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Large Scale Stochastic Dynamics

Organized by
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Abstract. The goal of this workshop was to explore the recent advances in the mathematical understanding of the macroscopic properties which emerge on large space-time scales from interacting microscopic particle systems. The talks addressed the following topics: randomness emerging from deterministic dynamics, hydrodynamic limits, Markov chain mixing times and cut-off phenomenon, superdiffusivity in out-of-equilibrium 2-dimensional systems.

Mathematics Subject Classification (2020): 82C05, 82C20, 82C24, 82C40, 60J10.

Introduction by the Organizers

The workshop *Large scale stochastic dynamics* is the continuation of the highly successful series of Oberwolfach workshops with the same title, whose organising team included along the years T. Bodineau, C. Landim, S. Olla, H. Spohn and two of the present organisers. This new edition, organised by P. Caputo (Roma Tre), F. Toninelli (TU Wien) and B. Tóth (Bristol/Budapest), was well attended with 56 participants (48 in person and 8 online) with broad geographic representation, including postdocs and graduate students, working in diverse intertwining areas of probability and statistical mechanics.

The workshop was devoted to the wide mathematical problem of understanding emergent structures on large space-time scales in the evolution of physical systems. These are modelled by particle systems, namely high-dimensional Markov processes and/or by systems of particles with deterministic (Hamiltonian) dynamics where randomness comes only with the initial conditions. With respect to the previous editions of this series of workshops, there was a larger focus on the presently very active topic of Markov chain mixing and total variation cutoff phenomena.

Very interesting links with integrable systems and with singular SPDES/SDEs have been emphasized by several of the talks.

During the meeting, 12 talks of 50 minutes and 18 talks of 20 minutes were scheduled and an evening "open problem session" was organised with 12 more short informal presentations of 10 minutes, many of which by younger participants. In our choice of 30 talks, we tried to illuminate major recent advances in the field and to expose and address at least some aspects of the works for each of the participants. The chosen schedule format (with a long afternoon break until 5pm, intended to favour discussions and interactions) was unanimously appreciated by the participants. The evening session was the occasion to learn both about intriguing open problems in this area, and about the recent results of early career participants. Both the talks and the evening session triggered further discussions afterwards.

A summary account of the 50- and 20-minute presentations is given below, grouped in thematic units.

1. Stochastic behaviour of large scale deterministic (Hamiltonian) systems with random initial data

- New *exponential decay bounds of correlations in a locally randomised periodic Lorentz gas* in 2-dimensions open the way towards the invariance principle for the Lorentz gas in non-periodic setting. [Liverani]
- Computation of the *generalised Gibbs ensemble for the Calogero fluid* provide one of the rare instances where equilibrium statistical physics of an integrable system with infinitely many conservation laws is fully formulated. [Spohn]
- Another integrable interacting particle system with an abundance of conservation laws is the *one-dimensional hard rod gas whose equilibrium large scale fluctuation fields were computed.* [Olla]
- The *equilibrium fluctuation field of a classical Hamiltonian system of hard-sphere gas, in the Boltzmann-Grad limit* converges weakly to the Ornstein-Uhlenbeck process which solves the so-called fluctuating linear Boltzmann equation. [Simonella]
- The *two-dimensional Lorentz gas with randomly placed scatterers and constant transverse magnetic field* in the Boltzmnn-Grad limit shows strikingly different behaviour from the same system with no external field. [Saffirio]
- A statistical mechanics approach to point vertex model in 2D turbulence allows to obtain Law of Large Numbers and CLT type results [Geldhauser]

2. Equilibrium statistical physics of lattice systems

- The *arboreal gas model on* $\mathbb{Z}^d$ is bond percolation conditioned on having no cycles. It shows, however, surprising differences from the percolation model: in $d = 2$ it is subcritical at any density, in $d \geq 3$ in the supercritical regime it shows signs of self-organized criticality. [Helmuth]
- The *random interlacement process on* $\mathbb{Z}^d$*,* $d \geq 3$*, is factor of i.i.d.* That is, it can be realised as a measurable and shift-invariant mapping of a collection of i.i.d. random variables indexed by $\mathbb{Z}^d$. The result extends to similar constructions on Cayley graphs of finitely generated groups. [Ráth]
- The solid-on-solid model of phase separating interfaces delocalizes in 2-dimensions. A new hands-on proof of this known fact gives fresh insight to this phenomenon. [Ott]
- Computing the thermodynamical free energy of statistical physics models with random interactions (like spin glasses or the perceptron) is notoriously difficult even in the mean field setting. Conditioning on trickily chosen sub-sigma-algebras where some form of concentration of measure is available helps in some relevant cases. [Bolthausen]

3. SPDEs, KPZ growth and random media

- The 2-d Stochastic Heat Equation (SHE) with multiplicative noise is closely related to the 2-d KPZ equation and to directed polymers in random environment. New results concern the high moments of the SHE in the weak disorder regime [Cosco], the scaling limit of SHE at the critical point [Caravenna] and the case where the noise is not Gaussian but a Lévy white noise [Lacoin]
- Last passage percolation (LPP) is intimately related to 1-d KPZ growth. Of particular interest are Busemann functions [Balázs] that provide new insights [Busani] on infinite geodesics in the directed landscape, the scaling limit of LPP.
- 2-dimensional (self)-interacting diffusions are expected to exhibit logarithmic superdiffusivity. Recent rigorous results confirm this for the AKPZ equation and for a Brownian particle in the curl of the GFF [Cannizzaro, Haunschmid-Sibitz, de Lima Feltes]
- The simple exclusion process is perhaps the most well-studied interacting particle system. New tools allow to obtain its hydrodynamic limit when the process has random conductances and the underlying geometry is a very general quenched random graph [Faggionato]

4. Interacting particle systems

- "First-passage percolation in hostile environment" is an interacting particle system with two competing types on the integer lattice. A novel multi-scale approach was presented which allows one to prove a phase transition regarding coexistence of the two types. [Stauffer]

- The contact process (CP) is a classical interacting particle system showing a transition from a unique stationary measure to multiple ones. Percolation ideas from Duminil-Copin et al. have recently allowed to deepen our understanding of very general related models [Hartarsky]
- The hydrodynamic limit for non-gradient interacting particle systems is a notoriously difficult problem. A new approach that borrows ideas from algebraic geometry and solves several open cases was presented [Sasada]
- The study of dynamical large deviations for interacting particle systems, i.e. the study of the probability of anomalous deviations of the current or density profile evolutions, has uncovered new phenomena, such as *dynamical phase transitions*. New results for the weakly asymmetric exclusion process have been presented [Landim]

5. Markov chain mixing and cutoff phenomena

- The abrupt convergence to stationarity of Markov chains on the relevant time scale is known as the cutoff phenomenon. Despite much effort this special type of phase transition remains elusive. Important progress was reported for reversible exclusion processes with reservoirs [Salez], for random walks in finite dimensional geometries [Quattropani,Sau] and for generic bistochastic matrices [Ben-Hamou].
- A non-robustness result for random walks on graphs under quasi-isometries was presented [Kozma]
- A result on the convergence to stationarity for low-temperature stochastic Ising models with suitable random initializations sheds some new light on a notoriously challenging problem for Glauber dynamics [Gheissari]
- The relative entropy decay in Markov chains encodes key feature of the convergence to stationairity. Remarkable advances based on functional inequalities were presented for various types of spin systems [Dagallier,Liu]

Summary. The workshop helped to update the participants on the state of the art and on the important pending open problems in the fields related to their domain of research. This was especially needed and welcome, after 2 years of Covid. The workshop triggered interactions between people working in different fields: probability, (stochastic and deterministic) PDEs, dynamical systems, theoretical computer science. It was also the perfect occasion to initiate and pursue collaborations. The scientific presentations proved that this research field is still very active and is absorbing new ideas from other branches of mathematics (e.g. integrable systems, SPDEs, group theory). Altogether, the atmosphere was extremely pleasant and constructive.

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Workshop: Large Scale Stochastic Dynamics

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Abstracts

Random Lorentz gas

Carlangelo Liverani
(joint work with Mark Demers)

The Lorentz gas was introduced by Lorentz in 1905 as a model for conduction in metals. The model consists of a gas of independent particles interacting elastically with fixed scatterers. Given the independence of the particles, the model is naturally reduced to the study of the motions of a single particle colliding elastically with the scatterers.

In the last fifty years many results have been obtained for the periodic Lorentz gas starting with the seminal works of Bunimovich and Sinai [3] and Bunimovich Sinai and Chernov [4] which establish the Central Limit Theorem in the diffusive scaling. Generalizations and more refined theorems have been recently established [4, 12] and [26], the latter being in the Boltzmann-Grad limit (low density limit).

On the contrary, understanding the aperiodic Lorentz gas, in the high density regime, is almost completely an open problem. Results on the statistical properties exist only for the case of locally perturbed gases (the obstacles are nonperiodic only in a bounded region) [13]. For the random Lorentz gas is known only recurrence in tubes [5, 6, 17, 18, 19, 20, 27].

While statistical properties are understood in the Boltzmann-Grad limit (e.g. see [14, 25, 24]), in the high density situation nothing is known. In particular, the problem of establishing a CLT is wide open, and very few ideas are on the table.

Given this state of affairs, it is important to relate the random Lorentz gas with probabilistic models that may be more easily understood. In particular, one would like to bring to bear the huge literature on random work in a random environment. In [1] it has been shown that some simple classes of deterministic walk in a random environment (the deterministic dynamics consists simply of expanding maps of the interval) are equivalent to a random walk in a random environment with a uniformly exponentially decaying memory, there called GRWRE (Gibbs random walk in random environment). In [11] the same result is obtained for some classes of Lorentz gases.

Namely, we consider Lorentz gases in which the obstacles are arranged on a square lattice, forming periodic cells, but in each cell there is a central obstacle whose position varies (randomly) from one cell to the next. To study the system we introduce a Poincarè section M consisting of the boundary of the central (random) obstacle and of lines (that we call *gates*) that the particle must necessarily cross to transit from one cell to another. Finally, for technical reasons, we limit ourselves to *lazy* gases. This means that the gates are normally closed (that is the particle reflects elastically against them) and open only each n_* collisions, for some n_* large enough (depending only on the class of possible cell configurations).

For the above class of lazy random Lorentz gasses, in [11] we establish equivalence with a GRWRE. To be more precise we need to set some notation. Let

$z_n \in \mathbb{R}^2$, $n \in \mathbb{N}$, be the cell in which is situated the particle at time n. Then we can have the jumps $w_k = z_{k+1} - z_k \in \{0, \pm e_1, \pm e_2\} =: \mathfrak{W}$. If $\mathcal{O}_*$ are the possible configurations of the center obstacle in a cell, then the space of obstacle configuration is $\Omega := \mathcal{O}_*^{\mathbb{Z}^2}$. Let $\mathbb{P}^e$ be a $\mathbb{Z}^2$ translation invariant and ergodic probability measure. Then $\mathbb{P}^e$ describes the random obstacle configuration of the Lorentz gas.

We assume that the process starts from the cell $z = 0$ with an initial position x in the cell described by a smooth density. Accordingly, for each configuration $\omega \in \Omega$ of the random obstacles, the increments of the walk $\{w_0, w_1, \dots\}$ are random variables described by a probability distribution $\mathbb{P}_\omega$. With this notation we prove the following Theorem.

Theorem 1. *There exist $C_* > 0$ and $\vartheta \in (0,1)$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, if x is distributed according to $f \in \mathcal{C}^1$ with $\int_M f = 1$ and $z_0 = (0,0)$, for all $n > m \geq 0$ and all $w \in \mathfrak{W}^{\mathbb{N}}$,*

$$\left|\mathbb{P}_\omega(w_n \mid w_{k_0} \dots w_{n-1}) - \mathbb{P}_{\xi_{z_m}\omega}(w_n \mid w_m \dots w_{n-1})\right| \leq C_* \vartheta^{n-m}.$$

That is, the transition probabilities at time n depend only on the near past. Note that to compute the probability of a trajectory conditioned to the path $z_1, \dots, z_n$ one must consider all the gates that allow for a different cell trajectory as holes. Hence the computation of such probability is similar to the study of escape rates in open systems.

To establish Theorem 1 we combine several ideas: the cone technique introduced in [21, 22]; the functional spaces first proposed in [2, 15, 16] and adapted to systems with discontinuities, and then billiards, in [7, 8, 9, 10]; the work on open systems with large holes [23] and, finally, the general ideas already put forward in [1].

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Non-equilibrium multi-scale analysis and coexistence in competing first-passage percolation

Alexandre Stauffer

We consider the following model of growth process with competition on $\mathbb{Z}^d$. There are two growth processes, which we call type 1 and type 2. Type 1 starts from the origin, whereas for each site in $\mathbb{Z}^d \setminus \{0\}$ with probability p we place a so-called *type 2 seed*, otherwise we leave the site empty. Then type 1 starts spreading through $\mathbb{Z}^d$ as a first passage percolation process at rate 1 (that is, with passage times distributed as independent exponential random variables of rate 1). Type 2 initially does nothing. Whenever a process (type 1 or type 2) attempts to occupy a site which hosts a type 2 seed, the occupation does not happen and that seed is *activated*. Activated seeds are regarded as being occupied by type 2. From that moment on, type 2 starts spreading from that seed as a first passage percolation process of rate λ. As other type 2 seeds are activated, more and more clusters of

type 2 will start to grow. Each site of $\mathbb{Z}^d$ will be occupied by the type that arrives to each first, and will never switch types afterwards.

The above process was introduced by Sidoravicius and Stauffer [8] under the name of *first passage percolation in a hostile envronment* (FPPHE). We say that a type *survives* if in the limit as time goes to ∞ the type occupies an infinite *cluster*; that is, it occupies an infinite *connected* set of sites. Our definition of survival differs from what is commonly used in the literature: the occupation of an infinite set of sites. The reason is that type 2 will eventually occupy all the seeds, which already consist of an infinite set of sites. We refer to a cluster of type 2 as a maximal connected set of sites that are occupied by type 2, regardless of whether they were all occupied from the activation of the same type 2 seed or from several type 2 seeds. We say that a type *dies out* if it does not survive.

There are three possible outcomes for FPPHE: *extinction* (meaning that type 1 does not survive), *strong survival* (meaning that, with positive probability, type 1 survives and type 2 dies out), and *coexistence* (meaning that both types 1 and 2 simultaneously produce infinite clusters with positive probability).

Let η_t^1 and η_t^2 be the set of sites occupied by type 1 and type 2, respectively, at time t (where non-active seeds of type 2 are not included in η_t^2). With Vladas Sidoravicius we showed the following result.

Theorem 1 (Small p regime [8])**.** *For any $d \geq 2$ and any $\lambda < 1$, there exists $p_0 > 0$ such that for all $p \in (0, p_0)$ there exists a constant $c > 0$ for which*

$$\mathbb{P}(\eta_t^1 \supset \mathcal{B}_{ct} \text{ for all } t \geq 0) > 0,$$

where $\mathcal{B}_r$ stands for the collection of sites within distance r from the origin.

In other words, the above theorem shows the existence of a strong survival regime for all $\lambda < 1$, provided p is small enough.

Intuitively, one expects that increasing p or λ favors type 2, however there is no proof of monotonicity for FPPHE on $\mathbb{Z}^d$. In fact, it is easy to show that the standard coupling that could be used to show monotonicity fails; moreover, there are graphs for which FPPHE is not monotone at all.

Theorem 2 (Non-monotonicity [2])**.** *There is a connected, infinite, quasi-transitive graph of bounded degrees and values p_1, p_2 with $0 < p_1 < p_2 < 1$ such that for λ small enough*

$$\mathbb{P}_{p_1,\lambda}(\textit{type 1 survives}) = 0 \quad \textit{and} \quad \mathbb{P}_{p_2,\lambda}(\textit{type 1 survives}) > 0,$$

where $\mathbb{P}_{p,\lambda}$ stands for the probability measure induced by FPPHE with parameters p, λ.

In other words, the theorem above shows that the probability that type 1 survives can increase if p is increased. The result in [2] shows three phase transitions: $\mathbb{P}_{p_1,\lambda}$(type 1 survives) is positive for all p small enough, then become 0 for an interval containg p_1, then turn positive again for an interval containing p_2 and then finally become 0 for all p large enough. It is implicit in the proofs that it is possible to elaborate on the graph in [2] to construct quasi-transitive graphs with a larger number of phase transitions.

FPPHE has also been studied in hyperbolic and non-amenable graphs, where the results are shown to change drastically [3]. For example, in such graphs type 2 survives with positive probability for any p and λ. Moreover, a similar version of Theorem 2 can be established.

Returning to the case of $\mathbb{Z}^d$, it is not difficult to see that when $p > 1 - p_c^{\text{site}}$, where p_c^{site} is the critical probability for site percolation on $\mathbb{Z}^d$, there is extinction because almost surely the origin is confined to a finite cluster of sites not hosting a type 2 seed. One can also show that when $\lambda \geq 1$ there is extinction for all p. This follows since even at $\lambda = 0$ we obtain that the growth of type 1 is slowed down by the presence of seeds. This can be made formal by using a result of van den Berg and Kesten [1], which applied to our case gives that there exists a value $\epsilon = \epsilon(d, p) > 0$ such that the growth of type 1 is contained in first passage percolation of rate $1 - \epsilon$. Then, extinction follows for any $\lambda > 1 - \epsilon$.

In Theorem 2 we studied the case of p small. Even though monotonicity is not known, it is natural to believe that type 1 can survive also when it is λ that is made small. This is confirmed by the theorem below.

