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The Renormalization Group

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Abstract. The renormalization group was originally introduced as a multi-scale approach to quantum field theory and the theory of critical phenomena, explaining in particular the universality observed e.g. in critical exponents. Over the years it has become a powerful tool in the mathematical analysis of systems with infinitely many interacting degrees of freedom. Its applications include quantum field theories, classical and quantum statistical mechanics, (stochastic) partial differential equations, operator theory, and probability theory. For some important problems, it is the only known tool for mathematical proofs. The last few years have seen further important developments, in particular in the application of the method to probabilistic questions, and to equilibrium and non-equilibrium quantum statistical mechanics. This workshop has given an account of the most important new developments in the last five years, including methodical progress, current applications, relations to other approaches, and identified new challenges that may be tackled in future work with the help of the renormalization group.

Mathematics Subject Classification (2020): 81T15, 81T16, 81T17, 82B05, 82C10.

Introduction by the Organizers

Invented by Kadanoff and Wilson in the late 1960s as a means for understanding universality in critical phenomena, the renormalization group (RG) has changed the way theoretical physicists approach the study of large systems. In particular, the RG phenomenology of investigating a sequence of effective models associated to different length scales, hence considering a dynamical system on “the space of all theories” instead of regarding fixed models, has completely changed the concepts and techniques used, far beyond the original application to critical phenomena.

Soon after that, the first attempts to use the RG in mathematical studies started. After meeting initial difficulties, the method led to the mathematical construction of models from theoretical and mathematical physics such as quantum field theories and closely related models of statistical mechanics, and to the determination of some key properties. Indeed, most presently known mathematically rigorous constructions of quantum field theories in dimensions three or higher involve the RG.

The basic concepts behind the RG are of a very general nature and therefore have potential applications, in one way or another, to essentially any system or mathematical problem wherein degrees of freedom associated with multiple spatio-temporal scales are coupled. Consequently, the RG has become a powerful method in the study of systems with infinitely many degrees of freedom in general. Examples include the construction and analysis of non-Gaussian measures on infinite-dimensional spaces – typically associated with random objects such as fluctuating fields or paths – but also e.g. to blow-up phenomena associated with certain types of partial differential equations (PDEs), to spectral theory, to non-equilibrium phenomena, or to systems driven by noise such as stochastic PDEs. The philosophy of the RG furthermore suggests that fixed points of the flow, often endowed with new emerging symmetries such as scale invariance, are particularly interesting points in theory space to be investigated in their own right. The study of such conformal field theories (CFTs) is now a very substantial and important subject in its own right.

The 2022 Oberwolfach workshop *The Renormalization Group*, the fifth of its kind at Oberwolfach, organised by Roland Bauerschmidt (Cambridge), Margherita Disertori (Bonn), Stefan Hollands (Leipzig), and Manfred Salmhofer (Heidelberg), brought 45 participants to the Forschungszentrum, and another 6 participants joined online. The majority of the people present were from Europe and adjacent regions, five from North America and two from South America.

The talks and discussions covered most of the above-mentioned topics. A group of talks concerned renormalization techniques in Euclidian and Lorentzian quantum field theory (V. Mastropietro, K. Rejzner, V. Rychkov, J. Zahn). Several contributions focussed on a particular variant of Wilsonian renormalization, the Polchinski flow equation, both in its original form as adapted to various problems in quantum field theory but also some variants adapted to stochastic PDEs and variational techniques (C. Kopper, M. Borji, N. Barashkov, W. Kroschinsky, P. Duch, M. Gubinelli). Hierarchical models were studied, from multicomponent spin models (G. Slade, D. Marchetti) to the nonlinear hyperbolic supersymmetric sigma model (S. Rolles and F. Merkl, L. Fresta). First mathematical results on the tensor network RG were discussed (T. Kennedy). Further topics included: non-abelian correlation inequalities (A. Abdesselam); a rigorous approach to the problem of emergence of histories (here: particle tracks) in quantum theory (J. Fröhlich); quantum scaling limits for anyon systems (A. Stottmeister); a new characterization of singular continuous spectrum in random Schrödinger operators via a different type of symmetry-breaking in the associated supersymmetric nonlinear sigma model (M. Zirnbauer).

After the 2016 Oberwolfach workshop on this topic, this was the first occasion for a topical conference on mathematical renormalization. The possibility to meet in person, as well as the excellent service of the MFO, were very much appreciated by all participants.

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Workshop: The Renormalization Group

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Abstracts

Fermionic models with long-range interactions: Wilsonian RG flows and fixed points

Slava Rychkov
(joint work with Alessandro Giuliani, Vieri Mastropietro, Giuseppe Scola)

These models are characterized, in physics jargon, by the action:

$$S(\psi) = \int d^d x \Omega^{ab} \psi_a (-\Delta)^\sigma \psi_b + V(\psi) \tag{1}$$

where we are in $\mathbb{R}^d$, $\psi = (\psi_a)_{a=1}^N$, N even, are Grassmann-valued fields, Ω is the symplectic matrix, $\sigma \in \mathbb{R}$ determines the scaling dimension of the field ψ in the Gaussian theory, and V is a local interaction potential, the simplest case being: $V = \nu\psi^2 + \lambda(\psi^2)^2$, $\psi^2 \equiv \Omega^{ab}\psi_a\psi_b$ i.e quadratic plus quartic preserving $Sp(N)$ invariance of the kinetic term.

This model is an ideal laboratory for Wilsonian RG. The parameter σ can be tuned so that the scaling dimension of ψ is $d/4 - \varepsilon$, $\varepsilon \ll 1$, while the quartic interaction is then close to marginal. RG flow of the effective action of model (1), appropriately defined with a momentum cutoff, can then be studied through convergent Feynman diagram expansions. In [1] we rigorously constructed a non-gaussian RG fixed point for small ε (similar results were obtained by Gawedzki and Kupiainen in 1985). Unlike for bosonic models, this fixed point is analytic in a small disk $|\varepsilon| < \varepsilon_0$. We are working on its various further properties: critical exponents, scale invariant effective action, scaling operators, universality. Conformal invariance is also expected and is an interesting open problem.

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Transseries and the forest formula

Razvan Gurau

The purpose of this talk is to give a detailed description of the transseries expansion for the free energy of a ϕ^4 zero-dimensional theory, written in the Loop Vertex Expansion. The latter provides an expansion for the logarithm of the partition function as a convergent series of multidimensional integrals, which can be analytically continued all the way to the cut on the negative axis in the complex plane of the coupling constant, as well as beyond the cut on the next sheet of the Riemann surface. We show that performing an asymptotic expansion of such series term by term a transseries is gradually reconstructed. We discuss the resurgence phenomenon in this context.

On asymptotically free scalar fields

Christoph Kopper

The mean field flow equations of the renormalization group admit smooth solutions for the n-point Schwinger functions which are asymptotically free in the ultraviolet. These solutions correspond to nonpolynomial bare interactions. They are constructed inductively from the mean field flow equations, starting from a class of two-point functions which are smooth with respect to the flow parameter. The work is to be put into perspective with the triviality theorems on scalar field theory in four dimensions by Fröhlich (1982) and Aizenman (1982), sharpened recently by Aizenman & Duminil-Copin (2019, 2021).

The bare actions considered in the triviality theorem, i.e. those of symmetric φ_4^4 theory, are lattice regularised versions of

$$L_0 = \int d^4x \Big[a\,\varphi^2 + b\,(\partial_\mu\varphi)^2 + \lambda\,\varphi^4\Big]\ ,\quad \lambda > 0\ .$$

It is a general belief that the trviality statements should be extendable in a suitable form to strictly renormalisable quantum field theories which are not asymptotically free (or safe) in the ultraviolet. Subsequently we shall consider more complicated forms of bare interactions which contain even monomials in φ of any order:

$$L_0(\varphi) = \int_{V\to\mathbb{R}^4} d^4x\,\Big[\sum_{n\in 2\mathbb{N}} a_{0,n}\varphi^{2n}\ +\ b_0(\partial_\mu\varphi)^2\Big]\ .$$

The renormalisation group setting is introduced through the flowing propagator

$$C^{\alpha_{\rm o},\alpha}(p) \int_{\alpha_{\rm o}}^{\alpha} d\alpha'\, e^{-\alpha'(p^2+m^2)}\ ,\quad 0\le\alpha_{\rm o}\le\alpha\le\infty$$

$$\text{with}\ \ \dot C^{\alpha}(p) \equiv \partial_\alpha C^{\alpha_{\rm o},\alpha}(p) = e^{-\alpha(p^2+m^2)}\ .$$

Here $0\le\alpha_{\rm o}\le\alpha\le\infty$, $\alpha_{\rm o}$ is an UV regulator, α generates the RG flow. Then we define the Wilson effective action

$$e^{-L^{\alpha_{\rm o},\alpha}(\varphi)} = \int d\mu_{\alpha_{\rm o},\alpha}(\phi)\ e^{-L_0(\varphi+\phi)}\ ,$$

where $\mu_{\alpha_{\rm o},\alpha}(\phi)$ is the normalised Gaussian measure with covariance $C^{\alpha_{\rm o},\alpha}$. The Wilson effective action at scale α

$$e^{-L^{\alpha_{\rm o},\alpha}(\varphi)}$$

is the generating functional of the free propagator amputated connected n-point Schwinger functions with free propagator $C^{\alpha_{\rm o},\alpha}$ and bare (inter)action L_0.

Using integration by parts one obtains the functional flow equation

$$\partial_\alpha L^{\alpha_{\rm o},\alpha} = \frac{1}{2}\langle\frac{\delta}{\delta\varphi}, \dot C^{\alpha}\frac{\delta}{\delta\varphi}\rangle L^{\alpha_{\rm o},\alpha}\ -\ \frac{1}{2}\langle\frac{\delta}{\delta\varphi}L^{\alpha_{\rm o},\alpha}, \dot C^{\alpha}\frac{\delta}{\delta\varphi}L^{\alpha_{\rm o},\alpha}\rangle\ .$$

$$\text{Here}\ \ \langle\frac{\delta}{\delta\varphi}, \dot C^{\alpha}\frac{\delta}{\delta\varphi}\rangle \equiv \int d^4x\,d^4y\,\frac{\delta}{\delta\varphi(x)}, \dot C^{\alpha}(x-y)\frac{\delta}{\delta\varphi(y)}\rangle\ \ \text{etc.}$$

Assuming that $L^{\alpha_\mathrm{o},\alpha}$ has an expansion in moments

$$L^{\alpha_0,\alpha}(\varphi) = \sum_{n\in 2\mathbb{N}} \int_{p_1,\ldots,p_n} \delta^4(p_1+\ldots p_n)\mathcal{L}_n^{\alpha_\mathrm{o},\alpha}(p_1,\ldots,p_n)\,\hat{\varphi}(p_1)\ldots\hat{\varphi}(p_n)$$

where we used translation invariance, we can write the flow equations (FEs) for the connected amputated Schwinger functions

$$(1)\qquad \partial_\alpha \mathcal{L}_n^{\alpha_\mathrm{o},\alpha}(p_1,\ldots,p_n) = \binom{n+2}{2}\int \dot{C}^\alpha(p)\,\mathcal{L}_n^{\alpha_\mathrm{o},\alpha}(p_1,\ldots,p_n,p,-p)$$

$$-\frac{1}{2}\sum_{\substack{n_i\in 2\mathbb{N}\\ n_1+n_2=n+2}} \Big[\mathcal{L}_{n_1}^{\alpha_\mathrm{o},\alpha}(p_1,\ldots,q')\dot{C}^\alpha(q')\mathcal{L}_{n_2}^{\alpha_\mathrm{o},\alpha}(-q',\ldots,p_n)\Big]_{\mathrm{sym}} .$$

The expression in square brackets has to be symmetrized w.r.t. the external momenta $p_1,\ldots,p_n$. Polchinski observed that these equations when expanded in the number of loops, provide an airtight inductive scheme to prove perturbative renormalisability. This programme has been extended to get more explicit (otherwise inaccessible) knowledge of the renormalised Schwinger functions.

The mean field approximation consists in replacing $\mathcal{L}_n(p_1,...,p_n)$ by $\mathcal{L}_n(0,...,0)$. Set

$$A_n^{\alpha_\mathrm{o},\alpha} \equiv \mathcal{L}_n^{\alpha_\mathrm{o},\alpha}(0,\ldots,0) = \lim_{V\to\mathbb{R}^4}\frac{1}{V}\int d^4x_1\ldots d^4x_n\ \hat{\mathcal{L}}_n^{\alpha_\mathrm{o},\alpha}(x_1,\ldots,x_n) .$$

For later convenience we perform a rescaling :

$$f_n(\mu) \equiv \alpha^{2-\frac{n}{2}}\ c^{\frac{n}{2}-1}\ n\ A_n^{\alpha_\mathrm{o},\alpha}\ ,\quad \mu := \ln(\frac{\alpha}{\alpha_\mathrm{o}}) .$$

We restrict μ to the interval $[0,\mu_{\max}]$ with $\mu_{\max} = \ln(\frac{1}{\alpha_\mathrm{o}})$. The maximal value of α, $\alpha=1$, then corresponds to an IR cutoff. For simplicity we therefore may set instead $m=0$. With these ingredients we get from (1) the mean field FEs:

$$f_{n+2}(\mu) = \frac{1}{n+1}\sum_{\substack{n_i\in 2\mathbb{N}\\ n_1+n_2=n+2}} f_{n_1}(\mu)\,f_{n_1}(\mu)\ + \frac{n-4}{n(n+1)}f_n(\mu) + \frac{2}{n(n+1)}\partial_\mu f_n(\mu) .$$

Note that $f_n(0)$ are the bare amplitudes and $f_n(\mu_{\max})$ are the "physical" ones.

It is then easy to see that if $f_2(\mu)$ is given together with all its derivatives, the mean field FEs determine inductively all $f_n(\mu)$ together with their derivatives.

The proof of the following statement is then straightforward

Proposition : *Let $f_2(\mu)$ be given such that*
i) $f_2(\mu)\in C^\infty[0,\mu_{\max}]$
ii) $|\partial_\mu^\ell f_2(\mu)| \le K^\ell\,\delta^{\ell+1}\,\ell!$ *for* $K>0,\ 0<\delta<1,\ \mu\in[0,\mu_{\max}]$.
Then there exist smooth solutions $f_n(\mu)\in C^\infty[0,\mu_{\max}]$ of the mean field FEs such that for $\mu\in[0,\mu_{\max}]$

$$|\partial_\mu^\ell f_n(\mu)|\ \le\ K^{n+\ell}\,\delta^{\ell+1}\,\frac{(n+\ell)!}{n!} .$$

Example : set

$$f_2(\mu) = -\delta(\mu)\ , \quad \delta(\mu) := \frac{\delta}{1 + (\mu_{\max} - \mu)\beta\delta}\ , \quad \beta > 0\ .$$

We then find

$$\partial_\mu \delta(\mu) = \beta\, \delta^2(\mu)\ .$$

This choice for $f_2(\mu)$ satisfies the assumptions of the Proposition. We find

$$f_4(\mu) = \frac{1}{3}\delta(\mu)(1 - \delta(\mu)) + \partial_\mu \delta(\mu) > 0\ .$$

We also find *Asymptotic Freedom* :

$$\lim_{\mu_{\max}\to\infty} f_n(0) \to 0 \ \ \forall n\ .$$

The bare interaction terms vanish logarithmically in the ultraviolet region.

The proof of the subsequent theorem is more delicate. We have

Theorem *(asymptotically free solutions of bounded action):*
There exist smooth solutions of the mean field FEs satisfying the previous proposition such that the bare mean field action can be written as

$$L_0^{\mathrm{mf}}(x) = \sum_{n\in 2\mathbb{N}} A_{0n} \sin(\alpha_{\mathrm{o}}^{n/2} x)\, \alpha_{\mathrm{o}}^{2-\frac{n}{2}}\ ,$$

where the $A_{0,n}$ *satisfy the bounds*

$$|A_{0,n}| \le \varepsilon\, n^{-5/4} \alpha_{\mathrm{o}}^{\frac{n}{2}-2}\ .$$

Here $\varepsilon > 0$ *has to be chosen sufficiently small (in fact very small).*

Among these solutions there are nontrivial asymptotically free solutions of the type discussed before. The solutions considered also satisfy

$$|\partial_\mu^\ell A_{0,n}| \ge \varepsilon^{\ell+1}\, n^{-5/4}\, \frac{(n+\ell)!}{n!}\ .$$

The bare action satisfies

$$|L_0^{\mathrm{mf}}(x)| \le \varepsilon \sum_{n\in 2\mathbb{N}} n^{-5/4}\ .$$

Finite-size scaling for the 4-dimensional multicomponent hierarchical $|\phi|^4$ model

Gordon Slade
(joint work with Emmanuel Michta, Jiwoon Park)

This report is based on joint work in progress with Emmanuel Michta (UBC) and Jiwoon Park (Cambridge) [3].

