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Universality: Random Matrices, Random Geometry and SPDEs

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Abstract. The postulate that large random systems can be described by limiting objects whose characteristic do not depend on the exact details of the models one started from is central in probability theory, under the name of universality. This workshop was aimed at uncovering the latest developments of this concept in the various topics where it is relevant, namely statistical physics, stochastic partial differential equations, random geometries and random matrices. It was in particular the occasion to feature some important recently introduced universal objects like the stochastic quantization of the Yang-Mills measure in dimensions 2 and 3, the KPZ fixed point, Liouville quantum gravity metrics and other objects connected to the Gaussian free field.

Mathematics Subject Classification (2020): Primary: 60-XX; Secondary: 81-XX, 83-XX.

Introduction by the Organizers

The workshop *Universality: Random Matrices, Random Geometry and SPDEs*, organised by Martin Hairer (London), Grégory Miermont (Lyon) and Horng-Tzer Yau (Cambridge, MA), was initially planned to happen in 2020 but had to be postponed due to the Covid pandemic. Faithful to a long tradition of workshops around the general theme of *stochastic analysis*, it aimed a presenting a panoramic view of modern probability theory and interacting topics. It focused on the important concept of *universality*, according to which a wide variety of large random systems are attracted to a small number of limiting random objects, which are often prone to explicit calculations and allow to shed light on the systems of interest.

The week started with a thematic morning on the Yang-Mills model. Ajay Chandra and Ilya Chevyrev reported their recent work with Hairer and Shen on the stochastic quantization of the Yang-Mills measure in dimensions 2 and 3. Among the highlights of this work are fact that the renormalization of the stochastic Yang-Mills heat flow can be made compatible with the action of the gauge group, and the construction of a Markov process in the gauge orbit space onto which the stochastic Yang-Mills equation projects. Hao Shen closed this morning by discussing his work with Smith, Zhu and Zhu on the dynamical lattice Yang-Mills measure, allowing to obtain properties of the lattice Yang-Mills measure via spectral and geometric techniques from Markov process theory. The Monday afternoon talks focused on statistical physics. Christophe Garban presented his ongoing work with Aru and Sepulveda on vector-valued Gaussian free field and their connection with planar $O(n)$ models, showing how this may help understand the absence of phase transition in $O(3)$ spin models predicted by Polyakov via a connection with XY models. Alain Sznitman presented his work on the fine analysis of the spectral gap in the Kac-Luttinger model of a Bose gas interacting with a Poisson cloud of hard obstacles and its consequences for the density of states of the model, and on Bose-Einstein condensation above a critical density. Antti Knowles discussed his work with Fröhlich, Schlein and Sohinger on interacting Bose gas converging to a Euclidean Φ^4_2 theory, by making rigorous a number of formal field-theoretic integral representation in terms of models of Brownian bridges.

On Tuesday, we focused on the various guises of the KPZ universality class. Alessandra Occelli described the integrable aspects of new last-passage percolation growth models in $\mathbb{Z}^2$. Jeremy Quastel showed how Skorokhod renormalization allows to give Feynman-Kac representations of solutions of the stocahstic heat equation, from which KPZ fluctuations can be easily deduced. Balint Virag gave an overview of the *directed landscape*, a universal object of crucial importance as it is one way to represent the universal *KPZ fixed point* as a random directed distance function on the plane. Amol Aggarwal presented his recent work with Gorin and Huang on the universality of lozenge tiling statistics, another model in the KPZ class, showing convergence to GUE corner process or Airy line ensembles for a wide class of tileable domains. In a related spirit, Benjamin Landon discussed his work with Noack and Sosoe on interacting diffusions via a class of potentials for which KPZ-type fluctuations can be observed, hinting again at the universality of these models. We closed this day with the different topic of random matrices. Tatyana Shcherbina, who gave the only remote talk of the conference, presented her work with Mariya Shcherbina on extreme singular values for deformed Ginibre ensembles via supersymmetry techniques, generalizing recent results of Cipolloni, Erdős and Schröder.

Wednesday started with a talk by Nicolas Curien on the parking model on the infinite binary tree, joint with Aldous, Contat and Hénard, showing a surprisingly general criterion for (sub)criticality of the model, and interesting connections with Tutte's quadratic equations typical of the theory of enumeration of maps. Ofer Zeitouni discussed directed polymers in weak disorder. In joint work with Cosco, he shows high moment bounds for the partition function, as a first step towards the goal of showing convergence to Gaussian multiplicative chaoses. Ewain Gwynne closed this morning session with his work with Ding on supercritical Liouville quantum gravity metrics, a family of random metrics on the complex plane that complete the subcritical LQG metrics. The afternoon was the opportunity to take a walk through the forest to Oberwolfach.

On Thursday, Scott Sheffield discussed an ongoing project with Chandgotia and Wolfram on the three-dimensional dimer model and the generalization of arctic curves phenomena supported by simulations. In a joint talk, Alice Guionnet and Raphaël Ducatez presented their recent results with Augeri and Cook dealing with large deviation principles for the edge of the spectrum of random Wigner matrices with sub-Gaussian entries, based on the analysis of spherical integrals and showing a delicate modification of the usual LDP functional in the form of a phase transition in the sub-Gaussianity constant. Giuseppe Cannizzaro then presented his work with Haunschmidt and Toninelli on the singular stochastic differential equation for a Brownian motion drifted by the curl of a GFF, showing the Tóth–Valkó conjecture that this process is superdiffusive with logarithmic divergence away from diffusivity. In the afternoon, Pierre-François Rodriguez presented exciting results with Drewitz and Prévost on percolation models with long-range dependence derived from GFF level lines on $\mathbb{Z}^d$. Jean-François Le Gall presented an intriguing property for the local times of Brownian motion indexed by the Brownian tree, showing that the joint process of local times and its derivative forms a Markov process with explicit transition probabilities. Finally, Jason Miller showed that chemical distances inherited from conformal loop ensembles rescale to a new family of random geodesic metrics in the plane, that are yet different from the LQG metrics he characterized with Gwynne.

Friday morning continued with random planar metrics with the talk by Ahmed Bou-Rabee, who showed in joint work with Gwynne how to construct a notion of harmonic balls for the LQG measures, i.e. sets E satisfying that averages of harmonic functions f on E have average $f(0)$, and motivated by the study of internal DLA in random maps. He showed how weak Harnack inequalities allow to uniquely construct a family of such balls, which in turn are the limits of DLA on Tutte-embedded mated-CRT maps. Dmitry Beliaev surveyed his recent joint work with McAuley and Muirhead on the number of level sets of smooth Gaussian field and their asymptotics. Florian Schweiger, in joint work with Zeitouni, extended the class of extreme theory for log-correlated fields by studying the case of log-correlated fields in random environment. To wrap up in a cyclic way, Massimiliano Gubinelli closed this conference by presenting his views on stochastic quantization in quantum field theory.

The workshop was well-attended, with about 35 participants on-site. About 20 participants registered online.

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Workshop: Universality: Random Matrices, Random Geometry and SPDEs

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Abstracts

On the number of level sets of smooth Gaussian fields

Dmitry Beliaev
(joint work with Michael McAuley, Stephen Muirhead)

In a series of papers we have studied the number of level and excursion sets of smooth Gaussian fields. Throughout this text we assume that $F(x)$ is a stationary Gaussian field defined in $\mathbb{R}^n$. We also assume that with probability one F is sufficiently smooth, we always require it to be at least C^2, but for some results we require more derivatives. In reality, all examples that we really care about are real analytic functions, so the smoothness assumption is not too restrictive.

The number of level sets could be thought of as a higher dimensional version of the number of level crossings of Gaussian processes. Indeed, in one-dimensional case the level set $\{F(x) = \ell\}$ is just a collection of point and excursion sets $\{F(x) \geq \ell\}$ are intervals between them. In higher dimensional case, level sets i.e. connected components of $\{F(x) = \ell\}$ have non-trivial geometry and topology, so the question is more involved. In particular, the number of level sets inside a domain is not an additive function of a domain, which makes things even more complicated and makes many one-dimensional tools inapplicable.

The other reason for looking at the number of level sets is the connection to the percolation theory and works of physicists who have related the statistical behaviour of the number of level sets with quantum chaos questions.

The starting point in the investigation of the number of level/excursion sets is the breakthrough paper [1] where Nazarov and Sodin obtained the Law of Large Numbers for the number of nodal domains for spherical random harmonics. Later they extended their result to more general setting [2]. Although both papers are written in the case of nodal domains, the argument works verbatim for other levels. Roughly speaking, they have shown that under some mild non-degeneracy and regularity assumptions

$$\frac{N_{LS,ES}(R,\ell)}{R^n} \to c_{LS,ES}(\ell)$$

where $N(R,\ell)$ is the number of either level sets (LS) or excursion sets (ES) of the field F inside the cube of side-length R.

In papers [4] and [5] we have studied how $c(\ell)$ depends on the level ℓ in the planar case. Usual Morse theory arguments give that the level set changes smoothly as long as ℓ does not cross a critical level. There are also some boundary effects that we ignore for now. When we cross the level of a local maximum, then one component of the level set and one component of the excursion set disappear. When we cross the level of a local minimum, then one level set appears, but the number of excursion sets does not change. The behaviour at saddles is a bit more complicated, there are two types of saddles (we call them positively and negatively connected saddles). In the first case one excursion set is split into two and one level set is split into two (a 'barbell' domain is split into two 'disc' domains), in the other case, the number of excursion sets does not change, and two level sets merge into one (a 'ring' domain turns into a 'horseshoe' domain). We have shown that under mild regularity assumptions and assuming that the covariance kernel decays sufficiently fast all non-degenerate critical points can be split into these four categories and each type has a continuous density. We show that

$$c'_{ES}(\ell) = p_{m^+}(\ell) - p_{s^-}(\ell)$$
$$c'_{LS}(\ell) = p_{m^+}(\ell) - p_{s^-}(\ell) + p_{s^+}(\ell) - p_{m^-}(\ell),$$

where $p_{m^+}, p_{m^-}, p_{s^+}, p_{s^-}$ are densities of local maxima, minima, positive and negative connected saddles at level ℓ.

Using similar ideas, we have obtained a lower bound on the variance of $N(R,\ell)$. We have shown in [6] that if F satisfies similar non-degeneracy and regularity assumptions, the Fourier transform of the covariance kernel (so called spectral measure) is bounded and bounded from zero in some neighbourhood of the origin, there is a certain estimate of '4-arm saddles' and $c'(\ell) \neq 0$, then the variance of $N(R,\ell)$ is of order at least R^2. It is not immediately clear how to verify all assumptions of the theorem, but it has been shown that the required '4-arm' estimate hold for all fields with non-negative and sufficiently fast decaying correlation. It is believed to be true for a wide class of fields. Finally, it is believed that $c'(\ell) \neq 0$ for all but a few exceptional levels. This is known for many fields and levels. Notably, by symmetry $c'_{LS}(0) = 0$, so our theorem is not applicable to nodal level sets. It is interesting to compare this result to a complementary result of Nazarov and Sodin. In [3] they have shown that under extremely mild conditions the variance of the number of nodal sets grows at least as R^δ for some $\delta > 0$.

We also want to point out that our result is not applicable to a very important case of the random plane wave. Using slightly different methods we can show that in this case if $\ell \neq 0$ and $c'(\ell) \neq 0$ then the variance is of order at least R^3.

Finally, in [7] we have shown that for a field in $\mathbb{R}^n$ which satisfies rather strong assumption on the decay of correlations and smoothness there is a Central Limit Theorem. Namely $(N(R,\ell) - \mathbb{E}N(R,\ell))/R^{n/2}$ converges to the normal variable with variance $\sigma(\ell)$. This result holds both for level and excursion sets. Assuming additionally that the integral of the covariance kernel is strictly positive, we can show that $\sigma > 0$ for all levels ℓ.

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Harmonic balls in Liouville Quantum Gravity

Ahmed Bou-Rabee
(joint work with Ewain Gwynne)

Let μ be a Radon measure on $\mathbb{C}$. A *harmonic ball* is an open set $E \subset \mathbb{C}$ containing the origin which satisfies the mean-value property

$$f(0) = \frac{1}{\mu(E)} \int_E f(z)\mu(dz), \tag{1}$$

for all functions $f : \mathbb{C} \to \mathbb{R}$ which are harmonic in E. We construct and analyze harmonic balls when μ is the *Liouville Quantum Gravity measure* — a Radon measure informally given by

$$\mu_h(dz) = e^{\gamma h(z)} dz \tag{2}$$

where h is a Gaussian free field and $\gamma \in (0, 2)$ is a constant.