Theorem 3 (Small λ regime [6]). *For any $d \geq 2$ and any $p < 1 - p_c^{\text{site}}$, there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ there exists a constant $c > 0$ for which*

$$\mathbb{P}(\textit{type 1 survives}) > 0.$$

The above theorem has a curious consequence in dimensions $d \geq 3$. In this case, it is known that $p_c^{\text{site}} < 1/2$ and, hence, the interval $(p_c^{\text{site}}, 1 - p_c^{\text{site}})$ is not empty. When p is inside this interval, Theorem 3 gives that type 1 survives if λ is small enough. However, type 2 also survives since the set of seeds occupies an infinite cluster at time 0. Therefore, we obtain coexistence. Indeed, an even stronger form of coexistence occurs, where both type 1 and type 2 occupy infinite clusters of *positive density*. Such a strong form of coexistence is impossible in other growth processes with competition such as the two-type Richardson model [7].

It is an interesting open problem to establish whether coexistence occurs in $d = 2$, and to obtain a coexistence regime (in any dimension) for $p < p_c^{\text{site}}$.

The proof of Theorem 3 goes by a novel type of multi-scale analysis, which we call *multi-scale analysis with non-equilibrium feedback*. This gives a way to handle non-local events inside a multi-scale renormalization scheme, and we believe could have wide applicability.

We conclude by mentioning that FPPHE was introduced in [8] as a tool to understand a notoriously challenging model known as *multi-particle diffusion limited aggregation* (MDLA). In fact, FPPHE seems to be a way to understand growth processes or models with a moving front (such as aggregation models and spread of infection). The idea is that the growth of type 1 from the origin models the progress of the moving front through "typically good" regions of the environment. It is unavoidable that in such models the moving front will eventually passes through bad regions of the environment. The locations of such regions are randomly spreadout and are modelled by the seeds of FPPHE. When the moving front gets to such a bad region (i.e., when the corresponding seed gets activated), then the growth of

type 2 from the activated seed represents the effect that discovering such a bad region of the environment may have in nearby regions. The speed at which this effect spreads is modelled by the value of λ. If p and λ are small enough, then bad regions are rare (p small) and their effect do not spread quickly (λ small), creating the conditions for the moving front to advance through the typically good regions of the environment, leaving the bad regions behind.

The above strategy was employed in the study of MDLA in [8], and was further developed by Duncan Dauvergne and Allan Sly in the study of non-equilibrium models of spread of infection [4, 5], where they generalized FPPHE to more general passage times and showed that type 1 survives with positive probability provided p and λ are small enough.

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The critical 2d Stochastic Heat Flow

Francesco Caravenna

(joint work with Rongfeng Sun, Nikos Zygouras)

We consider the *Stochastic Heat Equation (SHE)*, formally written as follows:

$$\text{(SHE)} \qquad \partial_t u(t,x) = \tfrac{1}{4}\Delta u(t,x) + \beta\, \xi(t,x)\, u(t,x)\,, \qquad t \geq 0,\ x \in \mathbb{R}^d\,,$$

where $\beta > 0$ is a coupling constant and $\xi(t,x)$ denotes *space-time white noise*, i.e. the Gaussian generalized process on $[0,\infty) \times \mathbb{R}^d$ with formal covariance

$$\mathbb{C}\mathrm{ov}(\xi(t,x), \xi(s,y)) = \delta(t-s)\,\delta(x-y)\,.$$

In dimensions $d \geq 2$, this equation falls outside the scope of existing solution theories for singular stochastic PDEs, such as Regularity Structures or Paracontrolled Calculus, and it is highly non-obvious to give a rigorous meaning to its solution. We focus here on $d = 2$ which is the critical dimension for this equation.

We regularise the SHE by discretising space-time diffusively. More precisely, we fix a (large) scale parameter $N \in \mathbb{N}$ and we consider the solution $u_N(t,x)$ of a suitable *discretised SHE*, defined for $(t,x) \in \mathbb{T}_N := \frac{\mathbb{N}}{N} \times \frac{\mathbb{Z}^2}{\sqrt{N}}$ by

$$\text{(d-SHE)} \qquad \partial_t^N u_N(t,x) = \tfrac{1}{4}\Delta^N u_N(t,x) + N\,\xi_N^\beta(t+\tfrac{1}{N},x)\,\langle u_N(t,x)\rangle$$

where we give the following definitions (we write $x' \sim x$ to denote nearest neighbour vertices in the rescaled lattice $\frac{\mathbb{Z}^2}{\sqrt{N}}$, i.e. such that $|x'-x| = \frac{1}{\sqrt{N}}$):

- $\partial_t^N u_N(t,x) := N\{u_N(t+\frac{1}{N},x) - u_N(t,x)\}$ is a rescaled lattice derivative;
- $\Delta_t^N u_N(t,x) := \frac{N}{4} \sum_{x' \sim x} \{u_N(t,x') - u_N(t,x)\}$ is a rescaled lattice Laplacian;
- $(\xi_N^\beta(t,x))_{(t,x)\in\mathbb{T}_N}$ are i.i.d. centred random variables with variance $\sim \beta^2$ (so that $N\,\xi_N^\beta(\cdot,\cdot)$ converges to white noise $\beta\,\xi(\cdot,\cdot)$ as $N \to \infty$);
- $\langle u_N(t,x)\rangle := \frac{1}{4} \sum_{x' \sim x} u_N(t,x')$ is a local space average of $u_N(t,x)$.

Given any initial condition $u_N(0,\cdot)$, the difference equation (d-SHE) admits a well-defined unique solution $u_N(t,x)$ for all $(t,x) \in \mathbb{T}_N$ (the choice of evaluating the noise ξ_N^β at time $t+\frac{1}{N}$ ensures that $u_N(t,x)$ is a martingale).

We make the structural assumption $\xi_N^\beta > -1$, which ensures that $u_N(t,x) > 0$ for every $(t,x) \in \mathbb{T}_N$. We parametrise the noise distribution as follows:

$$(\star) \qquad \xi_N^\beta = \frac{\mathrm{e}^{\beta\omega}}{\mathbb{E}[\mathrm{e}^{\beta\omega}]} - 1$$

for some fixed centred random variables ω with unit variance and finite exponential moments (note that the r.h.s. of $(\star)$ is centred with variance $\sim \beta^2$ as $\beta \to 0$).

Consider for simplicity the flat initial condition $u_N(0,\cdot) \equiv 1$. The question is: *does the solution $u_N(t,x)$ of* (d-SHE) *admit a non-trivial limit $\mathcal{U}(t,x)$ as $N \to \infty$?* Our main result provides a positive answer, with two important caveats.

(1) We look at $u_N(t,\cdot)$ as a random *distribution on* $\mathbb{R}^2$, since the limit $\mathcal{U}(t,\cdot)$ is not expected to be a function. We actually prove vague convergence as a *locally finite random measure on* $\mathbb{R}^2$, namely

$$\int_{\mathbb{R}^2} \varphi(x)\,u_N(t,x)\,dx \xrightarrow[N\to\infty]{d} \int_{\mathbb{R}^2} \varphi(x)\,\mathcal{U}(t,dx)$$

for any continuous and compactly supported $\varphi : \mathbb{R}^2 \to \mathbb{R}$.

(2) We rescale the coupling constant $\beta = \beta_N$ as $N \to \infty$ as follows:

$$\beta \sim \frac{\hat\beta}{\sqrt{\log N}}$$

for the *critical value* $\hat\beta = \sqrt{\pi}$. This is natural because, see [2],

$$\mathbb{V}\mathrm{ar}\Big[\int_{\mathbb{R}^2} \varphi(x)\,u_N(t,x)\,dx\Big] \xrightarrow[N\to\infty]{} \begin{cases} 0 & \text{if } \hat\beta < \sqrt{\pi}\,, \\ \infty & \text{if } \hat\beta > \sqrt{\pi}\,, \end{cases}$$

and note that $\mathbb{E}[\int_{\mathbb{R}^2} \varphi(x)\, u_N(t,x)\, dx] = \int \varphi(x)\, dx$ is constant. This means that $u_N(t,x)\, dx$ converges for $\hat{\beta} < \sqrt{\pi}$ to a *deterministic limit*, the Lebesgue measure dx, with Gaussian fluctuations proved in [2].

We can now state our main result, where we actually explore a "*critical window*" around the critical parameter $\hat{\beta} = \sqrt{\pi}$. Let us denote by "f.d.d." convergence in the sense of finite-dimensional distributions.

Theorem 1 ([5])**.** *Let $u_N(t,x)$ solve* (d-SHE)*, with ξ_N^β as in* $(\star)$*, where $\beta = \beta_N$ is rescaled in the critical window*

$$\beta = \frac{\sqrt{\pi}}{\sqrt{\log N}} \left(1 + \frac{\theta}{\log N}\right) \qquad \text{for } \theta \in \mathbb{R}\,.$$

Then we have the convergence to a unique and non-trivial limit

$$(u_N(t,x)\, dx)_{t\geq 0} \xrightarrow[N\to\infty]{f.d.d.} \mathcal{U}^\theta = (\mathcal{U}^\theta(t,dx))_{t\geq 0}$$

which we call the critical $2d$ Stochastic Heat Flow*.*

Many properties of the limit $\mathcal{U}^\theta$ are known, e.g. the first and second moments

$$\mathbb{E}[\mathcal{U}^\theta(t,dx)] = dx\,, \qquad \mathbb{E}[\mathcal{U}^\theta(t,dx)\,\mathcal{U}^\theta(t,dy)] = K_t^\theta(x,y)\, dx\, dy\,,$$

for an explicit non-trivial kernel $K_t^\theta(x,y) \sim C \log \frac{1}{|x-y|}$ as $|x-y| \to 0$, see [1]. The third moment was first derived in [3], while higher order moments were obtained in [8] (but they grow too fast to uniquely determine the law).

More recently, we showed in [6] that the limiting random measures $\mathcal{U}^\theta(t,dx)$ cannot be realised as a *Gaussian Multiplicative Chaos*, i.e. as the (renormalised) exponential of a generalised Gaussian field on $\mathbb{R}^2$. This result is interesting because it suggests that $\log u_N(t,x)$, which is the Cole-Hopf solution of a discretization of the $2d$ *KPZ equation*, might have a *non Gaussian limit* as $N \to \infty$.

We conclude by describing the strategy of the proof of Theorem 1. It was proved in [1] that $(u_N(t,x)\, dx)_{t\geq 0}$ is tight with bounded second moment, hence it admits non-zero limits along subsequences, but it is highly non trivial to prove *uniqueness* of the possible subsequential limits, due to the lack of a characterisation of the limiting distribution (e.g. via moments). This is, indeed, the main difficulty to prove Theorem 1. The strategy adopted in [5] is based on a *Cauchy argument*: we show that the random measures $u_N(t,x)\, dx$ and $u_M(t,x)\, dx$ are *close in distribution for large* $M, N \in \mathbb{N}$. This is obtained through four key tools:

(1) *coarse-graining*, which lets us approximate $u_N(t,x)\, dx$ in L^2 by a *coarse-grained model* $\mathcal{Z}_\epsilon(t,dx|\Theta_{N,\epsilon})$, which depends on N through a family $\Theta_{N,\epsilon}$ of (dependent) random variables built out of $u_N(\cdot,\cdot)$;
(2) a *renewal structure* which provides a probabilistic framework for second moments calculations, based on the so-called Dickman subordinator [4];
(3) a *Lindeberg principle* for dependent random variables, to show that the distribution of the coarse grained model $\mathcal{Z}_\epsilon(t,dx|\Theta)$ is insensitive to the distribution of the random variables Θ, if we keep mean and covariance;
(4) *functional inequalities* for the Green's function of multiple random walks, which are needed to bound the moments of the coarse-grained model.

Last but not least, a cornerstone of our proof is a *Feynman-Kac formula* for the solution $u_N(t,x)$ of (d-SHE), under the parametrisation ($\star$) for the noise ξ_N^β in terms of i.i.d. random variables ω. More precisely, if we fix $u_N(0,\cdot) \equiv 1$ for simplicity, writing $(t,x) = (\frac{n}{N}, \frac{z}{\sqrt{N}})$ for $n \in \mathbb{N}$ and $z \in \mathbb{Z}^2$ we have

$$u_N(t,x) = Z_N(n,z) = \frac{\mathrm{E}\big[\mathrm{e}^{\sum_{i=0}^{n-1} \beta\omega(n-i,S_i)} \big| S_0 = z\big]}{\mathbb{E}[\mathrm{e}^{\beta\omega}]^n},$$

where $S = (S_i)_{i\geq 0}$ denotes the simple random on $\mathbb{Z}^2$ with expectation E (while $\mathbb{E}$ denotes the expectation with respect to the variables ω). This means that the solution u_N of the discretized SHE is, up to a time reversal, the partition function Z_N of the much-studied model of *directed polymer in random environment* [7].

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A statistical physics view on gSQG point vortices

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(joint work with Marco Romito)

Point vortex models are a classical approach in 2D turbulence. We characterize a point vortex by its position X_j in $\mathbb{R}^2$, and its intensity $\gamma_j \in \mathbb{R}$. Point vortex models describe the evolution of vortex positions according to the system of equations

$$\begin{cases} \dot{X}_j = \sum_{k\neq j} \gamma_k \nabla^\perp G_m(X_j, X_k), \\ X_j(0) = x_j, \end{cases} \qquad j = 1,2,\dots,N, \tag{1}$$

where γ_k are real numbers and G_m is the Green function of the operator $(-\Delta)^{\frac{m}{2}}$. Here, we consider the case $m \in [1,2]$, where $m = 2$ is the Euler case and $m = 1$ the surface-quasigeostrophic case. The equations (1) form a Hamiltonian system with Hamiltonian $H_N(\gamma^N, X^N) = \frac{1}{2}\sum_{j\neq k} \gamma_j\gamma_k G_m(X_j, X_k)$, where $X^N = (X_1, X_2, \dots, X_N)$ and $\gamma^N = (\gamma_1, \gamma_2, \dots, \gamma_N)$.

Efforts have been made to consider the atomic measure $\theta_t(dx) = \sum_{i=1}^N \gamma_i \delta_{X_t^i}$ as a kind of weak solution to a generalized Euler PDE in vortex form.

In the case $m = 2$, Marchioro constructed Euler point vortices as limits of L^∞-vorticity patch solution to the Euler vortex equation, and this quality of being a true (although irregular) solutions makes them particularly interesting.

Onsager [4] suggested to use methods of statistical physics to investigate 2D turbulence. There, we look at the invariant distribution for the Hamiltonian dynamics (1), the measure

$$\mu_\beta^N(dX^N) = \frac{1}{Z_\beta^N} e^{-\beta H_N(X^N, \gamma^N)}\, d\ell^{\otimes N}, \tag{2}$$

where we denoted by ℓ the normalized Lebesgue measure.

For $m < 2$, Z_β^N is not finite, and therefore (2) does not make sense. Nevertheless, we are able to prove [2] a Law of Large Numbers and a Central Limit Theorem, by regularizing the Green function in a suitable way, and taking this regularization to zero in the limit as $N \to \infty$. However, the speed of convergence of $\epsilon = \epsilon(N)$ must be at least logarithmically slow in terms of N.

Under the conditions $\beta > 0$, propagation of chaos can be shown, namely vortices decorrelate and are independent in the limit. Our result holds for a fractional Green function on the torus, a common restriction to avoid issues of self-interaction between vortices, also used in the work of [1] for the Euler case. While recently it was possible to prove CLTs for bounded domains [3], this does not work (for now) in the gSQG case. Here, it would be necessary to build up the necessary theory on two-sided estimates on G_m, in order to have control on the partition function.

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Mixing times from random phase initializations

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(joint work with Alistair Sinclair)

1. Overview

It is well-known that local Markov chains (e.g., Glauber dynamics) for spin systems suffer an exponential slowdown in regimes of *phase coexistence*, referring to the emergence of multiple phases in the state space, separated by narrow bottlenecks. Much effort has been devoted to overcoming this and sampling using

non-local dynamics, dynamics for alternative representations of the system, and non-dynamical methods (e.g., [4, 5, 18, 10, 12, 14]).

There is a much more "obvious" proposed solution to this problem: initialize the standard Glauber dynamics to be in an appropriately weighted mixture of the ground states (one maximum likelihood configuration from each phase). Presumably, the main obstacle to rapid mixing is the slow transitions between phases, and it can converge rapidly "within each phase", at least when initialized nicely. Since the overall probability distribution on configurations is approximated by a mixture of the single-phase distributions, this should suffice for global convergence.

We describe recent progress with Alistair Sinclair towards the above folklore sampling approach in two classical contexts at which slow mixing is induced by phase coexistence: (1) the Ising model in its entire low-temperature regime, and (2) the random-cluster model at its critical point when q is large.

2. Low-temperature Ising dynamics

The Ising model at inverse temperature $\beta > 0$ on the torus $\mathbb{T}_n = (\mathbb{Z}/n\mathbb{Z})^d$ is the distribution over configurations in $\{-1,+1\}^{\mathbb{T}_n}$ with Gibbs weight given by $\exp(-\beta \sum_{v\sim w} \mathbf{1}\{\sigma_v \neq \sigma_w\})$. The (continuous-time) Glauber dynamics for the Ising model is well-known to undergo the following transition in its mixing times from a worst-case initialization in every $d \geq 2$. When $\beta < \beta_c(d)$, the mixing time is $O(\log n)$ [16], whereas when $\beta > \beta_c(d)$ the mixing time is exponential in n^{d-1} [2, 17]. This is due to the bottleneck between the two dominant phases of the Ising model—corresponding to the majority of sites taking state $+1$ versus -1. In [7], we established a version of the folklore paradigm described in the overview for the Ising dynamics on $\mathbb{T}_n$ in all $d \geq 2$ and all $\beta > \beta_c$.

