4$ in what follows[2]). These q-series physically has meaning of counting BPS stated in a 6d $\mathcal{N} = (2,0)$ theory on $Y \times \mathbb{C} \times \mathbb{R}$. The relation is then given by a sequence of dualities to Chern-Simons topological quantum field theory on Y partition function of which physically realizes the WRT invariant of Y [8].

[1]The convention is such that the quantum dimension of the representation of the highest weight $n-1$ is $[n] = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}$.

[2]Similar statements can be made for other values of r modulo 4.

It turns out that it is more natural to extend the relation between these q-series to the mod-2 cohomological refinement [6] of the WRT invariant $\mathrm{WRT}_r(Y,\omega)$, $\omega \in H^1(Y,\mathbb{Z}/2\mathbb{Z})$, and also the invariant of Costantino, Geer, and Patureau-Mirand (CGP) [1] $N_r(Y,\omega)$, defined for a choice of $\omega \in H^1(Y,\mathbb{C}/2\mathbb{Z}) \setminus H^1(Y,\mathbb{Z}/2\mathbb{Z})$.

Namely, let $\mathcal{T}(Y,\omega)$ be the appropriate version of Reidemeister-Turaev torsion (see [2] for details) and assume that it does not vanish[3] unless $\omega \in H^1(Y,\mathbb{Z}/2\mathbb{Z})$. Then define a combined invariant

$$\tilde{N}_r(Y,\omega) := \begin{cases} \mathrm{WRT}_r(Y,\omega), & \omega \in H^1(Y,\mathbb{Z}/2\mathbb{Z}) \\ \dfrac{N_r(Y,\omega)}{\mathcal{T}(Y,\omega)}, & \text{otherwise.} \end{cases}$$

for arbitrary $\omega \in H^1(Y,\mathbb{C}/2\mathbb{Z})$. One can then formulate the following[4]

Conjecture 1 ([3, 2])**.** *Assume Y is a rational homology sphere, that is[5] $b_1(Y) = 0$. There exists a family of q-series*

$$\hat{Z}_{\mathfrak{s}}(Y) \in \mathbb{Z}[[q]] \qquad (\mathfrak{s} \in \mathrm{Spin}^c(Y))$$

convergent for $|q| < 1$ and such that

$$\tilde{N}_r(Y,\omega) = \lim_{q \to e^{\frac{2\pi i}{r}}} \sum_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} C_{\mathfrak{s}}(Y,\omega)\, \hat{Z}_{\mathfrak{s}}(Y) \tag{1}$$

where $C_{\mathfrak{s}}(Y,\omega)$ are universal coefficients depending on r and certain simple homotopy-type invariants of Y (see [2] for details).

The conjecture was verified for a certain family of manifolds in [2], where it was also shown that the conjecture holds if its certain analog for links in a 3-sphere holds. An analogous conjecture can be made for the $\mathfrak{sl}(2|1)$ analogs of CGP invariants [5].

The existence of q-series as formulated in the conjecture above does not imply their uniqueness, since there exist non-trivial q-series that have zero limits at all roots of unity. Thus it would not be possible to define $\hat{Z}_s$ directly using formula (1). This issue in principle can be avoided by using elements of Habiro ring instead of q-series and modify the conjecture as follows:

Conjecture 2. *Assume $b_1(Y) = 0$. There exists a family*

$$I_{\mathfrak{s}}(Y) \in \widehat{\mathbb{Z}[q]} := \varprojlim_n \frac{\mathbb{Z}[q]}{\prod_{k=1}^n (1-q^k)} \qquad (\mathfrak{s} \in \mathrm{Spin}^c(Y))$$

such that

$$\tilde{N}_r(Y,\omega) = \sum_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} C_{\mathfrak{s}}(Y,\omega)\, I_{\mathfrak{s}}(Y)\big|_{q=e^{\frac{2\pi i}{r}}}$$

where $C_{\mathfrak{s}}(Y,\omega)$ are universal coefficients depending on r and certain simple homotopy-type invariants of Y (same as in Conjecture 1).

[3]This holds for a generic 3-manifold. However it would be interesting to see how can one avoid this assumption.

[4]The conventions differ slightly from the ones in [2].

[5]A version of the conjecture can be made for $b_1(Y) > 0$.

Taking into account that $C_{\mathfrak{s}}(Y,\omega)$ defines an invertible transformation between complex valued functions on $H^1(Y,\mathbb{C}/2\mathbb{Z})$ and on $\mathrm{Spin}^c(Y)$, and also that elements of the Habiro ring are uniquely fixed by their values on roots of unity, the existence of $I_{\mathfrak{s}}(Y)$ implies their uniqueness for a given 3-manifold Y.

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Semisimple topological quantum field theories and exotic smooth structure

David Reutter

(joint work with Christopher Schommer-Pries)

Motivated by a wealth of powerful field-theoretically-inspired 4-manifold invariants, a major open problem in quantum topology is the construction of a 4-dimensional topological quantum field theory (TQFT) in the sense of Atiyah-Segal which is sensitive to exotic smooth structure. More generally, how much manifold topology can a TQFT see?

In my talk, I outlined an answer to this question for even-dimensional field theories having a certain representation-theoretic property. This is based on [3] and joint work in progress with Christopher Schommer-Pries [4].

From semisimple topological quantum field theories...

A d-dimensional topological quantum field theory [1, 2] is a symmetric monoidal functor

$$Z : \mathrm{Bord}_{d,d-1} \to \mathrm{Vect}$$

from the category of closed $(d-1)$-manifolds and diffeomorphism classes of compact d-dimensional bordisms to the category of k-vector spaces and linear maps[1]. In other words, a topological quantum field theory is an assignment of vector spaces

[1]The results discussed in this extended abstract apply more generally to TQFTs on bordism categories with more general *tangential structures*, such as orientation, spin, etc. and *super TQFTs* valued in the symmetric monoidal category of super vector spaces.

to $(d-1)$-manifolds and linear maps to d-manifolds, which is compatible with gluing and taking disjoint union. In particular, topological quantum field theories lead to diffeomorphism invariants $Z(W) \in k$ for closed d-manifolds W. How strong are these invariants?

