Mathematisches Forschungsinstitut Oberwolfach

Report No. 17/2022

DOI: 10.4171/OWR/2022/17

Toric Geometry

Organized by
Jürgen Hausen, Tübingen
Milena Hering, Edinburgh
Nathan Ilten, Burnaby
Diane Maclagan, Coventry

27 March – 2 April 2022

Abstract. Toric geometry is a vibrant subfield of algebraic geometry that draws on strong connections to combinatorics. The 2022 workshop brought together a broad group of mathematicians both in-person and virtually to discuss aspects of the field, ranging from K-stability to machine learning.

Mathematics Subject Classification (2020): 14M25, 14B12, 14D06, 14L30, 52B20, 13F99, 14E15.

Introduction by the Organizers

The workshop *Toric Geometry* had 34 in-person participants, and 21 further virtual participants. Toric geometry is the rich interplay between algebraic and convex geometry relating varieties with almost transitive torus action to combinatorial and polyhedral data. The field continues to develop, with lots of recent activity connecting toric geometry to deformation theory, K-stability, birational geometry, resolution of singularities, among other topics.

A major theme of the workshop was degenerations to, and deformations of, toric varieties, including relations to K-stability. Petracci discussed the use of deformation theory of toric varieties to study the local geometry of the moduli space of K-stable Fano varieties. Blum reported on recent work that constructs a canonical two-step degeneration of any klt Fano variety to a klt Fano admitting a Kähler-Ricci soliton. Filip presented work-in-progress on the connection between mutations of Laurent polynomials and deformations of toric Gorenstein 3-folds with non-isolated singularities. Mutations of polytopes returned in the talk of Mohammadi, who described new techniques to obtain toric degenerations of Grassmannians. A major source of toric degenerations in recent years has come from finitely generated semigroups arising in Newton-Okounkov theory. Haase presented work showing the subtleties of this approach, giving a characterization of when these semigroups are finitely generated for non-toric flags on a toric variety.

Another theme was toric methods in resolution of singularities. Temkin gave an overview of the recent much improved algorithm, joint with Abramovich and Włodarczyk, for functorial resolution of singularities. In a second, example-oriented talk, Włodarczyk presented further improvements that exploit torus actions. Satriano proposed a framework for computing stringy Hodge numbers of singular varieties using resolutions via Artin stacks, and verified this framework for toric varieties.

The study of birational geometry and singularities played a prominent role in the talks of Batyrev, Wrobel, Süß, and Laface. Batyrev discussed how to construct minimal models of nondegenerate hypersurfaces in toric varieties via the Fine interior of Newton polytopes. Wrobel considered complete intersections in toric varieties, describing the various types of singularities of the minimal model program in terms of the anticanonical complex, and giving applications to classification of certain terminal Fano 3-folds. Süß bounded the possibilities for toric singularities via the normalized volume. Laface presented the recent breakthrough, joint with Castravet, Tevelev, and Ugaglia, on exhibiting toric surfaces whose blow-ups have non-polyhedral effective cones. This shows that $\overline{M}_{0,n}$ has non-polyhedral effective cone for $n \geq 10$.

Higher-complexity torus actions continue to play a prominent role in the subject. Using the geometry of $\mathbb{C}^*$-actions, Romano presented a detailed study of certain Picard-rank three Mori Dream Spaces. Monin described the cohomology ring of fiber bundles whose fibers are toric varieties. Kaveh extended the theory of toric vector bundles to equivariant vector bundles on toric schemes.

Other speakers developed core tools in toric geometry and made connections to other fields. Altmann continued the Oberwolfach tradition of progress on understanding the derived category of toric varieties by giving a description of all full exceptional collections of line bundles on Picard-rank two toric varieties. Bruce described a connection between multigraded regularity on products of projective spaces and quasilinear resolutions, generalizing the classical case. Machine learning starred in the talk of Hofscheier, who discussed his experiments with lattice polytopes. Sottile closed the conference by challenging us to return to underappreciated connections of toric geometry to mathematical physics.

On Tuesday evening there was lively session of five minute talks by both junior and senior participants. Speakers were:

(1) Jarek Wisniewski *Birational maps, $\mathbb{C}^*$-actions and Mori Dream Spaces*
(2) Oliver Clarke *Toric Degenerations via combinatorial mutations*
(3) Leonid Monin *Multi-polytopes and quasitoric manifolds*
(4) Julius Giesler *The plurigenera of minimal models of toric hypersurfaces*
(5) Juliette Bruce *Top weight cohomology of A_g*
(6) Marie Brandenburg *A problem from economics and lattice polytopes*
(7) Benjamin Nill *Thin polytopes*

(8) Andreas Bäuerle *Classification of some Fano 3-folds with torus action*
(9) Marti Salat Molti *Monomial ideals via Klyachko filtrations*
(10) Johannes Hofscheier *Generalised flatness constants*
(11) Fatemeh Mohammadi *Double Schubert polynomials*
(12) Kiumars Kaveh *An extension of Brianchon-Gram theorem and toric varieties*

In-person participants were happy to meet again for intense mathematical discussions after the pandemic pause, and many informal connections were created. We are grateful to Oberwolfach for hosting this workshop, and providing such excellent working conditions.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows".

Workshop: Toric Geometry

Table of Contents

Abstracts

The structure of exceptional sequences on toric varieties of Picard rank two

Klaus Altmann
(joint work with Frederik Witt)

We report on results obtained jointly with Frederik Witt, cf. [AW21].

1. Picard rank two

Smooth toric varieties X of Picard rank two provide the most basic class of projective toric varieties beyond projective spaces. According to Kleinschmidt's classification [Kle88] they are completely determined by integers $\ell_1,\ \ell_2 \geq 2$ and $0 = c^1 \geq c^2 \geq \ldots \geq c^{\ell_2}$ arising in the class map

$$\pi := \left(\begin{array}{ccc|ccc} 1 & \ldots & 1 & c^1 & \ldots & c^{\ell_2} \\ 0 & \ldots & 0 & 1 & \ldots & 1 \end{array} \right) : \mathbb{Z}^{\ell_1+\ell_2} = \mathrm{Div}_T(X) \to \mathrm{Pic}(X) = \mathbb{Z}^2.$$

The usual toric lattices M and N occur as $M = \ker \pi$ and $N = \operatorname{coker} \pi^*$. In other words, N is the quotient by the two rows of π, which leads to vectors $u^1, \ldots, u^{\ell_1}$ and $v^1, \ldots, v^{\ell_2} \in N$ as the images of the unit vectors of $\mathbb{Z}^{\ell_1+\ell_2}$. The fan $\Sigma \subseteq N_{\mathbb{R}}$ for X arises from the maximal cones $\sigma_{i,j}$ being generated by all rays except u^i and v^j. The classes of the divisors associated to the rays u^1 and v^1 generate the nef cone and identify the Picard lattice with $\mathbb{Z}^2$. The dimension of X is $d = \ell_1 + \ell_2 - 2$.

Actually, beyond ℓ_1, ℓ_2, only the invariants

$$\alpha := -c^{\ell_2} \geq 0 \quad \text{and} \quad \beta := -\textstyle\sum_i c^i \geq \alpha$$

will be important for us. The case $\alpha = \beta = 0$ means $X = \mathbb{P}^{\ell_1-1} \times \mathbb{P}^{\ell_2-1}$ and will be refered to as the *product case*; in the remaining *twisted case* we merely have a fibration $X \to \mathbb{P}^{\ell_1-1}$ with fiber $\mathbb{P}^{\ell_2-1}$.

2. Exceptional sequences

Assume that X is a smooth, projective variety with $\mathrm{H}^{\geq 1}(X, \mathcal{O}_X) = 0$. This applies, in particular, to all toric varieties.

Definition 1. A sequence $s = (\mathcal{L}^1, \ldots, \mathcal{L}^N)$ of line bundles on X is called *exceptional* if $\mathrm{Ext}^\bullet(\mathcal{L}_j, \mathcal{L}_i) = 0$ whenever $i < j$. Moreover, it is called *full* (abreviated by FES), if $\mathcal{L}^1, \ldots, \mathcal{L}^N$ generate the entire derived category $\mathcal{D}(X)$.

If, instead of line bundles, we allow arbitrary elements of $\mathcal{D}(X)$ it was shown in [Kaw06, Kaw13, Kaw16] that for toric varieties X FESs always exist. For the stronger version of the previous definition, however, this fails even for toric Fano varieties, cf. [Efi14].

On the other hand, for toric varieties of Picard rank two FESs of line bundles always exist. This is a well-known fact and follows, for instance, from the projective bundle structure. Indeed, one lifts a FES from the base $\mathbb{P}^{\ell_1-1}$ and defines a sequence from the union of several twists of this lifted FES. We shall refer to this sequence as being of Orlov type.

Hence, the goal of the present note is not to discuss existence questions but to shed light on the structure of *all* possible FESs.

3. The immaculate locus

What makes this goal accessible for the Picard rank two case is the explicit knowledge of the *immaculate locus*

$$I(X) := \{[\mathcal{L}] \in \operatorname{Pic} X \mid \mathrm{H}^\bullet(X, \mathcal{L}^{-1}) = 0\} \subset \operatorname{Pic} X = \mathbb{Z}^2.$$

By [ABKW20], it consists of the interior lattice points of the horizontal strip

$$\mathcal{H} = [0 \leq y \leq \ell_2] \subset \mathbb{R}^2$$

and of the parallelogram

$$\mathcal{P} = \{(x, y) \mid -\beta \leq x \leq \ell_1,\ 0 \leq \langle (1, \alpha), (x, y) \rangle \leq \ell_1 + \alpha\ell_2 - \beta\} \subset \mathbb{R}^2$$

as sketched in Figure 1. Now, setting $s^i := [\mathcal{L}^i]$ the sequence $s = (s^1, \ldots, s^N)$ is

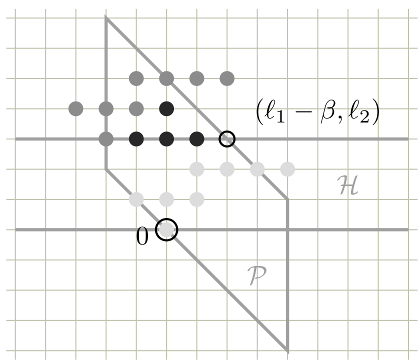


Figure 1. The sets R (dark), B (medium), and G (light) for $\ell_1 = 4$, $\ell_2 = 3$, $\alpha = 1$, and $\beta = 2$.

contained in $\mathbb{Z}^2$ and the condition for exceptionality turns into $s^j - s^i \in I(X)$. In particular, it becomes purely combinatorial. On algebro-geometric grounds $|s| = N$ is bounded by $|\Sigma(d)| = \ell_1\ell_2$.

4. The results

We call an exceptional sequence s *maximal* if $|s| = \ell_1\ell_2$. Using this terminology, we obtained the following results:

4.1. **Fullness.** For the present special case of toric varieties with Picard rank two, we can confirm Kuznetsov's conjecture that if there exists a FES at all, then every exceptional sequence of maximal length is full.

4.2. **Spatial constraints.** In the twisted case, FESs have a height of at most $2\ell_2$. In the product case we can guarantee this for either the height or the width. Interestingly (and almost strangely) enough, this fails if the exceptional sequence is not of maximal length. On $(\mathbb{P}^1)^3$ (which of course is of Picard rank three), this even fails for FES: Indeed, we present examples in [AA] spreading arbirarily far into all three directions.

4.3. **Lexicographic ordering.** In the twisted case, imposing a vertical lexicographic order preserves exceptionality. In the product case either the vertical or the horizontal order works. Again, this fails for non-maximal exceptional sequences or for FES on $(\mathbb{P}^1)^3$.

4.4. **Classification.** We produce FESs of height $\leq 2\ell_2$ as follows:
(i) Start with some so-called admissible set $R \subset \mathcal{P} \cap \mathbb{Z}^2$ sitting in the strip $\mathcal{H}' := [\ell_2 \leq y \leq 2\ell_2 - 1]$. Admissibility is explained in [AW21] – it does mainly say that R narrows with height, cf. Figure 1.
(ii) Inside $\mathcal{H}'$ supplement R with B such that at every height h the sets $B_h := B \cap [y = h]$ and $R_h := R \cap [y = h]$ arranged in this order consist of ℓ_1 consecutive points.
(iii) Define $G := B - (-\beta, \ell_2)$. Then $s := R \cup G$ together with lexicographic order defines a FES. Furthermore, any FES arises this way up to shifts and reordering, cf. Figure 1. Finally, the FESs of Orlov type correspond to the case of $R = \emptyset$.

4.5. **Exceptional posets.** By definition, FESs come with a particular total order. We can switch our viewpoint though and consider subsets $s \subset \mathbb{Z}^2$ of size $\ell_1\ell_2$ with a so-called exceptional partial order $P(s)$ such that a total ordering of s is exceptional if and only if it refines $P(s)$. This poset $P(s)$ can be explicitely determined out of the data in (4.4). It always contains $\leq_{\text{eff}}$ defined by $A \leq_{\text{eff}} B$ if and only if $B - A$ lies in the effective cone. To completely determine $P(s)$, however, we also need to take into account the relative positions of those layers B_h with $R_h = \emptyset$.

4.6. **Strongly exceptional sequences.** In connection with tilting bundles one looks for exceptional sequences which are strong, that is, they also satisfy

$$\operatorname{Ext}^{\geq 1}(s^i, s^j) = 0$$

for $0 \leq i,\ j \leq n$. It turns out that s is strong if and only if $P(s)$ equals the smallest possible choice, i.e., $P(s) = \leq_{\text{eff}}$.

References

[AA] Klaus Altmann and Martin Altmann. Exceptional sequences of 8 line bundles on $(\mathbb{P}^1)^3$. J. Algebr. Comb., 2022, DOI 10.1007/s10801-021-01112-z.

[ABKW20] Klaus Altmann, Jarosław Buczyński, Lars Kastner, and Anna-Lena Winz. Immaculate line bundles on toric varieties. *Pure Appl. Math. Q.*, 16(4):1147–1217, 2020.

[AW21] Klaus Altmann and Frederik Witt. The structure of exceptional sequences on toric varieties of Picard rank two. arXiv:2112.14637 [math.AG], 2021.

[Efi14] Alexander I. Efimov. Maximal lengths of exceptional collections of line bundles. *J. Lond. Math. Soc., II. Ser.*, 90(2):350–372, 2014.

[Kaw06] Yuriro Kawamata. Derived categories of toric varieties. *Mich. Math. J.*, 54(3):517–535, 2006.
[Kaw13] Yujiro Kawamata. Derived categories of toric varieties. II. *Mich. Math. J.*, 62(2):353–363, 2013.
[Kaw16] Yujiro Kawamata. Derived categories of toric varieties. III. *Eur. J. Math.*, 2(1):196–207, 2016.
[Kle88] Peter Kleinschmidt. A classification of toric varieties with few generators. *Aequationes Math.*, 35(2-3):254–266, 1988.

Multigraded Castelnuovo–Mumford Regularity on Products of Projective Space

Juliette Bruce
(joint work with Lauren Cranton Heller, Mahrud Sayrafi)

1. Castelnuovo–Mumford regularity on Projective Spaces

Before discussing multigraded Castelnuovo–Mumford regularity on products of projective spaces and our new work, we begin by briefly recalling the standard graded story of Castelnuovo–Mumford regularity on a single projective space. Introduced by Mumford in the mid-1960's Castelnuovo–Mumford regularity is defined in terms of cohomological vanishing.

Definition 1.1. A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is d-regular if and only if:

$$H^i(\mathbb{P}^n, \mathcal{F}(d-i)) = 0 \qquad \text{for all } i > 0.$$

The Castelnuovo–Mumford regularity of $\mathcal{F}$ is then

$$\operatorname{reg}(\mathcal{F}) \coloneqq \min\left\{d \in \mathbb{Z} \;\middle|\; \mathcal{F} \text{ is } d\text{-regular}\right\}.$$

Roughly speaking one should think about Castelnuovo–Mumford regularity as being a measure of geometric complexity. Mumford was interested in such a measure as it plays a key role in constructing Hilbert and Quot schemes. In particular, being d-regular implies that $\mathcal{F}(d)$ is globally generated. However, in the 1980's Eisenbud and Goto showed that being d-regular was also closely connected to interesting homological properties.

Theorem 1.2. [4] *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$ and $M = \bigoplus_{e\in\mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(e))$ the corresponding section ring. The following are equivalent:*

- *M is d-regular;*
- *$\beta_{i,j}(M) \coloneqq \dim_{\mathbb{K}} \operatorname{Tor}_i(M, \mathbb{K})_j = 0$ for all $i \geq 0$ and $j > d + i$;*
- *$M_{\geq d}$ has a linear resolution.*

The goal of our work is to try and understand how this theorem may be generalized to the multigraded setting, i.e. from coherent sheaves on a single projective space to sheaves on a product of projective spaces.

2. Multigraded Setting: Products of Projective Spaces

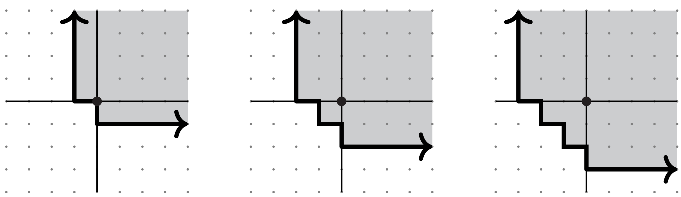
Shifting to the multgraded setting, we fix a dimension vector $\mathbf{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ and let $\mathbb{P}^{\mathbf{n}} := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$. We then let $S = \mathbb{K}[x_{i,j} \mid 1 \leq i \leq r, 0 \leq j \leq n_i]$ be the Cox ring of $\mathbb{P}^{\mathbf{n}}$ with the $\mathrm{Pic}(X) \cong \mathbb{Z}^r$-grading given by $\deg x_{i,j} = \mathbf{e}_i \in \mathbb{Z}^r$, where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbb{Z}^r$.

Maclagan and Smith generalized Castelnuovo–Mumford regularity to this setting in terms of certain cohomology vanishing. Before we can state their definition of multigraded regularity we need to fix some useful notation to described the regions in which we will require cohomology to vanish.

Notation 2.1. Given $\mathbf{d} \in \mathbb{Z}^r$ and $i \in \mathbb{Z}_{\geq 0}$ we let:

$$L_i(\mathbf{d}) := \bigcup_{\substack{\mathbf{v} \in \mathbb{N} \\ |\mathbf{v}|=i}} (\mathbf{d} - \mathbf{v}) + \mathbb{N}^r.$$

In order to get a sense for what these regions look like note when $r = 2$ the region $L_i(\mathbf{d})$ looks like a staircase with $(i + 1)$-corners. Below we've plotted the regions $L_1(0, 0)$, $L_2(0, 0)$, and $L_3(0, 0)$. Roughly speaking we are going to define regularity by require H^i to vanish on L_i.



With this notation in hand we recall the notion of multigraded Castelnuovo–Mumford regularity as introduced by Maclagan and Smith.

Definition 2.2. [5, Definition 6.1] A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{\mathbf{n}}$ is $\mathbf{d}$-regular if and only if

$$H^i(\mathbb{P}^{\mathbf{n}}, \mathcal{F}(\mathbf{e})) = 0 \qquad \text{for all } \mathbf{e} \in L_i(\mathbf{d}).$$

The multigraded Castelnuovo–Mumford regularity of $\mathcal{F}$ is then the set:

$$\mathrm{reg}(\mathcal{F}) := \{\mathbf{d} \in \mathbb{Z}^r \mid \mathcal{F} \text{ is } \mathbf{d}\text{-regular}\} \subset \mathbb{Z}^r.$$

Even for relatively simple examples the multigraded Castelnuovo–Mumford regularity does not necessarily have a unique minimal element (see Figure 1). That said $\mathrm{reg}(\mathcal{F})$ does have the structure of a module over the semi-group $\mathrm{Nef}(\mathbb{P}^{\mathbf{n}}) \cong \mathbb{N}^r$, i.e. if $\mathbf{d} \in \mathrm{reg}(\mathcal{F})$ then $\mathbf{d} + \mathbf{e} \in \mathrm{reg}(\mathcal{F})$ for all $\mathbf{e} \in \mathbb{N}^r$.

The obvious approaches to generalize Theorem 1.2 to a product of projective spaces turn out not to work. For example, the multigraded Betti numbers do not determine multigraded Castelnuovo–Mumford regularity [2, Example 5.1] With this in mind we focus on generalizing part (3) of Theorem 1.2.

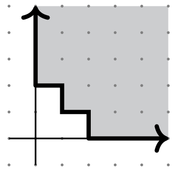


Figure 1. The multigraded Castelnuovo–Mumford regularity of $\mathcal{O}_X$ where $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ is the subscheme consisting of three distinct points $([1:1],[1:4])$, $([1:2],[1:5])$, and $([1:3],[1:6])$.

Definition 2.3. [2] Let $F_\bullet$ be a complex of $\mathbb{Z}^r$-graded free S-modules.

(1) We say that $F_\bullet$ is $\mathbf{d}$-linear if and only if F_0 is generated in degree $\mathbf{d}$ and each twist of F_i is contained in $L_i(\mathbf{d})$.
(2) We say that $F_\bullet$ is $\mathbf{d}$-quasilinear if and only if F_0 is generated in degree $\mathbf{d}$ and each twist of F_i is contained in $L_{i-1}(\mathbf{d}-\mathbf{1})$.

In order to see the difference between linear and quasilinear resolutions we note that on a product of projective spaces the irrelevant ideal generally will have a quasilinear resolution, not a linear resolution. For example, if we consider $\mathbb{P}^1 \times \mathbb{P}^2$ so that $S = \mathbb{K}[x_0, x_1, y_0, y_1, y_2]$ and $B = \langle x_0, x_1 \rangle \cap \langle y_0, y_1, y_2 \rangle$ then the minimal graded free resolution of S/B is:

$$S \longleftarrow S(-1,-1)^6 \longleftarrow \begin{matrix} S(-1,-2)^6 \\ \oplus \\ S(-2,-1)^3 \end{matrix} \longleftarrow \begin{matrix} S(-1,-3)^2 \\ \oplus \\ S(-2,-2)^3 \end{matrix} \longleftarrow S(-2,-3) \longleftarrow 0.$$

In particular, we see that the minimal graded free resolution S/B is not $(0,0)$-linear since $(-1,-1) \notin L_1(0,0)$, however, it is $(0,0)$-quasilinear.

It is not the case that M being $\mathbf{d}$-regular implies $M_{\geq \mathbf{d}}$ has a linear resolution [2, Example 4.2], however, we can characterize being $\mathbf{d}$-regular in terms of $M_{\geq \mathbf{d}}$ having a quasilinear resolution.

Theorem 2.4. [2, Theorem A] *Let M be a finitely generated $\mathbb{Z}^r$-graded S-module with $H^0_B(M) = 0$ then:*

$$M \textit{ is } \mathbf{d}\textit{-regular} \iff M_{\geq \mathbf{d}} \textit{ has a } \mathbf{d}\textit{-quasilinear resolution}$$

We briefly sketching the proof of the above theorem:

(1) Using a Fourier-Mukai argument we construct a complex $G_\bullet$ of free $\mathbb{Z}^r$-graded S-modules whose multigraded Betti numbers are given (in some range) as follows:

$$\beta_{i,\mathbf{a}}(G_\bullet) = \dim H^{|\mathbf{a}|-i}\left(\mathbb{P}^{\mathbf{n}}, \tilde{M} \otimes \Omega^{\mathbf{a}}_{\mathbb{P}^{\mathbf{n}}}(\mathbf{a})\right).$$

(2) Making use of a spectral sequence argument we show that even though $G_\bullet$ is not a priori a resolution of $M_{\geq \mathbf{d}}$ we have that:
$$\beta_{i,\mathbf{a}}\left(M_{\geq \mathbf{d}}\right) = \beta_{i,\mathbf{a}}\left(G_\bullet\right).$$
(3) Finally, we characterize M being $\mathbf{d}$-regular in terms of the vanishing of the cohomology in (1) above.

Note the complex $G_\bullet$ constructed in part (1) of the proof sketch above is a priori not a resolution of $M_{\geq \mathbf{d}}$, but instead is a virtual resolution of M [1]. That said as noted above it does have the same Betti numbers as $M_{\geq \mathbf{d}}$, and in all the examples we have done it turns out to be a resolution.

Conjecture 2.5. [2, Conjecture 6.7] *The complex $G_\bullet$ is the minimal free resolution of $M_{\geq \mathbf{d}}$.*

3. Further Questions

Since computing the minimal graded free resolution of $M_{\geq \mathbf{d}}$ can be effectively done via Gröbner basis methods, Theorem 2.4 provides an efficient algorithm for checking whether a module is $\mathbf{d}$-regular for a particular $\mathbf{d} \in \mathbb{Z}^r$. It would be interesting to know whether such an algorithm could be extended to computing all of the minimal elements of $\operatorname{reg}(M)$.

Question 3.1. Is there an effective algorithm for computing the multigraded Castelnuovo–Mumford regularity of a coherent sheaf or module on $\mathbb{P}^{\mathbf{n}}$?

This is equivalent to finding a finite box in $\mathbb{Z}^r$ that contains all of the minimal elements of $\operatorname{reg}(M)$. If such a finite box does exist, it is very special to the case of a product of projective spaces.