Theorem 1. *Fix $d \geq 2$ and $\beta > \beta_c(d)$. The Ising Glauber dynamics initialized from the all $+1$ configuration converges to the Ising distribution conditioned on having a majority of its spins being $+1$ in time at most $O(\log n)$ times the worst-case mixing time on a box of side-length $C \log n$ with all $+1$ boundary conditions.*

Plugging in the best known bounds on the mixing time with plus boundary conditions from [15], the above bound is an almost optimal $n^{o(1)}$ in $d = 2$ and quasi-polynomial in every $d \geq 3$. Stitching this together with its symmetric analogue from the all -1 initialization, we obtain that the mixing time of the Ising Glauber dynamics on $\mathbb{T}_n$ initialized from the $\frac{1}{2}$-$\frac{1}{2}$ mixture of ground states satisfies the same bound. It is a long-standing conjecture [13] that the mixing time in a box of side-length m with $+1$ boundary conditions is polynomial in m, which would make the above mixing time bounds $(\log n)^{O(1)}$.

We note that even restricted to the plus phase, the *worst-case* mixing time is much larger, exponentially so in $d \geq 3$, than the above bound, but the dynamics initialized from all $+1$ avoids all such secondary, geometric, bottlenecks.

3. Critical random-cluster dynamics

The random-cluster model is a model of dependent edge-percolation parametrized by (p, q), that at integer q is coupled to the Ising ($q = 2$) and more generally q-state Potts models with $p = 1 - e^{-\beta}$ [9].

The Glauber dynamics of the random-cluster model is both interesting in its own right, and is algorithmically salient because it is closely connected to the Swendsen–Wang algorithm for the Potts model. Not only do the random-cluster Glauber dynamics on $\mathbb{T}_n$ mix rapidly (from worst-case initialization) at all high temperatures $p < p_c(q, d)$ [11], they are also expected to mix rapidly at all low-temperatures $p > p_c(q, d)$: this is known in $d = 2$ [1], and at sufficiently large $p \gg p_c(q, d)$ [8]. The phase coexistence and slow mixing regime of the random-cluster dynamics are instead at the critical point $p = p_c(q, d)$ when q is large [3, 6].

This slow mixing around the critical point is induced by the coexistence of a *wired* phase, in which there is a giant connected component, and a *free* phase in which all connected components are microscopic. In [8], we proved analogues of our results of [7] for the random-cluster dynamics on $\mathbb{T}_n$ in a microscopic window about its critical point. Namely, if one starts from the all-wired initialization, the dynamics converges to the wired phase in quasi-polynomial time (and $n^{o(1)}$ in $d = 2$), and similarly to the free phase from the all-free initialization.

4. Open questions

In the above, we relied on the monotonicity of the Ising and random-cluster models quite strongly. It would be of significant interest to generalize the results to a non-monotone setting like the Potts Glauber dynamics.

Question 2. *Show that the Potts Glauber dynamics on $\mathbb{T}_n$ at $\beta > \beta_c(q, d)$ initialized from a $(\frac{1}{q}, ..., \frac{1}{q})$ mixture of the q ground states mixes in sub-exponential time.*

Another avenue of investigation is to study mixing times not from ground states, but from random initializations that don't need a priori knowledge of the phases of the system. The most natural choice is the uniform-at-random initialization.

Question 3. *Show that the Ising Glauber dynamics on $\mathbb{T}_n$ at β large, initialized uniformly over $\{\pm 1\}^{\mathbb{T}_n}$, mixes in polynomial time.*

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Random interlacement is a factor of i.i.d.

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(joint work with Márton Borbényi, Sándor Rokob)

Random interlacements, introduced in [1], describe the local distributional limit of the trace of a random walk on a d-dimensional discrete torus $(\mathbb{Z}/N\mathbb{Z})^d, d \geq 3$ if we run the random walk up to times comparable to the volume of the torus and let $N \to \infty$, cf. [3]. The notion of random interlacements was generalized to transient weighted graphs in [2].

Let us denote by W the space of doubly infinite transient nearest neighbour trajectories in G. We say that $w, w' \in W$ are equivalent modulo time-shift if there exists $k \in \mathbb{Z}$ such that for all $n \in \mathbb{Z}$ we have $w(n) = w'(n + k)$. Let us denote by W^* the set of equivalence classes of W with respect to time-shift

equivalence. Let us denote by $\pi^* : W \to W^*$ the function which maps to each $w \in W$ its equivalence class modulo time-shift. The random interlacement point process $\mathcal{Z} = \sum_{i \in I} \delta_{(w_i^*, t_i)}$ is a Poisson point process (PPP) on the space $W^* \times \mathbb{R}_+$ of labeled trajectories modulo time-shift with intensity measure $\nu \times \lambda$, where λ denotes the Lebesgue measure on $\mathbb{R}_+$ and ν is a certain σ-finite measure on W^*. The following property characterizes ν: for each finite subset K of the vertex set of G, an alternative way of generating a PPP on W^* with the same distribution as the point process of trajectories of $\mathcal{Z}$ that hit K and have a label in the interval $[0, u]$ is as follows: independently for each vertex v of K, let us start a POI(u) number of i.i.d. doubly infinite random walks from v indexed by $\mathbb{Z}$, throw away those trajectories that already visit K at a time indexed by a negative number and take the point process that consists of the equivalence classes of the remaining trajectories modulo time-shift.

Our main result is that we construct the interlacement Poisson point process $\mathcal{Z}$ from a family of i.i.d. random variables indexed by the vertex set of a locally finite, connected, transitive, transient infinite simple graph G via a measurable map which intertwines the action of the automorphism group Γ of G. In other words, we show that the interlacement point process is a factor of i.i.d.

It is relatively easy to check that if there exists a PPP $\mathcal{X} = \sum_{i \in I} \delta_{w_i}$ on W such that the law of $\mathcal{X}$ is invariant under the action of Γ and $\pi^*(\mathcal{X}) := \sum_{i \in I} \delta_{\pi^*(w_i)}$ is a PPP with intensity measure ν then $\mathcal{X}$, $\pi^*(\mathcal{X})$ and the interlacement point process are all factors of i.i.d. However, we also prove that such an $\mathcal{X}$ exists if and only if G is non-unimodular.

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Generalized Gibbs Ensembles for the Calogero Fluid

Herbert Spohn

In Statistical Mechanics of many classical particles, Gibbs ensemble refers to a probability measure on phase space of the form

$$Z^{-1} \exp\big[-\beta(H - \eta P - \mu N)\big]. \tag{1}$$

Here H is the hamiltonian, P the total momentum, and N the number of particles. The control parameters are β, η, μ and Z is the normalizing partition function. The parameter η controls the average momentum. Usually one sets $\eta = 0$. But to describe fluid flow such a parameter has to be included. Physically the rationale relies on a natural choice for spacetime stationary probability measures of large mechanical systems.

Generalized Gibbs ensembles (GGE) refer to mechanical systems with an extensive number of conserved fields. Such a property is exceptional and requires a fine-tuning of the interaction potential. Integrable many-body systems are mostly restricted to one space dimension. Following the same rationale as leading to (1), now *all* locally conserved fields must be written in the exponential. The three control parameters will have to be replaced by a function over $\mathbb{R}$. Such generalized Gibbs ensembles have surprising novel structures, which have been explored systematically only in recent years. In this note I discuss exclusively the Calogero fluid as a challenging example. As regards to prior work I mention the lecture notes by B. Doyon [1], the two recent special issues on the topic [2, 3], and the forthcoming book [4], which serve as an entry point to original research articles.

The Calogero fluid consists of particles on the real line, position q_j and momentum p_j of the j-th particle, $j = 1, ..., N$, see the book [5]. Particles interact through a repulsive potential of the form

$$V_{\mathrm{ca}}(x) = \frac{1}{(2\sinh(x/2))^2}, \tag{2}$$

Therefore Newton equations of motion are

$$\frac{d^2}{dt^2} q_j = \sum_{i=1, i\neq j}^{N} \tfrac{1}{4} \cosh(\tfrac{1}{2}(q_j - q_i)) \big(\sinh(\tfrac{1}{2}(q_j - q_i))\big)^{-3}, \tag{3}$$

$j = 1, ..., N$. The phase space is $\mathbb{R}^{2N}$. At short distances the potential has a repulsive $1/|x|^2$ singularity, which implies that the spatial ordering of particles is maintained throughout time. Thus alternatively the phase space can be taken as $\mathbb{W}_N \times \mathbb{R}^N$ with the Weyl chamber $\mathbb{W}_N = \{q_1 \leq ... \leq q_N\}$.

The dynamics admits a Lax pair consisting of the $N \times N$ matrices, L, M, through

$$L_{i,j} = \delta_{ij} p_j + \mathrm{i}(1 - \delta_{ij}) \big(2\sinh(\tfrac{1}{2}(q_i - q_j))\big)^{-1}, \tag{4}$$

$$\begin{aligned} M_{i,j} = \mathrm{i}\delta_{ij} \sum_{k=1, k\neq j}^{N} & \big(2\sinh(\tfrac{1}{2} q_j - q_k))\big)^{-2} \\ & -\mathrm{i}(1 - \delta_{ij}) \cosh(\tfrac{1}{2}(q_i - q_j)) \big(2\sinh(\tfrac{1}{2}((q_i - q_j))\big)^{-2}. \end{aligned} \tag{5}$$

The Lax matrix is hermitian, hence has real eigenvalues, while the partner matrix is anti-hermitian. Then, with L, M evaluated along trajectories of (3),

$$\frac{d}{dt} L(t) = [L(t), M(t)]. \tag{6}$$

Merely having a commutator implies that

$$\frac{d}{dt} \mathrm{tr}[L(t)^n] = 0, \quad n = 1, 2, \dots . \tag{7}$$

The eigenvalues of $L(t)$ do not change in time and the conserved charges are

$$Q^{[n]} = \mathrm{tr}[L^n] \tag{8}$$

with their density given by

$$Q^{[n]}(x) = \sum_{j=1}^{N} \delta(x - q_j)(L^n)_{j,j}, \tag{9}$$

$x \in \mathbb{R}$. The total momentum corresponds to $Q^{[1]}$ and total energy to $\frac{1}{2}Q^{[2]}$. As a general fact for Calogero type models, under the standard Poisson bracket,

$$\{Q^{[m]}, Q^{[n]}\} = 0. \tag{10}$$

Thus we have obtained a set of N conserved fields in involution. In addition there is the particle number

$$Q^{[0]} = N, \quad Q^{[0]}(x) = \sum_{j=1}^{N} \delta(x - q_j). \tag{11}$$

In the sense of hamiltonian systems the constant function is a trivial conservation law. But it must be included in a hydrodynamic context, since the particle density is a central physical observable.

Having obtained the conserved fields, a GGE can be written as minus the exponential of

$$\sum_{n=0}^{\infty} \mu_n \mathrm{tr}\big[L^n\big] = \mathrm{tr} V(L), \tag{12}$$

where the *confining potential* V equals to the power series on the left. Analyticity is not such a natural condition and one should think of V as being continuous, bounded from below, and having a at least linear increase at infinity. To understand the precise class of confining potentials is still an open problem.

With these preparations, the finite volume GGE is defined by

$$Z_{\mathrm{ca},N}(\ell, V)^{-1} \exp\Big(- \mathrm{tr}\big[V(L)\big] - \sum_{j=1}^{N} V_{\mathrm{box},\ell}(q_j) \Big) \prod_{j=1}^{N} \mathrm{d}q_j \mathrm{d}p_j. \tag{13}$$

We added the box potential $V_{\mathrm{box},\ell}$ to ensure that the positions of the particles are in essence constrained to lie in a box of length ℓ. Note that in general such a box potential breaks integrability. But here it is only a mathematical tool to properly construct the infinite volume GGE. This probability measure will then be spacetime stationary.

Our goal is a study of the infinite volume limit with $N, \ell \to \infty$ at fixed ratio $\ell/N = \nu$, hence $1/\nu$ the macroscopic density of particles. For notational simplicity we still use both N, ℓ, but their ratio is fixed as stated. Of interest is the generalized free energy per unit length

$$F_{\mathrm{ca}}(\nu, V) = \lim_{N \to \infty} -\frac{1}{\ell} \log Z_{\mathrm{ca},N}(\ell, V). \tag{14}$$

Physically even more important is density of states of the Lax matrix,

$$\varrho_{\mathrm{Q},N}(w) = \frac{1}{N}\sum_{j=1}^{N}\delta(\lambda_j - w), \tag{15}$$

where $\lambda_1, ..., \lambda_N$ are the real eigenvalues of L. Under a GGE, $\varrho_{\mathrm{Q},N}$ is a random measure on $\mathbb{R}$.

Using the transformation to scattering coordinates and the specific box potential

$$V_{\mathrm{box},\ell}(q) = \sum_{j=1}^{N} \mathrm{e}^{-\ell/2}\cosh(q_j), \tag{16}$$

we recently obtained the following two results.
(i) The infinite volume limit in (14) exists and is determined by a variational principle. We introduce the two-particle scattering shift of the Calogero fluid

$$\phi_{\mathrm{ca}}(p_1 - p_2) = -\log\Big(1 + \frac{1}{(p_1 - p_2)^2}\Big) \tag{17}$$

and the free energy functional

(18)

$$\mathcal{F}_{\mathrm{ca}}(\varrho) = \nu^{-1}\int_{\mathbb{R}} \mathrm{d}w\varrho(w)\Big(V(w) - 1 + \log\varrho(w) - \log\big(\nu + \int_{\mathbb{R}}\mathrm{d}w'\varrho(w')\phi_{\mathrm{ca}}(w - w')\big)\Big)$$

in case $\nu + \int_{\mathbb{R}}\mathrm{d}w'\varrho(w')\phi_{\mathrm{ca}}(w - w') > 0$ and $\mathcal{F}^{\circ}_{\mathrm{ca}}(\varrho) = \infty$ otherwise. Varying over all ϱ with $\varrho(w) \geq 0$ and $\int_{\mathbb{R}}\mathrm{d}w\varrho(w) = 1$, there is a unique minimizer, ϱ^*, and

$$F_{\mathrm{ca}}(\nu, V) = \mathcal{F}_{\mathrm{ca}}(\varrho^*). \tag{19}$$

(ii) When integrating (15) against smooth test functions, the density of states has a deterministic limit given by

$$\lim_{N\to\infty}\varrho_{\mathrm{Q},N}(w) = \varrho^*(w). \tag{20}$$

As an apparently rather general feature of integrable many-body systems the generalized free energy has a nonlinearity which involves the two-body scattering shift - an unusually close connection between free energy and dynamics. For non-integrable systems no such connection is available.

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Hydrodynamic limit of simple exclusion processes on point processes with random conductances

Alessandra Faggionato

We are interested in transport properties of disordered media. We consider a very large class of simple exclusion processes in random environments and show that the hydrodynamic limit is described by the heat equation with diffusion matrix given by the effective homogenized matrix of a single particle.

Let us define our model. Below $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space and the elements $\omega \in \Omega$ are called environments. The environment encodes all the microscopic randomness of the medium under consideration. We then fix a simple point process, i.e. a measurable map $\Omega \ni \omega \mapsto \hat{\omega} \in \{\text{locally finite subsets of } \mathbb{R}^d\}$ (see [3] also for the topology of the last space). We also fix a conductance field, i.e. a map $c : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \ni (\omega, x, y) \mapsto c_{x,y}(\omega) \in [0, +\infty)$ such that $c_{x,y}(\omega) = c_{y,x}(\omega)$. As it will be clear later, the relevant values of the conductance field are for $x \neq y$ in $\hat{\omega}$. At this point we can introduce the weighted undirected graph $\mathcal{G}(\omega)$ with vertex set $\hat{\omega}$, edge set $\{\, \{x, y\} \,:\, x \neq y \text{ in } \hat{\omega}\,,\ c_{x,y}(\omega) > 0 \,\}$ and weight of the edge $\{x, y\}$ given by $c_{x,y}(\omega)$, which is called the conductance of $\{x, y\}$ similarly to [2]. We assume that the abelian group $\mathbb{G} = \mathbb{R}^d$ or $\mathbb{G} = \mathbb{Z}^d$ acts both on the Euclidean space and on the probability space in a covariant way. To simplify the notation, we restrict here to the case $\mathbb{G} = \mathbb{R}^d$. The action $(\tau_g)_{g\in\mathbb{G}}$ of $\mathbb{G}$ on the Euclidean space is given by translations, and one can take $\tau_g x := x + g$. We denote by $(\theta_g)_{g\in\mathbb{G}}$ the action of $\mathbb{G}$ on the probability space. Roughly, $\theta_g\omega$ is the new environment when we apply the translation τ_{-g} on the Euclidean space and in particular on the medium. The covariant relation between the two actions is given by

$$\widehat{\theta_g\omega} = \tau_{-g}(\hat{\omega})\,,$$
$$c_{x,y}(\theta_g\omega) = c_{\tau_g x, \tau_g y}(\omega)\,.$$

We assume that the law $\mathcal{P}$ of the environment is stationary and ergodic for the action $(\theta_g)_{g\in\mathbb{G}}$. We call $\mathcal{P}_0$ the Palm distribution associated to $\mathcal{P}$. Roughly $\mathcal{P}_0 = \mathcal{P}(\cdot|0 \in \hat{\omega})$. We require some basic moment bounds:

$$\sum_{x\in\hat{\omega}:x\neq 0} c_{0,x}(\omega) \in L^1(\mathcal{P}_0)\,, \qquad \sum_{x\in\hat{\omega}:x\neq 0} c_{0,x}(\omega)|x|^2 \in L^1(\mathcal{P}_0)\,.$$

Although the above formalism can appear very abstract, it allows to describe several relevant models also with different geometrical features as explained in [1].