We consider critical scaling for the 4-dimensional n-component hierarchical $|\varphi|^4$ model for all $n \in \mathbb{N}$. This model has been extensively studied via a renormalisation group analysis in the recent book by Bauerschmidt, Brydges and Slade [2] which provides the basis for our work, though we require some extensions of the results of [2]. Our focus is on the universal finite-size scaling in the vicinity of the infinite-volume critical point.

Let $L > 1$ be fixed and let Λ_N be the subset of $\mathbb{Z}^4$ consisting of points in $[0, L^N)^4$. In particular, the volume of Λ_N is L^N. Let Δ_N be the hierarchical Laplacian on Λ_N (see [2, Chapter 4]). Given $n \in \mathbb{N}$, $g > 0$, $\nu \in \mathbb{R}$, and a spin field $\varphi : \Lambda_N \to \mathbb{R}^n$, we define the *Hamiltonian*

$$H_{g,\nu,N}(\varphi) = \frac{1}{2}(\varphi, (-\Delta_N)\varphi) + \frac{1}{2}\nu \sum_{x\in\Lambda_N} |\varphi_x|^2 + \frac{1}{4}g \sum_{x\in\Lambda_N} |\varphi_x|^4, \tag{1}$$

the *partition function*

$$Z_{g,\nu,N} = \int_{(\mathbb{R}^n)^{\Lambda_N}} e^{-H_{g,\nu,N}(\varphi)} d\varphi, \tag{2}$$

and its associated expectation

$$\langle F\rangle_{g,\nu,N} = \frac{1}{Z_{g,\nu,N}} \int_{(\mathbb{R}^n)^{\Lambda_N}} F(\varphi) e^{-H_{g,\nu,N}(\varphi)} d\varphi. \tag{3}$$

The finite-volume *susceptibility* is defined by

$$\chi_N(g,\nu) = \frac{1}{n} \sum_{x\in\Lambda_N} \langle \varphi_0 \cdot \varphi_x\rangle_{g,\nu,N}. \tag{4}$$

It is proved in [2] that for $n \in \mathbb{N}$, for L sufficiently large, and for $g > 0$ sufficiently small, there is a critical point $\nu_c(g) < 0$ such that the infinite-volume limit $\chi_\infty(g,\nu) = \lim_{N\to\infty} \chi_N(g,\nu)$ exists for all $\nu > \nu_c(g)$, and, as $\epsilon \downarrow 0$,

$$\chi_\infty(g, \nu_c(g) + \epsilon) \sim A_{g,n} \frac{1}{\epsilon} \big(\log \epsilon^{-1}\big)^{\hat\gamma} \tag{5}$$

with $\hat\gamma = \frac{n+2}{n+8}$ and, as $g \downarrow 0$,

$$A_{g,n} \sim \Big(\frac{Bg}{\log L}\Big)^{\hat\gamma}, \qquad B = (n+8)(1 - L^{-d}). \tag{6}$$

Similar results have also been proved for the $|\varphi|^4$ model on the Euclidean lattice $\mathbb{Z}^4$ in [1].

To state our main result, we need the following definitions. For $n \in \mathbb{N}$ and $s \in \mathbb{R}$ we define a probability measure on $\mathbb{R}^n$ by

$$d\sigma_{n,s} \propto e^{-\frac{1}{4}|x|^4 - \frac{1}{2}s|x|^2} dx. \tag{7}$$

We write the p^{th} moment of the above measure as $M_{n,p}(s) = \int_{\mathbb{R}^n} |x|^p d\sigma_{n,s}$. We define the *window scale*

$$w_N = \frac{1}{L^{2N} N^{\hat{\theta}}} \frac{A_{g,n}(\log L^2)^{\hat{\gamma}}}{B^{1/2}}, \qquad \hat{\theta} = \frac{1}{2} - \hat{\gamma} = \frac{4-n}{2(n+8)}. \tag{8}$$

Let $\Phi_N = \sum_{x \in \Lambda_N} \varphi_x$ denote the total field and let $\hbar_N = (BN)^{1/4} L^{-N}$. Our main result is the following theorem which provides a universal scaling profile within the high-temperature ($s \geq 0$) side of the critical window.

Theorem 1. *Let $n \in \mathbb{N}$. For L sufficiently large and for $g > 0$ sufficiently small, the following statements hold for the n-component 4-dimensional hierarchical model in the limit $N \to \infty$.*

(1) *For any $h \in \mathbb{R}^n$ and any $s \geq 0$,*

$$\lim_{N \to \infty} \big\langle e^{h \cdot \Phi_N / \hbar_N} \big\rangle_{g, \nu_c + s w_N, N} = \int_{\mathbb{R}^n} e^{h \cdot x} \, d\sigma_{n,s}. \tag{9}$$

(2) *Let $p \geq 1$, let $A > 0$ be any positive number, let $a > 0$ be sufficiently small, and let (s_N) be either a bounded sequence $s_N \in [-aN^{-1/2}, A]$ or a divergent sequence $s_N \to \infty$ with $s_N = o(N^{\frac{3}{2(n+2p)}})$. Then*

$$\big\langle |\Phi|^{2p} \big\rangle_{g, \nu_c + s_N w_N, N} = \hbar_N^{2p} M_{n,2p}(s_N)\big(1 + o(1)\big). \tag{10}$$

Theorem 1(i) is a statement of convergence of moment generating functions and implies that $\Phi_N / \hbar_N$ converges in distribution to a random variable on $\mathbb{R}^n$ with the universal distribution $d\sigma_{n,s}$ for $s \geq 0$. In other words,

$$(BN)^{-1/4} L^N \Phi_N \Rightarrow d\sigma_{n,s}. \tag{11}$$

As a corollary of Theorem 1(ii), we obtain the following universal scaling profile of the susceptibility in the vicinity of the infinite-volume critical point.

Corollary 2. *Let $n \in \mathbb{N}$, let L be sufficiently large, let $g > 0$ be sufficiently small, let A be any positive number, let a be sufficiently small, and let (s_N) be either a bounded sequence $s_N \in [-aN^{-1/2}, A]$ or a divergent sequence $s_N \to \infty$ with $s_N = o(N^{\frac{3}{2(n+2)}})$. Then the susceptibility obeys*

$$\chi_N(\nu_c + s_N w_N) = \frac{1}{n} M_{n,2}(s_N) L^{2N} (BN)^{1/2} [1 + o(1)]. \tag{12}$$

Our methods have the potential to extend from the hierarchical model to the Euclidean model with periodic boundary conditions; it is an open problem to carry out this extension.

As $n \downarrow 0$, the ratio $n^{-1} M_{n,2}(s)$ has limit $I_1(s) = \int_0^\infty r e^{-\frac{1}{4}r^4 - \frac{1}{2}sr^2} dr$. In [3], we provide evidence (but not yet proof) that $I_1(s)$ is the universal profile for the susceptibility for self-avoiding walk on a 4-dimensional discrete torus, with n set equal to 0 in the window scale w_N in (8).

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The Memory Effect and Infrared Divergences in Quantum Field Theory and Quantum Gravity

Robert Wald

It has been known since the earliest days of quantum field theory (QFT) that infrared divergences arise in scattering theory with massless fields [1]. These infrared divergences are manifestations of the memory effect: At order $1/r$ a massless field generically will not return to the same value at late retarded times ($u \to +\infty$) as it had at early retarded times ($u \to -\infty$). There is nothing singular about states with memory, but they do not lie in the standard Fock space. Infrared divergences are merely artifacts of trying to represent states with memory in the standard Fock space. If one is interested only in quantities directly relevant to collider physics, infrared divergences can be successfully dealt with by imposing an infrared cutoff, calculating inclusive quantities, and then removing the cutoff. However, this approach does not allow one to treat memory as a quantum observable and is highly unsatisfactory if one wishes to view the S-matrix as a fundamental quantity in QFT and quantum gravity, since the S-matrix itself is undefined. In order to have a well-defined S-matrix, it is necessary to define "in" and "out" Hilbert spaces that incorporate memory in a satisfactory way. Such a construction was given by Faddeev and Kulish [2] for quantum electrodynamics (QED) with a massive charged field. Their construction can be understood as pairing momentum eigenstates of the charged particles with corresponding memory representations of the electromagnetic field to produce states of vanishing large gauge charges at spatial infinity. (This procedure is usually referred to as "dressing" the charged particles.) We investigate this procedure for QED with massless charged particles and show that, as a consequence of collinear divergences, the required "dressing" in this case has an infinite total energy flux, so that the states obtained in the Faddeev-Kulish construction are unphysical. An additional difficulty arises in Yang-Mills theory, due to the fact that the "soft Yang-Mills particles" used for the "dressing" contribute to the Yang-Mills charge-current flux, thereby invalidating the procedure used to construct eigenstates of large gauge charges at spatial infinity. We show that there are insufficiently many charge eigenstates to accommodate scattering theory. In quantum gravity, the analog of the Faddeev-Kulish construction would attempt to produce a Hilbert space of eigenstates of supertranslation charges at spatial infinity. Again, the Faddeev-Kulish "dressing" procedure does not produce the desired eigenstates because the dressing contributes to the null memory flux.

We prove that there are no eigenstates of supertranslation charges at spatial infinity apart from the vacuum. Thus, analogs of the Faddeev-Kulish construction fail catastrophically in quantum gravity. We investigate some alternatives to Faddeev-Kulish constructions but find that these also do not work. We believe that if one wishes to treat scattering at a fundamental level in quantum gravity — as well as in massless QED and Yang-Mills theory — it is necessary to approach it from an algebraic viewpoint on the "in" and "out" states, wherein one does not attempt to "shoehorn" these states into some pre-chosen "in" and "out" Hilbert spaces. We outline the framework of such a scattering theory, which would be manifestly infrared finite. The research reported here was done in collaboration with Kartik Prabhu and Gautam Satishchandran [3].

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Perturbative renormalization of the massive ϕ_4^4 theory on the half-space $\mathbb{R}^+ \times \mathbb{R}^3$

Majdouline Borji
(joint work with Christoph Kopper)

In quantum field theory, a simple model to study surface effects alone (separated from finite size effects) is the semi-infinite scalar field model which first appeared in 1971 [5]. It is defined starting from the massive ϕ_4^4 model in infinite space, with the difference that it is defined on a half space bounded by a plane. In this model, three types of boundary conditions are considered in the litterature, namely Dirichlet, Neumann and Robin boundary conditions (b.c.). Each of these boundary conditions corresponds to a self-adjoint extension of the Laplacian in $\mathbb{R}^+ \times \mathbb{R}^3$. The self-adjointness of the Laplacian is required in order to define the propagator of a quantum field theory.

Using the path integral formalism, we consider the semi-infinite scalar field model and prove perturbative renormalizability of this theory using the Polchinski flow equations. The point of departure is the regularized free propagator

$$C_{\bullet}^{\Lambda,\Lambda_0}(p;z,z') = \int_{\frac{1}{\Lambda_0^2}}^{\frac{1}{\Lambda^2}} d\lambda\ e^{-\lambda(p^2+m^2)} p_{\bullet}(\lambda;z,z'),$$

where $\bullet \in \{D,R,N\}$ for respectively Dirichlet, Robin and Neumann boundary conditions. In our work, we consider the general case of the Robin boundary condition, but similar arguments hold for other boundary conditions. Since the translation invariance is broken in the z-direction (the semi-line), we work in the

pz-representation which consists in taking the partial Fourier transformation with respect to the variable $x \in \mathbb{R}^3$. The Robin heat kernel is given by

$$p_R(\lambda; z, z') := p_N(\lambda; z, z') - 2 \int_0^\infty \frac{dw}{\sqrt{2\pi\lambda}}\; e^{-w}\, e^{-\frac{\left(z+z'+\frac{w}{c}\right)^2}{2\lambda}} \;,$$

where p_N denotes the one-dimensional Neumann heat kernel

$$p_N(\lambda; z, z') := \frac{1}{\sqrt{2\pi\lambda}} \left(\frac{e^{-\frac{(z-z')^2}{2\lambda}} + e^{-\frac{(z+z')^2}{2\lambda}}}{2} \right).$$

Given the regularized propagator C_R^{Λ,Λ_0}, we proved that the support of its associated Gaussian measure is included in the set

$$\bigcap_{n\geq 1} \left\{ \left(-\Delta_R + m^2\right)^{-n} L^2\left(\mathbb{R}^+ \times \mathbb{R}^3\right) \right\},$$

where Δ_R is the Robin self-adjoint extension of Δ on the half-space $\mathbb{R}^+ \times \mathbb{R}^3$. This implies that the field is smooth and verifies the Robin boundary condition. From Wilson's differential equation follows the system of flow equations relating the connected amputated Schwinger (CAS) distributions $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}\left((z_1,p_1),\cdots,(z_n,p_n)\right)$, $n \in \mathbb{N}$, after a formal loop expansion $l \geq 1$. To establish bounds on the CAS, being distributions, they have to be folded first with test functions. A suitable class of test functions is introduced, together with tree structures that will be used in the bounds to be derived on the CAS. We state and prove inductive bounds on the Schwinger functions which, being uniform in the cutoff, guarantee the existence of finite Schwinger distributions $\lim_{\Lambda\to 0,\Lambda_0\to\infty} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}\left((z_1,p_1),\cdots,(z_n,p_n)\right)$. Since translation invariance is broken in the z-direction (the semi-line), all counter-terms can be z-dependent. In general, the constraints on the bare action result from the symmetry properties of the theory which are imposed, on its field content and on the form of the propagator. In [3], we considered the general bare interaction

$$\begin{aligned} L^{\Lambda_0,\Lambda_0}(\phi) = \frac{\lambda}{4!}\int_V \phi^4(z,x) + \frac{1}{2}\int_V \Big(& a^{\Lambda_0}(z)\phi^2(z,x) - b^{\Lambda_0}(z)\phi(z,x)\Delta_x\phi(z,x) \\ & - d^{\Lambda_0}(z)\phi(z,x)\partial_z^2\phi(z,x) + s^{\Lambda_0}(z)\phi(z,x)(\partial_z\phi)(z,x) + \frac{2}{4!}c^{\Lambda_0}(z)\phi^4(z,x)\Big), \end{aligned}$$

where the functions $a^{\Lambda_0}(z),\ b^{\Lambda_0}(z),\ c^{\Lambda_0}(z),\ d^{\Lambda_0}(z)$ and $s^{\Lambda_0}(z)$ are smooth.

In a second work, still in preparation, we prove that for a particular choice of the renormalization conditions, there exists a "minimal form" of the bare interaction for which the counter-terms are z-independent and can be related to the bulk counter-terms (i.e. needed to renormalize ϕ^4 in $\mathbb{R}^4$)

$$L_R^{\Lambda_0,\Lambda_0}(\phi) = \frac{\lambda}{4!}\int_V \phi^4(z,x) + \frac{1}{2}\int_V \Big(a^{\Lambda_0}\phi^2(z,x) - b^{\Lambda_0}\phi(z,x)\Delta\phi(z,x) + \frac{2}{4!}c^{\Lambda_0}\phi^4(z,x)\Big) \tag{1}$$

$$+ \int_S \left(m^{\Lambda_0} + c\; e^{\Lambda_0}\right)\phi^2(0,x)\;. \tag{2}$$

The counter-terms (c.t.) a^{Λ_0}, b^{Λ_0} and c^{Λ_0} denote the bulk counter-terms. We prove also that for Neumannn boundary condition (i.e. $c = 0$) the bare interaction has a similar form to $L_R^{\Lambda_0,\Lambda_0}$ with five counter-terms: three given by the bulk c.t. and two independent surface counter-terms. However for Dirichlet b.c., only the bulk counter-terms are required to make the theory finite.

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Lorentzian sine-Gordon in (perturbative) algebraic QFT

Kasia Rejzner

I presented the results obtained in [5, 3] concerning the sine-Gordon model with Minkowski signature in the framework of perturbative algebraic quantum field theory.

Perturbative algebraic quantum field theory (pAQFT) — see [12] for a review — is an approach to perturbation theory in quantum field theory that follows the paradigm of local quantum physics proposed by Haag and Kastler [10, 9]. The key feature of this framework is that one separates the construction of the algebra of observables (local aspects of the theory) from the choice of a state (global features). This is particularly important when generalizing the framework to quantum field theory on curved spacetime, see [1, 4, 11]. The name "perturbative" refers to the fact that instead of C^*-algebras (as in traditional AQFT framework), one uses forml power series with coefficients in topological *-algebras.

The pAQFT framework has been applied to a wide class of physical problems including quantization of a bosonic string [6, 13] and effective quantum gravity [2]. However, it is important to keep in mind that the ultimate goal of mathematical QFT is to find non-perturbatively constructed models.

The presented result (convergence results of the massless sine-Gordon model on the 2-dimensional Minkowski space in the model's ultraviolet-finite regime) is a proof of concept, showing that in some simple (yet interacting) cases perturbation theory leads to convergent quantities.