In particular, one may consider multigraded Castelnuovo–Mumford regularity of sheaves and modules on other toric varieties [5]. It turns out that there are examples of finitely generated modules on Hirzebruch surfaces whose multigraded Castelnuovo–Mumford regularity does not lie in finite a box [3]. This naturally leads one to ask what assumptions one needs to avoid such potential issues.

Question 3.2. Let X be a smooth projective toric variety, and M a finitely generated $\operatorname{Pic}(X)$-graded $\operatorname{Cox}(X)$-module. Under what assumptions is $\operatorname{reg}(M)$ finitely generated as a module over the semi-group $\operatorname{Nef}(X)$?

References

[1] Christine Berkesch, Daniel Erman, and Gregory G. Smith, *Virtual resolutions for a product of projective spaces*, Algebr. Geom. **7** (2020), no. 4, 460–481.

[2] Juliette Bruce, Lauren Cranton Heller, and Mahrud Sayrafi, *Characterizing Multigraded Regularity on Products of Projective Spaces* (2021). ArXiv pre-print: `https://arxiv.org/abs/2110.10705`.

[3] ———, *Bounds on the Multigraded Regularity of Modules and Powers of Ideals* (2022). To appear.

[4] David Eisenbud and Shiro Goto, *Linear free resolutions and minimal multiplicity*, Journal of Algebra **88** (1984), no. 1, 89–133.

[5] Diane Maclagan and Gregory G. Smith, *Multigraded Castelnuovo–Mumford regularity*, J. Reine Angew. Math. **571** (2004), 179–212.

Resolution of singularities in characteristic zero: new tools and methods

Michael Temkin
(joint work with Dan Abramovich, Jarosław Włodarczyk)

1. Intoduction

In a series of joint projects with Abramovich and Włodarczyk, we discovered logarithmic and stack-theoretic versions of the classical embedded desingularization algorithm. These methods are simpler, faster, possess better functorial properties and apply to morphisms too. They are based on extending the pool of geometric objects one works with by log schemes and stacks. As a result more basic birational operations are available for the new algorithms, including various versions of log blow ups and weighted blow ups.

1.1. **Classical desingularization.** All canonical (or functorial) resolution methods known so far are *embedded*: to desingularize a space X (variety, scheme, analytic space, etc.) of characteristic zero one (locally) embeds it into a smooth or regular space M, that will be called an *ambient manifold*, and principalizes the ideal $\mathcal{I}_X \subset \mathcal{O}_M$ of X by finding a sequence of *smooth blow ups* $M_n \to \cdots \to M_0 = M$ such that $I_n = I\mathcal{O}_{M_n}$ is invertible and supported on the exceptional divisor. In the classical situation smooth blow ups are blow ups with smooth centers. This guarantees that each M_i is a manifold and the last non-empty strict transform $X_l \hookrightarrow M_l$ of X is a component of the l-th center, and hence is a resolution of X. Usage of more general centers is not possible because they usually produce non-smooth modifications of M and, excluding the case of smooth blow ups, there is no simple criterion which guarantees that the blown up space is again smooth.

The recent progress is related to the discovery that one can realize analogous principalization algorithms in wider settings of log smooth log schemes or their morphisms, smooth stacks or log smooth log stacks. In this case, the pool of basic admissible blow ups increases, and one can use this to construct better algorithms. To simplify notation we will only discuss the case of varieties over a field k of characteristic zero, though everything works when the schemes have enough derivations in an appropriate sense.

2. Log methods

2.1. **Relative log manifolds.** Relative principalization takes place in the following situation: $M \to B$ is an exact log smooth morphism of DM log stacks over k. We call such a morphism a B-manifold. In our paper [2] we only consider the case, when B is log smooth over k (or log regular). So, this is also assumed in the sequel. Étale locally a B-manifold can be modelled on charts of the form

$\mathrm{Spec}(A_P[Q][t_1,\ldots,t_n]) \to \mathrm{Spec}(A)$, where P and Q give rise to the log structures and $A_P[Q] = A \otimes_{k[P]} k[Q]$. Any B-submanifold can be modelled by $V(t_1,\ldots,t_r)$ in an appropriate chart.

2.2. **Kummer blow ups.** By a Kummer submonomial center we mean a Kummer monomial ideal on a submanifold. Étale locally it can be described as the vanishing locus of $\mathcal{J} = (t_1,\ldots,t_r, m_1^{1/d},\ldots,m_s^{1/d})$ for appropriate coordinates $t_1,\ldots,t_n$ and monomials $m_1,\ldots,m_s$. One can think about this as a weighted generalization of ideal with weights $(1,\ldots,1,d,\ldots,d)$. Formally, this is an ideal in the Kummer étale topology of M, that is, an ideal $\mathcal{J} \subset \mathcal{O}_{M_{ket}}$. We introduce the *Kummer blow up* $M' = [\mathrm{Bl}_{\mathcal{J}}(M)] \to M$ along such a center. Its source is a stack which can be described as follows: find a Kummer G-cover $\overline{M} \to M$ such that $\mathcal{J}\mathcal{O}_{\overline{M}_{ket}}$ is a usual ideal on $\overline{M}$, blow up $\mathcal{J}\mathcal{O}_{\overline{M}_{ket}}$ and divide the obtained log scheme by G stack-theoretically. Then M' is a partial coarsening of the quotient (we will describe more general weighted blow ups later). It turns out that M' is again a B-manifold and in this setting one can extend the classical principalization algorithm to morphisms and log schemes.

2.3. **Main results.** Our main principalization result is as follows:

Theorem 2.3.1 ([2]). *There exists a relative principalization method $\mathcal{P}$ which obtains as an input a relative log manifold $f\colon M \to B$ with a log smooth B and an ideal $\mathcal{I} \subset \mathcal{O}_M$, and outputs either the empty set (fails) or a sequence of Kummer blow ups $M_n \to \cdots \to M_0 = M$ such that the ideal $\mathcal{J}_n = \mathcal{J}\mathcal{O}_{M_n}$ is monomial. This $\mathcal{P}$ satisfies the following properties:*

(i) Non-failure up to refining the base: *for any input there exists a modification $B' \to B$ such that $\mathcal{P}$ does not fail for $X' = X \times_B B' \to B'$ and $\mathcal{J}' = \mathcal{J}\mathcal{O}_{X'}$.*

(ii) Log smooth functoriality: *if $\mathcal{P}(f,\mathcal{J})$ is non-empty and $g\colon M' \to M$ is log smooth, then $\mathcal{P}(f \circ g, g^{-1}\mathcal{J}) = \mathcal{P}(f,\mathcal{J}) \times_M M'$.*

(iii) Base change functoriality: *if $\mathcal{P}(f,\mathcal{J})$ is non-empty, then $\mathcal{P}(f',\mathcal{J}\mathcal{O}_{M'}) = \mathcal{P}(f,\mathcal{J}) \times_M M'$. for any base change $g\colon B' \to B$ with a log smooth B' and the base change $f'\colon M' = M \times_B B' \to B'$.*

As in the classical case, principalization implies desingularization:

Theorem 2.3.2 ([2]). *There is a method $\mathcal{F}$ which assigns to a dominant morphism $f\colon X \to B$ of integral log varieties (or log DM stacks) over k with a log smooth B either a non-representable modification $X_{\mathrm{res}} \to X$ or a "fail output" $X_{\mathrm{res}} = \emptyset$ such that $X_{\mathrm{res}} \to B$ is log smooth and*

(i) Non-failure up to refining the base: *for any f there exists a modification $B' \to B$ with a log smooth B' such that $(X \times_B B')_{\mathrm{res}}$ is non-empty.*

(ii) Log smooth functoriality: *if X_{res} is non-empty and $X' \to X$ is log smooth, then $X'_{\mathrm{res}} = X_{\mathrm{res}} \times_X X'$.*

(iii) Base change functoriality: *if $X_{\mathrm{res}} \neq \emptyset$, then $(X \times_B B')_{\mathrm{res}} = X_{\mathrm{res}} \times_B B'$ for any base change $B' \to B$ with a log smooth source.*

Remark 2.3.3. (i) This theorem is the first canonical semistable reduction for any dimension of X and B (a non-canonical version was obtained by Abramovich-Karu). Even when B is a trait this provides the first known version of semistable reduction compatible with ramified extensions of the base. Log smooth functoriality is new even when B is the spectrum of a field (see [1]).

(ii) The choice of the modification $B' \to B$ in the non-failure claims of the above results can also be done canonically (and even smooth-functorially), but this is still a work in progress.

(iii) The algorithm outputs a stack-theoretic modification, but one can then find a further modification which is representable over the initial M or X. This is achieved by a destackification operation, which is smooth-functorial, but not log smooth-functorial.

3. The dream algorithm

Using stacks and certain weighted blow ups seems inevitable already for the sake of constructing log smooth-functorial methods. The weights used in these case are of a special form and the centers can be interpreted as relatively classical object – Kummer ideals. However, it is very natural as a next step to try to use arbitrary centers like $(x_1^{d_1}, \dots, x_n^{d_n})$, especially, because it was known for decades that such weighted blow ups provide the fastest way to resolve quasi-homogeneous singularities. For example, the minimal resolution of the elliptic singularity $X = V(x^2 + y^3 + z^6) \subset M = \mathbf{A}_k^3$ is obtained by the weighted blow up along (x^2, y^3, z^6). To make this idea a working tool for resolution we should formalize the notion of weighted centers and define corresponding blow ups.

3.1. ***h*-ideals.** We introduce generalized ideals of the form $\mathcal{I} = (t_1^{1/w_1}, \dots, t_n^{1/w_n})^l$ as ideals for the h-topology, or simply ideals on fine enough alterations of Y (an alternative approach is to define them using the Zariski-Riemann space of X). This idea resembles Kummer ideals, but there is one difference which is not essential for our applications: different ideals can become equivalent after h-localization, in particular, $(t_1^{1/w_1}, \dots, t_n^{1/w_n})^l = (t_1^{l/w_1}, \dots, t_n^{l/w_n})$ as h-ideals. In addition, any ideal on Y is invertible as an h-ideal since it becomes invertible on an appropriate modification.

3.2. **Weighted blow ups.** Given a weighted ideal $\mathcal{I} = (t_1^{1/w_1}, \dots, t_n^{1/w_n})$ on M we consider the associated Rees algebra $R_\mathcal{I} = \oplus_l \mathcal{I}^{[l]}$, where $\mathcal{I}^{[l]}$ is generated by all monomials $\prod_i t_i^{m_i}$ with $\sum_{i=1}^n w_i m_i \le l$. Then define the stack-theoretic weighted blow up $[\mathrm{Bl}_\mathcal{I}(M)]$ to be the stack-theoretic proj

$$\mathrm{Proj}_M(R_\mathcal{I}) = [(\mathrm{Spec}_M(R_\mathcal{I}) \setminus V)/G_m],$$

where V is the vanishing locus of the ideal $\oplus_{l>0}\mathcal{I}^{[l]}$ and the quotient is stack-theoretic. The coarse moduli space is the classical blow up of $t_1, \dots, t_n$ with weights $w_1, \dots, w_n$.

3.3. **$\mathcal{J}$-admissible centers.** By a $\mathcal{J}$-admissible center we mean an h-ideal $\mathcal{I}$ locally given by $(t_1^{d_1},\ldots,t_n^{d_n})$ with $d_1 \le d_2 \le \ldots \le d_n$ and such that $\mathcal{J} \subseteq \mathcal{I}$. The associated weighted blow up is along $\mathcal{J}^{1/N} = (t_1^{1/w_1},\ldots,t_n^{1/w_n})$, where N is such that $w_i = N/d_i$ are integral and $(w_1,\ldots,w_n) = 1$.

3.4. **The dream algorithm.** Using the new tools we obtained in [3] a simplest possible principalization algorithm, which is arguably impossible in the classical setting. It was independently discovered by McQuillan in [4] using a bit different language.

Theorem 3.4.1. *Let Y be a smooth stack of finite type over k and $\mathcal{J} \subseteq \mathcal{O}_Y$ an ideal, then*

(i) There exists a unique $\mathcal{J}$-admissible center $\mathcal{I} = (t_1^{d_1},\ldots,t_n^{d_n})$ such that $\mathrm{inv}(\mathcal{I}) := (d_1,\ldots,d_n)$ *is maximal possible with respect to the lexicographic order. In particular, the invariant* $\mathrm{inv}(\mathcal{J}) := \mathrm{inv}(\mathcal{I})$ *is well defined.*

(ii) Consider the weighted blow up $Y' = \mathrm{Bl}_{\mathcal{I}}(Y)$ and the transform $\mathcal{J}' = (\mathcal{J}\mathcal{O}_{Y'})(\mathcal{I}\mathcal{O}_{Y'})^{-1}$. Then $\mathrm{inv}(\mathcal{J}') < \mathrm{inv}(\mathcal{J})$. *In particular, iteratively blowing up such centers one obtains a principalization sequence $Y_n \to \cdots \to Y$ for $\mathcal{J}$.*

(iii) The above construction is smooth-functorial.

Once again, this implies existence of a non-embedded resolution algorithm, which has no history.

4. Future research

4.1. **Logarithmic dream algorithm.** Weighted log blow ups with arbitrary weights should give rise to a relative analog of the dream algorithm. In the absolute case this algorithm has been constructed by Quek in [5], and a similar method should apply to morphisms.

4.2. **Arbitrary bases.** It seems probable that similar algorithms work over arbitrary base log schemes, including log points or their thickenings. This is a topic of current research.

4.3. **Factorization.** It is natural to expect that similarly to the classical case, the new desingularization algorithms can be used to obtain factorization results for modifications of log smooth schemes or stacks which possess stronger factorization properties. It is an interesting question if using the new blow ups one can even obtain a strong factorization, which is still open in the classical case.

References

[1] D. Abramovich, M. Temkin, J. Włodarczyk, *Principalization of ideals on toroidal orbifolds*, JEMS **22** (2020), 3805–3866.

[2] D. Abramovich, M. Temkin, J. Włodarczyk, *Relative desingularization and principalization of ideals*, arXiv[2003.03659].

[3] D. Abramovich, M. Temkin, J. Włodarczyk, *Functorial embedded resolution via weighted blowings up*, arXiv[1906.07106].

[4] M. McQuillan, *Very fast, very functorial, and very easy resolution of singularities*, Geom. Funct. Anal. **30** (2020), 858–909.
[5] M.-H. Quek, *Logarithmic resolution via weighted toroidal blow-ups*, arXiv[2005.05939], to appear in Algebraic Geometry.

K-moduli spaces of Fano varieties and toric geometry

Andrea Petracci
(joint work with Anne-Sophie Kaloghiros)

We work over an algebraically closed field of characteristic zero, denoted by $\mathbb{C}$. If X is a projective reduced variety with mild singularities, one can consider its *canonical divisor* K_X, which is a $\mathbb{Q}$-Cartier divisor. If, in addition, X is smooth, then K_X is the divisor associated to the canonical sheaf $\omega_X = \det \Omega^1_X$. One can also consider the *anticanonical divisor* which is $-K_X$. If X has pure dimension n, then its canonical (resp. anticanonical) *volume* (also called degree) is the self-intersection K_X^n (resp. $(-K_X)^n$). X is called *canonically polarised* if K_X is ample, whereas it is called *Fano* if $-K_X$ is ample.

The most nagging obsession of algebraic geometers is to construct *moduli spaces*, in particular moduli spaces of varieties. One usually considers the set of projective varieties with fixed dimension, with fixed sign of the canonical divisor (i.e. ample or antiample), with fixed (anti)canonical volume and with certain classes of singularities and hopes to get a well-behaved moduli stack. (One can also consider moduli of varieties with trivial canonical divisor, but we avoid to discuss this in this note.)

In dimension 1, the Fano case is obvious thanks to Riemann's uniformisation, which says that $\mathbb{P}^1$ is the unique smooth projective curve of genus 0, hence the moduli space is just a point. The situation of canonically polarised curves has been clarified by Deligne and Mumford:

Theorem 1 (Deligne–Mumford [14])**.** *For each integer $g \geq 2$, smooth projective curves of genus g form a smooth Deligne–Mumford stack denoted by $\mathcal{M}_g$. This stack $\mathcal{M}_g$ has a compactification $\overline{\mathcal{M}}_g$ which is a smooth proper Deligne–Mumford stack and parametrises projective curves with at most nodes and with arithmetic genus g. Moreover, $\mathcal{M}_g$ (resp. $\overline{\mathcal{M}}_g$) admits a coarse moduli space M_g (resp. $\overline{M}_g$) which is a normal quasi-projective (resp. projective) variety with finite quotient singularities.*

Moduli of canonically polarised surfaces has been studied by Gieseker [16]. This was generalised in any dimension by Viehweg [35]. The analogue of Theorem 1 for canonically polarised varieties of dimension ≥ 2 is:

Theorem 2 (Kollár–Shepherd-Barron [22], Alexeev [2])**.** *For every $n \in \mathbb{Z}_{\geq 1}$ and $v \in \mathbb{Q}_{>0}$, there is a proper Deligne–Mumford stack $\overline{\mathcal{M}}^{\mathrm{KSBA}}_{n,v}$ which parametrises canonically polarised n-dimensional projective varieties with canonical volume v*

and with at most semi-log-canonical[1] *singularities. Moreover,* $\overline{\mathcal{M}}^{\mathrm{KSBA}}_{n,v}$ *admits a coarse moduli space* $\overline{M}^{\mathrm{KSBA}}_{n,v}$ *which is projective.*

We refer the reader to the book [23] for a thorough and updated account. Note that $\overline{\mathcal{M}}^{\mathrm{KSBA}}_{1,2g-2} = \overline{\mathcal{M}}_g$ for every $g \in \mathbb{Z}_{\geq 2}$.

The case of Fano varieties of dimension ≥ 2 has been elusive for decades, since moduli of Fanos are very non-separated. The most easy example that shows this is the family $\{x_0^2 + x_1^2 + x_2^2 + tx_3^2 = 0\} \subset \mathbb{P}^3$ over $\mathbb{A}^1$, where the central fibre is the quadric cone $\mathbb{P}(1,1,2)$ and all the other fibres are the smooth quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$. One can consider many more examples, e.g. degenerations of $\mathbb{P}^2$ [17, 29] or toric degenerations of toric Fano varieties [19].

One recent and spectacular discovery is that *K-stability* [15, 33], i.e. the study of Kähler–Einstein metrics on Fano varieties [12, 32], exactly selects a class of Fano varieties with well-behaved moduli. We will not even try to write down the definition of K-stability — we refer the reader to [38]; we just mention that one can define the notions of K-stable, K-polystable, and K-semistable for a normal Fano variety. The following implications hold:

$$\text{K-stable} \Rightarrow \text{K-polystable} \Rightarrow \text{K-semistable} \Rightarrow \text{klt}.$$

So, if one throws away the Fano varieties which are not K-semistable, one gets reasonable moduli. More precisely:

Theorem 3 (K-moduli [4, 9–11, 20, 25, 28, 36, 37])**.** *For every* $n \in \mathbb{Z}_{\geq 1}$ *and* $v \in \mathbb{Q}_{>0}$, *there exists an Artin stack of finite type* $\mathcal{M}^{\mathrm{Kss}}_{n,v}$ *which parametrises K-semistable n-dimensional Fano varieties with anticanonical volume v and which admits a good moduli space*[2], *denoted by* $M^{\mathrm{Kps}}_{n,v}$. *Moreover,* $M^{\mathrm{Kps}}_{n,v}$ *is a projective scheme whose closed points are in a natural 1-to-1 correspondence with K-polystable n-dimensional Fano varieties with anticanonical volume v.*

$\mathcal{M}^{\mathrm{Kss}}_{n,v}$ is called the K-moduli stack and $M^{\mathrm{Kps}}_{n,v}$ is called the K-moduli space.

In dimension 1 we get $\mathcal{M}^{\mathrm{Kss}}_{1,2} = \mathrm{B}\ \mathrm{PGL}_2$ (i.e. this is the point with isotropy group $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$) and $M^{\mathrm{Kps}}_{1,2} = \operatorname{Spec}\mathbb{C}$.

It is a very interesting (and quite open) question to study the geometry of $\mathcal{M}^{\mathrm{Kss}}_{n,v}$ and of $M^{\mathrm{Kps}}_{n,v}$. The first problem is to understand whether a Fano variety is K-(poly/semi)stable (Calabi problem). $\mathbb{P}^n$ (and more generally a product of projective spaces) is K-polystable, but not K-stable. The Calabi problem for smooth Fano varieties of dimension 2 is completely solved [33]. The Calabi problem for smooth Fano 3-folds is almost completely solved [6]. There is an easy criterion to understand the K-stability of toric varieties [8]: a toric Fano variety is K-polystable if and only if the polytope of its toric boundary has barycentre in the origin. Similar results exist for complexity-1 Fano T-varieties [18] and for spherical Fano varieties [13].

[1] Semi-log-canonical (slc, for short) singularities are defined in [24, Chapter 5]. Varieties with slc singularities can be reducible.

[2] The notion of good moduli space is a generalisation of coarse moduli space [3].

In general, the *global* geometry of K-moduli are understood for very few varieties, e.g. smooth(able) Fano varieties of dimension 2 [30], cubic 3-folds [26], cubic 4-folds [27], double covers of $\mathbb{P}^3$ branched in quartic surfaces [7].

In the rest of this note we concentrate on the *local* properties of moduli. Let $\mathcal{M}$ be one of $\mathcal{M}_g$, $\overline{\mathcal{M}}_g$, $\overline{M}^{\mathrm{KSBA}}_{n,v}$, $\mathcal{M}^{\mathrm{Kss}}_{n,v}$ and let M denote the good moduli space of $\mathcal{M}$. Let X be a variety which corresponds to a closed point $[X]$ of $\mathcal{M}$. Let $\mathrm{Def}(X)$ denote the base of miniversal deformation of X: this is the formal spectrum of a local noetherian $\mathbb{C}$-algebra with residue field $\mathbb{C}$ — one can also work in the analytic category. The automorphism group $\mathrm{Aut}(X)$ acts on $\mathrm{Def}(X)$. Then the quotient stack $[\mathrm{Def}(X)/\mathrm{Aut}(X)]$ gives an étale neighbourhood of $[X]$ in $\mathcal{M}$, and the quotient scheme $\mathrm{Def}(X)/\mathrm{Aut}(X)$ gives an étale neighbourhood of $[X]$ in M. Therefore the local geometry of $\mathcal{M}$ and of M is clear once understands the action of $\mathrm{Aut}(X)$ on $\mathrm{Def}(X)$.

If X is a nodal curve, then $\mathrm{Def}(X)$ is smooth; if, in addition, the arithmetic genus of X is $g \geq 2$, then $\mathrm{Aut}(X)$ is finite, hence $\overline{\mathcal{M}}_g$ is smooth and $\overline{M}_g$ is normal with finite quotient singularities. Therefore, for nodal curves, the moduli spaces are "almost smooth". This is completely false for canonically polarised varieties of dimension ≥ 2:

Theorem 4 (Murphy's law by Vakil [34]). *Let S be an analytic germ whose equations are polynomials with coefficients in $\mathbb{Z}$. Then, for every integer $n \geq 2$, there exists a smooth projective canonically polarised n-fold X such that* $\mathrm{Def}(X)$ *and S have the same singularity type, i.e. either there exists a smooth map* $\mathrm{Def}(X) \to S$ *or a smooth map $S \to \mathrm{Def}(X)$.*

This implies that the moduli space/stack of canonically polarised can be very singular!

Now the question is: what happens for K-moduli of Fano varieties? A very easy consequence of deformation theory says that if X is a smooth Fano variety then $\mathrm{Def}(X)$ is smooth. The same is true also for singular Fano varieties of dimension 2 [1]. This implies that $\mathcal{M}^{\mathrm{Kss}}_{n,v}$ (resp. $M^{\mathrm{Kps}}_{n,v}$) is smooth (resp. normal) in a neighbourhood of every point corresponding to a K-polystable Fano variety that is either smooth or 2-dimensional. It was natural to ask whether the K-moduli stack is always smooth and the K-moduli space is always normal. In joint work with Kaloghiros, we have shown that this is not the case:

Theorem 5 (Kaloghiros–P. [21]). *For every integer $n \geq 3$, there exist $v \in \mathbb{Q}_{>0}$ and a K-polystable toric Fano variety X such that at least one of the following statements holds:*

(1) $\mathcal{M}^{\mathrm{Kss}}_{n,v}$ and $M^{\mathrm{Kps}}_{n,v}$ have ≥ 2 local branches at $[X]$,
(2) $\mathcal{M}^{\mathrm{Kss}}_{n,v}$ and $M^{\mathrm{Kps}}_{n,v}$ are non-reduced at $[X]$.

In [31] we show that K-moduli of Fano varieties can be quite singular (but we don't know if Murphy's law in the sense of Vakil holds). In particular, we show that the number of local branches can be arbitrarily high:

Theorem 6 (P. [31]). *For every integer $n \geq 3$ and for every integer $m \geq 0$, there exist a number $v \in \mathbb{Q}_{>0}$ and a K-polystable n-dimensional toric Fano variety X such that $(-K_X)^n = v$ and the number of local branches of* $\mathrm{Def}(X)$, *of* $\mathcal{M}_{n,v}^{\mathrm{Kss}}$ *at* $[X]$, *and of* $M_{n,v}^{\mathrm{Kps}}$ *at* $[X]$ *is* $\geq m$.

The proof of this has the following three ingredients.

