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Diophantische Approximationen

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Abstract. This workshop was focused on a large variety of problems which have seen important progress during the last few years, such as extensions and refinements of Schmidt Subspace Theorem, together with new applications, works on the Zilber-Pink conjecture and on unlikely intersections, geometry of numbers, simultaneous Diophantine approximation, theory of heights, continued fractions, and arithmetic dynamics.

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Introduction by the Organizers

The workshop *Diophantische Approximationen* (Diophantine approximations), organised by Yann Bugeaud (Strasbourg), Pietro Corvaja (Udine), Laura DeMarco (Harvard), and Philipp Habegger (Basel) was held April 17th - April 23rd, 2022. There were 59 participants (29 on site and 30 online) with broad geographic representation and a large variety of mathematical backgrounds. Young researchers were very well represented, including among the speakers. Below we briefly recall the topics discussed, thus outlining some of the modern lines of investigation in Diophantine approximation and closely related areas. We refer the reader to the abstracts for more details. In total we had 30 talks, each lasting 35 minutes. The majority of talks were in-person.

Diophantine approximation is a branch of Number Theory that can be described as the study of the solvability of inequalities in integers, though this main theme of the subject is often generalized greatly. Classical examples involve rational approximation to irrational numbers. Topics of current interest in Diophantine approximation include irrationality and transcendence statements, which have been discussed in the talks of Hirata-Kohno, Moshchevitin, and Viola. Badziahin and Poëls described new progress towards a deep understanding of the uniform simultaneous rational approximation to a real number and its successive integral powers. The powerful parametric geometry of numbers, which has been recently introduced by W. M. Schmidt and Summerer, was at the heart of the talk of Roy. Among his results was a new advance on uniform simultaneous rational approximation to real n-tuples. Zudilin spoke of Apéry sequences and their limits, Coons of regular sequences and associated probability measures.

Metric Diophantine approximation has seen great advances during the last decade and was present in the talks of Beresnevich, Ghosh, and Breuillard, who explained an unexpected extension of Schmidt's Subspace Theorem in the context of metric Diophantine approximation. This allows him to recover and generalize the main results of Kleinbock and Margulis on Diophantine exponents of submanifolds. This powerful theorem of W. M. Schmidt was also at the heart of the talks of Fuchs and Mello.

The conjecture of Zilber-Pink on unlikely intersections encompasses many classical results and conjectures from Diophantine geometry including the Mordell and André–Oort Conjectures. These problems have seen spectacular developments within the last two or three years, and a number of the talks addressed work in this direction. Application of André's work on G-functions was at the center of Daw's presentation on the Zilber-Pink Conjecture in the modular setting. Orr presented semi-effective aspects of the Borel Harish-Chandra reduction theory. Gao presented work on the Manin-Mumford Conjecture in a family of abelian varieties, and Kühne spoke on the Bogomolov Conjecture variant of the problem in such a family. Masser explained ramifications of these ideas to elementary integration. Barroero and Dill spoke on problems in mixed characteristic, the latter motivated by an application of Bugeaud-Corvaja-Zannier of Schmidt's Subspace Theorem. Height upper bounds as in Amoroso's talk play an important part in understanding unlikely intersections.

Additional talks illustrated the breadth of disciplines connected to the traditional Diophantine Approximation themes. Demeio and Wilms spoke on arithmetic equidistribution, the former in the setting of arithmetic dynamics and the latter from an Arakelov-geometric point of view. Checcoli and Pazuki each spoke on intrinsic properties of the Weil height. Ostafe and Stewart studied the count of multiplicative dependent elements with bounded height. Capuano addressed the classical continued fraction expansion and how it carries over to the p-adic setting. Finally, Dimitrov presented a recent breakthrough, together with Calegari and Tang, towards the unbounded denominators conjecture.

As an indication of the evolution of the field, we remark that interest in Diophantine equations for their own sake seems to be declining, as they were present only in the talk of Evertse. But the applications of the classical theory to the broader areas of arithmetic geometry, arithmetic dynamics, metric geometry, groups, and dynamical systems show the importance of the topic and its central role in current mathematical research.

Workshop: Diophantische Approximationen

Table of Contents

Abstracts

Apéry sequences and limits: new perspectives

Wadim Zudilin

Apéry's proof of the irrationality of $\zeta(3)$ in 1978 displayed two particular phenomena. To be specific, consider the recurrence equation

$$(n+1)^3v_{n+1} - (2n+1)(17n^2+17n+5)v_n + n^3v_{n-1} = 0 \quad \text{for } n = 1, 2, \dots .$$

Then starting from $u_0 = 1$, $u_1 = 5$, the generated solution $\{u_n\}_{n=0}^{\infty}$, *a priori* a sequence of rational numbers, is in fact integer-valued. Thus, the first — *integrality* — phenomenon is about existence of non-trivial solution $\{u_n\}_{n=0}^{\infty} \subset \mathbb{Z}$. As the equation above has order 2, there is another linearly independent solution $\{v_n\}_{n=0}^{\infty} \subset \mathbb{Q}$, which starts from $v_0 = 0$, $v_1 = 6$ (the choice of 6 is for 'cosmetic purposes'). This solution is not any more integer-valued but the denominators grow exponentially, $\operatorname{lcm}(1,2,\dots,n)^3 v_n \in \mathbb{Z}$ (rather than $n!^3 v_n \in \mathbb{Z}$, which is also true but is much weaker). The second phenomenon is now related to the fact that the quotient v_n/u_n tends to a meaningful number, $\zeta(3)$, and it does at a very good rate! Namely, $u_n\zeta(3) - v_n \to 0$ as $n \to \infty$ (and even $\operatorname{lcm}(1,2,\dots,n)^3(u_n\zeta(3) - v_n) \to 0$ as $n \to \infty$). This limiting value, which is $\zeta(3)$ in our case, is called the Apéry limit, and its meaningfulness is precisely the *Apéry-limit* phenomenon.

In general, the integrality can be replaced with the global boundedness of a solution (meaning that we can scale the equation to get an integer-valued solution), and there are more Apéry limits when the order is more than 2. These Apéry limits, of course, depend on the initial data but in an easily controlable way; in the example above, the switch to a different pair of linearly independent (but rational-valued!) solutions will produce $(a\zeta(3)+b)/(c\zeta(3)+d)$ in place of $\zeta(3)$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$.

My first theme is a general result about properties of recursions for the so-called generalized Franel numbers $u^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^s$, where $s = 1, 2, \dots$ is fixed. Note that $u^{(1)}(n) = 2^n$ and $u^{(2)}(n) = \binom{2n}{n}$, which manifests that each of the sequences satisfies a first-order difference equation. Franel gave explicitly Apéry-type recursions for $s = 3, 4$ in 1894 and 1895. He also conjectured that the recursion has order $\lfloor (s+1)/2 \rfloor$ in general but it took almost a century for the next entry for $s = 5$ to be written up. Finally, a recursion (with polynomial in n coefficients) was algorithmically constructed in the 1990s, and this construction indeed leads to order at most $\lfloor (s+1)/2 \rfloor$.

What about bounds on the order from below? In 2008, it was shown that the order is at least 2 for $s \ge 3$. In our recent work [1] with Armin Straub we demonstrate that any recursion constructed for $u(n) = u^{(s)}(n)$ by *creative telescoping* for the term $\binom{n}{k}^s$, that is,

$$P(n, N)\binom{n}{k}^s = b(n, k+1) - b(n, k) \quad \text{for all } n, k,$$

where $N\colon f(n)\mapsto f(n+1)$ is the shift operator, $P(n,N)\in\mathbb{Z}[n,N]$, has order (= degree in N) at least $\lfloor (s+1)/2\rfloor$. One should take into account that the algorithm of creative telescoping guarantees the existence of a difference operator $P(n,N)$ and $b(n,k)=C(n,k)\binom{n}{k}^s$ with some rational function $C(n,k)$, a *certificate.* Our proof with Straub constructs *explicitly* $\lfloor (s+1)/2\rfloor$ solutions, say $u_0(n),u_1(n),\dots,u_{\lfloor (s-1)/2\rfloor}(n)$, to the same recurrence equation $P(n,N)u(n)=0$ and computes the Apéry limits

$$\lim_{n\to\infty}\frac{u_j(n)}{u(n)}\in\mathbb{Q}\zeta(2j)\quad\text{for } j=0,1,\dots,\left\lfloor\frac{s-1}{2}\right\rfloor.$$

The minimality of the order is then a consequence of the linear independence of the solutions constructed, in turn implied by the linear independence of the powers of π^2 over $\mathbb{Q}$.

My second theme is a far-going variation on Apéry's original work and its extension to the values of the dilogarithmic function $\mathrm{Li}_2(z)$. In joint work [2] with Christoph Koutschan we construct an Apéry-type recursion for the integrals

$$L_n(z)=\int_0^1\int_0^1\frac{x^{n-1/2}(1-x)^{2n-1/2}(1-xz)^{1/2}y^n(1-y)^{n-1/2}}{(x(1-y)+y/z)^{n+1}}\,\mathrm{d}x\,\mathrm{d}y\quad (z^{-1}\in\mathbb{Z}^\times),$$

which are linear combinations (with coefficients in $\mathbb{Q}(z^{-1})$) of

$$\rho_1(z)=\int_0^1\frac{\mathrm{d}x}{\sqrt{x(1-x)(1-xz)}},\ \ \rho_2(z)=\int_0^1\frac{\sqrt{1-xz}\,\mathrm{d}x}{\sqrt{x(1-x)}},\ \ \sigma_1(z)=L_0(z)$$

and

$$\sigma_2(z)=\int_0^1\int_0^1\frac{x^{-1/2}(1-x)^{1/2}(1-xz)^{1/2}(1-y)^{1/2}}{x(1-y)+y/z}\,\mathrm{d}x\,\mathrm{d}y.$$

The construction allows us to isolate any two of the four quantities and realise, for example, σ_1/ρ_1 as an Apéry limit (though with no irrationality implications).

Why would one care about such strange looking numbers? It happens that $\rho_1(z)$ is a period of the corresponding z-member $E: y^2=x(1-x)(1-zx)$ of Legendre's family of elliptic curves, hence related to its (twisted) L-value at 1. The value of $\sigma_1(z)$ when $z^{-1}\in\mathbb{Z}^\times$ is related to the L-value of the same curve at 2 (for most of such z conjecturally, based on Boyd's famous observations for two-variable Mahler measures). In this way our construction leads to rational approximations to the Apéry limit $L(E,2)/(\pi L(E,\chi,1))$ (where the choice of an *odd* quadratic character χ is arbitrary).

My third (and final) theme is about a particular generating function of the Legendre polynomials (themselves generated by $\sum_{n=0}^\infty P_n(y)z^n=1/\sqrt{1-2yz+z^2}$), namely, about

$$F(y,z)=\sum_{n=0}^\infty\binom{2n}{n}P_n(y)^2z^n.$$

What is special about it? The function satisfies a fourth-order linear differential equation, viewed as a function of either y or z, and both equations have (the Zariski closure of) the monodromy group O_4. In similar situations we always get such a function represented as a product of solutions of second-order differential equations, with each solution realised as a hypergeometric ${}_2F_1$ function (or as a period of an elliptic curve). Here we obtain in joint work [3] with Mark van Hoeij and Duco van Straten the following decomposition:

$$F(y,z) = w\, I_+(4z, w^2)\, I_-(4z, w^2),$$

where $w = \sqrt{(1+4z)^2 - 16y^2 z} + 4y\sqrt{-z}$ and

$$I_\pm(u,x) = \frac{1}{\pi}\int_0^1 \frac{1 - uv \pm v\sqrt{2u^2-2u}}{\sqrt{v(1-v)((1-v)(1-u^2v)(1+uv)^2 + x\,v(1-uv)^2)}}\,\mathrm{d}v$$

are generically *hyperelliptic* integrals. Furthermore, for each $u \in \mathbb{C}$, the function $I_\pm(u,x)$ satisfies a second-order differential equation with coefficients from $\mathbb{Q}(u, \sqrt{2u^2-2u})[x]$. Surprisingly enough, such second-order equations (and there are infinitely many of them because of the extra parameter u), not reducible to elliptic integrals, were not recorded in the literature; they are reasonably simple counterexamples to a 1990 conjecture of Dwork (which is already disproven *finitely* many times through examples based on Shimura and Teichmüller curves defined over quadratic extensions of $\mathbb{Q}$).

As an outcome of our proof, the following infinite family of Apéry-type recursions shows up: Define degree $4n$ polynomials $u_n = u_n(t)$ by $u_n = 0$ for $n < 0$, $u_0 = 1$ and

$$\begin{aligned}
&(n+1)^2 u_{n+1} \\
&\quad - 2^2(16(t^4 - 6t^3 - 4t^2 + 6t - 1)(n^2+n) + 4t^4 - 24t^3 - 12t^2 + 20t - 3)u_n \\
&\quad - 2^{11}t(t-1)^3(t+1)(8(t^2+2t-1)n^2 - 2t^2 - 6t + 3)u_{n-1} \\
&\quad + 2^{18}t^2(t-1)^6(t+1)^2(2n+1)(2n-3)u_{n-2} = 0 \quad \text{for } n = 0, 1, 2, \dots .
\end{aligned}$$

Then $u_n \in \mathbb{Z}[t]$.

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Rational approximations to Catalan's constant

Carlo Viola
(joint work with Raffaele Marcovecchio)

About 150 years ago, E. Catalan considered the constant

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}\,.$$

G is supposed to be transcendental, but even the irrationality of G is a famous open problem. In recent years, this problem has been studied by several mathematicians. Yu. V. Nesterenko [1] defined the double integral

$$J = \int_0^1\!\!\int_0^1 \frac{x^{a_1-\frac{1}{2}}(1-x)^{b_1-a_1}\, y^{a_2}(1-y)^{b_2-a_2-\frac{1}{2}}}{(1-xy)^{a_3+1}}\,\mathrm{d}x\,\mathrm{d}y$$

for positive integer parameters a_1, a_2, a_3, b_1, b_2 satisfying

$$b_1 \geq b_2 \geq a_1 \geq a_2 \geq a_3, \qquad a_1 + a_2 + a_3 \leq b_1 + b_2.$$

He proved that J is a linear form in 1 and G with rational coefficients having controlled denominators. Choosing, in the integral J, $a_1 = a_2 = a_3 = n$, $b_1 = b_2 = 2n$, and defining p_n, q_n through

$$4^{2n} d_{2n}^2 J = q_n G - p_n$$

where d_ν denotes the least common multiple of $1, 2, \ldots, \nu$, Nesterenko proved that p_n, $q_n \in \mathbb{Z}$ and

$$(1) \qquad 0 < \left| G - \frac{p_n}{q_n} \right| < q_n^{-0.5242\ldots} \quad \text{for all sufficiently large } n.$$

Nesterenko's sequence (p_n/q_n) is the best sequence of effective rational approximations to G available in the literature, but of course it does not suffice to prove the irrationality of G.

Using a different choice of the parameters in the integral J, Nesterenko [1] also found a sequence $u_n \in \mathbb{Q}$ and a sequence $v_n \in \mathbb{Z}$ such that

$$(2) \qquad 0 < \left| G - \frac{u_n}{v_n} \right| < v_n^{-\frac{11}{20}}.$$

Computer analysis shows that $u_n \in \mathbb{Z}$ for all $n \leq 350$. Since $-11/20 = -0.55 < -0.5242\ldots$, if the conjectural inclusion $u_n \in \mathbb{Z}$ holds for all n, the sequences u_n and v_n would yield an improvement upon the above estimate (1).

In joint work with R. Marcovecchio (to appear) we apply the Rhin–Viola permutation group method [2] to give new effective sequences of integers r_n and s_n satisfying

$$(3) \qquad 0 < \left| G - \frac{r_n}{s_n} \right| < s_n^{-0.6293\ldots}.$$

Since $-0.6293\ldots < -11/20 < -0.5242\ldots$, our sequences r_n and s_n improve both Nesterenko's result (1) and his conjectural estimate (2).

As I remarked in my talk during the conference in Moscow (June 10-14, 2019) dedicated to the 100th anniversary of N. I. Feldman's birth, if we write Nesterenko's integral J in the form

$$J(h,j,k,l,m) = \int_0^1\int_0^1 \frac{x^h(1-x)^j\, y^l(1-y)^k}{(1-xy)^{j+k-m}}\, \frac{\mathrm{d}x\,\mathrm{d}y}{\sqrt{x}\,\sqrt{1-y}\,(1-xy)}$$

where h, j, k, l, m are any nonnegative integers, it is plain that $J(h,j,k,l,m)$ is analogous with the Rhin–Viola double integral related to $\zeta(2)$ (see [2]), except for the crucial difference arising from the square roots in the measure

$$\frac{\mathrm{d}x\,\mathrm{d}y}{\sqrt{x}\,\sqrt{1-y}\,(1-xy)} \tag{4}$$

compared with the measure $\mathrm{d}x\,\mathrm{d}y/(1-xy)$ for $\zeta(2)$. In order to prove (3) we use Cauchy's theorem to express s_n in terms of double contour integrals associated with $J(h,j,k,l,m)$, by suitably getting rid of the two-valued square roots in $\mathbb{C}$. One may try to employ as extensively as possible the methods in [2] leading to irrationality measures of $\zeta(2)$, by writing $J(h,j,k,l,m)$ and the related contour integrals as periods, i.e., by means of substitutions (such as, e.g., $x=u^2$, $y=1-v^2$) transforming them into integrals of rational functions with rational coefficients over domains defined by polynomial inequalities with rational coefficients. To achieve this, it is essential to use one of the 10 birational transformations occurring in the Rhin–Viola group for $\zeta(2)$, namely the involution $(x,y) \longleftrightarrow (X,Y)$ defined by

$$\begin{cases} X = 1-xy \\ Y = \dfrac{1-x}{1-xy}\,, \end{cases} \tag{5}$$

since the measure (4) is easily seen to be an invariant (up to sign) of the involution (5). Accordingly, we get

$$J(h,j,k,l,m) = J(m,l,k,j,h)$$

for all h, j, k, l, m. In order to generate the relevant permutation group acting on the integers h, j, k, l, m, $k+l-h$, $l+m-j$, $m+h-k$, $h+j-l$, $j+k-m$, one has to combine the involution (5) with the Euler integral representation of the hypergeometric function ${}_2F_1$.

For $h=j=k=l=m=0$ Nielsen's formula (see [1], p. 155) yields

$$J(0,0,0,0,0) = 8G.$$

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On sets of exact approximation order

ANISH GHOSH
(joint work with Prasuna Bandi, Debanjan Nandi)

The set of exactly approximable numbers, E_ψ, is the set of real numbers x such that

$$|x - p/q| \leq \psi(q) \text{ infinitely often}$$

and

$$|x - p/q| \geq c\psi(q) \text{ for any } c < 1 \text{ and any } q \geq q_0(c, x).$$

In [2] Bugeaud, resolving a conjecture of Beresnevich, Dickinson and Velani from [1] proved that $\dim(E_\psi) = 2/\lambda$. Here λ denotes the lower order at infinity of $1/\psi$. We develop a general framework to investigate exactly approximable sets. Our results apply in particular to actions of discrete groups of isometries on boundaries of hyperbolic spaces for which there is a well developed theory of Diophantine approximation [3]. In fact we calculate the Hausdorff dimension of exactly approximable sets for any well-distributed set of points in a proper metric space which has a measure satisfying some natural metric properties. Our results also apply to Diophantine approximation on the Heisenberg group, and provide a new proof of Bugeaud's theorem.

Our main result is the following. Let (X, d, μ) be an (α, β)-regular metric measure space. Let $Q \subset X$ be well distributed with respect to the radius function R. Let $\psi : (0, \infty) \to (0, \infty)$ be a non-decreasing function such that $\sum_{\xi \in Q} \left(\frac{\psi(R(\xi))}{R(\xi)}\right)^\alpha$ converges. Then

$$\dim E_\psi(Q, R) = \frac{\alpha}{\lambda(\psi)}.$$

The notion of an (α, β)-regular metric measure space is a natural variation of the notion of Ahlfors regular metric spaces, namely one imposes an additional regularity condition for annuli.

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Measures associated to regular sequences and their characterisation

Michael Coons
(joint work with Michael Baake, James Evans, Neil Mañibo)

A sequence f is k-automatic if and only if its k-kernel,

$$\ker_k(f) := \left\{ (f(k^\ell n + r))_{n \geqslant 0} : \ell \geqslant 0, 0 \leqslant r < k^\ell \right\},$$

is finite. An automatic sequence takes only a finite number of values. A natural generalisation to sequences that can be unbounded was given in the early nineties by Allouche and Shallit [2]; a real sequence f is called *k-regular* if the $\mathbb{R}$-vector space $V_k(f) := \langle \ker_k(f) \rangle_{\mathbb{R}}$ generated by the k-kernel of f is finite-dimensional.

The study of automatic sequences is rich from both number-theoretical and dynamical viewpoints. Much of the number-theoretic literature on automatic sequences mirrors that of the rational-transcendental dichotomies of integer power series proved in the first third of the twentieth century, such as those of Fatou, Carlson and Szegő, and the more recent celebrated result by Adamczewski and Bugeaud [1]. The dynamical literature has focussed on the study of automatic sequences through their related substitution systems [4, 8].

Transitioning to regular sequences, the number-theoretic story is much the same as automatic sequences, mirroring that of rational-transcendental dichotomies. The generalisation of the Cobham–Loxton–van der Poorten Conjecture for regular sequences was proved by Bell, Bugeaud and Coons [5]. But, in contrast to automatic sequences, the study of the long-range order of unbounded regular sequences f, and so also the related spaces $V_k(f)$, is not so straight-forward; neither diffraction nor spectral measures can be associated with them in a natural way.

As a first step of addressing the long-range order of such objects, we [3] introduced a natural probability measure associated with Stern's diatomic sequence. Recently, we generalised this result to apply to a large class of regular sequences [6, 7]. Note that for a regular sequence, there are vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{d\times 1}$, and matrices $\mathbf{A}_0, \dots, \mathbf{A}_{k-1} \in \mathbb{R}^{d\times d}$ such that $f(n) = \mathbf{u}^T \mathbf{A}_{(n)_k} \mathbf{v}$ for all $n \geqslant 0$, where $(n)_k = i_s \cdots i_1 i_0$ is the base-k expansion of n and $\mathbf{A}_{(n)_k} := \mathbf{A}_{i_s} \cdots \mathbf{A}_{i_1} \mathbf{A}_{i_0}$. We call the tuple $f = (\mathbf{u}, \mathbf{A}_0, \dots, \mathbf{A}_{k-1}, \mathbf{v})$ a *linear representation* of f; note that, here, one can reduce to a minimal representation. Set

$$\Sigma_f(N) := \sum_{m=k^N}^{k^{N+1}-1} f(m) \quad \text{and} \quad \mu_N := \frac{1}{\Sigma_f(N)} \sum_{m=0}^{k^{N+1}-k^N-1} f(k^N + m)\, \delta_{m/k^N(k-1)},$$

where δ_x denotes the unit Dirac measure at x. We can view $(\mu_N)_{N\in\mathbb{N}_0}$ as a sequence of probability measures on the 1-torus, the latter written as $\mathbb{T} = [0, 1)$ with addition modulo 1. If the (weak) limit $\mu_f := \lim_{N\to\infty} \mu_N$ exists, we call μ_f the *ghost measure* of the regular sequence f.