As it turns out, despite working with hyperbolic signature, we can still base our proofs of convergence on the estimates established in [8] for a Euclidean version of the model. This way we test the robustness of the pAQFT framework, but also provide the first construction of the formal S-matrix (seen as a unitary element of an abstract algebra, without referring to a concrete Hilbert space representation) in the massless Sine-Gordon model on $\mathbb{R}^2$ (in the ultraviolet-finite regime) that is performed directly in the Lorentzian signature. This is important for generalization to curved spacetimes, which we believe is relatively straightforward, given the presented result.

In [3] we have shown that a very convenient class of states is provided by the Dereziński-Meissner construction [7]. In these states, we have shown strong convergence of our S-matrices.

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Field Theory of Random Schrödinger Operators

Martin Zirnbauer

The spectrum of a self-adjoint operator is known to decompose into three parts, which are called pure point (pp), absolutely continuous (ac), and singular continuous (sc). In the traditional physics approach to Anderson (de)-localization for random Schrödinger operators, only the first two types of spectrum are featured: ac spectrum comes with spatially extended eigenstates (a.k.a. metallic regime), while the eigenstates for eigenvalues in the pp spectrum are localized (a.k.a. insulating regime). Now, over the last few years there have been various predictions of a possible third regime, called NEE (for non-ergodic extended), where the eigenstates are multifractal, matching expectations for the case of sc spectrum. There exists, however, an ongoing controversy as to whether NEE can be a true thermodynamic phase (instead of just a finite-size effect). In this talk, I will first review the standard field-theory approach, developed by Wegner, Efetov and others, for random Schrödinger operators in the metallic and insulating regimes. Motivated by a recent proposal for the conformal field theory of the integer quantum Hall transition, I will then put forward a field-theoretic scenario for the elusive case of random Schrödinger operators with sc spectrum (NEE phase). Distinct from the usual sigma model, the proposed formulation is supported by an (unpublished) exact solution of Wegner's ($N = 1$)-orbital model on a Bethe lattice.

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Luttinger Liquids at the Edge of Quantum Hall Systems

Marcello Porta

(joint work with Vieri Mastropietro)

1. Introduction

The integer quantum Hall effect is a paradigmatic example of topological transport phenomenon in condensed matter physics. The transverse, of Hall, conductivity of a two-dimensional electron gas at low temperature, exposed to a transverse magnetic field, takes integer values in units of e^2/h. For non-interacting fermions, this phenomenon has been understood in a mathematically rigorous way, starting from the work of [27] for translation invariant systems. Quantization of the Hall conductivity has then been proved for disordered systems with a spectral gap in [7, 4], and later for disordered systems with a mobility gap [1]. In the translation invariant case, quantization follows by identifying the Hall conductivity with a Chern number [27, 4]; in the disordered case, the Hall conductivity turns out to be equal to a non-commutative Chern number [7], or equivalently to the index of a pair of projections [4, 1].

Concerning interacting systems, the rigorous proof of quantization of the Hall conductivity has been open for a while [3]. It has been recently established in [19], following an early suggestion of [5], for gapped many-body quantum systems. An independent proof, based on cluster expansion methods and Ward identities, has been given in [16]. The proof of [16] allows to consider many-body perturbations of gapped, noninteracting quantum systems. Combined with renormalization group methods, it has been extended in [15, 17] to the study of the topological phase transitions for the Haldane-Hubbard model.

All these results apply to lattice models defined on domains without boundary. In the presence of an edge, a non-zero Hall conductivity in the bulk is associated to the emergence of robust edge modes on the boundary of the system, which allow for the propagation of edge currents. In particular, the integer labelling the bulk Hall conductivity is equal to the signed number of edge states. This is the celebrated *bulk-edge duality* for the quantum Hall effect. The emergence of robust edge modes in quantum Hall systems has been predicted in [18]. The bulk-edge duality has been proven for translation invariant non-interacting models in [20], in [26, 11] for disordered systems with a spectral gap, and in [12] for disordered systems with a mobility gap. From an effective field theory viewpoint, edge currents can be described via the chiral Luttinger model [28]. A general, field-theoretic approach to bulk and edge transport in quantum Hall systems, based on anomaly cancellations, has been introduced in [13, 14].

Concerning rigorous results about interacting edge transport, in [2] we proved the quantization of the edge conductance for many-body perturbations of lattice fermions displaying one edge mode, which we extended to the case of two counterpropagating edge modes with opposite spins in [24]. Both results have been obtained via rigorous renormalization group methods; in particular, the results allow to quantify the emergence of the Luttinger model, in its chiral [2] or helical [24] version, for the large-scale behavior of the edge currents. Universality arises as a combination of the emergent and anomalous Ward identities for the Luttinger liquid, combined with lattice Ward identities implied by current conservation. The main limitation of these works is that they consider models with rather special edge state configurations. In particular, in the presence of multiple edge states, backscattering is allowed, and the model belongs to a different universality class. This is expected to have important consequences for edge state transport, in particular for the two-terminal conductance of the Hall bar [21, 22].

2. Main result

We consider a system of interacting fermions on the cylinder Λ_L:

$$\Lambda_L = (\mathbb{Z}/L\mathbb{Z}) \times (\mathbb{Z} \cap [0, L]) \,. \tag{1}$$

In second quantization, we consider many-body Hamiltonians of the form:

$$\mathcal{H}_L = \sum_{x,y} \sum_{\rho,\rho'} a^+_{x,\rho} H_{\rho,\rho'}(x;y) a^-_{y,\rho'} + \lambda \sum_{x,y} \sum_{\rho,\rho'} n_{x,\rho} v_{\rho,\rho'}(x;y) n_{y,\rho'} \,, \tag{2}$$

with H and v short-ranged, and $n_{x,\rho} = a^+_{x,\rho} a^-_{x,\rho}$ ($a^{\pm}_{x,\rho}$ are the usual fermionic creation and annihilation operators). The sum on x, y runs over Λ_L, while the sum on ρ, ρ' runs over possible internal degrees of freedom, labelled by $1, \dots, M$. We assume that the system is translation invariant along the $(1, 0)$ direction: that is,

$$H(x; y) \equiv H(x_1 - y_1; x_2, y_2) \,, \qquad v(x; y) \equiv v(x_1 - y_1; x_2, y_2) \,. \tag{3}$$

Let $\hat{H}(k)$ be the Bloch Hamiltonian, which depends smoothly on $k \in S^1$ as $L \to \infty$. We suppose that $\hat{H}(k)$ has a bulk spectral gap, and that it supports an arbitrary number of chiral edge modes at the Fermi level μ (which is chosen within the bulk gap).

The two-body scattering at the Fermi level can only happen if the Fermi momenta of the incoming and outgoing fermions satisfy the conservation rule:

$$k_F^{\omega_1} - k_F^{\omega_2} = k_F^{\omega_3} - k_F^{\omega_4} \qquad \text{mod } 2\pi, \tag{4}$$

where ω labels (locally) the edge modes crossing the Fermi energy. Since $k_F^{\omega} \equiv k_F^{\omega}(\mu)$ is nonconstant in μ, in the absence of special symmetries the relation (4) is generically false unless $\omega_1 = \omega_2$ and $\omega_3 = \omega_4$ or $\omega_1 = \omega_3$ and $\omega_2 = \omega_4$. We shall assume that (4) can only hold for these choices of edge mode labels.

We focus on the response of the edge current at the boundary $x_2 = 0$ after introducing a local perturbation. The edge response function is:

$$\widehat{G}^{\ell}(\eta, p) = -i \lim_{a\to\infty} \lim_{\beta,L\to\infty} \int_{-\infty}^{0} dt\, e^{\eta t} \sum_{y: y_2 \le a} e^{ipy_1} \langle [n_y(t), \mathcal{J}_0^{\ell}] \rangle \tag{5}$$

where: $n_y = \sum_\rho n_{y,\rho}$ is the density operator at y, $A(t) = e^{i\mathcal{H}_L t} A e^{-i\mathcal{H}_L t}$ and $\mathcal{J}_0^{\ell}$ is the edge current along the direction $(1, 0)$ in a strip of width ℓ, which is expressed in terms of the lattice current density $j_{1,x}$ as:

$$\mathcal{J}_0^{\ell} = \sum_{x_2 \le \ell} j_{1,(0,x_2)} \,. \tag{6}$$

The function $\widehat{G}^{\ell}(\eta, p)$ describes the linear response of the edge current to a slowly varying space-time perturbation, with rate of variation specified by η (in time) and by $|p|$ (in space). The next theorem is our main result, proven in [25].

Theorem 2.1. *For $|\lambda|$ small enough, and under the above assumptions of the Hamiltonian, the edge conductance is, for $\underline{p} = (\eta, p)$ and $\theta > 0$:*

$$\widehat{G}^\ell(\underline{p}) = \sum_\omega g_\omega(\underline{p}) \frac{v_\omega p}{-i\eta + v_\omega p} \frac{sgn(v_\omega)}{2\pi} + O(\ell|\underline{p}|^\theta) + O(e^{-\ell})$$

where the sum runs over the edge modes at $x_2 = 0$, v_ω are their Fermi velocities, and

$$g_\omega(\underline{p}) = \Big((1 + \frac{1}{4\pi|v|}\Lambda) \frac{1}{1 + \frac{1}{4\pi|v|}\omega(\underline{p})\Lambda} \Big)_{\omega\omega}$$

with $|v| = diag(|v_\omega|)$, $\Lambda_{\omega\omega'} = O(\lambda)$, $\omega(\underline{p}) = diag\Big(\frac{-i\eta+v_\omega p_1}{i\eta+v_\omega p_1}\Big)$. In particular,

$$\lim_{\ell\to\infty} \lim_{p\to 0} \lim_{\eta\to 0^+} \widehat{G}^\ell(\underline{p}) = \sum_\omega \frac{sgn(v_\omega)}{2\pi} \,. \tag{7}$$

Remark 2.2.

- *The matrix Λ is given by a convergent series in λ, and it depends on all microscopic details of the systems. The velocities v_ω are also interaction dependent, and given by a convergent series in λ.*
- *The edge response function becomes universal, and quantized, only in the limit of Eq. (7).*
- *The expression (7), combined with the universality of the Hall conductivity* [16] *and with the bulk-edge correspondence for non-interacting systems* [20], *allows to rigorously lift the bulk-edge duality to weakly interacting lattice models.*

The proof of the theorem is based on rigorous RG methods, as in [2, 24]. In particular, the proof allows to quantitatively show the emergence of the *multi-channel Luttinger liquid,* for the large-scale behavior of the edge currents. This is an effective QFT, describing interacting chiral fermions in $1 + 1$ dimensions, with an arbitrary number of chiralities. See [10] for a review and for applications to the quantum Hall effect. A key technical ingredient of the proof of Theorem 2.1 is the vanishing of the beta function of the multi-channel Luttinger model, that we establish via a generalization of the approach of [8, 9, 23]. In particular, the method allows to fully control the backscattering between counterpropagating edge modes, corresponding to marginal couplings in the renormalization group sense. Universality arises as a consequence of the combination of emergent and lattice Ward identities, which is here substantially more involved to exploit than in [2, 24], due to the presence of an arbitrary number of edge channels.

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Anomalies and Renormalization Group

Vieri Mastropietro

The predictions of the Electroweak Theory (EWT), that is the $U(1) \times SU(2)$ sector of the Standard Model, have been spectacularly confirmed by experiments; notable examples are the discovery of the W or Higgs particles. However, it is a theory defined only perturbatively by series which are known to diverge, which cannot by themselves lead to infinitely precise predictions. Moreover a non-perturbative definition without cut-off would be probably Gaussian (triviality), hence one needs to see the EWT as an effective theory. Both problems would be solved by a lattice formulation of EWT, valid up to a cut-off (the inverse lattice step) exponentially large in the inverse coupling, that is of the order of the Landau pole.

The mathematical control of functional integrals is however increasingly difficult (if possible) with cut-off. There is an expected connection between renormalizability and size of cut-off. In Fermi theory (non renormalizable) one expects control up to a cut-off order of the inverse of the coupling ($\sim 80Gev$ below modern experiments) . The EWT is renormalizable; in principle a mathematical construction is possible up to a cut-off exponentially large in the inverse coupling, a very large energy. A major difficulty in obtaining this result is that the perturbative renormalizability for a chiral gauge like EWT is not simply power counting but based on subtle properties; in particular

- the reduction of the degree of divergence with respect to power counting
- the cancellation of the chiral anomalies.

In QED adding a mass to a gauge field produce a non-renormalizable power counting; cancellations due to conservation of current ensure however perturbative renormalizability. Much more subtle is what happens in EWT; current are generically non conserved due to anomalies. At lowest order in perturbation theory the anomalies cancel under a condition which essentially fixes the elementary particle charges; at higher orders one needs to use the perturbative Adler-Bardeen property based on dimensional regulariztion.

It is interesting to see if such properties hold at a non-perturbative level and with finite cut-off, in view of possible construction of QED or EWT on a lattice with exponentially small lattice. I present results in simpler but related models.

- A massive vector boson-fermion model in d=4 with lattice cut-off with step of the order of the inverse coupling. The non-renormalization of the chiral anomaly is proven to be valid even with finite lattice.
- The Sommerfield model on a lattice, with massive vector boson-fermion model in d=2; the continuum limit can be taken with a finite wave number renormalization and the AB property hold for any value of the cut-off. The model show a reduction of divergence degree similar to the one appearing in higher dimensional gauge theories.

- An effective electroweak model on a lattice in d=4 in which the W, Z interaction are described by a quartic interaction. With cut-off of the order of the gauge masses, the anomaly vanishes under the anomaly cancellation condition and the charges are not renormalized.

The proofs are based on regularity properties obtained by Renormalization Group and lattice Ward Identities. The above results are natural prerequisite to a construction of lattice EWT with exponentially large cut-off.

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A Variational Perspective on the Polchinski Equation

Nikolay Barashkov

I gave an overview over the stochatic control approach to the Renormalization Group developed in [1] . Our aim is to study Euclidean Quantum Field Theories, which are measures the form

$$\mathrm{d}\nu^V = \exp\left(-\int_\Lambda V(\varphi)\mathrm{d}x\right)\mathrm{d}\mu$$

where $V : \mathbb{R} \to \mathbb{R}$ is a nonlinear function and μ is the GFF on some domain $\Lambda \subseteq \mathbb{R}^d$, $d < 4$. We will moslty be interested in the case $V(\phi) = \lambda\phi^4$ and $d = 2, 3$. For $d \geq 2$ the Gaussian Free Field is supported on genuine distributions, so the nonliarity V is apriori ill defined and requires renormalization. We look at the solution of the Polchinski equation

$$(\partial_t - \Delta_{\dot{C}_t})V_{t,T} + \frac{1}{2}\|\nabla V_{t,T}\|^2_{L^2} = 0 \qquad V_{T,T} = V_T,$$

where $V_{t,T} : S'(\Lambda) \to \mathbb{R}$, is a functional, with $S'(\Lambda)$ being the Schwarz distributions and V_T is a suitable regularization of $\int V(\varphi)\mathrm{d}x$. Moreover

$$\int_0^\infty \dot{C}_s \mathrm{d}s = (m^2 - \Delta)^{-1}$$

and t is a time paramter which corresponds to the scale where one studies the theory. $V_{t,T}$ is then an "effective potential" describing the behaviour of the theory at scale t. $V_{t,T}$ can be interpreted as the value function of a control problem:

$$V_{t,T}(\varphi) = \inf_{u\in\mathbb{H}_a} \mathbb{E}\left[V_T(Y^{\varphi}_{t,T}) + \frac{1}{2}\int_t^T \|u_s\|^2_{L^2}\mathrm{d}s\right] \tag{1}$$

where

- $$\mathrm{d}Y^{\varphi}_{t,s} = \dot{C}_s^{1/2}u_s\mathrm{d}s + \dot{C}_s^{1/2}\mathrm{d}X_s \qquad Y^{\varphi}_{t,t} = \varphi$$
- X_s is a cylyndrical Brownian motion on $L^2(\Lambda)$, $\mathbb{H}_a$ is the space of processes adapted to X_s which satisfy $\mathbb{E}\left[\int_t^T \|u_s\|^2_{L^2}\mathrm{d}s\right] < \infty$.

The advantage of the representation (1) is that one is able to leverage the sign of V_T. We demostrate this by establishing bounds for the Φ^4_2 model in finite volume. In this case one is able to show uniform upper and lower bounds for $V_{t,T}(\varphi)$ uniformly in the cutoff φ which will imply existence of the measure. One can also carry this out in the case of the Φ^4_3 model. Also that case one can give meaning to (1) in the $T\to\infty$ limit which proviedes one with an intrinstic description of the measure in terms of the Gaussian Free Field. This is of some interest since contrary to the Φ^4_2 model the Φ^4_3 model is not absolutely continuous with respect to the Gaussian Free Field.