Specifically, we construct a family of harmonic balls $\{\Lambda_t\}_{t>0}$ satisfying the following properties:

- $\mu_h(\Lambda_t) = t$ and $\mu_h(\partial\Lambda_t) = 0$ for each t
- $\{\Lambda_t\}_{t>0}$ is continuous and monotone in t.

We show that this family is essentially unique; that is, any other harmonic balls $\{\Lambda'_t\}_{t>0}$ satisfying the above properties coincide with $\{\Lambda_t\}_{t>0}$ up to a set of μ_h-measure zero. We also show that Λ_t is neither convex nor an LQG-metric ball for Lebesgue almost every t.

A key tool in proving these results is (a minor variation of) the following weak Harnack-type estimate:

$$\mu_h(\Lambda_t \cap B_r(x_0)) \le \alpha\mu_h(B_r(x_0)) \implies \Lambda_t \cap B_{r/2}(x_0) = \emptyset, \tag{3}$$

where $B_r(x_0)$ is the Euclidean ball of radius r centered at x_0, $t > 0$ is arbitrary, and $\alpha \in (0, 1)$ is small. The estimate (3) may be thought of as a continuum analogue of the no-thin-tentacles lemma of Jerison-Levine-Sheffield (2012).

Our main motivation for studying harmonic balls is that they are the conjectured scaling limit of internal diffusion limited aggregation on random planar maps.

Diffusion in the curl of the two-dimensional Gaussian Free Field

Giuseppe Cannizzaro
(joint work with L. Haunschid, F. Toninelli)

In the present talk, we study the motion of a Brownian particle in $\mathbb{R}^2$, subject to a random, time-independent drift ω given by the curl of the two-dimensional Gaussian Free Field ($2d$ GFF). Namely, we look at the SDE which is (formally) given by

$$\mathrm{d}X(t) = \omega(X(t))\mathrm{d}t + \mathrm{d}B(t), \tag{1}$$

where $B(t)$ is a standard two-dimensional Brownian motion and the two-dimensional vector field ω is defined as

$$x = (x_1, x_2) \mapsto \omega(x) = (\partial_{x_2}\xi(x), -\partial_{x_1}\xi(x)), \tag{2}$$

with ξ the $2d$ GFF. As written, (1) is ill-posed due to the singularity of the drift ω. In fact, not only classical stochastic analytical tools would fail in characterising (even) its law but it would also be *critical* for the recent techniques established in [4] as its spatial regularity is way below the threshold identified therein. Nevertheless, we are interested in its large time behaviour and hence we regularise ξ by convolving it with a C^∞ bump function, so that ω is well-defined pointwise and smooth. Note that the vector field ω is everywhere orthogonal to the gradient of the field ξ, and therefore parallel to its level lines. As a consequence, the particle is subject to two very different mechanisms: the drift tends to push the motion *along the level lines of the GFF* (which are finite as we smoothened ξ, but have long tails), while the Brownian noise tends to make it diffuse *isotropically.* Our main theorem is a *sharp* superdiffusivity result: the mean square displacement $\mathbf{E}[|X(t)|^2]$, under the joint law of the Brownian noise and of the random drift, is of order $t\sqrt{\log t}$ for $t \to \infty$, up to multiplicative loglog corrections. This proves a conjecture of B. Tóth and B. Valkó [13] and, in a broader perspective, it is the first proof of the $\sqrt{\log t}$*-superdiffusivity phenomenon* conjectured to occur in a large class of (self-)interacting diffusive systems in dimension $d = 2$, including self-repelling random walks and polymers [2, 13], lattice gas models [1, 10], and, more recently, the two-dimensional Anisotropic KPZ equation ($2d$ AKPZ) [5].

To put the model and the result into context, let us observe first that the vector field ω is divergence-free and that its law is translation-invariant and ergodic. Brownian diffusions in ergodic, divergence-free vector fields have been introduced in the Physics and Mathematics literature as a (toy) model for a tracer particle evolving in an incompressible turbulent flow. If the energy spectrum of the vector field (i.e. the Fourier transform $e(p)$ of the trace of the covariance matrix $R(x - y) = \{\mathbb{E}(\omega_a(x)\omega_b(y))\}_{a,b\leq d}$, with d the space dimension) satisfies the integrability condition

$$\int_{\mathbb{R}^d} \frac{e(p)}{|p|^2}\mathrm{d}p < \infty, \tag{3}$$

the behaviour of the particle is known to be diffusive on large scales [7, 12]. In the robustly superdiffusive case, where the integral in (3) has a power-law divergence for small p, it turns out that $\mathbf{E}[|X(t)|^2]$ grows like t^ν for some $\nu > 1$ [8]. The case under consideration in this work instead, where $d = 2$ and ω is the curl of the GFF, is precisely at the boundary between the two - $e(p)$ is essentially constant for p small, the integral (3) diverges logarithmically at small momenta and logarithmic corrections to diffusivity are expected.

In order to state the main result, we need some notations. Let $U\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth function in $C^\infty(\mathbb{R}^2)$, radially symmetric, decaying sufficiently fast at infinity (say, exponentially) and such that $\int_{\mathbb{R}^2} U(x)\mathrm{d}x = 1$, and set $\tilde{\xi} := \xi * U$, where ξ is a Gaussian free field on $\mathbb{R}^2$. Let X be the solution to (1) but with ω defined according to (2) but with $\tilde{\xi}$ replacing the ξ therein. For any $\lambda > 0$, we define the Laplace transform of the *(bulk) diffusion coefficient* of X according to

$$D(\lambda) := \int_0^\infty e^{-\lambda t}\mathbf{E}(|X(t)|^2)\mathrm{d}t\,. \tag{4}$$

Then, we have the following theorem.

Theorem 1. [6, Theorem 2.2] *For every $\varepsilon > 0$ there exists constants $C_\pm(\varepsilon)$ such that, for every $0 < \lambda < 1$,*

$$C_-(\varepsilon)\big(\log|\log\lambda|\big)^{-1-\varepsilon} \le \lambda^2 \frac{D(\lambda)}{\sqrt{|\log\lambda|}} \le C_+(\varepsilon)\big(\log|\log\lambda|\big)^{1+\varepsilon}. \tag{5}$$

where D is defined according to (4).

Remark 1. *By a well-established argument (see* [11]*) the upper bound in* (5) *implies an upper bound for the diffusivity in real time of the form*

$$\mathbf{E}(|X(t)|^2) \le O\left(t\sqrt{\log t}\,(\log\log t)^{1+\varepsilon}\right).$$

Deducing a pointwise (in time) lower bound on $\mathbf{E}(|X(t)|^2)$ from the behaviour for $\lambda \to 0$ of the Laplace transform is much more delicate. That said, one can easily get (applying for instance [3, Theorem 1.7.1]*) the following*

$$\limsup_{t\to\infty} \frac{\mathbf{E}(|X(t)|^2)}{t\sqrt{\log t}\,(\log\log t)^{-1-\varepsilon}} > 0\,.$$

The argument we exploit in order to prove the theorem above is based on an iterative analysis of the resolvent of the generator of the Markov process given by the environment seen from the particle, i.e. the process $t \mapsto \omega_t(\cdot) := \omega(X_t + \cdot)$. The method is inspired by that introduced in [9] and later employed by H.-T. Yau [14] to prove $(\log t)^{2/3}$ corrections to the diffusivity of the two-dimensional Asymmetric Simple Exclusion Process ($2d$ ASEP) and, more closely, by the techniques developed in [5] to determine logarithmically superdiffusive behaviour for the $2d$ AKPZ equation. Note that the exponent $2/3$ of the logarithmic corrections of $2d$ ASEP is different from the exponent $1/2$ in the present setting, reflecting the fact that the two models belong to two different universality classes, as emphasized already in [10, 13]. From a technical point of view, a crucial difference between the two models is that for $2d$ ASEP the iterative method in [14] provides, at each step k of the recursion, upper/lower bounds for $D(t)$ of the form $(\log t)^{\nu_k}$, with

ν_k converging exponentially fast to $2/3$ as $k \to \infty$. In our case, on the other hand, at step k the method naturally provides lower (resp. upper) bounds of order $(\log\log t)^k/k!$ (resp. $k!\log t/(\log\log t)^k$) and we have to run the iteration for a number of steps of order $k = k(t) \approx \log\log t$ (instead of $k(t) \approx \log\log\log t$ as in [14]) to reach the final result. As a consequence, in contrast with [14], we cannot afford to lose a multiplicative constant at each step of the iteration (such multiplicative constants are responsible for the sub-optimal result in the first version of [5]), and a much finer analysis of the resolvent is needed. Further, we get a significantly sharper control of sub-leading corrections to $D(t)$ with respect to $2d$ ASEP, namely, a multiplicative correction that is polynomial in $\log\log t$, to be compared with the corrections of order $\exp((\log\log\log t)^2)$ for $2d$ ASEP [14].

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Stochastic quantisation of Yang–Mills: state space and Markov process

Ilya Chevyrev
(joint work with Ajay Chandra, Martin Hairer, Hao Shen)

We report on results from [3, 4] which analyse the stochastic quantisation equation of Yang–Mills (YM) on the 2- and 3-dimensional torus. Following ideas of Parisi–Wu, the mathematical goal is to make rigorous sense of the YM measure on $\mathbb{T}^d$ for $d = 2, 3$ by studying the associated Langevin dynamic

$$\text{(SYM)} \qquad \partial_t A = -\,\mathrm{d}_A F_A - \mathrm{d}_A\, \mathrm{d}^* A + \xi\ .$$

Here $A = (A_1, \dots, A_d)\colon \mathbb{T}^d \to \mathfrak{g}^d$ is a $\mathfrak{g}$-valued 1-form, F_A is the curvature of A, $\mathfrak{g}$ is the Lie algebra of a compact Lie group G, and ξ is a canonical white noise.

One of the main results of [3, 4] is that there exists a natural solution to (SYM) obtained as the limit of renormalised smooth solutions when ξ is mollified; see the abstract of Ajay Chandra (see also [5] for a survey). This solution projects down to a Markov process on a suitable space of gauge orbits for which the YM measure is conjecturally invariant. We note that the construction of the YM measure on $\mathbb{T}^2$ is well-known but that on $\mathbb{T}^3$ is open.

In this abstract, we summarise the construction of the Markov process on gauge orbits along with a suitable state space for the solution to (SYM). In 2D, a natural state space is a Banach space $(\Omega^1_\alpha, |\cdot|_\alpha)$ of distributional $\mathfrak{g}$-valued 1-forms that embeds into the Hölder–Besov space $\mathcal{C}^{\alpha-1}$, where $\alpha \in (\frac{2}{3}, 1)$ is a regularity parameter. We define Ω^1_α as the closure of smooth 1-forms under the norm

$$(1) \qquad |A|_\alpha = \sup_\ell |\ell|^{-\alpha} |A(\ell)| + \sup_P |P|^{-\alpha/2} |A(\partial P)|\ ,$$

where the first supremum is taken over line segments ℓ in $\mathbb{T}^2$ with length $0 < |\ell| < 1$ and the second supremum is over oriented triangles $P = (\ell_1, \ell_2, \ell_3)$ with area $|P| > 0$. Here $A(\ell)$ is the line integral of A along ℓ, and $A(\partial P) = \sum_{i=1}^3 A(\ell_i)$ is the integral around the boundary of P.

The space Ω^1_α has a number of desirable properties, including that holonomies (and thus Wilson loops) are well-defined along $\mathcal{C}^2$ curves for all $A \in \Omega^1_\alpha$ and that the gauge group $\mathcal{C}^\alpha(\mathbb{T}^2, G)$ acts continuously on Ω^1_α by $(A, g) \mapsto A^g := gAg^{-1} - (\mathrm{d}g)g^{-1}$. Classical gauge equivalence $\sim$ thus extends to Ω^1_α via $A \sim B \Leftrightarrow \exists g \in \mathcal{C}^\alpha,\ A^g = B$. The quotient space of gauge orbits $\Omega^1_\alpha/{\sim}$ is furthermore Polish. Finally, one has a simple but useful estimate following from Young ODE theory

$$(2) \qquad |g|_{\mathcal{C}^\alpha} \le C_\alpha (|A|_\alpha + |A^g|_\alpha)\ .$$

The construction of the associated Markov process on $\Omega^1_\alpha/{\sim}$ proceeds by considering two solutions A_t and B_t to (SYM) with gauge-equivalent initial conditions $A_0 \sim B_0$. It is possible to couple A_t and B_t so that $B_t = A_t^{g_t}$, where g solves

$$(3) \qquad g^{-1}\partial_t g = \partial_j (g^{-1}\partial_j g) + [A_j, g^{-1}\partial_j g]\ .$$

One then defines a sequence of stopping times $\tau_1, \tau_2, \dots$ which stop the processes A and B when one of them, say A, becomes significantly larger than $\inf_{X\sim A} |X|_\alpha$. At each stopping time, we find a representative of the equivalence class of A which nearly attains this infimum and restart the process. By construction, it is clear that $A_t^{g_t} = B_t$ for all t up until blow up of $|A|)\alpha \wedge |B|_\alpha$, where g_t solves (3) on each interval (τ_i, τ_{i+1}), which proves the Markov property for the projected process $[A_t] \in \Omega^1_\alpha/\sim$. Crucial in this construction is the estimate (2) which ensures that g does not blow up before A or B does.