Theorem 1. *Let f be a nonnegative real-valued k-regular sequence with reduced representation $g = (\mathbf{w}, \mathbf{B}_0, \dots, \mathbf{B}_{k-1}, \mathbf{x})$. If the spectral radius $\rho(\mathbf{B})$ is the unique simple maximal eigenvalue of $\mathbf{B}$ and there is a linear cone K fixed by each $\mathbf{B}_i$, then $\mu_f = \mu_g$ exists.*

Recall that any finite real Borel measure μ on $\mathbb{T}$ has a *Lebesgue decomposition*; that is, μ is the sum of three mutually singular measures μ_{pp}, μ_{sc} and μ_{ac}, where, with respect to Lebesgue measure λ, μ_{pp} is pure point (the so-called Bragg part), μ_{sc} is singular continuous and μ_{ac} is absolutely continuous.

Theorem 2. *The measure μ_f provided by Theorem 1 is spectrally pure. That is, μ_f is either pure point, or singular continuous, or absolutely continuous.*

Moreover, one can determine this spectral type by inspecting properties of the underlying finite set of matrices. Recall that the *joint spectral radius* of a finite set of matrices is defined by

$$\rho^* = \rho^*(\{\mathbf{B}_0, \dots, \mathbf{B}_{k-1}\}) := \lim_{n\to\infty} \max_{0\leqslant i_1, i_2, \dots, i_n \leqslant k-1} \left\|\mathbf{B}_{i_1}\mathbf{B}_{i_2}\cdots\mathbf{B}_{i_n}\right\|^{1/n},$$

where $\|\cdot\|$ is any (submultiplicative) matrix norm.

Theorem 3. *If $\mathbf{B}_0, \dots, \mathbf{B}_{k-1}$ is a finite set of matrices associated to a ghost measure that exists via Theorem 1, then the measure μ_f is pure point if and only if $\rho^* = \rho$, the measure μ_f is singular continuous if $\rho/k < \rho^* < \rho$, and if $\rho^* = \rho/k$ and there is a $d > 0$ such that $\max_{1\leqslant i_1, i_2, \dots, i_n \leqslant \ell} \left\|\mathbf{B}_{i_1}\mathbf{B}_{i_2}\cdots\mathbf{B}_{i_n}\right\| \leqslant d(\rho^*)^n$, for each $n \geqslant 1$, then μ_f is absolutely continuous.*

As a final remark, we note that in the standard number-theoretic way of understanding regular sequences—via their generating functions subsequent classification in the diffeo-algebraic hierarchy—one usually proves a transcendence result. Here, if one is interested in the structure of regular sequences and more specific questions, a transcendence result is not extremely helpful; it is a negative result and does not provide a structural classification. The upshot in the context we present here is that the trichotomy of spectral type offers a way to view regular sequences which provides an actual structural result. We hope that this added structure can aid in the future understanding of regular sequences, especially regarding questions related to properties of finitely generated semigroups of matrices.

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Effective equidistribution of torsion parameters in elliptic fibrations

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(joint work with Myrto Mavraki)

The work presented here is still in progress. Theorem 1 below should also have an analogue for small points and for non-archimedean valuations, but for the sake of simplicity we expose here only the case of torsion parameters and archimedean valuations.

Let K be a number field with a fixed embedding in $\mathbb{C}$, and $\pi : E \to \mathbb{P}^1_K$ be a nice elliptic surface over $\mathbb{P}^1_K$, i.e. E is a nice K-variety, and the smooth fibers of π are elliptic curves (in particular we assume the existence of a zero-section O). Let P be a non-torsion section of π. We aim at understanding the *torsion parameters* $T(P) \subseteq \mathbb{P}^1(\overline{K})$, i.e. the parameters $t \in \mathbb{P}^1(\overline{K})$ such that P_t is torsion in E_t, and their Galois equidistribution.

Through the so-called Betti map, one may define a positive $(1,1)$-form $\mu = \mathrm{d}\,\beta_1 \wedge \mathrm{d}\,\beta_2$ on $\mathbb{P}^1(\mathbb{C})$ associated to the section P. This $(1,1)$-form is smooth away from the singular parameters of π. An equidistribution result of DeMarco and Mavraki [6, Corollary 1.2] gives as a special case that, for any $\psi \in C^0(\mathbb{P}^1(\mathbb{C}))$ and sequence of torsion parameters $t_n \in \mathbb{P}^1(\overline{K})$ of degree $d_n \to \infty$:

$$\frac{1}{d_n} \sum_{t^\sigma \text{conjugate of } t} \psi(t^\sigma) \to \int \psi \cdot \mu. \tag{6}$$

(Actually in [6] the limit measure is described through other means, better suited to the context. However it is proven in [4, Section 4] that the limit measure that they have is precisely μ for archimedean valuations.)

The proof of their result employs the equidistribution results of Chambert-Loir, Thuillier or Yuan [3, 13, 14] on a well-constructed adelic line bundle on the base (in our case $\mathbb{P}^1_K$) arising from the relative setting. To obtain "the continuity of potentials" of the measure $\mathrm{d}\,\beta_1 \wedge \mathrm{d}\,\beta_2$ (an essential hypothesis in all of the cited works) DeMarco and Mavraki employ some careful computations that rely on Silverman's results for the variation of local heights [10, 11, 12] in elliptic fibrations (see [6, Theorem 1.1]).

Mavraki and I give an explicit error term as follows:

Theorem 1. *The error term in* (6) *is* $\leq C \cdot d_n^{-1/4} \cdot Lip(\psi)$, *where* $Lip(\psi)$ *is the Lipschitz constant of* ψ, *and* C *is a constant depending on* E *and* P.

To prove Theorem 1, we adapt the proof of [6, Corollary 1.2]. Namely, we use a modification of the "quantitive" equidistribution result [7] instead of the "qualitative" equidistribution results [3, 13, 14] used by DeMarco and Mavraki. The need to modify [7] arises because Favre and Rivera-Latelier's result requires that the potentials of the limit measure are Hölder-continuous, while one can show, relying on Silverman's work [10, 11, 12], that the potentials of μ are not. Through Silverman's work one still manages to deduce, however, that they have a modulus of continuity proportional to $\frac{1}{\log d(x,y)}$, where d denotes the distance with respect to a smooth metric on $\mathbb{P}^1(\mathbb{C})$. One can still make Favre and Rivera-Latelier's proof work even in this less controlled case (with the necessary modifications), at the price of the exponent with which d_n appears: the factor $d_n^{-1/2}$ that would come up in the Hölder case is instead substituted by $d_n^{-1/4}$.

As an application of Theorem 1 we prove:

Theorem 2. *Fix a finite set of places S of K. Let $\alpha \in \mathbb{P}^1(\overline{K}) \setminus T(P)$ be of good reduction for π. Then there are only finitely many torsion parameters $\lambda \in T(P)$ that are S-integral with respect to α.*

Analogous theorems in the constant setting of $\mathbb{G}_m$ and of an elliptic curve were already proven by Baker, Ih and Rumely in [2]. Their main ingredient, as is ours, is "logarithmic equidistribution", i.e. an equidistribution result (in the sense of (6)) of Galois orbits of torsion points (or parameters, in our case) where the test function ψ is allowed to have finitely many singularities of logarithmic growth. To prove this logarithmic equidistribution one main obstacle is to make sure that torsion points do not approximate too well any given algebraic point in any given completion. To prove this in the archimedean case, Baker, Ih and Rumely use linear forms in logarithms: Baker's theorem [1, Theorem 3.1, p.22] for $\mathbb{G}_m$, and [5] for elliptic curves. In the non-archimedean case, they just use the discreteness of torsion points. In our setting, for the archimedean places we use the Betti map to translate the problem of bounding accumulation of torsion parameters near α to bounding accumulation of torsion points near P_α on the elliptic curve E_α, and there we use [5] as in [2]. For the non-archimedean places we use the discreteness of torsion parameters on the smooth reduction locus (proven by Lawrence and Zannier [9]). Once the obstacle is overcome, Baker, Ih and Rumely use Serre's open image theorem and the classical "non-singular" equidistribution to deduce the "logarithmic equidistribution". In our relative setting, Theorem 1 plays the role played by Serre's open image theorem in [2].

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On the distribution of algebraic numbers

Robert Wilms

A natural approach to the study of the distribution of algebraic numbers is to consider increasing sequences of finite sets of integer polynomials with bounded degree and bounded norm. We do this for the Bombieri norm by defining

$$\mathcal{P}_{n,r} = \left\{ \sum_{k=0}^{n} a_k X^k \in \mathbb{Z}[X] \;\middle|\; \max_{0\le k\le n} \frac{|a_k|}{\sqrt{\binom{n}{k}}} \le r \right\}.$$

for any positive integer $n \in \mathbb{Z}_{>0}$ and any positive real $r \in \mathbb{R}_{>0}$. We obtain the following equidistribution result for the zeros of the polynomials in these sets.

Theorem 1 ([5]). *For any continuous function $f\colon \mathbb{C} \to \mathbb{C}$ with compact support and any sequence $(r_n)_{n\in\mathbb{Z}_{>0}}$ of positive real numbers satisfying*

$$1 < \liminf_{n\to\infty} r_n^{1/n} \le \limsup_{n\to\infty} r_n^{1/n} < \infty$$

it holds

$$\lim_{n\to\infty} \frac{1}{\#\mathcal{P}_{n,r_n}} \sum_{P\in\mathcal{P}_{n,r_n}\setminus\{0\}} \left| \frac{1}{n} \sum_{\substack{z\in\mathbb{C}\\ P(z)=0}} f(z) - \frac{i}{2\pi}\int_{\mathbb{C}} f(z)\frac{dz d\overline{z}}{(1+|z|^2)^2} \right| = 0.$$

The study of the distribution of algebraic numbers has a long history and there are several equidistribution results for example by Erdős–Turán [3], Bilu [2] and Pritsker [4]. Let us point out the main differences of Theorem 1 to these results.

- Theorem 1 does not require that a height of the zeros tends to 0. For example, Bilu considered the Mahler measure $M(P)$ of a polynomial P instead of its Bombieri norm and his result is restricted to sequences with $\lim_{n\to\infty} M(P_n)^{1/n} = 1$. If P_n is the minimal polynomial of an algebraic number α_n of degree n, its height is given by $h(\alpha_n) = \frac{1}{n}\log M(P_n)$.
- The distribution of the zeros tends to the Fubini–Study measure instead of the uniform measure on $S^1 \subset \mathbb{C}$ as in all other results mentioned.

- Theorem 1 only states that for almost all sequences $(P_n)_{n\in\mathbb{Z}_{>0}}$ of polynomials $P_n \in \mathcal{P}_{n,r_n}$ the distribution of the zero set of P_n tends to the Fubini–Study measure on $\mathbb{C}$. But we do not know a good criterion for an individual sequence $(P_n)_{n\in\mathbb{Z}_{>0}}$ to satisfy this equidistribution property.

We can formulate and proof Theorem 1 in a much broader setting using Arakelov theory. For this purpose, let $f\colon \mathcal{X} \to \operatorname{Spec}(\mathbb{Z})$ be a projective arithmetic variety. That means $\mathcal{X}$ is an integral scheme, f is projective, flat, separated and of finite type and the generic fiber $\mathcal{X}_{\mathbb{Q}}$ is smooth. Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. By a *hermitian metric* on $\mathcal{L}$ we mean a family of hermitian metrics $h = (h_x)_{x\in\mathcal{X}(\mathbb{C})}$ on the fibers $(\mathcal{L}_x)_{x\in\mathcal{X}(\mathbb{C})}$ such that $|s(x)|^2 = h_x(s(x), s(x))$ is a smooth function on any open subvariety $U \subseteq \mathcal{X}_{\mathbb{C}}$ and for all sections $s \in H^0(U, \mathcal{L}_{\mathbb{C}})$. Moreover, we assume that h is compatible with complex conjugation. We call $\overline{\mathcal{L}} = (\mathcal{L}, h)$ a *hermitian line bundle.*

For all $s \in H^0(\mathcal{X}(\mathbb{C}), \mathcal{L}(\mathbb{C})) \setminus \{0\}$ the Poincaré–Lelong formula states

$$\tfrac{\partial\overline{\partial}}{2\pi i} \log |s|^2 = [c_1(\overline{\mathcal{L}})] - \delta_{\operatorname{div}(s)}$$

as currents. For every positive real number $r \in \mathbb{R}_{>0}$ we define the finite set

$$\widehat{H}^0_{\le r}\left(\mathcal{X}, \overline{\mathcal{L}}\right) = \left\{ s \in H^0(\mathcal{X}, \mathcal{L}) \;\middle|\; \sup_{x\in\mathcal{X}(\mathbb{C})} |s(x)| \le r \right\}$$

and analogously $\widehat{H}^0_{<r}\left(\mathcal{X}, \overline{\mathcal{L}}\right)$. We call $\overline{\mathcal{L}}$ *arithmetically ample* if $c_1(\overline{\mathcal{L}}) > 0$, $\mathcal{L}$ is ample and $H^0(\mathcal{X}, \mathcal{L}^{\otimes p})$ is generated by $\widehat{H}^0_{<1}(\mathcal{X}, \overline{\mathcal{L}}^{\otimes p})$ for infinitely many $p \in \mathbb{Z}_{>0}$.

Theorem 2 ([5])**.** *Let $\mathcal{X}$ be a projective arithmetic variety of dimension $d \ge 2$. Let $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ be any arithmetically ample hermitian line bundles on $\mathcal{X}$, $(r_p)_{p\in\mathbb{Z}_{>0}}$ any sequence of positive real numbers and Φ any $(d-2, d-2)$ C^0-form on $\mathcal{X}(\mathbb{C})$.*

(1) *If $\lim_{p\to\infty} r_p^{1/p} = 1$, then it holds*

$$\lim_{p\to\infty} \frac{1}{\#\widehat{H}^0_{\le r_p}\left(\mathcal{X}, \overline{\mathcal{L}}^{\otimes p}\right)} \sum_{s\in\widehat{H}^0_{\le r_p}\left(\mathcal{X},\overline{\mathcal{L}}^{\otimes p}\right)\setminus\{0\}} \frac{1}{p} \int_{\mathcal{X}(\mathbb{C})} |\log|s||\, c_1(\overline{\mathcal{M}})^{d-1} = 0.$$

(2) *If $1 \le \liminf_{p\to\infty} r_p^{1/p} \le \limsup_{p\to\infty} r_p^{1/p} < \infty$, then it holds*

$$\lim_{p\to\infty} \frac{1}{\#\widehat{H}^0_{\le r_p}\left(\mathcal{X}, \overline{\mathcal{L}}^{\otimes p}\right)} \sum_{s\in\widehat{H}^0_{\le r_p}\left(\mathcal{X},\overline{\mathcal{L}}^{\otimes p}\right)\setminus\{0\}} \left| \frac{1}{p} \int_{\operatorname{div}(s)(\mathbb{C})} \Phi - \int_{\mathcal{X}(\mathbb{C})} \Phi \wedge c_1(\overline{\mathcal{L}}) \right| = 0.$$

We get Theorem 1 from Theorem 2 (2) by setting $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}}$ and $\mathcal{L} = \mathcal{M} = \mathcal{O}(1)$ equipped with the Fubini–Study metric multiplied by $e^{-\epsilon}$ for a sufficiently small $\epsilon > 0$. In Theorem 2 we can deduce (2) from (1) by Stokes theorem. To apply (1) one may restrict to a subsequence such that $r_p^{1/p}$ converges to some $\tau \ge 1$ and multiply the metric of $\overline{\mathcal{L}}$ with $\frac{1}{\tau}$ to get $\lim_{p\to\infty} r_p^{1/p} = 1$.

The proof of Theorem 2 (1) is based on a distribution result of sections in complex analysis by Bayraktar, Coman and Marinescu [1]. It allows us to deduce a similar result for the sections in convex sets $K_p \subseteq H^0(\mathcal{X}, \mathcal{L}^{\otimes p})_{\mathbb{R}}$. To get from

K_p to the lattice points $K_p \cap H^0(\mathcal{X}, \mathcal{L}^{\otimes p})$ we use geometry of numbers. One of the main difficulties is to prove: If two sequences of sections $s_p, s'_p \in H^0(\mathcal{X}, \mathcal{L}^{\otimes p})_{\mathbb{C}}$ satisfy $\lim_{p\to\infty} \sup_{x\in\mathcal{X}(\mathbb{C})} |s_p - s'_p|^{1/p} < 1$, then the vanishing of the limit of the integral in Theorem 2 (1) is equivalent for the sequences $(s_p)_{p\in\mathbb{Z}_{>0}}$ and $(s'_p)_{p\in\mathbb{Z}_{>0}}$.

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Padé approximation for hypergeometric functions and S-unit equations

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(joint work with Sinnou David and Makoto Kawashima)

We study values of the generalized hypergeometric G-functions, by Padé approximations of Type II. We provide a new general linear independence criterion for the values of these functions at several distinct points, over a given algebraic number field of any degree. Our criterion shows the linear independence of values at algebraic points of contiguous hypergeometric functions [5], as was previously proven for values at algebraic numbers of generalized Lerch functions with different shifts [3] [4]. Kawashima and A. Poëls [8] applied these Padé approximants to obtain new precise irrational exponents by combining with an *effective* version of the Poincaré-Perron theorem and relying on parametric geometry of numbers. Their sharp estimates for cubic binomial functions give an improvement for the number of the solutions to the S-unit equation [7].

Let K be an algebraic number field of any degree over $\mathbb{Q}$. Let r be an integer with $r \geq 2$ and $a_1, \dots, a_r, b_1, \dots, b_{r-1} \in \mathbb{Q} \setminus \{0\}$, not being negative integers. We define the generalized hypergeometric function by

$$ {}_rF_{r-1}\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_{r-1} \end{matrix} \,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_{r-1})_k} \frac{z^k}{k!} $$

where $(a)_k$ is the Pochhammer symbol: $(a)_0 = 1$, $(a)_k = a(a+1)\cdots(a+k-1)$. For a rational number x, let us define

$$ \mu(x) = \prod_{\substack{q:\text{prime} \\ q|\text{den}(x)}} q^{q/(q-1)} \quad . $$

Suppose neither a_i nor a_i+1-b_j be strictly positive integers for $1 \le i \le r$ and $1 \le j \le r-1$. Under the assumption, it is proven essentially by Yu. Nesterenko, that these functions are linearly independent over the function field.

Consider now $\alpha_1, \dots, \alpha_m \in K \setminus \{0\}$ pairwise distinct and put $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_m)$. Let v_0 be a place of K and define a real number

$$\begin{aligned} V_v(\boldsymbol{\alpha}, \beta) = \log|\beta|_{v_0} - rm\mathrm{h}(\boldsymbol{\alpha}, \beta) - (rm+1)\log\|\boldsymbol{\alpha}\|_{v_0} + rm\log\|(\boldsymbol{\alpha}, \beta)\|_{v_0} \\ - \left(rm\log(2) + r\left(\log(rm+1) + rm\log\left(\frac{rm+1}{rm}\right)\right)\right) \\ - \sum_{j=1}^{r}\left(\log\mu(a_j) + 2\log\mu(b_j) + \frac{\mathrm{den}(a_j)\mathrm{den}(b_j)}{\varphi(\mathrm{den}(a_j))\varphi(\mathrm{den}(b_j))}\right) \quad , \end{aligned}$$

where φ is the Euler's totient function, the norm $\|\cdot\|_v$ denotes that of the supremum and h is the logarithmic absolute height.

Theorem 1 (contiguous hypergeometric function)**.** *Assume $V_{v_0}(\boldsymbol{\alpha}, \beta) > 0$. Then the $rm+1$ numbers :*

$${}_rF_{r-1}\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_{r-1} \end{matrix} \,\middle|\, \frac{\alpha_i}{\beta}\right) \quad , \quad {}_rF_{r-1}\left(\begin{matrix} a_1+1, \dots, \dots, \dots, a_r+1 \\ b_1+1, \dots, b_{r-s}+1, b_{r-s+1}, \dots, b_{r-1} \end{matrix} \,\middle|\, \frac{\alpha_i}{\beta}\right) \quad ,$$

for $1 \le i \le m$, $1 \le s \le r-1$ and 1 are linearly independent over K.

The assumption $V_{v_0}(\boldsymbol{\alpha}, \beta) > 0$ requires the points $\dfrac{\alpha_i}{\beta}$ are sufficiently closed to the origin, with respect to the metric v_0 (then it depends on K). The ingredient relies on our *term-wise formal* construction of Padé approximants with a new non-vanishing property for the generalized Wronskian of Hermite-type.

In [8], Kawashima and Poëls show the irrationality exponent of the values of certain hypergeometric functions containing both G-functions and E-functions, using Padé approximants constructed above, together with an effective version of the Poincaré-Perron theorem. Consider for instance the binomial function:

$$f(z) = \sum_{k=0}^{\infty} \frac{(-\omega)_k}{k!} \frac{1}{z^{k+1}} = \frac{1}{z} \cdot {}_2F_1\left(\begin{matrix} -\omega, 1 \\ 1 \end{matrix} \,\middle|\, \frac{1}{z}\right) = \frac{1}{z}\left(1 - \frac{1}{z}\right)^{\omega}.$$

For a given $z \in \mathbb{C}$ with $|z| > |\alpha|$, denote by $\rho_1(\alpha, z) \le \rho_2(\alpha, z)$ the moduli of the two roots $2z - \alpha \pm 2\sqrt{z^2 - z\alpha}$ of the characteristic polynomial

$$P(X) = X^2 - 2(2z - \alpha)X + \alpha^2.$$

The condition $|z| > |\alpha|$ implies that $\rho_1(\alpha, z) \ne \rho_2(\alpha, z)$. Let $\omega \in \mathbb{Q} \setminus \mathbb{Z}$ and $\beta \in \mathbb{Q}$ with $|\beta| > 1$. For $n \in \mathbb{Z}$, $n \ge 0$, put

$$\nu_n(\omega) = \prod_{\substack{q:\text{prime} \\ q|\mathrm{den}(\omega)}} q^{n + \lfloor n/(q-1) \rfloor},$$

$$G_n(\omega) = \mathrm{GCD}\left(\nu_n(\omega)\binom{n+k-1}{k}\binom{n-\omega-1}{n-k},\ \nu_n(\omega)\binom{n+k'}{k'}\binom{n+\omega}{n-1-k'}\right)_{\substack{0 \le k \le n \\ 0 \le k' \le n-1}},$$

$$\Delta = \Delta(\omega, \beta) = \nu(\omega) \cdot \mathrm{den}(\beta) \cdot \limsup_{n\to\infty} G_n(\omega)^{-1/n},$$
$$Q = \rho_2(1, \beta) \cdot \Delta, \quad \text{and } E = \rho_2(1, \beta) \cdot \Delta^{-1}.$$

They obtain the following result.

Theorem 2 (binomial function)**.** *Assume* $E > 1$. *The number* $(1 - 1/\beta)^\omega \notin \mathbb{Q}$ *has the irrationality exponent* $\mu_{\mathrm{eff}} \leq 1 + \dfrac{\log(Q)}{\log(E)}$. *In particular,*

$$\mu_{\mathrm{eff}}((1 - 1/\beta)^\omega) \leq 1 + \frac{\log \rho_2(1, \beta) + \log \nu(\omega) + \log \mathrm{den}(\beta)}{\log \rho_2(1, \beta) - \log \nu(\omega) - \log \mathrm{den}(\beta)}.$$

We then apply their Padé approximations for the cubic binomial function in the non-archimedean and algebraic case, to give an improvement for the number of the solutions due to J. -H. Evertse in 1984 [6], by adapting a refined estimate for $G_n(1/3)$, recently obtained by M. A. Bennett [2], instead of that in [1].

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Bost's slopes method and arithmetic holonomy bounds

Vesselin Dimitrov

(joint work with Frank Calegari, Yunqing Tang)

Let $D(0, 1) := \{|z| < 1\}$ denote the open unit complex disc. We introduce a Diophantine problem based on the following datum:

- A nonnegative integer $\sigma \in \mathbb{N}_0$.
- A meromorphic mapping $\varphi : \overline{D(0, 1)} \to \mathbb{CP}^1$ taking $\varphi(0) = 0$.
- A formal power series $x(t) \in t + t^2\mathbb{Q}[[t]]$.