The equation (1) is of special interest in the case $t=0, T=\infty, \varphi=0$. In this case the left hand side is equal to the logarithm of the (renormalized) partition function. It can be shown that in that case the minimizer on the r.h.s provides a coupling with the Free Field:

$$\nu^V = \mathrm{Law}(W_\infty + I_\infty(\bar{u}))$$

where

- $W_\infty = \int_0^\infty \dot{C}_s^{1/2}\mathrm{d}X_s$ and one can check that $\mathrm{Law}(W_\infty)$ is the Gaussian Free Field
- $I_\infty(\bar{u}) = \int_0^\infty \dot{C}_s^{1/2}\bar{u}\mathrm{d}s$, I_∞ has the property $\|I_\infty(\bar{u})\|_{H^1} \leq \left(\int_0^\infty \|\bar{u}_s\|^2_{H^1}\mathrm{d}s\right)^{1/2}$.
- $\bar{u}$ minimizes the r.h.s of (1)

In the case of Φ^4_3 one can show $I_\infty(\bar{u}) \in H^{1/2-\delta}$, while for Φ^4_2 one gets significantly better behaivour: $I \in I_\infty(\bar{u}) \in H^{2-\delta}$. By Sobolev embeddings this implies that $I_\infty(\bar{u}) \in C^{1-\delta}$ i.e $I_\infty(\bar{u})$ is Hoelder continous. For the Guassian Free Field, Bramsom Ding and Zeituni investigated the law of the renomalized maximum of the field m. They found that it is given by a randomly shifted Gumbel distribution

$$\mathbb{P}(m \leq x) = \mathbb{E}\exp(-\,\mathrm{Z}\exp(-cx))$$

where Z is a random variable related to the critical Gaussian Multiplicative Chaos of the Field. Our approach allows us to deduce similar results for the Φ^4_2 model following the blueprint develped by Bauerschmidt and Hofstetter [4]: We can couple the Φ^4_2 field to the Gaussian via a Hoelder continuous function and then transfer the techniques from the Gaussian case to Φ^4_2 measure [5].

We conclude with some open Problems:

- I would be intersting to find an extension for Fermions (see M.Gubinelli's report contained in this volume)
- It would be intersting to develop techniques to automatize the renormalization, for instance to construct $\Phi^4_{d-\varepsilon}$ and $\cos(\beta\varphi), \beta^2 < 8\pi$ models.
- We expect that one can use the techniques outlined above to construct invarant measures for PDE's with rough potentials (This is work in progress with F. De Vecchi and I. Zachhuber).
- The variational approach can be used to study EQFT's also in infinite volume [2, 6, 7]. It would be intersting to see if one can obtain exponential decay of correlations for $\Phi^4_{2,3}$ models in the small coupling regime $\lambda \ll 1$.

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Some results related to non-Abelian correlation inequalities for invariant observables

Abdelmalek Abdesselam

This talk reports on the results of the recent work [1] where more details can be found, and where some open problems are formulated. Let G be a connected simple graph with vertex set V and edge set E. The famous Kirchhoff polynomial of G is $K(m) = \sum_T \prod_{e\in T} m_e$ where T is summed over spanning trees in G which connect all the vertices. The polynomial is a function of the collection of variables $m = (m_e)_{e\in E}$ indexed by the edges of the graph. The following is an easy consequence of the Rayleigh property of K.

Theorem 1. *For all u, a, b in $[0,\infty)^E$, $K(u+a+b)K(u) \le K(u+a)K(u+b)$.*

This has the following trivial consequence.

Corollary 2. *For all $\eta > 0$ and all u, a, b in $(0,\infty)^E$, we have $K(u+a+b)^{-\eta}K(u)^{-\eta} \ge K(u+a)^{-\eta}K(u+b)^{-\eta}$.*

Now a much less trivial consequence of Corollary 2 is the following result.

Corollary 3. *For all $N \in \mathbb{N}_{>0}$, for all even elements u, a, b in $\mathbb{N}^E_{>0}$, the correlation functions or basic invariant observables of the $O(N)$ and the $\mathbb{CP}^{N-1}$, at zero ferromagnetic coupling (J=0) are such that*

$$\lim_{\lambda\to\infty} \frac{\langle \mathscr{O}^{\lambda(u+a+b)}\rangle\langle \mathscr{O}^{\lambda u}\rangle}{\langle \mathscr{O}^{\lambda(u+a)}\rangle\langle \mathscr{O}^{\lambda(u+b)}\rangle}$$

exists and belongs to $[1, \infty)$.

A similar result holds for a variety of non-Abelian models of statistical mechanics such as the principal chiral model, σ-models with spins taking values in Grassmannian and even flag varieties, etc. [3, 4]. In the last corollary, λ is a positive integer which is taken to infinity. The notation for expectations is as follows, in the case of the complete graph on p vertices, with an edge e being the same as a pair (i, j) with $1 \le i < j \le p$. For the $O(N)$-model,

$$\langle \mathscr{O}^a \rangle := \left\langle \prod_{1\le i<j\le p} (\sigma_i \cdot \sigma_j)^{a_{ij}} \right\rangle := \int_{(\mathbb{S}^{N-1})^p} \mathrm{d}\mu(\sigma) \prod_{1\le i<j\le p} (\sigma_i \cdot \sigma_j)^{a_{ij}}$$

with integration over the product of spheres $(\mathbb{S}^{N-1})^p$, with respect to μ, the product of $O(N)$-invariant probability measures. The basic invariant observables are the inner products of unit spins $\mathscr{O}_{ij} := \sigma_i \cdot \sigma_j$, and $\mathscr{O}^a$ is just a notation for general monomials in such basic observables. For the $\mathbb{CP}^{N-1}$ model,

$$\langle \mathscr{O}^a \rangle := \left\langle \prod_{1\le i<j\le p} |\langle z_i, z_j\rangle|^{2a_{ij}} \right\rangle := \int_{(\mathbb{S}^{2N-1})^p} \mathrm{d}\mu(z) \prod_{1\le i<j\le p} |\langle z_i, z_j\rangle|^{2a_{ij}}$$

withintegration over the product of complex spheres $(\mathbb{S}^{2N-1})^p$, with respect to μ, now the product of $U(N)$-invariant probability measures. The basic invariant observables are the squared moduli of Hermitian inner products of unit spins $\mathscr{O}_{ij} := |\langle z_i, z_j\rangle|^2$. This is only a small example of a large collection of asymptotic correlation inequalities established in [1]. The conjectural GKS inequalities for non-Abelian spin models, after clearing denominators and expanding in the ferromagnetic couplings $J_{ij} \ge 0$, amount to saying that convergent power series in the J couplings are nonnegative when all the couplings J_{ij} are nonnegative. A sufficient condition for this to hold is that all coefficients of these power series are nonnegative. The resulting infinite hierarchy of inequalities is what we call the coefficientwise GKS (or CGKS) inequalities. Our results such as Corollary 3 say that the CGKS inequalities hold in the limit where one takes a large power λ of the integrands that one takes the expectation of. In addition to the above property of the Kirchhoff polynomial (an example of real stable polynomial with a determinantal representation), the reason behind the last statement is that expectations $\langle \mathscr{O}^{\lambda a}\rangle$ are asymptotically equivalent to a harmless factor times $K(a)^{-\eta}$. For the $O(N)$-model, $\eta = \frac{N-1}{2}$, whereas for the $\mathbb{CP}^{N-1}$ model, $\eta = N - 1$. This suggests the above should also make sense and hold for real non-integer number of components $N \ge 1$. In this talk we also presented a purely combinatorial result [2] on Young subgroups of the symmetric group S_n which implies that, for arbitrary couplings $J_{ij} \geq 0$ and in finite volume, the $\mathbb{CP}^{N-1}$ models is perturbatively well defined for any real number of components $N \geq 1$. Moreover, the corresponding correlations of invariant observables satisfy the GKS inequalities in the zero coupling case ($J = 0$).

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Renormalization and Speed of Convergence to Equilibrium

Benoit Dagallier

The aim of this communication is to report on work started by Roland Bauerschmidt and Thierry Bodineau a few years ago, that I joined afterwards. Informally speaking, the main objective of this work is to quantify how fast the Langevin dynamics associated with a statistical of field theory model converges to its invariant measure, in the continuum and/or large system-size limit. The key aspect is that we are interested in models for which a certain renormalisation procedure is required to make sense of these limits. Quantification of the speed of convergence is done by establishing certain functional inequalities.

To make this informal description more precise, let me start with some notations. A statistical mechanics or field theory model is a probability measure $\mu^{\Lambda}_{A,V} = \mu_{A,V}$ defined on $\mathbb{R}^{\Lambda}$, with Λ a finite lattice, say $\Lambda = L\mathbb{T}^d \cap \epsilon\mathbb{Z}^d$, $d \geq 1$, where $\mathbb{T} = [0, 1)$ is the unit torus, the side-length L is large, and ϵ is either 1 or should be thought of as a small regularisation parameter. The measure reads:

$$(1) \qquad \mu_{A,V}(d\varphi) \propto \exp\Big[-\frac{\epsilon^d}{2}\big(\varphi, A\varphi\big) - \epsilon^d V_0(\varphi)\Big].$$

Above, $A \in \mathbb{R}^{\Lambda\times\Lambda}$ is a positive semi-definite matrix, $(\varphi, A\varphi) = \sum_{x,y} \varphi_x A_{x,y}\varphi_y$ for $\varphi \in \mathbb{R}^{\Lambda}$ and $V_0 : \mathbb{R}^{\Lambda} \to \mathbb{R}$ is a single site potential, of the form:

$$(2) \qquad V_0(\varphi) = \epsilon^d \sum_{x\in\Lambda} V(\varphi_x), \qquad V : \mathbb{R} \to \mathbb{R}, \quad \varphi \in \mathbb{R}^{\Lambda}.$$

A typical example for A and V, corresponding to the continuum φ^4 theory, would be $A = -\Delta^{\epsilon}$, with Δ^{ϵ} the lattice Laplacian given by:

$$(3) \qquad (\Delta\varphi)_x = \epsilon^{-2} \sum_{\substack{y\in\Lambda \\ |y-x|=\epsilon}} \big(\varphi_y - \varphi_x\big).$$

The continuum φ^4 potential in dimension $d \in \{2,3\}$ reads:

$$V(z) = V^\epsilon(z) = \frac{\lambda}{4} z^4 + \frac{1}{2}\big(\mu + a^\epsilon(\lambda)\big) z^2, \qquad z \in \mathbb{R}. \tag{4}$$

Above, $a^\epsilon(\lambda)$ is a counterterm, diverging to $-\infty$ when ϵ vanishes, that is required for $\mu_{A,V}$ to have a well-defined, non-Gaussian limit point when $\epsilon \downarrow 0$:

$$a^\epsilon(\lambda) = \begin{cases} -c_1 \log(\epsilon^{-1}) & \text{if } d = 2, \\ -c_1' \epsilon^{-1} + c_2' \log(\epsilon^{-1}) & \text{if } d = 3. \end{cases} \tag{5}$$

The Langevin dynamics associated with $\mu_{A,V}$ is the following stochastic partial differential equation (SPDE):

$$d\varphi_t = -\epsilon^d \big(A\varphi_t + \nabla V_0(\varphi_t)\big) dt + \sqrt{2} dW_t^\epsilon, \tag{6}$$

where W^ϵ is space-time white noise on $\mathbb{R}_+ \times \Lambda$. This SPDE admits $\mu_{A,V}$ as its only invariant reversible measure. The goal is then to understand how fast convergence takes place, and in particular how this speed depends on the parameters ϵ, L. One way to quantify this speed of convergence is via a log-Sobolev inequality: the measure $\mu_{A,V}$ satisfies LSI(γ) if, for any sufficiently nice test function F:

$$\mathrm{Ent}_{\mu_{A,V}}[F] \le \frac{2}{\gamma} \cdot \epsilon^{-d} \mathbb{E}_{\mu_{A,V}} \big[|\nabla F|^2 \big], \tag{LSI(γ)}$$

where $\mathrm{Ent}_{\mu_{A,V}}[F] = \mathbb{E}_{\mu_{A,V}}[F \log F] - \mathbb{E}_{\mu_{A,V}}[F] \log \mathbb{E}_{\mu_{A,V}}[F]$. One interest of proving (LSI(γ)) is that it implies an exponential decay of the entropy of the law $f_t d\mu_{A,V}$ of the dynamics at time $t \ge 0$:

$$\mathrm{Ent}_{\mu_{A,V}}(f_t) \le e^{-\gamma t} \mathrm{Ent}_{\mu_{A,V}}(f_0). \tag{7}$$

In this sense, understanding how the speed of convergence of the Langevin dynamics depends on ϵ, L amounts to asking the same thing about the parameter γ.

Traditional methods to bound γ rely on convexity considerations, following the seminal idea of Bakry and Emery [1]: if

$$\mathrm{Hess}((\varphi, A\varphi) + V_0(\varphi)) \ge c\, \mathrm{id}, \qquad c > 0, \tag{8}$$

then LSI(γ) holds with $\gamma = c$. Although very general and powerful, the bound (8) cannot be applied in many models of interest, such as the continuum φ^4 model (3)–(4). Indeed, due to the counterterm $a^\epsilon(\lambda)$, the smallest eigenvalue of the Hessian would go to $-\infty$ when ϵ vanishes.

To remedy this situation, Bauerschmidt and Bodineau [2] combined the convexity criterion (8) with a renormalisation group procedure known as the Polchinski equation. Informally, the idea is to progressively integrate out small scales and obtain an effective measure at each scale that has nicer convexity property. Scales are defined through the choice of a covariance decomposition for $(C_t)_{t\in[0,\infty]}$ for A:

$$0 = C_0 \le C_s \le C_t \le C_\infty = A^{-1}, \qquad 0 \le s \le t. \tag{9}$$

One can then interpolate between the full measure $\mu_{A,V} = \nu_0$ and a trivial measure $\nu_\infty = \delta_0$ by defining, for each scale $t \geq 0$, the renormalised measure at scale $t \geq 0$:

$$\nu_t(d\varphi) \propto \exp\Big[-\frac{\epsilon^d}{2}\big(\varphi,(C_\infty - C_t)^{-1}\varphi\big) - \epsilon^d V_t(\varphi)\Big]d\varphi, \tag{10}$$

where the renormalised potential V_t is the central object:

$$\exp\big[-\epsilon^d V_t(\varphi)\big] = \mathbf{E}_{C_t}\Big[\exp\big[-\epsilon^d V_0(\varphi+\zeta)\big]\Big], \qquad \varphi \in \mathbb{R}^\Lambda, \tag{11}$$

with $\mathbf{E}_{C_t}$ the Gaussian measure with covariance C_t. This potential follows the so-called Polchinski equation:

$$\partial_t V_t = \frac{1}{2}\sum_{x,y}\dot{C}_t(x,y)\partial^2_{\varphi_x\varphi_y}V_t - \frac{\epsilon^d}{2}\big(\nabla V_t, \dot{C}_t\nabla V_t\big). \tag{12}$$

Above, $\dot{C}_t$ stands for the component-wise derivative of C_t. Combining the Bakry-Emery argument with the decomposition (ν_t), Bauerschmidt and Bodineau arrive at a multiscale log-Sobolev criterion ([2, Theorem 2.5]): if for each $t > 0$ there is a number $\dot{\ell}_t \in \mathbb{R}$ with:

$$\forall \varphi \in \mathbb{R}^\Lambda, \qquad \dot{C}_t \mathrm{Hess} V_t(\varphi)\dot{C}_t - \frac{\ddot{C}_t}{2} \geq \dot{\ell}_t \dot{C}_t, \tag{13}$$

then LSI(γ) holds with:

$$\frac{1}{\gamma} := |\dot{C}_0| \int_0^\infty \exp\Big[-2\int_0^t \dot{\ell}_s\, ds\Big], \tag{14}$$

with $\dot{C}_0$ the largest eigenvalue of $\dot{C}_0$.

One can see from the above that the main difficulty when trying to use this multiscale criterion is to estimate the Hessian of the renormalised potential V_t, which also reads:

$$\mathrm{Hess} V_t(\varphi) = C_t^{-1}\Big(C_t - \mathrm{Cov}(\varphi)\Big)C_t^{-1}, \tag{15}$$

where $\mathrm{Cov}(\varphi)$ is the covariance matrix for the measure $\mu^{C_t^{-1}\varphi}_{C_t,V}$, defined as in (1) but with an external field $C_t^{-1}\varphi$. At present, we have no general method to do so. One major point of interest would be to better understand the Polchinski equation (12) at the PDE level, and in particular the kind of convexity property of the initial potential V_0 that are propagated to larger scales. In contrast, in models treated so far, little information was directly extracted from the PDE, the input being mostly probabilistic. Results include log-Sobolev inequalities with optimal dependence on the parameters ϵ and/or L for the following models:

- The high-temperature $O(n)$ and SK models [3].
- The continuum sine-Gordon model [2].
- The continuum φ^4 model in dimensions $2, 3$ [4].
- The critical and near-critical Ising model in dimensions $d \geq 5$ [5].

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The Majorant Method for Fermions

Wilhelm Kroschinsky

(joint work with Domingos H. U. Marchetti, Manfred Salmhofer)

1. General Definitions

Let U and W be two complex finite-dimensional vector spaces with basis vectors given by $\mathscr{U} := \{\psi_1, ..., \psi_{2n}\}$ and $\mathscr{W} := \{\theta_1, ..., \theta_{2n}\}$, respectively. An element $f \in \bigwedge(U \oplus W)$ is said to be $\mathscr{U}$-homogeneous (resp. $\mathscr{W}$-homogeneous) if it depends only on the generators $\psi_1, ..., \psi_{2n}$ (resp. $\theta_1, ..., \theta_{2n}$). Recall that an element of a Grassmann algebra is called even (resp. odd) if only those terms associated to even (resp. odd) coefficients are nonzero.