We now turn to the 3D case. Unlike in 2D, even the Gaussian free field (GFF) in 3D cannot be integrated along lines, which means we need a different strategy. We instead first define a suitable space $\mathcal{I}$ of initial conditions for $\{\mathcal{F}_t\}_{t>0}$ which is the solution map of the DeTurck–YM heat flow – this flow is given by the PDE (SYM) but without ξ. We define $\mathcal{I}$ as the closure of smooth 1-forms under the metric

$$\Theta(A,B) = |A-B|_{\mathcal{C}^{-\frac{1}{2}-\kappa}} + \sup_{t\in(0,1)} t^{1-\kappa}|e^{t\Delta}A\otimes\nabla e^{t\Delta}A - e^{t\Delta}B\otimes\nabla e^{t\Delta}B|_{\mathcal{C}^{-3\kappa}}$$

for $\kappa>0$ sufficiently small. One can show that $\mathcal{F}$ is well-posed and locally Lipschtiz on $\mathcal{I}$. By gauge covariance, $\sim$ extends to $\mathcal{I}$ via $A\sim B \Leftrightarrow \exists t>0,\ \mathcal{F}_t(A)\sim\mathcal{F}_t(B)$.

The second term in the definition of Θ controls the quadratic terms in (SYM). The GFF furthermore takes values in $\mathcal{I}$, which proves well-posedness of the YM heat flow for GFF. Similar well-posedness results and state space $\mathcal{I}$ independently appeared in [1, 2]. Note that $\mathcal{I}$ is non-linear, and this cannot be avoided: there is norm inflation, and thus ill-posedness, for $\mathcal{F}$ in every Banach space that supports the GFF on $\mathbb{T}^3$ [6] (and even the endpoint space $\mathcal{C}^{-1/2}$).

Unfortunately, for the space $\mathcal{I}$, there appears to be no analogue of the estimate (2). We therefore define the final state space in 3D as the closure of smooth 1-forms under the strengthened metric

$$\Sigma(A,B) = \Theta(A,B) + \sup_{t\in(0,1)}\sup_{|\ell|<t^\kappa} |\ell|^{-\frac{1}{2}+\kappa}|e^{t\Delta}A(\ell) - e^{t\Delta}B(\ell)|\,.$$

The final term substitutes the first term in (1), with the natural weighting $|\ell|^{-\frac{1}{2}+\kappa}$. The catch is that we now need to restrict to short line segments $|\ell|<t^\kappa$. One should think of t^κ as a characteristic length scale on which $e^{t\Delta}A$ integrates lines to give $(\frac{1}{2}-\kappa)$-Hölder continuous paths. One can then prove the crucial estimate $|g|_{\mathcal{C}^\nu} \le C(1+\Sigma(A,0)+\Sigma(A^g,0))^q$, for some $\nu\in(0,\frac{1}{2})$ and $C,q>0$, which substitutes (2). This allows one to use the same strategy as in 2D to build a Markov process associated to (SYM) on the space of gauge orbits $\mathcal{S}/\sim$ in 3D.

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Parking on the infinite binary tree

Nicolas Curien
(joint work with David Aldous, Alice Contat and Olivier Hénard)

Consider the full planar rooted binary tree $\mathbb{B} = \cup_{n \geq 0}\{0,1\}^n$, with $\{0,1\}^0 = \emptyset$ being the root of the tree. Those vertices will be interpreted as free parking spots, each spot accommodating at most 1 car. On top of that tree, we consider a non-negative integer labeling $(A_u : u \in \mathbb{B})$ representing the number of cars arriving on each vertex $u \in \mathbb{B}$. Each car tries to park on its arrival vertex, and if the spot is occupied, it travels downwards in direction of the root of the tree until it finds an empty vertex to park. If there is no such vertex on the path towards the root $\emptyset$, the car exits the tree, contributing to the *flux* of cars at the root that we denote by X. An easy Abelian property show that the value of X does not depend upon the order chosen to park the cars.

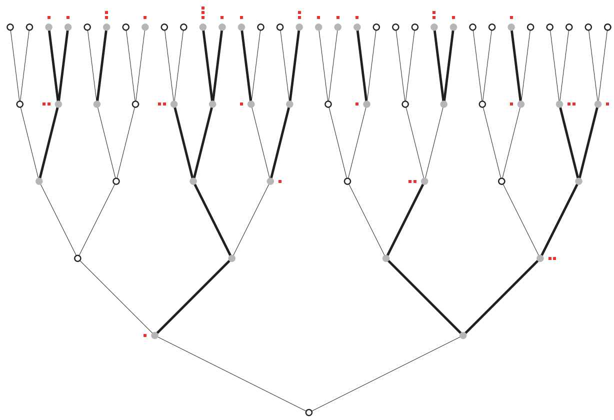

Figure 1. Illustration of the parking process in the first 5 levels of the full binary tree. The car arrivals are represented by red squares (including the cars that may come from higher levels on top of the tree). After the parking process, the vertices accommodating a car are displayed in gray, whereas the free spots are displayed in white. In this case the flux of outgoing cars at is null.

We suppose that the car arrivals $(A_u : u \in \mathbb{B})$ are i.i.d. with a given distribution $\mu = (\mu_k : k \geq 0)$ on $\{0, 1, 2, 3, ...\}$ with finite support (and suppose that $\mu(\{0,1\}) < 1$ to avoid trivialities). We let

$$G(x) = \sum_{k \geq 0} \mu_k x^k$$

be the generating function of the law μ.

Theorem 2. *Then the flux of outgoing car X is almost surely finite if and only if at*

$$t_c = \min\{t \geq 0 : \ 2(G(t) - tG'(t))^2 = t^2 G(t)G''(t)\} \in (0, \infty),$$

we have

$$(t_c - 2)G(t_c) \geq t_c(t_c - 1)G'(t_c).$$

Otherwise $X = \infty$ almost surely.

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Large deviations for the spectrum of random matrices

Raphaël Ducatez, Alice Guionnet
(joint work with Nicholas Cook)

We consider $H = \frac{1}{\sqrt{N}}X$ a Hermitian random matrix with entries $(X_{ij})_{i<j}$ iid where $\mathbb{E}X_{ij} = 0$, $\mathbb{E}(|X_{ij}|^2) = 1$. We denote the empirical spectal measure

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ are the eigenvalues of H. We have the well known Theorem of Wigner [10](Semi-circle law)

$$\lim_{N\to\infty} \mu_N([a,b]) = \sigma_{sc}([a,b]) = \frac{1}{2\pi}\int_a^b \sqrt{4-y^2}1_{|y|\leq 2}dy \quad a.s.$$

Moreover if $\mathbb{E}X_{ij}^4 < \infty$ then we also have the convergence of the largest eigenvalue [7, 5]

$$\lim_{N\to\infty} \lambda_1 = 2 \quad a.s.$$

The goal of this presentation is to give large deviation estimates for these results.

The Gaussian Case (Beta-Ensemble). When the entries X_{ij} are gaussian, the system has much more symmetries and one can write the law of the eigenvalues explicitely

$$d\mathbb{P}_\beta^N = \frac{1}{Z_\beta^N}\exp\left(\beta\sum_{i<j}\log|\lambda_i - \lambda_j| - \frac{\beta}{4}\sum_{i=1}^{N}\lambda_i^2\right)\prod_{1\leq i\leq N} d\lambda_i.$$

We also define $\xi(\mu) = \beta(J(\mu) - \inf J)$ where

$$J(\mu) := \frac{1}{8}\iint (x^2 + y^2 - 4\log|x-y|)d\mu(x)d\mu(y)$$

Here it is possible to compute the large deviation functions and we have the following.

Theorem 1. [9, 2] *(Large deviation for the distribution of the eigenvalues)For any "nice" $A \subset \mathcal{P}(\mathbb{R})$*

$$\lim_{N\to\infty}\frac{1}{N^2}\log \mathbb{P}_\beta^N(\mu_N \in A) = -\inf_{\mu\in A}\xi(\mu).$$

[1] *(Large deviation for the largest eigenvalue)For $x \geq 2$,*

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{1}{N}\log \mathbb{P}_\beta^N(|\lambda_1 - x| \leq \epsilon) = -I_{GOE}(x) := -\int_2^x \sqrt{y^2-4}dy$$

Remark that the rate of large deviation is N^2 for spectral measure and N for the largest eigenvalue. This is not so suprising as one should "move" $O(N)$ eigenvalues to have a $O(1)$ perturbation on μ_N.

The general case. The general non-gaussian case is more complicated. For the distribution of the eigenvalues the rate of the large deviation should still be N^2 if (X_{ij}) are bounded (or log-concave) and indeed we have

$$\limsup_{N\to\infty}\frac{1}{N^2}\log \mathbb{P}^N(d(\mu_N,\sigma_{sc}) > \delta) < 0.$$

However we do not have a large deviation function here. For the largest eigenvalue, the probability estimate depends on the tails of the law of the X_{ij}. Define $A = 2\sup\frac{\log \mathbb{E}e^{tX}}{t^2}$ and remark that $A = 1$ in the Gaussian case. We say that the law of the entries is not sub-gaussian if $A = \infty$. In that case, the largest eigenvalue is created by a very large entry in the matrix and as a consequence the rate of the large correspond to the tail of the law [3, 6]. In the case $A < \infty$, one obtains that the rate is N as in the gaussian case and write an large deviation function.

Theorem 2. [8, 4]*[Cook,Ducatez,Guionnet] There exists $C \leq C' \in (2,\infty)\cup\{\infty\}$ such that for $x \geq 2$,*

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{1}{N}\log \mathbb{P}^N(|\lambda_1 - x| \leq \epsilon) = \begin{cases} -I_{GOE}(x) & \text{if } x < C \\ -I(x) > -I_{GOE}(x) & \text{if } x > C' \end{cases}$$

with $C = \infty$ iff $A = 1$.

Here $I(x)$ is not universal but depends on the law of X, In general it doesn't have a simple expression but can be formulated as the solution of a minimising problem. One has $C = C'$ adding a few technical hypothesis.

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Exit sets for vector valued GFF and interplay with planar $O(N)$ models

Christophe Garban

(joint work with Juhan Aru, Avelio Sepúlveda)

The first motivation of this work is to understand the *exit sets* of (planar) vector-valued GFF. Given an instance of a vector valued GFF

$$\Phi = (\varphi_1, \dots, \varphi_N) : [-n,n]^2 \subset \mathbb{Z}^2 \to \mathbb{R}^N ,$$

and a magnitude $R > 0$, the exit set A_R is obtained by exploring the field starting from the boundary and by stopping the exploration at each vertex x where $\|\Phi_x\| > R$. In the scalar case (i.e. "one-component GFF"), such exit sets are now well understood (at least for the continuum GFF) thanks for example to the works [1, 2]. They happen to be intriguing fractal sets which can be described by versions of CLE_4 loop ensembles.

The first main result of this work is that for the N-component GFF, as long as $N \geq 2$, such exit sets A_R are *degenerate* for all values of R in the sense that the exploration process starting from $\partial[-n,n]^2$ does not enter the bulk of the domain further than $o(n)$ away from the boundary.

When $N = 2$, the main idea behind the proof uses the absence of long-range order of a suitable XY model with random long-range conductances.

The second main motivation of this work is to revisit a series of works by Patrascioiu-Seiler ([3, 4]) which argued against Polyakov's celebrated prediction in

1975 ([5]) that planar classical Heisenberg model should exhibit exponential decay at all positive temperatures.

We first prove that the (properly rescaled) transverse fluctuations of a classical Heisenberg model converge as the temperature goes to 0 to a two-component GFF $\Phi = (\varphi_1, \varphi_2)$. Using the first part of this work, it implies that if one projects the classical Heisenberg model down to a XY model in random environment (following [5]), then locally around pole-pointing spins, the "warm" regions of the induced random environment (where the XY model will decouple more rapidly) do not percolate. The percolation of such regions would have provided a convincing mechanism for exponential decay. In particular if Polyakov's prediction were to hold, a mass needs to be generated by the XY model defined on a strongly percolating set of high conductances. This possible scenario is excluded in Patrascioiu-Seiler works and the last contribution of this present work is to build an explicit example of an XY model in random conductances which has a mass gap despite a percolating set of (arbitrary) high conductances.

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What is stochastic quantisation?