We consider a formal power series $p \in \mathbb{Q}[[x]]$ which we assume to combine an *integrality property*

(7) $$x^* p \in \mathbb{Z}[[t]]$$

and an *analytic continuation property*

$$\varphi^* p \in \mathcal{M}(\overline{D(0,1)}) \quad \text{is meromorphic on the closed unit disc.}$$

Then we define a $\mathbb{Z}[p]$-module $\mathcal{H}(\sigma, \varphi, x(t))$ to consist of those formal functions $f \in \mathbb{Q}[[x]]$ that combine—similarly to p—a *relaxed integrality property*

$$x^* f = \sum_{n=0}^{\infty} a_n \frac{t^n}{[1, \ldots, n]^\sigma}, \quad a_n \in \mathbb{Z}, \quad \forall n \in \mathbb{N},$$

and again an *analytic continuation property*

$$\varphi^* f \in \mathcal{M}(\overline{D(0,1)}) \quad \text{is meromorphic on the closed unit disc.}$$

It turns out that $\mathcal{H}(\sigma, \varphi, x(t))$ has cardinality the continuum if $|\varphi'(0)| \leq e^\sigma$; there is not much else that can be said in this case, and we are not interested in it. In contrast, when $|\varphi'(0)| > e^\sigma$, the $\mathbb{Z}[p]$-module $\mathcal{H}(\sigma, \varphi, x(t))$ has a finite rank, and more precisely:

Theorem 1. *Suppose $|\varphi'(0)| > e^\sigma$. Then $\mathcal{H}(\sigma, \varphi, x(t))$ has $\mathbb{Z}[p]$-rank at most*

$$\inf_{\lambda \in \mathbb{R}} \left\{ \frac{2T(e^{-\lambda} \varphi^* p) + \lambda}{\log |\varphi'(0)| - \sigma} \right\},$$

where

$$T(g) = T(1, g) := \int_0^1 \log^+ |g(e^{2\pi i t})| \, dt + \sum_{\rho \in D(0,1)} \operatorname{ord}_\rho^+ (1/g) \cdot \log(1/|\rho|)$$

is the Nevanlinna characteristic function *of a meromorphic function $g : \overline{D(0,1)} \to \mathbb{C}$ at the limiting radius $r = 1$.*

In my talk, I outlined a simple approach to this basic holonomy bound whose applications simultaneously cover:

(a) With the parameter choice $\lambda = 0$, and then the setting $\sigma := 0$, $t := q^{1/N} = e^{\pi i \tau / N}$, $p(x) := x^N$, $\varphi : D(0,1) \to \mathbb{C} \setminus 16^{-1/N} \mu_N$ a (suitable restriction of) the universal covering map, and $x(t) = \sqrt[N]{\lambda(t^N)/16}$ with

$$\lambda(q) := \Big(\sum_{n \text{ odd}} q^{n^2} \Big/ \sum_{n \text{ even}} q^{n^2} \Big)^4$$

the modular lambda function, this result recovers the holonomy bound by $O\big(\int_0^1 \log^+ |\varphi^N(e^{2\pi i t})| \, dt \big/ \log |\varphi'(0)|\big)$ which suffices as a substitute for our § 2 in our recent paper [1] proving the "unbounded denominators conjecture."

(b) With $x(t) := t$, $p(x) := x$, and the parameter choice $\lambda := \sup_{D(0,1)} \log|\varphi|$, this $\mathbb{Z}[x]$-rank bound reduces to

$$\leq \frac{\sup_{D(0,1)} \log|\varphi|}{\log|\varphi'(0)| - \sigma},$$

which in turn with $\sigma = 5$ and with φ equal to the restriction of the $X_0(2)$ Hauptmodul $q\prod_{n=1}^{\infty}(1+q^n)^{24}$ to the q-disc of diameter $[-1, 1/7]$ turns out sufficient for an irrationality proof of the 2-adic avatar of $\zeta(5) \in \mathbb{Q}_2$.

The proof of the theorem follows the line of Bost's slopes inequality

$$\widehat{\deg}(\overline{E}_D) \leq \sum_{n=0}^{\infty} \operatorname{rank}(E_D^{(n)}/E_D^{(n+1)})\big(\mu_{\max}(\overline{F^{(n)}/F^{(n+1)}}) + h(\psi_D^{(n)})\big) \tag{8}$$

attached to the *injective* linear $\psi_D : E_D^{\bullet} \hookrightarrow F^{\bullet}$ of (split) filtered hermitian vector bundles over $\operatorname{Spec}\mathbb{Z}$, where $F := \mathbb{Q}[[t]]$ is the polynomial ring graded by the $t = 0$ order of vanishing and given the trivial metric $\|t^n\| := 1$ to its one-dimensional graded quotient pieces, and E_D is the free $\mathbb{Z}$-module $y_1\,\mathbb{Z}[p(x)]_{<D} + \cdots + y_m\,\mathbb{Z}[p(x)]_{<D}$ of rank $r = mD$, metricized by the positive-definite quadratic form $\langle y_i\,p(x)^k, y_j\,p(x)^l\rangle := \delta_{i,j}\delta_{k,l}\cdot e^{\lambda(k+l)}$ (using here the Kronecker delta notation). Finally, ψ_D is the *evaluation map* sending $y_i \mapsto f_i(x)$ for an m-tuple of presumed $\mathbb{Q}(p(x))$-linearly independent elements $f_1(x), \ldots, f_m(x) \in \mathcal{H}(\sigma, \varphi, x(t))$.

Here the $\mu_{\max}$ (maximal slope) terms are zero, and the rank filtration terms $r_D^{(n)} := \operatorname{rank}(E_D^{(n)}/E_D^{(n+1)}) \in \{0,1\}$ are constrained by $\sum_{n=0}^{\infty} r_D^{(n)} = r = mD$ and, hence, $\sum_{n=0}^{\infty} n\,r_D^{(n)} \geq \binom{mD}{2}$. The proof of the theorem then falls out from (8) in the $D \to \infty$ asymptotic by computing $\widehat{\deg}(\overline{E}_D) = -m\lambda D^2/2 + o(D^2)$ and the respective finite and Archimedean evaluation height pieces as follows:

$$h_{\text{fin}}(\psi_D^{(n)}) \leq \sigma \log[1,\ldots,n] = \sigma n + o(n),$$
$$h_{\infty}(\psi_D^{(n)}) \leq D\,T(e^{-\lambda}\,\varphi^* p) - n\,\log|\varphi'(0)| + o(D).$$

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Some unlikely intersection problems over finite fields

Fabrizio Barroero

(joint work with Laura Capuano, László Mérai, Alina Ostafe, Min Sha)

Multiplicative dependence of algebraic numbers and of rational functions has been extensively studied in recent years from various points of view. In particular, a result of Bombieri, Masser and Zannier [1] in the context of unlikely intersections in tori says that, given $\varphi_1, \ldots, \varphi_m \in \overline{\mathbb{Q}}(X)$ multiplicatively independent modulo constants, there are at most finitely many $\alpha \in \overline{\mathbb{Q}}$ such that $\varphi_1(\alpha), \dots, \varphi_m(\alpha)$ satisfy two independent multiplicative relations, i.e., the set

$$\mathcal{S}_1 = \left\{ \alpha \in \overline{\mathbb{Q}} : \prod_{i=1}^{m} \varphi_i(\alpha)^{k_i} = \prod_{i=1}^{m} \varphi_i(\alpha)^{\ell_i} = 1 \text{ for some linearly independent } (k_1, \dots, k_m), (\ell_1, \dots, \ell_m) \in \mathbb{Z}^m \right\}$$

is finite. In defining $\mathcal{S}_1$, we implicitly exclude the poles and zeros of $\varphi_1, \dots, \varphi_m$.

The independence modulo constants condition has been replaced by the necessary hypothesis that $\varphi_1, \dots, \varphi_m$ are multiplicatively independent by Maurin [4]. An effective proof of this result has been obtained by Bombieri, Habegger, Masser and Zannier [2].

For a prime p, we note that any element of $\overline{\mathbb{F}}_p^*$ has finite order, so Maurin's finiteness result in characteristic 0 does not hold in full generality in positive characteristic. In this context, Masser proposed some conjecture in positive characteristic putting more restrictive hypotheses on the rational functions in order to recover Maurin's finiteness result [4], and proved it for $n = 3$ [3, Theorem 1.1].

For positive integers $K, L \geq 1$ and a prime number p, we define the set

$$\mathcal{A}_\varphi(p, K, L) = \left\{ \alpha \in \overline{\mathbb{F}}_p : \prod_{i=1}^{m} \varphi_i(\alpha)^{k_i} = \prod_{i=1}^{m} \varphi_i(\alpha)^{\ell_i} = 1 \text{ for some linearly independent } (k_1, \dots, k_m), (\ell_1, \dots, \ell_m) \in \mathbb{Z}^m, \max_{i=1,\dots,m} |k_i| \leq K, \max_{i=1,\dots,m} |\ell_i| \leq L \right\}.$$

In defining $\mathcal{A}_\varphi(p, K, L)$, we implicitly assume that the reductions of the rational functions $\varphi_1, \dots, \varphi_m$ modulo p are all well-defined, and also we implicitly exclude the poles and zeros of the reductions of $\varphi_1, \dots, \varphi_m$ modulo p.

We have the following results.

Theorem 1. *Let $\varphi = (\varphi_1, \dots, \varphi_m) \in \mathbb{Q}(X)^m$ whose components are non-zero multiplicatively independent rational functions. Then, there exists an effectively computable constant c_1 depending only on φ such that for arbitrary integers $K, L \geq 1$, and any prime $p > \exp(c_1 KL)$, we have*

$$\#\mathcal{A}_\varphi(p, K, L) \leq \#\mathcal{S}_1,$$

where $\#\mathcal{S}_1$ is effectively upper bounded, and the elements of $\mathcal{A}_\varphi(p, K, L)$ come from the reduction modulo p of elements of $\mathcal{S}_1$.

Taking $m = 2$ and $K = L = \lfloor c_3 (\log p)^{1/2} \rfloor$ for some effectively computable constant c_3 depending only on $\varphi = \varphi_1$ and $\varrho = \varphi_2$, we directly obtain:

Corollary 2. *Let $\varphi, \varrho \in \mathbb{Q}(X)$ be non-zero rational functions such that φ, ϱ are multiplicatively independent. Then, there are three effectively computable constants c_1, c_2, c_3 depending only on φ, ϱ such that for any prime $p > c_1$, for all but c_2*

elements $\alpha \in \overline{\mathbb{F}}_p$ *we have*

$$\max\{\mathrm{ord}_p(\varphi(\alpha)), \mathrm{ord}_p(\varrho(\alpha))\} \geq c_3(\log p)^{1/2}.$$

We give a brief sketch of the proof of Theorem 1. The set of $\alpha \in \overline{\mathbb{Q}}$ such that $\prod_{i=1}^m \varphi_i(\alpha)^{k_i} = 1$ is the zero set of some polynomial P_k, therefore the elements of $\mathcal{S}_1$ are the union of the solutions of the systems $P_k = P_l = 0$ for varying independent $k, l \in \mathbb{Z}^m$, i.e., the union of the sets of zeroes of the resultants $\mathrm{Res}(P_k, P_l)$. On the other hand, if p divides $\mathrm{Res}(P_k, P_l)$ we have an element of $\mathcal{A}_\varphi(p, K, L)$, for appropriate K and L. We produce estimates for the degree and the height of the polynomials P_k and P_l and this gives a bound on $|\mathrm{Res}(P_k, P_l)|$ in terms of k and l. If p is larger than this bound, it cannot divide $\mathrm{Res}(P_k, P_l)$ if the latter is not zero. Thus, for p large compared to K and L, the elements of $\mathcal{A}_\varphi(p, K, L)$ must all come from the reduction of elements of $\mathcal{S}_1$.

Similar results can be obtained for powers of E^n and $E^n \times \mathbb{G}_m^m$, for some elliptic curve E defined over $\mathbb{Q}$.

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Yet another application of the Subspace theorem to Diophantine problems involving power sums

Clemens Fuchs

Roth's theorem says that for any algebraic number α and arbitrary $\varepsilon > 0$ there is a positive constant $c(\alpha, \varepsilon)$, depending on ε and the approximated number α, such that for all rational integers p, q with $q > 0$ and $p/q \neq \alpha$ the lower bound

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c(\alpha, \varepsilon)}{q^{2+\varepsilon}}$$

holds. We are interested in a moving version in which α *is replaced by a sequence depending on an integer* n for which we want an explicit constant c depending on n and ε (at the cost of enlarging the power of q).

To define our α's, we denote by $\mathcal{E}$ the ring of power sums whose characteristic roots belong to the set of positive integers $\mathbb{N}$ and whose coefficients belong to $\mathbb{Q}$, i.e. functions $G : \mathbb{N} \to \mathbb{Q}$ given by

$$G(n) = b_1 c_1^n + \cdots + b_h c_h^n$$

with $c_1, \ldots, c_h \in \mathbb{N}$ and $b_1, \ldots, b_h \in \mathbb{Q}$. We are interested in approximating $\alpha : \mathbb{N} \to \overline{\mathbb{Q}}$, which is a root of $f(G, y) = 0$ resp. $f(G^{(0)}, \ldots, G^{(d)}, y) = 0$ (i.e. for each n

we choose $\alpha(n)$ such that $f(G(n),\alpha(n))=0$ resp. $f(G^{(0)}(n),\dots,G^{(d)}(n),\alpha(n))=0$), where $G,G^{(0)},\dots,G^{(d)}\in\mathcal{E}$.

Using the Subspace theorem Corvaja and Zannier [2] proved that for power sums $G^{(1)}$ and $G^{(2)}$ a positive constant k exists such that under suitable assumptions for all but finitely many positive integers n and for integers p,q with q positive and not too large we have

$$\left|\frac{G^{(1)}(n)}{G^{(2)}(n)}-\frac{p}{q}\right|\geq\frac{1}{q^k}e^{-n\varepsilon}.$$

They used this approximation result to deduce that under the given conditions the length of the continued fraction for $G^{(1)}(n)/G^{(2)}(n)$ tends to infinity. A similar approximation statement for the square root of a power sum was given shortly afterwards by Bugeaud and Luca [1] in order to prove, again under suitable assumptions, that the length of the period of the continued fraction expansion of $\sqrt{G(n)}$ tends to infinity. Scremin [4] combined these two approximation results and showed, giving a final answer, that under assumptions which are very similar to those below for three power sums $G^{(1)},G^{(2)},G^{(3)}$ the lower bound

$$\left|\frac{\sqrt{G^{(1)}(n)}+G^{(2)}(n)}{G^{(3)}(n)}-\frac{p}{q}\right|\geq\frac{1}{q^k}e^{-n\varepsilon}$$

holds with finitely many exceptions. By another application of the Subspace theorem we get:

Theorem (F.-Heintze [3]**).** *Let $f\in\mathbb{Q}[x,y]$ be absolutely irreducible and $G\in\mathcal{E}$ with $G(n)=b_1c_1^n+\cdots+b_hc_h^n$, where $c_1>c_2>\cdots>c_h>0$ and $c_1>1$. Moreover, let α be a root of $f(G,y)=0$. Then there exists a positive integer s such that for each fixed $r\in\{0,1,\dots,s-1\}$ we get the subsequent approximation result. There exists an integer $k\geq 2$ such that for any $\varepsilon>0$ satisfying $\varepsilon<\min\{1/(2(s+2)),1/(2k)\}$ we have the following: If there is no power sum $\eta\in\mathcal{E}$ such that*

$$|\alpha(sm+r)-\eta(m)|\leq e^{-(sm+r)\varepsilon}$$

for infinitely many values of m, then for all but finitely many values of m and for $p,q\in\mathbb{Z}$ with $0<q<e^{(sm+r)\varepsilon}$ it holds

$$\left|\alpha(sm+r)-\frac{p}{q}\right|>\frac{1}{q^k}e^{-(sm+r)\varepsilon}.$$

We remark that the integer s does only depend on the polynomial f. The integer k depends on the power sum G, the polynomial f as well as on the integers s and r, but it is independent of n. In general k will be much larger than 2, but since we get the (weaker) lower bound $q^{-k}e^{-(sm+r)\varepsilon}>e^{-(sm+r)(k+1)\varepsilon}=e^{-(sm+r)\varepsilon'}$ the concrete value of k is not that important. The theorem does only say something about small values of q, which is not that restrictive since the bound in the theorem is not significant for large values of q.

A similar theorem holds if α is assumed to be a root of $f(G^{(0)},\dots,G^{(d)},y)=0$ or, more precisely (and equivalently under the conditions we work in), $F(g_1^n,\dots,g_h^n,\alpha(n))=0$, where $F(x_1,\dots,x_h,y)=l_d(x_1,\dots,x_h)y^d+\cdots+l_0(x_1,\dots,$

x_h) for linear polynomials $l_0, \dots, l_d \in \mathbb{Q}[x_1, \dots, x_h]$ and where $g_1, \dots, g_h$ are rational numbers satisfying $1 > g_1 > \cdots > g_h > 0$, where we now (additionally) assume that $l_d(0, \dots, 0) \neq 0$ and that $F(0, \dots, 0, y)$ has neither a multiple nor a rational zero (which automatically excludes the existence of η in this setup) as a polynomial in y.

This gives results in the direction of the Final remark (b) in the paper of Corvaja and Zannier [2]. However, our original motivation was to consider families of Thue equations $f(x, y) = c$ with an irreducible form $f \in \mathcal{E}[x, y]$ of degree ≥ 3 and with $c \in \mathbb{Z}, c \neq 0$ and to show that under suitable conditions *if there are infinitely many* $(n, x_n, y_n) \in \mathbb{N} \times \mathbb{Z}^2$ *such that* $f(n)(x_n, y_n) = c$, *then there are power sums* G *and* H *and an arithmetic progression* $\mathcal{P}$ *such that* $(n, x_n, y_n) = (n, G(n), H(n))$ *for all* $n \in \mathcal{P}$. Unfortunately, the bound from the theorem above is not strong enough to get the proof.

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Multiplicatively dependent vectors of algebraic numbers

Cameron L. Stewart

Let n be a positive integer, G be a multiplicative group and let $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n)$ be in G^n. We say that $\boldsymbol{\nu}$ is multiplicatively dependent if there is a non-zero vector $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ for which

$$\boldsymbol{\nu}^{\mathbf{k}} = \nu_1^{k_1} \cdots \nu_n^{k_n} = 1. \tag{9}$$

We denote by $M_n(G)$ the set of multiplicatively dependent vectors in G^n.

For instance, the set $M_n(\mathbb{C}^*)$ of multiplicatively dependent vectors in $(\mathbb{C}^*)^n$ is of Lebesgue measure zero, since it is a countable union of sets of measure zero. Further, if we fix an exponent vector $\mathbf{k}$ the subvariety of $(\mathbb{C}^*)^n$ determined by (9) is an algebraic subgroup of $(\mathbb{C}^*)^n$.

We shall be interested in counting the number of multiplicatively dependent n-tuples whose coordinates are algebraic numbers of fixed degree, or within a fixed number field, and bounded height.

Equivalently we shall count n-tuples of algebraic numbers in a fixed algebraic number field, or of fixed degree, and given height which occur in some proper algebraic subgroup of the algebraic group G_m^n, where G_m is the multiplicative group of an algebraic closure of $\mathbb{Q}$.

For any positive integer n, we denote by $L_{n,K}(H)$ the number of multiplicatively dependent n-tuples whose coordinates are algebraic integers of height at most H, and we denote by $L^*_{n,K}(H)$ the number of multiplicatively dependent n-tuples whose coordinates are algebraic numbers of height at most H.

With Pappalardi, Sha, and Shparlinski, we proved in 2018 that if K is a number field of degree d over $\mathbb{Q}$ and n is an integer with $n \geq 2$ then $L_{n,K}(H)$ is asymptotic to $C(n,K)H^{d(n-1)}(\log H)^{r(n-1)}$ where $C(n,K)$ is an explicitly given positive number which depends on n and K and r is the rank of the group of units in the ring of algebraic integers of K. We also proved that $L^*_{n,K}(H)$ is asymptotic to $C_1(n,K)H^{2d(n-1)}(\log H)^{r(n-1)}$ where $C_1(n,K)$ is an explicitly given positive number which depends on n and K.

It is natural to ask how the multipicatively dependent n-tuples are distributed in $\mathbb{R}^n$ or in $\mathbb{C}^n$.

With Konyagin, Sha and Shparlinski we have recently proved the following result. Suppose that $n \geq 2$ and let K be a number field. If the ring of algebraic integers of K is different from $\mathbb{Z}$ then the set of multiplicatively dependent vectors in $\mathbb{R}^n$ whose coordinates are from the elements of the ring of algebraic integers of K in $\mathbb{R}$ is dense in $\mathbb{R}^n$.

Let $\|\mathbf{x}\|$ be the Euclidean norm of $\mathbf{x} = (\mathbf{x_1}, \dots, \mathbf{x_n}) \in \mathbb{R}^{\mathbf{n}}$, that is,

$$\|\mathbf{x}\| = \sqrt{\mathbf{x_1^2} + \dots + \mathbf{x_n^2}}.$$

In order to study the distribution of multiplicatively dependent vectors with integer coordinates we define , for $H > 1$,

$$\rho_n(H;\mathbb{Z}) = \sup_{\substack{\mathbf{x}\in\mathbb{R}^{\mathbf{n}} \\ \|\mathbf{x}\|\leq \mathbf{H}}} \inf_{\mathbf{v}\in\mathbf{M_n}(\mathbb{Z})} \|\mathbf{x}-\mathbf{v}\|.$$

With Konyagin, Sha and Shparlinski we have recently proved the following result. For $n \geq 3$

$$H/(\log H)^{C_0(n)} \ll \rho_n(H;\mathbb{Z}) \ll H\frac{(\log\log H)^{n-1}}{(\log H)^{n-2}},$$

where $C_0(n)$ is a positive number which is effectively computable in terms of n.

Integer matrices with a given characteristic polynomial and multiplicative dependence of matrices

Alina Ostafe

(joint work with Igor Shparlinski)

For a positive integer n let $\mathcal{M}_n(\mathbb{Z})$ denote the set of all $n\times n$ matrices. Furthermore, for a real $H \geq 1$ we use $\mathcal{M}_n(\mathbb{Z};H)$ to denote the set of matrices

$$A = (a_{ij})_{i,j=1}^n \in \mathcal{M}_n(\mathbb{Z})$$

with integer entries of size $|a_{ij}| \leq H$.

We say that an s-tuple of matrices $(A_1, \dots, A_s)$ is *multiplicatively dependent* if there is a non-zero vector $(k_1, \dots, k_s) \in \mathbb{Z}^s$ such that

$$A_1^{k_1} \dots A_s^{k_s} = I_n, \tag{10}$$

where I_n is the $n \times n$ identity matrix.

In particular, our motivation to study multiplicatively dependent matrices comes from recent work on multiplicatively dependent integers (and also algebraic integers), see [5, 4, 9]. The matrix version of this problem is however of very different spirit and requires different methods. The most obvious distinction between the matrix and scalar cases is of course the non-commutativity of matrix multiplication. In particular, the property of multiplicatively dependence may change if the entries of $(A_1, \dots, A_s)$ are permuted. Another important distinction is the lack of one of the main tools of [5], namely the existence and uniqueness of prime number factorisation.

We note that the notion of multiplicative dependence given by (10) is also motivated by the notion of *bounded generation*. Namely, we says that a subgroup $\Gamma \subseteq \mathrm{GL}_n(\mathbb{Z})$ is boundedly generated if for some $A_1, \dots, A_s \in \mathrm{GL}_n(\mathbb{Z})$ we have

$$\Gamma = \{A_1^{k_1} \dots A_s^{k_s} : \ k_1, \dots, k_s \in \mathbb{Z}\},$$

see [1] and references therein.