Given a realization of the environment ω, the particles of the exclusion process lie on the vertex set $\hat{\omega}$. In particular, the particle configuration is described by an element $\eta \in \{0, 1\}^{\hat{\omega}}$, where $\eta(x)$ is the occupation number at the vertex x. Particles can jump only along the edges of the graph $\mathcal{G}(\omega)$ with rates $c_{x,y}(\omega)$. More precisely, the infinitesimal generator is formally given by

$$\mathcal{L}_\omega f(\eta) = \sum_{x\in\hat{\omega}} \sum_{y\in\hat{\omega}} c_{x,y}(\omega)\eta(x)\big(1 - \eta(y)\big)\left[f(\eta^{x,y}) - f(\eta)\right]\,.$$

Equivalently,

$$\mathcal{L}_\omega f(\eta) = \sum_{\{x,y\}\subset\hat{\omega}} c_{x,y}(\omega)\big[f(\eta^{x,y}) - f(\eta)\big]\,.$$

Examples are given (cf. [1, 6]) by the symmetric exclusion process with random jump rates on $\mathbb{Z}^d$ (or another lattice), on the supercritical percolation cluster on $\mathbb{Z}^d$ (or another lattice), on the Delaunay triangulation defined as the dual graph of the Voronoi tessellation of $\hat{\omega}$, the symmetric exclusion process obtained by adding the exclusion interaction to the Mott variable range hopping.

To describe the hydrodynamic behavior of the above exclusion process we introduce the effective homogenized matrix as the $d\times d$ nonnegative symmetric matrix D such that, for all $a\in\mathbb{R}^d$,

$$a\cdot Da = \inf_{f\in L^\infty(\mathcal{P}_0)} \frac{1}{2}\int d\mathcal{P}_0(\omega)\sum_{x\in\hat{\omega}} c_{0,x}(\omega)\left(a\cdot x - \nabla f(\omega,x)\right)^2\,,$$

where $\nabla f(\omega,x) := f(\theta_x\omega) - f(\omega)$. In [5] we have proved that the above matrix describes indeed the homogenization of the massive Poisson equation for a single particle, hence performing a random walk on $\hat{\omega}$ with jump rates $c_{x,y}(\omega)$. We point out that D can be degenerate.

We can now describe our main result. Let $\mathcal{M}$ be the space of Radon measures on $\mathbb{R}^d$ with the vague topology. Let $(P_t)_{t\ge 0}$ be the Markov semigroup of the Brownian motion on $\mathbb{R}^d$ with diffusion matrix $2D$. Given $\mathfrak{m}\in \mathrm{Prob}(\{0,1\}^{\hat{\omega}})$, let $\mathbb{P}_{\mathfrak{m},\omega}$ be the law of the symmetric exclusion process on $\hat{\omega}$ with initial distribution $\mathfrak{m}$. Finally, given $\epsilon>0$ and $\eta\in\{0,1\}^{\hat{\omega}}$, let $\pi_\epsilon(\eta)$ be the empirical measure $\pi_\epsilon(\eta) := \epsilon^d\sum_{x\in\hat{\omega}}\eta(x)\delta_{\epsilon x}$.

Then, under few additional minor technical assumptions, we have:

Theorem 1 ([1]). *For $\mathcal{P}$–a.a. ω the following holds. Let $\rho_0:\mathbb{R}^d\to[0,1]$ be measurable and define $\rho:\mathbb{R}^d\times[0,\infty)\to[0,1]$ as $\rho(x,t) := P_t\rho_0(x)$. For $\epsilon>0$ fix $\mathfrak{m}_\epsilon\in \mathrm{Prob}\left(\{0,1\}^{\hat{\omega}}\right)$ such that, when $\eta\overset{\mathcal{L}}{\sim}\mathfrak{m}_\epsilon$, it holds $\pi_\epsilon(\eta)\overset{\epsilon\downarrow 0}{\to}\rho_0(x)dx$ in probability in $\mathcal{M}$. Then, for all $T>0$, when $(\eta_s)_{s\ge 0}\overset{\mathcal{L}}{\sim}\mathbb{P}_{\mathfrak{m}_\epsilon,\omega}$ we have*

$$\left(\pi_\epsilon(\eta_{\epsilon^{-2}t})\right)_{0\le t\le T}\overset{\epsilon\downarrow 0}{\to}\left(\rho(x,t)dx\right)_{0\le t\le T}\ \text{in probability}$$

in $D([0,T],\mathcal{M})$.

Above the symbol $A\overset{\mathcal{L}}{\sim}B$ means that A and B have the same law. In [1] and [6] applications of the above theorem are discussed to several important exclusion processes, also getting improved results for the hydrodynamics of the symmetric exclusion process with random jump rates on supercritical percolation clusters of $\mathbb{Z}^d$ (see [4] for previous results). The proof of the above theorem is based on homogenization and duality. While homogenization heavily relies on [5], duality can be treated by techniques initially introduced in [10] and improved in [4], or equivalently by means of the corrected empirical measure [7, 8]. We point out that, although for a fixed environment the system is non-gradient [9], the above methods allow to avoid the heavy machinery of non-gradient systems.

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Large deviations of the empirical current over long time intervals

Claudio Landim

(joint work with L. Bertini, D. Gabrielli)

Stochastic lattice gases, that describe the evolution of interacting random particles on a lattice of mesh $1/N$, have been an instrumental tool in the development of non-equilibrium statistical mechanics. Their macroscopic behavior, usually referred to as hydrodynamic scaling limit, is described as follows. Given a microscopic realization of the process, the empirical density $\boldsymbol{\pi}_N$ is defined by counting locally the average number of particles while the empirical current $\boldsymbol{J}_N$ is defined by counting the net flow of particles. By the local conservation of the number of particles, $\boldsymbol{\pi}_N$ and $\boldsymbol{J}_N$ satisfy the continuity equation. The content of the hydrodynamical limit is the law of large numbers for the pair $(\boldsymbol{\pi}_N, \boldsymbol{J}_N)$ in the limit $N \to \infty$. For driven-diffusive systems the limiting evolution is given by

$$\begin{cases} \partial_t \boldsymbol{\rho} + \nabla \cdot \boldsymbol{j} = 0, \\ \boldsymbol{j} = -D(\boldsymbol{\rho}) \nabla \boldsymbol{\rho} + \sigma(\boldsymbol{\rho}) E, \end{cases} \tag{1}$$

where $E = E(x)$ is the applied external field, D is the diffusion matrix, and σ is the mobility. In particular, the density profile $\boldsymbol{\rho} = \boldsymbol{\rho}(t, x)$ solves the non-linear driven diffusive equation

$$\partial_t \boldsymbol{\rho} + \nabla \cdot \big(\sigma(\boldsymbol{\rho}) E\big) = \nabla \cdot \big(D(\boldsymbol{\rho}) \nabla \boldsymbol{\rho}\big). \tag{2}$$

The large deviations with respect to the hydrodynamic limit in the time window $[0,T]$ are characterized by the rate function

$$A_T(\boldsymbol{\rho},\boldsymbol{j}) = \int_0^T dt \int dx \, \frac{\left|\boldsymbol{j} + D(\boldsymbol{\rho})\nabla\boldsymbol{\rho} - \sigma(\boldsymbol{\rho})E\right|^2}{4\,\sigma(\boldsymbol{\rho})}, \tag{3}$$

that is at the base of the Macroscopic Fluctuations Theory and it is widely used in non-equilibrium statistical mechanics.

A significant problem is the behavior of the average of empirical current over the time interval $[0,T]$ in the limit when $N \to \infty$ and then $T \to \infty$. By the hydrodynamical large deviations principle and contraction principle, this amounts to analyze the behavior as $T \to \infty$ of the minimizers to (3) with the constraint $\frac{1}{T}\int_0^T dt\,\boldsymbol{j} = J$. This problem has been initially raised in [4] while in [1] it has been pointed out that the minimizers can exhibit a non-trivial time dependent behavior. In [2, 5] it has been then shown that this is actually the case for the weakly asymmetric exclusion process and the Kipnis-Marchioro-Presutti model where, for suitable value of the parameters, traveling waves are more convenient than constant profiles.

Denote by $I^{(2)}(J)$ the limiting value as $T \to \infty$ of the minimum to $T^{-1}\,A_T$ with the constraint $\frac{1}{T}\int_0^T dt\,\boldsymbol{j} = J$. Varadhan [6] proposed the following representation for $I^{(2)}$

$$I^{(2)}(J) = \inf\left\{\int dP \int dx \, \frac{\left|\boldsymbol{j}(t) + D(\boldsymbol{\rho}(t))\nabla\boldsymbol{\rho}(t) - \sigma(\boldsymbol{\rho}(t))E\right|^2}{4\sigma(\boldsymbol{\rho}(t))};\ \int dP\,\boldsymbol{j}(t) = J\right\} \tag{4}$$

where the infimum is carried out over the probabilities P invariant by time translations on the set of paths $(\boldsymbol{\rho},\boldsymbol{j})$ satisfying the continuity equation $\partial_t\boldsymbol{\rho} + \nabla\cdot\boldsymbol{j} = 0$. Note that $I^{(2)}$ is convex and that, by the stationarity of P, the actual value of t on the right hand side of (4) is irrelevant.

The validity of the representation (4), in the context of the weakly asymmetric exclusion process for which $D = 1$ and $\sigma(\rho) = \rho(1-\rho)$, is the content of [3] and of the talk presented at this workshop. This is achieved both when the limit $T \to \infty$ is carried out after the hydrodynamic limit $N \to \infty$ and when the limits are carried out in the opposite order. In fact, the representation (4) is deduced by the contraction principle from a large deviation result at the level of the empirical processes.

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Discrete harmonic analysis on the configuration space of large scale interacting systems

Makiko Sasada
(joint work with Kenichi Bannai, Yukio Kametani)

The theory of the hydrodynamic behavior of interacting particle systems has been well developed in the last decades, and the hydrodynamic limits have been shown for various models. However, we still do not have a sufficient understanding of the diffusive scaling limit for non-gradient models. In the seminal paper [3], Varadhan introduced the so-called Varadhan's method to prove the hydrodynamic limit for non-gradient models and derived a variational formula for the macroscopic diffusion coefficient. Since then, the method has been applied to several models and a common expression of the variational formula in terms of the conserved quantity appeared repeatedly. Nevertheless, it was not certain how common the variational formula for the diffusion coefficient holds, in particular for models with multiple conserved quantities, because we did not know the origin of this universal structure.

To apply Varadhan's method for a specific model, we need so-called the decomposition theorem of shift-invariant closed forms, and the variational formula for the diffusion coefficient is a direct consequence of the decomposition theorem. In [1], we introduced a universal geometric structure, which we call a configuration space with transition structure, and proved a general decomposition of shift-invariant closed "uniform" forms in terms of a group action on the configuration space and conserved quantities. It reveals that the group action on the configuration space was the origin of the common structure. Varadhan's decomposition theorem can be understood as L^2 forms version, namely the Hodge decomposition ([2]). Moreover, we recently found that the macroscopic diffusion coefficient matrix is the inverse of the periodic matrix with respect to topological forms and appropriate harmonic forms. This relation between the diffusion matrix and the periodic matrix holds not only for interacting particle systems but also in the case of one-particle random walks and in homogenization problems. In my talk, I discussed this connection between the scaling limits for random processes and the group cohomology, as well as the period matrix. I also gave examples of interacting systems, whose decomposition theorem was newly established by our theory, such as exclusion processes/generalized exclusion processes with finite range jumps or on general crystal lattices, and multi-species exclusion processes.

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Universality of cutoff for exclusion with reservoirs

Justin Salez

1. Setup

Let V be a finite set, equipped with a collection of vertex rates $(\kappa_i)_{i\in V}$ and a symmetric collection of edge rates $(c(i,j))_{i,j\in V}$. Let also (S,ν) be a finite probability space. The $S-$colored exclusion process with reservoir law ν on the network $G=(V,c,\kappa)$ is the continuous-time Markov chain $(X_t)_{t\geq 0}$ on $\mathcal{S}^V$ where

- each pair of vertices $\{i,j\}$ exchange contents at rate $c(i,j)$;
- each vertex i resamples its content afresh according to ν at rate $\kappa(i)$.

The product measure $\pi=\otimes_{i\in V}\nu$ is clearly reversible under this dynamics, and we are here interested in quantifying the rate of convergence to equilibrium, as measured by the so-called worst-case total-variation distance:

$$\mathrm{d}_{\mathrm{TV}}(t) \quad := \quad \max_{x\in S^V}\max_{A\subseteq S^V} \left|\mathbb{P}_x(X_t\in A)-\pi(A)\right|.$$

In particular, we seek to estimate the time at which this function drops below a given precision $\varepsilon\in(0,1)$, known as the mixing time of the process:

$$\mathrm{t}_{\mathrm{MIX}}(\varepsilon) \quad := \quad \inf\left\{t\geq 0\colon \mathrm{d}_{\mathrm{TV}}(t)\leq\varepsilon\right\}.$$

2. Main result

Quite remarkably, the mixing properties of our high-dimensional interacting particle system turn out to be dictated by those of a much simpler object, namely the Laplacian matrix $\Delta\in\mathbb{R}^{V\times V}$ of the network G:

$$\Delta(i,j) \quad := \quad \begin{cases} c(i,j) & \text{if } j\neq i \\ -\kappa(i)-\sum_{k\in V}c(i,k) & \text{if } j=i. \end{cases}$$

In words, Δ is the generator of a killed random walk G which, when alive at a site $i\in V$, jumps to another site $j\in V$ at rate $c(i,j)$ and is killed at rate $\kappa(i)$. Set

$$\Psi(t) \quad := \quad \langle \mathbf{1}, e^{2t\Delta}\mathbf{1}\rangle,$$

which is just $|V|$ times the probability that the walk is still alive at time t, when starting from the uniform law. Our main result asserts that the function $t\mapsto \mathrm{d}_{\mathrm{TV}}(t)$ is controlled, in a universal way, by the single-particle statistics $t\mapsto\Psi(t)$.

Theorem 1 (Main reduction). *Writing $\nu_\star := \min_{s\in S} \nu(s)$, we have for all $t \geq 0$,*

$$\frac{\Psi(t)}{4+\Psi(t)} \;\leq\; \mathrm{d}_{\mathrm{TV}}(t) \;\leq\; \sqrt{e^{\frac{\Psi(t)}{\nu_\star}} - 1}.$$

The first inequality shows that our interacting particle system is far from equilibrium ($\mathrm{d}_{\mathrm{TV}}(t) \approx 1$) as long as $\Psi(t)$ is large. Conversely, the second inequality shows that the system is completely mixed ($\mathrm{d}_{\mathrm{TV}}(t) \approx 0$) as soon as $\Psi(t)$ is small. Thus, the mixing time is essentially the time at which $\Psi(t)$ is of order 1. This reduction has a number of important consequences, which we now enumerate.

3. Implications

The most fundamental parameter of a reversible Markov generator L is its *spectral gap* or Poincaré constant (second smallest eigenvalue of $-L$). Our first application is the following non-conservative analogue of Aldous' spectral gap conjecture, proved by Caputo, Liggett and Richthammer [2].

Corollary 2 (Spectral gap). *The spectral gap of the $S-$colored exclusion process on a network G coincides with the smallest eigenvalue λ of the matrix $-\Delta$.*

A second notable consequence of Theorem 1 concerns the so-called cutoff phenomenon, a remarkable but still mysterious phase transition in the convergence to equilibrium of certain chains [3]. Specifically, we say that a sequence of Markov chains (indexed by n) exhibits cutoff if the first-order $n \to \infty$ asymptotics of the mixing time $\mathrm{t}^{(n)}_{\mathrm{MIX}}(\varepsilon)$ is independent of the required precision $\varepsilon \in (0,1)$:

$$\forall \varepsilon, \varepsilon' \in (0,1), \qquad \frac{\mathrm{t}^{(n)}_{\mathrm{MIX}}(\varepsilon')}{\mathrm{t}^{(n)}_{\mathrm{MIX}}(\varepsilon)} \;\xrightarrow[n\to\infty]{}\; 1. \tag{1}$$

A simple necessary condition for cutoff is the so-called *product condition*

$$\lambda_n \times \mathrm{t}^{(n)}_{\mathrm{MIX}}(1/4) \;\xrightarrow[n\to\infty]{}\; +\infty, \tag{2}$$

where λ_n denotes the spectral gap of the chain. While this criterion is too simple to be sufficient in general, it has been shown to imply cutoff for all birth-and-death chains and, more generally, all random walks on trees [1]. Our next result adds the colored exclusion process with reservoir to this short list.

Corollary 3 (Characterization of cutoff). *Consider the $S-$colored exclusion process with reservoir law ν on an arbitrary sequence of networks $(G_n)_{n\geq 1}$. Then, the cutoff phenomenon (1) occurs if and only if the product condition (2) holds.*

Finally, our main estimate easily leads to explicit formulae for the spectral gap and mixing time of the $S-$colored exclusion process when the underlying Laplacian matrix Δ is simple enough to be amenable to Fourier analysis. We illustrate this by generalizing the one-dimensional results recently obtained in [4, 5].