Massimiliano Gubinelli

I regularly teach a course called "Stochastic Analysis", in which the students are introduced to the use of stochastic integration and Itô's formulas to study several problems in probability, including existence, conditioning, uniqueness, convergence of diffusion processes. However it is quite difficult for me to give a succinct definition of what stochastic analysis *is*. It is usually however quite easy to classify a result as a result in stochastic analysis or not. As D. Stroock puts it in the introduction to [16]:

> *[...] there now exists a reasonably well-defined amalgam of probabilistic and analytic ideas and techniques that, at least among the cognoscenti, are easily recognized as stochastic analysis. Nonetheless, the term continues to defy a precise definition, and an understanding of it is best acquired by way of examples.*

According to Stroock [15], Itô arrived to his calculus while trying to understand Feller's theory of diffusions an evolution in the space of probability measures and he introduced stochastic differential equations as maps which send Wiener measure to the law of a diffusion. Malliavin's calculus, another example of stochastic analysis, is also based on reducing a probabilistic problem to the analysis of certain random maps and of their smoothness according to an infinite-dimensional calculus of variations. Lyons' rough path theory shift the focus on more robust properties of Itô's solution map and identify specific topologies in which it can be extended to a larger class of driving signals. Controlled rough paths are a variation on Itô's idea of using Brownian motion as "universal" building block for continuous diffusions, by replacing it with a general rough path in the sense of Lyons.

While technically all these approaches use an analysis based on one-dimensional random fields, the concerns of analysing the behaviour of certain maps which send a probability measure to another in not intrinsically limited to (possibly infinite-dimensional) systems evolving along a one-dimensional time parameter. Schramms' stochastic Lowener equation is a map which sends Wiener measure into a random growing domain in the complex plane whose law has natural probabilistic properties like diffeomorphism covariance and domain Markovianity. Hairer's regularity structures and paracontrolled calculus bring forward Itô's and Lyons' ideas to the realm of multidimensional random fields which are solutions to stochastic partial differential equations. In many cases these random fields are distribution-valued and the associated non-linear equation have to be understood to be posed in some kind of non-smooth jet space.

From this point of view, stochastic analysis is not really *just* a part of probability, in the same way as probability is not *just* a part of measure theory. While the technical background and the arena come from another discipline, both probability and stochastic analysis evolved quite characteristic features which sets them apart from their foundationals predecessors.

This long introduction was meant to set the stage for an attempt to an answer to the question in the title. Stochastic quantisation has been introduced, from different perspectives, by Nelson and by Parisi–Wu to provide a construction of Euclidean quantum field theories (EQFTs) using diffusion processes. EQFTs are probability measures on the space of Schwarz distributions on the Euclidean space $\mathbb{R}^d$ which enjoy particular properties (e.g. locality, reflection positivity, covariance under the Euclidean group, integrability). These properties allows to use them as building block to construct an Hilbert space and an algebra of local fields for a relativistic quantum theory in the sense of Wightman.

Parisi–Wu's construction uses the Itô map of a Langevin equation to push forward the law of a multidimensional white noise to a target EQFT, explicitly providing a stochastic construction of an (Euclidean) quantum theory. We can call it a *stochastic quantisation map*. The use of a Langevin equation framed, quite rapidly, the idea of stochastic quantisation in the area of the theory of Markov processes, to which stochastic analysis is naturally related. However I want to argue that stochastic quantisation is a stochastic analysis of Euclidean quantum fields in a more fundamental way, and that is in the very basic sense of an analysis of maps which send some convenient *source* probability measures μ to a target measure ν which has the structure and properties of an EQFT.

In order to articulate this idea and "flesh it out", together with various collaborators, we set out to explore this analysis along a range of directions. For example one can produce such a stochastic quantisation map in various ways, i.e. there are various kind of stochastic quantisation that we know of. One can use parabolic equations [10] or elliptic equations [2, 3, 11, 5], or wave equations [13, 12], or forward-backwards stochastic differential equations (FBSDEs) [7, 4] which in some cases are related to a stochastic control problem as in [6] and to renormalization group ideas, in the continuous formulation of Wilson and Polchinski.

Fermionic quantum field theories are associated to probability measures on Grassmann variables and there is an associated stochastic analysis [1] which give then rise to a stochastic quantisation, e.g. via FBSDEs coupled to an approximate RG flow as in [9]. In this case there is no measure theory or probability in conventional sense, but stochastic analysis retain is characteristic aspects and delivers useful results.

The stochastic quantisation map give rise to a natural coupling of the Gaussian free field and the "interacting" Euclidean quantum field which can be exploited to study pathwise properties of the EQFTs and related functional inequalities as in e.g. [8, 14].

These results are the beginning of a systematic exploration of these new examples of stochastic analysis which, I hope, could bring some new insights in constructive Euclidean quantum field theory at least in the superrenormalizable and asymptotically free cases where the Gaussian free field plays the role of a basic building block on small scales.

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The critical and supercritical Liouville quantum gravity metrics

Ewain Gwynne
(joint work with Jian Ding)

We discuss the construction of the **Liouville quantum gravity (LQG)** metric associated with the planar Gaussian free field in the subcritical ($\gamma \in (0,2)$), critical ($\gamma = 2$) and supercritical ($\gamma \in \mathbb{C}$, $|\gamma| = 2$) regimes.

To define this metric, let h be the planar Gaussian free field and let $\xi > 0$ be a parameter. For $\varepsilon > 0$, let h_ε be the convolution $h * p_{\varepsilon^2/2}$, where $p_t(z) = \frac{1}{2\pi t} e^{-|z|^2/2t}$ is the planar heat kernel. We define a family of approximating metrics $\{D_h^\varepsilon\}_{\varepsilon>0}$ by

$$D_h^\varepsilon(z,w) := \inf_{P: z \to w} \int_0^1 e^{\xi h_\varepsilon^*(P(t))} |P'(t)|\, dt, \quad \forall z, w \in \mathbb{C}$$

where the infimum is over all piecewise continuously differentiable paths $P : [0,1] \to \mathbb{C}$ from z to w.

To extract a non-trivial limit of the metrics D_h^ε, we need to re-normalize. We (somewhat arbitrarily) define our normalizing factor by

$$\mathfrak{a}_\varepsilon := \text{median of } \inf \left\{ \int_0^1 e^{\xi h_\varepsilon^*(P(t))} |P'(t)|\, dt : P \text{ is a left-right crossing of } [0,1]^2 \right\},$$

where a left-right crossing of $[0,1]^2$ is a piecewise continuously differentiable path $P : [0,1] \to [0,1]^2$ joining the left and right boundaries of $[0,1]^2$. We do not know the value of $\mathfrak{a}_\varepsilon$ explicitly. Nevertheless, it was shown by Ding and Gwynne (2020) that for each $\xi > 0$, there exists $Q = Q(\xi) > 0$ such that

$$\mathfrak{a}_\varepsilon = \varepsilon^{1 - \xi Q + o(1)} \quad \text{as } \varepsilon \to 0.$$

Furthermore, Q is a continuous, non-increasing function of ξ.

We define the **critical value** ξ_{crit} to be the unique $\xi > 0$ for which $Q(\xi) = 2$. In the subcritical case $\xi < \xi_{\mathrm{crit}}$, it was shown in works by Ding-Dunlap-Dubédat-Falconet (2019) and Gwynne-Miller (2019) that the re-scaled metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ converge in probability to a limiting random metric D_h on $\mathbb{C}$. This limiting metric induces the same topology as the Euclidean metric, but has very different geometric properties. For example, the Hausdorff dimension of $(\mathbb{C}, D_h)$ is strictly greater than two. The metric D_h can be interpreted as the Riemannian distance function associated with γ-LQG, where $\gamma \in (0,2)$ satisfies $Q(\xi) = 2/\gamma + \gamma/2 > 2$.

In the supercritical and critical cases, i.e., when $\xi \geq \xi_{\text{crit}}$, the convergence of $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ was established by Ding-Gwynne (2021). In the critical case $\xi = \xi_{\text{crit}}$, the limit is a random fractal metric on $\mathbb{C}$ which induces the Euclidean topology (like in the subcritical case). It can be interpreted as the distance function associated with critical ($\gamma = 2$) LQG.

In the supercritical case, however, the limiting metric D_h is allowed to take the value $+\infty$. More precisely, we say that $z \in \mathbb{C}$ is a **singular point** if

$$D_h(z,w) = \infty, \quad \forall w \in \mathbb{C} \setminus \{z\}.$$

For $\xi > \xi_{\text{crit}}$, a.s. the set of singular points is uncountable and Euclidean-dense, but has zero Lebesgue measure. Furthermore, any two non-singular points lie at finite distance from each other. These properties imply that D_h does not induce the Euclidean topology and that D_h-metric balls have positive Lebesgue measure but empty Euclidean interior.

For $\xi > \xi_{\text{crit}}$, we have $Q(\xi) \in (0,2)$. If we choose γ so that $Q(\xi) = 2/\gamma + \gamma/2$ then $\gamma \in \mathbb{C}$ with $|\gamma| = 2$. Hence, the limiting metric for $\xi > \xi_{\text{crit}}$ can be interpreted as the distance function associated with γ-LQG for such a complex value of γ, or equivalently with **matter central charge** $\mathbf{c}_{\mathrm{M}} = 25 - 6(2/\gamma + \gamma/2)^2 \in (1,25)$.

The $\mathbf{c}_{\mathrm{M}} \in (1,25)$ phase for LQG is rather mysterious, even at a physics level of rigor. Although we can now construct a metric in this phase, it remains an open problem to extend various other features of LQG from the case when $\gamma \in (0,2]$ ($\mathbf{c}_{\mathrm{M}} \leq 1$) to the case when $\mathbf{c}_{\mathrm{M}} \in (1,25)$. Such features include the construction of the volume measure, connections to SLE, and various formulas from conformal field theory.

The Euclidean ϕ_2^4 theory as a limit of an interacting Bose gas

Antti Knowles

(joint work with Jürg Fröhlich, Benjamin Schlein, Vedran Sohinger)

A Euclidean field theory of a scalar field on a the d-dimensional torus $\Lambda := \mathbb{T}^d$ is specified, at least formally, by a probability measure on a space of fields $\phi : \Lambda \to \mathbb{R}^N$ given by

$$\mu(\mathrm{d}\phi) = \frac{1}{c}\, \mathrm{e}^{-S(\phi)}\, \mathrm{D}\phi\,, \tag{1}$$

where $\mathrm{D}\phi = \prod_{x \in \Lambda} \mathrm{d}\phi(x)$ is the formal uniform measure on the space of fields, and S is the action. The latter is typically the integral over Λ of a local function of the field ϕ and its gradient. One of the simplest field theories with nontrivial interaction is the N-component *Euclidean ϕ_d^4 theory*, whose action is given by

$$S(\phi) := -\int_\Lambda \mathrm{d}x\, \phi(x) \cdot (\theta + \Delta/2)\phi(x) + \frac{\lambda}{2} \int_\Lambda \mathrm{d}x\, |\phi(x)|^4\,, \tag{2}$$

where θ is a constant, λ is a coupling constant, Δ is the Laplacian on Λ, and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$.

The rigorous construction of μ is a classical topic in mathematical physics; we refer for instance to [4, 9] for seminal references. The starting point is to regard μ as a perturbation of the Gaussian free field measure, obtained by setting $\lambda = 0$ in the definition of μ. For $d = 1$, under the Gaussian free field, the field ϕ is almost surely a continuous function, so that the density $\mathrm{e}^{-S(\phi)}$ is well defined, and hence μ can be rigorously constructed. For $d > 1$, under the Gaussian free field, the field ϕ is almost surely a distribution of negative regularity, and hence the interaction term $V(\phi) := \frac{\lambda}{2}\int_\Lambda \mathrm{d}x\, |\phi(x)|^4$ in (2) is ill-defined. This is an *ultraviolet* problem: a divergence for large wave vectors (i.e. spatial frequencies) producing small-scale singularities in the field. As the dimension $d = 1, 2, 3$ increases, the difficulty of making sense of the measure in (1) increases significantly.

To handle the ultraviolet problem, it is necessary to regularize the interaction $V(\phi)$ by subtracting infinite mass and energy counterterms. Formally, the renormalized interaction is $\frac{\lambda}{2}\int_\Lambda \mathrm{d}x\,(|\phi(x)|^4 - \infty|\phi(x)|^2 + \infty)$. More precisely, one replaces the field ϕ with its mollified version ϕ_η obtained by convolving it with an approximate delta function of range η. One then chooses M_η and E_η such that, as $\eta \downarrow 0$, the measure μ_η with interaction $V_\eta(\phi) := \frac{\lambda}{2}\int_\Lambda \mathrm{d}x\,(|\phi(x)|^4 - M_\eta|\phi(x)|^2 + E_\eta)$ converges to a limiting measure.