Associated to a topological quantum field theory is the algebra of *point operators* $Z(S^{d-1})$ with multiplication and unit given by the bordisms

$$D^d \backslash (D^d \sqcup D^d) : S^{d-1} \sqcup S^{d-1} \to S^{d-1} \qquad D^d : \emptyset \to S^{d-1}.$$

More generally, for a closed $(d-k)$-manifold M there is the associated algebra $Z(S^{k-1} \times M)$ with multiplication and unit bordisms

$$\big(D^k \backslash (D^k \sqcup D^k)\big) \times M \qquad D^k \times M.$$

We refer to this latter algebra as the *fusion algebra of M-shaped operators*, since given a closed d-manifold W with a normally framed embedded $M \hookrightarrow W$, and a vector $v \in Z(S^{k-1} \times M)$, we may compute the *partition function* of W with the operator v inserted along M as the composite

$$k \xrightarrow{v} Z(S^{k-1} \times M) \xrightarrow{Z(W \backslash (D^k \times M))} Z(\emptyset) \cong k$$

where $D^k \times M \hookrightarrow W$ is a tubular neihghborhood of $M \hookrightarrow W$ with boundary parametrization determined by the normal framing of M.

Definition. *We say that a $d = 2n$-dimensional TQFT is* semisimple *if the algebra of point operators $Z(S^{2n-1})$ and the fusion algebra of S^{n-1}-shaped operators $Z(S^n \times S^{n-1})$ are semisimple.*

In dimension strictly greater than 2, all examples of functorial TQFTs we are aware of are semisimple and hence subject to our results. As shown in [3], this includes all invertible field theories, unitary field theories and once-extended field theories (i.e. theories which also assign values to manifolds of dimension $d-2$) valued in the symmetric monoidal bicategory of algebras, bimodules and bimodule maps, or the symmetric monoidal bicategory of additive and idempotent complete k-linear categories, linear functors and natural transformations.

... to stable diffeomorphisms. The main premise of our work is to relate the representation-theoretic property of semisimplicity with the following key notion [5, 6] from surgery theory. Two closed connected $2n$-manifolds $W, \widetilde{W}$ are *stably diffeomorphic* if there is a positive integer $k \geq 0$ and a diffeomorphism between the connected sums

$$W \#^k (S^n \times S^n) \cong \widetilde{W} \#^k (S^n \times S^n).$$

Theorem A ([3] in 4 dimensions, [4] in arbitrary even dimensions)**.** *Even-dimensional semisimple TQFTs lead to stable diffeomorphism invariants.*

An upshot of this theorem is that stable diffeomorphism is a much coarser equivalence relation than diffeomorphism and stable diffeomorphism classes of manifolds are much better understood than diffeomorphism classes (see e.g. [6]). In particular, the following result is a direct consequence of the classical result [7].

Corollary ([3]). *Semisimple four-dimensional oriented topological quantum field theories cannot see exotic smooth structure.*

In upcoming work with Christopher Schommer-Pries, we show that this 'upper bound' provided by Theorem A on the sensitivity of topological quantum field theories is optimal.

Theorem B ([4]). *If two closed 4-manifolds with finite fundamental group are stably diffeomorphic, then they can be distinguished by a semisimple (super) topological quantum field theory.*

Theorem B also generalizes to arbitrary even dimensions subject to a somewhat more involved finiteness condition. As a corollary of Theorem B, it follows for example that there exist semisimple 4-dimensional topological quantum field theories which can see unoriented exotic smooth structure (i.e. can distinguish certain unoriented homeomorphic but not diffeomorphic 4-manifolds) and that oriented semisimple higher-dimensional TQFT can detect certain higher-dimensional exotic spheres.

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Volumes of supergrassmannians

Vera Serganova

(joint work with Alexander Sherman)

The supergrassmannian $Gr(r|s,m|n)$ is the superscheme representating the functor of $(r|s)$-dimensional subspaces in $\mathbb{C}^{m|n}$. If $G = GL(m|n)$, $K = GL(r|s)$ and $P \subset G$ a maximal parabolic subgroup of G containing K then the homogeneous space G/P is isomorphic to $Gr(r|s,m|n)$. The underlying algebraic variety is a product of two classical grassmannians $Gr(r|m) \times Gr(s|n)$. If $p \in Gr(r|s,m|n)$ is the point with stabilizer P then the tangent space $T_pGr(r|s,m|n)$ can be identified with $\mathfrak{g}/\mathfrak{p}$. That immediately implies the formulas for dimension and superdimension of $Gr(r|s,m|n)$

$$\dim Gr(r|s,m|n) = (r(m-r)+s(n-s)|r(n-s)+s(m-r)), \tag{1}$$

(2) $$\operatorname{sdim} Gr(r|s,m|n) = (r-s)((m-r)-(n-s)).$$

Recall that G has a compact real form $\mathcal{U} = U(m|n)$, it is called unitary supergroup. The Lie algebra $\mathfrak{u}(m|n)$ is the set of fixed points of antilinear involution θ defined by

$$\theta \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -\bar{A}^t & \mathbf{i}\bar{C}^t \\ \mathbf{i}\bar{B}^t & -\bar{D}^t \end{pmatrix}.$$

Let $\mathcal{K} := \mathcal{U} \cap K = \mathcal{U} \cap P \simeq U(r|s) \times U(m-r|n-s)$.