Let $A = (a_{ij}) \in \mathbb{M}_{2n}(\mathbb{C})$ be a $2n \times 2n$ skew-symmetric and invertible matrix with complex entries. Our main object of study is the so-called effective action

$$\tilde{\varphi}(\Psi; A, V) := -\log \int d\mu_A(\Theta) e^{-V(\Theta+\Psi)} \tag{1}$$

where μ_A is the Grassmann Gaussian measure associated to the covariance A and V is an even $\mathscr{U}$-homogeneous element of $\bigwedge(U \oplus W)$. The mapping

$$V \mapsto (T_A V)(\Psi) := \tilde{\varphi}(\Psi; A, V) \tag{2}$$

is called the Renormalization Group Transformation.

The Renormalization Group Transformation has the semigroup property: if $A_1, A_2 \in \mathbb{M}_{2n}(\mathbb{C})$ are both skew-symmetric and invertible, then

$$(T_{A_1+A_2} V)(\Psi) = ([T_{A_1} \circ T_{A_2}] V)(\Psi). \tag{3}$$

This property also holds if we continuously decompose the covariance in terms of a parameter $t \in [t_0, T]$, i.e. if we set

$$A = \int_{t_0}^{T} \dot{A}(\tau) d\tau \tag{4}$$

where $0 \le t_0 < T \le \infty$ are fixed parameters such that $A(T) = A$ and $A(t_0) = 0$. In this case, setting

$$A_{[s,k]} := \int_s^k \dot{A}(\tau) d\tau \tag{5}$$

we can write

$$A = A_{[t_0,t]} + A_{[t,T]} \tag{6}$$

so that the semigroup property leads to

$$\tilde{\varphi}(\Psi; A, V) = -\log \int d\mu_{A_{[t,T]}}(\Theta') e^{-V(t,\Theta'+\Psi)} \tag{7}$$

where we have defined

$$V(t,\Psi) := (T_{[t_0,t]}V)(\Psi) = \tilde{\varphi}(\Psi; A_{[t_0,t]}, V). \tag{8}$$

Notice that the above formulas imply

$$V(T,\Psi) = (T_{[t_0,T]}V)(\Psi) = \tilde{\varphi}(\Psi; A, V), \tag{9}$$

so that we can recover $\tilde{\varphi}$ by evaluating $V(t,\cdot)$ at $t = T$.

It is actually more convenient to work with the normalized $\tilde{\varphi}$, which we will denote by φ

$$\varphi(t,\Psi) := -\log \frac{\int d\mu_{A_{[t_0,t]}}(\Theta) e^{-V(\Theta+\Psi)}}{\int d\mu_{A_{[t_0,t]}}(\Theta) e^{-V(\Theta)}}. \tag{10}$$

If we define

$$F(t,\Psi) := e^{-\varphi(t,\Psi)} \tag{11}$$

we can prove that F satisfies

$$\frac{\partial F}{\partial t}(t,\Psi) = \frac{1}{2}\Delta_{\dot{A}} F(t,\Psi) - \frac{1}{2} F(t,\Psi) \left[\Delta_{\dot{A}} F(t,\Psi) \Big|_{\Psi=0} \right] \tag{12}$$

with initial conditions $F_0(\Psi) := F(0,\Psi)$ and $F(t,0) = 0$ for every t. In (11), $\Delta_{\dot{A}}$ is just shorthand notation for

$$\Delta_{\dot{A}} := \sum_{i,j=1}^{2n} \dot{a}_{ij} \frac{\partial^2}{\partial \psi_i \partial \psi_j}. \tag{13}$$

Using (12) we obtain the following partial differential equation (PDE) for φ

$$\frac{\partial \varphi}{\partial t}(t,\Psi) = \frac{1}{2}\Delta_{\dot{A}}\varphi(t,\Psi) - \frac{1}{2}\sum_{i,j=1}^{2n} \frac{\partial \varphi(t,\Psi)}{\partial \psi_i} \dot{a}_{ij} \frac{\partial \varphi(t,\Psi)}{\partial \psi_j} - \frac{1}{2}\Delta_{\dot{A}}\varphi(t,\Psi)\Big|_{\Psi=0}. \tag{14}$$

2. The Majorant Method

In the late 80s, D. Brydges and J. D. Wright published a paper [1] in which a majorant method for fermions is proposed. This work is based on a previous work by D. Brydges and T. Kennedy (see [2]) in which the same method is developed and applied to bosonic systems, where special attention was given to use this tool to study the convergence of the Mayer series for a (classical) gas interacting via Yukawa potential. The new version for fermions was intended to used as bound estimates on the irrelevant parts of the effective action.

The majorant method developed in both cited papers has the great advantage of of studying the dynamics generated by the renormalization group transformation (2) using PDEs, which are well-known powerful tools in the mathematics literature. However, as noted in [3], the fermionic version of this method as originally published by D. Brydges and J. Wright contains a gap, which was confirmed by the authors in a later errata [4], but the mistake was never addressed ever since.

One of our main targets is to persue a fixed version of this method. Before stating our results, we need to introduce some auxiliary concepts.

Definition 2.1. *Let $g = g(t,z)$ be an analytic function on the open disc*

$$D_R := \{z \in \mathbb{R} : \ |z| < R\}$$

for every $t \in [a,b]$, $a < b$ fixed. Let

$$g(t,z) := \sum_{m=1}^{\infty} g_m(t) z^m$$

be a power series expansion of g on $[a,b] \times D_R$. Let $f = f(t,z)$ be given by a formal power series expansion

$$f(t,z) = \sum_{m=1}^{\infty} f_m(t) z^m$$

for every $t \in [a,b]$. Then g is said to be a uniform majorant, or simply a majorant, of f if

$$|f_n(t)| \leq |g_n(t)|$$

holds for every $t \in [a,b]$.

Note that if g is a majorant of f, then we readily see that f is also analytic on D_R for every $t \in [a,b]$. In particular, because g is absolutely convergent, we get

$$\sum_{m=1}^{\infty} |f_m(t)| z^m \leq \sum_{m=1}^{\infty} |g_m(t)| z^m < +\infty$$

for all $t \in [a,b]$ and $z \in U$ for some open set $U \subseteq D_R$, so that f is also analytic at zero. This is the basic strategy of the majorant method to establish the convergence of the Mayer series when the method is applied for bosons.

To study convergence, we need topology. If $f \in \bigwedge U$, we define:

$$f_m := \sup_{i\in\{1,\dots,2n\}} \frac{1}{2m} \sum_{J\ni i,|J|=2m} \left| \frac{\partial f(\Psi)}{\partial \Psi_J}\Big|_{\Psi=0} \right| \tag{15}$$

where, if $J = \{i_1 < \cdots < i_p\}$,

$$\frac{\partial}{\partial \Psi_J} = \frac{\partial^{|J|}}{\partial \psi_{i_p} \cdots \partial \psi_{i_1}}.$$

The norm associated to f is then defined to be

$$||f||_z := \sum_{m=1}^{\infty} f_m z^{2m} \tag{16}$$

with $z \in \mathbb{R}$ being a norm parameter. In addition, we have to introduce norms for the covariance, so we set

$$||A|| := \sup_i \sum_{j=1}^{n} |a_{ij}| \quad \text{and} \quad \sigma^2_{(s,t)} := \int_s^t ||\dot{A}(s')||ds'. \tag{17}$$

We are now ready to state our main Theorem.

Theorem 2.2. *Suppose that*

$$\phi(0,z) := ||\varphi(0)||_z = \sum_{m=1}^{\infty} \varphi_m(0) z^{2m} \tag{18}$$

has a nonzero radius of convergence, with $\varphi = \varphi(t,\Psi)$ defined by (10). If $t > 0$ and $|z|$ is sufficiently small, $\phi(0,z)$ can be extended to a power series

$$\phi(t,z) = \sum_{m=1}^{\infty} \phi_m(t) z^{2m} \tag{19}$$

satisfying

$$\varphi_m(t) \le \phi_m(t) \tag{20}$$

for every t in which $\phi(t,z)$ exists around $z = 0$. Hence, ϕ is a uniform majorant for $||\varphi(t)||_z$. Moreover, if $s \in [t_0, T]$, ϕ is obtained evaluating $\tilde{\phi} = \tilde{\phi}(s,z;t)$ at $s = t$, with $\tilde{\phi}$ being the solution of

$$\frac{\partial \tilde{\phi}}{\partial s}(s,z;t) - \frac{1}{2}||\dot{A}(s)|| \left(\frac{\partial \tilde{\phi}}{\partial s}(s,z;t) \right)^2 = 0 \tag{21}$$

with initial condition

$$\tilde{\phi}(0,z;t) = \frac{1}{2}[\phi(0,\sigma_{(0,t)} + z) + \phi(0,\sigma_{(0,t)} - z)]. \tag{22}$$

By specifying V, one can (at least in principle) obtain $\tilde{\phi}$ explicitly or prove its existence within some range $|z| \le z_{max}$. One can also check whether the effective action has bounds which are uniform in the size of the lattice. By taking the thermodynamic limit one can study analiticity of this effective action in z and in V, the latter supposed to belong to a suitable Banach space of interactions.

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Prelude to the renormalization of oscillatory SUSY field theories

Luca Fresta

(joint work with Giovanni Antinucci, Marcello Porta)

Oscillatory SUSY field theories provide a dual description of fermionic non-interacting disordered systems, as was first understood by Wegner [10] and by Efetov [6]. In the simplest framework, such systems are described by means of random Schrödinger operators of the form

$$H_\omega = H + \gamma V_\omega \qquad \text{on} \quad \ell^2(\mathbb{Z}^d)\ ,$$

where H is deterministic with absolutely continuous spectrum while V_ω is a multiplication operator with random amplitudes, e.g., $(V_\omega u)_x = \omega_x u_x$ with $(\omega_x)_{x\in\mathbb{Z}^d}$ i.i.d. random variables. The SUSY formalism allows one to express the disorder-average of any product of Green's functions $(H_\omega - z)^{-1}$, $z \in \mathbb{C}\setminus\mathbb{R}$, on a finite domain $\Lambda \subset \mathbb{Z}^d$ in terms of the expectations of suitable oscillatory SUSY field theories. For example, if the disorder is Gaussian, the duality relation for the disorder-averaged Green's function reads

$$\mathbb{E}_\omega\Big(\frac{1}{H_{\omega,\Lambda} - \mu - \mathrm{i}0^+}\Big)_{x,y} = \mathrm{i}\langle F\rangle_{\Lambda,H,\gamma,\mu}\ ,$$

where $H_{\omega,\Lambda}$ denotes the restriction of H_ω on Λ, and where we have introduced the SUSY "measure"

$$\text{(1)} \quad \langle F\rangle_{\Lambda,H,\gamma,\mu} := \int \mathrm{d}\Phi_\Lambda \mathrm{e}^{-\mathrm{i}(\Phi^+,H\Phi^-)}\mathrm{e}^{-\frac{\gamma^2}{2}\sum_{x\in\Lambda}(\Phi_x^+\Phi_x^-)^2+\mathrm{i}\mu\sum_x \Phi_x^+\Phi_x^-}F(\Phi)\ ,$$

where $\Phi_x^\pm := (\phi_x^\pm, \psi_x^\pm)$ are the superfields, namely $\phi_x^\pm := \phi_x^{(1)} \pm \mathrm{i}\phi_x^{(2)}$ are bosonic fields, involving pairs of real fields $\phi_x^{(1)}, \phi_x^{(2)} \in \mathbb{R}$, whereas $\psi_x^\pm$ are Grassmann fields, $\{\psi_x^\pm, \psi_{x'}^\pm\} = \{\psi_x^+, \psi_{x'}^-\} = 0$. Above, $\int \mathrm{d}\Phi_\Lambda$ is the shorthand for $\int \Big[\prod_{x\in\Lambda}\frac{\mathrm{d}\phi_x^{(1)}\mathrm{d}\phi_x^{(2)}}{\pi}\Big]\Big[\prod_{x\in\Lambda}\frac{\partial}{\partial\psi_x^+}\frac{\partial}{\partial\psi_x^-}\Big]$.

The study of oscillatory SUSY measures of the kind in (1) is still in its infancy. The typical approach in the literature is to resort to suitable effective non-linear sigma models, which capture the correct properties of the system and are able to describe the metal-insulator transition [4, 5]. On the other hand, in [2] we initiate the rigorous construction of the oscillatory SUSY measure in (1) at weak disorder ($|\gamma|$ small) and at suitable μ inside of the spectrum of H. This regime is the most interesting from a physical point of view, because one expects to observe the metallic phase, in sufficiently high dimension [1]. However, having $\mu \in \mathrm{spec}(H)$ makes the covariance decay too slowly, so that the measure is infrared-singular and this requires the application of renormalization group (RG) techniques. Because standard methods are not applicable to the case of purely oscillatory Gaussians, we considered a block-spin hierarchical approximation of the measure as a preliminary problem [8]. Specifically, we considered a hierarchical model for single-cone Weyl semimetals [9, 3]. These are well-studied materials that have a conical dispersion relation around the critical points and can be thought of as the three-dimensional analogue of graphene.

Let us define our hierarchical model more precisely. Let $N \in \mathbb{N}$, $L \in 2\mathbb{N}$ and set $\Lambda_N := [0, L^N]^3 \cap \mathbb{Z}^3$ and $\Lambda_N^h := [0, L^{N-h})^3 \cap \mathbb{Z}^3$. The block-spin hierarchical superfield Φ is realized as follows

$$\Phi_x = \sum_{h=0}^{N-1} L^{-h} A_{\lfloor L^{-h} x \rfloor} \zeta^{(h)}_{\lfloor L^{-h-1} x \rfloor} , \qquad \sum_{\|x\|_\infty \leqslant L-1} A_x = 0 \qquad (A_x \in \{\pm 1\}) ,$$

where $(\zeta_x^{(h)})_{x \in \Lambda_N^{h+1}}$ are independent identically "distributed" superfields with oscillatory measure $\mu(\mathrm{d}\zeta_x^{(h)}) = \mathrm{d}\zeta_x^{(h)} \mathrm{e}^{-\mathrm{i}\zeta_x^{(h)+} \cdot \zeta_x^{(h)-}}$. Notice that, because of the coefficients A's, the fluctuation fields $\zeta^{(h)}$ have zero block-average. Set

$$\langle\!\langle \cdot \rangle\!\rangle_{N,L,\lambda,\mu} := \int \Big[\prod_{h=0}^{N-1} \prod_{x \in \Lambda_N^{h+1}} \mu(\mathrm{d}\zeta_x^{(h)}) \Big] \mathrm{e}^{-\lambda \sum_x (\Phi_x^+ \cdot \Phi_x^-)^2 + \mathrm{i}\mu \sum_x \Phi_x^+ \cdot \Phi_x^-} .$$

The hierarchical model captures the correct large-distance properties in the sense that $\langle\!\langle \psi_x^- \psi_y^+ \rangle\!\rangle_{N,L,0,0} \sim \frac{-\mathrm{i}}{\mathrm{d}(x,y)^2}$, $\mathrm{d}(x,y)$ denoting the hierarchical distance. With this notation in place, our main result is as follows.

Theorem 1. *There exist $L_0 \in \mathbb{N}$ and $\lambda_0 = \lambda_0(L) > 0$, such that for $L > L_0$ and $0 < \lambda < \lambda_0(L)$, there exists a unique renormalized Fermi energy $\mu_{\mathrm{crit}} = \mu_{\mathrm{crit}}(\lambda)$, satisfying $|\mu_{\mathrm{crit}}(\lambda)| \lesssim \lambda$, such that uniformly in N and for all $\theta \in (0, 1/2)$*

$$\Big| \langle\!\langle \psi_x^- \psi_y^+ \rangle\!\rangle_{N,L,\lambda,\mu_{\mathrm{crit}}(\lambda)} - \langle\!\langle \psi_x^- \psi_y^+ \rangle\!\rangle_{N,L,0,0} \Big| \lesssim \frac{\lambda^\theta}{\mathrm{d}(x,y)^{2+\theta}} .$$

In plain words, our theorem establishes the robustness of the algebraic decay of the disorder-averaged Green's function as the disorder is turned on. This is in contrast with the insulator phase, where the disorder-averaged Green's function decays exponentially instead [7].

To conclude, one of the main technical challenges of our analysis is the large-field problem in the case of oscillatory Gaussians. The crucial ingredient for its solution is the block-spin decomposition, which allows us to extract sufficient decay at large fields. The application of the block-spin RG to the non-hierarchical case requires the use of suitable cluster expansions that exploit the oscillatory nature of the SUSY integrals, see [7], and is deferred to future work.