In [3], we prove that the complex (i.e. $N = 2$) Euclidean ϕ^4_2 theory arises as a high density limit of an interacting Bose gas, in a regime where the range of the interactions is much smaller than the linear size of the system but much larger than the typical interparticle separation. This extends the recent results [1, 2, 5, 6, 7, 8] on the mean-field limit of the Bose gas at positive temperature to local interactions.

To describe our result more precisely, we recall that a quantum system of n spinless non-relativistic bosons of mass m in Λ is described by the Hamiltonian

$$\mathbb{H}_n := -\sum_{i=1}^n \frac{\Delta_i}{2m} + \frac{g}{2}\sum_{i,j=1}^n v(x_i - x_j)$$

acting on the space $\mathcal{H}_n$ of square-integrable wave functions that are symmetric in their arguments $x_1, \dots, x_n$ and supported in Λ^n. Here Δ_i is the Laplacian in the variable x_i, g is a coupling constant, and v is a repulsive (i.e. with nonnegative Fourier transform) two-body interaction potential. We consider a system in the grand canonical ensemble at positive temperature, characterized by the density matrix

$$\frac{1}{Z}\bigoplus_{n\in\mathbb{N}} \mathrm{e}^{-\beta(\mathbb{H}_n - \theta n)} \tag{3}$$

acting on Fock space $\mathcal{F} = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$, where $\beta < \infty$ is the inverse temperature, θ is the chemical potential, and Z is a normalization factor.

The limiting regime of this paper is obtained by introducing two parameters, $\nu, \varepsilon > 0$, where $\nu = \frac{\beta}{m} = \sqrt{\beta g}$, and the potential v is taken to be an approximate delta function of range ε. We suppose that $\nu, \varepsilon \to 0$. This can be interpreted as a high-density limit, where the range of the interactions tends to zero. Our main result is the following.

Theorem. Let $d = 2$. Suppose that $\nu, \varepsilon \to 0$ under the constraint

$$\varepsilon \geq \exp(-(\log \nu^{-1})^{1/2-c}),$$

for some constant $c > 0$. Then, there exists a suitable renormalization of the chemical potential $\theta \equiv \theta_\nu^\varepsilon$ such that, for any $p \in \mathbb{N}$, the reduced p-particle density matrices of the quantum Bose gas in the grand canonical state (3) converge to the correlation functions of the complex ϕ_d^4 theory. The convergence holds in the L^p-norm for any $p < \infty$.

We also show convergence in the L^∞ norm of the renormalized density matrices to the corresponding renormalized correlation functions of ϕ_2^4. This yields a link between the distribution of the local particle density of the Bose gas and the mass density of ϕ_2^4.

The proof is based on three main ingredients: (a) a quantitative analysis of the infinite-dimensional saddle point argument for the functional integral introduced in [2] using continuity properties of Brownian paths, (b) a Nelson-type estimate for a general nonlocal field theory in two dimensions, and (c) repeated Gaussian integration by parts in field space to obtain uniform control on the renormalized correlation functions.

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KPZ-type fluctuation exponents for interacting diffusions in equilibrium

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(joint work with Christian Noack, Philippe Sosoe)

Consider the family of diffusions $\{u_j(t)\}_{j=1}^N$ interacting via the following system of stochastic differential equations,

$$\begin{aligned} \mathrm{d}u_1 &= -V'(u_1)\mathrm{d}t + \mathrm{d}B_0 - \theta \mathrm{d}t + \mathrm{d}B_1 \\ \mathrm{d}u_j &= (V'(u_{j-1}) - V'(u_j))\mathrm{d}t + \mathrm{d}B_j - \mathrm{d}B_{j-1}, \qquad j \geq 2 \end{aligned}$$

where $\{B_j\}_{j\geq 1}$ are a family of independent Brownian motions, $V : \mathbb{R} \to \mathbb{R}$ is a smooth, convex potential, and the initial data of the system $\{u_j(0)\}_{j=1}^N$ are iid with density proportional to $\mathrm{e}^{-V(u)-\theta u}$ (which turns out to be the unique invariant measure for the above system).

We study the following *height function* of this class of interacting diffusions,

$$W_{N,t} := \sum_{j=1}^{N} u_j(t) - B_0(t) + \theta t.$$

For the special case $V(u) = \mathrm{e}^{-u}$, this height function has the same distribution as the free energy of a stationary version of the O'Connell-Yor polymer [4], which is an important example of a random growth model lying in the KPZ universality class and one of the central models of integrable probability.

Due to its exactly solvable nature, much is known about the O'Connell-Yor polymer. For N and t tuned in certain *characteristic directions*, Seppäläinen-Valko [5] showed that the fluctuations of the free energy are of the order $N^{1/3}$. Borodin-Corwin-Ferrari [1] and Imamura-Sasamoto [3] found the limiting distribution of the O'Connell-Yor polymer free energy and its stationary analog (the Tracy-Widom and Baik-Rains distributions, respectively) confirming the conjecture that this polymer lies in the KPZ universality class.

While the case $V(u) = \mathrm{e}^{-u}$ admits exact formulas for its distribution, the KPZ universality of the above system is conjectured to hold for general classes of potentials going beyond the exactly solvable cases [2].

In our work we consider the above system for a general class of potentials $V(u)$ which include, but are not limited to, the exactly solvable O'Connell-Yor case $V(u) = \mathrm{e}^{-u}$. In the same characteristic directions as in [5], we show that the height function $W_{N,t}$ obeys,

$$cN^{2/3} \leq \mathrm{Var}(W_{N,t}) \leq CN^{2/3},$$

which are the expected exponents for models lying in the KPZ universality class. In particular, our methods are dynamical and analytic in nature, and make no use of exact formulas or the polymer representation associated to the O'Connell-Yor polymer.

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On the Markov property of local times of Brownian motion indexed by the Brownian tree

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The Brownian tree may be defined as the tree coded by a Brownian excursion. To make this precise, write $\mathbf{n}(\mathrm{d}e)$ for the Itô measure of positive Brownian excursions, so that $\mathbf{n}(\mathrm{d}e)$ is a σ-finite measure. Given an excursion $(e(t), 0 \leq t \leq \sigma)$, we introduce the equivalence relation $\sim_e$ on $[0, \sigma]$ defined by

$$s \sim_e t \;\text{ if and only if }\; e(s) = e(t) = \min\{e(r) : s \wedge t \leq r \leq s \vee t\}.$$

Then the tree coded by e is the quotient space $\mathcal{T}_e := [0, \sigma]/\sim_e$, which is equipped with the distance induced by

$$d_e(s,t) := e(s) + e(t) - 2\min\{e(r) : s \wedge t \leq r \leq s \vee t\}.$$

The tree $\mathcal{T}_e$ under the measure $\mathbf{n}(\mathrm{d}e)$ is called the Brownian tree (under the conditional probability $\mathbf{n}(\mathrm{d}e \mid \sigma = 1)$, this is Aldous' CRT, up to an unimportant scaling factor). The volume measure Vol on $\mathcal{T}_e$ is the pushforward of Lebesgue measure on $[0, \sigma]$ under the canonical projection.

In what follows, we assume that the excursion e is distributed according to $\mathbf{n}(\mathrm{d}e)$. To define Brownian motion indexed by the Brownian tree, we consider, conditionally on the excursion $(e(t), 0 \leq t \leq \sigma)$, the centered Gaussian process $(Z_s, 0 \leq s \leq \sigma)$ whose distribution is characterized by the properties $Z_0 = 0$ and

$$\mathbf{E}[(Z_s - Z_t)^2] = d_e(s,t)\,, \quad \forall s,t \in [0,\sigma]$$

Then $(Z_s, 0 \leq s \leq \sigma)$ has a modification with continuous sample paths. Moreover the property $s \sim_e t$ implies that $Z_s = Z_t$. It follows that we may view Z as indexed by $\mathcal{T}_e$, just by setting $Z_a = Z_t$ whenever $a \in \mathcal{T}_e$ is the equivalence class of $t \in [0, \sigma]$.

The total occupation measure of the process $(Z_a, a \in \mathcal{T}_e)$ is the measure Θ on $\mathbb{R}$ defined by

$$\langle \Theta, \varphi \rangle = \int_{\mathcal{T}_e} \varphi(Z_a)\,\mathrm{Vol}(\mathrm{d}a),$$

for any continuous function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}_+$.

Proposition 1. *Almost everywhere, the measure Θ has a continuous density on $\mathbb{R}$, which is denoted by $(\ell_x)_{x\in\mathbb{R}}$, and the mapping $x \mapsto \ell_x$ is continuously differentiable.*

This essentially follows from Sugitani [4] and the relations between super-Brownian motion and Brownian motion indexed by the Brownian tree (see also Bousquet-Mélou and Janson [2]). We write $\dot{\ell}_x$ for the derivative of $x \mapsto \ell_x$.

In a way analogous to the classical Ray-Knight theorems for Brownian local times, one may expect the process $(\ell_x)_{x>0}$ to be Markov. This is not true (informally, to predict the future of this process after "time" y, one needs the value of ℓ_y, but clearly also the value of the derivative $\dot{\ell}_y$). However, we have the following statement

Theorem 2. *The process $(\ell_x, \dot{\ell}_x)_{x>0}$ is Markov.*

This makes sense because, although we are dealing with a σ-finite measure (the excursion e is distributed according to $\mathbf{n}(\mathrm{d}e)$), the event where $\sup\{\ell_x : x \geq \varepsilon\} > 0$ has finite measure, for every $\varepsilon > 0$.

The proof of the theorem heavily relies on the excursion theory developed in [1]. We also refer to [3] for closely related results in a discrete setting.

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Stationarity and space-time covariance for KPZ growth models in half-space

Alessandra Occelli

We consider two models of last passage percolation (LPP) with exponential weights in half-space geometry. A last passage percolation is a directed up-right path in a random environment that attempts to maximize its energy, which is given by the sum of the random variables visited by the path. We consider LPPs on the octant $\{(i,j) \in \mathbb{Z}^2_{>0}, i \geq j\}$ and we place independent random variables $\omega_{i,j}$ on each site of the lattice. Thus, the last passage time between $(1,1)$ and (n,m) is defined as

$$L_{(1,1)\to(n,m)} = \max_{\pi:(1,1)\to(n,m)} \sum_{(i,j)\in\pi} \omega_{i,j}. \tag{1}$$

Paths which maximize (1) are called geodesics.

As in full-space, also the half-space LPP has an interpretation in terms of an exclusion process, the totally asymmetric simple exclusion process (TASEP) on the half-line with a reservoir of particles in the origin. The connection between TASEP and LPP is as follows. If $\omega_{i,j}$, $i > j$ is an exponential random variable, then it represents the waiting time of particle j to jump from site $i-j-1$ to site $i-j$, while the diagonal weights $\omega_{i,i}$ regulate the rate of creation of particles in the origin. It follows that

$$\mathbb{P}(N_{m-n+1}(t) \leq n) = \mathbb{P}(L_{m,n} \geq t),$$

where $N_x(t)$ is the current at site x.

The first model is a point-to-point last passage percolation in half-space, i.e. with weights being exponentially distributed of parameter 1 in the bulk and of parameter ρ on the diagonal. It was proved in [1],[4],[5] that the fluctuations of the last passage time are influenced by the diagonal weights and that the limit distribution interpolates between the GSE Tracy–Widom ($\rho > 1/2$), the GOE Tracy–Widom ($\rho = 1/2$), the Gaussian ($\rho < 1/2$) and the GUE Tracy–Widom distributions (end point far away from the diagonal). The limit process for this model was defined in [1] as a marginal of Pfaffian Schur processes.

The second model is a stationary last passage percolation in half-space, which is defined starting from the point-to-point model by adding an extra boundary condition on the horizantal line, i.e. weights exponentially distributed of parameter $1-\rho$, and by putting 0 in the origin. This model is stationary in the sense of [6]: the increments of the last passage time along the horizontal and vertical directions are given by sums of i.i.d. random variables. For this model, in [8] we determine the limiting distribution of the last passage time in a critical window close to the origin. The result is a new two-parameter family of distributions: one parameter for the strength of the diagonal bounding the half-space (strength of the source at the origin in the equivalent TASEP language) and the other for the distance of the point of observation from the origin. The distribution is expressed in terms of the product of a Fredholm pfaffian (limit distribution of the point-to-point model, in other words, the half-space equivalent of the GUE Tracy–Widom distribution) and certain explicit functions that can be written as contour integrals of Airy-like functions.