- We make use of Altmann's description of the deformation theory of isolated Gorenstein toric 3-fold singularities [5]. In particular, we construct a sequence of lattice polygons F_m such the deformation space of U_{F_m} (which is the isolated Gorenstein toric 3-fold singularity associated to F_m) has $\geq m$ local branches.
- We pick a 3-dimensional polytope P_m which is quite symmetric (this implies that the corresponding toric Fano 3-fold X_{P_m} is K-polystable) and has some facets isomorphic to F_m.
- Since U_{F_m} is an open affine subscheme of X_{P_m} we consider the restriction map $\mathrm{Def}(X_{P_m}) \to \mathrm{Def}(U_{F_m})$ and we show that is surjective.

References

[1] Akhtar, M., Coates, T., Corti, A., Heuberger, L., Kasprzyk, A., Oneto, A., Petracci, A., Prince, T. & Tveiten, K. Mirror symmetry and the classification of orbifold del Pezzo surfaces. *Proc. Amer. Math. Soc.*. **144**, 513-527 (2016)

[2] Alexeev, V. Moduli spaces $M_{g,n}(W)$ for surfaces. *Higher-dimensional Complex Varieties (Trento, 1994)*. pp. 1-22 (1996)

[3] Alper, J. Good moduli spaces for Artin stacks. *Ann. Inst. Fourier (Grenoble)*. **63**, 2349-2402 (2013)

[4] Alper, J., Blum, H., Halpern-Leistner, D. & Xu, C. Reductivity of the automorphism group of K-polystable Fano varieties. *Invent. Math.*. **222**, 995-1032 (2020)

[5] Altmann, K. The versal deformation of an isolated toric Gorenstein singularity. *Invent. Math.*. **128**, 443-479 (1997)

[6] Araujo, C., Castravet, A., Cheltsov, I., Fujita, K., Kaloghiros, A., Martinez-Garcia, J., Shramov, C., Süss, H. & Viswanathan, N. The Calabi problem for smooth Fano threefolds., https://www.maths.ed.ac.uk/cheltsov/pdf/Fanos.pdf

[7] Ascher, K., DeVleming, K. & Liu, Y. K-stability and birational models of moduli of quartic K3 surfaces., arXiv:2108.06848

[8] Berman, R. K-polystability of $\mathbb{Q}$-Fano varieties admitting Kähler-Einstein metrics. *Invent. Math.*. **203**, 973-1025 (2016)

[9] Blum, H. & Xu, C. Uniqueness of K-polystable degenerations of Fano varieties. *Ann. Of Math. (2)*. **190**, 609-656 (2019)

[10] Blum, H., Halpern-Leistner, D., Liu, Y. & Xu, C. On properness of K-moduli spaces and optimal degenerations of Fano varieties. *Selecta Math. (N.S.)*. **27**, Paper No. 73, 39 (2021)

[11] Blum, H., Liu, Y. & Xu, C. Openness of K-semistability for Fano varieties., To appear in Duke Math. J., arXiv:1907.02408

[12] Chen, X., Donaldson, S. & Sun, S. Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities. II: Limits with cone angle less than 2π. III: Limits as cone angle approaches 2π and completion of the main proof. *J. Amer. Math. Soc.*. **28**, 183-278 (2015)

[13] Delcroix, T. K-stability of Fano spherical varieties. *Ann. Sci. Éc. Norm. Supér. (4)*. **53**, 615-662 (2020)

[14] Deligne, P. & Mumford, D. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*., 75-109 (1969)

[15] Donaldson, S. Scalar curvature and stability of toric varieties. *J. Differential Geom.*. **62**, 289-349 (2002)
[16] Gieseker, D. Global moduli for surfaces of general type. *Invent. Math.*. **43**, 233-282 (1977)
[17] Hacking, P. & Prokhorov, Y. Smoothable del Pezzo surfaces with quotient singularities. *Compos. Math.*. **146**, 169-192 (2010)
[18] Ilten, N. & Süss, H. K-stability for Fano manifolds with torus action of complexity 1. *Duke Math. J.*. **166**, 177-204 (2017)
[19] Ilten, N. Mutations of Laurent polynomials and flat families with toric fibers. *SIGMA Symmetry Integrability Geom. Methods Appl.*. **8** pp. Paper 047, 7 (2012)
[20] Jiang, C. Boundedness of $\mathbb{Q}$-Fano varieties with degrees and alpha-invariants bounded from below. *Ann. Sci. Éc. Norm. Supér. (4)*. **53**, 1235-1248 (2020)
[21] Kaloghiros, A. & Petracci, A. On toric geometry and K-stability of Fano varieties. *Trans. Amer. Math. Soc. Ser. B.* **8** pp. 548-577 (2021)
[22] Kollár, J. & Shepherd-Barron, N. Threefolds and deformations of surface singularities. *Invent. Math.*. **91**, 299-338 (1988)
[23] Kollár, J. Families of varieties of general type., Book in preparation, available at `https://web.math.princeton.edu/~kollar/`
[24] Kollár, J. Singularities of the minimal model program. (Cambridge University Press, Cambridge,2013), With a collaboration of Sándor Kovács
[25] Li, C., Wang, X. & Xu, C. Algebraicity of the metric tangent cones and equivariant K-stability. *J. Amer. Math. Soc.*. **34**, 1175-1214 (2021)
[26] Liu, Y. & Xu, C. K-stability of cubic threefolds. *Duke Math. J.*. **168**, 2029-2073 (2019)
[27] Liu, Y. K-stability of cubic fourfolds., To appear in J. Reine Angew. Math., arXiv:2007.14320
[28] Liu, Y., Xu, C. & Zhuang, Z. Finite generation for valuations computing stability thresholds and applications to K-stability., To appear in Ann. of Math., arXiv:2102.09405
[29] Manetti, M. Normal degenerations of the complex projective plane. *J. Reine Angew. Math.*. **419** pp. 89-118 (1991)
[30] Odaka, Y., Spotti, C. & Sun, S. Compact moduli spaces of del Pezzo surfaces and Kähler-Einstein metrics. *J. Differential Geom.*. **102**, 127-172 (2016)
[31] Petracci, A. On deformation spaces of toric singularities and on singularities of K-moduli of Fano varieties., To appear in Trans. Amer. Math. Soc., arXiv:2105.01174
[32] Tian, G. K-stability and Kähler-Einstein metrics. *Comm. Pure Appl. Math.*. **68**, 1085-1156 (2015)
[33] Tian, G. Kähler-Einstein metrics with positive scalar curvature. *Invent. Math.*. **130**, 1-37 (1997)
[34] Vakil, R. Murphy's law in algebraic geometry: badly-behaved deformation spaces. *Invent. Math.*. **164**, 569-590 (2006)
[35] Viehweg, E. Quasi-projective moduli for polarized manifolds. (Springer-Verlag, Berlin,1995)
[36] Xu, C. & Zhuang, Z. On positivity of the CM line bundle on K-moduli spaces. *Ann. Of Math. (2)*. **192**, 1005-1068 (2020)
[37] Xu, C. A minimizing valuation is quasi-monomial. *Ann. Of Math. (2)*. **191**, 1003-1030 (2020)
[38] Xu, C. K-stability of Fano varieties: an algebro-geometric approach. *EMS Surv. Math. Sci.*. **8**, 265-354 (2021)

Non-toric valuation semi-groups for toric surfaces

Christian Haase

(joint work with Klaus Altmann, Alex Küronya, Karin Schaller, Lena Walter)

We provide a combinatorial criterion for the finite generation for valuation semi-groups associated with an ample divisor on a smooth toric surface and certain non-toric valuations of maximal rank.

The main idea behind Newton–Okounkov theory is to attach combinatorial/convex-geometric objects to situations in algebraic geometry in order to facilitate their analysis [KK12, LM09]. In other words, one tries to partially replicate the setup of toric geometry in settings without any useful group action. By now applications of Newton–Okounkov theory range from combinatorics and representation theory through birational geometry [KL17a, KL17b, KL19, KL18a] to mirror symmetry [9] and geometric quantization in mathematical physics.

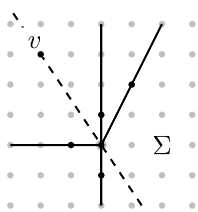
The question of finite generation of the involved valuation semigroups appears to be difficult in general, with little progress beyond the completely toric situation, but potentially great benefits such as the existence of toric degenerations [And13] and completely integrable systems [HK15] to name but a few. In this talk, we take a few steps away from the situation where every participant is toric: we consider valuation semigroups associated to torus-invariant divisors on toric surfaces with respect to a slightliy non-toric valuation.

Setup. We are concerned with a polarized toric surface (X, D) given by a polytope P in a 2-dimensional lattice M with normal fan Σ in the dual lattice N. A primitive $v \in N$ determines a toric morphism $\iota\colon \mathbb{P}^1 \to X$. For our flag, we choose $Y_1 = \iota(\mathbb{P}^1)$ and $Y_2 = \iota(1 \in \mathbb{P}^1)$. (If we chose $Y_2 = \iota(0)$ or $Y_2 = \iota(\infty)$ instead, we would be in the situation of [IM19] and finite generation would follow.) For $\ell \in \mathbb{Z}_{\geq 0}$ and any non-trivial section $s \in \Gamma(X, \mathcal{O}_X(\ell D))$ we define

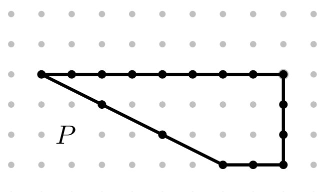
$$\mathrm{val}_{Y_\bullet}(s) := (\ell, \mathrm{val}_1(s), \mathrm{val}_2(s))$$

where $\mathrm{val}_1(s) = \mathrm{ord}_{Y_1}(s)$ and $\mathrm{val}_2(s) = \mathrm{ord}_{Y_2}(\widetilde{s}|_{Y_1})$, with $\widetilde{s} := s/f^{\mathrm{val}_1(s)}$ for a local equation f of Y_1 around Y_2. The valuation semigroup $S_{Y_\bullet}(D)$ is then

$$S_{Y_\bullet}(D) := \{\mathrm{val}_{Y_\bullet}(s) \mid s \in \Gamma(X, \mathcal{O}_X(\ell D)) \setminus \{0\}\} \subseteq \mathbb{N}^3.$$



(A) Σ, v in $N_{\mathbb{R}}$



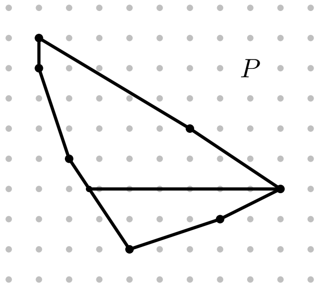
(B) P in $M_{\mathbb{R}}$

Main Result. Given the combinatorial data P and v, we define two cones, $\sigma_\pm$ based on the longest segment in P perpendicular to v.

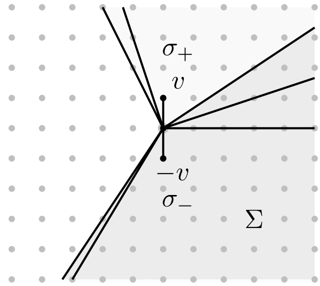
They are unions of cones in Σ with $\pm v \in \mathrm{int}(\sigma_\pm)$.

Theorem. *The following are equivalent.*

(1) The semigroup $S_{Y_\bullet}(D)$ is finitely generated.
(2) $\pm v$ lies on the boundary of $\mathrm{conv}(\mathrm{int}\, \sigma_\pm \cap N)$, respectively.
(3) If X' is the toric variety with fan generated by $\sigma_\pm$, the morphism $\mathbb{P}^1 \to X'$ given by v is a smooth embedding.



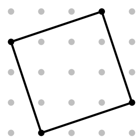
(A) The longest segment in P perpendicular to v.



(B) The cones $\sigma_\pm$ are dual to the edge directions adjacent to the longest segment.

We leave it as an exercise to the reader to convince herself that in our first example, $S_{Y_\bullet}(D)$ is not finitely generated while in our second example (taken from [CLTU20]) $S_{Y_\bullet}(D)$ will be finitely generated for any choice of D.

We close with an example of a polarized toric surface (X, D) for which no $v \in N$ yields a finitely generated semigroup.



References

[AHKSW22] Klaus Altmann, Christian Haase, Alex Küronya, Karin Schaller, Lena Walter, *On the Finite Generation of Valuation Semigroups for Toric Surfaces with Non-Toric Flags.*

[And13] Dave Anderson. Okounkov bodies and toric degenerations. *Math. Ann.*, 356(3):1183–1202, 2013.

[CLTU20] Ana-Maria Castravet, Antonio Laface, Jenia Tevelev, and Luca Ugaglia. Blown-up toric surfaces with non-polyhedral effective cone. 2020. arXiv:2009.14298v3.

[HKW20] Christian Haase, Alex Küronya, Karin Schaller, Lena Walter, *Toric Newton-Okounkov Functions with an Application to the Rationality of Certain Seshadri Constants on Surfaces*, arXiv:2008.04018 [math.AG]

[HK15] Megumi Harada and Kiumars Kaveh. Integrable systems, toric degenerations and Okounkov bodies. *Invent. Math.*, 202(3):927–985, 2015.

[IM19] Nathan Ilten and Christopher Manon. Rational complexity-one T-varieties are well-poised. *IMRN*, 13:4198–4232, 2019.

[KK12] Kiumars Kaveh and Askold G. Khovanskii. Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory. *Ann. Math. (2)*, 176(2):925–978, 2012.

[KL17a] Alex Küronya and Victor Lozovanu. Infinitesimal Newton-Okounkov bodies and jet separation. *Duke Math. J.*, 166(7):1349–1376, 2017.

[KL17b] Alex Küronya and Victor Lozovanu. Positivity of line bundles and Newton-Okounkov bodies. *Doc. Math.*, 22:1285–1302, 2017.

[KL18a] Alex Küronya and Victor Lozovanu. Geometric aspects of Newton-Okounkov bodies. In *Phenomenological approach to algebraic geometry*, volume 116 of *Banach Center Publ.*, pages 137–212. Polish Acad. Sci. Inst. Math., Warsaw, 2018.
[KL18b] Alex Küronya and Victor Lozovanu. Local positivity of linear series on surfaces. *Algebra Number Theory*, 12(1):1–34, 2018.
[KL19] Alex Küronya and Victor Lozovanu. A Reider-type theorem for higher syzygies on abelian surfaces. *Algebr. Geom.*, 6(5):548–570, 2019.
[LM09] Robert K. Lazarsfeld and Mircea Mustaţă. Convex bodies associated to linear series. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):783–835, 2009.
[RW19] K. Rietsch and L. Williams. Newton-Okounkov bodies, cluster duality, and mirror symmetry for Grassmannians. *Duke Math. J.*, 168(18):3437–3527, 2019.

Toric vector bundles over a discrete valuation ring and affine buildings

Kiumars Kaveh
(joint work with Christopher Manon, Boris Tsvelikhovsky)

Let T be an n-dimensional split algebraic torus over a field $\mathbf{k}$ with cocharacter lattice $N \cong \mathbb{Z}^n$. Let $X = X_\Sigma$ be a T-toric variety associated to a fan Σ in $N_\mathbb{R} = N \otimes \mathbb{R} \cong \mathbb{R}^n$. A *torus equivariant vector bundle* (or *toric vector bundle* for short) on X is a vector bundle $\mathcal{E}$ on X together with a linear action of T that lifts that of X. Toric vector bundles have been classified by Kaneyama [Kan75] and by Klyachko [Kly89]. In the talk we discussed extending Klyachko's classification to toric schemes over a discrete valuation ring $\mathcal{O}$. Our classification, on the other hand, extends the known classification of toric line bundles on toric schemes (see [KKMS73, §IV.3(e)], [BPS14, Section 3.6]). The classification is in terms of the *piecewise affine maps to Bruhat-Tits buildings* (also called *affine buildings*) of general linear groups.

The present result can be considered as a continuation of ideas in [KM] where classifying torus equivariant principal bundles on toric varieties over a field $\mathbf{k}$ (or *toric principal bundles* for short) is connected with the theory of Tits buildings of algebraic groups. The main result in [KM] states that for a reductive group G, toric principal G-bundles on X_Σ are classified by *piecewise linear maps* from the fan Σ to the (cone over the) Tits building of G.

Let $\mathcal{O}$ be a discrete valuation ring with field of fractions K and residue field $\mathbf{k}$. We let $\mathrm{val} : K \to \mathbb{Z} \cup \{\infty\}$ be the corresponding discrete valuation. The scheme $\mathrm{Spec}(\mathcal{O})$ has two points: the *generic point* η corresponding to the prime ideal $\{0\}$ and the *special point* o corresponding to the unique maximal ideal $\mathfrak{m}$. For a scheme $\mathfrak{X}$ over $\mathrm{Spec}(\mathcal{O})$, we let $\mathfrak{X}_\eta$ and $\mathfrak{X}_o$ denote the fibers of $\mathfrak{X}$ over η and o respectively.

Let T be a split torus over $\mathrm{Spec}(\mathcal{O})$. A *toric scheme* over $\mathrm{Spec}(\mathcal{O})$ is a normal integral separated scheme of finite type $\mathfrak{X}$ over $\mathrm{Spec}(\mathcal{O})$ equipped with a dense open embedding $T_K \hookrightarrow \mathfrak{X}_\eta$ and an action of T on $\mathfrak{X}$ over $\mathrm{Spec}(\mathcal{O})$ which extends the translation action of T_K on itself.

While toric varieties are classified by fans in $N_\mathbb{R} \cong \mathbb{R}^n$, toric schemes are classified by fans $\widetilde{\Sigma}$ in $N_\mathbb{R} \times \mathbb{R}_{\geq 0}$ (see [KKMS73, §IV.3] and [BPS14, Section 3.5]). If we intersect the fan $\widetilde{\Sigma}$ with the hyperplane $N_{\mathbb{R}} \times \{1\}$ we get a polyhedral complex Σ. The fan $\widetilde{\Sigma}$ can be recovered from Σ by taking cones over the polyhedra in Σ.

The insight from [KM] is that the right gadgets for classification of toric principal bundles are buildings of algebraic groups. Buildings are special kind of simplicial complexes arising in the classification theory of algebraic groups over arbitrary fields. Abstractly speaking, a building is an (infinite) simplicial complex together with certain distinguished subcomplexes called *apartments* that satisfy a list of axioms.

Buildings come in two flavors: *spherical buildings* and *affine buildings*. In a spherical building each apartment is a triangulation of a sphere while in an affine building each apartment is a triangulation of an affine (or Euclidean) space. To a reductive algebraic group over a field one associates its *Tits building* which is the typical example of a spherical building. Moreover, to a reductive algebraic group over a discretely valued field K one associates its *Bruhat-Tits building* which is the typical example of an affine building.

There is a nice descriptions of the Bruhat-Tits building of $\mathrm{SL}(r)$ and $\mathrm{GL}(r)$ as we briefly recall. Let $E \cong K^r$ be an r-dimensional vector space over a discretely valued field K.

Recall that an *additive norm* on E is a function $v : E \to \mathbb{R} \cup \{\infty\}$ that satisfies the following axioms.

(a) $v(e_1 + e_2) \geq \min\{v(e_1), v(e_2)\}$, for all $e_1, e_2 \in E$
(b) $v(\lambda e) = \mathrm{val}(\lambda) + v(e)$, for all $e \in E$ and $\lambda \in K$
(c) $v(e) = \infty$ if and only if $e = 0$.

We say that v is *adapted* to a basis $B = \{b_1, \ldots, b_r\}$ for E if for any $e = \sum_i \lambda_i b_i$ we have:

$$v(e) = \min\{\mathrm{val}(\lambda_i) + v(b_i)\}.$$

An additive norm is *integral* if it attains values in $\mathbb{Z} \cup \{\infty\}$. The integral additive norms on E are in one-to-one correspondence with $\mathcal{O}$-lattices in E, that is, the full rank $\mathcal{O}$-submodules in E.

Two prevaluations are said to be *equivalent* if their difference is a constant. The set of all equivalence classes of additive norms on E can be identified with the Bruhat-Tits building of $\mathrm{GL}(E)$. An *apartment* in the building consists of equivalence classes of additive norms adapted to a basis B. Each apartment is naturally an affine space.

We denote the set of all additive norms on E by $\widetilde{\mathfrak{B}}_{\mathrm{aff}}(E)$ and call it the *extended Bruhat-Tits building* of $\mathrm{GL}(E)$. We also denote the set of additive norms adapted to a basis B by $\widetilde{A}_B$ call it the *extended apartment* associated to B.

Definition 1 (Piecewise affine map to a Bruhat-Tits building)**.** Let Σ be a (rational) polyhedral complex in $N_{\mathbb{R}}$. We say that a map $\Phi : |\Sigma| \to \widetilde{\mathfrak{B}}_{\mathrm{aff}}(E)$ is a *piecewise affine map* if the following holds:

(a) for every polyhedron $\Delta \in \Sigma$, there is an extended apartment $\widetilde{A}_\Delta$ in $\widetilde{\mathfrak{B}}_{\mathrm{aff}}(E)$ such that $\Phi(\Delta)$ lands in $\widetilde{A}_\Delta$.

(b) The map $\Phi_{|\Delta} : \Delta \to \widetilde{A}_\Delta$ is the restriction of an affine map from $N_\mathbb{R}$ to the affine space $\widetilde{A}_\Delta$.

We say that Φ is *integral* if, for every $\Delta \in \Sigma$, $\Phi_{|\Delta}$ is the restriction of an integral affine map from $N_\mathbb{R}$ to $\widetilde{A}_\Delta$.

Our main result is the following:

Theorem 2. *Let $\mathfrak{X} = \mathfrak{X}_\Sigma$ be a toric scheme over* $\mathrm{Spec}(\mathcal{O})$ *associated to a polyhedral complex Σ in $N_\mathbb{R}$. There is a one-to-one correspondence between the isomorphism classes of toric vector bundles on $\mathfrak{X}_\Sigma$ and the integral piecewise affine maps from $|\Sigma|$ to $\widetilde{\mathfrak{B}}_{\mathrm{aff}}(E)$, where E is the fiber over the distinguished point $1 \in T_\eta \hookrightarrow \mathfrak{X}_\eta$. This correspondence in fact, gives an equivalence of categories.*

Moreover, the linear part Φ_0 of the piecewise affine map Φ is a piecewise linear map from the recession fan of Σ to the cone over the Tits building of $\mathrm{GL}(E)$. This is equivalent to the Klyachko data of the toric vector bundle over the generic fiber of $\mathfrak{X}_\eta$. This fits nicely with the general construction in building theory that the "boundary at infinity" of an affine building is a spherical building.

The morale of Theorem 2 is that the simplicial complex $\widetilde{\mathfrak{B}}_{\mathrm{aff}}(E)$ can be considered as a kind of classifying space for rank r toric vector bundles over toric schemes.

Remark 3. The theory of complexity-one T-varieties has similarities with the theory of toric schemes on a discrete valuation ring. During the workshop the speaker and N. Ilten and H. Süss had fruitful conversations and discussed connections with the results in [IS15].

It was mentioned to the speaker by Michel Brion that, in light of recent results of V. Balaji and Y. Pandey [BP], it might be possible to extend the above classification result to valuation rings of higher rank.

References

[BP] Balaji, V.; Pandey, Y. *On Bruhat-Tits theory over a higher dimensional base.* arXiv:2203.09431

[BDP18] Biswas, I.; Dey, A.; Poddar, M. *On equivariant Serre problem for principal bundles.* Internat. J. Math. 29 (2018), no. 9, 1850054, 7 pp.

[BPS14] Burgos Gil, J. I.; Philippon, P.; Sombra, M. *Arithmetic geometry of toric varieties. Metrics, measures and heights.* Astérisque No. 360 (2014).

[IS15] Ilten, N.; Süss, H. *Equivariant vector bundles on T-varieties.* Transform. Groups 20 (2015), no. 4, 1043–1073.

[Kan75] Kaneyama, T. *On equivariant vector bundles on an almost homogeneous variety.* Nagoya Math. J. 57 (1975), 65–86.

[KM] Kaveh, K.; Manon, C. *Toric principal bundles and Tits buildings.* arXiv.

[KKMS73] Kempf, G.; Knudsen, F.; Mumford, D.; Saint-Donat, B. *Toroidal embeddings. I.* Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin-New York, 1973.

[Kly89] Klyachko, A. A. Equivariant bundles over toric varieties. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 5, 1001–1039, 1135; translation in Math. USSR-Izv. 35 (1990), no. 2, 337–375.

Degenerations of unstable Fano varieties and Kahler-Ricci soliton

Harold Blum
(joint work with Yuchen Liu, Chenyang Xu, and Ziquan Zhuang)

A central problem in complex geometry is to find canonical metrics on complex projective varieties. On a smooth Fano variety X, the natural metrics to consider are Kähler-Einstein metrics, which are Kähler metrics $\omega \in c_1(X)$ such that

$$\mathrm{Ric}(\omega) = \omega.$$

While such a metric does not always exist, the Yau-Tian-Donaldson Conjecture, which is a theorem by [4, 15], states that a Fano variety admits a Kähler-Einstein metric if and only if it is K-polystable. The latter notion is an algebraic criterion introduced by Tian and Donaldson to characterize the existence of such metrics. More recently, K-stability has received interest from algebraic geometers due to its use in constructing compact moduli spaces of Fano varieties.