Corollary 4 (Discrete Euclidean boxes)**.** *Fix an integer $d \geq 1$ and let G_n be the network obtained by restricting the lattice $\mathbb{Z}^d$ to the box $V = \{1, \dots, n\}^d$, with edge rates $c(i,j) = 1_{\|i-j\|=1}$ and vertex rates $\kappa(i) = \sum_{j \in \mathbb{Z}^d \setminus V} 1_{\|i-j\|=1}$. Then,*

$$\lambda_n = 2d \left[1 - \cos\left(\frac{\pi}{n+1}\right)\right], \quad \textit{and} \quad \mathrm{t}_{\mathrm{MIX}}^{(n)}(\varepsilon) = \frac{n^2 \log n}{2\pi^2} + O(n^2).$$

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Recent developments on the mixing of the Averaging process and its discrete dual

Matteo Quattropani, Federico Sau

We introduce the Averaging process as presented in the expository article [1], review some of its main properties, and present some recent quantitative results on its mixing times on various geometries.

The Averaging process is a Markovian model of mass redistribution among nearest-neighboring sites of a graph. Informally, the Averaging process may be described as follows: after initially assigning some real values to each site, at exponentially distributed times neighboring sites are selected and, then, split equally among themselves their total mass. The process takes place on a growing sequence of graphs which we assume to be finite-dimensional, in the sense that the random walk on those geometries satisfies a family of Nash inequalities. Along with the Averaging process, we introduce a related particle system, analyzed in detail in [4] and referred to as Binomial splitting process. The Binomial Splitting process is a natural particle analogue of the Averaging process, in which pairs of sites split particles rather than mass, according to a Binomial distribution. The Binomial Splitting process shares some features with some classical symmetric particle systems. Among these, the presence of dualities and intertwinings play a prominent role in our analysis. By means of such dual descriptions, on the one hand we derive sharp upper bounds for the Averaging process from properties of the Binomial Splitting with few particles; on the other hand, we also deduce results on the many-particle Binomial Splitting through the analysis of the Averaging dynamics.

Our main results amount to showing that the L^p convergence to equilibrium of the Averaging process occurs gradually, i.e., without exhibiting cutoff, on all finite dimensional geometries on the timescale of the relaxation time of the simple

random walk on the underlying graph. By means of a finer analysis of the Avearging process mixing behavior, we further establish total-variation distance cutoff for the Binomial splitting on such geometries, as soon as the number of particles diverges and is at most of the order of the size of the graph squared.

The works [2], [4], and [3], are the main references of our talks.

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Attractive probabilistic cellular automata

Ivailo Hartarsky

1. A classical example

Let us start by recalling a classical result on the contact process. The latter is a continuous time Markov process on $\{0,1\}^{\mathbb{Z}^d}$ defined by the following graphical construction (see [8, 9] for background). Each site $x \in \mathbb{Z}^d$ goes from 1 to 0 at rate 1 and from 0 to 1 at rate λ times the current number of nearest neighbours of x in state 1, where $\lambda \in [0,\infty]$ is the parameter of the model. We denote by $\eta_x(t)$ the state of the process at site x at time t and by $\mathbb{P}_\lambda^{\mathbf{1}}$ its law, starting from the all-1 configuration $\mathbf{1}$. The following was proved by Bezuidenhout and Grimmett in 1991 [2].

Theorem 1. *There exists* $\lambda_c \in [0,\infty]$ *such that*

$$\forall \lambda > \lambda_c, \qquad \lim_{t\to\infty} \mathbb{P}_\lambda^{\mathbf{1}}(\eta_0(t)=1) > 0,$$

$$\forall \lambda < \lambda_c, \exists C, c > 0, \forall t \geq 0, \qquad \mathbb{P}_\lambda^{\mathbf{1}}(\eta_0(t)=1) \leq Ce^{-ct}.$$

2. Setting and result

We seek to generalise Theorem 1. Our setting of interest is slightly different, as we will work in discrete time, but it is an interesting open problem to extend our results to the continuous time setting, so as to include Theorem 1. We consider the class of all attractive probabilistic cellular automata (PCA), which we define next.

Fix a *range* $R \in [0,\infty)$ and let $\Omega_R = \{0,1\}^{[-R,R]^d \cap \mathbb{Z}^d}$. An *up-set* $\mathcal{U} \subset \Omega_R$ is a set such that

$$\forall x \in U, \forall y \in \Omega_R, x \leq y \Rightarrow y \in \mathcal{U},$$

where $\leq$ holds pointwise. Let υ be a probability measure on the up-sets of Ω_R. Let $\mathcal{U}_{x,t}$ for $x \in \mathbb{Z}^d$ and $t \in \mathbb{N}$ be a family of i.i.d. random up-sets with law υ. We define the υ-PCA inductively, starting from an initial condition $\eta(0)$, by

$$\eta_x(t) = \begin{cases} 1 & \eta_{x+[-R,R]^d \cap \mathbb{Z}^d}(t-1) \in \mathcal{U}_{x,t}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathcal{U}_{x,t} = \emptyset$ corresponds to a *death* (the state becomes 0 regardless of everything else), while $\mathcal{U}_{x,t} = \Omega_R$ is a *birth*.

The main result of this talk is the following (see [6] for more generality).

Theorem 2. *Denote by $\mathbb{P}_p^{\mathbf{1}}$ the law of the $p\upsilon + (1-p)\delta_\emptyset$-PCA. There exists $p_c \in [0,1]$ such that*

$$\forall p > p_c, \qquad \lim_{t\to\infty} \mathbb{P}_p^{\mathbf{1}}(\eta_0(t) = 1) > 0,$$

$$\forall p < p_c, \exists C, c > 0, \forall t \geq 0, \qquad \mathbb{P}_p^{\mathbf{1}}(\eta_0(t) = 1) \leq Ce^{-ct}.$$

The proof applies the method of [3].

3. Further remarks

3.1. **Bootstrap percolation.** It turns out that when υ is a Dirac measure, there is a one-to-one correspondence between attractive PCA and a class of bootstrap percolation models (see [6]). When pushed via this correspondence, Theorem 2 yields an important exponetial decay property in this class. It remains open to show the exponential decay of the tail of the infection time above criticality for bootstrap percolation in full generality. By [5] this is equivalent to the positivity of the spectral gap above criticality of the corresponding kinetically constrained models.

3.2. **Non-trivial transition.** Let us note that for the contact process it is classical that $\lambda_c \in (0, \infty)$. Moreover, when υ is a Dirac measure on the up-set $\mathcal{U}$, there is a caracterisation of those $\mathcal{U}$ which yield a non-trivial p_c [11]. For more general, albeit not as complete results in this direction and in continuous time we refer to [4, 10].

3.3. **Supercritical and cooperative survival phases.** While Theorem 2 settles the subcritical regime, it tells us very little about the supercritical one. Bezuidenhout and Gray [1] studied general attractive PCA and their continuous time analogues in the "supercritical" regime. They showed that the phase transition is continuous and that one can renormalise, based on which one can obtain many results (see [7] for this program in a restricted case). However, the phase transition they studied is different, namely, they ask for a positive probability that the process is not absorbed in the $\mathbf{0}$ state starting from a single site in state 1. While for the contact process and, more generally, additive processes, the two transitions coincide, in general this is not the case. It is a wide open problem to study the behaviour of attractive PCA in the intermediate regime of cooperative survival, but individual death.

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Busemann process and infinite geodesics in the directed landscape

Ofer Busani

(joint work with Timo Seppäläinen, Evan Sorensen)

In Last Passage Percolation (LPP) on the lattice, to each site of $\mathbb{Z}^2$ we assign a random positive weight so that the weights are i.i.d. across the lattice. To any two points such that one is located north-east to the other, we look for an up-right path that maximizes the weight between the two points. Such paths are called geodesics and have been much studied in this model as well as other LPP models. One of the reasons for the interest in such objects is that they can be used as tools to study models in the KPZ universality class. An infinite geodesic in an LPP model is and infinite up right path such that its restrictions to between any to points on it is a geodesic. Since their introduction in the 90's by Newman and Hoffman, the Busemann function and the Busemann process, have been proven to be an important tool in the study of infinite geodesics in LPP as well.

The directed landscape, constructed in the breakthrough work of Dauvergen, Ortman and Virag 18', is believed to be the universal scaling limit of all metric-like (LPP, FPP etc.) models in the KPZ universality class. In a recent work, Rahman and Virag showed that infinite geodesics exist in the directed landscape as well. In this talk I will discuss the construction of the Busemann process on the directed

landscape and show how it can be used to obtain new results about the infinite geodesics in the directed landscape.

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Spectral Independence: A New Tool to Analyze Markov Chain Mixing Times

Kuikui Liu

(joint work with Dorna Abdolazimi, Nima Anari, Zongchen Chen, Shayan Oveis Gharan, Eric Vigoda, Cynthia Vinzant, June Vuong)

A central problem in the analysis of Markov chains for algorithmically sampling from Gibbs distributions of physical systems is estimating its *mixing time*. Classical techniques based on path coupling (and variants thereof) break down dramatically as the parameters of the distribution approach "phase transition points". We present a new and versatile technique for analyzing Markov chains called *spectral independence* which has broken these longstanding barriers, allowing us to give sharp mixing time bounds for many distributions of interest. Our notion is essentially a "limited correlations" property of the stationary distribution which is simple to describe, and is inspired by recent developments in the theory of *high-dimensional expanders*.

Definition 1 (Spectral Independence; Informal)**.** *Let μ be a probability distribution over configurations $\sigma \in \{-1,1\}^n$, and define the* ***influence matrix*** *$\Psi_\mu \in \mathbb{R}^{n\times n}$ by*

$$\Psi_\mu(i,j) = \Pr_{\sigma\sim\mu}[\sigma_j = 1 \mid \sigma_i = 1] - \Pr_{\sigma\sim\mu}[\sigma_j = 1 \mid \sigma_i = -1].$$

*For $\eta \geq 0$, we say μ is η-**spectrally independent** if the maximum eigenvalue of Ψ_μ is at most $1+\eta$.*[1]

Theorem 2 (Spectral Independence $\implies$ Fast Mixing; Informal)**.** *Let μ be a probability distribution over $\{-1,1\}^n$, and assume μ, as well as all conditional distributions of μ, are all η-spectrally independent for some $\eta \geq 0$. Then the Glauber dynamics for sampling from μ has mixing time at most $O(n^{2+\eta})$. If, in addition, μ satisfies the* global Markov property *(i.e. conditional independence) w.r.t. some underlying n-vertex graph of maximum degree $\Delta \leq O(1)$, then the Glauber dynamics mixes in $O_{\eta,\Delta}(n\log n)$ steps.*

[1] A priori, it isn't so clear that Ψ_μ even has real eigenvalues, since it is asymmetric in general, and has both positive and negative entries. However, one can prove this by showing it is an appropriate normalization of the *covariance matrix* of μ. There are also natural generalizations of this notion beyond the Boolean hypercube $\{-1,1\}^n$.

It is in general a nontrivial task to establish spectral independence, especially since one requires a bound on the influence matrix for all conditional distributions. However, the strength of this approach is that it has connections to a wide variety of different areas of mathematical research. In particular, there are now four well-known techniques for establishing spectral independence, all originating from different fields.

(1) Oppenheim's "Trickle-Down" Theorem: This result says that sufficiently strong spectral independence for conditional distributions $\mu \mid \tau$ over only partial configurations τ on $n-2$ many vertices imply spectral independence for all conditional distributions. This was first observed by Oppenheim in the study of expansion phenomenon in high-dimensional simplicial complexes, building on the works of Garland. This technique proved decisive in analyzing the random cluster model when $0 < q < 1$.
(2) Exponential Decay of Correlations/Spatial Mixing: This technique was pivotal to establish spectral independence for the hardcore gas model all the way up to the critical threshold, beyond which it is hard to sample from the Gibbs distribution under standard complexity-theoretic hypotheses.
(3) Geometry of Polynomials: Algebraic and analytic properties of the multivariate generating polynomial of μ such as zero-freeness/stability and log-concavity imply spectral independence bounds.
(4) Coupling: Finding contractive couplings for local dynamics, or constructing "local couplings" between conditional distributions, yield bounds on spectral independence.

Diffusive Fluctuations in Hard Rods System

Stefano Olla
(joint work with Pablo Ferrari)

Consider a system of one dimensional hard rods of variable length in the dynamics considered in [8]: when two rods collide they exchange positions. The equilibrium dynamics is constructed as follows. We start with $X^\varepsilon = (x, v, r)$ the Poisson process on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+$ with intensity $\varepsilon^{-1}\rho\, dx\, d\mu(v, r)$, with μ a positive probability measure on $\mathbb{R}^2$ with finite second moments. The usual hard rods case is given by $d\mu(v, r) = \delta_a(dr)d\mu(v)$, $a > 0$. We define

$$\begin{aligned} \sigma &= \rho \iint r d\mu(v, r), \qquad \text{volume density} \\ \pi &= \rho \iint rv d\mu(v, r), \qquad \text{momentum density.} \end{aligned} \tag{1}$$

and

$$m_a^b(X^\varepsilon) = \begin{cases} \sum_{(x,v,r)\in X^\varepsilon, x\in[a,b]} \varepsilon r & b > a \\ -\sum_{(x,v,r)\in X^\varepsilon, x\in[b,a]} \varepsilon r & b < a. \end{cases} \tag{2}$$

By the law of large numbers we have

$$m_a^b(X^\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} (b-a)\sigma, \qquad \text{a.s. as} \tag{3}$$

To each configuration X^ε there is a dilated configuration of the rods

$$Y^\varepsilon = \{(y = x + m_0^x(X^\varepsilon), v, r) : (x, v, r) \in X^\varepsilon\}$$

For a given test function $\varphi(y, v, r)$ we have the law of large numbers

$$\begin{aligned} \varepsilon \sum_{(y,v,r)\in Y^\varepsilon} r\varphi(y, v, r) &= \varepsilon \sum_{(x,v,r)\in X^\varepsilon} r\varphi(x + m_0^x(X^\varepsilon), v, r) \\ &\underset{\varepsilon\to 0}{\longrightarrow} \rho \iiint r\varphi(x(1+\sigma), v, r)dx\, d\mu(v, r) \\ &= \frac{\rho}{1+\sigma} \iiint r\varphi(y, v, r)dy\, d\mu(v, r) = \frac{1}{1+\sigma}\langle\varphi\rangle. \end{aligned} \tag{4}$$

i.e. $\bar\rho = \frac{\rho}{1+\sigma}$ is the density of the hard rods.

The fluctuation field for the rods is defined by

$$\xi^{Y,\varepsilon}(\varphi) = \varepsilon^{-1/2}\left[\varepsilon \sum_{(y,v,r)\in Y^\varepsilon} r\varphi(y, v, r) - \mathbb{E}\left(\varepsilon \sum_{(y,v,r)\in Y^\varepsilon} r\varphi(y, v, r)\right)\right]. \tag{5}$$

It is not hard to prove, using the underlying CLT of the Poisson process, that

$$\xi^{Y,\varepsilon}(\varphi) \underset{\varepsilon\to 0}{\overset{\text{law}}{\longrightarrow}} \xi^Y(\varphi) \tag{6}$$

where ξ^Y is the centered gaussian field with covariance

$$< \xi^Y(\varphi)\xi^Y(\psi) >= \bar\rho \iiint r^2 C\varphi(y, v, r)C\psi(y, v, r)dyd\mu(v, r). \tag{7}$$

where $C = I - \frac{\sigma}{1+\sigma}P$ and P is the *projection* operator

$$P\varphi(x) = \frac{\rho}{\sigma} \iint r\varphi(x, v', r')\, d\mu(v', r'). \tag{8}$$

In the dynamics that we consider, in the Euler scaling, the position at time t of the hard rod (y, v, r) that at initial time is at the position y that is the dilated image (wrt 0) of the point x is given by

$$y_t = x + m_0^x(X^\varepsilon) + vt + j_{X^\varepsilon}(x, v, t). \tag{9}$$

The flux $j_{X^\varepsilon}(x, v, t)$ is defined by

$$j_{X^\varepsilon}(x, v, t) = \varepsilon \sum_{(x',v',r')\in X^\varepsilon} r'\left(1_{[v'<v]}1_{[x<x'<x+(v-v')t]} - 1_{[v'>v]}1_{[x+(v-v')t<x'<x]}\right). \tag{10}$$

By the law of large numbers, for any tagged rod $(y, v, r) \in Y^\varepsilon$ we have that

$$(y_t - y) \underset{\varepsilon\to 0}{\longrightarrow} v^{\text{eff}}(v)t, \qquad \text{a.s.} \tag{11}$$

where the effective velocity is given by

$$v^{\text{eff}}(v) := v + \rho \iint r(v-w)d\mu(w,r) = v(1+\sigma) - \pi. \tag{12}$$

The fluctuation field in the Euler scaling is defined by

$$\xi_t^{Y,\varepsilon}(\varphi) = \varepsilon^{-1/2}\left[\varepsilon \sum_{(y,v,r)\in Y^\varepsilon} r\varphi(y_t,v,r) - \frac{1}{1+\sigma}\langle\varphi\rangle\right]. \tag{13}$$

It is proven in [3] that $\xi_t^{Y,\varepsilon}$ converges in law to

$$\xi_t^Y(\varphi) = \xi_0^Y(\varphi_t), \qquad \varphi_t(y,v,r) = \varphi(y + v^{\text{eff}}t, v, r). \tag{14}$$

i.e.

$$\partial_t \xi_t^Y(\varphi) = \xi_0^Y(v^{\text{eff}}\partial_x\varphi_t) = \xi_t^Y(v^{\text{eff}}\partial_x\varphi), \tag{15}$$

that in the hard rods with deterministic length correspond to the linerized equation of the Euler hydrodynamics proven in [1, 2]. Our work concerns the fluctuation field recentered on the effective velocities under a diffusive rescaling:

$$\Xi_t^{Y,\varepsilon}(\varphi) = \varepsilon^{-1/2}\left[\varepsilon \sum_{(y,v,r)\in Y^\varepsilon} r\varphi\left[y_{\varepsilon^{-1}t} - v^{\text{eff}}(v)\varepsilon^{-1}t, v, r\right] - \frac{1}{1+\sigma}\langle\varphi\rangle\right]. \tag{16}$$

We prove the following convergence in law:

$$\Xi_t^{Y,\varepsilon}(\varphi) \xrightarrow[\varepsilon\to 0]{\text{law}} \Xi_t^Y(\varphi) = \Xi^Y\left(\varphi(\cdot + \sqrt{\mathcal{D}}W_t)\right). \tag{17}$$

This means that an initial fluctuation of rods of velocity v, after *recentering* around the effective velocity, evolves in the diffusive scale by random rigid translations driven by a Brownian motion with diffusivity $\mathcal{D}(v)$ explicitely defined by

$$\mathcal{D}(v) = \rho \iint r^2|v-\bar{v}|d\mu(\bar{v},r). \tag{18}$$

In the case of the usual hard rods with deterministic length this coincide with the diffusivity that appears in the Navier-Stokes corrections of the hydrodynamics [4, 5] (see also more recent [7]). This rigidity in the evolution of the fluctuations in the diffusive scaling is in contrast with expected results for chaotic systems where fluctuation hydrodynamics predict an evolution driven by an additive space-time white noise [13]. On the other hand, in the case of the ususal hard rods with fixed size, it is in agreement with previous calculations of the space-time covariance [10, 12]. We expect this rigid evolution of the fluctuations in other completely integrable system such as the Ball-Box dynamics [6, 9] or the Toda lattice [14].