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The Appearance of Particle Tracks in Detectors

Jürg Fröhlich

In this lecture I consider regimes of Quantum Mechanics (QM) that can be described in classical terms. Such regimes constitute what I call the "Classical Periphery/Skin of Quantum Mechanics."

I won't develop the general theory, but illustrate it in a study of tracks left behind by charged quantum-mechanical particles propagating in detectors. These tracks are close to classical particle trajectories.

I begin my lecture with some general comments on the notion of "events" in Quantum Mechanics and their role in understanding "state reduction", as manifested in measurements and observations. My discussion is cast in the language of the "ETH - Approach to QM", a presumed cornerstone of Quantum Geometry.

This lecture touches upon the *foundations of Quantum Mechanics*, a subject that, in my opinion, *ought to occupy center stage of contemporary theoretical physics.* Unfortunately, many people who think about quantum foundations prefer to endlessly talk about puzzles and paradoxes and the "weirdness" of Quantum Mechanics – rather than to sit down and try to actually solve some of the most pressing open problems, such as the so-called *"Measurement Problem"*, the *"Information Paradox"*, or a coherent account of *Relativistic Quantum Theory* – problems that, I am convinced, can actually be straightened out.

The main subject of today's lecture is, however, a concrete example in the *theory of indirect measurements,* namely the problem of tracks left by charged particles moving through a detector. In QM, information about a physical system,

S, of interest is gained by measurements describable in the *classical periphery* of QM. Often, information on properties of S is gathered by *indirect measurements* involving *probes* (photons, neutrons, atoms, etc.) that interact with S, their states getting entangled with the state of S. After their interaction with S the probes are subjected to *projective measurements* resulting from sequences of *actual events* that are describable within the ETH - Approach to QM. Because of entanglement, a long sequence of (possibly very boring) projective measurements of probes yields (possibly very interesting) information on the *state* of S.

Given a theory of *projective measurements*, as provided by the ETH - Approach to QM, the general theory of *indirect measurements* – pioneered by Kraus, Maassen and Kümmerer, and others – is well developed, and it is not repeated in this lecture. Instead, I illustrate it by explaining how, in QM, classical-looking tracks of charged particles interacting with the degrees of freedom of a detector that are then subjected to projective measurements can be understood to appear. My understanding of (a solution of) this problem is based on semi-classical analysis (a Egorov-type theorem, among other things) and some results in statistics (concerning the reconstruction of precise information from large sets of noisy data). Precise results and proofs can be found in a recent paper with Tristan Benoist and Martin Fraas ([1], and references given there).

A little history: At the 1927 Solvay conference, in a famous debate with Bohr and Born, the problem of the classical periphery of QM, and in particular the problem of particle tracks, was raised by Einstein. Born sketched an insightful answer due to Heisenberg. Later this phenomenon was studied by C. G. Darwin and N. Mott, whence the name *"Mott tracks"*. However, their analysis was quite far from being totally convincing. There was therefore continued interest in the problem. Relatively recent work on it is due to Blasi et al.; O. Steinmann; Dürr et al.; R. Figari and A. Teta (who, in their recent little book, have included a very nice account of the history of work on the problem); Ballesteros, Benoist, Fraas and myself; Benoist, Fraas and myself; and others.

Papers on these matters co-authored by JF can be found on arXiv, and slides of lectures are available from him on request.

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The Statistical Mechanics of (multi-)Forests and Fermionic σ-models

Nick Crawford

Given an undirected finite graph $G = (\Lambda, E)$, a forest $F = (\Lambda, E(F))$ is an acyclic subgraph of G having the same vertex set as G. Given an edge weight $\beta > 0$ (later to be interpreted as an inverse temperature) and a vertex weight $h \geq 0$ (external field), the probability of a forest F under the arboreal gas measure is

$$\mathbb{P}^G_{\beta,h}[F] := \frac{1}{Z^G_{\beta,h}} \beta^{|E(F)|} \prod_{T \in F} (1 + h|V(T)|) \tag{1}$$

where $T \in F$ denotes that T is a tree in the forest, i.e., a connected component of F, $|E(F)|$ is the number of edges in F, and $|V(T)|$ is the number of vertices in T. With R. Bauerschmidt and T. Helmuth, [1], we showed that when formulated over the graph $\mathbb{Z}^d, d \geq 3$, the arboreal gas exhibits a percolative phase transition and has massless decay of truncated correlations in its supercritical phase (for β large enough).

The departure point for our proof is a beautiful observation of Caracciolo et. al. [3, 2] that the arboreal gas is the graphical representation of an $\mathbb{H}^{0|2}$ nonlinear σ-model - i.e. in which the target space is one half of the degenerate sphere $\mathbb{S}^{0|2}$. In our work it is more natural to interpret the target space as the degenerate hyperbolic plane $\mathbb{H}^{0|2}$. The latter model is defined as follows, see [1, Section 2] for further details. For every vertex $x \in \Lambda$, there are two (anticommuting) Grassmann variables ξ_x and η_x and we then set

$$z_x := \sqrt{1 - 2\xi_x\eta_x} := 1 - \xi_x\eta_x. \tag{2}$$

The second equality follows from Taylor expanding the square root around $x_0 = 1$ and nilpotency of ξ_x, η_x. Thus the z_x commute with each other and with the odd elements ξ_x and η_x. The formal triples $u_x := (\xi_x, \eta_x, z_x)$ are supervectors with two odd components ξ_x, η_x and an even component z_x. These supervectors satisfy the sigma model constraint $u_x \cdot u_x = -1$ for the super inner product

$$u_x \cdot u_y := -\xi_x\eta_y - \xi_y\eta_x - z_x z_y. \tag{3}$$

The constraint is reminiscent of the embedding of the hyperbolic space $\mathbb{H}^2$ in $\mathbb{R}^3$ equipped with the standard quadratic form with Lorentzian signature $(1, 1, -1)$. Indeed, $-\xi_x\eta_y - \xi_y\eta_x$ is the fermionic analogue of the Euclidean inner product on $\mathbb{R}^2$.

For any F which is a polynomial in the ξ_x, η_x, its 'expectation' in the $\mathbb{H}^{0|2}$ model is

$$\langle F \rangle_{\beta,h} := \frac{1}{Z_{\beta,h}} \int (\prod_{x \in \Lambda} \partial_{\eta_x} \partial_{\xi_x} \frac{1}{z_x}) e^{\frac{\beta}{2}(u,\Delta u) - h(1, z-1)} F. \tag{4}$$

In this expression, $\int \prod_{x\in\Lambda} \partial_{\eta_x}\partial_{\xi_x}$ denotes the Grassmann integral (i.e., the coefficient of the top degree monomial of the integrand), $Z_{\beta,h}$ is a normalising constant, and

(5)

$$(u, \Delta u) = -\frac{1}{2} \sum_{xy \in E(\Lambda)} (u_x - u_y) \cdot (u_x - u_y) = \sum_{xy \in E(\Lambda)} (u_x \cdot u_y + 1), \qquad (1, z) = \sum_{x \in \Lambda} z_x,$$

where $xy \in E(\Lambda)$ denotes that x and y are nearest neighbours (counting every pair once), and the inner products are given by (3). The factors $1/z_x$ in (4) are the canonical fermionic volume form invariant under the symmetries associated with (3).

As explained in [1, 2], connection and edge probabilities of the arboreal gas are equivalent to correlation functions of the $\mathbb{H}^{0|2}$ model. The following proposition summarises the relations we use.

Proposition 1. *For any finite graph G, any $\beta \geq 0$ and $h \geq 0$,*

$$\mathbb{P}_{\beta,h}[0 \leftrightarrow \mathfrak{g}] = \langle z_0 \rangle_{\beta,h}, \tag{6}$$

$$\mathbb{P}_{\beta,h}[0 \leftrightarrow x, 0 \not\leftrightarrow \mathfrak{g}] = \langle \xi_0 \eta_x \rangle_{\beta,h}, \tag{7}$$

$$\mathbb{P}_{\beta,h}[0 \leftrightarrow x] + \mathbb{P}_{\beta,h}[0 \not\leftrightarrow x, 0 \leftrightarrow \mathfrak{g}, x \leftrightarrow \mathfrak{g}] = -\langle u_0 \cdot u_x \rangle_{\beta,h}, \tag{8}$$

and the normalising constants in (1) *and* (4) *are equal. In particular,*

$$\mathbb{P}_{\beta,0}[0 \leftrightarrow x] = -\langle u_0 \cdot u_x \rangle_{\beta,0} = -\langle z_0 z_x \rangle_{\beta,0} = \langle \xi_0 \eta_x \rangle_{\beta,0} = 1 - \langle \xi_0 \eta_0 \xi_x \eta_x \rangle_{\beta,0}. \tag{9}$$

This correspondence allows us to study the $\mathbb{H}^{0|2}$ lattice field theory to understand the behavior of the arboreal gas. We proceed by considering (4) defined on boxes $\Lambda_N \subset \mathbb{Z}^d$ of sidelength L^N and with periodic boundary conditions. In this new language, the main theorem we prove is

Theorem 1. *Let $d \geq 3$ and $L \geq L_0(d)$. There exists $\beta_0 \in (0, \infty)$ and constants $\theta_d(\beta) = 1 + O(1/\beta)$ and $\mathsf{c}_i(\beta) = \mathsf{c}_i + O(1/\beta)$ and $\kappa > 0$ (all dependent on d) such that for $\beta \geq \beta_0$,*

$$\lim_{h \downarrow 0} \lim_{N \to \infty} \langle z_0 \rangle_{\beta,h} = \theta_d(\beta) \tag{10}$$

$$\lim_{h \downarrow 0} \lim_{N \to \infty} \langle \xi_0 \eta_x \rangle_{\beta,h} = \frac{\mathsf{c}_1(\beta)}{\beta |x|^{d-2}} + O(\frac{1}{\beta |x|^{d-2+\kappa}}) \tag{11}$$

$$\lim_{h \downarrow 0} \lim_{N \to \infty} \mathbb{P}\langle z_0 z_x \rangle_{\beta,h} - \langle z_0 \rangle_{\beta,h} \langle z_x \rangle_{\beta,h} = -\frac{\mathsf{c}_2(\beta)}{\beta^2 |x|^{2d-4}} + O(\frac{1}{\beta^2 |x|^{2d-4+\kappa}}). \tag{12}$$

In particular,

$$\lim_{h \downarrow 0} \lim_{N \to \infty} \langle u_0 \cdot u_x \rangle_{\beta,h} = -\theta_d(\beta)^2 - \frac{2\mathsf{c}_1(\beta)}{\beta |x|^{d-2}} + O(\frac{1}{\beta |x|^{d-2+\kappa}}). \tag{13}$$

In spite of appearances, Grassman variables are more conveniant from an analytic perspective. In particular, there are a number of well developed tools for lattice Grassmann variable field theories unavailable on the combinatorial side. The two which are key to our work are 1) Ward identities. 2) the rigorous Renormalization Group.

In statistical and quantum physics, Ward identities simply mean identities between expectations of observables derived from symmetries of the model. For us, the key Ward identity, with symmetry generator

$$T = \sum_{x\in\Lambda} z_x \partial_{\xi_x}, \tag{14}$$

is

$$\langle T \cdot F \rangle_{\beta,h=0} = 0$$

for any expression F in the joint Grassman algebra. This leads to

$$\langle z_0 \rangle_{\beta,h} = \langle T\xi_0 \rangle_{\beta,h} = -\sum_{x\in\Lambda} h\xi_0 \bar{T} z_{x\,\beta,h} = h \sum_{x\in\Lambda} \xi_0 \bar{\eta}_{x\,\beta,h}, \tag{15}$$

for ANY $h \in \mathbb{C}$. How this identity is used in our proof will be explained below.

Regarding the use of the renormalization group, by expanding z_x as above and rescaling generators of the Grassman algebra by $1/\sqrt{\beta}$, the $\mathbb{H}^{0|2}$ model can be viewed as a quartic Grassman field theory with density

$$\exp\left[-(\psi, -\Delta\bar{\psi}) - \frac{1}{\beta}(1+h)\sum_{x\in\Lambda} \psi_x \bar{\psi}_x - \frac{1}{2\beta}\sum_{x\in\Lambda} \psi_x \bar{\psi}_x \nabla\psi_x \cdot \nabla\bar{\psi}_x\right]. \tag{16}$$

The scaling dimensions of $\psi, \bar{\psi}$ are the conventional $(d-2)/2$. With this definition, note that the quartic term is *irrelevant* in this model and that no nonquadratic higher order relevant terms exist consistent with lattice symmetries.

The RG implementation we use was laid down systematically in the series of papers [5, 6, 7, 8, 9]. Slightly generalizing (16), with $m^2 \geq 0$, we consider densities

$$\exp\left[-(\psi, [-\Delta + m^2]\bar{\psi}) - s_0(\psi, -\Delta\bar{\psi}) - a_0 \sum_{x\in\Lambda} \psi_x \bar{\psi}_x - b_0 \sum_{x\in\Lambda} \psi_x \bar{\psi}_x \nabla\psi_x \cdot \nabla\bar{\psi}_x\right]. \tag{17}$$

A relatively straightforward, but lengthy, adaptation of [5, 6, 7, 8, 9] then gives the following theorem.

Theorem 2. *Let $d \geq 3$ and $L \geq L_0(d)$. For b_0 sufficiently small and $m^2 \geq 0$, there are $s_0 = s_0^c(b_0, m^2)$ and $a_0 = a_0^c(b_0, m^2)$ independent of N so that the following hold: The functions s_0^c and a_0^c are continuous in both variables, differentiable in b_0 with uniformly bounded b_0-derivatives, and satisfy the estimates*

$$s_0^c(b_0, m^2) = O(b_0), \qquad a_0^c(b_0, m^2) = O(b_0) \tag{18}$$

uniformly in $m^2 \geq 0$. There exists $\kappa > 0$ such that if the torus sidelength satisfies $L^{-N} \leq m$,

$$\sum_{x\in\Lambda_N} \langle \bar{\psi}_0 \psi_x \rangle_{m^2, s_0, a_0, b_0} = \frac{1}{m^2} + \frac{O(b_0 L^{-(2+\kappa)N})}{m^4}. \tag{19}$$

Moreover, there are functions

$$\lambda_c = \lambda_c(b_0, m^2) = 1 + O(b_0), \qquad \gamma_c = \gamma_c(b_0, m^2) = (-\Delta^{\mathbb{Z}^d} + m^2)^{-1}(0,0) + O(b_0), \tag{20}$$

having the same continuity properties as s_0^c and a_0^c such that

$$\langle \bar\psi_0 \psi_0 \rangle_{m^2,s_0,a_0,b_0} = \gamma_c + O(b_0 L^{-\kappa N}), \tag{21}$$

$$\langle \bar\psi_0 \psi_x \rangle_{m^2,s_0,a_0,b_0} = (-\Delta + m^2)^{-1}(0,x) + O(b_0 |x|^{-(d-2)-\kappa}) + O(b_0 L^{-\kappa N}), \tag{22}$$

$$\langle \bar\psi_0 \psi_0 ; \bar\psi_x \psi_x \rangle_{m^2,s_0,a_0,b_0} = -\lambda_c^2 (-\Delta + m^2)^{-1}(0,x)^2 + O(b_0 |x|^{-2(d-2)-\kappa}) + O(b_0 L^{-\kappa N}). \tag{23}$$

Here $\langle A; B\rangle = \langle AB \rangle - \langle A \rangle \langle B \rangle$.

The statement (19) should be interpreted as saying that s_0, a_0 can be chosen as functions of (m^2, b_0) so that the model lies on the 'critical manifold' in the space of lattice Grassman field theories. Now, given (m^2, b_0, a_0, s_0), there is a change of variables which maps the Grassman field theory with density (17) to an $\mathbb{H}^{0|2}$ model of the form (4), provided $\beta = (1+s_0)^2/b_0$ and $(1+h)/\beta = (a_0+m^2)/(1+s_0)$. The Ward identity is used to show that if s_0^c, a_0^c are given as in Theorem 2, then $h \to 0$ as $m^2 \to 0$, which allows us then to conclude that for $\beta \gg 0$, the arboreal gas has an infinite volume state with infinite clusters. On the other hand, it is relatively easy to see that the sizes of clusters have exponentially decaying tail probabilities if $0 < \beta \ll 1$, thus demonstrating a percolation transition.