To understand Fano varieties that are K-unstable and hence, do not admit Kähler-Einstein metrics, it is necessary to look at more general metrics. One class of metric to consider are Kähler-Ricci solition, which are the data of a Kähler metric $\omega \in c_1(X)$ and a vector field $\xi \in H^0(T_X)$ such that

$$\mathrm{Ric}(\omega) = \omega + L_\xi \omega,$$

where L_ξ denotes the Lie derivative. Similar to the case of Kähler-Einstein metrics, the existence of a Kähler-Ricci solition is equivalent to a version of K-stability for Fano varieties with vector fields (X, ξ) [5, 8].

To produce Kähler-Ricci soliton, one can fix an initial metric $\omega_0 \in c_1(X)$ and study the long term behavior of the normalized Kähler-Ricci flow

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) + \omega_t.$$

The Tian-Hamilton Conjecture, now a theorem by [1, 6], states that, up to taking a subsequence, the Gromov-Hausdorff limit of (X, ω_t) as $t \to \infty$ is naturally a klt Fano variety with a Kähler-Ricci soliton (Y, ω_Y). Chen, Sun, and Wang observed that the degeneration $X \rightsquigarrow Y$ can be achieved in two steps

$$X \rightsquigarrow (Z, \xi_Z) \rightsquigarrow (Y, \xi_Y),$$

and conjectured that this degeneration process is uniquely determined by X, and independent of the choice of initial metric [5]. The latter was recently confirmed in [11].

A natural problem is to try to construct the two step degeneration algebraically and for all singular Fano varieties. This is achieved in recent joint work with Liu, Xu, and Zhuang that builds on work of Han and Li.

Main Theorem. [2, 11] *Any klt Fano variety X admits a canonical two step degeneration*

$$X \rightsquigarrow (Z, \xi_Z) \rightsquigarrow (Y, \xi_Y)$$

Furthermore, (Y, ξ_Y) admits a Kähler-Ricci soliton.

The first degeneration in the above theorem is the unique $\mathbb{R}$-degeneration minimizing the H-functional of Dervan and Szekelyhidi and the pair (Z, ξ_Z) is K-semistable. The pair (Y, ξ_Y) is the unique K-polystable degeneration of (Z, ξ_Z).

The approach to proving this theorem is through valuations and their relation to degenerations, which has been particularly effective in the algebraic study of K-stability, see e.g. [3, 10, 12]. In particular, Han and Li defined a function

$$H : \mathrm{Val}_X \to \mathbb{R} \cup \{+\infty\},$$

where Val_X is the space of real valuations of X [11]. Constructing the first degeneration $X \rightsquigarrow (Z, \xi_Z)$ and proving it is canonical amounts to showing

(1) There exists a valuation v minimizing H [11],
(2) The minimizer v is unique [2, 11], and
(3) the associated graded ring of $\mathrm{gr}_v R$ of the ring $R := \oplus_{m \in \mathbb{N}} H^0(X, -mK_X)$ is finitely generated[2].

Step (3) relies on using recent powerful finite generation results of Liu, Xu, and Zhuang in [13]. With these steps complete, $Z := \mathrm{Proj}(\mathrm{gr}_v R)$ and there is a natural torus action $\mathbb{T}$ on Z and vector field $\xi \in N_{\mathbb{R}} := \mathrm{Hom}(\mathbb{G}_m, \mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}$.

A deficiency in the above theory is that it is entirely theoretical. While there do exist Fano varieties where the first degeneration is non-trivial and $X \neq Z$ (e.g. this will happen if X is a K-unstable Fano variety and $\mathrm{Aut}(X)$ does not contain $\mathbb{G}_m$) very few interesting examples have been explicitly computed. An important problem is to explicitly describe non-trivial examples and, hopefully, find one where the valuation v is not divisorial.

References

[1] R. Bamler *Convergence of Ricci flows with bounded scalar curvature*, Ann. of Math. **188** (2018), 753–831.

[2] H. Blum, Y. Liu, C. Xu, and Z. Zhuang. *The existence of the Kähler-Ricci soliton degeneration*, `arXiv:2103.15278`.

[3] S. Boucksom, T. Hisamoto, and M. Jonsson. *Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs*, Ann. Inst. Fourier **67** (2017), no. 2, 743–841.

[4] X.X. Chen, S. K. Donaldson and S. Sun. *Kähler-Einstein metrics on Fano manifolds, I-III*, J. Amer. Math. Soc. **28** (2015), 183–197, 199–234, 235–278.

[5] X.X. Chen, S. Sun, and B. Wang. *Kähler-Ricci flow, Kähler-Einstein metric, and K-stability*, Geom. Topol. **22** (2018).

[6] X.X. Chen and B. Wang. *Space of Ricci flows (II)–Part B: Weak compactness of the flows*, J. Differential Geom. **116** (2020), 1–123.

[7] S. K. Donaldson. *Scalar curvature and stability of toric varieties*, J. Differential Geom. **62** (2002), no. 2, 289–349.

[8] V. Datar and G. Székelyhidi. *Kähler-Einstein metrics along the smooth continuity method*, Geom. Funct. Anal. 26 (2016) no. 4, 975–1010

[9] R. Dervan and G. Székelyhidi. *The Kähler-Ricci flow and optimal degenerations*, J. Differential Geom. **116** (2020), 187–203.

[10] K. Fujita. *A valuative criterion for uniform K-stability of $\mathbb{Q}$-Fano varieties*, J. Reine Angew. Math. **751** (2019), 309–338.

[11] J. Han and C. Li. *Algebraic uniqueness of Kähler-Ricci flow limits and optimal degenerations of Fano varieties*, `arXiv:2009.01010`.

[12] C. Li. *K-semistability is equivariant volume minimization*, Duke Math. J. **166** (2017), 3147–3218.
[13] Y. Liu, C. Xu, and Z. Zhuang. *Finite generation for valuations computing stability thresholds and applications to K-stability*, `arXiv:2102.09405`. To appear in Ann. of Math.
[14] G. Tian. *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. **130** (1997), 239–265.
[15] G. Tian. *K-stability and Kähler-Einstein metrics*, Comm. Pure Appl. Math. **68** (2015), 1085–1156.

The anticanonical complex for non-degenerate toric complete intersections

Milena Wrobel
(joint work with Jürgen Hausen, Christian Mauz)

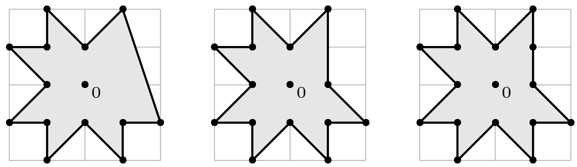
The idea behind anticanonical complexes is to extend the correspondence between the toric Fano varieties and the Fano polytopes to wider classes of varieties. After introducing their general construction, in this talk, we work out the case of non-degenerate toric complete intersections.

For an n-dimensional toric Fano variety Z, one defines the *Fano polytope* to be the convex hull $A \subseteq \mathbb{Q}^n$ over the primitive ray generators of the describing fan of Z. For any toric resolution of singularities $\pi\colon Z' \to Z$, the exceptional divisors E_ϱ are given by rays of the fan of Z' and one obtains the discrepancies as

$$\operatorname{disc}_Z(E_\varrho) \;=\; \frac{\|v_\varrho\|}{\|v'_\varrho\|} - 1,$$

where $v_\varrho \in \varrho$ is the shortest non-zero lattice vector and $v'_\varrho \in \varrho$ is the intersection point of ϱ with the boundary ∂A of the Fano polytope. In particular, the toric Fano polytope encodes the singularity type of a toric Fano variety in terms of lattice points; see for example [1, 6, 7, 9] for work making use of this fact.

This principle has been extended to $\mathbb{Q}$-Gorenstein varieties X with sufficiently nice toric embedding $X \subseteq Z$ by replacing the Fano polytope with a suitable starshaped region, named the *anticanonical region*, which is supported on the tropical variety of $X \subseteq Z$; see [4]. Such an anticanonical region is called an *anticanonical complex* if it can be endowed with the structure of a polyhedral complex, and in this case, it encodes the singularity type in full analogy to the toric Fano polytope. For instance, all $\mathbb{Q}$-Gorenstein (not necessarily Fano) toric varieties have anticanonical complexes. One obtains for example the following ones, corresponding to a log terminal, canonical and terminal (hence smooth) $\mathbb{Q}$-Gorenstein toric variety, defined by the complete fan having the bullets different from the origin as its primitive ray generators:



Further classes of varieties with torus action having anticanonical complexes can be found in [2, 4, 5].

Leaving the setting of varieties with torus action, in [3], we investigate the case of non-degenerate toric complete intersections. In this talk, we restrict ourselves to the following construction, delivering general members of this class of varieties; see [3, Def. 3.4] for the precise definition. Let $B_1, \dots, B_s \subseteq \mathbb{Q}^n$ be integral polytopes and let $f_i \in \mathbb{K}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$ be Laurent polynomials with Newton polytopes $B(f_i) = B_i$ and general coefficients. Then $F := (f_1, \dots, f_s)$ forms a non-degenerate system of Laurent polynomials in the sense of Khovanskii; see [8]. Let Σ be any fan in $\mathbb{Z}^n$ refining the normal fan of the Minkowski sum $B_1 + \dots + B_s$ and let Z denote the toric variety associated with Σ. The *non-degenerate toric complete intersection* defined by F and Σ is the subvariety

$$X := \overline{V(f_1, \dots, f_s)} \subseteq Z.$$

All varieties X arising this way are normal, and smoothness of the variety Z implies smoothness of X. As a first step towards describing the structure of the anticanonical region in this case, we have a look at the tropical variety of $X \subseteq Z$. For this, we denote by $\mathbb{T}^n$ the dense open torus in Z and by z_σ the distinguished point in the torus orbit defined by $\sigma \in \Sigma$.

Proposition. *Let $X \subseteq Z$ be an irreducible non-degenerate toric complete intersection and set*

$$\Sigma_X := \{\sigma \in \Sigma;\ X \cap \mathbb{T}^n \cdot z_\sigma \neq \emptyset\}.$$

Then we have $\mathrm{Supp}(\Sigma_X) = \mathrm{trop}(X)$.

We denote with Z_X the toric variety associated to Σ_X. Then the toric variety Z_X is $\mathbb{Q}$-Gorenstein, if and only if X is so. Moreover, one obtains the following connection between the anticanonical complexes of X and Z_X:

Theorem 1. *Let $X \subseteq Z$ be a $\mathbb{Q}$-Gorenstein, irreducible non-degenerate toric complete intersection. Then X has an anticanonical complex*

$$\mathcal{A}_X = \mathcal{A}_{Z_X},$$

where $\mathcal{A}_{Z_X}$ denotes the anticanonical complex of the $\mathbb{Q}$-Gorenstein toric variety Z_X.

We observe, that in this setting each vertex of $\mathcal{A}_X$ is a primitive ray generator of the fan Σ. In particular, all vertices of the anticanonical complex are integral vectors; this does not necessarily hold for anticanonical complexes in general, see [2, 4]. Regarding the singularity types, we obtain the following result:

Corollary. *Let $X \subseteq Z$ be a $\mathbb{Q}$-Gorenstein irreducible non-degenerate toric complete intersection.*

(i) X has at most log-terminal singularities.
(ii) X has at most terminal (canonical) singularities if and only if Z_X has at most terminal (canonical) singularities.

As an application of our concrete description of the anticanonical complex, we gain the following classification result in the terminal setting.

Theorem 2. *There are 42 families of non-toric terminal Fano general non-degenerate toric complete intersection threefolds in fake weighted projective spaces.*

References

[1] Victor. V. Batyrev. Toric Fano threefolds. *Izv. Akad. Nauk SSSR Ser. Mat.*, 45(4):704–717, 927, 1981

[2] Benjamin Bechtold, Jürgen Hausen, Elaine Huggenberger, and Michele Nicolussi. On terminal Fano 3-folds with 2-torus action. *Int. Math. Res. Not. IMRN,* no. 5:1563–1602,2016.

[3] Jürgen Hausen, Christian Mauz, and Milena Wrobel. The anticanonical complex for non-degenerate toric complete intersections, arXiv:2006.04723, 2020.

[4] Christoff Hische, and Milena Wrobel. On the anticanonical complex, arXiv:1808.01997, 2018.

[5] Christoff Hische, and Milena Wrobel. On Fano threefolds with C^*-action, arXiv:1912.08184, 2018.

[6] Alexander M. Kasprzyk. Toric Fano three-folds with terminal singularities. *Tohoku Math. J. (2)*, 58(1):101–121, 2006.

[7] Alexander M. Kasprzyk. Canonical toric Fano threefolds. *Canad. J. Math.*, 62(6):1293–1309, 2010.

[8] Askold G. Khovanskiĭ. Newton polyhedra, and toroidal varieties, *Funkcional. Anal. i Priložen*, 11(4):56–64, 1977 (Russian).

[9] Maximilian Kreuzer and Benjamin Nill. Classification of toric Fano 5-folds. *Adv.Geom.*, 9(1):85–97, 2009.

Bounding toric singularities via normalized volume

Hendrik Süss
(joint work with Joaquín Moraga)

We study the normalized volume of toric singularities. As it turns out, there is a close relation to the notion of Mahler volume from convex geometry. This observation allows us to use standard tools from convex geometry, such as the Blaschke-Santaló inequality to prove non-trivial facts about the normalized volume in the toric setting.

For a normal singularity (X, x) and a valuation v on X with centre in $x \in X$ the volume was defined in [2] as

$$\operatorname{vol}(v) := \lim_{m\to\infty} \frac{\ell(\mathcal{O}_{X,x}/a_m(v))}{m^n/n!}.$$

Here, $a_m(v) := \{f \in \mathcal{O}_{X,x} \mid v(f) \geq m\}$ and ℓ denotes the Artinian length of the module. For a log terminal singularity Li in [5] then defined its normalised volume $\widehat{\operatorname{vol}}(X, x)$ to be the infimum taken over all volumes of valuation with log discrepancy equal to 1. From results in [1] we know that the infimum in the definition of the normalised volume is actually a minimum.

The normalised volume plays an important role in the context of K-stability, see [5]. In particular, K-semistability can be seen as an algebraic version and far-reaching generalisation of the principle of volume minimisation known from Sasaki-Einstein geometry [6].

We now turn to the case of a toric Gorenstein singularity (X, x) corresponding to a cone $\mathbb{R}_{\geq 0}(P \times \{1\}) \subset N_{\mathbb{R}} \times \mathbb{R}$, where P is a lattice polytope in the real vector space $N_{\mathbb{R}}$. In this case the (torus invariant) valuation centred at the singularity correspond to interior elements of the cone and the ones with log discrepancy equal to 1 are exactly those which lie in the interior of the polytope $P \times \{1\}$. Hence, we may identify any such valuation with an interior element $v \in P$. Now, the volume of such a valutation can be expressed in terms of the dual polytope

$$P^v := (P - v)^* = \{u \in N_{\mathbb{R}}^* \mid \forall_{w \in P-v} \colon \langle u, w \rangle \geq -1\}.$$

Namely, we have $\operatorname{vol}(v) = (\dim P + 1)! \operatorname{vol} P^v$ and the normalised volume of (X, x) is given by $\min_{v \in P} \operatorname{vol}(v)$. The unique element $v \in P$ realising this minimum is known as Santaló point in convex geometry, see [7]. The minimal dual volume gives also rise to the notion of the Mahler volume of a context body, which is defined as $\operatorname{vol} P \cdot \operatorname{vol} P^v$, where $v \in P$ is the Santaló point. The Mahler volume of a convex body is known to be invariant under affine transformations and to be bounded from above by the square of the volume of unit ball of the same dimension. The latter result is known as the Blaschke-Santaló Inequality, see [7]. On the other hand, the conjectural lower bound is $(n + 1)^{n+1}/(n!)^2$, the Mahler volume of the simplex. This inequality is known as the Mahler Conjecture which plays a prominent role in convex geometry.

As a consequence of the Blaschke-Santaló inequality we see that every lower bound for the normalised volume of the toric singularity implies an upper bound for the volume of P. On the other hand, there are (up to unimodular equivalence) only finitely many polytopes for every upper bound on the volume [4]. Hence, we obtain the following result.

Theorem. *In each dimension and for any $\epsilon > 0$ there are only finitely many toric $\mathbb{Q}$-Gorenstein singularities with $\widehat{\operatorname{vol}}(X) > \epsilon$.*

This can be seen as a boundedness result for toric singularities in terms of the normalised volume. Such a boundedness statement had been conjectured more generally for log terminal singularities in [3]. For an alternative proof for the case of toric singularities see [8].

Note, that also the conjectural lower bound for the Mahler volume has an interpretation in terms of the normalised volume. The Mahler conjecture would imply that the inequality

$$\widehat{\operatorname{vol}}(X) \geq \frac{d^d}{\chi(\widetilde{X})} \tag{1}$$

holds, where $\chi(\widetilde{X})$ denotes the number of fixed points in a (toric) crepant resolution $\widetilde{X}$ of the toric Gorenstein singularity X. It would be interesting to know whether there is an interpretation of that number which leads to a reasonable generalisation of the conjectured inequality (1) to the non-toric case.

References

[1] H. Blum. Existence of valuations with smallest normalized volume. *Compos. Math.*, 154(4):820–849, 2018.
[2] L. Ein, R. Lazarsfeld, and K. E. Smith. Uniform approximation of Abhyankar valuation ideals in smooth function fields. *Amer. J. Math.*, 125(2):409–440, 2003.
[3] J. Han, Y. Liu, and L. Qi. Acc for local volumes and boundedness of singularities, 2020.
[4] J. C. Lagarias and G. M. Ziegler. Bounds for lattice polytopes containing a fixed number of interior points in a sublattice. *Can. J. Math.*, 43(5):1022–1035, 1991.
[5] C. Li. Minimizing normalized volumes of valuations. *Math. Z.*, 289(1-2):491–513, 2018.
[6] D. Martelli, J. Sparks, and S.-T. Yau. Sasaki-Einstein manifolds and volume minimisation. *Commun. Math. Phys.*, 280(3):611–673, 2008.
[7] L. Santaló. Un invariante afin para los cuerpos convexos del espacio de n dimensiones. *Portugaliae mathematica*, 8(4):155–161, 1949.
[8] Z. Zhuang. On boundedness of singularities and minimal log discrepancies of Kollár components, 2021. In Preparation.

Laurent polynomials and deformations of toric Gorenstein varieties

Matej Filip

Mirror symmetry suggests that there is a relationship between Fano manifolds and certain Laurent polynomials, cf. [2]. More precisely, if a Laurent polynomial f is mirror to a Fano manifold Y, it is expected that a Fano manifold Y admits a Q-Gorenstein degeneration to a singular toric variety, whose fan is the spanning fan of the Newton polytope $\Delta(f)$. In [3] the above conjectural relationship was extended to Q-factorial terminal Fano varieties and their mirrors called rigid maximally mutable Laurent polynomials, see also [5, Conjecture 29].

An affine Gorenstein toric variety X is given by a cone

$$\sigma = \text{cone}\{P\} \subset N \oplus \mathbb{Z},$$

where P is a polytope in the lattice N, lying on height one. We want to understand deformation theory of X. By comparison theorem it is enough to understand deformation theory of affine toric varieties in order to understand deformations of projective toric varieties.

In the case of isolated singularities the miniversal deformation was constructed by Altmann [1]. We are going to try to understand deformation theory of non-isolated Gorenstein toric varieties using mutations of Laurent polynomials which have Newton polytope equal to P.

A *combinatorial mutation* of a Laurent polynomial f with $\Delta(f) = P$ is for us a pair (m, g), where $m \in M \oplus \mathbb{Z}$ is an affine function on N defined by $m(n) := \langle m, n + e\rangle$, where $e = (\underline{0}, 1)$, and g is a Laurent polynomial such that we can write $f = \sum_{i\in\mathbb{Z}} f_i$ with $f_i \in \mathbb{C}[(m = i)] \subset \mathbb{C}[N]$ and that $\frac{f_i}{g^i}$ is a Laurent polynomial;

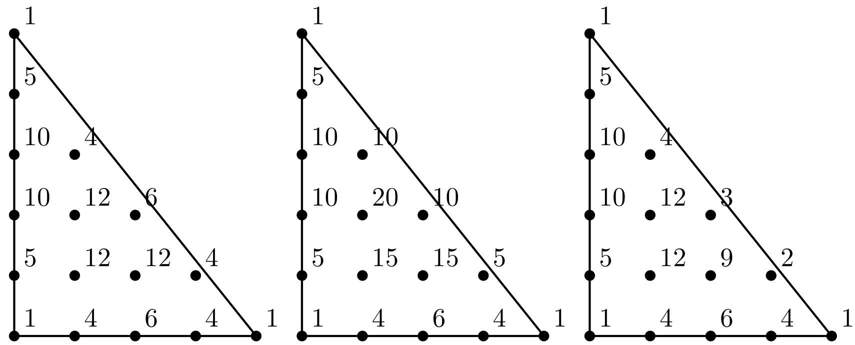


Figure 1. Rigid maximally mutable Laurent polynomials

here $(m = i)$ is the set of elements $n \in N$ with $m(n) = i$. If the above is the case we say that f is (m, g)-mutable and denote its mutation by $\mathrm{mut}_m^g f := \sum_{i\in\mathbb{Z}} \frac{f_i}{g^i}$, which is a Laurent polynomial. We denote its Newton polytope by $P_{(m,g)} := \Delta(\mathrm{mut}_m^g f)$.

The following result of Ilten [4] shows that X and $\mathrm{TV}(\mathrm{cone}\{P_{(m,g)}\})$, which is a toric affine Gorenstein variety defined by the polytope $P_{(m,g)}$, lie in the same deformation component: let f be (m, g)-mutable with $\pm m \notin \sigma^\vee \cap (M \oplus \mathbb{Z})$. Then there exists a flat family $\pi : \widetilde{X} \to \mathbb{P}^1$ with $\pi^{-1}(0) = X$ and $\pi^{-1}(\infty) \cong \mathrm{TV}(\mathrm{cone}\{P_{(m,g)}\})$.

Note that there is a correspondence between mutations of f and one-parameter deformation of X. This correspondence is obtained by observing that $\Delta(g)$ is a Minkowski summand of the cross-cut

$$\sigma \cap \{v \in N \oplus \mathbb{Z} \mid \langle v, m\rangle = 1\}.$$

This is a one-parameter deformation of X over $\mathbb{A}^1 = \mathrm{Spec}\,\mathbb{C}[t_{(m,g)}]$, where $t_{(m,g)}$ is an element of T_X^1 obtained by restricting the one parameter deformation to $\mathrm{Spec}\,\mathbb{C}[t_{(m,g)}]/(t_{(m,g)}^2)$.

The subspace

$$\mathcal{T}(f) := \{t_{(m,g)} \mid f \text{ is } (m, g)\text{-mutable}\} \subset T_X^1,$$

build from a Laurent polynomial f is crucial for understanding deformation theory of X.

Conjecture: Affine toric Gorenstein variety X is unobstructed in $\mathcal{T}(f)$, i.e. there exists a flat family over $\mathbb{C}[[\mathcal{T}(f)]] \subset \mathbb{C}[[T_X^1]]$ with the fiber over zero equal to X. Moreover, if f is rigid maximally mutable, then the generic fiber is terminal.

We provide some evidence in the three-dimensional case and state that f being rigid maximally mutable roughly means that f is uniquely determined by its mutations up to a scalar (see also [3]). We conclude the talk by describing all rigid maximally mutable polynomials on the following polytope (see Figure 1).

References

[1] K. Altmann: *The versal deformation of an isolated, toric Gorenstein singularity*, Invent. Math. **128** (1997), 443–479.

[2] T. Coates, A. Corti, S. Galkin, V. Golyshev, A. M. Kasprzyk, *Mirror symmetry and Fano manifolds*, In European congress of mathematics, Eur. Math. Soc. (2013), 285–300.

[3] T. Coates, A. M. Kasprzyk, G. Pitton, K. Tveiten, *Maximally mutable Laurent polynomials*, arXiv:2107.14253.

[4] N.O. Ilten, *Mutations of Laurent polynomials and flat families with toric fibers*, SIGMA Symmetry Integr. Geom. Methods Appl. **8** (047), (2012).

[5] A. M. Kasprzyk, V. Przyjalkowski: *Laurent polynomials in mirror symmetry: why and how?*, arXiv:2112.15339.

Blown-up toric surfaces with non-polyhedral effective cone

Antonio Laface

(joint work with Ana-Maria Castravet, Jenia Tevelev, Luca Ugaglia)

Let X be a normal projective variety defined over an algebraically closed field. The *effective cone* of X is the cone $\mathrm{Eff}(X) \subseteq \mathrm{N}^1(X)_{\mathbb{R}}$ generated by classes of effective divisors. Its closure in the euclidean topology is the *pseudoeffective cone* of X, denoted by $\overline{\mathrm{Eff}}(X)$. A cone is *polyhedral* if it is generated by finitely many vectors. The main application of our results is the following.