The basic argument behind the proof of (17) is that the fluxes $j_{X^\varepsilon}(x,v,\varepsilon^{-1}t)$ and $j_{X^\varepsilon}(\bar{x},v,\varepsilon^{-1}t)$ corresponding to two particles at initial macroscopic distance $x-\bar{x}$ are completely correlated in the limit as $\varepsilon\to 0$.

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A "quenched=annealed" approach in spin glasses

Erwin Bolthausen

Let Z_N be the random partition function of a random spin system, for instance

$$Z_N(\omega) := \sum_{\sigma} \exp\left[\beta H_{N,\omega}(\sigma)\right],$$

the σ running over some set of Ising spins, and the ω in $H_{N,\omega}$ indicating the random dependence of the Hamiltonian. N stands for the size of the system. The free energy is given as

$$f = \lim_{N\to\infty} \frac{1}{N} \log Z_N \overset{\text{often}}{=} \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_N.$$

$\mathbb{E}$ stands for the expectation with respect to randomness. The second equality is the so-called self-averaging, which has to be proved, but often follows easily from a concentration of measure argument.

In the best of all worlds $\mathbb{E}Z_N^2 \le \mathrm{e}^{o(N)} \left(\mathbb{E}Z_N\right)^2$. If one has a good concentration of measure property for Z_N, then

$$f = f^{\text{ann}} := \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E} Z_N,$$

and $\mathbb{E}Z_N$ is in many cases easy to evaluate. This however is true only in very special situations.

A more sophisticated method: Find σ-fields $\mathcal{F}_N$ such that

$$\mathbb{E}\left(Z_N^2|\mathcal{F}_N\right) \leq \mathrm{e}^{o(N)}\left(\mathbb{E}\left(Z_N|\mathcal{F}_N\right)\right)^2$$

and prove again a suitable concentration of measure property. Then

$$f = \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left(Z_N|\mathcal{F}_N\right). \tag{1}$$

Task 1. *Take $\mathcal{F}_N$ large enough that the estimate is correct, but small enough that you can compute $\mathbb{E}\left(Z_N|\mathcal{F}_N\right)$.*

This has recently been used in spin glasses in [2, 5, 6].

The following result for the so-called perceptron has recently been proved in [4] extending earlier results by Talagrand [8].

Let $u : \mathbb{R} \to [-\infty, \infty)$ be a measurable function which is bounded above and define

$$Z_N := \sum_{\sigma\in\{-1,1\}^N} \exp\left[\sum\nolimits_{j=1}^{[\alpha N]} u\left(y_j\right)\right],\ y_{\sigma,i} := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} J_{ij}\sigma_i,$$

where J_{ij} are i.i.d. standard Gaussian random variables. α is a positive constant where "small α" is equivalent to "high temperature".

The construction of the $\mathcal{F}_N$ uses Mézard's TAP equations (see [7]) for $m_i := \langle\sigma_i\rangle$: Define

$$F_q\left(x\right) := \frac{1}{\sqrt{1-q}} \frac{E_Z\left[Z\mathrm{e}^{u\left(x+\sqrt{1-q}Z\right)}\right]}{E_Z\left[\mathrm{e}^{u\left(x+\sqrt{1-q}Z\right)}\right]},\ q \in (0,1),$$

Z standard Gaussian, and then the fixed point equations for $q, \psi \in \mathbb{R}^+$

$$\begin{aligned} q &= E_Z \tanh^2\left(\sqrt{\psi}Z\right) \\ \psi &= \alpha E_Z\left[F_q\left(\sqrt{q}Z\right)\right]^2. \end{aligned}$$

Mézard's TAP equations are:

$$\begin{aligned} m_i &= \tanh\left(\sum\nolimits_{j=1}^{[\alpha N]} \frac{J_{ij}}{\sqrt{N}} n_j - \alpha E_Z F_q'\left(\sqrt{q}Z\right) m_i\right) \\ n_j &= F_q\left(\sum\nolimits_{i=1}^{N} \frac{J_{ij}}{\sqrt{N}} m_i - (1-q)\, n_k\right). \end{aligned}$$

For small α, they can iteratively be solved (see [1, 3]), leading to a sequence $m_i^{[k]},\ n_j^{[k]}$, generating the σ-fields

$$\mathcal{F}_N^{[k]} := \sigma\left(m_i^{[s]},\ n_j^{[s]} : s \leq k\right),$$

and if α is small enough, one gets under a weak additional condition the equation (1) in the following form

$$f = \lim_{k\to\infty} \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left(Z_N | \mathcal{F}_N^{[k]}\right).$$

With this method, we obtain in [4]

Theorem 2. *Assume*

$$\sup_{x\in\mathbb{R},\ c\in[1/2,2]} \frac{E_{Z,Z'}\left[(Z-Z')^2 \exp\left[u\left(x+cZ\right)+u\left(x+cZ'\right)\right]\right]}{E_{Z,Z'}\left[\exp\left[u\left(x+cZ\right)+u\left(x+cZ'\right)\right]\right]} < \infty.$$

Then for small enough $\alpha > 0$

$$\begin{aligned} f_{\alpha,u} &= \mathrm{RS}\left(\alpha,u\right) := -\frac{\psi\cdot(1-q)}{2} + E_Z\left(\log 2\cosh\sqrt{\psi}Z\right) \\ &\quad +\alpha E_Z \log E_{Z'} \exp\left[u\left(\sqrt{q}Z+\sqrt{1-q}Z'\right)\right]. \end{aligned}$$

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Sensitivity of mixing times for groups

Gady Kozma

(joint work with Jonathan Hermon)

For two metric spaces X and Y, we say that X and Y are K-quasi isometric if there exists a map $\varphi : X \to Y$ with the following two properties.

(1) For all $x, x' \in X$,

$$\frac{1}{K} d(x,x') - K \le d(\varphi(x),\varphi(x')) \le K d(x,x') + K.$$

(2) For every $y \in Y$ there exists an $x \in X$ such that

$$d(\varphi(x), y) \le K.$$

We discussed which properties of random walk are invariant to quasi isometries. We mentioned the result of Carlen, Kusuoka and Stroock [1] that polynomial upper bounds for the heat kernel are invariant, and the result of Terry Lyons [2] simplified by Benjamini [3] that the Liouville property is not.

We then defined the (total variation) mixing time of a finite, connected graph G,

$$t_{\text{mix}} := \max_x \min\{t : ||p_t(x,\cdot) - \pi||_{\text{TV}} < \tfrac{1}{4}\},$$

where $p_t(x,y)$ is the probability that a random walker starting at vertex x arrives at time t to vertex y, and where π is the stationary distribution. We mentioned results of Ding and Peres [4] and Hermon and Peres [5] which show that the mixing time is not invariant to quasi isometries, and the give the optimal amount by which it may vary.

Finally, we discussed a relatively new result of Hermon and the speaker [6]. The result is a construction of two sequence of finite Cayley graphs G_n and H_n which are uniformly quasi-isometric, namely for some K (in fact for $K = 3$) the graphs G_n and H_n are K-quasi isometric. But, on the other hand, the mixing times differ as follows,

$$t_{\text{mix}}(G_n) \geq \log\log\log|G_n| t_{\text{mix}}(H_n).$$

We concluded with two open problems. Is it possible to give such an example which, in addition, satisfies that the graphs G_n and H_n have uniformly bounded degrees? And can the rate $\log\log\log n$ be improved?

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The Arboreal Gas

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(joint work with Roland Bauerschmidt, Nick Crawford, Andrew Swan)

The *arboreal gas*, defined below, is a discrete probability model of random forests with a variety of motivations. These include being a model for gelation transitions of branched polymers [9]; being the $q \downarrow 0$ limit of the q-random cluster model [8]; and falling naturally into a class of models that act as toy models for the Anderson (metal-insulator) transition [3].

In this extended abstract we put these motivations aside in favour of contrasting the behaviour of the arboreal gas with that of the much better understood model of Bernoulli bond percolation.

We begin with bond percolation, which is a probability measure $\mathbb{Q}_p$ on subsets A of edges of a graph $G = (V, E)$. Fixing $p \in [0, 1]$, each $A \subset E$ is assigned probability $p^{|A|}(1-p)^{|E\setminus A|}$. Much of the fundamental behaviour of the model on $\mathbb{Z}^d$ can be summarized in a single equation: there is a $p_c = p_c(d)$ such that for $p \neq p_c$,

$$\mathbb{Q}_p[0 \leftrightarrow x] \approx (\theta_d(p))^2 + \exp\left(-c_{d,p}\|x\|\right) \quad \text{as } \|x\| \to \infty, \tag{1}$$

where $c_{d,p} > 0$ and $\theta_d(p) \equiv \mathbb{Q}_p[0 \leftrightarrow \infty]$ is strictly positive if $p > p_c$ and zero if $p < p_c$. The event $0 \leftrightarrow x$ means that vertices 0 and x are connected in the random subgraph (V, A), and $0 \leftrightarrow \infty$ indicates that 0 is contained in an infinite connected component. It is well-known that $0 < p_c(d) < 1$ for $d \geq 2$, while $p_c(1) = 1$. The informal use of $\approx$ in (1) hides some minor technicalities that are inessential here; for a precise formulation see [6].

The interpretation of (1) is that if $p < p_c$, then distant vertices are in separate components with high probability. Above p_c distant vertices are in the same component with essentially probability $(\theta_d(p))^2$: they independently have a chance of being in the unique giant component. The exponentially small correction indicates that components other than this giant are small, so it is unlikely two distant vertices both belong to such a component. The behaviour of the model precisely at the critical point p_c is more subtle and we will not discuss this further.

We now turn to the arboreal gas. For $\beta > 0$ the arboreal gas is the probability measure

$$\mathbb{P}_\beta[\cdot] = \mathbb{Q}_p[\cdot \mid (V, A) \text{ a forest}], \qquad p = \frac{\beta}{1+\beta}, \tag{2}$$

where we recall that a forest is a graph that contains no cycles. In plain words, $\mathbb{P}_\beta$ assigns a probability proportional to β^k to a forest that contains k edges. The conditioning in the definition (2) does not make sense on infinite graphs, but it is well-defined on finite graphs. Henceforth we will restrict to this finite setting and aim for estimates that are uniform in the size of the graph. From the definition (2) and the FKG inequality for $\mathbb{Q}_p$, one can deduce that the arboreal gas is stochastically dominated by bond percolation. This implies that for small β the arboreal gas is in a subcritical phase where all trees (connected components of the random forest) have vanishing density. Perhaps surprisingly, this is always the case in two dimensions in the following sense:

Theorem 1 ([2]). *For all $\beta > 0$ there is a $c_\beta > 0$ such that for all $\Lambda \subset \mathbb{Z}^2$,*

$$\mathbb{P}_\beta[0 \leftrightarrow x] \leq \|x\|^{-c_\beta}. \tag{3}$$

In particular, the density of the tree containing 0 is zero.

On the other hand, the arboreal gas does have a phase transition in $d \geq 3$. For $L \in \mathbb{N}$ let Λ_N^d be the d-dimensional torus $(\mathbb{Z}/L^N\mathbb{Z})^d$ of side-length L^N. We leave L implicit in the notation.

Theorem 2 ([1]). *Consider Λ_N^d with $d \geq 3$ and $L \geq L_0(d)$. If β is sufficiently large then there are constants $\kappa > 0$, $\theta_d(\beta) = 1 - O(1/\beta)$ and c_β of order one such that*

$$\mathbb{P}_\beta [0 \leftrightarrow x] = (\theta_d(\beta))^2 - \frac{c_\beta}{\beta \|x\|^{d-2}} + O(\frac{1}{\beta \|x\|^{d-2+\kappa}}) + O(\frac{1}{\beta L^{\kappa N}}). \tag{4}$$

The polynomial correction in (4) should be contrasted with the exponentially small correction in (1). Informally, this correction suggests that the arboreal gas is critical in the complement of the giant component(s). Rigorous statements in this direction on $\mathbb{Z}^d$ would be very interesting. Rather precise statements to this effect are known in the geometrically simpler settings of the complete graph [10, 11] and the wired regular tree [7, 12].

The proofs of Theorems 1 and 2 both rely on the fact that connection probabilities for the arboreal gas can be expressed in terms of correlation functions of the so-called $\mathbb{H}^{0|2}$ spin system [5, 2]. A key point is that the $\mathbb{H}^{0|2}$ spin system has a continuous (hyperbolic) symmetry; from the point of view Theorems 1 and 2 are natural. In particular, the proof of Theorem 1 is by a Mermin–Wagner type argument [2]. Non-rigorous renormalization group heuristics suggest that Theorem 1 is far from sharp, and that the truth is that the arboreal gas connection probabilities always decay exponentially in two dimensions [5]. The proof of Theorem 2 is via a rigorous renormalization group argument, using in part techniques developed by Brydges and Slade [1, 4].

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Existence of solution and localization for the stochastic heat equation with multiplicative Lévy white noise

Hubert Lacoin
(joint work with Quentin Berger and Carsten Chong)

We consider the following stochastic partial differential equation in $\mathbb{R}^d$

$$\partial_t u = \Delta u + \xi \cdot u$$

where the unknown u is a function of space and time. The operator Δ denotes the usual Laplacian in $\mathbb{R}^d$ and ξ is a space-time Lévy white noise. This equation has been extensively studied in the case where ξ is a Gaussian White noise. In that case, it is known that the equation is well posed only when the space dimension d is equal to one [1].

In our presentation, we consider the case where ξ is a Lévy white noise with no diffusive part and only positive jumps. We identify necessary and sufficient conditions on the Lévy measure λ associated with ξ for having existence and uniqueness of solutions to the equation. In dimension one and two the necessary condition and the sufficient one are the same while for $d \geq 3$ they differ only by a third order factor [2].

We further discuss the connection between the SHE and continuum directed polymer models, more precisely, how the solution of the SHE with α-stable noise can be obtained as the limit of a directed polymer with heavy tail [3, 4].

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log-Sobolev inequalities and renormalisation

Benoit Dagallier
(joint work with Roland Bauerschmidt)

The aim of this communication is to report on work started by Roland Bauerschmidt and Thierry Bodineau a few years ago, that I joined afterwards. Informally speaking, the main objective of this work is to quantify how fast the Langevin dynamics associated with a statistical field theory model converges to its invariant measure, in the continuum and/or large system-size limit. The key

aspect is that we are interested in models for which a certain renormalisation procedure is required to make sense of these limits. Quantification of the speed of convergence is done by establishing certain functional inequalities.