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The non-linear supersymmetric hyperbolic sigma model on a complete graph with hierarchical interactions

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(joint work with Margherita Disertori)

The non-linear supersymmetric hyperbolic sigma model, also called $H^{2|2}$ model for short, was introduced by Zirnbauer [3]. We study it on a complete graph with hierarchical interactions. The model associates a superspin variable $\sigma_i = (x_i, y_i, z_i, \xi_i, \eta_i)$ to every vertex i. Here, the three variables x_i, y_i, z_i are commuting, whereas the two variables ξ_i, η_i are anticommuting. They are subject to the constraint $\sigma_i \in H^{2|2}$, i.e. $x_i^2 + y_i^2 - z_i^2 + 2\xi_i\eta_i = -1$ and $z_i > 0$. In addition, there is the constraint $\sigma_\rho = (0, 0, 1, 0, 0)$ for a "pinning vertex" ρ. The supersymmetric Hamiltonian of the model is given by

$$H(\sigma) = -\sum_{i\sim j} W_{ij}\langle\sigma_i, \sigma_j\rangle$$

with the supersymmetric inner product

$$\langle\sigma_i, \sigma_j\rangle = x_i x_j + y_i y_j - z_i z_j + \xi_i\eta_j - \eta_i\xi_j$$

and weights $W_{ij} > 0$. The $H^{2|2}$-model is described by the following superintegration form acting on test superfunctions f:

$$\langle f\rangle_{\Lambda,\rho}^W := \int_{(H^{2|2})^{\Lambda\setminus\{\rho\}}} \mathcal{D}\sigma \exp\left\{\sum_{i\sim j} W_{ij}(1 + \langle\sigma_i, \sigma_j\rangle)\right\} f(\sigma)$$

with the following canonical supermeasures on $H^{2|2}$ and $(H^{2|2})^{\Lambda\setminus\{\rho\}}$, respectively:

$$\mathcal{D}\sigma_i = \frac{1}{2\pi} dx_i\, dy_i\, \partial_{\xi_i}\partial_{\eta_i} \circ \frac{1}{z_i} \quad \text{and} \quad \mathcal{D}\sigma = \prod_{i\in\Lambda\setminus\{\rho\}} \mathcal{D}\sigma_i$$

where $z_i = \sqrt{1 + x_i^2 + y_i^2 + 2\xi_i\eta_i}$.

We consider the complete graph with the vertex set $\Lambda_N = \{0,1\}^N \cup \{\rho\}$, viewed as the set of leaves of a binary tree with the additional pinning vertex ρ. The hierarchical distance $d(i, j)$ of vertices i and j equals their distance to the least common ancestor in the binary tree. We take $\epsilon > 0$ and a function $w : \mathbb{N} \to (0, \infty)$ and consider the weights $W_{ij} := w(d(i, j))$, $W_{i\rho} = \epsilon$ for $i, j \in \Lambda_N \setminus \{\rho\}$.

Following [2], we transform cartesian coordinates $\sigma_i \in H^{2|2}$ to horospherical coordinates u_i, s_i (even) and $\overline{\psi}_i, \psi_i$ (odd) by

$$x_i = \sinh u_i - \left(\frac{1}{2}s_i^2 + \overline{\psi}_i\psi_i\right)e^{u_i}, \qquad y_i = s_i e^{u_i},$$

$$z_i = \cosh u_i + \left(\frac{1}{2}s_i^2 + \overline{\psi}_i\psi_i\right)e^{u_i}, \qquad \xi_i = \overline{\psi}_i e^{u_i}, \qquad \eta_i = \psi_i e^{u_i}.$$

After this change of coordinates, we integrate over all Grassmann variables $\overline{\psi}_i$, ψ_i. This yields a probability measure $P_{\Lambda,\rho}^W(du\, ds)$. For interactions which do not decrease too fast in the hierarchical distance, the following tightness result holds uniformly in the pinning and in the size of the graph:

Theorem [1]: Assume that the rescaled weights $\beta_l := 2^{2l+1}w(l+2)$, $l \in \mathbb{N}_0$, fulfill

$$\sum_{l=0}^{\infty} \sqrt{\frac{\log \max\{\beta_l, e\}}{\beta_l}} < \infty.$$

Then one has

$$\lim_{M\to\infty} \sup_{N\in\mathbb{N}} \sup_{\epsilon>0} \sup_{p,q\in\Lambda_N} P^W_{\Lambda_N,\rho}(|u_p - u_q| \geq M) = 0.$$

Antichains. Let $\mathcal{T}^N$ denote the set of vertices of the binary tree with set of leaves $\{0,1\}^N$. The notation $i \preceq j$ means that the vertex $i \in \mathcal{T}^N$ is on the path from j to the root in the tree. A set $A \subseteq \mathcal{T}^N$ is called an antichain if $i \not\preceq j$ holds for all $i, j \in A$, $i \neq j$.
Block spins. Let $\mathcal{B}_i$ denote the set of leaves above $i \in \mathcal{T}^N$. The level $\ell(i)$ of i is its distance from the leaves. Introduce block spin variables by $\sigma_i := |\mathcal{B}_i|^{-1} \sum_{j\in\mathcal{B}_i} \sigma_j$. Weights and pinning are extended to all $i, j \in \mathcal{T}^N$ by

$$W_{ij} := 2^{\ell(i)+\ell(j)} w(\ell(i \wedge j)), \quad \epsilon_i := |\mathcal{B}_i|\epsilon,$$

where $i \wedge j$ means the least common ancestor of i and j.

The proof of the tightness result relies on the following reduction to an effective model; its size is logarithmic in the size of the original model:

Theorem [1]: Let A be a maximal antichain. The following superexpectations are well-defined on a non-empty open set of parameters $a_j = (x_{a_j}, y_{a_j}, z_{a_j}, \xi_{a_j}, \eta_{a_j})$, $j \in A$, and coincide there:

$$\underbrace{\left\langle e^{\sum_{j\in A}\langle a_j,\sigma_j\rangle} \right\rangle^W_{\Lambda_N,\rho}}_{\text{on the complete graph}} = \underbrace{\left\langle e^{\sum_{j\in A}\langle a_j,\sigma_j\rangle} \right\rangle^W_{A,\rho}}_{\text{on the antichain}}.$$

Here, on the l.h.s. the σ_j mean block spins, but on the r.h.s. the σ_j mean spin variables in $H^{2|2}$ of the model.

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Stochastic Analysis and Euclidean Fermions

Massimiliano Gubinelli
(joint work with Francesco De Vecchi, Luca Fresta)

Progresses in stochastic analysis allowed to have a rigorous formulation of Parisi-Wu stochastic quantisation for Boson Euclidean quantum field theories. In this talk I report on the extension these ideas to the realm of Fermion Euclidean theories. The Euclidean approach to Fermionic models gives rise to a non-commutative probabilistic model involving Grassmann analogs of Gaussian variables and associated Gibbsian measures. In recent work [1] with Albeverio, Borasi and De Vecchi, we proposed a simple and rigorous setting to discuss the stochastic quantisation of these theories. The approach via Berezin integral is problematic in this context because stochastic quantisation relies on a infinite dimensional source of (Grassmann) noise, in the form of a Grassmann Brownian motion. As such it cannot be reliably modelled via a finite-dimensional Grassmann algebra and its Berezin functional. A more abstract approach, dating back essentially to Osterwalder and Schrader [5], is however effective and more in line with the general point of view of non-commutative probability. Within this framework, Grassmann equivalents of the usual Brownian motion, Ito formula, diffusions and invariant measure are easily developed, at least for the concrete situations needed in stochastic quantisation to the extent to control the large scale limit of certain Grassmann Gibbs measures.

Together with De Vecchi and Fresta [4], we extended further this approach by tackling the removal of the ultraviolet cutoff and the related renormalisation in a large class of subcritical fermionic models. In order to do so we introduced a Grassmann analog of the variational approach [2, 3] developed by Barashkov and myself in the Bosonic setting by identifying certain forward-backward stochastic differential equation (FBSDE) which construct the law of the Euclidean fields under the Gibbs measure. The Grassmann FBSDE can be studied in the ultraviolet (UV) limit using ideas from the renormalisation group (RG) and in particular the continuous flow equation of Polchinski. The key observation is that RG coupled with the FBSDE is a flexible tool which makes the extraction of UV singularities and their non-perturbative control much easier, at least for subcritical theories. As a result we complete the stochastic quantisation program for this class of models, including control of correlations in the small coupling regime without cluster expansion.

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Strong cosmic censorship and quantum fields

Jochen Zahn

Strong cosmic censorship, first formulated somewhat vaguely by Penrose [1], is the hypothesis that for generic initial data in General Relativity, no Cauchy horizons form, beyond which the evolution of fields (and the spacetime) is not determined. Hence, it encodes *determinism*. However, physically relevant spacetimes such as the Kerr spacetime do possess Cauchy horizons. Due to the blue shift effect, one expects fields on such spacetimes (including perturbations of the metric), to generically become singular at the would-be Cauchy horizon, prohibiting the extension of the fields beyond the Cauchy horizon (or the passage of an observer through it). Several mathematically precise formulations of strong cosmic censorship have been put forward, which specify the desired degree of irregularity at the Cauchy horizon of generic finite energy smooth initial data. The most popular of these is due to Christodoulou [2] and requires that the stress tensor of generic (matter or gravitational) perturbations is not locally integrable near the Cauchy horizon (so that the metric can not be extended as a weak solution of the Einstein equation across the Cauchy horizon).

While there is strong evidence that the Christodoulou formulation of strong cosmic censorship holds in the asymptotically flat case [3], it was shown [4], based on the mathematical analysis of [5], that it is violated for scalar fields in near extremal Reissner-Nordström-de Sitter (RNdS) spacetimes, i.e., spacetimes describing a static charged black hole in the presence of a positive cosmological constant.

Together with S. Hollands and B. Wald [6], we investigated the behaviour of a free scalar quantum field at the Cauchy horizon of RNdS. We found that in any state Ψ which is Hadamard, i.e., suitably regular, across the event and the cosmological horizon, the expectation value of the renormalized stress tensor component T_{VV}, with V a Kruskal type coordinate in which the metric can be analytically continued across the Cauchy horizon (situated at $V = 0$), diverges as

$$\langle T_{VV} \rangle_\Psi \sim C V^{-2}, \tag{1}$$

up to sub-leading terms which behave as for classical fields. Interestingly, the coefficient C is *universal*, i.e., dependent only on the parameters of the spacetime, but not on the state Ψ. The coefficient C has to be computed numerically and is found to allow for both positive and negative sign [7], corresponding to physically different types of singularity at the Cauchy horizon if backreaction on the metric is taken into account (infinite stretching or squashing of observers approaching the singularity). The backbone of the proof of (1) are the mathematical results on the behaviour of classical fields near the Cauchy horizon [5].

In any case, the divergence (1) is strong enough to rescue (the Christodoulou formulation of) strong cosmic censorship, so in this sense quantum effects save determinism! Our results also demonstrate that even free quantum fields can show interesting behaviour related to the global geometry of spacetime.

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Renormalization of singular stochastic PDEs using flow equation

Paweł Duch

1. Introduction

Stochastic partial differential equations (SPDEs) describe macroscopic or mesoscopic behavior of many physical systems with some random or chaotic microscopic forcing. The randomness is typically modelled by the white noise and the self-interaction of the system – by a non-linear term in the equation. As a result of the lack of regularity of the white noise, many interesting random PDEs arising in physics, such as the KPZ equation describing the motion of a growing interface or the stochastic quantization equation of the Φ^4 Euclidean QFT, are ill posed in the classical sense. One calls such equations singular in order to distinguish them from other SPDEs that can be solved using standard tools from PDE theory. The first technique that gives a rigorous meaning to a large class of singular SPDEs was developed by Hairer in his breakthrough papers [7, 8] about the theory of regularity structures. A different approach using paracontrolled calculus was proposed later by Gubinelli, Imkeller and Perkowski [6]. Yet another approach, based on discrete Wilsonian renormalisation group analysis, was given by Kupiainen [9]. Due to tremendous progress in past few years [1, 2, 3] the renormalization of singular sub-critical (i.e. super-renormalizable) SPDEs with local differential operators is now well-understood.

2. Main result

In recent papers [4, 5] we developed a new solution theory for singular SPDEs. Our method is inspired by the work of Kupiainen [9]. It is based on a certain continuous renormalization group flow equation that plays an analogous role to the Polchinski equation [10] in QFT. The method is applicable to a large class of semi-linear parabolic or elliptic SPDEs with fractional Laplacian, additive noise and polynomial non-linearity. A nice feature of the method is that it does not use the Feynman diagrams, and hence allows to avoid completely the algebraic and combinatorial problems arising in different approaches. Similarly to other methods, the technique requires the assumption of sub-criticality. For concreteness, consider the following parabolic SPDE

$$(\partial_t + (-\Delta_x)^{\sigma/2})\Phi(t,x) = \xi(t,x) - \Phi(t,x)^3 + \infty\,\Phi(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T}^4, \tag{1}$$

where $\mathbb{T}^4$ is the four-dimensional torus, $(-\Delta_x)^{\sigma/2}$ is the fractional Laplacian of order $\sigma > 0$ and ξ is the so-called white noise, which is a rough and random object. If $\sigma > 4$, then Eq. (1) is not singular and can be solved using standard PDE tools. If $\sigma \leq 4$, then the solution of Eq. (1) is not expected be be a function but only a distribution. In order to make sense of the non-linear term on the RHS of Eq. (1) we replace the white noise ξ by some smooth regularized noise ξ_κ which converges to ξ as $\kappa \searrow 0$ and study the following regular SPDE

$$(\partial_t + (-\Delta_x)^{\sigma/2})\Phi_\kappa(t,x) = \xi_\kappa(t,x) - \Phi_\kappa(t,x)^3 + c_\kappa\,\Phi_\kappa(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T}^4. \tag{2}$$

The main result can be informally stated as follows.

Theorem 1. *Let $\sigma \in (2,4]$. There exists a choice of the constant c_κ diverging as $\kappa \searrow 0$ and a stochastic process Φ_0 such that $\lim_{\kappa\searrow 0}\Phi_\kappa = \Phi_0$ in some space of spacetime distributions, where Φ_κ is the solution of Eq. (2) with a suitable initial condition. Moreover, Φ_0 does not depend on the choice of an approximation ξ_κ of the white noise ξ.*

For $\sigma \leq 2$ the singular SPDE (1) is not sub-critical and cannot be made sense of using existing techniques. The convergence holds up to some possibly finite random explosion time. The initial condition is assumed to be sufficiently close to the equilibrium. See [4] for details.

3. Idea of the proof

To simplify the exposition, we ignore the initial condition. We first rewrite Eq. (2) in the mild form

$$\Phi_\kappa = G * F_\kappa[\Phi_\kappa], \tag{3}$$

where $*$ denotes the spacetime convolution, the kernel G is the inverse of the differential operator $\partial_t + (-\Delta)^{\sigma/2}$ and

$$F_\kappa[\psi](t,x) := \xi_\kappa(t,x) - \lambda\,\psi(t,x)^3 + \sum_{i=1}^{i_\sharp} \lambda^i\, c_\kappa^{[i]}\,\psi(t,x), \quad i_\sharp = \lfloor \sigma/(2\sigma-4) \rfloor, \tag{4}$$

is called the force. Observe that we introduced a book-keeping parameter λ that is set to 1 at the end. Let G_μ, $\mu \in (0,\infty)$, be a family of smooth kernels depending smoothly on μ such that $G_\mu(t,x) = G(t,x)$ for $t \in (2\mu,\infty)$ and $G_\mu(t,x) = 0$ for $t \in (-\infty,\mu)$. The basic object of the flow equation approach to singular SPDEs is the so-called effective force $F_{\kappa,\mu}[\psi]$, which depends on the UV cutoff $\kappa \in (0,1]$, the flow parameter $\mu \in [0,\infty)$ and a test function $\psi \in \mathcal{S}(\mathbb{M})$, where $\mathbb{M}$ is the spacetime. The effective force is defined by the flow equation

$$\text{(5)}\quad \partial_\mu F_{\kappa,\mu}[\psi](t,x) = \int_{\mathbb{M}^2} \frac{\delta F_{\kappa,\mu}[\psi](t,x)}{\delta\psi(s,y)}\,\partial_\mu G_\mu(s-u,y-z)\,F_{\kappa,\mu}[\psi](u,z)\,\mathrm{d}s\mathrm{d}y\mathrm{d}u\mathrm{d}z$$

together with the initial condition $F_{\kappa,0}[\psi] = F_\kappa[\psi]$, where $F_\kappa[\psi]$ is the force (4). Using the flow equation one show that $\Phi_\kappa = G * F_{\kappa,T}[0]$ is a solution of Eq. (3) in the time interval $[0,T]$. In order to construct the effective force $F_{\kappa,\mu}[\psi]$ we make the following ansatz

$$\text{(6)}\qquad F_{\kappa,\mu}[\psi](x) = \sum_{i=0}^{\infty}\lambda^i \sum_{m=0}^{\infty} F^{i,m}_{\kappa,\mu}(x;y_1,\ldots,y_m)\,\psi(y_1)\ldots\psi(y_m)\,\mathrm{d}y_1\ldots\mathrm{d}y_m,$$

where $F^{i,m}_{\kappa,\mu} \in \mathcal{S}'(\mathbb{M}^{1+m})$, $i,m \in \mathbb{N}_0$, are called effective force coefficients. Eq. (5) implies a flow equation for the effective force coefficients. The latter equation allows to express $\partial_\mu F^{i,m}_{\kappa,\mu}$ as a linear combination of $F^{j,k}_{\kappa,\mu}$ with $j<i$, or $j=i$ and $k>m$. Using this fact we construct the effective force coefficients recursively:

- $F^{0,0}_{\kappa,\mu} = \xi_\kappa$ and $F^{i,m}_{\kappa,\mu} = 0$ if $m > 3i$,
- $\partial_\mu F^{i,m}_{\kappa,\mu}$ is defined using the flow equation,
- $F^{i,m}_{\kappa,\mu} = F^{i,m}_{\kappa} + \int_0^\mu \partial_\eta F^{i,m}_{\kappa,\eta}\,\mathrm{d}\eta$.