In this work in progress we use a different approach to establish nontrivial upper and lower bounds on the cardinality of the set $\mathcal{N}_{n,s}(H)$ of multiplicatively dependent s-tuples $(A_1, \dots, A_s) \in \mathcal{M}_n(\mathbb{Z}; H)^s$.

Furthermore, the non-commutativity of matrices suggests yet another variation of the above questions. Namely, we say that an s-tuple $(A_1, \dots, A_s) \in \mathcal{M}_n(\mathbb{Z}; H)^s$ is *free* if

$$A_{i_1}^{\pm 1} \cdots A_{i_L}^{\pm 1} \neq I_n$$

for any nontrivial word in $A_1^{\pm 1}, \dots, A_s^{\pm 1}$, that is, a word without occurrences of the form $A_i A_i^{-1}$, of any length $L \geq 1$. Unfortunately we do not have nontrivial upper bounds on the number of non-free s-tuples $(A_1, \dots, A_s) \in \mathcal{M}_n(\mathbb{Z}; H)^s$. However we can estimate the number of such s-tuples with the additional condition that if

$$A_{i_1}^{\pm 1} \cdots A_{i_L}^{\pm 1} = I_n, \quad i_1, \dots, i_L \in \{1, \dots, s\}, \tag{11}$$

then for at least one $i = 1, \dots, s$ the ± 1 exponents of A_i in (11) do not sum up to zero.

Some of our estimates depend on the quality of an upper bound on the number $R_n(H; f)$ of matrices $A \in \mathcal{M}_n(\mathbb{Z}; H)$ with a given characteristic polynomial $f \in \mathbb{Z}[X]$, which is of course a question of independent interest.

If $f \in \mathbb{Z}[X]$ is a monic irreducible polynomial, Eskin, Mozes and Shah [2] give an asymptotic formula for a variant $\widetilde{R}_n(H; f)$ of $R_n(H; f)$, where the matrices are ordered by the L_2-norm rather than by the L_∞-norm, but this should not be very essential and plays no role in our context as we are only interested in upper bounds for $R_n(H; f)$. Namely, by [2, Theorem 1.3],

$$\widetilde{R}_n(H; f) = (C(f) + o(1)) H^{n(n-1)/2}. \tag{12}$$

with some constant $C(f) > 0$ depending on f, when a monic irreducible polynomial $f \in \mathbb{Z}[X]$ is fixed. Unfortunately, this result of [2] as well as its variants obtained via different approaches in [6, 10] (see also the result of [7] in the case when f splits completely over $\mathbb{Q}$) are not sufficient for our purposes because we need an upper bound which:

- holds for arbitrary $f \in \mathbb{Z}[X]$, which is not necessary irreducible;
- is uniform with respect to the coefficients of f.

The results of [2, 6, 10] lead us towards the following conjecture.

Conjecture 1. *Uniformly over polynomials f we have*

$$R_n(H; f) \le H^{n(n-1)/2+o(1)},$$

as $H \to \infty$.

Unconditionally, only counting matrices with a given determinant and applying [8, Theorem 4], we instantly obtain

$$R_n(H; f) \le H^{n^2-n+o(1)}. \tag{13}$$

Furthermore, we also show that Conjecture 1 holds for $n = 2$. This motivates us formulating our results conditionally up on the following.

Assumption 2. *There is some $\gamma_n \ge 0$ such that uniformly over polynomials f we have*

$$R_n(H; f) \le H^{n^2-n-\gamma_n+o(1)},$$

as $H \to \infty$.

Clearly the value $\gamma_n = n(n-1)/2$ corresponds to Conjecture 1 while by (13) it always holds with $\gamma_n = 0$.

In particular, we are able to prove that Assumption 2 holds with some explicit sequence $\gamma_n > 0$ (with $\gamma_3 = \gamma_4$=1). In fact, for $n \ge 4$ we bound the cardinality of the following set,

$$\mathcal{S}_n(H; d, t) = \{A \in \mathcal{M}_n(\mathbb{Z}; H) : \ \det A = d \text{ and } \mathrm{Tr} A = t\}.$$

We denote $S_n(H; d, t) = \#\mathcal{S}_n(H; d, t)$, and obviously one has

$$R_n(H; f) \le S_n(H; d, t),$$

where $d = (-1)^n f(0)$ and $-t$ is the coefficient of X^{n-1} in f.

Theorem 3. *For $n \ge 4$, uniformly over d and t we have*

$$S_n(H; d, t) \ll \begin{cases} H^{11+o(1)}, & \text{if } n = 4, \\ H^{n^2-n-\gamma_n}, & \text{if } n \ge 5, \end{cases}$$

where $\gamma_n = \frac{2n-4}{(n-3)^3}$.

The proof of this result is based on some techniques from the geometry of numbers and relies on some ideas which we have borrowed from the work of Katznelson [3] and then adjusted to our settings.

We can state now one of our main result, which gives nontrivial upper and lower bounds on the cardinality of the set $\mathcal{N}_{n,s}(H)$. For this we define

$$w(n) = \max \left\{ \sum_{j=1}^{h} \varphi(k_j)^2 : \ n = \sum_{j=1}^{h} \varphi(k_j) \right\},$$

where φ is the Euler function and the maximum is taken over all such representations of all possible lengths $h \geq 1$.

Theorem 4. *Under Assumption 2, we have*

$$H^{sn^2-n-\min\{n,\gamma_n\}+o(1)} \geq \#\mathcal{N}_{n,s}(H) \gg H^{(s-1)n^2+w(n)/2-n/2}.$$

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Parametric geometry of numbers over a number field and extension of scalars

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(joint work with Anthony Poëls)

1. Parametric geometry of numbers. We fix an integer $n \geq 2$, a number field K and a non-trivial place w of K. Let $d = [K : \mathbb{Q}]$ denote the degree of K over $\mathbb{Q}$. For each non-trivial place v of K, we normalize the absolute value $|\ |_v$ so that it extends one of the usual absolute values on $\mathbb{Q}$. We denote by K_v the completion of K with respect to that absolute value and by $d_v = [K_v : \mathbb{Q}_v]$ its local degree.

Then the product formula reads $\prod_v |a|_v^{d_v} = 1$ for each non-zero $a \in K$, and the (absolute) Weil height of a non-zero point $\mathbf{x} = (x_1, \dots, x_n) \in K^n$ is given by

$$H(\mathbf{x}) = \prod_v \|\,\mathbf{x}\,\|_v^{d_v/d} \quad \text{where} \quad \|\,\mathbf{x}\,\|_v = \begin{cases} \left(\sum |x_i|_v^2\right)^{1/2} & \text{if } v \mid \infty, \\ \max |x_i|_v & \text{else.} \end{cases}$$

For a non-zero point $\boldsymbol{\xi} \in K_w^n$, we further set

$$D_{\boldsymbol{\xi}}(\mathbf{x}) := |\,\mathbf{x} \cdot \boldsymbol{\xi}\,|_w^{d_w/d} \prod_{v \neq w} \|\,\mathbf{x}\,\|_v^{d_v/d} \quad \text{and} \quad D^*_{\boldsymbol{\xi}}(\mathbf{x}) := \|\,\mathbf{x} \wedge \boldsymbol{\xi}\,\|_w^{d_w/d} \prod_{v \neq w} \|\,\mathbf{x}\,\|_v^{d_v/d},$$

where the dot represents the standard bilinear form on K_w^n. In view of the product formula, the three numbers $H(\mathbf{x})$, $D_{\boldsymbol{\xi}}(\mathbf{x})$ and $D^*_{\boldsymbol{\xi}}(\mathbf{x})$ depend only on the class of $\mathbf{x}$ in the projective space $\mathbb{P}(K^n)$.

For each $j = 1, \dots, n$ and each $q \geq 0$, define $L_{\boldsymbol{\xi},j}(q)$ to be the smallest $t \geq 0$ for which the conditions

$$H(\mathbf{x}) \leq e^t \quad \text{and} \quad D_{\boldsymbol{\xi}}(\mathbf{x}) \leq e^{t-q}$$

admit at least j solutions $\mathbf{x} \in K^n$ that are linearly independent over K. Then, essentially all the information that we would like to know about approximation over K to the point $\boldsymbol{\xi}$ is encoded by the map $\mathbf{L}_{\boldsymbol{\xi}} : [0, \infty) \to \mathbb{R}^n$ given by

$$\mathbf{L}_{\boldsymbol{\xi}}(q) = (L_{\boldsymbol{\xi},1}(q), \dots, L_{\boldsymbol{\xi},n}(q)).$$

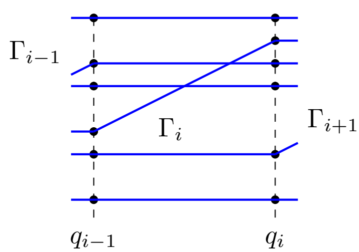
We extend to this setting the theory initiated by Schmidt and Summerer in [5].

Theorem A. *Modulo the additive group of bounded function from $[0, \infty)$ to $\mathbb{R}^n$, the set of maps $\mathbf{L}_{\boldsymbol{\xi}}$ attached to points $\boldsymbol{\xi} \in K_w^n$ having linearly independent coordinates over K coincides with the set of proper n-systems defined as follows.*

A proper n-system is a continous map $\mathbf{P}\colon [0, \infty) \to \mathbb{R}^n$ that has components $\mathbf{P}(q) = (P_1(q), \dots, P_n(q))$ satisfying the following properties:

- (S1) $0 \leq P_1(q) \leq \cdots \leq P_n(q)$ and $P_1(q) + \cdots + P_n(q) = q$ for each $q \geq 0$;
- (S2) there is an unbounded sequence $0 = q_0 < q_1 < q_2 < \cdots$ in $[0, \infty)$ such that, over each subinterval $[q_{i-1}, q_i]$ with $i \geq 1$, the union of the graphs of $P_1, \dots, P_n$ decomposes into horizontal line segments and one line segment Γ_i of slope 1 which all project down to $[q_{i-1}, q_i]$;
- (S3) for each $i \geq 1$, the line segment Γ_i ends strictly above the point where Γ_{i+1} starts, on the vertical line $q = q_i$;
- (S4) P_1 is unbounded.

As the reader will observe from the picture of a 6-system on the right, the graphs of the individual components $P_1, \dots, P_n$ of a proper n-system are more complicated than their union.



When $K = \mathbb{Q}$ and $\boldsymbol{\xi} \in \mathbb{R}^n$, Theorem A reduces to the main result of [4]. This is because any non-zero point of $\mathbb{Q}^n$ is proportional to a primitive integer point $\mathbf{x}$ of $\mathbb{Z}^n$. For such a point, we have $H(\mathbf{x}) = \|\,\mathbf{x}\|$ and $D_{\boldsymbol{\xi}}(\mathbf{x}) = |\mathbf{x}\cdot\boldsymbol{\xi}|$ and so $L_{\boldsymbol{\xi},j}(q)$ is simply the logarithm of the j-th minimum of the symmetric convex body $\mathcal{C}_{\boldsymbol{\xi}}(q)$ of $\mathbb{R}^n$ consisting of the points $\mathbf{x} \in \mathbb{R}^n$ with $\|\,\mathbf{x}\| \le 1$ and $|\,\mathbf{x}\cdot\boldsymbol{\xi}\,| \le e^{-q}$.

In the general case, the components of $\mathbf{L}_{\boldsymbol{\xi}}$ differ by a bounded function from the logarithms of the successice minima of a similarly defined one-parameter family of adelic convex bodies of $K_{\mathbb{A}}^n$ where $K_{\mathbb{A}}$ denotes the ring of adèles of K. Following [4, 5], our proof in [2] uses the adelic version of Minkowski's convex body theorem due to MacFeat and Bombieri-Vaaler, the adelic version of Mahler's theory of compound bodies due to Burger, and constructions over the ring of S-integers of K where S consists of w and all the archimedean places of K.

2. Exponents of approximation. For each non-zero $\boldsymbol{\xi} \in K_w^n$, we define $\omega(\boldsymbol{\xi})$ (resp. $\widehat{\omega}(\boldsymbol{\xi})$) to be the supremum of all ω for which the conditions

$$H(\mathbf{x}) \le Q \quad \text{and} \quad D_{\boldsymbol{\xi}}(\mathbf{x}) \le Q^{-\omega}$$

have a non-zero solution $\mathbf{x} \in K^n$ for arbitrarily large values of Q (resp. for each large enough Q). We also define the dual exponents $\lambda(\boldsymbol{\xi})$ (resp. $\widehat{\lambda}(\boldsymbol{\xi})$) to be the supremum of all λ for which the conditions

$$H(\mathbf{x}) \le Q \quad \text{and} \quad D^*_{\boldsymbol{\xi}}(\mathbf{x}) \le Q^{-\lambda}$$

have a non-zero solution $\mathbf{x} \in K^n$ for arbitrarily large values of Q (resp. for each large enough Q). When $K = \mathbb{Q}$ and $\boldsymbol{\xi} \in \mathbb{R}^n$, we may restrict to primitive points $\mathbf{x} \in \mathbb{Z}^n$ and we recover the usual exponents of approximation.

For an n-system $\mathbf{P} = (P_1, \dots, P_n)$ whose difference with $\mathbf{L}_{\boldsymbol{\xi}}$ is bounded, we have

$$\liminf_{q\to\infty} \frac{P_1(q)}{q} = \liminf_{q\to\infty} \frac{L_{\boldsymbol{\xi},1}(q)}{q} = \frac{1}{\omega(\boldsymbol{\xi})+1},$$

with similar formulas for the other three exponents, hence the following result.

Corollary. *The spectrum of the four exponents* $(\lambda, \widehat{\lambda}, \omega, \widehat{\omega})$*, namely the set of quadruples* $(\lambda(\boldsymbol{\xi}), \widehat{\lambda}(\boldsymbol{\xi}), \omega(\boldsymbol{\xi}), \widehat{\omega}(\boldsymbol{\xi}))$ *attached to points* $\boldsymbol{\xi} \in K_w^n$ *with* K*-linearly independent coordinates is the same for any choice of* K *and* w.

In particular, Jarník's identity $1/\widehat{\lambda}(\boldsymbol{\xi}) - 1 = 1/(\widehat{\omega}(\boldsymbol{\xi}) - 1)$ holds for any triple $\boldsymbol{\xi} \in K_w^3$ with linearly independent coordinates over K.

3. Extension of scalars. Suppose that the place w of K has relative degree one over $\mathbb{Q}$, namely that $K_w = \mathbb{Q}_\ell$ for the place ℓ of $\mathbb{Q}$ induced by w. Choose a basis $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$ of K over $\mathbb{Q}$. For any $\boldsymbol{\xi} \in K_w^n$ with linearly independent coordinates over K, the point

$$\Xi = \boldsymbol{\alpha} \otimes \boldsymbol{\xi} = (\alpha_1\boldsymbol{\xi}, \dots, \alpha_d\boldsymbol{\xi}) \in K_w^{dn} = \mathbb{Q}_\ell^{dn}$$

has linearly independent coordinates over $\mathbb{Q}$. We say that it is obtained from $\boldsymbol{\xi}$ by extending scalars from $\mathbb{Q}$ to K. Then the maps $\mathbf{L}_{\boldsymbol{\xi}}$ and $\mathbf{L}_{\Xi}$ are linked as follows.

Theorem B. *With the above notation and hypotheses, we have*

$$\sup_{q\geq 0}|L_{\Xi,d(i-1)+j}(dq)-L_{\boldsymbol{\xi},i}(q)|<\infty \quad \textit{for } i=1,\dots,n \textit{ and } j=1,\dots,d.$$

This is obtained by applying the underlying principle behind the alternative proof by Jeff Thunder of the adelic version of Minkowski's theorem based on the usual Minkowski's theorem in [6]. As a corollary, this yields formulas relating the various exponents of approximation of $\boldsymbol{\xi}\in K_w^n$ to those of $\Xi\in\mathbb{Q}_\ell^{dn}$. In particular, we find that

$$d\Big(\frac{1}{\widehat{\lambda}(\boldsymbol{\xi})}+1\Big)=\frac{1}{\widehat{\lambda}(\Xi)}+1. \tag{14}$$

In [1], Bel showed that the supremum of the numbers $\widehat{\lambda}(1,\xi,\xi^2)$ where ξ runs through the elements of K_w that are transcendental over K is $1/\gamma\simeq 0.618>1/2$, where $\gamma=(1+\sqrt{5})/2$ denotes the golden ratio. Combining this with (14) yields the first examples of very singular points on algebraic curves defined over $\mathbb{Q}$ of degree greater than 2.

Theorem C. $\sup\Big\{\widehat{\lambda}\big((\boldsymbol{\alpha},\xi\boldsymbol{\alpha},\xi^2\boldsymbol{\alpha}),\mathbb{Q},\ell\big)\ ;\ \xi\in\mathbb{Q}_\ell\setminus\overline{\mathbb{Q}}\Big\}=\dfrac{1}{d\gamma^2-1}>\dfrac{1}{3d-1}.$

For example, if we take $K=\mathbb{Q}(\sqrt{2})\subset\mathbb{R}$ and $\boldsymbol{\alpha}=(1,\sqrt{2})$, then the supremum of the numbers $\widehat{\lambda}\big(1,\sqrt{2},\xi,\sqrt{2}\xi,\xi^2,\sqrt{2}\xi^2\big)$ with $\xi\in\mathbb{R}\setminus\overline{\mathbb{Q}}$ is $1/(2\gamma^2-1)\simeq 0.236>1/5$. Note that, for the same values of ξ, the supremum of $\widehat{\lambda}\big(1,\sqrt{2},\xi\big)$ is $1/\gamma>1/2$ by [3]. So, we wonder if the supremum of $\widehat{\lambda}\big(1,\sqrt{2},\xi,\xi^2\big)$ is greater than 1/3.

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Diophantine exponents and Khintchine's theorem

Victor Beresnevich

(joint work with Lei Yang)

Let $\psi:(0,+\infty)\to(0,1)$ be decreasing. The point $\mathbf{y}=(y_1,\dots,y_n)\in\mathbb{R}^n$ is said to be ψ-*approximable* if there exist infinitely many $(q,p_1,\dots,p_n)\in\mathbb{N}\times\mathbb{Z}^n$ such that

$$\max_{1\leq i\leq n}|qy_i-p_i|<\psi(q)\,.$$

Let $\mathcal{S}_n(\psi)$ be the set of ψ-approximable points in $\mathbb{R}^n$. If $\psi(q) = q^{-v}$ for some $v \in \mathbb{R}$, we write $\mathcal{S}_n(v)$ for $\mathcal{S}_n(\psi)$. Given $\mathbf{y} \in \mathbb{R}^n$, $v_n(\mathbf{y}) := \sup\{v > 0 : \mathbf{y} \in \mathcal{S}_n(v)\}$ is called the *exponent* of simultaneous Diophantine approximations of $\mathbf{y}$.

The talk presented at the workshop described the main results of the paper [1] which addresses the problem of generalising Khintchine's theorem and the Jarník-Besicovitch theorem to submanifolds in $\mathbb{R}^n$.

The Jarník-Besicovitch theorem:

$$\dim_H \mathcal{S}_n(v) = \frac{n+1}{v+1} \qquad \text{if } v \geq 1/n\,,$$

where $\dim_H$ *denotes Hausdorff dimension.*

Khintchine's theorem: *Given any monotonic* ψ*,*

$$\mathcal{L}_n\big(\mathcal{S}_n(\psi) \cap [0,1]^n\big) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q)^n < \infty\,, \\ 1 & \text{if } \sum_{q=1}^{\infty} \psi(q)^n = \infty\,, \end{cases}$$

where $\mathcal{L}_n$ *stands for* n*-dimensional Lebesgue measure.*

Given a smooth map $\mathbf{f} = (f_1, \dots, f_n) : U \to \mathbb{R}^n$ defined on an open ball $U \subset \mathbb{R}^d$, let

$$\mathcal{S}_{\mathbf{f}}(\psi) := \mathbf{f}^{-1}\big(\mathcal{S}_n(\psi)\big) := \big\{(x_1, \dots, x_d) \in U : \mathbf{f}(x_1, \dots, x_d) \in \mathcal{S}_n(\psi)\big\}\,.$$

Main problem: *Generalise Khintchine's theorem and the Jarník-Besicovitch theorem to smooth submanifolds in* $\mathbb{R}^n$*. In other words, replace* $\mathcal{S}_n(\psi)$ *and* $\mathcal{S}_n(v)$ *in the above two theorems by* $\mathcal{S}_{\mathbf{f}}(\psi)$ *and* $\mathcal{S}_{\mathbf{f}}(v)$ *for suitably chosen smooth maps* $\mathbf{f}$*.*

We will be interested in non-degenerate maps $\mathbf{f}$. For simplicity we will assume that $\mathbf{f}$ is analytic. In this case $\mathbf{f}$ is non-degenerate if and only if $1, f_1, \dots, f_n$ are linearly independent over $\mathbb{R}$. Recall that Kleinbock and Margulis proved in 1998 that for any nondegenerate map $\mathbf{f} : U \to \mathbb{R}^n$, $\mathcal{L}_d\big(\mathcal{S}_{\mathbf{f}}(v)\big) = 0$ if $v > 1/n$. Thus the above problem aims to refine this celebrated result to a Khintchine-type theorem and a Jarník-Besicovitch type theorem. One of the main results in [1] is a full Khintchine type theorem for convergence for arbitrary nondegenerate submanifolds of $\mathbb{R}^n$. Note that the divergence case was known earlier for analytic manifolds and more generally. Regarding a Jarník-Besicovitch type theorem, we obtain the following refitment of the Khintchine type theorem for Hausdorff s-measures (the Khintchine case corresponds to $s = d$).

Theorem 1 (Theorem 2.8 in [1])**.** *Let* $n > d \geq 1$ *be integers,* $m = n - d$*,* $s > 0$*,* $l \in \mathbb{N}$*,* $\psi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ *be monotonic,* $U \subset \mathbb{R}^d$ *be an open ball and* $\mathbf{f} : U \to \mathbb{R}^n$ *be nondegenerate such that* $\mathcal{H}^s\left(\{\mathbf{x} \in U : \mathbf{f} \text{ is } l\text{–degenerate at } \mathbf{x}\}\right) = 0$*. Suppose that*

$$\sum_{q=1}^{\infty} q^n \Big(\frac{\psi(q)}{q}\Big)^{s+m} < \infty \tag{15}$$

and

$$\sum_{t=1}^{\infty} \left(\frac{\psi(e^t)}{e^{\frac{t}{2}}}\right)^{s-d} (\psi(e^t)^n e^{\frac{3t}{2}})^{-\frac{1}{d(2l-1)(n+1)}} < \infty\,. \tag{16}$$

Then

$$\mathcal{H}^s(\mathcal{S}_{\mathbf{f}}(\psi)) = 0\,, \tag{17}$$

where $\mathcal{H}^s$ is s-dimensional Hausdorff measure.

Amongst various consequences of this result is the following corollary regarding the spectrum of the exponent of simultaneous approximations to $x, x^2, \dots, x^n$:

$$\lambda_n(x) := v_n(x, \dots, x^n) = \sup\{v > 0 : (x, x^2, \dots, x^n) \in \mathcal{S}_n(v)\}\,.$$

Corollary 2. *For every $n \geq 3$ there is an explicitly computable number δ_n satisfying*

$$\frac{1}{2n^2+6n} < \delta_n < \frac{1}{2n^2+5n}$$

such that for any $\lambda \in \left[\frac{1}{n}, \frac{1}{n} + \frac{\delta_n}{n}\right]$ we have that

$$\dim_H\{x \in \mathbb{R} : \lambda_n(x) = \lambda\} = \frac{n+1}{\lambda+1} - n + 1\,.$$

This contributes to resolving two problems: one by Bugeaud and Laurent (2007) and the other by Bugeaud (2010) on the spectrum of the exponent λ_n.