This result is a generalisation of what is observed in full space, i.e. the one-parameter family giving the Baik–Rains distributions [3], which depends only on the distance from the diagonal. We also show that far away from the characteristic line, the distributions indeed converge to the Baik–Rains family. The strategy to prove this results heavily relies on the integrable structure of the point-to-point model. In fact, we derive our results using a related inhomogeneous integrable model having Pfaffian correlations, together with careful analytic continuation, and steepest descent analysis.

In [9] we study the multi-point distribution of the same model. We derive both finite-size and asymptotic results for this distribution: we observe a new one-parameter process we call *half-space* $Airy_{stat}$. It is a one-parameter generalization of the $\mathrm{Airy}_{\mathrm{stat}}$ process of [2], which is recovered far away from the diagonal.

We expect both the distribution and the process to be universal in the framework of stationary KPZ models in half-space geometry (e.g. compare with [7]).

In [11] we study several properties of the point-to-point half-space last passage percolation, in particular the two-time covariance. We extend the results obtained for the full space model in [10] and we show that, when the two end points are at small macroscopic distance, then the first order correction to the covariance for the point-to-point model is the same as the one of the stationary model. In this case, our strategy is based on more probabilistic arguments, such as polymer localization. In order to obtain this result, we first derive comparison inequalities of the last passage increments with respect to the stationary increments. This is used to prove tightness of the point-to-point process as well as localization of the geodesics. Unlike for the full-space case, for half-space we have to overcome the difficulty that the point-to-point model in half-space with generic start and end points is not known.

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KPZ fluctuations in the planar stochastic heat equation

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(joint work with Alejandro Ramirez, Balint Virag)

We study the stochastic heat equation

$$\partial_t u = \tfrac{1}{2}\Delta u + u\,\xi, \tag{4}$$

where ξ is a white noise on $\mathbb{R}^2$ that does not depend on time. The equation is taken in the *Skorokhod* sense, motivated by the interpretation as a polymer free energy. By the Feynman-Kac formula one expects the solution of the stochastic heat equation (4) to have a representation

$$u(t,x) = E_B \left[:\exp: \left\{ \int_0^t \xi(B(s))ds \right\} \right] p(t,x). \tag{5}$$

This is heuristic, and the colons around the exponential are simply to remind us that some sort of renormalization will be required. The x is hidden on the right side of equation in the fact that the expectation is over a 2d Brownian bridge B from 0 to x. If we tried to take the exponent literally, it would be $\int_{\mathbb{R}^2} \xi m$ where m is the occupation measure of the Brownian bridge. But $m \notin L^2(\mathbb{R}^2)$ so it doesn't make sense.

Let e_j be bounded functions forming an orthonormal basis of $L^2(\mathbb{R}^2)$, $\xi_j = \langle \mathrm{e}_j, \xi \rangle$, $m_j = \int_0^t \mathrm{e}_j(B(s))\, ds$ and

$$Z_n(\xi) = E_B e^{\sum_{j=1}^n m_j \xi_j - \frac{1}{2}\sum_{j=1}^n m_j^2}. \tag{6}$$

Z_n is a positive martingale w.r.t. $\sigma(\xi_1, \ldots, \xi_n)$. If it converges in L^1, i.e. is *uniformly integrable*, then

$$u(t,x) = p(t,x) \lim_{n\to\infty} Z_n \tag{7}$$

turns out to be the Skorokhod solution with Dirac initial data. Other initial data can be solved by convolving with it. The reader can consult [3] for a nice introduction to the Skorokhod integral.

Formally, $\sum_{j=1}^\infty m_j^2 = \int_0^t \int_0^t \delta_0(B(s) - B(s'))dsds'$, the *self-intersection local time*. It does not exist without a renormalization. If δ_0^ϵ is a smoothed out Dirac, by convolving with a heat kernel at time ϵ, then [2]

$$\int_0^t \int_0^t \delta_0^\epsilon(B(s) - B(s'))dsds' - c|\log \epsilon| t \to \gamma_t(B) \tag{8}$$

is the *renormalized self-intersection local time*. A more popular notion of solution to (4) is called PAM (for parabolic Anderson model). If we try to make sense of (5) by smoothing out the white noise to get $\int_{\mathbb{R}^2} \xi_\epsilon m$, and then subtracting something in the exponent to compensate as we remove the smoothing, PAM means we subtract $c|\log \epsilon| t$, while Skorokhod means we subtract $c|\log \epsilon| t + \gamma_t(B)$. We will also construct polymer measures corresponding to the solution of the stochastic heat equation. An advantage of the Skorokhod interpretation is that the expectation of the resulting polymer measure is the original Brownian bridge. On the other hand, the PAM interpretation results in a solution operator with the semi-group property, while Skorokhod does not. The choice is reminiscent of Itô vs. Stratonovich for stochastic integrals and is more of a modelling issue.

As we saw above, to obtain a solution we need to show uniform integrability of the martingale Z_n. Typically, one might try to bound $E[Z_n^2]$. It is fairly easy to see that this is given by $E_{B,B'}[e^{\sum_{j=1}^n m_j m_j'}]$, where B' is an independent copy of B, and m_j' are the coefficients of its occupation measure. Now

$$\sum_{j=1}^\infty m_j m_j' = \int_0^t \int_0^t \delta_0(B(s) - B'(s'))dsds', \tag{9}$$

the *mutual intersection local time* of the two Brownian bridges. Let's call it $\alpha_t(B, B')$. It is not hard to check that $E_{B,B'}[e^{\alpha_t(B,B')}]$ is only finite for $t \leq t_c$ (see [2]). The critical time t_c is related to the constant in the Gagliardo-Nirenberg-Sobolev inequality [1]. So we only have $\sup_n E[Z_n^2(t,x)] < \infty$ for $t \leq t_c$. Because of this, a solution theory was previously only known up to this time t_c.

We show that for any $c, \delta > 0$ there is a set of Brownian bridge paths A with $P(A) > 1 - \delta$ such that

$$E_{B,B'}[\mathbf{1}_{B,B' \in A} e^{c\alpha_1}] < \infty. \tag{10}$$

The set $A = \{\text{Holder norm}, M \leq L\}$ where

$$M = \frac{\epsilon^{-2}}{|\log \epsilon|} \max_{\mathbb{R}^2 = \dot{\cup}\{\epsilon \times \epsilon \text{ boxes}\}} \delta \sum_{i=0}^{\lfloor 1/\delta \rfloor} \mathbf{1}\left(B(\delta i) \in \text{box}\right). \tag{11}$$

This means that for the expectation with respect to the white noise ξ,

$$E_\xi[E_B[\mathbf{1}_{B \in A}(e^{\sum_{j=1}^n \xi_j m_j - \frac{1}{2} m_j^2}])^2] \leq E[\mathbf{1}_{B,B' \in A} e^{t\alpha_1}] \tag{12}$$

and

$$E_\xi[E_B[\mathbf{1}_{B \in A^C} |e^{\sum_{j=1}^n \xi_j m_j - \frac{1}{2} m_j^2}]|] = P(A^C) \tag{13}$$

which tells us that Z_n is uniformly integrable.

This produces a solution $Z \in L^1(\xi)$ for all time. The construction also yields the polymer measure. The method is also very good for convergence. If m^n are occupation measures converging to a limiting occupation measure m, then under reasonable conditions the corresponding $Z(m^n)$ converge in L^1 to $Z(m)$. In fact, all one needs is that the corresponding mutual intersection local times converge in L^2.

One also sees that the law of $\xi + m$, averaged over m, is absolutely continuous with respect to the law of the white noise ξ, and Z is the Radon-Nikodym derivative.

The one dimensional stochastic heat equation with space-time white noise,

$$\partial_t z = \partial_x^2 z + \xi z \tag{14}$$

fits into the same picture. Recall that $h = \log z$ is the Cole-Hopf solution of KPZ. Now the occupation measure is the occupation measure of the graph $(s, B(s))$ of the one dimensional Brownian bridge.

In particular, we show that on appropriate scales, the solution u of the planar stochastic heat equation (4) has KPZ fluctuations: *For any $t > 0$ and $a \in \mathbb{R}$ as $N \to \infty$, we have*

$$P(u(Nt, 0, N^{3/2}t) \times N e^{N^2 t/2} \sqrt{2\pi t} \leq a) \to F_{KPZ}(t, a), \tag{15}$$

where $F_{KPZ}(t, \cdot)$ are the KPZ crossover distributions

$$F_{KPZ}(t, a) = P(z(t, 0) \leq a).$$

It is actually not hard to see why this is true. The above constructions are just to provide renormalizations which make the following heuristic scaling rigorous:

$$\begin{aligned}
&E_{(0,0)\to(N^{3/2}t,N^{1/2}x)}\left[e^{\int_0^{Nt}\xi(B_1(s),B_2(s))ds}\right]\\
&=E_{(0,0)\to(N^{3/2}t,N^{1/2}x)}\left[e^{N\int_0^{t}\xi(B_1(Ns),B_2(Ns))ds}\right]\\
&=E^{\mathrm{cov}=\mathrm{diag}(N^{-2},1)}_{(0,0)\to(t,x)}\left[e^{N\int_0^{t}\xi(N^{3/2}B_1(s),N^{1/2}B_2(s))ds}\right]\\
&=E^{\mathrm{cov}=\mathrm{diag}(N^{-2},1)}_{(0,0)\to(t,x)}\left[e^{\int_0^{t}\tilde\xi(B_1(s),B_2(s))ds}\right]\\
&\sim E_{0\to x}\left[e^{\int_0^{t}\tilde\xi(s,B_2(s))ds}\right].
\end{aligned}$$

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Critical exponents for three-dimensional percolation models with long-range dependence

Pierre-François Rodriguez
(joint work with Subhajit Goswami, Franco Severo, Alexander Drewitz, Alexis Prévost)

Consider the Gaussian free field φ on $\mathbb{Z}^3$. Associated to φ are the following two percolation models:

- Model $(M1)$, introduced by Lebowitz and Saleur [4], consists of looking at the mountaintops in this landscape, i.e. study percolation of the excursion sets $\{x\in\mathbb{Z}^3 : \varphi_x \geq h\}$ for a varying height parameter $h\in\mathbb{R}$.
- Model $(\widetilde{M1})$ is the bond percolation model obtained by retaining each nearest-neighbor edge $\{x,y\}$ in $\mathbb{Z}^3$ conditionally on φ independently with probability $1-\exp\{-(\varphi_x-h)_+(\varphi_y-h)_+\}$, where $a_+=\max\{a,0\}$ denotes the positive part of a.

Thus, it is harder to percolate in $(\widetilde{M1})$, for not only must the field value at the endpoints x and y exceed value h, but also the ensuing 'disorder variable' on the edge survive. In fact, one knows that the critical parameters h_* and $\tilde h_*$ corresponding to percolation in $(M1)$ and $(\widetilde{M1})$, respectively, satisfy

$$0=\tilde h_* < h_* < \infty,$$

which reflects this intuitive picture, and also implies that the phase transition for both models is non-trivial. Note that the field φ induces long-range correlations, whereby the covariance between occupation variables around 0 and x decays like $|x|^{-a}$ with $a = 1$ as $|x| \to \infty$. As it turns out, the exponent a, which comes as part of the data, plays a central role in determining the scaling behavior of the models at and near criticality.

The goal of this talk is to present two related results, one for each of the models $(M1)$ and $(\widetilde{M1})$, aimed at exhibiting this scaling behavior. For simplicity, the focus will be on one specific quantity, the (truncated) one-arm observable $\tau_h^{\mathrm{tr}}(L)$, which is the probability that the open cluster of the origin at level h is bounded and intersects the boundary of the ball of radius $L \geq 1$ around 0. We write $\tilde{\tau}_h^{\mathrm{tr}}(L)$ for the corresponding quantity for $(\widetilde{M1})$. The first result is:

Theorem 1 ([3]). *For $(M1)$ and all $h \in \mathbb{R}$,*

$$-\tfrac{\log L}{L} \log \tau_h^{\mathrm{tr}}(L) \to \tfrac{\pi}{6}\xi(h)^{-1} \ \text{as } L \to \infty,$$

where $\xi(h) = |h - h_*|^{-\nu}$ with $\nu = 2$.

Among other things, Theorem 1 settles the leading-order decay of $\tilde{\tau}_h^{\mathrm{tr}}(\cdot)$ away from criticality, which is sub-exponential. The lower bounds are obtained by a change of measure argument, and the upper bounds require a delicate coarse-graining. The decay turns out to be entirely carried by deviations of harmonic averages on small macroscopic scales, as those derived in [6] in a different context, yielding a cost governed by the capacity induced by the Dirichlet form naturally associated to φ. Moreover, these bounds produce the sharp polynomial divergence of $\xi(h)$ as h approaches h_*. We return to this below.

The results for $(\widetilde{M1})$ refine this picture, by providing estimates which are quantitative in both h and L.