To make this informal description more precise, let me start with some notations. A statistical mechanics or field theory model is a probability measure $\mu^{\Lambda}_{A,V_0} = \mu_{A,V_0}$ defined on $\mathbb{R}^{\Lambda}$, with Λ a finite lattice, say $\Lambda = L\mathbb{T}^d \cap \mathbb{Z}^d$, $d \geq 1$, where $\mathbb{T} = [0,1)$ is the unit torus, and the side-length L is thought of as large. The measure reads:

$$\mu_{A,V_0}(d\varphi) \propto \exp\Big[-\frac{1}{2}\big(\varphi, A\varphi\big) - V_0(\varphi)\Big]. \tag{1}$$

Above, $A \in \mathbb{R}^{\Lambda\times\Lambda}$ is a positive semi-definite matrix, $(\varphi, A\varphi) = \sum_{x,y} \varphi_x A_{x,y}\varphi_y$ for $\varphi \in \mathbb{R}^{\Lambda}$ and $V_0 : \mathbb{R}^{\Lambda} \to \mathbb{R}$ is a single site potential, of the form:

$$V_0(\varphi) = \sum_{x\in\Lambda} V(\varphi_x), \qquad V : \mathbb{R} \to \mathbb{R}, \quad \varphi \in \mathbb{R}^{\Lambda}. \tag{2}$$

A typical example for A and V, corresponding to the lattice φ^4 theory, would be $A = -\Delta$, with Δ the lattice Laplacian given by:

$$(\Delta\varphi)_x = \sum_{\substack{y\in\Lambda \\ |y-x|=1}} \big(\varphi_y - \varphi_x\big). \tag{3}$$

The φ^4 potential reads:

$$V(z) = V(z) = \frac{\lambda}{4}z^4 + \frac{\mu}{2}z^2, \qquad z \in \mathbb{R}. \tag{4}$$

The Langevin dynamics associated with μ_{A,V_0} is the following stochastic partial differential equation (SPDE):

$$d\varphi_t = -\big(A\varphi_t + \nabla V_0(\varphi_t)\big)dt + \sqrt{2}dW_t, \tag{5}$$

where W is space-time white noise on $\mathbb{R}_+ \times \Lambda$. This SPDE admits μ_{A,V_0} as its only invariant reversible measure. The goal is then to understand how fast convergence takes place, and in particular how this speed depends on the size of the state space, parametrised by L. One way to quantify this speed of convergence is via a log-Sobolev inequality: the measure μ_{A,V_0} satisfies LSI(γ) if, for any sufficiently nice test function F:

$$\mathrm{Ent}_{\mu_{A,V_0}}[F^2] \leq \frac{2}{\gamma}\mathbb{E}_{\mu_{A,V_0}}\big[|\nabla F|^2\big], \tag{LSI(γ))}$$

where $\mathrm{Ent}_{\mu_{A,V_0}}[F^2] = \mathbb{E}_{\mu_{A,V_0}}[F^2 \log F^2] - \mathbb{E}_{\mu_{A,V_0}}[F^2]\log \mathbb{E}_{\mu_{A,V_0}}[F^2]$. One interest of proving (LSI(γ)) is that it implies an exponential decay of the entropy of the law $f_t d\mu_{A,V_0}$ of the dynamics at time $t \geq 0$:

$$\mathrm{Ent}_{\mu_{A,V_0}}(f_t) \leq e^{-\gamma t}\mathrm{Ent}_{\mu_{A,V_0}}(f_0). \tag{6}$$

In this sense, understanding how the speed of convergence of the Langevin dynamics depends on L amounts to asking the same thing about the parameter γ.

Traditional methods to bound γ rely on convexity considerations, following the seminal idea of Bakry and Emery [2]: if

$$\mathrm{Hess}\Big(\frac{1}{2}(\varphi, A\varphi) + V_0(\varphi)\Big) \geq c\,\mathrm{id}, \qquad c > 0, \tag{7}$$

then LSI(γ) holds with $\gamma = c$. Although very general and powerful, the bound (7) cannot be applied in many models of interest, such as the lattice φ^4 model (3)–(4). Indeed, it is expected that, for each $\lambda > 0$, there is a critical value $\mu_c(\lambda) < 0$ of the mass μ such that, for $\mu > \mu_c(\lambda)$, convergence of the dynamics to μ_{A,V_0} is fast, that is $\inf_L \gamma(L) > 0$. On the other hand, for $\mu < \mu_c(\lambda)$, relaxation is expected to be slow. Sufficiently far below the critical point, a Peierls-type argument should give $\gamma(L) \leq e^{-c(\lambda,\mu)L^{d-1}}$. However, the convexity-based Bakry-Emery criterion fails for any $\mu \leq 0$, since in that case:

$$\mathrm{Hess}\Big(\frac{1}{2}(\varphi, (-\Delta)\varphi) + \sum_x \Big[\frac{\lambda}{4}\varphi_x^4 + \frac{\mu}{2}\varphi_x^2\Big]\Big) \geq \mu\,\mathrm{id}. \tag{8}$$

One can understand this failure in two ways. First, the Bakry-Emery criterion requires convexity of the microscopic potential V_0. When L is large, one expects macroscopic details to not be crucial. A better requirement would then be to ask for the convexity of an effective potential, in which small scales have been integrated out. Secondly, and in a related manner, the Bakry-Emery criterion only involves the energy, while one generally thinks of phase transitions as determined by a competition between entropy and energy. The entropy should help, which is not visible here.

To remedy this situation, Bauerschmidt and Bodineau [3] combined the convexity criterion (7) with a renormalisation group procedure known as the Polchinski equation. Informally, the idea is to progressively integrate out small scales and obtain an effective measure at each scale that has nicer convexity property. Scales are defined through the choice of a covariance decomposition for $(C_t)_{t\in[0,\infty]}$ for A:

$$0 = C_0 \leq C_s \leq C_t \leq C_\infty = A^{-1}, \qquad 0 \leq s \leq t. \tag{9}$$

One can then interpolate between the full measure $\mu_{A,V} = \nu_0$ and a trivial measure $\nu_\infty = \delta_0$ by defining, for each scale $t \geq 0$, the renormalised measure at scale $t \geq 0$:

$$\nu_t(d\varphi) \propto \exp\Big[-\frac{1}{2}\big(\varphi, (C_\infty - C_t)^{-1}\varphi\big) - V_t(\varphi)\Big]d\varphi, \tag{10}$$

where the renormalised potential V_t is the central object:

$$\exp\big[-V_t(\varphi)\big] = \mathbf{E}_{C_t}\Big[\exp\big[-V_0(\varphi+\zeta)\big]\Big], \qquad \varphi \in \mathbb{R}^\Lambda, \tag{11}$$

with $\mathbf{E}_{C_t}$ the Gaussian measure with covariance C_t. This potential follows the so-called Polchinski equation:

$$\partial_t V_t = \frac{1}{2}\sum_{x,y} \dot{C}_t(x,y)\partial^2_{\varphi_x\varphi_y} V_t - \frac{1}{2}\big(\nabla V_t, \dot{C}_t \nabla V_t\big). \tag{12}$$

Above, $\dot{C}_t$ stands for the component-wise derivative of C_t. Combining the Bakry-Emery argument with the decomposition (ν_t), Bauerschmidt and Bodineau obtain

a multiscale log-Sobolev criterion ([3, Theorem 2.5]): if for each $t > 0$ there is a number $\dot{\ell}_t \in \mathbb{R}$ with:

$$\forall \varphi \in \mathbb{R}^\Lambda, \qquad \dot{C}_t \mathrm{Hess}\, V_t(\varphi) \dot{C}_t - \frac{\ddot{C}_t}{2} \geq \dot{\ell}_t \dot{C}_t, \tag{13}$$

then LSI(γ) holds with:

$$\frac{1}{\gamma} := |\dot{C}_0| \int_0^\infty \exp\Big[-2 \int_0^t \dot{\ell}_s \, ds \Big], \tag{14}$$

with $\dot{C}_0$ the largest eigenvalue of $\dot{C}_0$.

One can see from the above that the main difficulty when trying to use this multiscale criterion is to estimate the Hessian of the renormalised potential V_t, which also reads:

$$\mathrm{Hess}\, V_t(\varphi) = C_t^{-1} \Big(C_t - \mathrm{Cov}(\varphi) \Big) C_t^{-1}, \tag{15}$$

where $\mathrm{Cov}(\varphi)$ is the covariance matrix for the measure $\mu_{C_t, V_0}^{C_t^{-1}\varphi}$, defined as in (1) but with an external field $C_t^{-1}\varphi$. At present, we have no general method to do so. The miracle in the φ^4 case comes from a correlation inequality recently obtained by Ding, Song and Sun [1], which allows us to take $\dot{\ell}_t = \frac{1}{t} - \frac{\chi_t}{t^2}$ in (13), with $\chi_t = \sum_x (\mathrm{Cov}(0))_{0,x}$ the susceptibility at 0 field and parameters $\lambda, \mu + \frac{1}{t}$. Injecting this in the bound (14) of the log-Sobolev constant yields $\inf_L \gamma(L) > 0$ under the optimal condition that $\sup_L \chi_\infty < \infty$ through the following two observations (with χ_∞ the susceptibility of the original φ^4 model). First, for t small, $\chi_t \leq t/(t\mu + 1)$ (this is Brascamp-Lieb, as for t small the measure has mass $\mu + \frac{1}{t} > 0$ and is thus strictly convex). Secondly, the second Griffiths inequality implies that $\chi_t \leq \chi_\infty$ for each $t > 0$.

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Superdiffusivity and Weak Coupling: the Anisotropic KPZ equation

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(joint work with D. Erhard, F. Toninelli)

We present a novel approach to prove superdiffusivivity and weak coupling limits for critical systems at stationarity. In order to streamline the type of results which can be obtained with our methods, we focus on a specific example, namely an anisotropic version of the two-dimensional KPZ equation.

The KPZ equation is a stochastic PDE formally given by

$$\partial_t h = \tfrac{1}{2}\Delta h + \lambda\langle \nabla h, Q\nabla h\rangle + \xi\,, \tag{1}$$

where $h = h(t,x)$ for $t \geq 0$, $x \in \mathbb{T}^d$ (the d-dimensional unit torus), ξ is a space-time Gaussian white noise on $\mathbb{R}\times\mathbb{T}^d$, Q is a $d\times d$ symmetric matrix encoding how the growth mechanism depends on the slope and $\lambda \geq 0$ is the so-called *coupling constant*, tuning the strength of such dependence.

In the last years significant progress has been made in the study of the large-scale properties of h, at least in the *sub-* and *super-critical* regimes, corresponding to dimensions $d = 1$ and $d \geq 3$ respectively. In the first, it was shown that the solution h is *superdiffusive*, meaning that its fluctuations evolve non-trivially on time scales shorter than those of the stochastic heat equation (SHE), given by (1) with $\lambda = 0$. This can be quantified by means of the bulk diffusion coefficient $\mathrm{D}_{\mathrm{bulk}}$, which measures how the correlations of a process spread in space as a function of time and formally is such that the correlation length behaves like $\ell(t) \sim \sqrt{t \times \mathrm{D}_{\mathrm{bulk}}(t)}$[1]. For the 1-dimensional KPZ equation, $\mathrm{D}_{\mathrm{bulk}}(t)$ grows like $t^{1/3}$ for t large (see [1]), as opposed to SHE whose bulk diffusion coefficient is constant. Further, its fluctuations have been fully characterised in [16, 17]. In the supercritical regime instead, a recent series of works [15, 2, 14, 8] showed that, for Q the identity matrix and λ small enough, h is *diffusive* and its scaling limit is the solution of a SHE with *renormalised* coefficients.

Yet, the *critical* case $d = 2$ remains poorly understood. Formally, (1) is scale invariant under diffusive scaling and therefore it is expected that finer features of the equation, and in particular the nature of the slope dependence determined by the matrix Q, might qualitatively influence its properties. Wolf conjectured in [18] that its large scale behaviour will depend on the sign of $\det Q$ - in the Isotropic case, corresponding to $\det Q > 0$, $\mathrm{D}_{\mathrm{bulk}}(t) \sim t^{\beta}$ for some universal $\beta > 0$, while in the Anisotropic case, corresponding to $\det Q \leq 0$, $\mathrm{D}_{\mathrm{bulk}}(t) \sim t^{\beta}$ for $\beta = 0$.

The present paper focuses on the latter, and more specifically on the case of $Q = Q_{\mathrm{AKPZ}} = \mathrm{Diag}(1,-1)$. We will refer to (1) with Q_{AKPZ} as the Anisotropic KPZ (AKPZ) equation, which reads

$$\partial_t h = \tfrac{1}{2}\Delta h + \lambda\big((\partial_1 h)^2 - (\partial_2 h)^2\big) + \xi\,. \tag{2}$$

Let us stress that (2) is *critical* as $d = 2$ is the dimension at which Hairer's theory of Regularity Structures [12] and the other pathwise approaches break down for (1) and a local solution theory is not even expected to hold. One is therefore naturally led to first regularise the equation and consequently (try to) determine the large-scale properties of its solution as the regularisation is removed. Contrary to the folklore belief, in [5] we showed that (2) is not diffusive but *logarithmically superdiffusive*. Translating the result therein to the torus $\mathbb{T}^2$, we proved that for every $\lambda > 0$, the bulk diffusion coefficient $\mathrm{D}^N_{\mathrm{bulk}}$ of the solution h^N of (2) regularised

[1] see [5, eq. (1.6) and Appendix A] respectively, for the definition of $\mathrm{D}_{\mathrm{bulk}}$ and for a heuristic connecting the latter to $\ell(t)$

at level $N \in \mathbb{N}$ satisfies

$$\mathrm{D}^N_{\mathrm{bulk}}(t) \approx \sqrt{\log N}\,, \qquad \text{for } N \text{ large and } t \text{ fixed}, \tag{3}$$

where $\approx$ hides sub-dominant corrections, of the order of powers of $\log\log N$ and possibly depending on λ. We emphasise that the limit $N \to \infty$ corresponds to removing the regularisation.

As the linear part of (2) is clearly diffusive, the origin of the observed logarithmically superdiffusive behaviour must lie in the slope dependence. In order to find a regime in which the nonlinearity has a non-trivial but not divergent effect, it is therefore natural to tune the coupling constant $\lambda = \lambda_N$, which modulates the strength of the nonlinearity, together with the regularisation parameter N. In other words, the present paper is devoted to the study the AKPZ equation (2) under the *weak coupling scaling*, i.e. for $\hat{\lambda} > 0$ consider

$$\partial_t h^N = \tfrac{1}{2}\Delta h^N + \lambda_N \mathcal{N}^N[h^N] + \xi\,, \qquad \lambda_N \stackrel{\text{def}}{=} \frac{\hat{\lambda}}{\sqrt{\log N}} \tag{4}$$

where $\mathcal{N}^N$ is a regularisation of the nonlinearity in (2) at level N.

In [6], it was shown the following theorem in which the $N \to \infty$ limit of (4) has been derived and the choice of λ_N proved to be meaningful.

Theorem 1 (Theorems 1.3 and 1.4 [6])**.** *For any $\hat{\lambda} > 0$, the solution h^N of (4) under the weak coupling scaling converges in distribution in $C([0,T], \mathcal{D}'(\mathbb{T}^2))$ to h which solves*

$$\partial_t h = \tfrac{\nu_{\mathrm{eff}}}{2}\Delta h + \sqrt{\nu_{\mathrm{eff}}}\xi\,, \qquad \textit{for} \qquad \nu_{\mathrm{eff}} \stackrel{\text{def}}{=} \sqrt{2\frac{\hat{\lambda}^2}{\pi} + 1}\,, \tag{5}$$

where the effective diffusivity ν_{eff} is the limit for $N \to \infty$ of the bulk diffusion coefficient $\mathrm{D}^N_{\mathrm{bulk}}$ of h^N.

The previous theorem shows on the one hand that the solution h^N of (4) is *asymptotically diffusive* for *any* value of $\hat{\lambda}$ as the limit of $\mathrm{D}^N_{\mathrm{bulk}}$ is constant. On the other, since the effective bulk diffusion coefficient $\nu_{\mathrm{eff}} > 1$, it also proves that the nonlinearity, even though tuned down by a logarithmic factor, does not vanish but actually produces a new noise (and a new Laplacian) in the limit where the regularisation is removed.

The scaling regime in (4) and the phenomenon observed above have already appeared for the Isotropic case in [3, 7, 11]. In all of these works, the matrix Q is chosen to be the identity matrix so that the nonlinearity in (1) becomes $|\nabla h|^2$. There is a major difference between the AKPZ and IKPZ (in any dimension). Indeed, the latter can be *linearised* via a nonlinear transformation, the so-called Cole-Hopf transform, which turns IKPZ into the linear stochastic heat equation with multiplicative noise. Once in possession of a linear SPDE, it is then possible to obtain an explicit representation of its solution thanks to the Feynman-Kac formula and therefore reduce the analysis of IKPZ to a problem of directed polymers in random environment. For the AKPZ equation there exists neither a transformation that linearises the equation nor (up to the authors' knowledge) an explicit

representation of its solution, so that we need to resort to a completely alternative set of tools. What we can and will exploit is that, as shown in [4], (4) admits an invariant measure which is a Gaussian Free Field χ. Our approach, partly inspired by [10], is based on a thorough analysis of the action of the generator $\mathcal{L}^N$ of the solution of (4) on the space $L^2(\chi)$ of square integrable functions with respect to χ. The idea is that the distribution of the solution of (5) is fully determined by a number of observables which is a (very small!) subset S of $L^2(\chi)$. Loosely speaking, our goal is therefore to identify a (N-dependent) subset S^N of $L^2(\chi)$ big enough to be able to characterise the fluctuations of h^N and such that for each $b \in S$ there exists $b^N \in S^N$ for which both b is well-approximated by b^N and the action of the generator of (5) on b is well-approximated by $\mathcal{L}^N b^N$. As we will see, while S admits an easy description, the choice of S^N is rather subtle and $S^N \cap S = \emptyset$! In other words, the structure of the b^N's needs to be sufficiently rich to be able to capture the roughness of the nonlinearity which is encoded in $\mathcal{L}^N$.

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$\sqrt{\log t}$-superdiffusivity for a Brownian particle in the curl of the 2d GFF

Levi Haunschmid-Sibitz
(joint work with Giuseppe Cannizzaro, Fabio Toninelli)

In the recent article [2], we showed that the mean square displacement of a Brownian particle diffusing in a field given by the curl of the two-dimensional Gaussian Free Field is of order $t\sqrt{\log t}$, which was conjectured by B. Toth and B. Valko in [3]. There are a number of two-dimensional models for which this behaviour has been conjectured based on heuristics from [1], but, to the best of the authors' knowledge, this is the first model for which it has been rigorously established. The talk introduces the model and gives a sketch of the method, which is inspired by the method used by H.-T. Yau [4] to proof $(\log t)^{\frac{2}{3}}$ superdiffusivity for 2d ASEP.