The key ingredient in the proof of Theorem 1 is a new technique of 'major and minor arcs' based on geometric and dynamical ideas. In particular, we establish sharp upper bounds for the number of rational points of bounded height lying near 'major arcs' and give explicit exponentially small bounds for the measure of 'minor arcs'. The latter uses a result of Bernik, Kleinbock and Margulis.

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Finiteness conditions for $\mathfrak{P}$-adic continued fractions over number fields

Laura Capuano

(joint work with Nadir Murru and Lea Terracini)

The classical continued fraction algorithm provides an integer sequence $[a_0, a_1, \dots]$ that represents a real number α_0 by means of the following recursive algorithm:

$$\begin{cases} a_n = \lfloor \alpha_n \rfloor \\ \alpha_{n+1} = \dfrac{1}{\alpha_n - a_n} \quad \text{if } \alpha_n - a_n \neq 0, \end{cases} \tag{18}$$

for all $n \geq 0$, where $\lfloor \cdot \rfloor$ denotes the integral part of a real number. The Euclidean algorithm ensures that, for classical continued fractions, the procedure eventually stops if and only if α_0 is a rational number.

Motivated by this property, Rosen [8] posed the problem of finding more general definitions of continued fraction expansions characterizing all the elements of an algebraic number field K by means of finite expansions and providing approximations of elements not in the field by means of elements in K, giving explicit constructions in the case $K = \mathbb{Q}(\sqrt{5})$. This question has been then studied more deeply in [7], where the authors studied the number fields of the form $\mathbb{Q}(\beta)$ where β is a Perron number.

The problem of Rosen can be naturally translated into the context of p–adic numbers. In this context, however, there is no natural definition of a p–adic continued fraction, since there is no canonical definition for a p-adic floor function. The two main definitions of a p–adic continued fraction algorithm are due to Browkin [1] and Ruban [9]; they are both based on the definition of a p–adic floor function

$$s(\alpha) = \sum_{n=k}^{0} x_n p^n \in \mathbb{Q}, \quad \text{where} \quad \alpha = \sum_{n=k}^{\infty} x_n p^n \in \mathbb{Q}_p,$$

where the x_n's are the representatives modulo p in the interval $(-p/2, p/2)$ for Browkin definition and in the interval $[0, p-1]$ for Ruban definition. It has been proved that rational numbers have always finite Browkin continued fraction expansion [2] and finite or eventually periodic Ruban continued fraction expansion [6].

We are interested here in the p-adic analogue of Rosen question. Given a number field K and a prime ideal $\mathfrak{P}$ in its ring of integers $\mathcal{O}_K$, lying over an odd prime p. Let $\mathcal{M}_K$ be a set of representatives for the places of K. For every rational prime q and every $v \in \mathcal{M}_K$ above q let K_v be the completion of K w.r.t. the v-adic valuation and $\mathcal{O}_v$ be its valuation ring; we put $d_v = [K_v : \mathbb{Q}_q]$, and let $|\cdot|_v$ denote the unique extension of $|\cdot|_q$ to K_v.

Let $v_0 \in \mathcal{M}_K$ be the place corresponding to $\mathfrak{P}$. We define

$$\mathcal{O}_{K,\{v_0\}} = \{\alpha \in K \mid |\alpha|_v \leq 1 \text{ for every non archimedean } v \neq v_0 \text{ in } \mathcal{M}_K\}.$$

Definition 1. A $\mathfrak{P}$*-adic floor function* for K is a function $s : K_{v_0} \to K$ such that

a) $|\alpha - s(\alpha)|_{v_0} < 1$ for every $\alpha \in K_{v_0}$;
b) $|s(\alpha)|_v \leq 1$ for every non archimedean $v \in \mathcal{M}_K \setminus \{v_0\}$;
c) $s(0) = 0$;
d) $s(\alpha) = s(\beta)$ if $|\alpha - \beta|_{v_0} < 1$.

The choice of a $\mathfrak{P}$-adic floor function amounts to choose a set $\mathcal{Y}_s$ of representatives of the cosets of $\mathfrak{P}\mathcal{O}_{v_0}$ in K_{v_0} containing 0 and contained in $\mathcal{O}_{K,\{v_0\}}$.

We shall call the data $\tau = (K, \mathfrak{P}, s)$ (or $(K, \mathfrak{P}, \mathcal{Y}_s)$) a *type*.

In the case where $\mathfrak{P}$ is principal, there is a more natural way of defining a floor function associated to $\mathfrak{P}$. Indeed, let $\pi \in \mathcal{O}_K$ be generator and let $\mathcal{R}$ be a complete set of representatives of $\mathcal{O}_K/\mathfrak{P}$ containing 0. Then, every $\alpha \in K_{v_0}$ can be expressed uniquely as a Laurent series $\alpha = \sum_{j=-n}^{\infty} c_j \pi^j$, where $c_j \in \mathcal{R}$ for every j. It is possible to define a $\mathfrak{P}$-adic floor function by

$$s(\alpha) = \sum_{j=-n}^{0} c_j \pi^j \in K.$$

We shall denote the types $\tau = (K, \mathfrak{P}, s)$ obtained in this way by $\tau = (K, \pi, \mathcal{R})$, and we will call them *special types*.

Given a type $\tau = (K, \mathfrak{P}, s)$, we can consider the continued fractions associated to τ. Namely, given $\alpha \in K_{v_0}$, we can apply the classical algorithm (18) with s in place of the classical floor function. Notice that, for every $n \geq 0$, $a_n \in \mathcal{Y}_s$, and $|a_n|_{v_0} > 1$ for every $n \geq 1$. As proved in [3], this is a well posed definition of continued fractions; indeed, if we consider the sequence of truncated fractions $Q_n = a_0 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}$, this converges $\mathfrak{P}$-adically to $\alpha \in K_{v_0}$.

We are interested in giving necessary and sufficient conditions on τ in order to ensure that every element of K have finite continued fraction expansion of type τ. In this case, say that the type τ satisifes the Finiteness Condition Property (CFF). Moreover, we say that K satisfies the $\mathfrak{P}$-adic Continued Fraction Finiteness Property (CFF) if there is a type $\tau = (K, \mathfrak{P}, s)$ satisfying the CFF property.

Firstly, we have a strong necessary condition for CFF:

Proposition 2. [3, Proposition 7.1 and Corollary 7.2] *Assume that the field K satisfies the $\mathfrak{P}$-adic* CFF *property. Then, the ideal class group of K is cyclic generated by $[\mathfrak{P}]$. In particular:*

- *if $\mathfrak{P}$ is principal, then $\mathcal{O}_K$ is a* PID*;*
- *if K satisfies the $\mathfrak{P}$-adic* CFF *property for all but finitely many prime ideals $\mathfrak{P}$, then $\mathcal{O}_K$ is a* PID.

Moreover, in [3] we proved a sufficient condition for a type τ to satisfy the CFF involving the $\mathfrak{P}$-adic absolute values of θ on the elements in the image of s and on their conjugates, where for $x \in \mathbb{C}$ we define

$$\theta(x) = \frac{1}{2}\left(|x|_\infty + \sqrt{|x|_\infty + 4}\right).$$

The result is the following.

Theorem 3. [3, Theorem 4.5] *Let $\tau = (K, \mathfrak{P}, s)$ be a type. Let Σ be the set of embeddings of K in $\mathbb{C}$, and let us denote by*

$$\nu_\tau = \sup\left\{ \frac{\prod_{\sigma\in\Sigma}\theta(a^\sigma)}{|a|_{v_0}^{d_{v_0}}} \mid a \in \mathcal{Y}_s^1 \right\}.$$

If $\nu_\tau < 1$, then τ satisfies the CFF.

This result allows us to study the $\mathfrak{P}$-adic CFF property when the field is norm Euclidean field; in particular, we prove that a norm Euclidean field with Euclidean minimum < 1 satisfies the $\mathfrak{P}$-adic CFF property for all but finitely many prime ideals $\mathfrak{P}$, by applying a result of Cerri [4]. Furthermore, for certain Euclidean quadratic fields K we provide some more effective constructions by exploiting the form of unitary neighborhoods covering a fundamental domain of $\mathcal{O}_K$ as done in [5].

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Simultaneous rational approximation to successive powers of a real number

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(joint work with Damien Roy)

For each $\xi \in \mathbb{R}$ and each integer $n \geq 1$, let $\widehat{\lambda}_n(\xi)$ denote the supremum of all $\lambda \geq 0$ such that the system

$$|x_0| \leq X \quad \text{and} \quad \max_{1 \leq k \leq n} |x_0 \xi^k - x_k| \leq X^{-\lambda}$$

admits a non-zero integer solution $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$ for each sufficiently large X. Further, let $\tau_{n+1}(\xi)$ denote the supremum of all $\tau \geq 0$ for which there exist infinitely many algebraic integers α of degree at most $n+1$ with

$$|\xi - \alpha| \leq H(\alpha)^{-\tau},$$

where $H(\alpha)$ stands for the height of α, namely the largest absolute value of the coefficients of its irreducible polynomial over $\mathbb{Z}$. Dirichlet's theorem ensures that $\widehat{\lambda}_n(\xi) \geq 1/n$. In their seminal 1969 paper [3], Davenport and Schmidt proved the following transference inequality

$$\tau_{n+1}(\xi) \geq 1 + 1/\widehat{\lambda}_n(\xi).$$

Thus any upper bound on $\widehat{\lambda}_n(\xi)$ yields a lower bound on $\tau_{n+1}(\xi)$. Assuming that ξ is not itself an algebraic number of degree at most n, they further show that

$\widehat{\lambda}_1(\xi) = 1$ and

$$\widehat{\lambda}_n(\xi) \le \begin{cases} (-1+\sqrt{5})/2 = 0.618\cdots & \text{if } n = 2 \\ 1/2 & \text{if } n = 3 \\ 1/\lfloor n/2 \rfloor & \text{if } n \ge 4. \end{cases}$$

In the special case $n = 2$, the upper bound $\widehat{\lambda}_2(\xi) \le (-1+\sqrt{5})/2$ is best possible by [9, Theorem 1.1], and the corresponding lower bound $\tau_3(\xi) \ge (3+\sqrt{5})/2 \cong 2.618$ is also best possible by [10, Theorem 1.1]. Furthermore, by [11, Corollary] the values $\widehat{\lambda}_2(\xi)$ with ξ real and transcendental form a dense subset of the interval $[1/2, (-1+\sqrt{5})/2]$. We know large families of transcendental real numbers ξ with $\widehat{\lambda}_2(\xi) > 1/2$. In chronological order, they are the extremal numbers ξ of [9], the Sturmian continued fractions of [2], Fischler's numbers from [4], the Fibonacci type numbers of [11] and the Sturmian type numbers of [6], all contained in the very general class of numbers studied in [7]. By contrast, for each $n \ge 3$ the existence of transcendental real numbers ξ satisfying $\widehat{\lambda}_n(\xi) > 1/n$ is still conjectural.

Question. Given an integer $n \ge 3$, does it exist a transcendental real number ξ such that $\widehat{\lambda}_n(\xi) > 1/n$?

Since the paper of Davenport and Schmidt [3], refined upper bounds for $\widehat{\lambda}_n$ have also been established by Laurent [5] (for n odd), by Roy [12] (for $n = 3$), by Schleischitz [13, 14] and by Badziahin [1] (both for n even), all of the form

$$\widehat{\lambda}_n(\xi) \le \frac{1}{n/2 + c_n}, \qquad \text{with } 0 < c_n < 1.$$

Our main result [8] below improves significantly on this when n is large.

Theorem. *For any integer $n \ge 2$ and any real number ξ satisfying $[\mathbb{Q}(\xi) : \mathbb{Q}] > n$, we have*

$$\widehat{\lambda}_n(\xi) \le \frac{1}{n/2 + a\sqrt{n} + 1/3}, \qquad \text{where } a = \frac{1-\log 2}{2} \cong 0.1534.$$

Note that the multiplicative constant a in the denominator is not optimal and could be improved with additional work. The same applies to the additive constant $1/3$ given the actual choice of a. In view of the Davenport and Schmidt's transference inequality, this gives

$$\tau_{n+1}(\xi) \ge n/2 + a\sqrt{n} + 4/3$$

for the same n and ξ, which improves the prior bound $\tau_{n+1}(\xi) \ge n/2 + \mathcal{O}(1)$.

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On $\widehat{\lambda}_n(x)$

Dzmitry Badziahin

Given $\xi \in \mathbb{R}$, a simultaneous Diophantine exponent $\lambda_n(\xi)$ (respectively, $\widehat{\lambda}_n(\xi)$) is defined as the supremum of values λ such that the following inequalities

$$\max_{1\le i\le n} ||q\xi^i|| < Q^{-\lambda}; \quad 1 \le q \le Q,$$

have solutions in integer q for some (respectively, all) arbitrarily large values of Q.

In a similar way, a dual Diophantine exponent $w_n(\xi)$ (resp., $\widehat{w}_n(\xi)$) is defined as the supremum of w such that the inequalities

$$P(\xi) \le H^{-w}, \quad H(P) \le H$$

have solutions in integer polynomials $P(x), \deg(P) \le n, P(x) \not\equiv 0$ for some (resp., all) arbitrarily large value of H.

In this report we will mostly concentrate on the exponent $\widehat{\lambda}_n(\xi)$. That is because it is one of the most mysterious Diophantine exponents. We know very little about it. For $n \ge 3$ we even do not know if $\widehat{\lambda}_n(\xi)$ can take any values apart from the generic $1/n$. On the other hand, this exponent is closely related with some questions about the distribution of algebraic numbers. For example, if one shows that for transcendental ξ, $\widehat{\lambda}_n(\xi)$ can not any values apart from $1/n$ then the famous Wirsing conjecture is true.

From the classical Dirichlet and Minkowski theorems we know that

$$\frac{1}{n} \le \widehat{\lambda}_n(\xi) \le \lambda_n(\xi).$$

In 1969, Sprindzhuk [5] verified that the bounds above are in fact equalities for almost all $\xi \in \mathbb{R}$ in terms of Lebesgue measure.

The only case when we know something about the spectrum of $\widehat{\lambda}_n(\xi)$ is when $n = 2$. It was initially proven in 1969 by Davenport and Schmidt [2] that $\widehat{\lambda}_2(\xi) \le \frac{\sqrt{5}+1}{2}$. In 2003, Roy showed [4] that this upper bound is sharp and provided a family of numbers $\xi \in \mathbb{R}$ with $\widehat{\lambda}_2(x) = \frac{\sqrt{5}+1}{2}$. Later, various authors (Roy, Bugeaud, Laurent and others) showed that the spectrum of $\widehat{\lambda}_2(\xi)$ contains many numbers between $\frac{1}{2}$ and $\frac{\sqrt{5}+1}{2}$.

In further discussion, we will always assume that ξ is transcendental. The values of $\widehat{\lambda}_n(\xi)$ are well understood, due to the Schmidt Subspace Theorem, and therefore are not interesting to us.

One of the standard ways to investigate the simultaneous Diophantine exponents is through the set of the so called minimal points $\mathbf{x}_i$. They satisfy the following conditions: $||\mathbf{x}_1|| < ||\mathbf{x}_2|| < ||\mathbf{x}_3|| < \cdots$ and for $L_i = L(\mathbf{x}_i) := \max_{1\le j\le n} |x_{i,0}\xi^j - x_{i,j}|$ one has $L_1 > L_2 > L_3 > \cdots$. Finally, for any $\mathbf{x}$ with $||\mathbf{x}|| \le ||\mathbf{x}_{i+1}||$, $||\mathbf{x}|| \ne ||\mathbf{x}_i||$, one has $L(\mathbf{x}) > L_i$. One can easily verify that the condition $\widehat{\lambda}_n(\xi) > \lambda$ is equivalent to $L_i > ||\mathbf{x}_{i+1}||^{-\lambda}$ for all large enough i.

The first non-trivial upper bound for $\widehat{\lambda}_n(\xi)$ was provided by Davenport and Schmidt in 1969 [2]. They consider multivectors of the form $\mathbf{x}_i^{(0,k)} \wedge \mathbf{x}_i^{(2,k)} \wedge \cdots \wedge \mathbf{x}_i^{(n-k-1,k)}$, where $\mathbf{x}_i^{(j,k)} = (x_{i,j}, x_{i,j+1}, \ldots, x_{i,j+k})$. For example, one can show that the norm of such a multivector does not exceed $CX_iL_i^{n-k}$ and therefore for λ large enough it must be zero, which gives us a bunch of linearly dependent vectors. By exploring this idea, Davenport and Schmidt manage to show that

$$\widehat{\lambda}_n(\xi) \le \frac{1}{\lceil n/2 \rceil}.$$

The next incremental improvement of the upper bound for $\widehat{\lambda}_n(\xi)$ is due to Laurent in 2003 [3]. He showed that $\widehat{\lambda}_n(\xi) \le \frac{1}{\lfloor n/2 \rfloor}$. Many other people improved the upper bound since then (Schleischitz, B., Roy and Poëls) but asymptotically, the best upper bound is still of the form

$$\widehat{\lambda}_n(\xi) \le \frac{2}{n} - o(n^{-1}).$$

In the last couple of years, several groups of Mathematicians, including myself, actively worked on this problem. One of the surprising results, which was achieved by me in 2021 [1], relates $\widehat{\lambda}_n(\xi)$ and $w_k(\xi)$ where k is much smaller than n. In particular, it was shown that if $\widehat{\lambda}_n(\xi)$ is far enough from its generic value $1/n$ then $w_k(\xi)$ must also be far enough from its generic value k. more exactly, one of my results is

Theorem 1. *Assume that ξ is transcendental and for a given $k \in \mathbb{N}$,*

$$\delta_k := \frac{k}{w_k(\xi) + 1 - k} \ge 1. \tag{19}$$

Then one has

$$\widehat{\lambda}_n(\xi) \le \begin{cases} \dfrac{1}{n-k} & \text{for } 2k+1 \le n < 2k+1+\delta_k, \\ \min\left\{ \dfrac{1}{n - \left\lceil \frac{n-\delta_k-1}{2} \right\rceil}, \dfrac{1}{\left\lfloor \frac{n-\delta_k-1}{2} \right\rfloor + 1 + \delta_k} \right\} & \text{for } n \ge 2k+1+\delta_k. \end{cases}$$

In particular, this result shows that if $w_k(\xi) = k$, which is satisfied for almost all ξ, then $\widehat{\lambda}_{3k+1} \frac{1}{2k+1} \asymp \frac{3}{2n}$, which is essentially better than the best unconditional upper bound of $2/n - o(n^{-1})$.

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Irrationality measure functions for several real numbers

Nikolay Moshchevitin

Let α be real irrational number. By $||\cdot||$ we denote the distance to the nearest integer. We consider irrationality measure function

$$\psi_\alpha(t) = \min_{1\le q\le t,\, q\in\mathbb{Z}} ||q\alpha||, \quad t \ge 1.$$

By Lagrange's theorem if we consider the representation of α as an ordinary continued fraction

$$\alpha = [a_0; a_1, a_2, ..., a_n, ...]$$

and its convergents and approximations

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, ..., a_n], \;\; \xi_n = |q_n\alpha - p_n|,$$

we see that

$$\psi_\alpha(t) = \xi_n \;\text{ for }\; q_n \le t < q_{n+1}.$$

It is clear that for any t we have $\psi_\alpha(t) < t^{-1}$.

In 2010 I.D. Kan and N. Moshchevitin [1] proved the following result.

Theorem 1. *Suppose $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ such that $\alpha \pm \beta \notin \mathbb{Z}$. Then the difference function*

$$\psi_\alpha(t) - \psi_\beta(t)$$

changes its sign infinitely many times as $t \to +\infty$.

Here we introduce two recent results which generalise Theorem 1. For n-tuple $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$ of pairwise incommensurable numbers (that is $\psi_{\alpha_i}(t) \neq \psi_{\alpha_j}(t)$ for all t large enough) consider permutation

$$\boldsymbol{\sigma}(t) : \{1, 2, 3, ..., n\} \mapsto \{\sigma_1, \sigma_2, \sigma_3, ..., \sigma_n\}$$

with

$$\psi_{\alpha_{\sigma_1}}(t) > \psi_{\alpha_{\sigma_2}}(t) > \psi_{\alpha_{\sigma_3}}(t) > ... > \psi_{\alpha_{\sigma_n}}(t).$$

We define $\mathfrak{k}$-index $\mathfrak{k}(\boldsymbol{\alpha}) = \mathfrak{k}(\alpha_1, ..., \alpha_n)$ as

$$\mathfrak{k}(\boldsymbol{\alpha}) = \max\{k : \text{there exist different pemutations } \boldsymbol{\sigma}_1, ..., \boldsymbol{\sigma}_k$$

$$\text{with the following property} : \forall\, j\ \forall t_0 > 0\ \exists\ t > t_0 \text{ such that } \boldsymbol{\sigma}(t) = \boldsymbol{\sigma}_j\}$$

In particular, Kan-Moshchevitin's Theorem 1 states that $\mathfrak{k}(\alpha_1, \alpha_2) = 2$ if $\alpha_1 \pm \alpha_2 \notin \mathbb{Z}$.

Very recently a progress was obtained in determining the set of admissible valued of $\mathfrak{k}$-index. The following result was proven by V.O. Manturov and N. Moshchevitin [2].

Theorem 2. *Let $k \geq 3$ and $n = \frac{k(k+1)}{2}$. Then there exists a pairwise incommensurable n-tuple $\boldsymbol{\alpha}$ with*

$$\mathfrak{k}(\boldsymbol{\alpha}) = k.$$

The following result was very recently obtained by V. Rudykh [3].

Theorem 3. *The size of an n-tuple $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$ of pairwise independent numbers with $\mathfrak{k}(\boldsymbol{\alpha}) = k$ is*

$$n \leq \frac{k(k+1)}{2}.$$

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Around the support problem for Hilbert class polynomials

Gabriel A. Dill
(joint work with Francesco Campagna)

The starting point of our collaboration is the following theorem by Bugeaud, Corvaja, and Zannier.

Theorem 1 (Theorem 1 in [1]). *Let a, b be multiplicatively independent integers ≥ 2, and let $\varepsilon > 0$. Then, provided n is sufficiently large, we have*

$$\gcd(a^n - 1, b^n - 1) < \exp(\varepsilon n).$$

Here, the left-hand side measures the size of the intersection of the Zariski closure in $\mathbb{G}_{m,\mathbb{Z}}^2$ of the singleton $\{(a,b)\} \subseteq \mathbb{G}_{m,\mathbb{Q}}^2(\mathbb{Q})$ with the kernel of the raising-to-the-n-th-power morphism. This is an intersection of two 1-dimensional schemes inside a 3-dimensional scheme. Thus, Theorem 1 fits into the framework of "unlikely intersections". As remarked by Zannier, Theorem 1 can be regarded as an arithmetical analogue of results about unlikely intersections over fields of characteristic 0.

In our joint work, we replace $\mathbb{G}_m$ by the coarse moduli space of elliptic curves $Y(1) = \mathbb{A}^1$ and we study the analogue of Theorem 1 and related questions in this context. This venture is inspired by the well-known fact that there is a notion of special subvarieties in both the realm of semiabelian varieties and the realm of mixed Shimura varieties. Let F be a field. A *special subvariety* of $\mathbb{G}_{m,F}^n$ is an irreducible component of an algebraic subgroup of $\mathbb{G}_{m,F}^n$. If $X_1, \ldots, X_n$ are affine coordinates on $Y(1)_F^n$, then a *special subvariety* of $Y(1)_F^n$ is an irreducible component of the intersection of the zero loci of finitely many modular polynomials $\Phi_{N_k}(X_{i_k}, X_{j_k})$ $(k = 1, \ldots, K)$. In particular, a special point of $\mathbb{G}_{m,\mathbb{C}}$ is a root of unity and a special point of $Y(1)_\mathbb{C}$ is a *singular modulus*, *i.e.*, the j-invariant of an elliptic curve with complex multiplication.