Theorem 1 ([4]). *The cone $\overline{\mathrm{Eff}}(\overline{M}_{0,n})$ is not polyhedral for $n \geq 10$ in any characteristic.*

Proof. Recall that a *rational contraction* is a dominant rational map $X \dashrightarrow Y$ of projective varieties that can be decomposed into a sequence of small $\mathbb{Q}$-factorial modifications and surjective morphisms. There exist rational contractions and surjections [1]

$$\mathrm{Bl}_e\overline{LM}_{n+1} \dashrightarrow \overline{M}_{0,n} \to \mathrm{Bl}_e\overline{LM}_n \dashrightarrow \mathrm{Bl}_e\mathbb{P}, \tag{1}$$

where $\overline{LM}_n$ is the toric variety named Losev-Manin space, $\mathbb{P}$ is a (not unique) toric surface and $e \in \mathbb{P}$ is a point in the big torus orbit. When $n = 10$ the surface $\mathbb{P}$ can be chosen so that $\overline{\mathrm{Eff}}(\mathrm{Bl}_e\mathbb{P})$ is not polyhedral, as shown in Theorem 3. This can be done for any characteristic. Then one uses the fact that given a surjective morphism of normal projective varieties $f\colon X \to Y$ if $\overline{\mathrm{Eff}}(X)$ is polyhedral then $\overline{\mathrm{Eff}}(Y)$ is also polyhedral. Indeed $\overline{\mathrm{Eff}}(X)$ is dual to the cone of moving curves $\overline{\mathrm{Mov}}_1(X)$ and proper pushforward of the latter cone is $\overline{\mathrm{Mov}}_1(Y)$, by [3]. □

Using (1) it is possible to show that $\overline{M}_{0,n}$ is not a Mori dream space by producing a nef class in $\mathrm{Bl}_e\mathbb{P}$ which is not semiample. This has been done in [6–8] producing each time a better lower bound for n until reaching $n = 10$. Concerning effective cones the main criterion is the following.

Proposition 2. *Let X be a normal $\mathbb{Q}$-factorial projective surface with $\varrho(X) \geq 3$. If $\overline{\mathrm{Eff}}(X)$ is polyhedral then the following hold.*

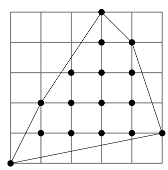
(1) $\overline{\mathrm{Eff}}(X)$ is generated by finitely many classes of negative curves [9].

(2) If $C \in \mathrm{Pic}(X)$ is a nef class with $C^2 = 0$, then $[C]$ is in the relative interior of a maximal face of $\overline{\mathrm{Eff}}(X)$. In particular $nC \sim \sum_i a_i C_i$ with $n, a_i \in \mathbb{Z}_{>0}$ for any i where each curve C_i is irreducible with $C_i \cap C = \emptyset$.

Before stating the next theorem, let us recall that a *lattice polygon* $\Delta \subseteq \mathbb{Q}^2$ is a polygon with integer vertices. Any such polygon defines a pair $(\mathbb{P}_\Delta, H_\Delta)$ consisting of a normal toric variety $\mathbb{P}_\Delta$ whose fan $\Sigma(\mathbb{P})$ is the normal fan to Δ and an ample divisor $H_\Delta := -\sum_{\rho \in \Sigma(\mathbb{P})} \min_{u \in \Delta} \langle u, \rho \rangle D_\rho$ on $\mathbb{P}_\Delta$.

Theorem 3 ([4])**.** *There exists a lattice polygon $\Delta \subseteq \mathbb{Q}^2$ such that $\overline{\mathrm{Eff}}(\mathrm{Bl}_e \mathbb{P}_\Delta)$ is non-polyhedral for all but a finite number of positive characteristics.*

Proof. (Char. $= 0$). Let $\Delta \subseteq \mathbb{Q}^2$ be a lattice polygon with at least 4 sides, $\mathrm{Vol}(\Delta) = m^2$ and $|\partial\Delta \cap \mathbb{Z}^2| = m$, like e.g.



Let $X := \mathrm{Bl}_e(\mathbb{P}_\Delta)$ and let $C \subseteq X$ be the strict transform of a curve of the linear system $|H_\Delta|$ which has multiplicity m at e. Such a curve exists because by Pick's formula $\#(\Delta \cap \mathbb{Z}^2) = \binom{m+1}{2} + 1$. In the example a defining equation for this curve in the torus is given by (underlined monomials correspond to vertices of the polygon)

$$-\underline{u^5v} - \underline{3u^4v^4} + 6u^4v^3 - 4u^4v^2 + 6u^4v - \underline{u^3v^5} + 8u^3v^4 - 10u^3v^3 +$$
$$4u^3v^2 - 11u^3v - 6u^2v^3 + 6u^2v^2 + 10u^2v + \underline{4uv^2} - 9uv + \underline{1} = 0.$$

The numerical conditions on Δ are equivalent to $C^2 = C \cdot K_X = 0$. So, if C is smooth and the Newton polygon of its defining Laurent polynomial is Δ, then C is a Cartier divisor and it has genus one. In the previous example a minimal equation for C is $y^2 + y = x^3 - x^2 - 2x + 2$. This is the curve 57.a1 from the LMFDB database of elliptic curves with Mordell-Weil group $\mathrm{Pic}^0(C)(\mathbb{Q}) \simeq \mathbb{Z}$. Since

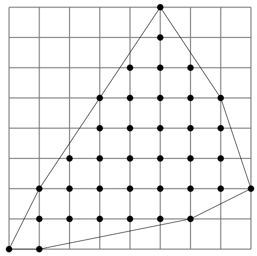
$$\mathrm{res}(C) := \mathcal{O}_C(C) \in \mathrm{Pic}^0(C)(\mathbb{Q})$$

it follows that, in the example, either $\mathrm{res}(C)$ is trivial or it is not torsion. On the other hand, from the long exact cohomology sequence of

$$0 \longrightarrow \mathcal{O}_X((n-1)C) \longrightarrow \mathcal{O}_X(nC) \longrightarrow \mathcal{O}_C(nC) \longrightarrow 0$$

one deduces that $\mathrm{res}(C)$ has order n if and only if $\dim H^0(X, nC) > 1$. Thus in the example $\mathrm{res}(C)$ cannot be trivial because one easily computes that $\dim H^0(X, C) = 1$. So $\mathrm{res}(C)$ must be non-torsion, or equivalently $\dim H^0(X, nC) = 1$ for any $n > 0$ and one concludes that C cannot be linearly equivalent to $\sum_i a_i C_i$ with all the curves C_i disjoint from C. By Proposition 2 we deduce that $\overline{\mathrm{Eff}}(X)$ is not polyhedral. □

Proof. (Char. > 0). Let $\Delta \subseteq \mathbb{Q}^2$ be the following lattice polygon.



One can show that the corresponding curve $C \subseteq \mathrm{Bl}_e(\mathbb{P}_\Delta)$ is irreducible, smooth and with Newton polygon Δ for all characteristics $p \neq 2, 3, 5, 7, 11, 19, 71$. In characteristic 0 a minimal equation for C is $y^2 + xy + y = x^3 + x^2 - 520x + 4745$. This is the curve labelled 2130.j4 in the LMFDB database. The Mordell-Weil group is $\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and one can show that $\mathrm{res}(C)$ is 4-torsion. The divisor $K_X + C$ is linearly equivalent to an effective divisor by the long exact cohomology sequence of

$$0 \longrightarrow \mathcal{O}_X(K_X) \longrightarrow \mathcal{O}_X(K_X + C) \longrightarrow \mathcal{O}_C(K_X + C) \longrightarrow 0$$

and the fact that $h^0(X, K_X) = h^1(X, K_X) = 0$. A direct calculation shows that the Zariski decomposition of $K_X + C$ has no positive part. Then, by running the $(K_X + C)$-MMP, one gets a sequence of contractions:

$$X = X_0 \to X_1 \to \cdots \to X_s = Y,$$

where all the exceptional loci consist of irreducible curves disjoint from C because $C \cdot (K_X + C) = 0$. Thus, with abuse of notation, we will denote by the same letter C the image of the curve C in Y. Observe that $-K_Y \sim C$ is Cartier so that Y has at most du Val singularities. Since $-K_Y$ is effective, the same holds for the anticanonical class of the minimal resolution Z of Y. Summarizing we have the following diagram

$$Z \searrow \; Y \; \swarrow X$$

where Z is a smooth rational surface of Picard rank 10 with nef anticanonical class and each of the three surfaces contains a copy of C as a Cartier divisor disjoint from the exceptional locus. Being Y an anticanonical rational surface with nef Cartier anticanonical class, by Mori theory it follows that $\overline{\mathrm{Eff}}(Y)$ is polyhedral if and only if $K_Y^\perp$ is generated by classes of (-2)-curves and the same statement holds for Z. Since the map $Z \to Y$ only contracts (-2)-curves, one deduces that $\overline{\mathrm{Eff}}(Y)$ is polyhedral if and only if $\overline{\mathrm{Eff}}(Z)$ is polyhedral and both conditions are equivalent to the fact that $K_Z^\perp = C^\perp \subseteq \mathrm{Pic}(Z)$ is generated by classes of (-2)-curves. It is not difficult to show that a class $R \in \mathrm{Pic}(Z)$ such that

$$R^2 = -2, \quad R \cdot K_Z = 0, \quad \mathrm{res}(R) = 0$$

is effective and it is union of (-2)-curves. On the other hand any class of (-2)-curve satisfies the above equalities. Thus $\overline{\mathrm{Eff}}(Z)$ is polyhedral if and only if $K_Z^\perp$ is generated by classes satisfying the above condition. By changing R with $R + nC$ we get a class with the same numerical properties, so the third condition can be replaced by $\mathrm{res}(R) \in \langle \mathrm{res}(C) \rangle$. One can show that, when $\mathrm{res}(C)$ is torsion, the latter condition is equivalent to ask that the kernel of the following surjection

$$\mathbb{E}_8 \simeq \frac{K_Z^\perp}{\langle C \rangle} \xrightarrow{\mathrm{res}} \frac{\mathrm{res}(K_Z^\perp)}{\langle \mathrm{res}(C) \rangle}$$

is generated by roots. In the example $\mathbb{E}_8$ is not generated by roots in ker(res) for any $p \neq 2, 3, 5, 7, 11, 19, 71$. So for these primes $\overline{\mathrm{Eff}}(Z)$ and $\overline{\mathrm{Eff}}(Y)$ are not polyhedral and thus $\overline{\mathrm{Eff}}(X)$ is not polyhedral as well. The remaining prime characteristics are analyzed by means of a similar argument applied to other lattice polygons. □

References

[1] Ana-Maria Castravet and Jenia Tevelev, *$\overline{M}_{0,n}$ is not a Mori dream space*, Duke Math. J. **164** (2015), no. 8, 1641–1667.

[2] Steven Dale Cutkosky and Kazuhiko Kurano, *Asymptotic regularity of powers of ideals of points in a weighted projective plane*, Kyoto J. Math. **51** (2011), no. 1, 25–45, DOI 10.1215/0023608X-2010-019. MR2784746

[3] Mihai Fulger and Brian Lehmann, *Zariski decompositions of numerical cycle classes*, J. Algebraic Geom. **26** (2017), no. 1, 43–106.

[4] Ana-Maria Castravet, Jenia Tevelev, Antonio Laface, and Luca Ugaglia, *Blown-up toric surfaces with non-polyhedral effective cone* (2020). arXiv:2009.14298.

[5] Javier González-Anaya, José Luis González, and Kalle Karu, *Non-existence of negative curves* (2022). arXiv:2110.13333.

[6] José Luis González and Kalle Karu, *Some non-finitely generated Cox rings*, Compos. Math. **152** (2016), no. 5, 984–996, DOI 10.1112/S0010437X15007745. MR3505645

[7] Shiro Goto, Koji Nishida, and Keiichi Watanabe, *Non-Cohen-Macaulay symbolic blow-ups for space monomial curves and counterexamples to Cowsik's question*, Proc. Amer. Math. Soc. **120** (1994), no. 2, 383–392, DOI 10.2307/2159873. MR1163334

[8] Jürgen Hausen, Simon Keicher, and Antonio Laface, *On blowing up the weighted projective plane*, Math. Z. **290** (2018), no. 3-4, 1339–1358, DOI 10.1007/s00209-018-2065-6. MR3856856

[9] Viacheslav V. Nikulin, *A remark on algebraic surfaces with polyhedral Mori cone*, Nagoya Math. J. **157** (2000), 73–92.

Exploring lattice polytopes through machine learning

Johannes Hofscheier

(joint work with Jiakang Bao, Yang-Hui He, Edward Hirst, Alexander Kasprzyk, Suvajit Majumder)

Algebraic geometers and combinatorialists have a long-established culture of producing and interrogating classification datasets. These datasets can be at the limit of current computing resources. For example, the list of 4-dimensional Gorenstein toric Fano varieties in terms of the 473 800 776 reflexive 4-polytopes [9] can be a challenge to work with using modern (consumer) hardware. Another example is the ongoing Fanosearch programme [6], a ground-breaking new approach to the classification of Fano varieties via Mirror Symmetry, that has already produced over 2 petabytes[1] of data, and is still rapidly growing.

With this "explosion" of mathematical classification datasets, the time is ripe to look for suitable tools to investigate such "big data". Here, we explore to what extent tools from data science (more precisely, from machine learning (ML)) can:

- efficiently (i.e., performantly with only a small error) predict invariants of mathematical objects,
- "find" new mathematical results and theorems, and
- generate or approximate examples with prescribed properties.

The ML tools that have been used in the explorations [2, 3] include principal component analysis (PCA), random forests, support vector machines, and artificial neural networks (NNs). We refer the interested reader to [8] for further details and references on ML. The book [7] gives a hands-on introduction to some industry-leading software libraries [1, 5, 10] that implement the aforementioned tools.

To get a better idea about the potential of ML, consider the following question: if $P \subseteq \mathbb{R}^d$ is a lattice polytope, can ML predict the dimension of P from its Ehrhart sequence $(\mathrm{ehr}_P(k))_{k\in\mathbb{Z}_{\geq 0}}$ (recall that $\mathrm{ehr}_P(k) = |kP \cap \mathbb{Z}^d|$ for $k \in \mathbb{Z}_{\geq 0}$)? Clearly, in practice a truncation of the Ehrhart sequence to finitely many terms will be used as input. Depending on the maximum dimension of lattice polytopes in the investigated dataset, relatively few entries of the Ehrhart sequence suffice. In our experiments we used $(\mathrm{ehr}_P(0), \ldots, \mathrm{ehr}_P(30))$ but could reduce the number of entries further. Any of the above ML algorithms predict the dimension to an accuracy of $\geq 99.8\%$ when trained on a relatively small dataset, i.e., the training set is 10% of the whole set (in this case study a total of $\sim 6\,000$ samples). Indeed, the high performance of ML on this task has a simple mathematical explanation. Recall the n-th *forward differences* $\Delta^n\mathrm{ehr}_P(k)$ that are inductively defined:

$$\Delta^n\mathrm{ehr}_P(k) = \begin{cases} \mathrm{ehr}_P(k) & \text{if } n = 0, \\ \Delta^{n-1}\mathrm{ehr}_P(k+1) - \Delta^{n-1}\mathrm{ehr}_P(k) & \text{otherwise.} \end{cases}$$

Since the degree of $\mathrm{ehr}_P(t)$ coincides with $\dim(P)$, the dimension can be recovered form the Ehrhart sequence $(\mathrm{ehr}_P(k))_{k\in\mathbb{Z}_{\geq 0}}$ (respectively from the forward differences $(\Delta^n\mathrm{ehr}_P(0))_{n\in\mathbb{Z}_{\geq 0}}$). More precisely, by [4, Theorem 1.1], we get that

$$\binom{\dim(P)}{n} < \Delta^n\mathrm{ehr}_P(0) \qquad \text{for } 0 \leq n \leq \dim(P),$$

and $\Delta^n\mathrm{ehr}_P(0) = 0$ for $n > \dim(P)$. By using the fundamental idea of data science to consider samples of the input as points in space, we consider Ehrhart sequences $(\mathrm{ehr}_P(k))_{k\in\mathbb{Z}_{\geq 0}}$ as points in the vector space ℓ^∞ of bounded sequences. For $(x_k)_{k\in\mathbb{Z}_{\geq 0}} \in \ell^\infty$, the n-th forward differences $\Delta^n x_0 = \sum_{j=0}^{n}(-1)^j\binom{n}{j}x_{n-j}$ give linear forms on ℓ^∞. Then the linear subspaces $V_i := \bigcap_{n=i}^{\infty}\ker(\Delta^n x_0) \subseteq \ell^\infty$

[1]According to Wikipedia this is the same as roughly 4 000 years of MP3 encoded music at a constant bitrate of 128 kbit/s.

yield a filtration of ℓ^∞, i.e., $V_0 \subseteq V_1 \subseteq V_2 \subseteq \ldots$ It follows that, given an Ehrhart sequence $x = (\mathrm{ehr}_P(k))_{k\in\mathbb{Z}_{\geq 0}}$, the dimension of the corresponding polytope P is the smallest $n \in \mathbb{Z}_{\geq 0}$ such that $x \in V_{n+1}$. Hence, the Ehrhart sequences of different dimensions are separated by hyperplanes. The idea to separate input sample points by hyperplanes is at the heart of most (state-of-the-art) ML algorithms which explains the high performance of ML on this task.

On the flip side, we want to argue that whenever ML performs well on a task, a mathematical argument is very likely to be found explaining the phenomenon. To illustrate this idea consider the following second question: can ML predict the normalised volume when polytopes themselves are used as input for the ML? This raises the question of how to represent polytopes appropriately for use in ML. Possible solutions include vertex or facet normal representations. Those representations have the downside that the input can no longer be considered as a "homogeneous" point in space. For example, when considering the representation of a polytope in terms of its vertices, the input is a matrix whose columns consist of the vertices of the polytope. Clearly, the matrix can be considered as a point in space, however, then distinct coordinates might correspond to distinct vertices, and thus the vector isn't "homogeneous" any more (this can be partly mitigated by using convolutional NN layers which regard the input as a matrix).

We suggest to use another more homogeneous approach as follows. For simplicity, let us assume that the vertices of the polytope span the ambient lattice, i.e., if $P \subseteq \mathbb{R}^d$ is a lattice polytope with vertices $V \subseteq \mathbb{Z}^d$ then $\mathrm{span}_{\mathbb{Z}} V = \mathbb{Z}^d$. By ordering the vertices of P, say $V = \{v_i\}$, we obtain an exact sequence

$$0 \to \mathbb{Z}^{|V|-d} \to \mathbb{Z}^{|V|} \xrightarrow{\varphi} \mathbb{Z}^d \to 0$$

where the linear map φ is uniquely determined by assigning the i-th standard basis vector of $\mathbb{Z}^{|V|}$ to the i-th vertex $v_i \in V$. Then the map on the left-hand side is the integer kernel of φ. Notice that, since the vertices of V span the ambient lattice $\mathbb{Z}^d$, it follows that $\ker\varphi \subseteq \mathbb{R}^d$ uniquely describes P. Indeed, P is equivalent to the image of the standard simplex in $\mathbb{Z}^{|V|}$ under the natural projection map $\mathbb{Z}^{|V|}/(\ker\varphi \cap \mathbb{Z}^{|V|})$. By classical algebraic geometry, the linear subspace $\ker\varphi$ is uniquely given by its *Plücker coordinates* in $\mathbb{P}^{N-1}$ where $N = \binom{|V|}{|V|-d}$. The advantage of this approach is that the Plücker coordinates can be regarded as "homogeneous", i.e., no obvious distinction of coordinates is given.

We use this approach to study ML's efficiency in predicting invariants of polytopes. To our surprise, ML predicted the volume of 3-dimensional polytopes from their Plücker coordinates to a high accuracy (see [2, Section 4.3]). In forthcoming work, we will give a mathematical explanation for this phenomenon (for any dimensions). We also let ML predict the normalised volume of the polar dual polytope from the Plücker coordinates of the original polytope which, however, turned out to be not as successful (see [2, Section 4.3.2] for further details).

In conclusion, the availability of big datasets of mathematical objects is worth to be further explored for new invariants and relations between objects. Here, ML can be a valuable tool to observe new phenomena. However, it's important to keep in mind that those tools have their limitations and there is an upfront investment in developing representations of the mathematical objects suitable for ML. Ultimately, the dream would be to reverse the above approach, i.e., instead of predicting invariants, build an "oracle" that can suggest (approximate) examples that have a given set of properties/invariants.

References

[1] M. Abadi, A. Agarwal, P. Barham, E. Brevdo, Z. Chen, C. Citro, G. S. Corrado, A. Davis, J. Dean, M. Devin, S. Ghemawat, I. Goodfellow, A. Harp, G. Irving, M. Isard, Y. Jia, R. Jozefowicz, L. Kaiser, M. Kudlur, J. Levenberg, D. Mané, R. Monga, S. Moore, D. Murray, C. Olah, M. Schuster, J. Shlens, B. Steiner, I. Sutskever, K. Talwar, P. Tucker, V. Vanhoucke, V. Vasudevan, F. Viégas, O. Vinyals, P. Warden, M. Wattenberg, M. Wicke, Y. Yu, and X. Zheng, *TensorFlow: Large-scale machine learning on heterogeneous systems*, Software available from `tensorflow.org`, 2015.

[2] J. Bao, Y.-H. He, E. Hirst, J. Hofscheier, A. Kasprzyk, and S. Majumder, *Polytopes and machine learning*, arXiv:2109.09602 (2021).

[3] J. Bao, Y.-H. He, E. Hirst, J. Hofscheier, A. Kasprzyk, and S. Majumder, *Hilbert series, machine learning, and applications to physics*, Phys. Lett. B **827** (2022), Paper No. 136966.

[4] M. Beck, J. A. De Loera, M. Develin, J. Pfeifle, and R. P. Stanley. *Coefficients and roots of Ehrhart polynomials*, in Integer points in polyhedra—geometry, number theory, algebra, optimization, Contemp. Math. **374** (2005), pages 15–36. Amer. Math. Soc., Providence, RI.

[5] F. Chollet et al, *Keras*, `https://keras.io`, 2015.

[6] T. Coates, A. Corti, S. Galkin, V. Golyshev, and A. Kasprzyk. *Mirror symmetry and Fano manifolds*, in European Congress of Mathematics, pages 285–300. Eur. Math. Soc., Zürich, 2013.

[7] A. Géron, *Hands-On Machine Learning with Scikit-Learn and TensorFlow: Concepts, Tools, and Techniques to Build Intelligent Systems*, O'Reilly Media, Incorporated, Sebastopol, 2017.

[8] G. James, D. Witten, T. Hastie, and R. Tibshirani, *An introduction to statistical learning—with applications in R*, Springer Texts in Statistics. Springer, New York, 2021, Second edition.

[9] M. Kreuzer and H. Skarke, *Complete classification of reflexive polyhedra in four dimensions*, Adv. Theor. Math. Phys. **4** No. 6 (2000), 1209–1230.

[10] F. Pedregosa, G. Varoquaux, A. Gramfort, V. Michel, B. Thirion, O. Grisel, M. Blondel, P. Prettenhofer, R. Weiss, V. Dubourg, J. Vanderplas, A. Passos, D. Cournapeau, M. Brucher, M. Perrot, and E. Duchesnay, *Scikit-learn: Machine learning in Python*, Journal of Machine Learning Research **12** (2011), 2825–2830.

Resolution of singularities by torus actions

Jarosław Włodarczyk

We show a simple and fast embedded desingularization of varieties and principalization of ideals in the language of torus actions on ambient smooth schemes with or without SNC divisors. The canonical functorial resolution of varieties in characteristic zero is given by, the introduced here, operations of *cobordant blow-ups* of smooth weighted centers.

Recall that a functorial resolution without SNC divisors by stack-theoretic weighted blow-ups in characteristic zero was given first in the paper by McQuillan [2], and the joint work of the author with Abramovich and Temkin [1].

I will discuss a variant of this construction which uses slightly different approach. The idea is to represent a weighted blow-up by a birational cobordism that means a smooth scheme with the torus action. The geometric quotient of this space defines a usual weighted blow-up. The stack-theoretic quotient determines the stack-theoretic weighted blow-up.

As the result of the resolution procedure we obtain a smooth variety with a torus action and the exceptional divisor having simple normal crossings. Moreover, its geometric quotient is birational to the resolved variety, has abelian quotient singularities, and can be desingularized directly by combinatorial methods.

As an application of the method we show the resolution of a certain class of isolated singularities in positive and mixed characteristic. In fact, the operation of coborodant blow-up does not require the use the Artin stacks in nonzero characteristic which makes it a very convenient resolution tool in this case. Moreover it carries additional information which can be used for resolving some classes of varieties.

The results are written in the very recent paper of the author [3].

References

[1] Dan Abramovich, Michael Temkin, and Jarosław Włodarczyk, *Functorial embedded resolution via weighted blowings up*, arXiv e-prints (2019), arXiv:1906.07106.

[2] McQuillan, M. (2020). *Very functorial, very fast, and very easy resolution of singularities.* Geometric and functional analysis, 30(3), 858-909.

[3] J. Włodarczyk, *Functorial resolution by torus actions* arXiv:2203.03090.