Theorem 2 ([1]). *For $(\widetilde{M1})$, with $\xi = \xi(h) = |h|^{-\nu} (= |h - \tilde{h}_*|^{-\nu})$ and $\nu = 2$,*

$$\tilde{\tau}_h^{\mathrm{tr}}(L) \asymp \tilde{\tau}_0^{\mathrm{tr}}(L) \exp\big\{ - c\tfrac{L/\xi}{\log((L/\xi)\vee 2)} \big\}.$$

Here, $\asymp$ means that the two sides are comparable up to numerical factors, with the upper bound "$\leq$" holding for all $|h| < 1$ and $L \geq 1$ and the lower bound "$\geq$" for all $|h| < 1$ and $L \leq \xi$ or $L \geq \xi(\log \xi)^{\delta}$ where $\xi = \xi(h)$, for some $\delta \in (0, 1)$.

The results of Theorem 2 are nearly optimal. The bounds witness the transition from a near-critical picture $L \lesssim \xi$ where $\tilde{\tau}_h^{\mathrm{tr}}(L) \approx \tilde{\tau}_0^{\mathrm{tr}}(L)$, to an off-critical one at scales $L \gg \xi$ governed by the rapidly decaying factor. The system size is thus naturally measured in units of the (correlation) length scale ξ, which diverges polynomially at the critical point. The value of the associated exponent ν fits with a prediction of Weinrib and Halperin [7] based on an extended Harris criterion, which suggests that $\nu = \frac{2}{a}$ (recall that $a = 1$ here).

The talk will highlight the mechanisms underlying the formation of long clusters in Theorems 1 and 2. In the latter case, the critical cost $\tilde{\tau}_0^{\mathrm{tr}}(L)$, $L \lesssim \xi$, emerges when producing a "blob," i.e. forcing the cluster C^h of 0 to have large capacity. The latter is an integrable quantity for $(\widetilde{M1})$: as a result of [2], one knows that

$$\mathbb{P}[\mathrm{cap}(C^0) \geq r] \sim r^{-\frac{1}{2}}, \ \text{as } r \to \infty;$$

noteworthily, along with the fact that clusters in $(\widetilde{M1})$ at levels $h = -\varepsilon$ are carried by random interlacements at intensity $u = \varepsilon^2/2$, which automatically percolate, the previous estimate implies continuity of the phase transition. In the cross-over regime $L \approx \xi$, the cluster in $(\widetilde{M1})$ is carried by essentially one random walk trajectory, which emanates from the explored region and escapes to ∞, on account of a suitable signed isomorphism theorem [5]; forcing the capacity to be suitably large is exactly designed to give birth to this trajectory. Finally at scales $L \gg \xi$, one can prove a finite-size criterion to continue building the cluster by stacking good boxes. The criterion corresponds to a sharp quantitative version of some of the elements underlying the proof of Theorem 1.

Currently, all results generalize to any transient graph with polynomial volume growth of the form R^d for $d \geq a+2$ and Green's function decay with exponent $0 < a \leq 1$, with $\nu = 2/a$ and stretched exponential decay of the form $\exp\{-c(L/\xi)^a\}$ when $a < 1$. The presence of $\log((L/\xi) \vee 2)$ rather than the weaker $\log(L \vee 2)$ for the upper bound in Theorem 2 is part of ongoing work to extend these results, notably to higher values of a.

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The maximum of log-correlated Gaussian fields in random environments

Florian Schweiger

(joint work with Ofer Zeitouni)

We study logarithmically correlated random fields on a d-dimensional lattice. It is expected that many features of such fields are universal in the sense that they do not depend on the precise law of the field. An example of such a feature is the maximum of the field, when restricted to a box of sidelength N, say. Here it is expected that the maximum is of order $c_1 \log N - c_2 \log \log N$ for some explicit constants c_1, c_2 and that the recentered maximum even converges in distribution to a non-degenerate random variable.

For general (non-Gaussian) log-correlated fields, such precise results are often out of reach. The situation is better for Gaussian log-correlated fields. In particular, it is known that the recentered maximum of the two-dimensional discrete Gaussian free field converges in distribution [1], and due to work by Biskup, Louidor and others we have even much finer results on the extremal process. Regarding universality, there are results by Madaule and by Ding, Roy and Zeitouni. In particular, in [3] it is proved that all d-dimensional log-correlated Gaussian fields satisfying certain assumptions on their covariances share the property that their recentered maximum converges in distribution. These assumptions are satisfied by the two-dimensional discrete Gaussian free field and also by the four-dimensional discrete membrane model [4].

The focus of the talk was on log-correlated Gaussian fields in (quenched) random environments, and based on [5]. Unfortunately, such fields typically do not satisfy the assumptions from [3], because uniform bounds on the covariances as required there cannot hold at points near which the environment is atypical. Motivated by this, we presented a generalization of the result in [3] where the uniform assumptions on the covariances are replaced by non-uniform assumptions in terms of some random scales, together with bounds on the number of points where those scales are large. The proof is an adaption of the one in [3], with additional technicalities arising from the need to control the maximum of the field on the atypical points.

This general result should be applicable to various log-correlated Gaussian fields in random environments. One particularly interesting example is the discrete Gaussian free field on the infinite cluster of Bernoulli bond percolation with parameter $p > \frac{1}{2}$ in two dimensions. Here we can prove that if p is sufficiently close to 1, then the aforementioned general result applies, and consequently the recentered maximum of the field on a box of sidelength N converges in distribution. Our proof uses recent results by Armstrong, Dario and Gu on quantitative homogenization on percolation clusters (in particular [2]), as well as various large-deviation results for the local behavior of the cluster. It is an interesting open question whether this result still holds true for any $p > \frac{1}{2}$.

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Dimer model in 3D

Scott Sheffield
(joint work with Nishant Chandgotia, Catherine Wolfram)

The 3D dimer model is more complex than its 2D counterpart. Classical 2D tools that fail to apply in 3D include Kasteleyn matrices and determinants, spanning tree bijections, monotonicity, FKG inequalities, amoeba and Ronkin function constructions, and so on. Even the simplest "local move connectedness" results fall apart in dimensions higher than two, although several papers have been written with partial results.

Nonetheless, the 3D dimer model turns out to be surprisingly interesting as a model of a random divergence-free flow. There are few tools (such as Hall's marriage theorem and the non-intersecting path interpretation) that continue to apply in three dimensions, and we build on these to establish a large deviation principle for the random flow which is analogous to the 2D results of Cohn, Kenyon and Propp, with a unique rate function minimizer. We also present several interesting simulations and animations, including for higher dimensional Aztec diamonds and variants. There are many open problems here on which we would welcome assistance. Example: "Is there a finite set of local moves that, for all n, connect the tilings of a $2n$ by $2n$ by $2n$ box?"

The continuum analog of this model is also straightforward to describe. If one starts with a 3-vector-valued white noise, the orthogonal projection onto the space of "curl-free fields" gives the gradient of a Gaussian free field (GFF) while the orthogonal projection onto the space of "divergence-free fields" gives a very closely related object called the Gaussian divergence-free field (GDFF) which itself has many beautiful properties.

The least singular value of the deformed Ginibre ensemble

Tatyana Shcherbina
(joint work with Mariya Shcherbina)

We study the random $n\times n$ matrices $H = A + H_0$, where A is some general $n\times n$ matrix with complex entries, and H_0 is drawn from the complex Ginibre ensemble, i.e. H_0 has i.i.d. complex Gaussian entries $\{h_{ij}^{(0)}\}_{i,j=1}^n$ such that

$$\mathbb{E}[h_{ij}^{(0)}] = 0, \quad \mathbb{E}[|h_{ij}^{(0)}|^2] = 1/n, \quad \mathbb{E}[(h_{ij}^{(0)})^2] = 0.$$

Deformation A can be deterministic or random (but in this case it is independent of H_0).

Such matrices are important in communication theory, where A is considered as a *signal*, and H_0 as a *noise* matrix. In particular, one is interested in effective numerical solvability of a large system of linear equations $Hx = b$ which is determined by the behaviour of the smallest singular value $\sigma_1(H)$ of H.

The classical bound of Sankar, Spielman and Teng [5] states that the smallest singular value $\sigma_1(H)$ is of order not smaller than n^{-1} (equivalently, the smallest eigenvalue $\lambda_1(HH^*)$ of HH^* is of order not smaller that n^{-2}), i.e.

$$\mathbb{P}\Big(\lambda_1(HH^*) = (\sigma_1(H))^2 \le x/n^2\Big) \lesssim x, \quad x > 0 \tag{16}$$

up to logarithmic corrections, and uniformly in A.

The bound is proved to be optimal for the case of pure complex Ginibre ensemble (i.e. $A = 0$), see [1].

The matrix HH^* can be considered as the so-called deformed Laguerre ensemble, and its limiting eigenvalue distribution is well-known, see [4]. Moreover, almost surely no eigenvalues lie outside of any finite support neighbourhood of the limiting measure. Therefore, there are three possible situations: 0 is away of the support of the limiting spectral measure, and then $\lambda_1(HH^*)$ has a positive constant lower bound; 0 lies in the bulk of the spectrum, and then (16) is expect to be optimal; the intermediate regime when 0 is near the edge of the spectrum.

In the bulk regime the lower bounds on $\lambda_1(HH^*)$ with quite general A and even with the non-Gaussian H_0 (with iid elements) have been obtained in [7] (although not uniformly in A).

The edge regime is much less studied. However, as it was shown in [2] for the case of the constant diagonal shift of the Ginibre ensemble, i.e. $A = -zI$, the bound (16) can be improved in the edge regime $|z| \sim 1$.

Another important source of motivation is that an effective lower tail bound on the least singular value of $H - z$ is an essential ingredient for the study of eigenvalues distribution of large non-Hermitian matrices. In particular, the results of Cipolloni, Erdős, Schröder [2] was used in their subsequent work [3] to remove the four moment matching condition in the classical edge universality result for non-Hermitian random matrices with iid entries by Tao, Vu [8]. Better understanding of higher order correlation functions of the shifted Ginibre ensemble in the bulk is expected to help to do the same thing for the universality in the bulk.

The main result of the current presentation extends the result of [2] to the case $A = A_0 - zI$ with a complex deformation A_0 satisfying some rather general conditions ensuring that the support D of the limiting normalized counting measure of eigenvalues of $H = H_0 + A_0$ takes the nice form

$$D = \{z : \limsup_{\varepsilon\to 0} n^{-1}\mathrm{Tr}\left(Y_0(z) + \varepsilon^2\right)^{-1} \ge 1\}, \quad Y_0(z) = (A_0 - z)(A_0 - z)^*$$

and that ∂D is a set of piece-wise smooth closed curves enclosing $\sigma_0 = \{z : 0 \in \operatorname{supp}\nu_z\}$ (with a high probability, if A is random) where ν_z is the limiting spectral distribution of $Y_0(z)$ (see [6] for the technical details).

The main result is the following theorem

Theorem 1. *Under the conditions above, if $z \in D$, $dist\{z, \partial D\} = \tilde{\delta}^2 n^{-1/2}$ with $\tilde{\delta} \le C_0$, then (16) can be improved to*

$$\mathbb{P}\Big\{\lambda_1((H-z)(H-z)^*) \le xn^{-3/2}\Big\} \le Cx(1+\log x)$$

uniformly in $\tilde{\delta} \le C_0$.

The proof is based on the asymptotic analysis of the supersymmetric integral representation of the average trace of the resolvent of $(H-z)(H-z)^*$.

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A dynamical approach to lattice Yang-Mills

Hao Shen

Lattice Yang–Mills is a model of a collection of matrices. We consider its Langevin dynamic (stochastic quantization) which is a system of SDEs. Using this SDE system, we give a simple derivation of the Makeenko–Migdal loop equations, prove at strong coupling the uniqueness of infinite volume limit and ergodicity, log-Sobolev and Poincare inequalities, and apply these inequalities to prove large N limits and exponential decay of correlations. These are based on joint works with Scott Smith, Rongchan Zhu and Xiangchan Zhu.

The dynamics in continuum in 2D and 3D have been recently constructed by Chandra, Chevyrev, Hairer and the speaker [1, 2] – see the summaries of Chandra and Chevyrev's talk. Part of the motivation of the talk is to see if certain properties of the YM model can be extracted from the dynamics with a simplification of a lattice cutoff.