Theorem 1 is about values of the polynomials $T^n - 1$ $(n \in \mathbb{N} = \{1, 2, \ldots\})$. These have the property that their zeroes are all special points of $\mathbb{G}_{m,\mathbb{C}}$. Using the dictionary above, the analogue of this family (or more accurately: of the family of cyclotomic polynomials) in the $Y(1)$ case is precisely the family of *Hilbert class polynomials* $H_D(T)$ with $D \in \mathbb{D}$, where $\mathbb{D} = \{-3, -4, \ldots\}$ is the set of negative integers $\equiv 0, 1 \bmod 4$ and $H_D(T) \in \mathbb{Z}[T]$ is the minimal polynomial over $\mathbb{Q}$ of any j-invariant of an elliptic curve with complex multiplication by the imaginary quadratic order of discriminant D. Thus, we are led to studying how large the greatest common divisor of $H_D(a)$ and $H_D(b)$ can be, where $a, b \in \mathbb{Z}$.

This question as well as more general divisibility questions also make sense with rings of S-integers in a number field in place of $\mathbb{Z}$. The following is a modular counterpart of Theorem 1 and its generalization to arbitrary number fields by Corvaja and Zannier:

Theorem 2. *Let K be a number field and let S be a finite set of maximal ideals of $\mathcal{O}_K$. Consider two elliptic curves $E_{1/K}, E_{2/K}$ such that the j-invariant $j(E_i)$ belongs to the ring of S-integers $\mathcal{O}_{K,S} \subseteq K$ for $i = 1, 2$. Suppose that there exists* *a prime ideal $\mathfrak{p}$ of $\mathcal{O}_{K,S}$ at which both E_1 and E_2 have potential good supersingular reduction. Let p denote the rational prime lying under $\mathfrak{p}$. Then*

$$\limsup_{D\in\mathbb{D},\,|D|\to\infty} (\deg H_D)^{-1} \log N(H_D(j(E_1))\mathcal{O}_{K,S} + H_D(j(E_2))\mathcal{O}_{K,S}) \geq \frac{\log p}{p-1} > 0,$$

where $N(\cdot)$ denotes the ideal norm in $\mathcal{O}_{K,S}$.

In particular, Theorem 2 shows that the naive analogue of Theorem 1 over number fields is false. Namely, the condition in Theorem 1 that a and b are multiplicatively independent is equivalent to demanding that the point $(a, b) \in \mathbb{G}^2_{m,\mathbb{C}}(\mathbb{C})$ is not contained in any proper special subvariety of $\mathbb{G}^2_{m,\mathbb{C}}$. In the modular setting, this translates into the condition that a and b are both not singular moduli and that $\Phi_N(a, b) \neq 0$ for all $N \in \mathbb{N}$. However, there are infinitely many such pairs $(a, b) \in \mathcal{O}^2_{K,S}$ such that a and b are the j-invariants of elliptic curves with a fixed common prime ideal of potential good supersingular reduction.

When trying to understand the size of the greatest common divisor of $H_D(a)$ and $H_D(b)$ for a and b in some Dedekind domain R, we were led to consider the extreme case where every prime dividing $H_D(a)$ also divides $H_D(b)$ for all but finitely many $D \in \mathbb{D}$. For which a and b is this possible? This is the modular instance of the so-called *support problem.* If we again replace the polynomials $H_D(T)$ $(D \in \mathbb{D})$ by the polynomials $T^n - 1$ $(n \in \mathbb{N})$, this becomes the *multiplicative support problem.* In the case where R is the ring of S-integers in some number field, Corrales-Rodrigáñez and Schoof have solved this problem in [2]. Concerning the *modular support problem* in the number field case, we prove the following theorem:

Theorem 3. *Let K be a number field and let S be a finite set of maximal ideals of $\mathcal{O}_K$. Let $a, b \in \mathcal{O}_{K,S}$. Suppose that there exists $D_0 \in \mathbb{N}$ such that every prime ideal of $\mathcal{O}_{K,S}$ dividing $H_D(a)$ also divides $H_D(b)$ for every $D \in \mathbb{D}$ with $|D| > D_0$. Then either $a = b$ or there exists $\widetilde{D} \in \mathbb{D}$ such that $H_{\widetilde{D}}(a) = H_{\widetilde{D}}(b) = 0$.*

We do not know whether the conclusion of Theorem 3 can be strengthened to saying that $a = b$ always. However, this strengthened conclusion is certainly false if we assume that $\mathfrak{p} \mid H_D(a) \Rightarrow \mathfrak{p} \mid H_D(b)$ holds just for infinitely many D (and all $\mathfrak{p}$) instead of holding for all but finitely many D (and all $\mathfrak{p}$). For instance, if a and b are the two zeroes of $H_{-15}(T)$ in $\mathbb{C}$, we can show that $\mathfrak{p} \mid H_D(a)$ if and only if $\mathfrak{p} \mid H_D(b)$ for all prime ideals $\mathfrak{p}$ in the ring of integers of $\mathbb{Q}(a) = \mathbb{Q}(b)$ and for all $D \in \mathbb{D}$ such that $D \equiv 1 \bmod 8$.

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The relative Bogomolov conjecture for fibered products of elliptic families

Lars Kühne

In my talk, I summarized the progress made in [10] on the relative Bogomolov conjecture in the setting of fibered products of families of elliptic curves, exposing some of the new ideas.

For reasons of comparison, let me first state the classical version of the

Bogomolov conjecture. *Let A be an abelian variety defined over $\overline{\mathbb{Q}}$, X an irreducible subvariety of A, and $\widehat{h} : A(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$ the Néron-Tate height associated with an ample symmetric line bundle on A. Then,*

(1) *there exists some $\varepsilon(X) > 0$ such that the set*

$$\{x \in X(\overline{\mathbb{Q}}) \mid \widehat{h}(x) < \varepsilon(X)\}$$

is not Zariski-dense, or

(2) *X is a torsion coset (i.e., of the form $B + t$ where $B \subseteq A$ is an abelian subvariety and $t \in A$ is a torsion point).*

In this form, the conjecture is a theorem due to Ullmo [11] and Zhang [12], later reproven by David and Philippon [3]. The setting of the relative Bogomolov conjecture is a straightforward generalization: Instead of a single abelian variety we consider a family $\pi : A \to S$ of abelian varieties over an irreducible $\overline{\mathbb{Q}}$-variety S and irreducible subvarieties $X \subseteq A$. In this situation, the fiberwise Néron-Tate heights can be chosen such that they combine to a global height $\widehat{h} : A(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$. The assertion then becomes the

relative Bogomolov conjecture (RBC). *For each irreducible subvariety $X \subset A$ such that $\pi(X) = S$, either*

(1) *there exists some $\varepsilon(X) > 0$ such that the set*

$$\{x \in X(\overline{\mathbb{Q}}) \mid \widehat{h}(x) < \varepsilon(X)\}$$

is not Zariski-dense, or

(2) *there exists a horizontal torsion coset $Y \subset A$ such that X is a subvariety of codimension $\leq \dim(S)$ in Y.*

Up to pulling back by a generically finite map $S' \to S$ and projecting down again, a horizontal torsion coset $Y \subseteq A$ is just the spreading out of a torsion coset $B_\eta + t_\eta$ in the generic fiber A_η.

The new feature in the relative Bogomolov conjecture is that the second alternative does *no longer* assert that a subvariety X with a Zariski-dense set of points of small height – or even torsion points – is actually a (horizontal) torsion coset. Already for a non-isotrivial family of elliptic curves $\pi : E \to C$ over a base curve C, the image of a section $\sigma : C \to E$ contains infinitely many torsion points even if it is not a torsion section itself (by Manin's theorem of the kernel [2, 9]).

As it seems, the relative Bogomolov conjecture is still widely open, though a proof of the relative Manin-Mumford conjecture has been announced at this very same workshop by Gao and Habegger [8]. As the title suggests, I will concentrate on the case where the generic fiber is just a product of elliptic curves. The first Bogomolov-type result in this direction that I am aware of is due to DeMarco and Mavraki [5], and their work was indeed my main motivation.

Theorem. *(K. [10]) Let S be an irreducible (possibly non-proper) algebraic variety over $\overline{\mathbb{Q}}$ and let each $\pi_i : E_i \to S$, $1 \leq i \leq g$, be an elliptic curve over S. Writing A for the total space of the fibered product*

$$\pi = \pi_1 \times \cdots \times \pi_g : \ E_1 \times_S \cdots \times_S E_g \longrightarrow S,$$

the relative Bogomolov conjecture is true for every irreducible subvariety $X \subset A$.

My other motivation for working on this result was that it also implies (as pointed out by Dimitrov, Gao, and Habegger [6]) uniform results of Bogomolov-type, giving for example another proof of the main result in [4].

Finally, let me conclude by sketching some aspects of the proof. Equidistribution techniques from Arakelov theory and the (fibered) product structure imply easily that any irreducible subvariety containing a Zariski-generic sequence of points of sufficiently small height has to satisfy a (rather weird) set of real-analytic differential equations. Even worse, these equations are totally invariant under monodromy since they are derived from the monodromy-invariant Betti forms. However, they are also far from being holomorphic so that a separation argument that I learned from André, Corvaja, and Zannier [1, Subsection 5.2] yields another set of equations on which the monodromy group acts non-trivially. Exploiting monodromy for these and Gao's version of the mixed Ax-Schanuel conjecture [7] completes the proof.

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Quantitative reduction theory

Martin Orr
(joint work with Christopher Daw)

Let $\mathbf{H}$ be a reductive $\mathbb{Q}$-algebraic group and let $\Gamma_{\mathbf{H}} \subset \mathbf{H}(\mathbb{Q})$ be an arithmetic subgroup. Borel and Harish-Chandra [1] constructed a fundamental set for the action of $\Gamma_{\mathbf{H}}$ on $\mathbf{H}(\mathbb{R})$ by translations. These fundamental sets have the form $B\mathfrak{S}u^{-1} \cap \mathbf{H}(\mathbb{R})$ where $\mathfrak{S}$ is a Siegel set in $\mathbf{GL}_n(\mathbb{R})$, B is a finite subset of $\mathbf{GL}_n(\mathbb{Z})$ and $u \in \mathbf{GL}_n(\mathbb{R})$. This construction can be used to prove that $\Gamma_{\mathbf{H}}$ is finitely generated. The finite generating set for $\Gamma_{\mathbf{H}}$ is closely related to the finite set B appearing in the construction of the fundamental set. This gives one motivation for seeking quantitative information about the set B. A further motivation, which was responsible for our interest in the problem, comes from applications to the Zilber–Pink conjecture for Shimura varieties, in particular the parameterisation of special subvarieties step in the Pila–Zannier strategy.

The proof of Borel and Harish-Chandra relies on "soft" topological arguments and therefore is not well adapted to obtaining such quantitative information. Ideally (and somewhat vaguely) one would like to bound the heights of elements of the finite set B by a polynomial with respect to "arithmetic invariants" of the group $\mathbf{H}$. For example, in the case where $\mathbf{H}$ is an orthogonal group, Li and Margulis [4] obtained an effective version of the construction of fundamental sets, in which the heights of elements of B are polynomially bounded in terms of the coefficients of the quadratic form stabilised by the orthogonal group.

In order to describe the situation in which we seek bounded construction of fundamental sets, we fix an ambient reductive group $\mathbf{G}$ and consider $\mathbb{Q}$-algebraic subgroups $\mathbf{H} \subset \mathbf{G}$ which lie in a single $\mathbf{G}(\mathbb{R})$-conjugacy class. For example, the units of any order in any quaternion algebra over $\mathbb{Q}$ can be embedded as a $\mathbb{Q}$-algebraic subgroup of $\mathbf{GL}_4$, and these all lie in a single $\mathbf{GL}_4(\mathbb{R})$-conjugacy class, but not in a single $\mathbf{GL}_4(\mathbb{Q})$-conjugacy class. In fact, these quaternion unit groups are contained in $\mathbf{GSp}_4$ and lie in a single $\mathbf{GSp}_4(\mathbb{R})$-conjugacy class, which is important for our application to parameterising quaternionic Shimura curves inside the moduli space of principally polarised abelian surfaces. In this case, the relevant "arithmetic invariant" of $\mathbf{H}$ is the discriminant of the quaternion order.

Daw and I [2] have obtained a version of Borel and Harish-Chandra's construction of fundamental sets which gives some quantitative control of the finite set B as $\mathbf{H}$ varies through $\mathbb{Q}$-algebraic subgroups in a $\mathbf{G}(\mathbb{R})$-conjugacy class of reductive subgroups of $\mathbf{G}$. The bound in our theorem involves an auxiliary representation $\rho \colon \mathbf{G} \to \mathbf{GL}(V)$ of a type used in the construction of Borel and Harish-Chandra. In particular, the representation must contain a vector v whose stabiliser is $\mathbf{H}$ and whose $\mathbf{G}(\mathbb{R})$-orbit is closed in $V \otimes_{\mathbb{Q}} \mathbb{R}$.

Theorem 1. *Let $\mathbf{G}$ and $\mathbf{H}$ be reductive $\mathbb{Q}$-algebraic groups with $\mathbf{H} \subset \mathbf{G}$. Let $\rho \colon \mathbf{G} \to \mathbf{GL}(V)$ be a representation containing a vector v with the properties specified above. Let $\mathfrak{S}$ be a Siegel set in $\mathbf{G}(\mathbb{R})$. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup.*

For every $u \in \mathbf{G}(\mathbb{R})$ such that $\mathbf{H}_u := u\mathbf{H}_{\mathbb{R}}u^{-1}$ is defined over $\mathbb{Q}$, choose a vector $v_u \in V \otimes_{\mathbb{Q}} \mathbb{R}$ satisfying certain technical conditions linking v_u to the group $\mathbf{H}_u$. Then there is a finite set $B_u \subset \Gamma$ such that $B_u\mathfrak{S}u^{-1} \cap \mathbf{H}_u(\mathbb{R})$ is a fundamental set for $\Gamma \cap \mathbf{H}_u(\mathbb{R})$ in $\mathbf{H}_u(\mathbb{R})$, and every element $b \in B_u$ satisfies

$$|\rho(b^{-1}u)v_u| \leq C_1|v_u|^{C_2}$$

where C_1 and C_2 are independent of u (and ineffective).

The conclusion of the theorem is a polynomial bound, but both sides of this inequality require some explanation. On the right hand side, in special cases of interest [2, 3], we can relate $|v_u|$ to arithmetic invariants of $\mathbf{H}$. This requires careful construction of the representation ρ and a lengthy calculation and can probably be carried out for a wide range of $\mathbf{G}$ and $\mathbf{H}$.

The quantity on the left hand side of the inequality, $|\rho(b^{-1}u)v_u|$, should be thought of as a poor substitute for the height of b. It seems rather artificial but it is sufficient for our applications to the Zilber–Pink conjecture, as well as for obtaining a polynomial bound on the number of elements of B_u. It should be noted that the vector $\rho(b^{-1}u)v_u$ has integer coordinates, so a bound on its length also bounds its height, and its stabiliser is $b^{-1}\mathbf{H}_u b$, that is, a Γ-conjugate of $\mathbf{H}_u$. In the application, we form a definable set in which vectors of the form $\rho(b^{-1}u)v_u$ occur as rational points, then use the Pila–Wilkie theorem to control the number of rational vectors in this definable set.

One would like to improve this theorem to obtain polynomial bounds for the heights of elements of B_u themselves. This might be possible by combining the proof of this theorem with the methods used to prove Li and Margulis's theorem for orthogonal groups, namely homogeneous dynamics and spectral gaps for automorphic representations. I would be interested to hear from anyone knowledgeable in homogeneous dynamics who would like to work on this question.

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A subspace theorem for manifolds

Emmanuel Breuillard
(joint work with Nicolas de Saxcé)

We revisit Schmidt's subspace theorem in terms of the dynamics of diagonal flows on homogeneous spaces and describe how the exceptional subspaces arise from certain rational Schubert varieties associated to the family of linear forms through the notion of Harder-Narasimhan filtration and an associated slope formalism.

This geometric understanding opens the way to a natural generalization of Schmidt's theorem to the setting of diophantine approximation on submanifolds of GL_d, which is our main result. In turn this allows us to recover and generalize the main results of Kleinbock and Margulis regarding diophantine exponents of submanifolds.

The above-mentioned slopes correspond to the rates of exponential growth of the successive minimas of the dilated lattice $a_t L\mathbb{Z}^d$, where $L \in GL_d(\mathbb{R})$ and a_t is a diagonal one-parameter subgroup. Starting with a connected analytic submanifold M of $GL_d(\mathbb{R})$ we show:

Theorem 1. [1] *If the Zariski-closure of M is defined over $\overline{\mathbb{Q}}$, then for almost every L in M, the lattices $a_t L\mathbb{Z}^d$ assume a fixed asymptotic shape as t tends to $+\infty$.*

This means that the slopes are well-defined. When M is reduced to a point, this statement is equivalent to Schmidt's parametric subspace theorem. When M is contained in a certain unipotent subgroup, this allows to recover the main result of [2] and some of its generalizations to matrices and Lie groups.

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Zilber–Pink in $Y(1)^n$

Christopher Daw
(joint work with Martin Orr)

By $Y(1)$ we denote the modular curve $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ and (by abuse of notation) the underlying algebraic variety, the affine line over $\mathbb{Q}$. As such, $Y(1)$ (and, therefore, $Y(1)^n$) is a Shimura variety and, hence, the subject of the Zilber–Pink conjecture.

In 2012, Habegger and Pila obtained a major result in the direction of the Zilber–Pink conjecture for $Y(1)^n$ [5]. Indeed, they showed that, if $C \subset Y(1)^n$ is an irreducible curve defined over $\overline{\mathbb{Q}}$, not contained in any proper special subvariety (that is, Hodge generic), then the Zilber–Pink conjecture holds for C provided C is *asymmetric* (more on the latter momentarily). In particular (and this is the statement for which asymmetry is needed), for any two distinct pairs $(i_1, i_2), (i_3, i_4)$ of distinct coordinates, the union over all $M, N \in \mathbb{N}$ of the sets

$$\Theta_{M,N} = \{(s_1, \ldots, s_n) \in C(\overline{\mathbb{Q}}) : \Phi_M(s_{i_1}, s_{i_2}) = \Phi_N(s_{i_3}, s_{i_4}) = 0\}$$

is finite. (As usual, $\Phi_N(X, Y) \in \mathbb{Z}[X, Y]$ denotes the classical *modular polynomial.*)

The asymmetry condition can be described as follows: let $\pi_i : Y(1)^n \to Y(1)$ denote the natural i-th projection map and let $\pi_i|_C$ denote its restriction to C; in this way, we obtain a set of integers $D = \{\deg \pi_i|_C : i = 1, \ldots, n\}$ (where we define $\deg \pi_i|_C = 0$ if $\pi_i|_C$ is constant) and we say that C is asymmetric if $|D| \geq n - 1$.

This condition is used to obtain the following height bound: there exists $c = c(C) > 0$ such that, for any $s = (s_1, \ldots, s_n) \in \Theta_{M,N}$, the logarithmic height of the s_{i_j} $(j = 1, \ldots, 4)$ is at most $c \max\{\log M, \log N\}$. Via isogeny estimates, this yields Galois lower bounds (that is, lower bounds for $[\mathbb{Q}(s) : \mathbb{Q}]$), and then Zilber–Pink follows from the now famous Pila–Zannier strategy.

Unfortunately, the assumption of asymmetry appears essential to the method. Moreover, the method is apparently limited to the "product situation".

In light of this, we have pursued other techniques with the potential to yield suitable height bounds. One such technique was originally devised by André [1]. Consider an abelian scheme $f : \mathcal{A} \to C$ of relative dimension $n > 1$ over a curve, all defined over a number field $K \subset \mathbb{C}$. One obtains two associated gadgets: the algebraic relative de Rham cohomology $W_{\mathrm{dR}} = H^1_{\mathrm{dR}}(\mathcal{A}/C)$ (which we assume to be free), and the local system $W = R_1 f^{\mathrm{an}}_{\mathbb{C}*} \mathbb{Q}_{\mathcal{A}^{\mathrm{an}}_{\mathbb{C}}}$ (which becomes constant over an open subset $U \subset C^{\mathrm{an}}_{\mathbb{C}}$). Choosing a basis $\{\omega_i\}$ for W_{dR} and a frame $\{\gamma_j\}$ for $W|_U$, we obtain holomorphic "period" functions

$$p_{ij} : U \to \mathbb{C}, \; s \mapsto \frac{1}{2\pi i} \int_{\gamma_j(s)} \omega_i(s).$$

André highlighted that, at "nice" points $s_0 \in C(\overline{\mathbb{Q}})$ (or possibly $C'(\overline{\mathbb{Q}})$ for C' a curve over which $\mathcal{A}$ extends to a semiabelian scheme), the Taylor series of *some of* the p_{ij} (with respect to a local parameter $x \in K(C)$) are, in fact, "G–functions". That is, they are elements of $K[[X]]$, satisfying a linear differential equation over $K[X]$, whose first n coefficients have height at most c^n for some constant c. Such functions are susceptible to the following theorem of Bombieri: let $g_1, \ldots, g_m \in K[[X]]$ denote G–functions and, for $d \in \mathbb{N}$, define Σ_d to be the set of $\xi \in \overline{\mathbb{Q}}$ such that $g_1(\xi), \ldots, g_m(\xi)$ satisfy a "global, nontrivial relation of degree d". Then, any $\xi \in \Sigma_d$ has logarithmic height at most $c_1 d^{c_2}$ for constants c_1, c_2 independent of d. (A global relation is a homogeneous polynomial $P \in K[X_1, \ldots, X_m]$ such that, for all places ν of K, and each corresponding embedding $\iota_\nu : K \hookrightarrow K_\nu$ such that $\iota_\nu(g_i)(\iota_\nu(\xi))$ converges for all $i = 1, \ldots, m$,

$$\iota_\nu(P)(\iota_\nu(g_1)(\iota_\nu(\xi)), \ldots, \iota_\nu(g_m)(\iota_\nu(\xi))) = 0.$$

A relation is non-trivial if it is not the specialization at ξ of a homogeneous relation over $K[X]$ of the same degree between the g_i.)

André's suggestion was to choose s_0 to be a point of purely multiplication degeneration of $\mathcal{A}$ (that is, a point at which the fiber degenerates to $\mathbb{G}_m^n$). This has two important consequences: first, half of the p_{ij} provably give rise to G–functions (indeed, this is the case whenever γ_j is "locally invariant"); second, for points $s \in C(\overline{\mathbb{Q}})$ such that $\mathrm{End}(\mathcal{A}_s)$ has no ring embedding into $\mathrm{M}_n(\mathbb{Q})$, one can ignore all finite places of K in the construction of global relations. Then, using the action of $\mathrm{End}(\mathcal{A}_s)$ on W_{dR} and W, one can construct linear or quadratic relations between the "locally invariant periods" at all archimedean places. Multiplying these relations together yields a global, non-trivial relation of degree polynomially bounded in terms of $[K(s) : K]$ and, hence, height bounds for s as above. Our generalization of André's result [4], in combination with our work on "Quantitative Reduction Theory" [2, 3] and the Pila–Zannier strategy, has allowed us to prove various instances of the Zilber–Pink conjecture in $\mathcal{A}_g$ for curves intersecting the 0-dimensional stratum of the Baily–Borel compactification.