The recurrence proceeds from i to $i+1$ and for fixed i from m to $m-1$.

The crucial part of the proof involves demonstrating existence of the limit $\kappa \searrow 0$ of the effective force coefficients $F^{i,m}_{\kappa,\mu}$ and absolute convergence of the series on the RHS of Eq. (6). In particular, we establish the following bounds

$$\text{(7)}\qquad \|F^{i,m}_{\kappa,\mu}\|_\mu \lesssim \mu^{\varrho(i,m)/\sigma}, \qquad \|\partial_\mu F^{i,m}_{\kappa,\mu}\|_\mu \lesssim \mu^{\varrho(i,m)/\sigma-1}.$$

uniform in both the UV cutoff κ and the flow parameter μ, where $\varrho(i,m) \in \mathbb{R}$ is determined using dimensional analysis. There are only finitely many $i,m \in \mathbb{N}_0$ such that $\varrho(i,m) \le 0$. The corresponding effective force coefficients $F^{i,m}_{\kappa,\mu}$ are called relevant. The remaining effective force coefficients $F^{i,m}_{\kappa,\mu}$ are called irrelevant. Assuming that the bounds (7) hold for the relevant coefficients we prove these bounds for the irrelevant coefficients by induction. The second of the bounds (7) is a consequence of the flow equation and the induction hypothesis. The first bound follows then from the second one, the equality $F^{i,m}_\kappa = 0$ and the estimate

$$\text{(8)}\qquad \|F^{i,m}_{\kappa,\mu}\|_\mu \le \int_0^\mu \|\partial_\mu F^{i,m}_{\kappa,\eta}\|_\eta\,\mathrm{d}\eta \lesssim \int_0^\mu \eta^{\varrho(i,m)/\sigma-1}\mathrm{d}\eta \lesssim \mu^{\varrho(i,m)/\sigma}.$$

Note that the last of the above bounds is false for $\varrho(i,m) \le 0$. As a result, the proof of the bounds (7) for the relevant coefficients requires a special treatment. We first establish uniform bounds for moments of the relevant coefficients and subsequently apply a Kolmogorov-type argument. The uniform bounds for moments follow from uniform bounds for cumulants. The inductive proof of the bounds for cumulants is based on a certain flow equation for cumulants and uses the strategy borrowed from the proof of perturbative renormalizability of QFT models given by Polchinski [10].

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The Scaling Limit of the 2*D* Discrete Gaussian Model

Jiwoon Park
(joint work with Roland Bauerschmidt, Pierre-François Rodriguez)

We consider the statistical physics model given by probability measure

$$\mathbb{P}^{\mathrm{DG}}_{\beta,\Lambda}(\sigma) = \frac{1}{Z^{\mathrm{DG}}_{\beta,\Lambda}} \exp\Big(-\frac{1}{4\beta} \sum_{x\sim y\in\Lambda} (\sigma_x - \sigma_y)^2 \Big) \tag{1}$$

on $\Omega_\Lambda = \{\sigma \in (2\pi\mathbb{Z})^\Lambda : \sigma(0) = 0\}$, where Λ is a two dimensional torus with side length L^N. This defines the Discrete Gaussian model in the finite volume with periodic boundary condition, and infinite volume Discrete Gaussian model can also be obtained by taking $|\Lambda| \to \infty$. When $\beta > 0$ is sufficiently small, $\sup_{x\in\mathbb{Z}^2} \mathrm{Var}^{\mathrm{DG}}_{\beta,\mathbb{Z}^2}[\sigma_x] < \infty$, but as β increases, a phase transition occurs, characterised by the divergence of the variance. This is also called the localisation-delocalisation phase transition or the Kosterlitz-Thouless phase transition. In the delocalised phase ($\beta \gg 1$), a Gaussian-like behaviour is expected from [5, 4], where the moment generating function of the Discrete Gaussian model is bounded below and above by that of the Gaussian free field. The main result establishes that the scaling limit of the Discrete Gaussian model is equal to the Gaussian free field on both finite torus and $\mathbb{R}^2$.

The strategy is to observe the connection between the Discrete Gaussian model and the lattice sine-Gordon model. If we consider $\sum_{\sigma\in\Omega_\Lambda}$ as a integral, the partition function is

$$Z^{\mathrm{DG}}_{\beta,\Lambda} = \sum_{\sigma\in\Omega_\Lambda} e^{-\frac{1}{2\beta}(\sigma,-\Delta\sigma)} \propto \mathbb{E}^{\sigma}\Big[\prod_{x\in\Lambda}\sum_{m\in\mathbb{Z}}\delta_m(\sigma(x))\Big] \tag{2}$$

where $\sigma \sim \mathcal{N}(0,\beta(-\Delta)^{-1})$ is a centred Gaussian random variable with covariance $(-\Delta)^{-1}$. Since $(-\Delta)$ is a bounded operator on $\mathbb{Z}^2$, one may find $\gamma > 0$ such that $C := (-\Delta)^{-1} - \gamma\,\mathrm{id}$ is a positive operator (as operators on the gradient fields). Hence we may make decomposition $\beta^{-1/2}\sigma = \varphi + Y$ into independent Gaussian random variables $(\varphi, Y) \sim \mathcal{N}(0, C \oplus \gamma\,\mathrm{id})$, giving

$$Z^{\mathrm{DG}}_{\beta,\Lambda} \propto \mathbb{E}^{\varphi,Y}\Big[\prod_{x\in\Lambda}\sum_{m\in\mathbb{Z}}\delta_m(\varphi(x)+Y(x))\Big]. \tag{3}$$

Using independence of $(Y(x))_{x\in\Lambda}$, the integral in Y can be computed explicitly, giving

$$Z^{\mathrm{DG}}_{\beta,\Lambda} \propto \mathbb{E}^{\varphi}\big[e^{\sum_{x\in\Lambda}\sum_{q\geq 1} z^{(q)}(\beta)\cos(q\beta^{1/2}\varphi(x))}\big] \tag{4}$$

for some $z^{(q)}(\beta) = O(e^{-c\beta})$. Therefore, when β is sufficiently large, this can be considered as a lattice sine-Gordon model with small activity and infinitely many cosine terms. The renormalisation group flow for the lattice sine-Gordon model was constructed in [3, 1, 2], so a similar type of analysis can be applied to study the scaling limit of the Discrete Gaussian model.

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Quantum scaling limits of anyon chains

Alexander Stottmeister

Recently, so-called anyon chains [1, 3, 19] which generalize well-known quantum spin chains, such as the Heisenberg chain or the transverse-field Ising chain [4, 16, 7, 2], have drawn a lot of attention in theoretical physics – not least due to their proposed connection to the fractional quantum Hall effect and conformal field theory [13, 12, 8].

There exists compelling evidence that scaling limits of (critical) anyon chains provide a wealth of conformal field theories (CFTs), and it is an interesting mathematical challenge to put the existence and construction of such limits on rigorous grounds. Moreover, precise mathematical control of such scaling limits is also relevant in the context of quantum simulations of the associated complex physical systems.

Together with T. J. Osborne [10, 11, 17], and more recently also with D. Cadamuro, I have explored the construction of said scaling limits in a Hamiltonian framework [18, 9], coined operator algebraic renormalization.

In the arguably simplest case, the transverse-field Ising model (TIM) intimately connected to the 2-dimensional classical Ising model [14, 15],

$$H^{(N)}_{\mathrm{TIM}} = -\varepsilon_N^{-1} \sum_{x\in\Lambda_N} \big(t_x\sigma^{(1)}_x + t_{x+\varepsilon_N+1}\sigma^{(3)}_x\sigma^{(3)}_{x+\varepsilon_N}\big),$$

we have been able construct the scaling limit CFT from the quantum spin chain in a controlled manner. Conformal invariance of the scaling limit is established using a proposal of Koo and Saleur [6], i.e. we show the convergence of their lattice approximation of the Virasoro generators in the scaling limit. In this sense our work complements recent results by Hongler et al. [5] in the Euclidean setting. Although we have also shown the convergence of the lattice disorder parameter,

$$\mu_x = \prod_{y<x} \sigma^{(1)}_y,$$

to its continuum counterpart in the sense of an automorphism of the chiral fermion algebra, the convergence at the level of the implementing quadratic form has not been settled yet.

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Renormalization group maps for Ising models and tensor networks

Tom Kennedy
(joint work with Slava Rychkov)

Tensor networks can be defined in any number of dimensions for any lattice. Here we restrict our attention to the two-dimensional square lattice. Given a tensor A_{ijkl} with four indices (often called legs), we put a copy of A at each site in the square lattice. For each bond in the lattice we perform a contraction by taking the indices of the two tensor legs that form that bond to be equal, and summing over that common index. We impose periodic boundary conditions, so after performing all these contractions for a finite volume lattice we are left with a number which we call the value of the tensor network. Given any finite range Hamiltonian for a discrete spin system, one can construct a tensor A so that the value of the tensor network is the partition function of the spin model.

The nearest-neighbor Ising model is particularly easy to represent as a tensor network. We take the index on the tensor legs to range over just two values: $+$ and $-$. At the midpoint of each bond in the square lattice used for the tensor network we put an Ising spin. If $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are the four spins around a tensor A, then the components of the tensor are

$$A_{\sigma_1,\sigma_2,\sigma_3,\sigma_4} = \exp(\beta(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1) + \frac{h}{2}(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)) \tag{1}$$

(Note that the lattice for the Ising model is made up of the midpoints of the bonds for the tensor network lattice and so is rotated by 45 degrees with respect to the tensor network lattice.)

Renormalization group maps for tensor networks were first introduced in [4]. The simplest RG map for the tensor network is to divide the lattice into two by two blocks and contract the four tensors in each block into a single tensor. (The resulting tensor has eight legs rather than four, so we have to introduce a map that transforms it back to a four-leg tensor. This squares the number of values for the index on a leg, forcing us to work ultimately in an infinite dimensional space of tensors.)

We can obtain the high-temperature fixed point for this map by setting $\beta = 0$ in eq. (1). The result is the tensor in which all the components are equal to 1. Tensor networks have a form of gauge invariance. If G is a tensor with two legs (a matrix) that is invertible, then we can insert GG^{-1} into each leg of the network. The tensors G and G^{-1} can then be grouped with the nearest A to form a new tensor $\hat{A}$. The contraction of this new tensor network has the same value as the original tensor network. This gauge invariance can be used to transform the tensor whose components are all equal to 1 to the tensor A^* (after rescaling) which has one component A^*_{0000} equal to 1 and all other components equal to 0.

Unfortunately, simple RG maps such as this are not well behaved. The RG map has a trivial high-temperature fixed point, but the map is not a contraction about this fixed point. One can show there are eigenvectors of the linearization about the fixed point that have eigenvalue 1. To overcome this problem one must introduce "disentanglers" into the map. Details of the disentangler that we use may be found in [3]. It is loosely based on the disentangler introduced in [1].

In joint work with Slava Rychkov [3] we proved that in two dimensions we can use this disentangler to construct a RG map such that the high temperature fixed point is locally stable:

Theorem 1. *Let $A = A^* + \delta A$ be a tensor such that $||\delta A||$ is small and $A_{0000} = 1$. Let A' be the output of the RG map, normalized so that $A'_{0000} = 1$. Then $A' = A^* + \delta A'$ with*

$$\|\delta A'\| \le C \|\delta A\|^{3/2} . \tag{2}$$

We now discuss our work in progress on a RG map for tensor networks that represent the Ising model at low temperature. The low temperature fixed point is obtained from eq. (1) by rescaling the tensor by a factor of $e^{-4\beta}$ and letting $\beta \to \infty$. With $h = 0$ the resulting tensor is $A^+ + A^-$ where $A^\pm$ are the tensors which have one nonzero component: $A^+_{++++} = A^-_{----} = 1$. This is a fixed point of the simple RG map as are the individual tensors A^+ and A^-. These two tensors correspond to the two ground states in which the spins are all equal. If we include a nonzero magnetic field, then eq. (1) implies that at low temperatures the Ising model with a magnetic field h can be represented by a tensor network with tensor (after suitable rescaling)

$$A = A(\alpha, B) = \alpha A^{(+)} + (1 - \alpha) A^{(-)} + B, \tag{3}$$

where $\alpha = \frac{e^{2h}}{e^{2h} + e^{-2h}}$ and B is a tensor with $||B|| = O(e^{-4\beta})$.

In our work in progress we are developing an RG map which can be applied when the perturbation B is small, in particular for the low temperature Ising model. This map has the form

$$A = A(\alpha, B) \to A' = \mathcal{N}(\alpha, B)A(\alpha', B') \tag{4}$$

$$\mathcal{N}(\alpha, B) = \alpha^4 + (1-\alpha)^4 + \text{small corrections} \tag{5}$$

$$\alpha'(\alpha, B) = r(\alpha) + \text{small corrections}, \quad r(\alpha) = \frac{\alpha^4}{\alpha^4 + (1-\alpha)^4} \tag{6}$$

$$\|B'(\alpha, B)\| < \rho\|B\|, \qquad \rho < 1 \tag{7}$$

When $B = 0$ the RG map is just given by $\alpha \to r(\alpha)$. This map has stable fixed points at $\alpha = 0$ and $\alpha = 1$ corresponding to the two ground states and an unstable fixed point at $\alpha = 1/2$. For small non-zero B these fixed points persist, and we can use a stable manifold theorem to show that there is a function $\alpha_c(B)$ such that if we start with a tensor $A(\alpha, B)$ with $\alpha > \alpha_c(B)$ then the RG map flows to the stable $\alpha = 1$ fixed point. When $\alpha < \alpha_c(B)$ it flows to the stable $\alpha = 0$ fixed point. When $\alpha = \alpha_c(B)$ it flows along the stable manifold to the unstable fixed point at $\alpha = 1/2$. Thus the stable manifold corresponds to the phase co-existence surface. The RG group trajectory is discontinuous as we cross this surface since it changes from converging to one stable fixed point to the other. It is this jump in the behavior of the RG trajectory that gives rise to the discontinuity of the magnetization at the phase-coexistence surface.

The big important open problem here is to use the tensor network RG approach to study the second order phase transition in models like the Ising model. In contrast with the high and low temperature fixed points, the fixed point that describes the second order transition is expected to be infinite dimensional. Numerical studies which are inherently finite dimensional are in excellent agreement with the known critical behavior. (See [1, 2] and references therein.) Our goal is to find an approximate fixed point which is finite dimensional and then prove that there is an exact infinite dimensional fixed point nearby. We hope that the results described here are a modest start towards proving the existence of a non-trivial fixed point for a tensor network RG map which would correspond to the critical point of the Ising model.

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Hierarchical N–vectorial model for dimension $d \geq 4$ in the local potential approximation: limit laws of total spin by renormalization group

Domingos H. U. Marchetti
(joint work with William R. P. Conti)

The hierarchical N–vectorial model is a d-dimensional classical system of spins living in a N-sphere with hierarchical ferromagnetic interactions. One of the most attractive aspect of this model is, for N large, its closeness to the exactly analyzable spherical model introduced by Berlin and Kac in early fifties as a simple model exhibiting phase transition. The $N \to \infty$ limit is a well known subject whose tools might be useful for describing limit laws of the total spin by renormalization group for any $d > 2$ and any inverse temperature β including β_c.

Recently, in collaboration with William R. P. Conti, from Unifesp-Santos, we have revisited some issues raised in our 2008 preprint [1] . We established a dictionary between two different approaches to the limit laws of total spin in the hierarchical spherical model. One of them writes the characteristic function of the total spin as a ratio of partition functions which are evaluated by saddle point method. The other solves exactly the renormalization group equation for the characteristic function of the "a priori" N-vector spin measure at each hierarchical level k in the formal $N = \infty$ limit.

The partition function of the N–vectorial model can be bounded from above and below by partition functions of spherical models. Differently from the usual, the latter partition functions have two parameters associated with the spherical constraint, the dimension N of spin space and the hierarchical levels k of partition of $\mathbb{Z}^d$ into hypercubes of size L^{dk}, varying from 0 to ∞.

The present investigation requires to extend the analysis performed by Ben Arous, Hryniv and Molchanov on the phase transition for the hierarchical spherical model (c.f [2]), when the scalar spins are replaced by vector spins whose dimension space N tends to ∞ for any fixed hierarchical level k. The explicit expressions involved in both approaches are uniform on the block size L^d as the scale L tends to 1, referred by Felder as local potential approximation. In this limit the renormalization group for the N-vectorial model are governed by Polchinski partial differential equations and the lower and upper bounds for partition functions would provide sub and super (explicit) solutions to that equation. For more information on these and related topics see [3] and references therein.

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