Motivic Integration and Toric Stacks

Matthew Satriano

(joint work with Jeremy Usatine)

In 1995, Kontsevich [3] introduced Motivic Integration and applied his theory to prove that birational Calabi-Yau manifolds have equal Hodge numbers. An essential ingredient in his proof is a *motivic change of variables formula*, which relates the motivic measures of smooth projective varieties X and Y under a birational modification $\pi\colon X \to Y$. Building off of Kontsevich's theory as well as work of Denef–Loeser [2], Batyrev [1] introduced the *stringy Hodge numbers* $h^{p,q}_{\mathrm{st}}(Y)$ of any log-terminal scheme Y with $\mathbb{Q}$-Gorenstein singularities. His motivation came from Mirror Symmetry: one wants a mirror pair (Y, Y^*) of d-dimensional Calabi-Yau manifolds to satisfy $h^{p,q}(Y) = h^{d-p,q}(Y^*)$, however if Y^* is singular, one must replace the Hodge numbers by the stringy Hodge numbers. It is important to note that these stringy Hodge numbers are not defined as dimensions of cohomology groups but rather the coefficients of a certain polynomial build from a resolution of singularities; in particular, it is not clear that the stringy Hodge numbers are non-negative. In the case where Y admits a crepant resolution $\pi\colon X \to Y$, we have $h^{p,q}_{\mathrm{st}}(Y) = h^{p,q}(X) \geq 0$. Although crepant resolutions do not exist in general, Batyrev conjectured [1, Conjecture 3.10] that the stringy Hodge numbers always satisfy $h^{p,q}_{\mathrm{st}}(Y) \geq 0$.

Yasuda [7] proved Batyrev's conjecture when Y has finite quotient singularities. He did so by extending the theory of motivic integration to include smooth Deligne–Mumford stacks and establishing a change of variables formula for the coarse space map $\pi\colon \mathcal{X} \to Y$, where $\mathcal{X}$ is the canonical smooth Deligne–Mumford stack of Y; since $\mathcal{X}$ is smooth and the π is a proper coarse space map which is an isomorphism over the smooth locus Y^{sm}, one may think of π as a stacky crepant resolution of singularities. Using his motivic change of variables formula, Yasuda showed that $h^{p,q}_{\mathrm{st}}(Y)$ agrees with the orbifold Hodge numbers $h^{p,q}_{\mathrm{orb}}(\mathcal{X})$ of $\mathcal{X}$.

For varieties Y with worse than finite quotient singularities (e.g. many GIT quotients including non-simplicial toric varieties), there is never a smooth Deligne–Mumford stack with coarse space Y. As a result, one cannot apply Yasuda's results in this setting. One may still hope to study Y through a stacky resolution $\pi\colon \mathcal{X} \to Y$, however, one is forced to consider smooth *Artin* stacks. In characteristic 0, such stacks are non-separated, and so arcs of Y may have *many* lifts to arcs of $\mathcal{X}$.

In [5, 6], we defined motivic integration for smooth Artin stacks and proved a general motivic change of variables formula. Much of our initial intuition came from developing motivic integration for toric stacks. Like toric varieties, toric stacks [4] are a concrete class of objects where one can form intuition and test conjectures. We first state our general motivic change of variables formula and then focus on the toric setting.

Theorem 1 ([6, Theorem 1.3]). *Let k be an algebraically closed field of characteristic 0. Let $\mathcal{X}$ be a smooth irreducible finite type Artin stack over k with affine geometric stabilizers and separated diagonal, let Y be an irreducible finite type scheme over k with $\dim Y = \dim \mathcal{X}$, let $\pi\colon \mathcal{X} \to Y$ be a morphism, and let $\mathcal{U}$ be an open substack of $\mathcal{X}$ such that $\mathcal{U} \hookrightarrow \mathcal{X} \to Y$ is an open immersion. Let $\sigma : \mathcal{X} \to \mathcal{I}_{\mathcal{X}}$ be the identity section of the inertia stack $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$.*

Let $\mathcal{C} \subset |\mathscr{L}(\mathcal{X})| \setminus |\mathscr{L}(\mathcal{X} \setminus \mathcal{U})| \subset |\mathscr{L}(\mathcal{X})|$ and $D \subset \mathscr{L}(Y)$ be cylinders such that, for all field extensions k' of k, the map $\overline{\mathcal{C}}(k') \to D(k')$ is a bijection. Then

$$\mu_Y(D) = \int_{\mathcal{C}} \mathbb{L}^{\mathrm{ht}^{(0)}_{L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}} - \mathrm{ht}^{(0)}_{L_{\mathcal{X}/Y}}} \mathrm{d}\mu_{\mathcal{X}}.$$

The relative canonical bundle plays a key role in Kontsevich's change of variables formula. In Theorem 1, the relative canonical bundle is replaced by the relative cotangent complexes $L_{\mathcal{X}/Y}$ and $L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}$. Our formula also involves a new function that we call a *height function*: given a complex $\mathcal{F}^\bullet$ of coherent sheaves on $\mathcal{X}$, to every arc $\varphi\colon \mathrm{Spec}(k'[[t]]) \to \mathcal{X}$, the height function $\mathrm{ht}^{(i)}_{\mathcal{F}^\bullet}(\varphi)$ is defined as $\dim_{k'} L^i\varphi^*\mathcal{F}^\bullet$. Furthermore, it is worth mentioning that $L\sigma^* L_{\mathcal{I}_{\mathcal{X}}/\mathcal{X}}$ is a novel contribution that only appears when $\mathcal{X}$ is an Artin stack that is not Deligne–Mumford.

The remainder of this article focuses on the toric case. In this setting, we proved:

Theorem 2 ([5, Proposition 1.5, Theorem 1.7, and Theorem 1.11])**.** *Let X be a $\mathbb{Q}$-Gorenstein toric variety, let $\pi\colon \mathcal{X} \to X$ be its canonical smooth Artin stack, and assume $\mathcal{X}$ has connected stabilizers. Then*

$$\mu_X^{\mathrm{Gor}}(\mathscr{L}(X)) = \int_{\mathscr{L}(\mathcal{X})} \mathrm{sep}_{\mathcal{X}}\, \mathrm{d}\mu_{\mathcal{X}}$$

From the quantity $\mu_X^{\mathrm{Gor}}(\mathscr{L}(X))$, one may read off the stringy Hodge numbers. Thus, Theorem 2 encodes the stringy Hodge numbers of X as a motivic integral on the arc stack of $\mathcal{X}$. The formula makes use of a new function $\mathrm{sep}_{\mathcal{X}}$ which measures the degree of non-separatedness of an arc: if ψ is an arc of $\mathcal{X}$, then $\mathrm{sep}_{\mathcal{X}}(\psi) = n_\psi^{-1}$, where n_ψ is the number of isomorphism classes of arcs lifting $\pi \circ \psi$.

Let us sketch the main ideas that go into the proof of Theorem 2. To prove the toric motivic change of variables, we must understand the fibers of the map

$$\mathscr{L}_n(\pi)\colon \mathscr{L}_n(\mathcal{X}) \to \mathscr{L}_n(X)$$

on n-th order jet stacks. As a starting point, we begin with the easier problem of understanding the fibers of the map

$$\mathscr{L}(\pi)\colon \mathscr{L}(\mathcal{X}) \to \mathscr{L}(X)$$

of arc stacks. Since this is a local problem, we may assume $X = \operatorname{Spec} k[P]$ where $P = \sigma^\vee \cap M$ and σ is a pointed cone. We have the following *tropicalization map*

$$\mathrm{trop} : \mathscr{L}(X_\sigma)(k') \to \mathrm{Hom}(\sigma^\vee \cap M, \mathbb{N} \cup \{\infty\})$$

defined by $\mathrm{trop}(\varphi)(p) := \mathrm{ord}_t\, \varphi^*(\chi^p)$, where $\varphi\colon \operatorname{Spec} k'[[t]] \to X$ and ord_t denotes the order of vanishing at t. Since we may discard sets of measure zero, we may concentrate on arcs φ whose generic point lands in the torus $T \subset X$. For such φ, we have

$$\mathrm{trop}(\varphi) \in \mathrm{Hom}(\sigma^\vee \cap M, \mathbb{N}) = \sigma \cap N.$$

In a similar manner, we have a tropicalization map for $\mathcal{X}$ given by

$$\mathrm{trop} : \overline{\mathscr{L}(\mathcal{X})}(k') \to \mathrm{Hom}(\widetilde{\sigma}^\vee \cap \widetilde{M}, \mathbb{N} \cup \{\infty\}),$$

where $\mathcal{X} = [\operatorname{Spec} k[F]/\mathbb{G}_m^r]$, $\overline{\mathscr{L}(\mathcal{X})}$ denotes isomorphism classes of arcs, $F = \widetilde{\sigma}^\vee \cap \widetilde{M}$ is a free monoid, $\widetilde{\sigma}$ is a smooth pointed cone on a lattice $\widetilde{N}$, and $\widetilde{M} = \widetilde{N}^*$. We have an induced map

$$\beta\colon \widetilde{\sigma} \cap \widetilde{N} \to \sigma \cap N.$$

We proved that if $\varphi \in \mathscr{L}(X)(k')$ with $\mathrm{trop}(\varphi) = w \in \sigma \cap N$, then

$$\mathrm{trop}\colon \mathscr{L}(\pi)^{-1}(\varphi) \to \beta^{-1}(w)$$

defines a canonical bijection between $\beta^{-1}(w)$ and the isomorphism classes of arcs $\psi \in \mathscr{L}(\mathcal{X})(k')$ lifting φ.

Having now understood the fibers of the map of arc stacks, we turn to the fibers of the map of jet stacks. Let $\varphi \in \mathscr{L}(X)(k')$ and suppose $\mathrm{trop}(\varphi) = w \in \sigma \cap N$. For each $n \geq 0$, let $\varphi_n \in \mathscr{L}_n(X)(k')$ denote the n-th truncation of φ. We proved that for n sufficiently large, the fiber of $\mathscr{L}_n(\pi)$ over φ_n has connected components $\mathcal{F}_{\widetilde{w}}$ indexed by $\widetilde{w} \in \beta^{-1}(w)$. Furthermore,

$$(\mathcal{F}_{\widetilde{w}})_{\text{red}} \simeq [\mathbb{A}^{r_w}/\mathbb{G}_a^{s_{\widetilde{w}}}],$$

where $(\mathcal{F}_{\widetilde{w}})_{\text{red}}$ denotes the reduced structure. This is the key local computation that goes into proving Theorem 2. From this, one can see how the function $\text{sep}_{\mathcal{X}}$ shows up: the fiber of $\mathscr{L}_n(\pi)$ over φ_n has connected components indexed by $\beta^{-1}(w)$ and $|\beta^{-1}(w)|$ is precisely the number of isomorphism classes of arcs lifting φ.

References

[1] V. V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities.* Integrable systems and algebraic geometry, 1–32, World Sci. Publ., River Edge, NJ, 1998.

[2] J. Denef and F. Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. 135 (1999), no. 1, 201–232.

[3] M. Kontsevich, *String cohomology*, December 1995. Lecture at Orsay.

[4] A. Geraschenko and M. Satriano, *Toric stacks I: The theory of stacky fans.* Trans. Amer. Math. Soc. 367 (2015), no. 2, 1033–1071.

[5] M. Satriano and J. Usatine, *Stringy invariants and toric Artin stacks.* Forum Math. Sigma 10 (2022), Paper No. e9, 60 pp.

[6] M. Satriano and J. Usatine, *A motivic change of variables formula for Artin stacks.* `https://arxiv.org/pdf/2109.09800.pdf`, (2022).

[7] T. Yasuda, *Twisted jets, motivic measures and orbifold cohomology.* Compos. Math. 140 (2004), no. 2, 396–422.

Toric bundles

Leonid Monin

(joint work with Johannes Hofscheier, Askold Khovanskii, Hendrik Süß, and Milena Wrobel)

Toric bundles are certain equivariant compactifications of torus principle bundles. More precisely, let $p\colon E \to B$ be a *T-principal bundle* over a topological space B where $T \simeq (\mathbb{C}^*)^n$ is an algebraic torus. Let also X_Σ be a T-toric variety given by a fan Σ. The *associated toric bundle* is given as the quotient $E \times_T X_\Sigma := (E \times X_\Sigma)/T$. It is a fiber bundle over B with fiber X_Σ. We will denote $E \times_T X_\Sigma$ by E_Σ. One of the main examples of toric bundles are toric bundles over generalized flag varieties which are also called toroidal horospherical varieties. We are interested in topological and geometric properties of toric bundles. Another familiar class of toric bundles are toric bundles over toric varieties [7]. They have a structure of toric varieties themselves, thus they form a special class of toric varieties which for instance include Bott towers [2].

First, let me explain a computation of cohomology ring of a toric bundle. A generalization of the Stanley-Reisner description for the cohomology ring $H^*(E_\Sigma, \mathbb{R})$ was obtained by Sankaran and Uma [9]. Like in the classical toric case, the description by Sankaran and Uma implicitly contains an algorithm to compute products of cohomology classes in the top degree. In [3] we studied such top degree products and obtained a generalization of the BKK theorem. Moreover, using a version of

Macaulay inverse system theorem for graded-commutative algebras with Poincaré duality we obtained a Pukhlikov-Khovanskii presentation of cohomology ring of toric bundles. Let me talk about these results in more detail.

Similar to the case of classical toric variety, every virtual polytope whose normal fan coarsens Σ defines a class in $H^2(E_\Sigma, \mathbb{R})$. Moreover, by Leray-Hirsch theorem every class in $H^*(E_\Sigma, \mathbb{R})$ can be written as a combination of classes of type: $p^*(\gamma)\Delta_1 \ldots \Delta_r$, where $\gamma \in H^*(B, \mathbb{R})$ and Δ_i is a polytope.

We reduce the computation of intersection numbers on the toric bundle to intersection numbers on the base. This provides a BKK-type theorem for any choice of a cohomology class in the base $\gamma \in H^*(B, \mathbb{R})$.

Theorem 1 ([3])**.** *Let $E \to B$ be a $T \simeq (\mathbb{C}^*)^n$-principal bundle with $\dim_{\mathbb{R}} B = k$. Let Σ be a smooth projective fan and $p : E_\Sigma \to B$ be the corresponding toric bundle. Finally, let Δ be a polytope whose normal fan coarsens Σ and $\gamma \in H^{k-2i}(B, \mathbb{R})$. Then the intersection index $p^*(\gamma) \cdot \Delta^{n+i}$ can be computed as*

$$p^*(\gamma) \cdot \Delta^{n+i} = \frac{(n+i)!}{i!} \int_\Delta \gamma \cdot c(x)^i dx.$$

Here $c : M_{\mathbb{R}} \to H^2(B, \mathbb{R})$ is a linear map which depends on principle bundle $E \to B$ only.

It should be mentioned that as in a classical toric case Theorem 1 admits a polarized version. Also our statements above are true not only in the algebraic category, but more generally hold for smooth manifolds.

To obtain a computation of cohomology ring of toric bundles using Theorem 1, one needs to have a convenient description of graded commutative algebras with Poincaré duality. Such description was obtained in [3]. It turns out that the above description has a much nicer formulation if we focus on the *even cohomology rings*, let us focus on this slightly restrictive case. Even cohomology rings are examples of commutative algebras with Gorenstein duality. In [6] we provide an explicit form of Macauley inverse system for algebras with Gorenstein duality over field of characteristic 0. Our description is very general and is applicable to not-necessarily Artinian algebras.

Let A be a commutative algebra over field k of characteristic 0 with Gorenstein duality given by linear function $\ell : A \to k$. Let $V \subset A$ be a (possibly infinite dimensional) vector subspace which generates A as an algebra. Let us define Exp_ℓ to be the formal sum of polynomials on V via

$$\mathrm{Exp}_\ell(x) = \ell(1) + \ell(x) + \ell\left(\frac{x^2}{2!}\right) + \ell\left(\frac{x^3}{3!}\right) + \ldots.$$

The symmetric algebra $\mathrm{Sym}(V)$ can be identified with an algebra $\mathrm{Diff}(V)$ differential operators with constant coefficients on V by sending a vector v to the Gateaux derivative in direction of v. We extend the action of $\mathrm{Diff}(V)$ on polynomial functions to the action on formal sums of polynomial such as Exp_ℓ via termwise action.

Theorem 2 ([6]). *Let A be an algebra with Gorenstein duality as before, then*

$$A \simeq \mathrm{Sym}(V)/\mathrm{Ann}(\mathrm{Exp}_\ell),$$

where $\mathrm{Ann}(\mathrm{Exp}_\ell) = \{D \in \mathrm{Diff}(V) \mid D \cdot \mathrm{Exp}_\ell \equiv 0\}$*. In other words,* Exp_ℓ *is the inverse system of A.*

In the case of graded algebras generated in degree 1 and satisfying Poincaré duality, Theorem 2 specializes to the statement used in [5, 8] to obtain descriptions of cohomology ring of toric varieties and full flag varieties respectively.

Example. *Let $A = \bigoplus_{i=0}^n A_i$ be a commutative algebra generated in degree 1 and satisfying Poincaré duality. Then the potential* $\mathrm{Exp}^*\widetilde{\ell}$ *of A on A_1 is given by*

$$\mathrm{Exp}^*\widetilde{\ell}(v) = \widetilde{\ell}\left(\frac{v^n}{n!}\right).$$

In particular algebra A can be constructed as

$$A \simeq \mathrm{Diff}(A_1)/\mathrm{Ann}\left(\widetilde{\ell}\left(\frac{v^n}{n!}\right)\right).$$

Now we combine Theorems 1 and 2 to give a description of cohomology ring of a toric bundle E_Σ in case when $\dim_{\mathbb{R}} B$ is even. Let us denote by Exp_{E_Σ} and Exp_B the inverse systems of even cohomology rings $H^{2*}(E_\Sigma)$ and $H^{2*}(B,\mathbb{R})$ respectively.

Theorem 3 ([6]). *Let $E_\Sigma \to B$ be a toric bundle with $\dim_{\mathbb{R}} B = 2k$. Let Δ be a polytope those normal fan coarsens Σ and $\gamma \in H^*(X,\mathbb{R})$, then we have*

$$\mathrm{Exp}_{E_\Sigma}(\gamma, \Delta) = \int_\Delta \mathrm{Exp}_B(c(\lambda) + \gamma)dx,$$

where $c : M_{\mathbb{R}} \to H^2(B,\mathbb{R})$ is as before.

Our description of cohomology rings is well suited to compute cohomology rings of toric bundles over a fixed base manifold B. In particular, a computation of the ring of conditions of horospherical homogeneous spaces naturally follows. Recall that the ring of conditions is an intersection ring for (not necessarily complete) homogeneous spaces [1]. Furthermore, for a connected complex reductive group G a homogeneous space G/H is called *horospherical* if H is a closed subgroup in G containing a maximal unipotent subgroup.

Finally, in a work in progress [4] we study further geometric properties of toric bundles. We first obtain a combinatorial criterion for a line bundle on E_Σ to be ample. We also use it to give a combinatorial description of Fano toric bundles. Let $c : M \to \mathrm{Pic}(B)$ be the homomorphism of lattices defining the T-principal bundle $E \to B$.

Theorem 4 ([4]). *Let $p : E_\Sigma \to B$ be a toric bundle defined by a homomorphism $c : M \to \mathrm{Pic}(B)$ and a fan Σ. Then E_Σ is Fano if and only if*

- *Σ is a Fano fan;*
- *for every $\lambda \in \Delta_{-K_\Sigma} \cap M_{\mathbb{Q}}$ the divisor $c(\lambda) - K_B$ on B is $\mathbb{Q}$-ample, where Δ_{-K_Σ} is the anticanonical polytope of X_Σ.*

References

[1] C. De Concini, C. Procesi, *Complete symmetric varieties II*, Adv. Stud. Pure Math 6 (1985): 481-513.

[2] M. Grossberg, Y. Karshon, *Bott towers, complete integrability, and the extended character of representations*, Duke Mathematical Journal 76.1 (1994): 23-58.

[3] J. Hofscheier, A. Khovanskii, and L. Monin, *Cohomology rings of toric bundles and the ring of conditions*, arXiv preprint arXiv:2006.12043 (2020).

[4] J. Hofscheier, L. Monin, H. Süß, and M. Wrobel *Fano toric bundles*, to appear.

[5] K. Kaveh, *Note on Cohomology Rings of Spherical Varieties and Volume Polynomial*, Journal of Lie Theory 21 (2011): 263-283.

[6] A. Khovanskii, L. Monin, *Gorenstein algebras and toric bundles*, arXiv preprint arXiv: 2106.15562 (2021).

[7] A. Khovanskii, L. Monin, *Toric bundles over toric varieties*, to appear.

[8] A. Pukhlikov, A. Khovanskii, *The Riemann–Roch theorem for integrals and sums of quasipolynomials on virtual polytopes*, Algebra i Analiz 4.4 (1992): 188-216.

[9] P. Sankaran, V. Uma, *Cohomology of toric bundles*, Commentarii Mathematici Helvetici 78.3 (2003): 540-554.

$\mathbb{C}^*$-actions and Mori dream spaces

Eleonora A. Romano

(joint work with Gianluca Occhetta, Luis E. Solá Conde,
Jarosław A. Wiśniewski)

The relation of birational geometry with the theory of quotients of $\mathbb{C}^*$-actions has been acknowledged since the early days of the Minimal Model Program. In this context, we recall for instance the contributions by Thaddeus and Reid [4,10] which focused on describing birational transformations in terms of variation of stability conditions yielding geometric quotients. This concept gave rise to the notions of Mori Dream Space (MDS) introduced by Hu and Keel in [5], and Cox ring, whose spectrum gives, as GIT quotients, small $\mathbb{Q}$-factorial modifications (SQM for short) of an MDS. Furthermore, Włodarczyk in [12] used $\mathbb{C}^*$-actions to prove the Weak Factorization Conjecture, which asserts that a birational map of smooth projective varieties can be factored as a sequence of blowups and blowdowns in smooth centers. The key tool in his work was the notion of *birational cobordism*, constructed by Morelli in the toric case.

We deal with $\mathbb{C}^*$-actions from the viewpoint of birational geometry, and we introduce here some recent results contained in [8]; we refer to this paper for detailed proofs and examples, and to [6, 7] and references therein for an account on torus actions and some applications. Our purpose is to investigate equivariant birational modifications of smooth projective varieties with a $\mathbb{C}^*$-action. More precisely, when the action is *equalized*, which means that no point has finite isotropy, after blowing up the extremal fixed components we obtain a variety which admits a system of small $\mathbb{Q}$-factorial $\mathbb{C}^*$-equivariant modifications (see Theorem 3). We link these SQM with the GIT quotients of the $\mathbb{C}^*$-varieties. Each of these modifications is a projective version of a cobordism associated to the natural birational map between a pair of GIT quotients.

Let X be a complex smooth projective variety admitting a non-trivial $\mathbb{C}^*$-action. We denote by $\mathcal{Y}$ the set of irreducible fixed point components. Among these components there are two distinguished ones, called *sink* and *source* of the action, defined by the property of containing, respectively, the limiting points $\lim_{t\to 0} t^{-1}x$, $\lim_{t\to 0} tx$ where $x \in X$ is a general point.

Moreover, we take an ample line bundle L on X, and a linearization μ_L of the $\mathbb{C}^*$-action on it, so that for every $Y \in \mathcal{Y}$, $\mathbb{C}^*$ acts on $L_{|Y}$ by multiplication with a character $m \in M(\mathbb{C}^*) = \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*)$, that we call *weight of the linearization on* Y. By fixing an isomorphism $M(\mathbb{C}^*) \simeq \mathbb{Z}$, this linearization defines a map $\mu\colon \mathcal{Y} \to \mathbb{Z}$, sending every fixed point component to its weight. Considering all the weights $\mu_L(Y)$, $Y \in \mathcal{Y}$, in an increasing order, we obtain a chain of integers $a_0 < a_1 < \ldots < a_r$. We then define the *criticality* of the $\mathbb{C}^*$-action on the polarized pair (X, L) as the integer r. See [6, 9] for explicit examples of $\mathbb{C}^*$-actions on polarized pairs, and [4, 7] for some applications of classification results of varieties admitting $\mathbb{C}^*$-actions of small criticality in the context of the LeBrun-Salamon conjecture.

For every $Y \in \mathcal{Y}$ we consider the *Białynicki-Birula cells* of the action (cf. [2]):

$$X^+(Y) := \{x \in X \mid \lim_{t\to 0} tx \in Y\}, \quad X^-(Y) := \{x \in X \mid \lim_{t\to \infty} tx \in Y\}.$$

Following Mumford's Geometric Invariant Theory (GIT), given a reductive group G acting on a variety X, one may consider the problem of describing all the possible proper geometric and semi-geometric quotients of G-invariant open subsets of X. In the case in which X is normal and proper, and $G = \mathbb{C}^*$ the problem was treated in [3]; the solution was written in terms of the ordered set of fixed point components of X by means of *sections* and *semi-sections* of $\mathbb{C}^*$-actions. We recall the description introduced there by means of linearizations of the $\mathbb{C}^*$-action on (X, L), since it provides a very clear geometric insight on the construction of the quotients we will work with. Moreover, we will consider only a certain type of sections and semi-sections, whose quotients will not only be proper, but projective, since they will be standard GIT quotients of X. The construction is the following.