Let's start by recalling the definition of the lattice Yang–Mills model. Consider a square lattice (finite or infinite) of dimension d. Each edge xy, where x and y are nearest neighbor vertices, is associated with a Lie group element Q_{xy}. For the talk we simply assume that the Lie group is $SO(N)$. Here edges are directed, and $Q_{xy} = Q_{yx}^{-1}$. The lattice Yang–Mills model is defined by the measure $\exp(\mathcal{S}(Q))d\mu_{Haar}$

on $\mathcal{Q} = \{Q_e : e \in E\}$, where E is the set of all positively directed edges, μ_{Haar} is the Haar measure, and

$$\mathcal{S}(Q) = \beta N \sum_p Tr(Q_{xy}Q_{yz}Q_{zw}Q_{wx}).$$

Here, p sums over all the plaquettes, and we wrote x, y, z, w for the four vertices of the plaquette p. On a finite lattice this is clearly a well-defined measure. The importance of the model is that it has gauge symmetry: $\forall SO(N)$ valued function g on vertices, $\mathcal{S}$ is invariant under

$$Q_{xy} \mapsto g_x Q_{xy} g_y^{-1}.$$

In our "dynamical approach" we study the dynamic which leaves the lattice YM measure invariant (so called "stochastic quantization"). This yields interesting stochastic processes. The study of these dynamics may also lead to proofs for properties of the measure. We consider the following SDE (Langevin dynamic)

$$\mathrm{d}Q = \nabla\mathcal{S}(Q)\mathrm{d}t + \sqrt{2}\mathrm{d}\mathfrak{B}$$

where $Q = (Q_e : e \in E)$ and $\mathfrak{B} = (\mathfrak{B}_e : e \in E)$ where E is the set of all lattice edges. Each $\mathfrak{B}_e$ is a Brownian motion on $SO(N)$, and the Brownian motions on different edges are independent. The measure $\exp(\mathcal{S}(Q))d\mu_{Haar}$ is invariant.

We can give a more explicit form of this equation by some calculations, namely:

(1) Recall $\mathcal{S}(Q) = \beta N \sum_p Tr(Q_p)$ where we write $Q_p \stackrel{def}{=} Q_{e_1}Q_{e_2}Q_{e_3}Q_{e_4}$ for the rest of this summary. So, we can calculate the gradient $\nabla\mathcal{S}$ explicitly.
(2) It is well-known that the Brownian motion $\mathfrak{B}$ on $SO(N)$ and the Brownian motion B on the Lie algebra $so(N)$ are related by

$$\mathrm{d}\mathfrak{B} = \mathrm{d}B \circ \mathfrak{B} = \mathrm{d}B\,\mathfrak{B} + \frac{c}{2}\mathfrak{B}\mathrm{d}t,$$

where $\mathrm{d}B \circ \mathfrak{B}$ is the Stratonovich product and $\mathrm{d}B\,\mathfrak{B}$ is the Itô product, and $c = -\frac{N-1}{2}$ (Itô–Stratonovich correction) for $SO(N)$.

With these we can obtain a more explicit SDE system parametrized by edges e

$$\mathrm{d}Q_e = -\frac{1}{2}N\beta \sum_{p \succ e}(Q_p - Q_p^*)Q_e\mathrm{d}t - \frac{1}{2}(N-1)Q_e\mathrm{d}t + \sqrt{2}\mathrm{d}B_eQ_e$$

where $p \succ e$ means plaquettes starting from the edge e (namely if $e = xy$ then p must be of the form $xyzw$ for some z, w), and the last term is Itô.

The first application of this SDE is a simplified proof of Dyson–Schwinger equations. Fix a loop $\gamma = e_1e_2\cdots e_n$ in the lattice. The observable $W_\gamma = Tr(Q_{e_1}\cdots Q_{e_n})$ is called a Wilson loop. Expectations of W_γ satisfy recursive relations among different loops γ, which are called Dyson–Schwinger (a.k.a. Makeenko–Migdal equations or master loop equations). They state that $\mathbf{E}[W_\gamma]$ is equal to a sum of terms of the form $\sum_{\gamma'}\mathbf{E}[W_{\gamma'}]$ (with certain coefficients) where γ' are all possible 'deformations', 'splittings', 'twists', etc. of γ. This was first proved by Chatterjee in 2016 for $SO(N)$ lattice YM using Stein's method. Our new proof is extremely simple: since we have the SDE for Q_e, the proof just follows by applying Itô formula to $\mathbf{E}W_\gamma$, namely:

- The drift term $\sum_{p\succ e}(Q_p - Q_p^*)Q_e\mathrm{d}t$ yields deformations of γ, namely γ 'deformed' by one plaquette;
- The linear term $Q_e\mathrm{d}t$ yields the $\mathbf{E}W_\gamma$ on left-hand side;
- The martingale term $\mathrm{d}B_eQ_e$ disappears upon taking $\mathbf{E}$;
- Quadratic variation yields splittings and twists.

We refer to [3] for details.

For the following results, we write $G = SO(N)$ and we assume $|\beta| < \frac{N-2}{32(d-1)N}$. We have

Theorem. ([4]) The SDE on entire $\mathbf{Z}^d$ has a unique probabilistically strong solution in $C([0,\infty); G^E)$. Every tight limit of the finite volume YM measures is invariant under the above solution to the SDE (on $\mathbf{Z}^d$). Under the above smallness assumption for β the invariant measure of the SDE (on $\mathbf{Z}^d$) is unique.

The proof of the above results relies on a version on Kendall–Cranston coupling and applying Itô formula to a suitable distance between the SDE solutions with different initial conditions. See [4] for details.

Let's discuss a few other results on lattice YM. Fix a finite volume. Under the above smallness assumption for β, the Bakry–Émery condition holds: for any tangent vector v (of the product Lie group)

$$\mathrm{Ric}(v,v) - \mathrm{Hess}_{\mathcal{S}}(v,v) \geq K_{\mathcal{S}}|v|^2, \qquad K_{\mathcal{S}} > 0.$$

It is a standard fact that $\mathrm{Ric}(v,v) = C_{\mathrm{Ric},N}|v|^2$ with $C_{\mathrm{Ric},N} = \frac{N-2}{4}$ for $SO(N)$. By detailed calculations, $|\mathrm{Hess}_{\mathcal{S}}(v,v)| \leq 8(d-1)N|\beta||v|^2$. From these, $K_{\mathcal{S}} > 0$ leads to $|\beta| < \frac{N-2}{32(d-1)N}$. Note that this assumption is such that β still belongs to a non-empty open set when we take $N \to \infty$ later.

Note that Bakry–Émery $\Leftrightarrow$ Ergodicity $\Leftrightarrow$ log-Sobolev $\Rightarrow$ Poincaré inequality. Moreover, Ergodicity $\Rightarrow$ uniqueness of invariant measure on finite lattice. The log-Sobolev and Poincaré inequalities pass to infinite volume, since they are dimension independent.

We show several applications of the Poincaré inequality: for $F \in C^\infty(\mathcal{Q})$,

$$Var(F) \leq \frac{1}{K_{\mathcal{S}}}\mathbf{E}(|\nabla F|^2)\ .$$

Apply this inequality to $F = \frac{1}{N}W_\gamma$, and calculate the right-hand side. We get

$$Var\Big(\frac{1}{N}W_\gamma\Big) \leq \frac{1}{K_{\mathcal{S}}}\frac{n(n-3)}{N} \to 0 \qquad N \to \infty$$

which shows that the rescaled Wilson loop converges to a deterministic limit (so called master field). Studying fluctuations or higher order corrections would be interesting.

As another application of Poincare inequality, one has

Corollary (Mass gap): Let $f, g \in C^\infty_{cyl}(\mathcal{Q})$ with supports $\Lambda_f \cap \Lambda_g = \emptyset$. Then

$$Cov(f,g) \leq c_1 e^{-c_2 d(\Lambda_f,\Lambda_g)}$$

In particular f, g can be Wilson loops. The proof uses certain earlier ideas by Guionnet–Zegarlinski for other models.

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On the spectral gap in the Kac-Luttinger model and Bose-Einstein condensation

Alain-Sol Sznitman

In this talk we report on the results of the recent preprint [1].We consider the Dirichlet eigenvalues of the Laplacian among a Poissonian cloud of hard spherical obstacles of fixed radius in large boxes of R^d, $d \geq 2$. In a large box of side-length 2ℓ centered at the origin, the lowest eigenvalue is known to be typically of order $(\log \ell)^{-2/d}$. We show in [1] that with probability arbitrarily close to 1 as ℓ goes to infinity, the spectral gap stays bigger than $\sigma(\log \ell)^{-(1+2/d)}$, where the small positive number σ depends on how close to 1 one wishes the probability. Incidentally, the scale $(\log \ell)^{-(1+2/d)}$ is expected to capture the correct size of the gap. Our result involves the proof of new deconcentration estimates. Combining this lower bound on the spectral gap with the results of Kerner-Pechmann-Spitzer [2], we infer a type-I generalized Bose-Einstein condensation in probability for a Kac-Luttinger system of non-interacting bosons among Poissonian spherical impurities, with the sole macroscopic occupation of the one-particle ground state when the density exceeds the critical value.

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Directed polymers in the weak disorder regime, and exponential moments

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(joint work with Clement Cosco)

Given an (infinite) graph $G = (V, E)$ and an i.i.d. collection of centered standard Gaussian random variables $\{\omega(i,x)\}_{i\geq 0, x\in V}$, the *directed polymer at inverse temperature* β is defined as

$$\frac{d\hat{\mathrm{P}}^x}{d\mathrm{P}^x} = \frac{\left[e^{\sum_{n=1}^N \beta\omega(n,S_n) - N\beta^2/2}\right]}{W_N(x,\beta)},$$

where

$$W_N(x,\beta) = \mathrm{E}^x\left[e^{\sum_{n=1}^N \beta\omega(n,S_n) - N\beta^2/2}\right]$$

is the partition function, and P^x (with associated expectation E^x) is the law of simple random walk (S_n) on the graph, started at x. For fixed β, $W_N := W_N(0,\beta)$ is a positive martingale and thus converges to a limit $W_\infty(\beta)$. The *weak disorder* regime is when $W_\infty(\beta)$ is non-degenerate, i.e. $W_\infty(\beta) > 0$. The directed random polymer case has close relations with the stochastic heat equation (SHE) and the KPZ equation.

In the classical case, $G = \mathrm{Z}^d$, and then a weak disorder phase exists for $d \geq 3$. We refer to [2] for an extensive discussion and references, and to [4] for a discussion of the general case. For $d = 2$, such a weak disorder phase does not exists, but a meaningful rescaling was discovered in the context of the SHE by Bertini and Cancrini [1] and studied extensively in both the SHE and polymer setups by Caravenna, Sun and Zygouras [5, 6], see also [7], [8], [9] and [10].

To define the rescaling, introduce the mean intersection local time for two random walks (S_n^1, S_n^2) by

$$R_N = \mathrm{E}^{0\otimes 0}\left[\sum_{n=1}^N \mathbf{1}_{S_n^1 = S_n^2}\right] \sim \frac{\log N}{\pi},$$

and set

$$\beta_N = \frac{\hat{\beta}}{\sqrt{R_N}}, \quad \hat{\beta} \geq 0. \tag{17}$$

Then, see [5], one has that

$$\forall \hat{\beta} < 1: \quad \log W_N \xrightarrow{(d)} \mathcal{N}\left(-\frac{\lambda^2}{2}, \lambda^2\right), \quad \text{with} \quad \lambda^2(\hat{\beta}) = \log\frac{1}{1-\hat{\beta}^2}. \tag{18}$$

The convergence in (18) has recently been extended in [11] to the convergence of W_N to the exponential of a Gaussian, in all L^p.

The spatial behavior of $W_N(\beta_N, x)$ is also of interest. Indeed, one has, see [6],

$$G_N(x) := \sqrt{R_N}\Big(\log W_N(\beta_N, x\sqrt{N}) - \mathbb{E}\log W_N(\beta_N, x\sqrt{N})\Big) \xrightarrow{(d)} \sqrt{\frac{\hat\beta^2}{1-\hat\beta^2}}G(x),$$

with $G(x)$ a log-correlated Gaussian field on $\mathbb{R}^2$.

A natural question pertains to the study of extrema of $G_N(x)$. As a first step in this direction, we study the exponential moments of $G_N = G_N(0)$. Our main result reads as follows.

Theorem [3] *There exists $\hat\beta_0 \le 1$ so that if $\hat\beta < \hat\beta_0$ and*

$$\limsup_{N\to\infty} \frac{3\hat\beta^2}{\Big(1-\hat\beta^2\Big)} \frac{1}{\log N}\binom{q}{2} < 1, \tag{19}$$

then,

$$\mathbb{E}[W_N^q] \le Ce^{\binom{q}{2}\lambda^2(1+|\varepsilon_N|)}, \tag{20}$$

where $\varepsilon_N = \varepsilon(N, \hat\beta) \to 0$ as $N \to \infty$, and $C = C(\hat\beta) > 0$.

The proof shows that in the Theorem, $\hat\beta_0$ can be taken as 1/96, but we do not expect this to be optimal.

A complimentary lower bound, as well as multi-points exponential estimates, is part of work in progress.

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