Proposition. *The unitary supergroup $\mathcal{U}$ acts transitively on the supergrassmannian $Gr(r|s,m|n)$ and we have an isomorphism of real supermanifolds*

$$Gr(r|s,m|n) \simeq \mathcal{U}/\mathcal{K}.$$

Furthermore, $Gr(r|s,m|n)$ has a unique up to normalization $\mathcal{U}$-invariant volume form.

For many applications in representation theory of supergroups it is important to understand when the volume

$$\int_{Gr(r|s,m|n)} \omega$$

of $Gr(r|s,m|n)$ is not zero.

Theorem. *The volume of the supergrassmannian $Gr(r|s,m|n)$ is not zero if and only if* $\operatorname{sdim} Gr(r|s,m|n) \geq 0$.

The above theorem confirms the conjecture of Voronov, [2].

The proof of the theorem is based on the localization method of Schwarz and Zaboronsky, [1].

Let M be a compact real supermanifold with underlying manifold M_0 and a volume form ω. We fix an orientation on M_0. Let Q be a vector field on M. A point $p \in M$ is a zero of Q if for any smooth function $f \in \mathcal{C}^\infty M$ we have $Q(f) \subset I_p$ where I_p is the ideal of p. By $Z(Q)$ we denote the set of all zeros of Q. If $p \in Z(Q)$ then Q induces the linear operator in $T_p^* X = I_p/I_p^2$. We say that $p \in Z$ is isolated if $Q : T_p^* X \to T_P^* X$ is an isomorphism. Let us consider an odd vector field Q on M satisfying the following properties:

(1) $Z(Q)$ is finite and consists of isolated zeros;
(2) $Q\omega = 0$;
(3) Q^2 is a compact vector field, i.e.,there exists a compact Lie group $\mathcal{K}$ acting on M such that $Q \in \operatorname{Lie} \mathcal{K}$.

Let us note that the above conditions imply $\dim M = (2n|2n)$.

Theorem. *Let a supermanifold M, a volume form ω and an odd vector field Q satisfy the above assumptions. Then there exists an odd function σ such that $Q^2\sigma = 0$ and $Q\sigma \notin I_p$ for any $p \notin Z(Q)$. Furthermore,*

$$\int_M \omega = \frac{\pi^{2n}}{(2n)!} \sum_{p \in Z(Q)} \alpha(T_p M, \operatorname{Hess}_p(Q\sigma), \omega_p),$$

where $\mathrm{Hess}_p(f)$ *is the Hessian of the function* f *at* p *which can be considered as an even symmetric* Q*-invariant form on* T_pM *and the definition of* α *is given below.*

Let V be a finite-dimensional representation of the Lie superalgebra with basis Q, Q^2 such that Q^2 is compact. Let us fix a volume form ω on V and orientation ω_0 on V_0. Let B be a Q-invariant even symmetric bilinear form. Choose bases $e_1, \dots, e_{2n} \in V_0$ and $f_1, \dots, f_{2n} \in V_1$ such that

$$\omega(e_1, \dots, e_{2n}, f_1, \dots, f_{2n}) = 1, \omega_0(e_1, \dots, e_n) > 0.$$

Set

$$\alpha(V, B, \omega) := (-1)^n \exp\left(\frac{\pi \mathbf{i}(\dim V_0^+ - \dim V_0^-)}{4}\right) \frac{\mathrm{Pf}(B_1)}{\sqrt{|\det B_0|}},$$

where $\dim V_0^+ - \dim V_0^-$ is the signature of B_0.

Let $Q \in \mathfrak{u}(m|n)$ be a generic odd element, it induces an odd vector field on $Gr(r|s, m|n)$. If $\mathrm{sdim}\, Gr(r|s, m|n) < 0$ then $Z(Q) = \emptyset$ and hence the volume is zero. We reduce the case $\mathrm{sdim}\, Gr(r|s, m|n) \geq 0$ to the case $r = s$ and $m = n$ and then show that the contributions terms in the Schwarz-Zaboronsky formula are the same for all $p \in Z(Q)$.

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Tau-functions, coset constructions and free fermion conformal blocks

Jörg Teschner

The isomonodromic tau-functions encode solutions to certain nonlinear PDE appearing in many problems of mathematical physics. More recent developments have revealed profound connections to representation theory, conformal field theories (CFTs), and supersymmetric quantum field theories. The goal of my talk has been to stimulate some discussions at the workshop by pointing out some of these connections, and by proposing some natural conjectures.

The Gamayun-Iorgov-Lisovyy (GIL) formula conjectured in [GIL],

$$\mathcal{T}(\sigma, \eta; \mathbf{m}; q) = \sum_{n \in \mathbb{Z}} e^{2\pi \mathrm{i} n \eta} Z(\sigma + n, \mathbf{m}; q), \tag{1}$$

relates the isomonodromic tau-functions $\mathcal{T}(\sigma, \eta; \mathbf{m}; q)$ to the conformal blocks $Z(\sigma, \mathbf{m}; q)$ of the Virasoro algebra with central charge $c = 1$. The variables σ and η are certain distinguished coordinates for the monodromy data of a holomorphic connection on the the four-punctured sphere with regular singularities at all four punctures, and singular behaviour at the four punctures parameterised by $\mathbf{m} = (m_1, \dots, m_4)$. The variable q denotes the cross-ratio of the positions of the four punctures. A brief review of the GIL formula and its proof using CFT methods [ILT] can be found in [Te17].