Construction 1. Let (X, L) be a polarized pair with a nontrivial $\mathbb{C}^*$-action, and denote by $a_0 < \cdots < a_r$ the weights of the linearization on the fixed point components. We obtain a semi-section (respectively a section) of the action choosing an index $i \in \{0, \ldots, r\}$ (resp. $i \in \{0, \ldots, r-1\}$), and setting

$$\begin{aligned} &\mathcal{Y}_- := \{Y \in \mathcal{Y} \mid \mu_L(Y) \le a_{i-1}\}, \quad \mathcal{Y}_0 := \{Y \in \mathcal{Y} \mid \mu_L(Y) = a_i\}, \\ &\mathcal{Y}_+ := \{Y \in \mathcal{Y} \mid \mu_L(Y) \ge a_{i+1}\}, \end{aligned}$$

(resp. $\mathcal{Y}_- := \{Y \in \mathcal{Y} \mid \mu_L(Y) \le a_i\}$, $\mathcal{Y}_+ := \{Y \in \mathcal{Y} \mid \mu_L(Y) \ge a_{i+1}\}$). Let us denote by $X^{ss}(i,i)$ (resp. $X^{ss}(i,i+1)$) the open set $X \setminus \left(\bigcup_{Y\in\mathcal{Y}_-} X^+(Y) \cup \bigcup_{Y\in\mathcal{Y}_+} X^-(Y)\right)$, and by $\mathcal{GX}(i,i)$ (resp. $\mathcal{GX}(i,i+1)$) the corresponding proper semi-geometric (resp. geometric) quotients. See Figure 1 below.

Remark 2. Let us keep the above notation. Given a semi-section $\mathcal{Y} = \mathcal{Y}_- \sqcup \mathcal{Y}_0 \sqcup \mathcal{Y}_+$, then for every possible index i, the quotient of $X^{ss}(i,i)$ by the induced action

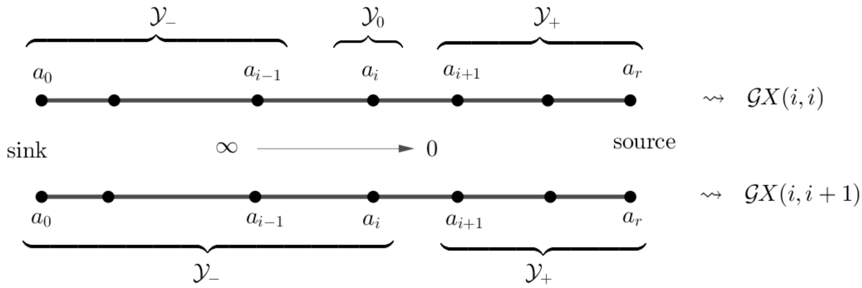


FIGURE 1. Weight representation of the semi-geometric and geometric GIT quotients of X.

of $\mathbb{C}^*$ is semi-geometric. Furthermore, if the semi-section is a section, then the quotient of $X^{ss}(i,i+1)$ by $\mathbb{C}^*$ is geometric (see also [3, Theorem 2.1]).

We summarize our main result as follows. For clarity of exposition, we state it only in the case in which the sink and the source are positive dimensional. Our arguments work also in the remaining cases; a complete description, examples and proofs can be found in [8, Sections 4–5].

Theorem 3. *Let X be a complex smooth projective variety of Picard number one admitting a nontrivial $\mathbb{C}^*$-action. Assume that the action is equalized of criticality r, and that its extremal fixed point components Y_0, Y_r are not isolated points. Denote by $\mathcal{GX}(i,i+1)$, $i=0,\dots,r-1$, the corresponding geometric quotients. Then:*

(1) The varieties $\mathcal{GX}(i,i+1)$ are smooth and the natural birational maps $\mathcal{GX}(0,1) \dashrightarrow \mathcal{GX}(1,2) \dashrightarrow \dots \dashrightarrow \mathcal{GX}(r-1,r)$ are flips.
(2) The blowup $X^\flat$ of X along Y_0, Y_r is a Mori dream space.
(3) Given a pair (i,j) of indices $i,j \in \{0,\dots,r\}$, $i \le j$, there exists a unique small $\mathbb{Q}$-factorial modification $X(i,j)$ of $X^\flat$ that is smooth and admits a $\mathbb{C}^*$-action with extremal fixed point components $\mathcal{GX}(i,i+1)$, $\mathcal{GX}(j-1,j)$.
(4) Every small $\mathbb{Q}$-factorial modification of $X^\flat$ is constructed as above.

The case of non-equalized actions is going to be investigated in [1], where we are studying torus actions in a more general setting, by allowing some mild singularities for the $\mathbb{C}^*$-varieties. Our next purpose is to establish a correspondence between Mori dream regions and certain $\mathbb{C}^*$-actions which are called *bordisms*.

REFERENCES

[1] L. Barban, E. A. Romano, L. E. Solá Conde, S. Urbinati, *Local Cox rings and $\mathbb{C}^*$ actions*, work in progress.
[2] A. Białynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. of Math. (2), 98:480–497, 1973.

[3] A. Białynicki-Birula, J. Święcicka, *Complete quotients by algebraic torus actions.* In Group actions and vector fields (Vancouver, B.C., 1981), volume 956 of Lecture Notes in Math., pages 10–22. Springer, Berlin, 1982.
[4] J. Buczyński, J. A. Wiśniewski, A. Weber, *Algebraic torus actions on contact manifolds*, to appear in J. Differ. Geom., arXiv:1802.05002, 2018.
[5] Y. Hu, S. Keel, *Mori dream spaces and GIT*, Michigan Math. J., **48**(2000), 331–348.
[6] G. Occhetta, E. A. Romano, L. E. Solá Conde, J. A. Wiśniewski, *Small bandwidth varieties with $\mathbb{C}^*$-actions and birational geometry*, arXiv:1911.12129, 43 pages, 2019.
[7] G. Occhetta, E. A. Romano, L. E. Solá Conde, J. A. Wiśniewski, *High rank torus actions on contact manifolds*, Sel. Math. New Ser. **27**(10)(2021), published online https://doi.org/10.1007/s00029-021-00621-w.
[8] G. Occhetta, E. A. Romano, L. E. Solá Conde, J. A. Wiśniewski, *Small modifications of Mori Dream Spaces arising from $\mathbb{C}^*$-actions*, to appear in the European Journal of Mathematics, arXiv:2103.07209, 29 pages, 2021.
[9] E. A. Romano, J. A. Wiśniewski, *Adjunction for varieties with a $\mathbb{C}^*$ action*, Transformation Groups (2020), published online https://doi.org/10.1007/s00031-020-09627-8.
[10] M. Thaddeus, *Complete collineations revisited*, Math. Ann. (1999), **315**(3), 469–495.
[11] M. Reid, *What is a flip?*, http://homepages.warwick.ac.uk/staff/Miles.Reid/3folds, 1992.
[12] J. Włodarczyk, *Birational cobordisms and factorization of birational maps*, J. Algebr. Geom., **9**(3)(2000), 425–449.

Minimal models of non-degenerate toric hypersurfaces

Victor V. Batyrev

1. Introduction

Let

$$f(\mathbf{t}) = \sum_{m \in M} a_m \mathbf{t}^m \in \mathbb{C}[M] \cong \mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}] = \mathbb{C}[M]$$

be a non-invertible Laurent polynomial. Denote by $Z_f := \{f = 0\}$ the affine toric hypersurface in d-dimensional algebraic torus $\mathbb{T}^d \cong (\mathbb{C}^*)^d$ with the character lattice M. Any primitive lattice vector $\nu \in N = Hom(M, \mathbb{Z})$ defines an elementary affine torus embedding $\mathbb{T}^d \hookrightarrow X_\nu \cong \mathbb{C}^{d-1} \times \mathbb{C}$ consisting of exactly two torus orbits: $\mathbb{T}^d$ and the closed torus invariant divisor $D_\nu \cong \mathbb{T}^{d-1}$.

Definition 1.1. *A Laurent polynomial $f \in \mathbb{C}[M]$ and the affine toric hypersurface $Z_f \subset \mathbb{T}^d$ are called* **non-degenerate** *if for any primitive lattice vector $\nu \in N$ the Zariski closure $\overline{Z}_{f,\nu} \subset X_\nu$ is smooth and the intersection two divisors $D_\nu \cap \overline{Z}_{f,\nu}$ is either emply or transversal.*

Another equivalent characterization of non-degenerate toric hypersurfaces Z_f can be more explicitly expressed via the **Newton polytope** P of f:

$$P := \text{Conv}\{m \in M \mid a_m \neq 0\}.$$

One can show that the property of the Laurent polynomial f to be **non-degenerate** is a Zariski open condition on the set of its coefficients $\{a_{\mathbf{m}}\}$. It means that the Zariski closure of Z_f in the projective toric variety V_P associated with the normal fan of P is transversal to all torus orbits in V_P [1, 2].

The talk is devoted to a very explicit algorithmic combinatorial construction of minimal projective birational models of non-degenerate toric hypersurfaces Z_f. The proposed construction runs in the **opposite way** if one compares it with the **traditional Mori program** based on "extremal contractions" and "flips". We show that if a non-degenerate toric hypersurface Z_f admits a minimal model then such a model $\widehat{Z_f}$ can be obtained by a crepant partial desingularization of a **uniquely** determined projective birational model $\widetilde{Z_f}$, so called **canonical model**, having at worst canonical singularities [8]. The canonical model $\widetilde{Z_f}$ is the Zariski closure of Z_f in some projective $\mathbb{Q}$-Gorenstein canonical toric variety $\widetilde{V}$ associated with a d-dimensional rational polytope $\widetilde{P}$ that depends only on the Newton polytope P.

2. The canonical toric variety $\widetilde{V}$

Let $P \subset M_{\mathbb{R}} \cong \mathbb{R}^d$ be a d-dimensional lattice polytope. We consider below P as Newton polytope of a non-degenerate Laurent polynomial $f(\mathbf{t})$.

Consider the dual lattice $N := Hom(M, \mathbb{Z}) \subset N_{\mathbb{R}} := Hom(M, \mathbb{R})$ and the piecewise linear function $\mathrm{ord}_P : N_{\mathbb{R}} \to \mathbb{R}, \ \ y \mapsto \mathrm{ord}_P(y) := \min_{x \in P} \langle x, y \rangle$, where $\langle *, * \rangle$ denotes the canonical pairing between M and N. We will use the function ord_Q also for some rational polytopes $Q \subset M_{\mathbb{R}}$.

Definition 2.1 ([3,4])**.** *The convex set*

$$F(P) := \{x \in M_{\mathbb{R}} \mid \langle x, n \rangle \geq \mathrm{ord}_P(n) + 1 \ \ \forall n \in N \setminus \{0\}\} \subset P \subset M_{\mathbb{R}}$$

is called **the Fine interior of** P.

Remark 2.2. One can show that $F(P)$ is a convex hull of finitely many rational points in $Int(P) \cap M_{\mathbb{Q}} := M \otimes \mathbb{Q}$, where $Int(P) := P \setminus \partial P$ denotes the usual interior of P. Note that the Fine interior $F(P) \subset P$ may happen to be empty.

Definition 2.3. [6,8] Assume that $F(P) \neq \emptyset$. Then we call the set

$$S_F(P) := \{n \in N \mid \mathrm{ord}_{F(P)}(n) = \mathrm{ord}_P(n) + 1\} \subset N \setminus \{0\}$$

the support of $F(P)$.

Remark 2.4. The set $S_F(P)$ consists of finitely many primitive lattice vectors in N such that $\mathbb{R}_{\geq 0} S_F(P) = N_{\mathbb{R}}$. We will use below the $S_F(P)$ as generating set $\widehat{\Sigma}[1]$ of 1-dimensional cones in a complete simplicial fan $\widehat{\Sigma}$ defining a projective simplicial toric variety $\widehat{V}$ that contains a minimal model $\widehat{Z_f}$ of Z_f.

Definition 2.5 ([8])**.** *Assume that $F(P) \neq \emptyset$. Then we call the d-dimensional rational polytope*

$$C(P) := \{x \in M_{\mathbb{R}} \mid \langle x, n \rangle \geq \mathrm{ord}_P(n) \ \ \forall n \in S_F(P)\}$$

the canonical hull of P. *It is clear that $P \subseteq C(P)$.*

Theorem 2.6 ([8]). *Assume that $F(P) \neq \emptyset$. Consider the Minkowski sum*

$$\widetilde{P} := C(P) + F(P).$$

Then 1-dimensional cones in the normal fan $\widetilde{\Sigma}$ to the full-dimensional polytope $\widetilde{P}$ are spanned by elements of $S_F(P)$, i.e., $\widetilde{\Sigma}[1] \subseteq S_F(P)$. The normal fan $\widetilde{\Sigma}$ defines a projective toric variety $\widetilde{V} = V(\widetilde{\Sigma})$ with at worst $\mathbb{Q}$-Gorenstein canonical singularities.

Proposition 2.7 ([8]). *There exists a maximal projective simplicial refinement $\widehat{\Sigma}$ of the complete fan $\widetilde{\Sigma}$ such that $\widehat{\Sigma}[1] = S_F(P)$. The corresponding birational toric morphism $\widehat{V} \to \widetilde{V}$ is crepant, i.e., $\widehat{V}$ is a maximal projective partial desingularization of the canonical toric variety $\widetilde{V}$.*

3. The canonical model $\widetilde{Z_f}$ and minimal models $\widehat{Z_f}$

The following result is due to S. Ishii:

Theorem 3.1 ([6]). *A nondegenerate toric hypersurface Z_f has a minimal model if and only if $F(P) \neq \emptyset$. Moreover, if $F(P) \neq \emptyset$, then a minimal model of $\widehat{Z_f}$ can be obtained as Zariski closure of Z_f in some projective simplicial torus embedding $\mathbb{T}^d \subset \widehat{V}$ having at worst terminal singularities whose defining simplical fan $\widehat{\Sigma}$ satisfies the condition $\widehat{\Sigma}[1] = S_F(P)$.*

Definition 3.2 ([8]). We call the Zariski closure $\widetilde{Z_f}$ of Z_f in the toric variety $\widetilde{V}$ **the canonical model of** Z_f.

The canonical model $\widetilde{Z_f}$ is our new main tool. We construct minimal models $\widehat{Z_f}$ via the canonical model $\widetilde{Z_f}$.

Theorem 3.3 ([8]). *The canonical model $\widetilde{Z_f}$ has at worst canonical singularities. Moreover, the Zariski closure $\widehat{Z_f}$ of Z_f in a maximal projective partial desingularization $\widehat{V} := V(\widehat{\Sigma})$ of $\widetilde{V}$ is* **a minimal model of** *Z_f which is a maximal projective crepant partial desingularization of the canonical model $\widetilde{Z_f}$.*

Remark 3.4. A refinement $\widehat{\Sigma}$ of $\widetilde{\Sigma}$ with $\widehat{\Sigma}[1] = S_F(P)$ in Proposition 2.7 is not unique in general. Therefore, there exist possibly many minimal models $\widehat{Z_f}$ of the canonical model $\widetilde{Z_f}$.

Theorem 3.5 ([8]). *Let $\kappa(\widehat{Z_f})$ be the Kodaira dimension of the minimal model $\widehat{Z_f}$. Then $\kappa(\widehat{Z_f}) = \min\{d-1, \dim F(P)\}$.*

Remark 3.6. On can show that a lattice polytope $P \subset M_{\mathbb{R}}$ is **reflexive** if and only if $F(P) = \{0\}$ and $C(P) = P$. In the latter case, we have $\widetilde{P} = P$ and $S_F(P) = \partial P^* \cap N$, where

$$P^* := \{y \in N_R \mid \langle x, y\rangle \geq -1 \ \ \forall x \in P\} \subset N_{\mathbb{R}}$$

is the polar dual reflexive polytope. Moreover, $\widetilde{V}$ is the Gorenstein toric Fano variety V_P, and the above general construction of minimal models gives rise to already known minimal Calabi-Yau hypersurfaces $\widehat{Z_f}$ used in the combinatorial Mirror Symmetry based on the polar duality $P \longleftrightarrow P^*$ for reflexive polytopes [5].

Remark 3.7. Recently (see [9]), the proposed method was applied to all $674,688$ three-dimensional canonical lattice polytopes classified by Kasprzyk [7].

References

[1] A. G. Khovanskiĭ, *Newton polyhedra and toroidal varieties*, Funct. Anal. Appl. **11** (1977), 289–296.

[2] A. G. Khovanskiĭ, *Newton polyhedra and the genus of complete intersections*, Funct. Anal. Appl. **12** (1978), 38–46.

[3] J. Fine, *Resolution and Completion of Algebraic Varieties*, Ph.D. University of Warwick 1983.

[4] M. Reid, *Young person's guide to canonical singularities*, Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 345–414.

[5] V. V. Batyrev, *Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties*, Journal of Algebraic Geometry **3** (1994), 493–535.

[6] S. Ishii, *The minimal model theorem for divisors of toric varieties*, Tohoku Math. J. (2) **51** (1999), 213–226.

[7] A. M. Kasprzyk, *Canonical toric Fano threefolds*, Can. J. Math. 62 (6) (2010), 1293–1309.

[8] V. V. Batyrev, *Canonical models of toric hypersurfaces*, arXiv:2008.05814 [math.AG]

[9] V. V. Batyrev, A. M. Kasprzyk, and K. Schaller, *On the Fine interior of three-dimensional canonical Fano polytopes*, In "Interactions with Lattice Polytopes", Benjamin Nill and Alexander Kasprzyk (eds.), Springer Proceedings in Mathematics & Statistics, Springer, (2022), 11-52.

Computing toric degenerations of varieties

Fatemeh Mohammadi

(joint work with Oliver Clarke, Francesca Zaffalon)

In this note, we focus on toric degenerations of Grassmannians, however, the given constructions also apply to the case of (partial) flag varieties [8].

A toric degeneration of an algebraic projective variety X is a flat family $\mathcal{F} \to \mathbb{A}^1$ whose fibers $\mathcal{F}_t$ over all points $t \in \mathbb{A}^1 \backslash \{0\}$ are isomorphic to X and whose fiber over 0 is a toric variety $X_0 := \mathcal{F}_0$. Toric degenerations are particularly useful because many important algebraic and geometric invariants of X, such as the Hilbert polynomial and degree, coincide with those of X_0. It is often practical to study these invariants using the toric variety X_0 because toric varieties are rich in combinatorial structure. This is due to a well-established dichotomy between toric varieties and discrete geometric objects, such as polyhedral fans and polytopes.

The study of toric degenerations is motivated by two natural questions:

(1) How do we compute toric degenerations of X?
(2) What is the relationship between two different toric degenerations of a fixed algebraic variety X?

To answer the first question, one general approach is through Gröbner degenerations [3, 11, 12]. Such toric degenerations of a closed subvariety $V(I)$ of a large space are produced by toric (binomial and prime) initial ideals of I. This is naturally linked to tropical geometry. More precisely, given a variety X as above, realized as $\mathrm{Proj}(A) \cong \mathrm{Proj}(\mathbb{C}[x_1, \ldots, x_n]/I)$ where A is its homogeneous coordinate ring and $\mathbb{C}[x_1, \ldots, x_n]/I$ a choice of presentation of A, the tropicalization of X, denoted by $\mathrm{trop}(X)$ is a subset of $\mathbb{R}^n$ consisting of those weight vectors whose corresponding initial ideals $\mathrm{in}_w(I)$ contain no monomials. In particular, $\mathrm{trop}(X)$ is a subfan of the Gröbner fan, hence carrying a lot of combinatorial structure. Given that primality is impossible if $\mathrm{in}_w(I)$ contains a monomial, the top-dimensional cones of the tropicalization provide a good search space for the weight vectors for which the initial ideal $\mathrm{in}_w(I)$ is toric. Hence, the first question boils down to computing the tropicalization of varieties, however, such computations are very challenging. For example for the Grassmannians, on the computational side [3,10], we currently only have the description for $\mathrm{Gr}(2, n)$, $\mathrm{Gr}(3, 6)$, and $\mathrm{Gr}(3, 7)$. For flag varieties, such computations are only available for FL_4 and Fl_5.

In the case of $\mathrm{Gr}(2, n)$, i.e. the tropical Grassmannian of 2-planes in n-space, it is shown by Speyer and Sturmfels that all the top-dimensional cones lead to toric degenerations, however, this is no longer true for higher Grassmannian. In [12], we used the correspondence between tropical line arrangements and the so-called coherent matching fields from [14] to provide a combinatorial characterization of the maximal cones in the case of $\mathrm{Gr}(3, n)$ which yields toric degenerations.

Extending this work, in [8] we study the tropicalization of $\mathrm{Gr}(k, n)$ and produce families of points in its top-dimensional cones. We prove that the associated initial ideal of every such point is binomial and prime, hence leading to toric degeneration.

The class of weight vectors that we study arises from matching fields. In the case of Grassmannians, the idea is the following. Consider the Plücker embedding, induced by the following polynomial map: $\phi : \mathbb{C}[P_I] \to \mathbb{C}[x_{i,j}] : P_I \mapsto \det(X_I)$, where $X = (x_{i,j})$ is a $k \times n$ matrix of variables and X_I is the submatrix on the columns indexed by I for each k-subset I of $\{1, \ldots, n\}$. A good candidate for a Gröbner degeneration of $\mathrm{Gr}(k, n)$ is given by deforming ϕ to a monomial map ϕ_Λ. This is done by sending each Plücker variable P_I to one of the summands of $\det(X_I)$. A matching field Λ is a combinatorial object which encodes this data.

Matching fields were originally introduced by Sturmfels and Zelevinsky to study Newton polytopes of products of maximal minors. A matching field is defined as a map that sends each variable P_I to a permutation. In particular, this gives a natural monomial map that corresponds to a toric subvariety of the projective space. In order for a matching field Λ to give rise to a toric degeneration of $\mathrm{Gr}(k, n)$, it is necessary for Λ to be coherent, i.e. induced by a weight matrix. Toric degenerations arising from matching fields have been studied in great detail in [5,7,8]. Such degenerations correspond to top-dimensional cones of $\mathrm{trop}(\mathrm{Gr}(k, n))$ as in [11]. There remain many open questions in this area. Notably, which matching fields produce toric degenerations? In studying this question, we use combinatorial mutations introduced in [1]. A combinatorial mutation is a special kind of piecewise linear map between polytopes. We say that two polytopes are mutation equivalent if there exists a sequence of combinatorial mutations between them.

For the second question above, we need to understand the properties of algebraic varieties that are preserved under toric degeneration. We do this by looking into the associated polytopes of the resulted toric varieties. Recall that each monomial map obtained from a coherent matching field Λ gives rise to a projective toric variety $V(\ker(\phi_\Lambda))$. In general, this is not a toric degeneration of $\mathrm{Gr}(k,n)$. We associate to Λ the matching field polytope P_Λ which is the toric polytope of $V(\ker(\phi_\Lambda))$. We prove in [8] that a matching field Λ inherits the property of giving rise to a toric degeneration from another matching field Λ' whenever P_Λ and $P_{\Lambda'}$ are mutation equivalent.

Theorem ([8]). Let Λ be a matching field for the Grassmannian $\mathrm{Gr}(k,n)$. If the matching field polytope P_Λ is combinatorial mutation equivalent to the Gelfand-Tsetlin polytope, then Λ gives rise to a toric degeneration of $\mathrm{Gr}(k,n)$.

We then construct a family of matching fields $\{\Lambda_\sigma\}$ indexed by permutations and show that:

Theorem ([8]). For each permutation $\sigma \in S_n$, the matching field polytope P_{Λ_σ} is a combinatorial mutation equivalent to the Gelfand-Tsetlin polytope. In particular, each Λ_σ gives rise to a toric degeneration of $\mathrm{Gr}(k,n)$.

Following up on the known results for the tropical Grassmannian, one may ask a more general question for arbitrary projective varieties X: what is the relationship between the different toric degenerations which arise from different maximal cones in $\mathrm{trop}(X)$? In [9], Escobar and Harada have studied this problem and showed that the toric degenerations of adjacent maximal cones are related by certain flip maps. In the case of $\mathrm{Gr}(2,n)$ the combinatorial mutation above is equivalent to the cluster mutation, and the flip map. The natural question is to understand such mutations for higher Grassmannians.

The connection between cluster mutation and Escobar–Harada's flip map was first observed in a discussion with Lara Bossinger and Megumi Harada during the MFO-Workshop on Toric geometry in 2019, see [2] and [4].

References

[1] Akhtar, Mohammad, Coates, Tom, Galkin, Sergey, and Kasprzyk, Alexander, *Minkowski polynomials and mutations*, SIGMA. Symmetry, Integrability, and Geometry: Methods and Applications 8 (2012): 094.

[2] Bossinger, Lara, Harada, Megumi, and Mohammadi, Fatemeh, *Toric degenerations of tropical Grassmannians*, Extended abstract in Mathematisches Forschungsinstitut Oberwolfach Workshop Report No. 43, 2019.

[3] Bossinger, Lara, Sara Lamboglia, Kalina Mincheva, and Fatemeh Mohammadi. *Computing toric degenerations of flag varieties*, In Combinatorial algebraic geometry, pp. 247-281. Springer, New York, NY, 2017.

[4] Bossinger, Lara, Mohammadi, Fatemeh, and Nájera Chávez, Alfredo, *Families of Gröbner degenerations, Grassmannians and universal cluster algebras*, SIGMA. Symmetry, Integrability and Geometry: Methods and Applications 17 (2021): 059.

[5] Clarke, Oliver, Higashitani, Akihiro, and Mohammadi, Fatemeh, *Combinatorial mutations and block diagonal polytopes*, Collectanea Mathematica (2021): 1-31.

[6] Clarke, Oliver, and Mohammadi, Fatemeh, *Toric degenerations of flag varieties from matching field tableaux*, Journal of Pure and Applied Algebra, (2021) 225(8), p.106624.