However, height bounds for the sets $\Theta_{M,N}$ lie outside the scope of this method, at least as described above. The problem, of course, is the condition that $\mathrm{End}(\mathcal{A}_s)$ has no ring embedding into $\mathrm{M}_n(\mathbb{Q})$. In order to circumvent this problem, one is forced to construct relations at finite places. In our forthcoming work, we show, using p-adic Tate uniformizations, that there exists a global relation of degree polynomially bounded in terms of $[K(s) : K]$ and $\max\{\log N, \log M\}$. This is sufficient to prove the Zilber–Pink conjecture for irreducible, Hodge generic $\overline{\mathbb{Q}}$–curves $C \subset Y(1)^n$ containing the point $(\infty, \infty, \dots, \infty)$ in their closure.

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Torsion points in families of abelian varieties

Ziyang Gao

(joint work with Philipp Habegger)

Let S be an irreducible quasi-projective variety defined over $\bar{\mathbb{Q}}$, and let $\pi\colon \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$. Let η be the generic point of S and fix an algebraic closure of the function field of S. For any subvariety X of $\mathcal{A}$, denote by $X_{\overline{\eta}}$ the geometric generic fiber of X. In particular, $\mathcal{A}_{\overline{\eta}}$ is an abelian variety.

Let $\mathcal{A}_{\mathrm{tor}}$ denote the set $\{P \in \mathcal{A}(\bar{\mathbb{Q}}) : [N]P$ is in the zero section for some $N \in \mathbb{Z}\}$; this is the set of fiberwise torsion points in $\mathcal{A}$.

The goal of this talk is to report a recent work in progress joint with Philipp Habegger, known as the *relative Manin–Mumford conjecture*.

Theorem 1. *Let X be an irreducible subvariety of $\mathcal{A}$ that dominates S. Assume that $X_{\overline{\eta}}$ is irreducible and not contained in any proper algebraic subgroup of $\mathcal{A}_{\overline{\eta}}$. If $X \cap \mathcal{A}_{\mathrm{tor}}$ is Zariski dense in X, then* $\dim X \geq g$.

This conjecture was proposed by Pink [15, Conj.6.2] and Zannier [18], which is proved when $\dim X = 1$ in a series of papers of Corvaja, Masser, and Zannier [11, 12, 13, 1, 14]. If we allow fiberwise *small points* instead of only torsion points, then the same result is proved by DeMarco–Mavraki [4] for sections in a fiber product of elliptic surfaces and by Kühne [10] for arbitrary X in a fiber product of elliptic surfaces.

It is worth pointing out that the conclusion of this theorem is the realm of algebraic geometry. Indeed, Let $Y(2) = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and let $\mathcal{E} \to Y(2)$ be the Legendre family, which is an abelian scheme defined over $\bar{\mathbb{Q}}$ with $g = 1$. Consider a non-torsion section $\sigma\colon Y(2) \to \mathcal{E}$, and let X be its image. By Manin's Theorem of the Kernel, X contains a dense subset of torsion points. So X satisfies the assumption of the theorem above, and $\dim X = 1$.

As a consequence, we reprove the following *Uniform Manin–Mumford Conjecture* for curves embedded in their Jacobians which is recently proved by Kühne [9, Thm.1.2].

Corollary 2. *For each integer $g \geq 2$, there exists a constant $c = c(g) > 0$ with the following property. Let C be a geometrically irreducible, smooth, projective curve of genus g defined over $\bar{\mathbb{Q}}$. Let $P_0 \in C(\bar{\mathbb{Q}})$, and let $C - P_0$ be the image of the Abel–Jacobi embedding based at P_0. Then*

$$(C(\bar{\mathbb{Q}}) - P_0) \cap \mathrm{Jac}(C)_{\mathrm{tor}} \leq c. \tag{20}$$

Before Kühne proved the full Uniform Manin–Mumford Conjecture for curves, the case of genus 2 bi-elliptic curves was proved by DeMarco–Krieger–Ye [3]. Another proof of this result is recently given by Yuan [16] based on the theory of adelic line bundles over quasi-projective varieties recently developed by Yuan–Zhang [17]. In all these proofs, equidistribution is used in a serious way.

Our approach does not use equidistribution. We prove Corollary 2 as an easy consequence of Theorem 1, and our proof of Theorem 1 is in spirit of the Pila–Zannier method to solve special point problems, which roughly speaking can be divided into four steps.

- (i) Large Galois Orbit (LGO) of the "special points" in question.
- (ii) A suitable version of the Pila–Wilkie counting theorem (semi-variant version by Habegger–Pila [8, Cor.7.2] in our case).
- (iii) A suitable functional transcendence result (mixed Ax–Schanuel by Gao [7] in our case).
- (iv) Study the corresponding weakly special or weakly anomalous locus.

Let us start with the LGO step to see what happens. The type of result in need is of the following form:

Claim: There exist $c = c(X) > 0$ and $\delta = \delta(X) > 0$ such that for any $x \in X(\bar{\mathbb{Q}}) \cap \mathcal{A}_{\mathrm{tor}}$, we have $|\mathrm{Gal}(\bar{\mathbb{Q}}/K)x| \geq cN(x)^{\delta}$. Here $N(x)$ is the order of the torsion point x.

In practice, it often suffices to prove this bound for x coming from a fixed Zariski open dense subset U. We start with a result of David [2].

Theorem 3 (S.David)**.** *Let A be an abelian variety of dimension g defined over K and $P \in A(\bar{\mathbb{Q}})_{\mathrm{tor}}$. Then*

$$|\mathrm{Gal}(\bar{\mathbb{Q}}/K)P| \geq cN(P)^{\delta}/h(A)^{\delta'}$$

for some constants c, δ, δ' depending only on g. Here $N(P)$ is the order of P and $h(A)$ is a suitable height of A (for example, the stable Faltings height or a theta height or the moduli height).

It is clear that in order to get the desired LGO result from David's theorem, one needs some tools to assure uniformity in all fibers. The tool we use is the height inequality of Dimitrov–Gao–Habegger. An adapted version to the current situation is as follows.

Theorem 4 ([5, Thm.1.6 and Thm.B.1])**.** *Assume X is non-degenerate. Then there exist a constant $c > 0$ and a Zariski open dense subset U of X such that*

$$h(\mathcal{A}_{\pi(x)}) < c(\hat{h}(x) + 1) \quad \textit{for all } x \in U(\bar{\mathbb{Q}}). \tag{21}$$

One equivalent way (as an application of mixed Ax–Schanuel) to define non-degenerate subvarieties is that $X \neq X^{\mathrm{deg}}$ for a certain (intrinsically defined) degeneracy locus $X^{\mathrm{deg}} \subseteq X$.

From these two results, it is immediately clear that we obtain the desired LGO result *if X is non-degenerate.* However, determining when a given subvariety X is non-degenerate or not is a highly non-trivial task. In previously applications of this height inequality [5, 9], one started with constructing a non-degenerate subvariety using results of [6].

In the current work, we apply [6] in a different way. Instead of constructing a non-degenerate subvariety to start with, we do part (iv) of the Pila–Zannier method more carefully to prove our theorem when X is degenerate and when X is non-degenerate. The key idea here is to introduce another degeneracy locus $X^{\mathrm{deg}}(1) \subseteq X$ which by definition contains X^{deg}, and reduce Theorem 1 to the following statement.

Theorem 5. *Under the assumptions of Theorem 1, we have $X^{\mathrm{deg}}(1) = X$.*

Now if X is degenerate, *i.e.* $X = X^{\mathrm{deg}}$, then we can conclude because $X^{\mathrm{deg}}(1) \supseteq X^{\mathrm{deg}}$. If X is non-degenerate, then we follow the Pila–Zannier method described above to prove $X^{\mathrm{deg}}(1) = X$. Hence we are done.

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Northcott numbers and applications

Fabien Pazuki

(joint work with Niclas Technau, Martin Widmer)

We start by recalling the definition of Northcott numbers. We then state a Bertini theorem involving Northcott numbers, obtained in earlier collaboration between the author and Widmer. We also state an undecidability criterion of Julia Robinson, based on Northcott numbers for totally real numbers. These two applications are used as motivation for the Northcott number inverse problem, and we conclude by stating recent results on this inverse problem, obtained in collaboration between the author, Technau, and Widmer. For $\alpha \in \overline{\mathbb{Q}}$, we denote by $\lceil\overline{\alpha}\rceil$ the house of α and we denote by $h(\alpha)$ the logarithmic Weil height of α.

Definition 1. Let $f : \overline{\mathbb{Q}} \to [0, \infty[$ be a function on $\overline{\mathbb{Q}}$. Let $S \subset \overline{\mathbb{Q}}$. For any real number t, we pose $S_t = \{s \in S \mid f(s) \leq t\}$. Then the Northcott number $\mathcal{N}_f(S)$ of S, with respect to f, is defined by

$$\mathcal{N}_f(S) = \inf\{t \geq 0 \mid S_t \text{ infinite}\}.$$

In the case of the logarithmic Weil height, we get $\mathcal{N}_h(\overline{\mathbb{Q}}) = 0$ and $\mathcal{N}_h(\mathbb{Q}) = +\infty$.

Motivation 1: Bertini and Northcott.

To draw a curve on a projective variety with a control on its genus, degree, and height, one may use the following result.

Theorem 2 (P. and Widmer, [4]). *Let k be a number field and $S \subset \overline{\mathbb{Q}}$ with $\mathcal{N}_h(S) < +\infty$. Let X be a smooth closed subvariety in $\mathbb{P}^N_k$, with $\dim X \geq 2$. There exists a finite set $s \subset S$ and a curve $C/k(s)$ on X which is smooth, geometrically irreducible, and such that*

- $g(C) \leq (\deg X)^2 + \deg X$,
- $\deg C \leq \deg X$,
- $h_{\mathbb{P}^N}(C) \leq h_{\mathbb{P}^N}(X) + (\dim X)^2(\deg X)\Big(\mathcal{N}_h(S) + 1 + \frac{1}{2}\log(N+1)\Big)$, *where $h_{\mathbb{P}^N}$ is a projective height.*

The proof of Theorem 2 relies on previous work of Philippon, Rémond, Cadoret, Tamagawa. The explicit dependance in S is new. It is very useful in the case where X is an abelian variety, as can be seen in [3, 4]. Being able to construct sets S with nice properties and bounded Northcott number $\mathcal{N}_h(S)$ is thus desirable.

Motivation 2: Undecidability after Julia Robinson.

Julia Robinson showed a deep link between undecidability questions and the Northcott numbers with respect to the house. More precisely, if $\mathcal{O}$ is a ring of algebraic integers of a subfield of the totally real algebraic numbers, write $\mathcal{O}_t = \{r \in \mathcal{O} \mid 0 < \sigma(r) < t, \quad \forall \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\}$. Define

$$A(\mathcal{O}) = \{t \in \mathbb{R} \cup \{+\infty\} \mid \mathcal{O}_t \text{ is infinite}\}.$$

Define the Julia Robinson number by $\mathrm{JR}(\mathcal{O}) = \inf A(\mathcal{O})$. We say that $\mathcal{O}$ has the JR property if

$$\mathrm{JR}(\mathcal{O}) \in A(\mathcal{O}).$$

Theorem 3 (Robinson, [6]). *Let $\mathcal{O}$ be a ring of algebraic integers of a subfield of the field of totally real algebraic numbers. If $\mathcal{O}$ has the JR property, then the semi-ring $(\mathbb{N}, 0, 1, +, \cdot)$ is first order definable in $\mathcal{O}$. In particular $\mathcal{O}$ has undecidable full theory.*

To progress on these undecidability questions, a possible path is thus to study Northcott numbers restricted to totally real algebraic numbers.

Northcott numbers: solution to the inverse problem.

Motivated by the preceding facts, we were interested in studying a question of Videaux and Videla in [7], namely: which real numbers can be realized as Northcott numbers? We obtain the following theorem.

Theorem 4 (P., Technau, Widmer, [5]). *The following three properties are satisfied.*

(i) Every $t \geq 1$ is the Northcott number of a ring of integers of a field, with respect to the house $\lceil\cdot\rceil$.

(ii) For each $t \geq 0$ there exists a field with Northcott number in $[t, 2t]$, with respect to the logarithmic Weil height $h(\cdot)$.

(iii) For any $\gamma > 0$, let h_γ be defined by $h_\gamma(\alpha) = \deg(\alpha)^\gamma h(\alpha)$, for all $\alpha \in \overline{\mathbb{Q}}$. For all $0 < \gamma \leq 1$ and $0 < \gamma' < \gamma$ there exists a field K with $\mathcal{N}_{h_{\gamma'}}(K) = 0$ and $\mathcal{N}_{h_\gamma}(K) = \infty$.

The proof of Theorem 4 uses sequences $((p_i)^{1/d_i})_{i\geq 1}$, where p_i and d_i are carefully chosen primes. This is reminiscent of previous work of Widmer [8].

In a recent preprint [2], Okazaki and Sano extend our result *(ii)* and completely solve the inverse problem for the logarithmic Weil height as well. They use sequences of the form $((p_i/q_i)^{1/d_i})_{i\geq 1}$, where p_i, q_i, d_i are primes as well. These constructions are also linked with the study of *Corps de Siegel* by Gaudron and Rémond [1].

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Bounded Height in Pencils of Subgroups of finite rank

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(joint work with David Masser and Umberto Zannier)

Let $\mathcal{C} \subseteq \mathbb{P}^n$ be an absolutely irreducible curve with function field $\mathbb{F} := \overline{\mathbb{Q}}(\mathcal{C})$. We fix a height on $\mathbb{P}^n(\overline{\mathbb{Q}})$. Recently, we non-trivially extend the main result of [5], which corresponds to the special case $r = 1$, $V = \{1\}$ of the result below.

Theorem ([1], AMZ'17)**.** *Let $\Gamma \subset \mathbb{G}_{\mathrm{m}}^r(\mathbb{F})$ be a finitely generated constant-free subgroup*[1] *and let V be a subvariety of $\mathbb{G}_{\mathrm{m}}^r$ defined over $\mathbb{F}$. Then the height of the points $t_0 \in \mathcal{C}(\overline{\mathbb{Q}})$, such that for some $\boldsymbol{\gamma} \in \Gamma \setminus V$ the value $\boldsymbol{\gamma}_{t_0}$ is defined and lies in V_{t_0}, is bounded above.*

In 1997 Beukers [4] proved that the solutions $t_0 \in \overline{\mathbb{Q}}$ of the equation

$$t^n + (1-t)^n = 1 \qquad \text{(Beukers)}$$

with n integer > 1, has uniformly bounded height, $h(t_0) \le \log(216)$. This is a special case of AMZ'17: take $\mathbb{F} = \overline{\mathbb{Q}}(t)$, $r = 2$, $\Gamma = \langle (t, 1-t) \rangle$ and V the line in $\mathbb{G}_{\mathrm{m}}^2$ of equation $x + y = 1$. It was the started point of our investigation.

As a new example, we find that the solutions $t_0 \in \overline{\mathbb{Q}}$ of the equation

$$t^n + (1-t)^n + (1+t)^n = 1 \qquad \text{(Denz)}$$

has uniformly bounded height (take $\mathbb{F} = \overline{\mathbb{Q}}(t)$, $r = 3$, $\Gamma = \langle (t, 1-t, 1+t) \rangle$ and V of equation $x+y+z = 1$). Denz [6] provides and explicit bound, $h(t_0) \le 856 \log 2$.

AMZ'17 allows also to solve some family of diophantine equations, see [2].

Here we deal with similar questions for the division group of a constant-free, finitely generated subgroup Γ of $\mathbb{G}_{\mathrm{m}}^r(F)$. The trivial equation $t^\lambda = 2$ already shows some new phenomena. Since $h(\sqrt[\lambda]{2}) = \frac{1}{\lambda}\log 2$, we *don't have* bounded height for a rational λ close to 0. Consider now Beukers' equation $t^\lambda + (1-t)^\lambda = 1$ with a rational exponent[2]λ Again, we could not expect uniformly bounded height for $\lambda \to 0$. Indeed, if $t_0^\lambda + (1-t_0)^\lambda = 1$ and t_0 is not a root of unity,

$$h(t_0) \ge \frac{1}{4\lambda} \log\left(\frac{1+\sqrt{5}}{2}\right) - \frac{1}{2}\log 2.$$

This follows from Zagier's lower bound [7] for the height on $x + y = 1$. Moreover, we obviously don't have bounded height when $\lambda = 1$ (since the equation becomes trivial). Our first result extends Beukers' result to a rational λ.

Theorem. *Let λ be a positive rational $\lambda \ne 1$. Then the solutions $t_0 \in \overline{\mathbb{Q}}$ of $t^\lambda + (1-t)^\lambda = 1$ satisfy:*

$$h(t_0) \le 100 \max(1, \lambda^{-1}).$$

We generalise this result for specialisation of elements in the *division group* of a finitely generated group of rational functions. Let $\mathcal{C}$, $\mathbb{F}$, Γ and V be as at the beginning (with now V defined over $\overline{\mathbb{Q}}$). For $n \in \mathbb{N}$, let $\Gamma^{1/n} = \{\boldsymbol{\gamma} \in \mathbb{G}_{\mathrm{m}}^r(\mathbb{F}) \mid \boldsymbol{\gamma}^n \in \Gamma\}$ and $\Gamma^{\mathrm{div}} = \bigcup_n \Gamma^{1/n}$ (division group). For $f \in \overline{\mathbb{F}}$, let $h_{\mathrm{geo}}(f) = \sum_P \max(-\mathrm{ord}_P(f), 0)$ (geometric height). We define a distance on $\mathbb{G}_{\mathrm{m}}^r(\overline{\mathbb{F}})/\mathrm{tors}$ by $\mathrm{dist}(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2) = h_{\mathrm{geo}}(\boldsymbol{\gamma}_1/\boldsymbol{\gamma}_2)$.

[1]That is, the image Γ' by any surjective homomorphism $\mathbb{G}_{\mathrm{m}}^r \to \mathbb{G}_{\mathrm{m}}$ satisfies $\Gamma' \cap \overline{\mathbb{Q}}^* = \Gamma'_{\mathrm{tors}}$

[2]If $\lambda = p/q$ with p, $q \in \mathbb{N}$, we say that t_0 is a solution of $t^\lambda + (1-t)^\lambda = 1$ if $\exists u, v$ such that $t_0 = u^q$, $1 - t_0 = v^q$ and $u^p + v^p = 1$.

Theorem ([3]). *Let $\varepsilon \in (0,1)$. Then the points $t_0 \in \mathcal{C}(\overline{\mathbb{Q}})$ such that for some $\boldsymbol{\gamma} \in \Gamma^{\mathrm{div}}$ the value $\boldsymbol{\gamma}_{t_0}$ is defined, non-zero, and lies in V, have $h(t_0) \le C_\varepsilon$, except possibly for $(\boldsymbol{\gamma}, t_0)$ with $h_{\mathrm{geo}}(\boldsymbol{\gamma}) \le C$ such that*

$$(\boldsymbol{\gamma}/\boldsymbol{\eta})_{t_0} \cdot \boldsymbol{\eta} \in V(\overline{\mathbb{F}})$$

for some $\boldsymbol{\eta} \in \Gamma^{1/n}$ with $\mathrm{dist}(\boldsymbol{\gamma}, \boldsymbol{\eta}) < \varepsilon$. The constants

$$C_\varepsilon = C(\Gamma, V, \varepsilon), \quad C = C(\Gamma, V) \text{ and } n = n(\Gamma, V) \in \mathbb{N}$$

are effective.

When $\boldsymbol{\gamma} \in \Gamma$ and ε is sufficiently small, we have $\mathrm{dist}(\boldsymbol{\gamma}, \boldsymbol{\eta}) < \varepsilon \Rightarrow \boldsymbol{\eta} = \boldsymbol{\gamma}$. Thus $(\boldsymbol{\gamma}/\boldsymbol{\eta})_{t_0} \cdot \boldsymbol{\eta} = \boldsymbol{\gamma}$. We recover AMZ'17.

As an example, consider again Denz's equation with now a rational exponent. Then, the solutions $t_0^\lambda + (1-t_0)^\lambda + (1+t_0)^\lambda = 1$ with $\lambda \in \mathbb{Q}$ have height $h(t_0) \le C_\varepsilon$, except possibly for (λ, t_0) for which $|\lambda| \le C$ and

$$t_0^{\lambda - a} t^a + (1-t_0)^{\lambda - a}(1-t)^a + (1+t_0)^{\lambda - a}(1+t)^a = 1 \qquad (\star)$$

for some integer a with $|a - \lambda| < \varepsilon$ (which correspond to $\boldsymbol{\gamma} = (t, 1-t, 1+t)^\lambda$, $\boldsymbol{\eta} = (t, 1-t, 1+t)^a$ in the theorem, with "denominator" $n = 1$). Note that the equation $(\star)$ has solutions only if $a = 0$ or 1. A close analysis of the solutions of $(\star)$ corresponding to $a = 1$ shows that they have still bounded height. Thus

Corollary. *Let $\varepsilon \in (0,1)$ and $\lambda \in \mathbb{Q}$ with $|\lambda| > \varepsilon$. Then the solutions $t_0 \in \overline{\mathbb{Q}}$ of*

$$t_0^\lambda + (1-t_0)^\lambda + (1+t_0)^\lambda = 1$$

satisfy $h(t_0) \le C_\varepsilon$.

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Small height and local degrees

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(joint work with Arno Fehm)

Let h be the absolute logarithmic Weil height on a fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. This is a non-negative function which, by Kronecker's theorem, takes value zero precisely at zero and at the roots of unity. While points of minimal height are well understood, there are nevertheless many interesting questions concerning points of non-zero small height.

A first result in this setting is Northcott's theorem, which states that any set of algebraic numbers having bounded degree and bounded height is finite (and, in principle, can be explicitly described), making the height and its generalizations a very important tool in diophantine geometry. A second inescapable statement is that of Lehmer's conjecture, asserting that there is a constant $c > 0$ such that, for any $\alpha \in \overline{\mathbb{Q}}$, the product $h(\alpha)[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is either zero or bigger than c. Lehmer's conjecture has been proven to be true for several classes of algebraic numbers, but stands open in general.

One can now ask whether there are sets of algebraic numbers for which the above statements hold without conditions on the degree. Such sets are precisely those satisfying the properties of Northcott and Bogomolov introduced by Bombieri and Zannier [4]: a set of algebraic numbers $\mathcal{A}$ has the *Northcott property (N)* if it contains finitely many elements of bounded height, while it has the *Bogomolov property (B)* if there exists some constant $C > 0$ such that, for all $\alpha \in \mathcal{A}$, $h(\alpha)$ is either 0 or at least C.

It is easy to see that property (N) implies property (B), while the proof that the converse is false is non trivial. The simplest counterexample is the field $\mathbb{Q}^{ab}$, the maximal abelian extension of $\mathbb{Q}$, which has property (B) by [1] and which clearly fails to have property (N) as it contains infinitely many roots of unity. By Northcott's theorem, both properties hold for number fields. On the other hand none of the properties hold for $\overline{\mathbb{Q}}$, as for instance $h(\sqrt[n]{2})$ tends to 0 as n increases. An interesting (and in general hard) problem is then, given an infinite algebraic extension of $\mathbb{Q}$, to decide whether it satisfies property (N) or (B).