Nekrasov's derivation of the GIL formula [Ne20] starts from the following blow-up formulae for instanton partition functions of the $N_f = 4$, $SU(2)$, $\mathcal{N} = 2$ SUSY gauge theory in the presence of a surface operator:

$$\begin{aligned}(2)\qquad &\Psi(a, \mathbf{m}; w, q; \epsilon_1, \epsilon_2) = \\ &= \sum_{n\in\mathbb{Z}} \Psi(a + \epsilon_1 n, \mathbf{m}; w, q; \epsilon_1, \epsilon_2 - \epsilon_1) Z(a + \epsilon_2 n, \mathbf{m}; q; \epsilon_1 - \epsilon_2, \epsilon_2)\end{aligned}$$

The variables have the following meaning in SUSY gauge theory: a is the Coulomb-branch parameter, $\mathbf{m} = (m_1, \dots, m_4)$ is the vector of four mass parameters, q is the exponential of the complexified UV gauge coupling, w is the defect parameter, and ϵ_1, ϵ_2 are the parameters of the Ω-deformation. From the blow-up formula (2) one may derive the GIL-formula (1) in the limit $\epsilon_1 \to 0$ [Ne20].

The author conjectures that the basis of Nekrasov's derivation, provided by equivariant localisation on instanton moduli spaces, can be replaced by identities between conformal blocks following from the coset construction in CFT. It should be possible to prove (2) as a consequence of the following identity among vertex operator algebra (VOA) representations [BFL, Theorem 3.3(b)],

$$(3)\qquad \mathcal{V}_{h,k} \otimes \mathcal{V}_{0,1} = \bigotimes_{n\in\mathbb{Z}} \mathcal{V}_{h+2n,k+1} \otimes \mathcal{W}_{P+nb,b}.$$

We are using the following notations (similar to [BFL]):

- $\mathcal{V}_{h,k}$: Representation of affine $\widehat{\mathfrak{sl}}_{2,k}$ with level k and highest weight h.
- $\mathcal{W}_{P,b}$: Representation of the Virasoro algebra with central charge $c = 1 + 6Q^2$, $Q = b + b^{-1}$, and highest weight $\Delta = Q^2/4 - P^2$.

In (3) we are assuming that the following relations hold[1]:

$$(4)\qquad b = \mathrm{i}\sqrt{\frac{k+3}{k+2}}, \quad Q = \frac{1}{\mathrm{i}}\frac{1}{\sqrt{(k+2)(k+3)}}, \quad P = \frac{1}{2\mathrm{i}}\frac{h+1}{\sqrt{(k+2)(k+3)}}.$$

We conjecture that it should be possible to derive (2) from (3) if one assumes the relations $\frac{\epsilon_1-\epsilon_2}{\epsilon_2} = b^2 = -\frac{k+3}{k+2}$. In order to derive (2) from the VOA relations (3) one will need to determine the precise relations between the normalisations of the conformal blocks involved in (2).

The following observation can be seen as support for this conjecture. A formal limit $k \to \infty$ of (3) for $h = 0$ would yield the relation of VOAs

$$(5)\qquad \mathcal{V}_{0,1} = \bigotimes_{n\in\mathbb{Z}} R_{2n} \otimes \mathcal{W}_{\mathrm{i}n,\mathrm{i}},$$

where R_j is the representation of $\mathfrak{sl}_2$ of dimension $2j + 1$. On the right side one identifies an extension of the Virasoro VOA at $c = 1$. In taking $k \to \infty$ of (3) we find on both sides the same Poisson-VOA representing the classical limit of $\mathcal{V}_{0,k}$

[1]Note that the parameter b used here is the inverse of the parameter b used in [BFL].

as a factor. This factor is omitted in (5). Tensoring with a free boson VOA $\mathcal{B}$ one obtains a VOA that has a well-known extension to the free fermion super VOA,

$$\mathcal{F} = \mathcal{B} \otimes \bigotimes_{j\in\mathbb{Z}} R_j \otimes \mathcal{W}_{\frac{\mathrm{i}}{2}j,\mathrm{i}}\ . \tag{6}$$

The relation between free fermions and Virasoro degenerate fields implied by (6) is equivalent to the bosonisation formulae used in [ILT, Section 5.1] to identify the GIL formula (1) as a relation between Virasoro and free fermion conformal blocks.

Twisting conformal blocks by local systems. There is an interesting family of VOAs having decompositions similar to (5). The members of this family are called Feigin-Tipunin algebras $\mathcal{FT}_k(\mathfrak{g})$, and they can be represented as[2]

$$\mathcal{FT}_k(\mathfrak{g}) = \bigoplus_{\lambda\in Q^+} R_\lambda \otimes \mathcal{W}^{1/k}_{\lambda,0}(\mathfrak{g}). \tag{7}$$

An important feature of the conformal blocks of the VOAs $\mathcal{FT}_k$ emphasised in [CDGG] is related to the appearance of the finite-dimensional irreducible $\mathfrak{g}$-modules R_λ in (7). This implies that a group G with Lie algebra $\mathfrak{g}$ represents a group of continuous automorphisms of $\mathcal{FT}_k$, allowing one to define conformal blocks of $\mathcal{FT}_k$ twisted by G-local systems.

The picture drawn in [CDGG] and references therein suggests that twisting the conformal blocks of $\mathcal{FT}_k$ by local systems will simplify the mapping class group action on the spaces of conformal blocks considerably by semi-simplifying the relevant VOA representation categories. Note that the feature of having continuous groups of automorphisms is shared with the free fermion VOA $\mathcal{F}$. The twisting of $\mathcal{F}$ by continuous automorphisms is the basis of the relations between isomonodromic tau functions and free fermion conformal blocks described in [ILT, CPT, CLT]. It should be interesting to study the dependence of conformal blocks of the algebras $\mathcal{FT}_k$ with respect to the choice of the twisting local systems.