[7] Clarke, Oliver, and Mohammadi, Fatemeh, *Toric degenerations of Grassmannians and Schubert varieties from matching field tableaux*, Journal of Algebra 559 (2020): 646-678.
[8] Clarke, Oliver, Mohammadi, Fatemeh, and Zaffalon, Francesca, *Toric degenerations of partial flag varieties and combinatorial mutations of matching field polytopes*, in preparation.
[9] Escobar, Laura, and Harada, Megumi, *Wall-crossing for Newton-Okounkov bodies and the tropical Grassmannian*, International Mathematics Research Notices 2022.7 (2022): 5152-5203.
[10] Herrmann, Sven, Jensen, Anders, Joswig, Michael, and Sturmfels, Bernd, *How to Draw Tropical Planes*, The Electronic Journal of Combinatorics (2009): R6-R6.
[11] Kaveh, Kiumars and Manon, Christopher, *Khovanskii Bases, Higher Rank Valuations, and Tropical Geometry*, SIAM Journal on Applied Algebra and Geometry **3** (2019), no. 2, 292–336.
[12] Mohammadi, Fatemeh, and Shaw, Kristin, *Toric degenerations of Grassmannians from matching fields*, to appear in Algebraic Combinatorics, Algebraic Combinatorics 2.6 (2019): 1109-1124.
[13] Speyer, David and Williams, Lauren, *The tropical totally positive Grassmannian*, J. Algebraic Combinatorics **22** (2005) no. 2, 189–210.
[14] Sturmfels, Bernd and Zelevinsky, Andrei, *Maximal minors and their leading terms*, Advances in Mathematics **98** (1993), no. 1, 65–112.

Toric Geometry and Discrete Periodic Operators

Frank Sottile
(joint work with Matthew Faust)

For background, see [2] and its references, including [5]. A fundamental problem in mathematical physics is to understand the spectrum $\sigma(L)$ of a Schrödinger operator L. For a function $f\colon \mathbb{R}^d \to \mathbb{C}$, we have

$$Lf \;=\; L(f) \;:=\; -\Delta f \;+\; Vf\,,$$

where $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$ is the Laplacian and $V\colon \mathbb{R}^d \to \mathbb{R}$ is a potential. As L is a self-adjoint operator on an appropriate Hilbert space, its spectrum $\sigma(L)$ consists of a union of intervals in $\mathbb{R}$, giving the familiar spectral bands and gaps.

Solid-state physics studies the Schrödinger operator in a crystalline material, with the Laplacian altered due to a periodic anisotropy and V periodic.

We consider a discrete version. Let Γ be a graph equipped with a free action of $\mathbb{Z}^d$ having finitely many orbits on its edges, $\mathcal{E}(\Gamma)$, and vertices, $\mathcal{V}(\Gamma)$. Fix parameters, functions $c\colon \mathcal{E}(\Gamma) \to \mathbb{R}$ and $V\colon \mathcal{V}(\Gamma) \to \mathbb{R}$ that are $\mathbb{Z}^d$-periodic. For a function $f\colon \mathcal{V}(\Gamma) \to \mathbb{C}$, our Schrödinger operator L is

$$Lf(v) \;:=\; V(v)f(v) \;+\; \sum_{(u,v)\in\mathcal{E}(\Gamma)} c_{(u,v)}(f(v) - f(u))\,.$$

Then L is an operator on the space $\ell^2(\Gamma)$ of square-summable functions on $\mathcal{V}(\Gamma)$.

Fourier transform (called Floquet transform) reveals more structure of the spectrum $\sigma(L)$. Let $\mathbb{T} \subset \mathbb{C}$ be the unit complex numbers. For $x \in \mathbb{T}$, $\overline{x} = x^{-1}$. Then $\mathbb{T}^d := \mathrm{Hom}(\mathbb{Z}^d, \mathbb{T})$ is the space of unitary characters of $\mathbb{Z}^d$. The evaluation of $z \in \mathbb{T}^d$ at $\gamma \in \mathbb{Z}^d$ is a monomial, $z(\gamma) = z_1^{\gamma_1} \cdots z_d^{\gamma_d} =: z^\gamma$. Fourier transform of $f\colon \mathcal{V}(\Gamma) \to \mathbb{C}$ is a function $\hat{f}(z,v)$ on $\mathbb{T}^d \times \mathcal{V}(\Gamma)$ satisfying $\hat{f}(z, \gamma + v) = z^\gamma \hat{f}(z,v)$.

If $W \subset \mathcal{V}(\Gamma)$ is a fundamental domain for the $\mathbb{Z}^d$-action, then Fourier transform is a linear isometry between $\ell^2(\Gamma)$ and $L^2(\mathbb{T}^d)^W$, the space of functions $\hat{f}\colon W \to L^2(\mathbb{T}^d)$. The action of the operator L on such functions $\hat{f}$ becomes

$$(1) \qquad L\hat{f}(v) \;:=\; V(v)\hat{f}(v) \;+\; \sum_{(\gamma+u,v)\in\mathcal{E}(\Gamma)} c_{(\gamma+u,v)}(\hat{f}(v) - z^{\gamma}\hat{f}(u))\,.$$

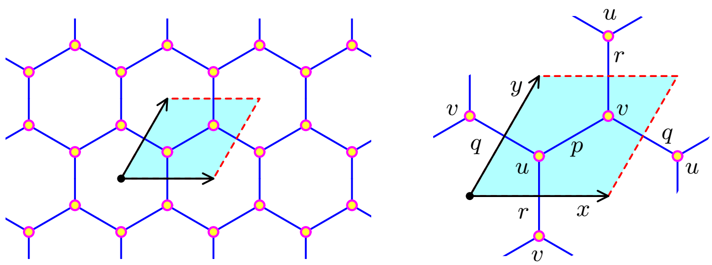


Figure 1. Graphene and $\mathbb{Z}^2$-periodic edge parameters

Example. On the left in Figure 1 is graphene, a $\mathbb{Z}^2$-periodic graph. Its fundamental domain W has two vertices, and there are three orbits of edges. Fix edge parameters p, q, r, as on the right. The operator L is

$$\begin{aligned} L\hat{f}(u) &= V(u)\hat{f}(u) + p(\hat{f}(u) - \hat{f}(v)) + q(\hat{f}(u) - x^{-1}\hat{f}(v)) + r(\hat{f}(u) - y^{-1}\hat{f}(v))\,,\\ L\hat{f}(v) &= V(v)\hat{f}(v) + p(\hat{f}(v) - \hat{f}(u)) + q(\hat{f}(v) - x\hat{f}(u)) \quad + r(\hat{f}(v) - y\hat{f}(u))\,. \end{aligned}$$

Collecting coefficients of $\hat{f}(u), \hat{f}(v)$, we represent L by the 2×2-matrix,

$$L \;=\; \begin{pmatrix} V(u)+p+q+r & -p-qx^{-1}-ry^{-1} \\ -p-qx-ry & V(v)+p+q+r \end{pmatrix}\,,$$

whose entries are Laurent polynomials in x, y. Observe that for $(x,y)\in\mathbb{T}^2$, $L^T = \overline{L}$, so that L is Hermitian. ◇

This holds in general. After Fourier transform, L is multiplication by a $W\times W$ matrix $L(z)$ of Laurent polynomials in z. As $z\in\mathbb{T}^d$, we have $L(z)^T = \overline{L(z)}$, so that $L(z)$ is Hermitian and hence has $|W|$ real eigenvalues.

These eigenvalues are the roots of $D(z,\lambda) := \det(L(z)-\lambda I)$. As a polynomial in z,λ, it is the dispersion relation which defines the Bloch variety, $\{(z,\lambda) \mid D(z,\lambda) = 0\} \subset \mathbb{T}^d\times\mathbb{R}$. Figure 2 shows two Bloch varieties for graphene with zero potential. On the left the edge parameters are 6, 3, 2, and on the right they are 1, 1, 1, giving the graph Laplacian. The spectrum $\sigma(L)$ is the image of the Bloch variety under projection to the vertical λ-axis. On the left there are two spectral bands with a gap in between, while there is one spectral band on the right.

Since Bloch varieties are defined by Laurent polynomials, we may complexify, allowing complex parameters c and V, $z\in(\mathbb{C}^\times)^d$, and $\lambda\in\mathbb{C}$. This gives complex Bloch varieties in $(\mathbb{C}^\times)^d\times\mathbb{C}$.

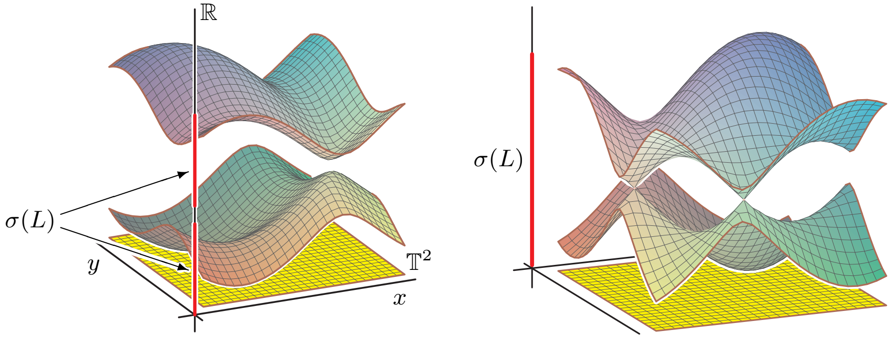


Figure 2. Bloch varieties with edge parameters 6,3,2 and 1,1,1

Several fundamental questions from physics may be understood in terms of the algebraic geometry of the Bloch variety. The spectral edges conjecture [5]*Conj. 5.25 posits that for L sufficiently general (e.g. c, V general), all extrema of the function λ on the real Bloch variety are nondegenerate. Important notions, such as effective masses in solid state physics, the Liouville property, etc., depend upon this conjecture. This is discussed in [2, Sect. 1.4].

This holds for the Bloch varieties in Figure 2. On the left, λ is a Morse function, and all critical points are nondegenerate. The critical value from the singularities on the Bloch variety of the Laplacian lies in the interior of the spectrum and the extrema are nondegenerate. A first step towards the spectral edges conjecture is to understand the critical points of λ. This was the approach in [2] to prove the spectral edges conjecture for the graph on the left of Figure 4.

Classical work. In 12992, Gieseker, Knörrer, and Trubowicz [4] settled a number of questions in the following situation. Let Γ be the grid graph, with vertices $\mathbb{Z}^2$ where $\alpha, \beta \in \mathbb{Z}^2$ adjacent if $\|\alpha - \beta\| = 1$. If we let the standard generators e_1, e_2 of $\mathbb{Z}^2$ act respectively as translation by ae_1 and be_2, for a, b coprime integers, then Γ is $\mathbb{Z}^2$-periodic with fundamental domain the integer points in any $a \times b$ box.

The Schrödinger operator with the graph Laplacian depends upon the ab values of the potential V on a fundamental domain. Gieseker, Knörrer, and Trubowicz [4] prove identifiability: if V and V' are general and give the same Bloch variety, then V and V' differ only by obvious symmetries of relabeling the fundamental domain.

They also show that there is a dense open set of $\mathbb{C}^{ab}$ consisting of potentials V such that the number of critical values of the function λ on the Bloch variety is

$$2a^2b^2 + 6ab(a+b) + 12ab - 12(a^2+b^2) = 2(a+b) - 12 \quad (2)$$

For potentials in this open set, the Bloch variety is smooth and irreducible.

These and other results were obtained by compactifying the Bloch variety in a natural toric variety, followed by a toric resolution of singularities.

Current work. We describe some work with Faust [3]. The dispersion relation $D(z, \lambda)$ is a Laurent polynomial which is an ordinary polynomial in λ. Let

$P_\Gamma \subset \mathbb{R}^{d+1}$ be its Newton polytope—the convex hull of the exponent vectors of all monomials in z, λ occurring in $D(z,\lambda)$. Figure 3 shows typical Newton polytopes for graphene and for the two graphs of Figure 4.

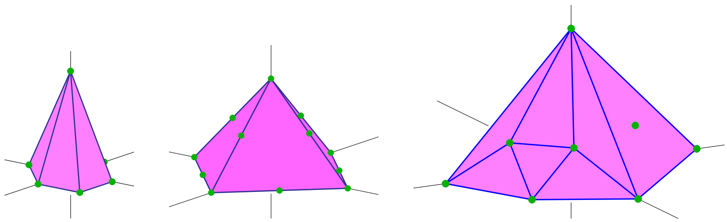


FIGURE 3. Three Newton Polytopes

Let $X^\circ = (\mathbb{C}^\times)^d \times \mathbb{C}$, the domain of the dispersion relation $D(z,\lambda)$ and the ambient space of the Bloch variety. This has a compactification given by the projective toric variety X_Γ corresponding to the polytope P_Γ, and the closure of the Bloch variety in X_Γ compactifies the Bloch variety.

Our main result is a generalization of the enumeration (2) of critical values. Implicit differentiation of $0 = D(z,\lambda)$ gives $0 = \frac{\partial}{\partial z_i} D(z,\lambda) + \frac{\partial}{\partial \lambda} D(z,\lambda) \cdot \frac{\partial \lambda}{\partial z_i}$. Thus $(z,\lambda) \in X^\circ$ is a critical point of λ if it is a common zero of the polynomials

$$D(z,\lambda)\,,\ z_1\frac{\partial}{\partial z_1} D(z,\lambda)\,,\ \ldots\,,\ z_d\frac{\partial}{\partial z_d} D(z,\lambda)\,. \tag{3}$$

Each polynomial has Newton polytope a subset of P_Γ. Consequently, the critical points are the common zeroes on X° of $d{+}1$ sections of the line bundle $\mathcal{O}(P_\Gamma)$ on X_Γ. We have the following bound.

Theorem A. *The number of critical points of the function λ on the Bloch variety is at most the degree of the toric variety X_Γ, which is $(d{+}1)!\mathrm{vol}(P_\Gamma)$.*

An application of Bernstein's Theorem B [1] informs us that if the number of critical points of λ on the Bloch variety is less than the degree of X_Γ, then the critical point equations (3) have solutions in $\partial X_\Gamma := X_\Gamma \setminus X^\circ$.

We obtain X_Γ from X° by adding divisors for each facet of P_Γ, except its base (as $\{\lambda = 0\} \subset X^\circ$). Each face F of P_Γ corresponds to a torus orbit X_F° on X_Γ. The intersection of X_F° with the compactified Bloch variety is defined by the restriction $D(z,\lambda)|_F$ of $D(z,\lambda)$ to the monomials whose exponent vectors lie in F.

Theorem B. *If the critical point equations* (3) *have a solution in X_F°, for F a face of P_Γ that is not its base, then either F is vertical or the hypersurface in X_F° defined by $D(z,\lambda)|_F$ is singular at that point.*

A periodic graph Γ is dense if it has all possible edges given its structure: if there is one edge between translates $\beta + W$ and $\gamma + W$ of the fundamental domain W then it has all edges between vertices in $(\beta + W) \cup (\gamma + W)$. Of the two graphs in Figure 4, the one on the left is dense.

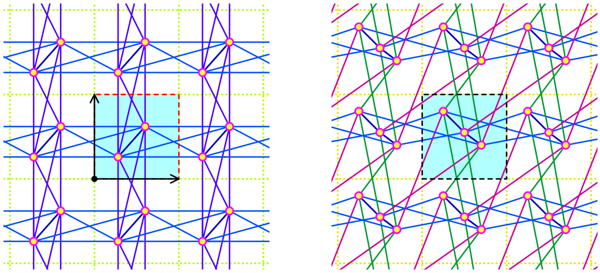


FIGURE 4. More periodic graphs

Given a graph Γ, let $A(\Gamma)$ be the set of $\gamma \in \mathbb{Z}^d$ such that Γ has an edge between W and $\gamma + W$. Let P be the convex hull of $A(\Gamma)$ and the point $(\mathbf{0}, 1)$, which is a pyramid over the convex hull Q of $A(\Gamma)$ with apex $(\mathbf{0}, 1)$. It has no vertical faces.

Theorem C. *Suppose that Γ is dense. Then there is a dense open subset U of the space of parameters consisting of parameters c, V in U such that the Newton polytope P_Γ of the dispersion relation $D(z, \lambda)$ is $|W| \cdot P$.*

When $d = 2, 3$, we may choose U such that for parameters c, V from U and all faces F of P_Γ, the hypersurface in X_F° defined by $D(z, \lambda)|_F$ is smooth.

Using the formula for the volume of a pyramid, we obtain the following result.

Corollary. *Suppose that Γ is dense and $d = 2, 3$. Then there is a dense open subset U of the space of parameters consisting of parameters c, V such that the function λ on the complex variety has exactly $|W|^{d+1} d! \mathrm{vol}(Q)$ critical points.*

References

[1] D.N. Bernstein, *The number of roots of a system of equations*, Functional Anal. Appl. **9** (1975), no. 3, 183–185.

[2] N. Do, P. Kuchment, and F. Sottile, *Generic properties of dispersion relations for discrete periodic operators*, J. Math. Phys. **61** (2020), no. 10, 103502, 19.

[3] M. Faust and F. Sottile, *Critical points of Discrete Periodic Operators*, In preparation, 2022.

[4] D. Gieseker, H. Knörrer, and E. Trubowitz, *The geometry of algebraic Fermi curves*, Perspectives in Mathematics, vol. 14, Academic Press, Inc., Boston, MA, 1993.

[5] P. Kuchment, An overview of periodic elliptic operators, Bulletin AMS **53** (2016), Number 3, 343–414.

Reporter: Oliver Clarke

Participants

Prof. Dr. Dan Abramovich
Department of Mathematics
Brown University
Box 1917
151 Thayer Street
Providence RI 02912
UNITED STATES

Prof. Dr. Klaus Altmann
Institut für Mathematik
Freie Universität Berlin
Arnimallee 6
14195 Berlin
GERMANY

Prof. Dr. Carolina Araujo
Instituto de Matematica Pura e
Aplicada (IMPA)
Estrada Dona Castorina 110
Rio de Janeiro RJ 22460-320
BRAZIL

Andreas Baeuerle
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Victor V. Batyrev
Fachbereich V - Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Christine Berkesch
Department of Mathematics
University of Minnesota
127 Vincent Hall
206 Church Street S. E.
Minneapolis, MN 55455
UNITED STATES

Prof. Dr. Harold Blum
Department of Mathematics
University of Utah
155 SOUTH 1400 EAST, JWB 233
84112 Salt Lake City Utah 84112
UNITED STATES

Marie-Charlotte Brandenburg
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig
GERMANY

Dr. Michel Brion
Laboratoire de Mathématiques
Université de Grenoble Alpes
Institut Fourier, Bureau 43C
100 rue des Maths
38610 Gières Cedex
FRANCE

Dr. Juliette Bruce
Department of Mathematics
University of California, Berkeley
970 Evans Hall
Berkeley CA 94720-3840
UNITED STATES

Dr. Jaroslaw Buczynski
Institute of Mathematics of the Polish
Academy of Sciences
Sniadeckich 8
00-656 Warszawa
POLAND

Prof. Dr. Cinzia Casagrande
Dipartimento di Matematica
Università degli Studi di Torino
Via Carlo Alberto, 10
10123 Torino
ITALY

Prof. Dr. Ana-Maria Castravet
Laboratoire de Mathématiques
Université Paris-Saclay, Versailles
Bâtiment Fermat 3304
45 Avenue des États Unis
78035 Versailles Cedex
FRANCE

Dr. Oliver Clarke
Department of Mathematics
University of Bristol
Woodland Road
Bristol BS8 1UG
UNITED KINGDOM

Prof. Dr. Alessio Corti
Department of Mathematics
Imperial College London
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Alicia Dickenstein
Departamento de Matemática (FCEN)
Universidad de Buenos Aires
Pabellon 1 - Ciudad Universitaria
1428 Buenos Aires EGA
ARGENTINA

Prof. Dr. Sandra Di Rocco
Department of Mathematics
Royal Institute of Technology
Lindstedtsvägen 25
100 44 Stockholm
SWEDEN

Dr. Matej Filip
Institute of Mathematics, Physics
and Mechanics
University of Ljubljana
1000 Ljubljana
SLOVENIA

Julius Giesler
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Christian Haase
Institut für Mathematik
Freie Universität Berlin
Arnimallee 3
14195 Berlin
GERMANY

Prof. Dr. Megumi Harada
Department of Mathematics and
Statistics
McMaster University
1280 Main Street West
Hamilton ON L8S 4K1
CANADA

Prof. Dr. Jürgen Hausen
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Dr. Milena Hering
School of Mathematics
The University of Edinburgh
James Clerk Maxwell Building
Peter Guthrie Tait Road
Edinburgh EH9 3FD
UNITED KINGDOM

Dr. Johannes Hofscheier
School of Mathematical Sciences
The University of Nottingham
University Park
Nottingham NG7 2RD
UNITED KINGDOM

Dr. Nathan Ilten
Department of Mathematics
Simon Fraser University
8888 University Drive
Burnaby BC V5A 1S6
CANADA

Dr. Joachim Jelisiejew
Institute of Mathematics, University of Warsaw
Banacha 2
02-097 Warszawa
POLAND

Prof. Dr. Elana Kalashnikov
Department of Pure Mathematics,
University of Waterloo
200 University Avenue West
Waterloo Ontario, N2L 3G1
CANADA

Prof. Dr. Kalle Karu
Department of Mathematics
University of British Columbia
121-1984 Mathematics Road
Vancouver BC V6T 1Z2
CANADA

Prof. Dr. Alexander M. Kasprzyk
School of Mathematical Sciences
The University of Nottingham
University Park
Nottingham NG7 2RD
UNITED KINGDOM

Prof. Dr. Kiumars Kaveh
Department of Mathematics
University of Pittsburgh
301 Thackery Hall
Pittsburgh PA 15260
UNITED STATES

Prof. Dr. Antonio Laface
Departamento de Matematica
Facultad de Ciencias Fisicas y Matematicas
Universidad de Concepción
Casilla 160-C
4070412 Concepción
CHILE

Dr. Eveline Legendre
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118, route de Narbonne
31062 Toulouse Cedex 9
FRANCE

Prof. Dr. Diane Maclagan
Mathematics Institute
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Christopher Manon
Department of Mathematics
University of Kentucky
715 Patterson Office Tower
Lexington, KY 40506-0027
UNITED STATES

Prof. Dr. Fatemeh Mohammadi
Department of Mathematics
Ghent University
Krijgslaan 281
9000 Gent
BELGIUM

Dr. Leonid Monin
MPI MiS
Inselstraße 22
Leipzig 04103
GERMANY

Prof. Dr. Benjamin Nill
Fakultät für Mathematik
Institut für Algebra und Geometrie
Otto-von-Guericke-Universität
Magdeburg
Postfach 4120
39016 Magdeburg
GERMANY

Prof. Dr. John Christian Ottem
Department of Mathematics
University of Oslo
Blindern
P.O. Box 1053
0316 Oslo
NORWAY

Prof. Dr. Sam Payne
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin, TX 78712-1082
UNITED STATES

Dr. Andrea Petracci
Dipartimento di Matematica
Universita degli Studi di Bologna
Piazza di Porta S. Donato, 5
40127 Bologna
ITALY

Dr. Elisa Postinghel
Dipartimento di Matematica
Università degli Studi di Trento
Via Sommarive, 14
38123 Povo (TN)
ITALY

Dr. Eleonora Romano
Dipartimento di Matematica
Universita di Genova
Via Dodecaneso 35
16146 Genova
ITALY

Martí Salat Moltó
Facultat de Matematiques
Universitat de Barcelona
Departament d'Algebra i Geometria
Gran Via 585
08007 Barcelona
SPAIN

Dr. Matthew Satriano
Department of Pure Mathematics
University of Waterloo
200 University Avenue West
Waterloo ON, N2L 3G1
CANADA

Prof. Dr. Henry K. Schenck
Department of Mathematics
Auburn University
Auburn, AL 36849
UNITED STATES

Prof. Dr. Gregory G. Smith
Department of Mathematics and
Statistics
Queen's University
Jeffery Hall
Kingston ON K7L 3N6
CANADA

Prof. Dr. Frank Sottile
Department of Mathematics
Texas A & M University
College Station, TX 77843-3368
UNITED STATES

Prof. Dr. Bernd Sturmfels
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig
GERMANY

Prof. Dr. Hendrik Süß
Mathematisches Institut
Universität Jena
Ernst-Abbe-Platz 2-4
07743 Jena
GERMANY

Dr. Bernard Teissier
IMJ - PRG
Bâtiment Sophie Germain
Case 7012
8, Place Aurélie Nemours
75205 Paris Cedex 13
FRANCE

Prof. Dr. Michael Temkin
Einstein Institute for Mathematics
The Hebrew University of Jerusalem
Givat Ram
Jerusalem 91904 01
ISRAEL

Prof. Dr. Jenia Tevelev
Department of Mathematics
University of Massachusetts
Lederle Graduate Research Tower
710 North Pleasant Street
Amherst, MA 01003-9305
UNITED STATES

Prof. Dr. Jaroslaw Wisniewski
Instytut Matematyki
Uniwersytet Warszawski
ul. Banacha 2
02-097 Warszawa
POLAND

Prof. Dr. Jaroslaw Wlodarczyk
Department of Mathematics
Purdue University
150 N. University Street
West Lafayette, IN 47907-1395
UNITED STATES

Dr. Milena Wrobel
Institut für Mathematik
Carl-von-Ossietzky-Universität
Oldenburg
Ammerländer Heerstraße 114-118
26129 Oldenburg
GERMANY