In recent decades there has been a lot of activity around property (B) and we now know many examples of fields that satisfy this property. It holds, for instance, as evoked earlier, for $\mathbb{Q}^{ab}$ (by [1]), for K^{ab}, the maximal abelian extension of a number field K (by [2], [3]) and for the field $\mathbb{Q}(E_{tor})$ obtained by adding to $\mathbb{Q}$ all coordinates of the torsion points of an elliptic curve E defined over $\mathbb{Q}$ (by [6]). All these examples are fields obtained by *adding torsion* to a base field. A second class of examples comes from field satisfying some local conditions: property (B) holds, for instance, for the field $\mathbb{Q}^{tr}$ of totally real numbers (by [7]) and for any Galois extension $L/\mathbb{Q}$ having *bounded local degrees at some prime* p i.e., that can be embedded in a finite extension of $\mathbb{Q}_p$ (by [4]). In particular, it holds for the field of totally p-adic numbers $\mathbb{Q}^{tp}$ and for the field $K^{(d)}$, the compositum of all extensions of a number field K of degree at most d.

Concerning property (N), results continue to be relatively rare. A first result, proved by Bombieri and Zannier ([4, Theorem 1]), is that it is satisfied by $K_{ab}^{(d)}$, the maximal subextension of $K^{(d)}$ being abelian over K, for all d and number fields K. A second result due to Widmer ([8, Theorem 3]) is that it holds for unions of infinite towers of number fields where, at each step, the discriminants grow *enough*. It is natural to ask if there are other examples of fields with (N).

Before stating our result, we need to take a step back to Property (B): in [4, Theorem 2] it is proved that it holds for Galois extensions of $\mathbb{Q}$ with bounded local degree at some prime and, more precisely, that if $L/\mathbb{Q}$ is Galois and $S(L) \neq \emptyset$ is the set of primes at which L has bounded local degrees, then

$$\liminf_{\alpha \in L} h(\alpha) \geq \beta(L) = \frac{1}{2} \sum_{p \in S(L)} \frac{\log p}{e_p(p^{f_p}+1)}$$

where e_p and f_p are, respectively, the ramification index and inertial degree of L at p. The authors of [4] remark also that if $\beta(L) = \infty$ then L has also (N) and that $\beta(L) = \infty$ if L is a number field. They then ask whether there are infinite extensions $L/\mathbb{Q}$ for which $\beta(L)$ is divergent. One can further ask whether there are infinite extensions $L/\mathbb{Q}$ for which $\beta(L)$ is divergent and that are not covered by the results of Bombieri-Zannier and Widmer.

In a joint work with Arno Fehm [5] we positively answer this question proving that there exist infinite Galois extensions $L/\mathbb{Q}$ such that $\beta(L) = \infty$, L is not contained in $\mathbb{Q}_{ab}^{(d)}$ and L does not satisfy Widmer's criterion. We also show that, if one is only interested in the divergence of $\beta(L)$, there is some freedom in choosing the Galois group i.e., that given any infinite product $G = \prod_{i=1}^{\infty} G_i$ of finite solvable groups G_i, there exists $L/\mathbb{Q}$ Galois such that $\mathrm{Gal}(L/\mathbb{Q}) = G$ and $\beta(L) = \infty$.

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Hermite equivalence of polynomials

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(joint work with Manjul Bhargava, Kálmán Győry, László Remete, Ashvin Swaminathan)

In 1857, Hermite [11] introduced an equivalence relation for univariate polynomials in $\mathbb{Z}[X]$, which has remained rather unnoticed. In our work, mostly contained in [4], we compare this with the much better known $\mathrm{GL}_2(\mathbb{Z})$-equivalence.

Let us first consider decomposable forms of degree $n \geq 2$ in n variables $F(\mathbf{X}) = F(X_1, \ldots, X_n) = c\prod_{i=1}^{n}(\alpha_{i,1}X_1 + \cdots + \alpha_{i,n}X_n)$, where the coefficients of F itself are in $\mathbb{Z}$ and the coefficients of its linear factors are algebraic numbers. The discriminant of F is $D(F) = c^2(\det(\alpha_{i,j}))^2$. Call two decomposable forms F, G as above $\mathrm{GL}_n(\mathbb{Z})$-equivalent if there is a matrix $U \in \mathrm{GL}_n(\mathbb{Z})$ such that $G(\mathbf{X}) = \pm F(U\mathbf{X})$. Then F, G have the same discriminant. Hermite proved [9, 10] that the decomposable forms with integer coefficients in n variables of degree $n \geq 2$ and discriminant $D \neq 0$ lie in finitely many $\mathrm{GL}_n(\mathbb{Z})$-equivalence classes. To a polynomial $f = c(X - \alpha_1)\cdots(X - \alpha_n)$ of degree n we associate a decomposable form $[f](\mathbf{X}) = c^{n-1}\prod_{i=1}^{n}(X_1 + \alpha_i X_2 + \alpha_i^2 X_3 + \cdots + \alpha_i^{n-1}X_n)$. If f has integer coefficients then so does $[f]$. Further, by Vandermonde's identity for determinants, $D([f]) = c^{2n-2}\prod_{1\leq i<j\leq n}(\alpha_i - \alpha_j)^2$, which is precisely the discriminant $D(f)$ of f. In accordance with Hermite's definition from 1857, we call two polynomials $f, g \in \mathbb{Z}[X]$ Hermite equivalent if $[f]$ and $[g]$ are $\mathrm{GL}_n(\mathbb{Z})$-equivalent. Then, as observed by Hermite, the polynomials in $\mathbb{Z}[X]$ of given degree $n \geq 2$ and discriminant $D \neq 0$ lie in finitely many Hermite equivalence classes.

A much better known equivalence relation for univariate polynomials is $\mathrm{GL}_2(\mathbb{Z})$-equivalence. We call two polynomials $f, g \in \mathbb{Z}[X]$ of degree n $\mathrm{GL}_2(\mathbb{Z})$-equivalent if there is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ such that $g(X) = \pm(cX + d)^n f((aX + b)/(cX + d))$. It is not hard to show that two $\mathrm{GL}_2(\mathbb{Z})$-equivalent polynomials are Hermite equivalent.

In 1972, Birch and Merriman [5] proved that for any $n \geq 2$ and $D \neq 0$ there are only finitely many $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of polynomials in $\mathbb{Z}[X]$ of degree n and discriminant D; their proof is ineffective. Evertse and Győry [7] gave in 1991 an effective proof, allowing in principle to determine a full system of representatives for the $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes. While in his work Hermite used reduction theory of positive quadratic forms, Birch and Merriman and Evertse and Győry had to use much deeper tools such as finiteness results for unit equations and Bakeer type estimates for logarithmic forms.

In what follows we restrict ourselves to polynomials in $\mathbb{Z}[X]$ that are primitive, i.e., whose coefficients have no common factor, and irreducible. Let $\mathcal{PI}(n)$ denote the set of these polynomials of degree n. Taking a root α of some $f \in \mathcal{PI}(n)$, we define the so-called *invariant order* of f, $\mathbb{Z}_\alpha := \mathbb{Z}[\alpha] \cap \mathbb{Z}[\alpha^{-1}]$. This is an order in the number field $\mathbb{Q}(\alpha)$, which is up to isomorphism uniquely determined by f. We define the *invariant ideal* of f to be the fractional ideal $I_\alpha := \mathbb{Z}_\alpha + \alpha\mathbb{Z}_\alpha$. We are now ready to state some of our results.

It follows from famous work of Delone and Faddeev [6] from 1940 that two polynomials $f, g \in \mathcal{PI}(3)$ are $\mathrm{GL}_2(\mathbb{Z})$-equivalent, if and only if they are Hermite equivalent, if and only if they have isomorphic invariant orders. We now mention some results for polynomials of degree $n \geq 4$.

Theorem 1 ([4])**.** *Let $n \geq 4$ and $f, g \in \mathcal{PI}(n)$. Then f, g are Hermite equivalent if and only if f has a root α and g a root β such that $\mathbb{Z}_\alpha = \mathbb{Z}_\beta$ and I_α and I_β are in the same ideal class of $\mathbb{Z}_\alpha$.*

For $n = 4$ we provide examples of polynomials f, g that have the same invariant order, but with invariant ideals from different ideal classes. Hence these polynomials are not Hermite equivalent.

From work of Bérczes, Evertse and Győry [1], it follows that if K is any number field of degree $n \geq 4$ and $\mathcal{O}$ any order in K, then the polynomials $f \in \mathcal{PI}(n)$ with invariant order isomorphic to $\mathcal{O}$ lie in at most $C(n)$ $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes. Recently, Bhargava [3] obtained the bound $C(4) = 10$, and Evertse and Győry [8] the bound $C(n) = 2^{5n^2}$ for $n \geq 5$. Of course this implies that any Hermite equivalence class of polynomials in $\mathcal{PI}(n)$ falls apart into at most $C(n)$ $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes.

As for lower bounds, at the moment we can do no better than the following.

Theorem 2 ([4])**.** *Let $n \geq 4$. Then $\mathcal{PI}(n)$ contains infinitely many Hermite equivalence classes that are the union of at least two $GL_2(\mathbb{Z})$-equivalence classes.*

We would like to pose the following problem. For a number field K, let $\mathcal{PI}(K)$ denote the set of primitive, irreducible polynomials in $\mathbb{Z}[X]$ having a root that generates K. Is it true that $\mathcal{PI}(K)$ has only finitely many Hermite equivalence classes that fall apart into more than one $\mathrm{GL}_2(\mathbb{Z})$-equivalence class?

From the work of Delone and Faddeev mentioned above, for cubic polynomials Hermite equivalence and $\mathrm{GL}_2(\mathbb{Z})$-equivalence coincide. It follows from work of Bérczes, Evertse and Győry [2] that if K is a number field of degree $n \geq 5$ whose Galois closure has as Galois group the full symmetric group S_n, then $\mathcal{PI}(K)$ contains only finitely many Hermite equivalence classes that are represented by a monic polynomial and fall apart into at least two $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes. Further, they showed that there are number fields K of degree 4 for which this is false.

As for now (unpublished work in progress by JHE) we can prove only the following. Let K be a number field of degree n. Call two Hermite equivalence classes $\mathcal{H}_1$ and $\mathcal{H}_2$ in $\mathcal{PI}(K)$ $\mathrm{GL}_2(\mathbb{Q})$-equivalent if there are $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$ such that $g(X) = \lambda(cX + d)^n f((aX + b)/(cX + d))$ for some $\lambda \in \mathbb{Q}^*$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$.

Theorem 3. *Let K be a number field of degree $n \geq 5$. Suppose that the Galois closure of K has Galois group S_n. Then the Hermite equivalence classes in $\mathcal{PI}(K)$ that fall apart into more than one $GL_2(\mathbb{Z})$-equivalence class lie in finitely many $GL_2(\mathbb{Q})$-equivalence classes.*

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Integral points in orbits under finitely many morphisms

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(joint work with Yu Yasufuku)

A famous theorem of Siegel says that any nonsingular curve of genus 1 defined by a polynomial equation $f(x,y)=0$ with rational coefficients has only finitely many solutions $(x,y)\in\mathbb{Z}^2$. Siegel's proof relies on diophantine approximation results, namely on versions of a celebrated theorem of Roth. We are interested in dynamical analogues of Siegel's results.

Let k be a number field, S be a finite set of places (including all archimedean ones) R_S be the ring of S-integers, X be an algebraic variety (smooth) over k, and $\phi : X\to X$ be a morphism defined over k. We denote the n-iterate of ϕ by $\phi^{\circ n}=\underbrace{\phi\circ\cdots\circ\phi}_{n \text{ times}}$ and the orbit of $P\in X$ by $\mathcal{O}_\phi(P)=\{\phi^{\circ n}(P) : n\geq 0\}$.

In a dynamical analogue of Siegel's problem, one can ask how big can the set of integral points in $\mathcal{O}_\phi(P)$ be.

On $X=\mathbb{P}^1$, finiteness will fail for polynomial mappings, and these form the main set of exceptions for having such sparsity according to the following

Theorem 1. *([5] Silverman 1993) Let $\phi(x)\in k(x)$ of degree $d\geq 2$ with $\phi^{\circ 2}(x)\notin k[x]$. Then $\#(\mathcal{O}_\phi(P)\cap R_S)$ is finite for all $P\in\mathbb{P}^1(k)$.*

For any divisor D of X defined over k and each place v of k, one can associate local heights $\lambda_v(D,\cdot) : X(k)\backslash|D| \to \mathbb{R}$ and a global Weil height $h(D,\cdot) : X(k) \to \mathbb{R}$ such that $\sum_v \lambda_v(D,P) = h(D,P) + O(1)$ for every $P \in X(k) \setminus |D|$.

Extending Silverman's result, Hsia and Silverman proved ([3]), under similar conditions, a quantitative upper bound for the size of the set

$$\{\phi^{\circ n}(P) : \sum_{v\notin S} \lambda_v((\infty), \phi^{\circ n}(P)) \leq (1-\epsilon)h((\infty), \phi^{\circ n}(P))\}$$

of ϵ-quasiintegral points in the orbit of a rational map ϕ on $\mathbb{P}^1$ and $0 < \epsilon < 1$. Among other applications, this result is useful to prove finiteness of points whose orbits have multiplicative dependent iterates [1].

In [4], finiteness of quasiintegral points in orbits on $\mathbb{P}^1$ was established in the context of semigroup dynamics. Namely, given $\phi_1,\dots,\phi_\ell : \mathbb{P}^1 \to \mathbb{P}^1$ morphisms defined over k, $\mathcal{F} = \langle \phi_1, ...\phi_\ell\rangle$ the semigroup generated by $\phi_1,\dots,\phi_\ell$ via composition, and $\mathcal{O}_{\mathcal{F}}(P) = \{\phi(P) : \phi \in \mathcal{F}\}$ the semigroup orbit of $P \in \mathbb{P}^1$, if we suppose that $\deg \phi_i \geq 2$ and ϕ_i is not totally ramified at any point of $\mathcal{O}_{\mathcal{F}}(\infty)$, then $\{\phi(P) : \phi \in \mathcal{F}, \sum_{v\notin S} \lambda_v((\infty), \phi(P)) \leq (1-\epsilon)h((\infty), \phi(P))\}$ is finite for each $0 < \epsilon < 1$. An ingredient to prove such results is a quantitative version of Roth's Theorem. In higher dimension, we have the following deep conjecture.

Conjecture 2. *(Vojta) Let X be a smooth projective variety, D be a reduced normal-crossing (NC) divisor, K be a canonical divisor, and A be an ample divisor, all def. over $\overline{k}$. Then for all $\epsilon > 0$, there exists a Zariski-closed $Z_\epsilon \neq X$ such that for all $P \in X(k)\backslash Z_\epsilon$,*

$$\sum_{v\in S} \lambda_v(D,P) + h(K,P) < \epsilon h(A,P) + O(1).$$

When $X = \mathbb{P}^n$ and D is a NC union of hyperplanes, this conjecture becomes Schmidt's subspace theorem. It also provides the context for generalizations to higher dimension. The one below recovers Theorem 1 in dimension 1.

Theorem 3. *([6] Yasufuku 2015) Let $\phi : \mathbb{P}^N(k) \to \mathbb{P}^N(k)$ be a morphism of degree $d \geq 2$. Let D be a divisor on $\mathbb{P}^N$ def. over k and $D^{\circ n}_{nc}$ be the red. normal-crossings part of $(\phi^{\circ n})^*(D)$. Let $c_n = \dfrac{\deg D^{\circ n}_{nc} - (N+1)}{\deg(D)d^n}$. Let $P \in \mathbb{P}^N(k)$ and $\epsilon > 0$. Then, assuming Conjecture 2 for $D^{\circ n}_{nc}$, the following set is Zariski-non-dense:*

$$\left\{\phi^{\circ m}(P) : \sum_{v\notin S} \lambda_v(D, \phi^{\circ m}(P)) \leq (c_n - \epsilon)h(D, \phi^{\circ m}(P))\right\}.$$

For projective varieties and semigroups, we obtain the following

Theorem 4. *(Mello, Yasufuku) Let X be a smooth projective variety, K a canonical divisor, A an ample divisor, $\phi_1,\dots,\phi_\ell$ endomorphisms of X, all def. over k, and $\mathcal{F} = \langle\phi_1,\dots,\phi_\ell\rangle$. Suppose there exist $\psi_1,\dots,\psi_m \in \mathcal{F}$ and an effective divisor D def. over k such that*

(i) $\psi_i^*(D) = D_i^{(nc)} + D_i'$, $D_i^{(nc)}$ is NC, and $D_i^{(nc)} + K$ is big for all i

(ii) $\mathcal{F} \setminus \bigcup_{i=1}^{m} (\psi_i \circ \mathcal{F})$ *is finite.*

Then, assuming Conjecture 2 for $(X, D_i^{(nc)})$*, there exists* $\epsilon > 0$ *such that for all* $P \in X(k)$*, the set below is not Zariski-dense:*

$$\left\{ \phi(P) : \phi \in \mathcal{F}, \sum_{v \notin S} \lambda_v(D, \phi(P)) \leq \epsilon h(A, \phi(P)) \right\}.$$

We also use a quantitative generalization of Schmidt's subspace theorem due to Evertse and Ferretti ([2]) to prove unconditional results.

Theorem 5. *(Mello, Yasufuku) Let* $\phi_1, \ldots, \phi_\ell$ *be endomorphisms of* $\mathbb{P}^N$ *def. over* k *and* $\mathcal{F} = \langle \phi_1, \ldots, \phi_\ell \rangle$*. Suppose there exist* $\psi_1, \ldots, \psi_m \in \mathcal{F}$ *and an effective divisor* D *of* $\mathbb{P}^N$ *def. over* k *such that*

(i) $\psi_i^*(D) = D_{i1} + D_{i2} + \cdots + D_{i,q_i} + D_i'$ *is effective,* $D_{i1}, \ldots, D_{i,q_i}$ *are in general position, and* $q_i > N + 1$ *for all* i

(ii) $\mathcal{F} \setminus \bigcup_{i=1}^{m} (\psi_i \circ \mathcal{F})$ *is finite.*

Then there exists $\epsilon \in (0, 1)$ *such that for any* $P \in \mathbb{P}^N(k)$*, the following set is not Zariski-dense:*

$$\left\{ \phi(P) : \phi \in \mathcal{F}, \sum_{v \notin S} \lambda_v(D, \phi(P)) \leq \epsilon h(\phi(P)) \right\}.$$

Example 6. Let $\phi_1 = [L_1 L_2 : L_3 L_4 : XY]$, and $\phi_2 = [G_0 : G_1 : L_5 L_6 L_7 L_8]$ on $\mathbb{P}^2(k)$, where each L_i is a line, $X = 0, Y = 0, L_1, \ldots, L_8$ are in general position, G_i is a homogeneous polynomial of degree 4, $L_5, \ldots, L_8$ do not go through any points of $(G_0 = 0) \cap (G_1 = 0)$, and $G_0 G_1 = 0$ contains at least 4 lines in general position. If $D = (Z = 0)$, then $\phi_2^*(D) = (L_5 L_6 L_7 L_8 = 0)$, $(\phi_1^{\circ 2})^*(D) = (L_1 L_2 L_3 L_4 = 0)$, and $(\phi_1 \circ \phi_2)^*(D) = (G_0 G_1 = 0)$. Each of these divisors contains a NC subdivisor consisting of lines of degree at least 4. In this case, theorems 4 and 5 reduce to Schmidt subspace theorem, and since $\mathcal{F} \setminus (\phi_1^{\circ 2} \circ \mathcal{F}) \cup (\phi_2 \circ \mathcal{F}) \cup ((\phi_1 \circ \phi_2) \circ \mathcal{F}) = \{\phi_1\}$, we unconditionally obtain that $\mathcal{O}_{\mathcal{F}}(P) \cap \{[a : b : 1] : a, b \in R_S\}$ is contained in a finite union of curves for any $P \in \mathbb{P}^2$.

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Some new elliptic integrals

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(joint work with Umberto Zannier)

In 1981 James Davenport published his Ph.D. thesis [2] about the computer-assisted integration of algebraic functions of a single variable. Most of the book concerned specific integrals like

$$\int \frac{\mathrm{d}x}{x-\tau},\ \int \frac{\mathrm{d}x}{(x^2-\tau^2)\sqrt{x^3-x}}. \tag{22}$$

The focus was on the possibility of doing the integration by elementary means; that is, by using functions built up from the field $\mathbf{C}(x)$ through taking algebraic extensions like $\mathbf{C}(x,\sqrt{x^3-x})$ or logarithmic extensions like $\mathbf{C}(x,\log x)$ or exponential extensions like $\mathbf{C}(x,\exp x)$. Or all three:

$$\mathbf{C}(x,\sqrt[3]{x^4-1},\sqrt[5]{\log\log x},\exp(\sqrt[6]{\log(1+\exp(x^7))})).$$

However Davenport did make some speculations about one-parameter families like

$$\int \frac{\mathrm{d}x}{x-t},\ \int \frac{\mathrm{d}x}{(x^2-t^2)\sqrt{x^3-x}}. \tag{23}$$

where now t is an independent variable. For a general $\int f(x,t)\mathrm{d}x$, where the integrand is algebraic over $\mathbf{C}(x,t)$, he surmised that there are at most finitely many complex values τ of t such that $\int f(x,\tau)\mathrm{d}x$ can be done by elementary means unless there is the obvious obstruction. Namely that $\int f(x,t)\mathrm{d}x$ itself cannot be done by elementary means using extensions as above but starting with $\mathbf{C}(x,t)$ instead of $\mathbf{C}(x)$, as for example in the first of (23), which is of course $\log(x-t)$ and so gives $\log(x-\tau)$ in the first of (22).

In 2020 Zannier and I [4] obtained two counterexamples to Davenport's surmise (and in principle we could characterize them all, at least over the algebraic closure of $\mathbf{Q}(x,t)$ - they are somewhat rare), namely

$$\int \frac{x\mathrm{d}x}{(x^2-t^2)\sqrt{x^3-x}},\ \int \frac{x\mathrm{d}x}{(x^2+tx+t^2)\sqrt{x^3-1}}, \tag{24}$$

the first not quite as in the second of (23) - indeed that is not a counterexample, so that there are indeed at most finitely many τ such that the second of (22) can be done by elementary means. Originally we had included

$$\frac{((5t^2+40t+62)x+t^3+8t^2+70t+144)\mathrm{d}x}{(x-t)((2t+8)x+t^2+4t+18)\sqrt{x^3-30x-56}} \tag{25}$$

as a third counterexample, but Detmar Welz expressed scepticism and then we found a mistake in our calculations. Going further we could show that there are at most 138 complex τ for (25).

Now (24) involves elliptic curves with complex multiplication by $\sqrt{-1}$ and $(-1+\sqrt{-3})/2$, while (25) has $\sqrt{-2}$. In [4] we then expressed the opinion that there are no counterexamples with $\sqrt{-2}$.

Recently in trying to streamline our argument for (25) we saw how to use the Weierstrass zeta function $\zeta(z)$ to reduce 138 to 0 (so no τ at all). At the same time we could then after all construct a counterexample simply by adding 1 to the rational part of the integrand, leading to

$$\frac{(x^2+(2t+10)x+2t+18)\mathrm{d}x}{(x-t)((2t+8)x+t^2+4t+18)\sqrt{x^3-30x-56}}. \tag{26}$$

We could even construct similar counterexamples for any (imaginary) $\sqrt{-d}$ and show that they are essentially unique. That for $\sqrt{-43}$ has degree 105 in t and coefficients with about 170 digits.

The relevant fact for ζ in (26) is

$$\zeta(\sqrt{-2}z)+\sqrt{-2}\zeta(z)=2\sqrt{-2}z-\frac{\sqrt{-2}}{4}\frac{\wp'(z)}{\wp(z)+4} \tag{27}$$

where of course $\wp'(z)^2=4(\wp(z)^3-30\wp(z)-56)$. Such things can already be found in my own Ph.D. thesis [3]. And there Lemma 3.2 hints why we got stuck at (24).

Analogues of (27) for the Weierstrass sigma function appear to be related to the phenomenon of Ribet curves discovered by Daniel Bertrand [1].

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