It is furthermore intriguing to observe that the key feature of the conformal blocks of the Virasoro VOA at $c = 1$ underlying the derivation of the GIL-formula given in [ILT] is a root-of-unity phenomenon: The parameter $q = e^{\pi \mathrm{i} b^2}$ characterising the braid group representation on Virasoro conformal blocks is equal to $q = e^{-\pi\mathrm{i}}$ for $c = 1$. Similar phenomena can be expected to occur for $\mathcal{W}^{1/k}_{\lambda,0}$. This suggests that (7) could imply relations between the conformal blocks of the VOAs appearing in this formula generalising the relations proven in [ILT].

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A universal approach to the spectral reflection equation

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(joint work with Andrea Appel)

A key question which led to the discovery of quantum groups by Drinfeld and Jimbo was the representation-theoretic origin of known matrix-valued formal Laurent series[1] $R_{VW}(z) \in \mathsf{End}(V \otimes W)((z))$ of the *spectral Yang-Baxter equation*:

$$R_{UV}(z_1) \cdot R_{UW}(z_2) \cdot R_{VW}(z_2/z_1) = R_{VW}(z_2/z_1) \cdot R_{UW}(z_2) \cdot R_{UV}(z_1), \tag{1}$$

in $\mathsf{End}(U \otimes V \otimes W)((z_1, z_2/z_1))$. Indeed, there is a universal approach to matrix solutions to (1) and hence to quantum integrable models with closed (periodic) boundary conditions: if $\mathfrak{g}$ is a complex simple Lie algebra, then the universal R-matrix $\mathcal{R}$ of the quantum affine algebra $U_q\widetilde{\mathfrak{g}}$ acts on z-shifted tensor products of finite-dimensional representations of the untwisted quantum loop algebra $U_qL\mathfrak{g}$. This yields solutions $R_{VW}(z) \in \mathsf{End}(V \otimes W)((z))$ of (1); moreover, if V and W are both irreducible then $R_{VW}(z)$ essentially depends rationally on z.

It is natural to extend this picture to models with open (reflecting) boundary conditions. Since the 1980s [Ch84, Sk88] the type-B analogue of the spectral Yang-Baxter equation, called *spectral reflection equation*, has been studied, namely the following equation in $\mathsf{End}(V \otimes W)((z_1, z_2/z_1))$:

$$\begin{aligned} &K_V(z_1) \otimes \mathsf{id} \cdot R_{WV}(z_1z_2)_{21} \cdot \mathsf{id} \otimes K_W(z_2) \cdot R_{VW}(z_2/z_1) = \\ &= R_{WV}(z_2/z_1)_{21} \cdot \mathsf{id} \otimes K_W(z_2) \cdot R_{VW}(z_1z_2) \cdot K_V(z_1) \otimes \mathsf{id} \end{aligned} \tag{2}$$

for the unknowns $K_V(z_1) \in \mathsf{End}(V)((z_1))$ and $K_W(z_2) \in \mathsf{End}(W)((z_2))$; here $R_{WV}(z)_{21} := (12) \cdot R_{WV}(z) \cdot (12)$. It is a consistency condition for an action of the Artin-Tits braid group of type B_n on vector-valued formal series (or functions) of n variables, which combines a "tensor action" and a "function action", the latter of which factorizes through the faithful action of the Coxeter group $W(\mathsf{B}_n)$ by permutations and inversions of variables. What is the universal approach that generates (the many known) solutions of (5)?

[1] We work over an algebraically closed field $\mathbb{F}$ containing $\mathbb{C}(q)$ for some formal parameter q. For finitely many formal parameters $z_1, z_2, \ldots$ and any $\mathbb{F}$-linear space M we use the following shorthands: $M(z_1, z_2, \ldots) := M \otimes \mathbb{F}(z_1, z_2, \ldots)$ and $M((z_1, z_2, \ldots)) := M \otimes \mathbb{F}((z_1, z_2, \ldots))$.

For the *constant* reflection equation, this is known: let $U_q(\mathfrak{g}^\theta)$ be Letzter's [Le02] coideal deformation of the fixed-point subalgebra of any involutive automorphism θ of $\mathfrak{g}$. Balagović and Kolb [BK19] showed that, up to completion with respect to $\mathsf{Mod}_{\mathsf{f.d.}}(U_q\mathfrak{g})$, the centralizer of $U_q(\mathfrak{g}^\theta)$ contains a solution $\mathcal{K}$ of

$$(\mathcal{K}\otimes 1)\cdot\mathcal{R}_{21}\cdot(1\otimes\mathcal{K})\cdot\mathcal{R}=\mathcal{R}_{21}\cdot(1\otimes\mathcal{K})\cdot\mathcal{R}\cdot(\mathcal{K}\otimes 1). \tag{3}$$

However, this construction does not directly generalize beyond finite type. Even if such a solution $\mathcal{K}$ existed, given a representation $\pi_{V,z}:U_qL\mathfrak{g}(z)\to\mathsf{End}(V)(z)$ with V finite-dimensional, the equation (3) *cannot* yield[2] (5).

This issue motivated our Kac-Moody generalization of [BK19]. Letzter's theory of q-deformed symmetric pairs extends to involutive automorphism θ of Kac-Moody algebras $\mathcal{L}$, see [Ko14], provided $\dim(\theta(\mathcal{L}^+)\cap\mathcal{L}^+)<\infty$ where $\mathcal{L}^+$ is the standard nilpotent subalgebra (i.e., θ is *of the second kind*). In [AV20] pairs $(\mathcal{K},\psi)$ were constructed where $\mathcal{K}$ is an invertible element of the completion of $U_q\mathcal{L}$ with respect to $\mathcal{O}^{\mathsf{int}}$ (integrable modules in category $\mathcal{O}$) and ψ an algebra automorphism of $U_q\mathcal{L}$. They satisfy $\mathcal{K}\cdot b=\psi(b)\cdot\mathcal{K}$ for all $b\in U_q(\mathcal{L}^\theta)$ and the *ψ-twisted universal reflection equation*:

$$(\mathcal{K}\otimes 1)\cdot(\mathcal{R}^\psi)_{21}\cdot(1\otimes\mathcal{K})\cdot\mathcal{R}=\mathcal{R}_{21}^{\psi\psi}\cdot(1\otimes\mathcal{K})\cdot\mathcal{R}^\psi\cdot(\mathcal{K}\otimes 1) \tag{4}$$

where $\mathcal{R}^\psi:=(\psi\otimes\mathsf{id})(\mathcal{R})$ and $\mathcal{R}^{\psi\psi}:=(\psi\otimes\psi)(\mathcal{R})$. Such $(\mathcal{K},\psi)$ naturally arise as "gauge transformations" $(u\cdot\mathcal{K}_0,\mathsf{Ad}(u)\circ\psi_0)$ defined in terms of an invertible element u of the completion of $U_q\mathcal{L}$ on a pair $(\mathcal{K}_0,\psi_0)$, where $\mathcal{K}_0$ lies in the $\mathcal{O}$-completion of $U_q(\mathcal{L}^+)$ and ψ_0 is a q-deformation of θ itself. If $\mathcal{L}$ is of finite type, u can be chosen, in terms of Lusztig braid group operators, so that ψ is a bialgebra automorphism, which recovers the Balagović-Kolb formalism.

In parallel to the story of the Yang-Baxter equation, in recent work [AV22] we show the following. If $\mathcal{L}$ is the untwisted affine Lie algebra $\widetilde{\mathfrak{g}}$ associated to the finite-dimensional Lie algebra $\mathfrak{g}$, then for suitable twists ψ, $\mathcal{K}$ acts on any (appropriately z-shifted) finite-dimensional $U_qL\mathfrak{g}$-module as a matrix-valued formal series $K_V(z)\in\mathsf{End}(V)((z))$, satisfying the generalized reflection equation

$$\begin{aligned}K_V(z_1)\otimes\mathsf{id}\cdot R_{\psi^*(W)V}(z_1z_2)_{21}\cdot\mathsf{id}\otimes K_W(z_2)\cdot R_{VW}(z_2/z_1)=\\ =R_{\psi^*(W)\psi^*(V)}(z_2/z_1)_{21}\cdot\mathsf{id}\otimes K_W(z_2)\cdot R_{\psi^*(V)W}(z_1z_2)\cdot K_V(z_1)\otimes\mathsf{id},\end{aligned} \tag{5}$$

cf. [Ch92]; here $\psi^*(V)$ indicates the pullback by ψ, i.e. $\pi_{\psi^*(V)}:=\pi_V\circ\psi$. Building on the irreducibility result [HJ12, Prop. 3.5], we show that if V is irreducible, then it is irreducible as a $U_q((L\mathfrak{g})^\theta)$-module. This implies the linear space

$$\{K(z)\in\mathsf{End}(V)((z))\,|\,K(z)\pi_{V,z}(b)=\pi_{\psi^*(V),1/z}(b)K(z)\text{ for all }b\in U_q((L\mathfrak{g})^\theta)\}$$

is one-dimensional so that, up to a scalar, $K_V(z)$ is a rational function of z. Hence this universal approach produces known and new "trigonometric" solutions of generalized reflection equations.

[2]Instead, it yields a consistency condition for an action of the braid group of B_n on vector-valued series where the "function" action of $W(\mathsf{B}_n)$ is non-faithful, namely the one trivially extending the permutation action of the symmetric group.

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A holomorphic approach to fivebranes

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(joint work with Surya Raghavendran and Ingmar Saberi)

The goal of this talk is to present an approach to characterizing the algebra of local operators of the infamous six-dimensional superconformal field theory associated to the Lie algebra $\mathfrak{g} = \mathfrak{sl}(N)$. The description is at the level of the *holomorphic* twist of this theory which is richer than previous approaches which utilize a further, more topological, twist of the theory. The notion of a twist of a supersymmetric theory is like taking the (derived) invariants, or cohomology, with respect to a single square-zero supercharge inside of the supersymmetry algebra. Though not as popular as their topological counterparts, holomorphic twists of supersymmetric theories are much more plentiful [1]. In fact, besides a few low dimensional examples, almost every supersymmetric Yang–Mills theory admits a holomorphic twist [2]. In this context, holomorphic refers to the fact that after taking derived invariants with respect to such a supercharge the theory only depends on the underlying complex structure of spacetime.[1]

The crux of the model I use to describe the class of theories uses the theory of factorization algebras which are perhaps more familiar in the topological situation:

$$\{\text{TFT}\} \xrightarrow{\text{Obs}} \{\text{Topological factorization algebras on } \mathbf{R}^n\} \leftrightarrow \{\mathbb{E}_n - \text{algebras}\}$$

or in the **one**-dimensional holomorphic situation:

$$\{\text{CFT}\} \xrightarrow{\text{Obs}} \{\text{Holomorphic factorization algebras on } \mathbf{C}\} \leftrightarrow \{\text{vertex algebras}\}.$$

Because we are working with the holomorphic twist, the factorization algebras described here are holomorphic in *three* directions so we can consider them on $\mathbf{C}^3$, or more generally an arbitrary complex three-fold.

[1] The notion of holomorphic twist needs to be slightly adjusted in the case of odd-dimensional spacetimes.

Infinitesimally, a translation invariant QFT is one which has a symmetry by the abelian Lie algebra of constant vector fields $\{\partial/\partial x_i\}$. A theory on $\mathbf{R}^n$ is (infinitesimally) *topological* if these vector fields act homotopically trivially. A *holomorphic* theory on complex affine space $\mathbf{C}^n$ is one for which only the anti-holomorphic translations act homotopically trivially. Globally, on a complex manifold X, a holomorphic field theory has space of fields $\Omega^{0,\bullet}(X,\mathcal{V})$ where $\mathcal{V}$ is a graded holomorphic vector bundle on X. The axiomatic approach of QFT developed in [3] is that the observables of a quantum field theory form a *factorization algebra*. The space of classical observables of such a holomorphic theory supported on an open set $U \subset X$ is the cochain complex

$$\mathrm{Obs}(U) = \mathrm{Sym}\left(\Omega^{0,\bullet}(U,\mathcal{V})^*\right).$$

Relatedly, the *local operators* at a point $p \in X$ are obtained as the limit $\mathrm{Obs}(p) = \lim_{U \ni p} \mathrm{Obs}(U)$. In dimension one, the local operators of a holomorphic factorization algebra on $\mathbf{C}$ have the structure of a vertex algebra [3]. An approach to the quantization of a holomorphic QFT can be employed at the level of factorization algebras using the Batalin–Vilkovisky (BV) formalism and has been carried out quite generally in [4].

There is a six-dimensional superconformal field theory $\chi(\mathfrak{g})$ associated to any simply laced Lie algebra $\mathfrak{g}$. In the case of $\mathfrak{g} = \mathfrak{sl}(N)$ this theory arises in M-theory as the worldvolume theory on a stack of $N > 1$ fivebranes. Six-dimensional $\mathcal{N} = (2,0)$ supersymmetry admits two types of twists: (1) Holomorphic. This leaves three directions invariant and the stabilizer is a double cover of $U(3)$. In particular, this theory can be placed on any complex three-fold equipped with a square-root of its canonical bundle. (2) Partially topological. This leaves five directions invariant and the stabilizer is $SO(4) \times U(1)$. In particular, this theory can be placed on a product manifold $M \times \Sigma$ where M is a smooth oriented four-manifold and Σ is a Riemann surface. I will pay particular attention to the first type, denoted $\chi^{hol}(\mathfrak{g})$ which has the structure of a holomorphic theory in the sense above. We remark that the partially topological twist can be obtained as a deformation of this.

The simplest case is the abelian theory associated to the Lie algebra $\mathfrak{u}(1)$—in M-theory this arises from a theory on a single fivebrane. The holomorphic twist $\chi^{hol}(\mathfrak{u}(1))$ exists on any complex three-fold X, equipped with $K_X^{1/2}$. In the BV formalism, the fields are given by [5]:

$$\Pi\Omega^{0,\bullet}(X, K_X^{1/2}) \otimes \mathbf{C}^2[1] \oplus \Omega^{2,\bullet}(X)[1] \xrightarrow{0 \oplus \partial} \Omega^{3,\bullet}(X).$$

In the ordinary BV formalism the observables are equipped with a bracket which is non-degenerate as it arises from a shifted symplectic structure on the space of fields. There is no shifted symplectic structure on the fields above; nevertheless, there is a BV bracket acting on observables. This endows the factorization algebra $\chi(\mathfrak{u}(1))$ with the structure of a shifted Poisson algebra, where we note that the bracket is degenerate (and so cannot arise from a shifted symplectic form). One outcome of this is that the theory does not admit an action functional in the usual sense.

It does however admit the following "non local" formulation as $\frac{1}{2}\int_X \alpha\overline{\partial}\partial^{-1}\alpha + \frac{1}{2}\int_X(\psi, \overline{\partial}\psi)$ where α is a closed $(2, \bullet)$ form and ψ is the fermion.

On $X = \mathbf{C}^3$ is not hard to see explicitly how the residual superconformal algebra $\mathfrak{osp}(6|2)$ acts on $\chi^{hol}(\mathfrak{u}(1))$. In fact, a much larger algebra acts after performing the holomorphic twist—the infinite-dimensional exceptional super Lie algebra $E(3|6)$ defined in [6, 7]. As a module for this exceptional algebra we can identify the local operators with a Verma-type module of the form $\mathrm{Sym}(I(0,0;1;-1)^*)$, see [7] for a definition. From this description, we compute the character of local operators as the plethystic exponential of

$$f_{\mathfrak{u}(1)}(t_1,t_2,r,q) = \frac{(r+r^{-1})q^{3/2} - (t_1 + t_1^{-1}t_2 + t_2^{-1})q^2 + q^3}{(1-t_1^{-1}q)(1-t_1t_2^{-1}q)(1-t_2q)}$$

By a general argument, one can identify this character with the partition function of the theory on the Hopf manifold $(\mathbf{C}^3 - 0)/\sim\, \simeq\; S^5 \times S^1$. From this perspective, our computation above readily agrees with results for the partition function of five-dimensional supersymmetric Yang–Mills theory on S^5.

A Maurer–Cartan element of $E(3|6)$ gives rise to a deformation of $\chi^{hol}(\mathfrak{g})$. A particular deformation localizes the theory to live on a fixed complex curve $\Sigma \subset X$ and is related to the famous equivariant localization in the context of the AGT correspondence.

Our proposal for the theory $\chi^{hol}(\mathfrak{sl}(N))$ on the holomorphic twist of a stack of $N > 1$ fivebranes is based off the formulation of *twisted holography* developed by Costello, Gaiotto, Li, and Paquette [8, 9, 10]. In [11] we have given a description of the holomorphic (minimal) twist of eleven-dimensional supergravity in terms of the exceptional super Lie algebra $E(5|10)$. There is a filtration $F^\bullet$ this Lie algebra whose Nth term we conjecture is related to the theory $\chi^{hol}(\mathfrak{sl}(N))$. The theory on a stack of $N = 2$ fivebranes then corresponds to the zeroth associated graded component of this filtration F^0/F^{-1} and is precisely the algebra $E(3|6)$ we found above. We present evidence for this description of $\chi^{hol}(\mathfrak{sl}(2))$ in the form of character computations and a careful analysis of the localization of this theory to complex dimension one—where it exactly presents the chiral part of Liouville theory.

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Differential graded modular functors and the Verlinde formula

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A topological field theory is defined as a symmetric monoidal functor from the category of cobordisms to the category of vector spaces. Especially in dimension three, topological field theories provide a powerful tool for the investigation of representation-theoretic objects, such as *modular categories*, i.e. finite ribbon categories with a non-degenerate braiding. Examples of modular categories can be obtained by taking modules over ribbon factorizable Hopf algebras (in particular quantum groups) or suitable vertex operator algebras. Given a *semisimple* modular category, one can apply the Reshetikhin-Turaev construction [1] to obtain a topological field theory. It is not possible to apply the Reshetikhin-Turaev construction directly to a non-semisimple modular category, but a construction of Lyubashenko [2] still gives us a *modular functor* in this case, i.e. a system of projective representations of mapping class groups of oriented surfaces that is compatible with gluing. The mapping class group representation assigned to a specific surface is also referred to as the *conformal block* of that surface.

Lyubashenko's modular functor assigns vector spaces to surfaces; the important homological quantities of modular categories such as the Hochschild complex or the Ext algebra of the monoidal unit do not appear. It is therefore a natural question whether there is a meaningful interaction between the homological algebra of a modular category and low-dimensional topology. Building on preparations in [3, 4, 5], this question is answered affirmatively in [6]: Any modular category $\mathcal{C}$ gives naturally rise to a *differential graded modular functor* $\mathfrak{F}_\mathcal{C}$ that assigns chain complex valued conformal blocks to oriented surfaces. These chain complexes come with projective actions of the respective mapping class groups up to coherent homotopy. The gluing of differential graded conformal blocks is implemented via homotopy coends over projective objects of $\mathcal{C}$. The differential graded conformal block for the sphere is the (dual of the) Ext algebra of the monoidal unit of $\mathcal{C}$; for the torus, it is the Hochschild complex of $\mathcal{C}$. In zeroth homology, Lyubashenko's construction is recovered.

The differential graded modular functor of a non-semisimple modular category $\mathcal{C}$ provides a topological access to the homological algebra of $\mathcal{C}$. Nonetheless, it is really unclear whether the key features of modular categories carry over from the linear to the differential graded setting. One of the most important results for semisimple modular categories is the *Verlinde formula* [7], a statement on the conformal block for the torus equipped with the algebra structure induced by the monoidal product (this is called the *Verlinde algebra*). The Verlinde formula, when phrased topologically, says that the algebra structure coming from the monoidal product is transformed into a diagonal multiplication through the action of an element $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ in the mapping class group of the torus. This is often summarized in the slogan that the 'S-matrix diagonalizes the fusion'. In [8] we prove a homotopy coherent version of the Verlinde formula for differential graded modular functors: To this end, one first needs a differential graded Verlinde algebra, i.e. an algebra structure on the differential graded conformal block of the torus. This algebra structure should be induced by the monoidal product. The algebras that we construct are E_2-algebras, i.e. homotopy commutative differential graded algebras whose commutativity behavior is controlled by braid groups; their homology is a Gerstenhaber algebra. In fact, it is easy to see that the torus conformal block, i.e. the Hochschild complex of $\mathcal{C}$, comes with the structure of a non-unital E_2-algebra induced by the monoidal product [4]. For the differential graded Verlinde formula, it turns out that the differential graded conformal block for the torus needs to be treated in tandem with its dual, the Hochschild *cochain* complex of $\mathcal{C}$. It also comes with the structure of an E_2-algebra induced by the monoidal product (compared to the dual result, this is much harder). This tells us that on the differential graded conformal block for the torus and its dual, one has indeed a multiplicative structure that can be interpreted as a differential graded analogue of the Verlinde algebra. Now we can state a Verlinde formula, the main result of [8]: Through the action of the mapping class group element S from above (we can talk about the action of this mapping class group element because we have the differential graded modular functor), the Verlinde algebra is transformed into a different E_2-algebra, namely *Deligne's E_2-structure*, the famous E_2-structure that lifts the standard Gerstenhaber bracket on Hochschild (co)homology. Since we need here Deligne's E_2-structure on the Hochschild chain *and* cochain complex simultaneously, we need the *cyclic* version of Deligne's E_2-structure that additionally requires a Calabi-Yau structure on the tensor ideal of projective objects of $\mathcal{C}$. This Calabi-Yau structure turns out to be the one induced by the modified trace, an important non-semisimple replacement of the quantum trace [9, 10].

In the semisimple case, the differential graded Verlinde formula collapses to degree zero and recovers the 'usual' Verlinde formula. In the cochain version of the non-semisimple case, the differential graded Verlinde formula recovers in zeroth cohomology a proposal for a non-semisimple Verlinde formula by Gainutdinov and Runkel [11]. There is however 'higher information' beyond degree zero; the differential graded Verlinde algebra generally has non-zero Gerstenhaber brackets. In contrast to that, the chain version of the differential graded Verlinde algebra has a non-trivial product only in degree zero. The Verlinde formula for this product in degree zero amounts to a *block diagonalization* of the fusion product. Interestingly, the quantum dimensions of simple objects that appear in the semisimple version of the Verlinde formula (they are part of the diagonal product that the fusion product in transformed into) are replaced with the modified dimensions coming from the modified trace. This is not built in, but really a consequence of the differential graded Verlinde formula.

Key ingredients in the proof of the differential graded Verlinde formula are

- a new description of Deligne's E_2-structure on the Hochschild cochain complex of a finite tensor category using the homotopy theory of braided operads and the canonical end of a finite tensor category [12],
- and the *trace field theory* of a finite tensor category, an open-closed topological conformal field theory that helps us relate modified traces and homological algebra [13].

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