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Brownian particle in the curl of 2-d stochastic heat equations

Guilherme L. Feltes
(joint work with Hendrik Weber)

We study the long time behaviour of a Brownian particle evolving in a dynamic random environment. Recently, Toninelli et al. [2] proved sharp $\sqrt{\log}$-super diffusive bounds for a Brownian particle in the curl of the 2-d Gaussian Free Field (GFF). Adapting their method, we show that if the environment evolves according to the stochastic heat equation, then the particle stays super diffusive, whereas if a fractional stochastic heat equation is considered, then the particle becomes diffusive. Our results agree with the Alder-Wainwright scaling argument (see [1]) used originally in [4] to predict the log-corrections to diffusivity.

Precisely, we consider $(X_t)_{t\geq 0}$ the solution to the Itô SDE

$$\begin{cases} dX_t = \omega_t(X_t)dt + \sqrt{2}dB_t, \quad t \geq 0, \\ X_0 = 0\,, \end{cases}$$

where $(B_t)_{t\geq 0}$ is a standard two-dimensional Brownian motion and $(\omega_t(x))_{t\geq 0, x\in\mathbb{R}^2}$ is a time-dependent random field which is independent from $(B_t)_{t\geq 0}$. the coordinates of $\omega = (\omega^1, \omega^2)$ satisfy

$$\begin{cases} d\omega_t^k = -(-\Delta)^s \omega_t^k dt + \sqrt{2}\partial_k^\perp (-\Delta)^{\frac{s-1}{2}} dW_t \ , \ t \geq 0 \ \ , \ \ k = 1, 2 \ \ , \\ \omega_0^k = \partial_k^\perp \eta \ \ , k = 1, 2 \ \ , \end{cases}$$

where $s \in [0, 1]$ and $\nabla^\perp = (\partial_1^\perp, \partial_2^\perp) := (\partial_{x_2}, -\partial_{x_1})$. Here, W is a mollified (in space) space-time white noise and $\nabla^\perp \eta$ is distributed according to the law of the curl of a mollified GFF. The environmental process $(\omega_t)_{t\geq 0}$ is Markovian, Gaussian and leaves the law of $\nabla^\perp \eta$ invariant, for every $s \in [0, 1]$. The parameter $s \in [0, 1]$ controls the speed of the environment on different scales: smaller values of s correspond to faster movement of the larger scales. The case $s = 1$ corresponds to the standard stochastic heat equation (SHE), whereas $s = 0$ is the infinite dimensional Ornstein Uhlenbeck process (O-U).

Our main theorem [3] shows that if $s = 1$ (SHE), then, (up to $\log\log$ corrections),

$$t(\log t)^{\frac{1}{8}} \lesssim \mathbb{E}[|X_t|^2] \lesssim t(\log t)^{\frac{7}{8}} \ . \tag{1}$$

And if $s \in [0, 1)$ (fractional SHE) + (O-U), then

$$\mathbb{E}[|X_t|^2] \approx t \ .$$

Both results obtained in a Tauberian sense, i.e., in terms of the Laplace transform of the mean square displacement $D(\lambda) := \int_0^\infty e^{-\lambda t}\mathbb{E}[|X(t)|^2]dt$, for $\lambda > 0$.

The scaling argument of [1] for 2-d for isotropic drift fields says that if we assume

$$\mathbf{E}[\omega_0(0)\omega_t(x)] \approx \beta(t)^{-2}\psi(\beta(t)^{-1}x) \text{ and } \mathbb{E}[|X_t|^2] \approx t^{2\nu}(\log t)^{2\gamma} \ \ ,$$

then $\frac{\beta(t)}{t^\nu (\log t)^\gamma} \leq C \ , \ t \geq 0 \Rightarrow \nu = \frac{1}{2}$ and $\gamma = \frac{1}{4}$. In other words, if the environment moves slower than the particle, one should expect a $\sqrt{\log t}$ correction to diffusivity. Note that for $s \in (0, 1]$, $\beta(t) = t^{\frac{1}{2s}}$, and the condition $\frac{t^{\frac{1}{2s}}}{t^{\frac{1}{2}}(\log t)^{\frac{1}{4}}} \leq C$ is only satisfied for $s = 1$, so our result agrees with this argument.

The proof of the sharp estimates obtained in [2] is based on Yau's method [5] of recursive estimates of iterative truncations of a resolvent equation. The suboptimal bounds in (1) are an effect that in the current proof we have, in contrast to [2], the dynamics of the environment (for $s = 1$) introduces an extra factor 2 (and 1/2) in the recursive estimates that propagates to every level, since the lower and upper bounds interact at every level. The exponent 1/8 in (1) is the sharp 1/2 corrected by the square of the extra factor 1/2. Obtaining the sharp exponent 1/2 for the case $s = 1$ is still work in progress.

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2D Lorentz gas for magnetotransport

Chiara Saffirio
(joint work with Alessia Nota, Sergio Simonella)

We consider a point particle moving in a random distribution of identical hard disks of radius $\varepsilon > 0$ under the action of a uniform, constant magnetic field $\mathbf{b} = (0, 0, B)$ orthogonal to the plane $\mathbb{R}^2$. The point particle with position $x \in \mathbb{R}^2$ and velocity $v \in \mathbb{R}^2$ moves according to Newton's law under the action of the Lorentz force $F(v) = v \times \mathbf{b}$. Without loss of generality we can therefore consider $v \in S^1$. When the test particle encounters an obstacle, a collision takes place modifying its velocity according to the scattering rule

$$v' = v - 2(v \cdot n)n$$

where $n \in S^1$ is the scattering vector.

We consider the case of Poisson distributed scatterers in the low-density regime, i.e. given $\varepsilon > 0$ small, the intensity is chosen to be proportional to ε^{-1}.

For $t > 0$, let

$$\Pi_t = \bigcup_{m \geq 0} \Pi_{t,m}$$

the path space of the particle in $[0, t]$. For $m = 0$, $\Pi_{t,0}$ denotes the circling path space (this happens when the point particle does not encounter any obstacle on its cyclotron orbit). For $m \geq 1$, $\Pi_{t,m}$ is the m-path space defined by the collection of ordered impact times and impact vectors with new obstacles encountered by the point particle.

On Π_t we define a path measure $P_{\varepsilon,t}^{(x,v)}$ induced by the Poisson distribution under the condition that the test particle does not start from an obstacle at the initial configuration $(x, v) \in \mathbb{R}^2 \times S^1$.

Let $\{\zeta^\varepsilon(s)\}_{s\in[0,t]}$ be the random trajectory starting from (x, v) and let refer to it as the Lorentz process. Let $\{\zeta(s)\}_{s\in[0,t]}$ be the generalized Boltzmann process starting from (x, v) with forward equation

$$\begin{aligned} &(\partial_t + v \cdot \nabla_x - (v \times \mathbf{b}) \cdot \nabla_v) f(t, x, v) \\ &\qquad = \sum_{k=0}^{[t/T]} e^{-2kT} \int_{S^1} (v \cdot n)_+ [\sigma_n - 1] f(t - kT, S_n^{(k)}(x, v)) \end{aligned} \tag{1}$$

where T is the Larmor time, $[\,\cdot\,]$ denotes the integer part, $[t/T]$ is the number of cyclotron periods T completed before time $t > 0$, $(\,\cdot\,)_+$ denotes the positive part,

$n \in S^1$ is the scattering vector, σ_n is the operator implementing the change of velocity in the scattering process and $S_n^{(k)}$ acts as $S_n^{(k)}(x,v) = (x, R_{k\theta}(v))$ with $R_{k\theta}$ the $k\theta$-rotation and θ the scattering angle.

In this setting Gallavotti [2] showed that if $\mathbf{b} = 0$ the Lorentz process converges towards the Boltzmann process with forward equation the linear Boltzmann equation as $\varepsilon \to 0^+$, that is equation (1) for k=0. Bobylev et al. observed that when $\mathbf{b} \neq 0$ the probability of circling orbits is non zero and they proposed equation (1) as limiting kinetic equation describing a non-Markovian process.

We address the rigorous derivation of the generalized Boltzmann equation providing the convergence of the particle process to the non-Markovian process governed by equation (1). More precisely,

Theorem 1. *For all $t > 0$, the Lorentz process $\{\zeta^\varepsilon(s)\}_{s\in[0,t]}$ converges as $\varepsilon \to 0^+$ to the generalized Boltzmann process $\{\zeta(s)\}_{s\in[0,t]}$ with forward equation* (1)*, in the sense of weak convergence of path measures on the Skorokhod space $D([0,t], \mathbb{R}^2 \times S^1)$.*

The proof is based on coupling the Lorentz process with the generalized Boltzmann process through a parametrization of the process in terms of scattering times and scattering vectors, as introduced by Gallavotti in [2]. The new difficulty with respect to Gallavotti's work is that we have to deal with memory terms which survive in the limit $\varepsilon \to 0^+$. The coupling method needs therefore more additional care to identify self-recolliding trajectories that survive in the limit. The key idea is to exploit the fact that there is an underlying Markovian process given by the sequence of path segments between two different obstacles. This is indeed the rationale of the definition of the path space Π_t.

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Cutoff for permuted Markov chains

Anna Ben-Hamou
(joint work with Yuval Peres)

Let Ω be a finite set of size $n \geq 1$. When uniform sampling on Ω is not directly feasible, it is often the case that there is a natural Markov kernel P on Ω, whose transitions are easy to simulate, and which converges to the uniform distribution on Ω. To quantify the speed of this convergence, a commonly used distance is the worst-case total-variation distance:

$$\mathcal{D}(t) = \max_{x\in\Omega} \max_{A\subset\Omega} \left\{ P^t(x,A) - \frac{|A|}{n} \right\},$$

and the mixing time is then defined as

$$t_{\mathrm{mix}}(\varepsilon) = \inf\{t \geq 0\,,\, \mathcal{D}(t) \leq \varepsilon\}\,,$$

for $\varepsilon \in (0,1)$.

In many situations, the natural kernel P is very slow to mix, and it is tempting to try and speed it up. Here we focused on a model which has been formalized by Chatterjee and Diaconis [1] and which consists in inserting deterministic jumps in the trajectory.

More precisely, let $\pi : \Omega \to \Omega$ a permutation on Ω, and let Π be the associated permutation matrix with entries $\Pi_{i,j} = \mathbf{1}_{\pi(i)=j}$. The permuted kernel is then given by $Q = P\Pi$. Note that since P as uniform stationary distribution, so does Q.

For instance, consider the lazy random walk on the circle $\mathbb{Z}_n$, for n prime, with uniform jumps on $\{-1, 0, 1\}$. The mixing time is of order n^2, which is very slow. Chung, Diaconis and Graham [2] considered the permuted version of this chain with $\pi(x) = 2x$, and established an upper bound of order $\log n$. Cutoff was then established by Eberhard and Varjú [3]. Another example of permutation on the circle which leads to such a speed-up is $\pi(x) = x^{-1}$ (where x^{-1} is the inverse in the fields $\mathbb{F}_n$), see [13].

In a more general setting, can we find simple conditions on the permutation π to guarantee a logarithmic mixing time?

In [1], it is shown that, under mild assumptions on the initial kernel P, if the permutation π satisfies some expansion condition with respect to the kernel P, then the mixing time of the permuted kernel $Q = P\Pi$ is logarithmic in n, and that this expansion condition is satisfied by almost all permutations (but interestingly, it is not satisfied by the Chung–Diaconis–Graham mentioned above).

In this talk, we presented a refinement of this result obtained in [4]: when the permutation π is chosen uniformly at random, then the mixing time can be characterized even more precisely: with high probability, there is cutoff at time $\frac{\log n}{\mathbf{h}}$, where $\mathbf{h}$ is the entropic rate of P:

$$\mathbf{h} = \frac{1}{n} \sum_{x,y \in \Omega} P(x,y) \log \frac{1}{P(x,y)}\,\cdot$$

This result fits into the more general topic of cutoff for random instances of Markov chains, see [5, 7, 6, 8, 9, 12, 11, 10].

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Global-in-time fluctuations of the empirical measure at low density

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(joint work with Thierry Bodineau, Isabelle Gallagher, Laure Saint-Raymond)

In this note we report a recent result on the large scale limit problem of a deterministic dynamics with random initial data. For such a problem the number of situations is limited, in which mathematical results can be proved; especially if we require our model to live in the physical - say, three-dimensional - space. We focus here on a traditional case: the classical gas at low density.

To fix ideas, we shall consider an overidealized system. Hard spheres of radius $\varepsilon/2$ are moving in $\mathbb{T}^3$, the unit 3-dimensional periodic box. All the particle variables are random variables, to be sampled at time zero according to a probability measure $\mathbb{P}_\varepsilon$: the positions $(x_i)_{i=1}^N$, the velocities $(v_i)_{i=1}^N$, and the total number of particles N. They are therefore random variables at time t, uniquely determined by the hard-sphere dynamics. The hard spheres move freely and interact at distance $\varepsilon > 0$, by updating their velocities instantaneously according to the conservation laws: $(v_i, v_j) \to (v_i', v_j')$ with $v_i' = v_i - \omega[\omega \cdot (v_i - v_j)]$, $v_j' = v_j + \omega[\omega \cdot (v_i - v_j)]$, where $\omega = (x_j - x_i)/\varepsilon \in \mathbb{S}^2$ is the scattering vector.

We can now choose a scaling ensuring that the mean free path between collisions is of the same order of magnitude of the macroscopic length (say, the size of the box). This scaling is known as Boltzmann-Grad limit (see [9]) and it amounts to the prescription: $\varepsilon \to 0$, $\frac{\mathbb{E}_\varepsilon[\mathcal{N}]}{\mu_\varepsilon} \sim 1$ where $\mathbb{E}_\varepsilon[\mathcal{N}]$ is the average number of particles and $\mu_\varepsilon := \varepsilon^{-2}$. The milestone is the work by Lanford [11] proving the law of large numbers for the empirical measure, under some reasonable assumption on the initial data. In our setting, denoting by $\left(\mathbf{x}_i^{\varepsilon 0}, \mathbf{v}_i^{\varepsilon 0}\right)_{i=1}^{\mathcal{N}} \longrightarrow (\mathbf{x}_i^\varepsilon(t), \mathbf{v}_i^\varepsilon(t))_{i=1}^{\mathcal{N}}$, $t \in \mathbb{R}$ the hard-sphere flow with initial (random) configuration $\left(\mathbf{x}_i^{\varepsilon 0}, \mathbf{v}_i^{\varepsilon 0}\right)_i$, we have that

$$\pi_t^\varepsilon(\varphi) := \frac{1}{\mu_\varepsilon} \sum_{i=1}^{\mathcal{N}} \varphi(\mathbf{x}_i^\varepsilon(t), \mathbf{v}_i^\varepsilon(t)) \longrightarrow \int_{\mathbb{T}^3\times\mathbb{R}^3} f(t)\varphi \,, \quad t \in [0,T],\ \varphi \in C_b^0$$

as $\varepsilon \to 0$, where f solves the Boltzmann equation $(\partial_t + v\cdot\nabla_x) f = Q(f,f)$.

This, and subsequent results in the field, are restricted to short times $t \le T$, where T is about one fifth of the mean free scattering time [15]. Technically, this limitation arises from the method (based on the traditional BBGKY hierarchy of equations for correlation functions) which is a perturbation theory around free transport. Although, for well prepared initial data, one expects to observe a fast convergence to global equilibrium for both π_t^ε and $f(t)$, the methods at disposal suffer of, and are unable to exclude, the development in time of local singularities.

Our purpose is to tackle the problem of the short time limitation in a simplified context. We propose to study a different class of observables, for which the existence (and explicit knowledge) of an invariant measure can be exploited more directly. We investigate the fluctuations of an equilibrium state

$$\xi_t^\varepsilon(\varphi) := \frac{1}{\varepsilon}\left(\pi_t^\varepsilon(\varphi) - \mathbb{E}_\varepsilon\left[\pi_t^\varepsilon(\varphi)\right]\right) \,, \quad t \ge 0$$

where the expectation is taken with respect to the grand canonical Gibbs measure with hard sphere potential. This measure is defined by a collection of density distributions $\{W_N^{\varepsilon,\mathrm{eq}} : \mathcal{D}_N^\varepsilon \to \mathbb{R}^+ \,;\ N \ge 0\}$ living on the accessible phase space $\mathcal{D}_N^\varepsilon := \left\{(x_i,v_i)_i \in \left(\mathbb{T}^3\times\mathbb{R}^3\right)^N \mid \forall i \ne j \,,\ |x_i - x_j| > \varepsilon\right\}$, where $W_N^{\varepsilon,\mathrm{eq}} := \mathcal{Z}_\varepsilon^{-1}\,\mu_\varepsilon^N\, M^{\otimes N}$ and $M(v) := (2\pi)^{-3/2}\exp\left(-v^2/2\right)$, with $\mathcal{Z}_\varepsilon$ the grand canonical partition function. Notice that $\lim_{\varepsilon\to 0}\mathbb{E}_\varepsilon\left[\mathcal{N}\right]/\mu_\varepsilon = 1$ according to the Boltzmann-Grad scaling condition. Moreover, as $\varepsilon \to 0$ one finds that $\pi_t^\varepsilon(\varphi) \longrightarrow \int M\varphi$ for all times $t \in \mathbb{R}$: the law of large numbers becomes then very simple providing an ideal gas with velocity distribution M.

Before stating the result, let us compare the problem with situations studied previously. There are two cases in which the time restriction has been lifted. The first one is a small cloud of gas expanding in the whole space [10, 8], which corresponds to a stringent assumption on the initial data ensuring that, after a finite number of collisions, each particle is free to move to infinity. In this case, the Boltzmann-Grad limit can be worked out globally in time, and the key ingredient in the proof is the dispersivity of the free flow. The second case (more related to ours) is the tracer particle in the equilibrium gas [1, 2]: one assumes that the hard sphere gas is initially distributed according to the Gibbs equilibrium measure, conditioned on particle 1 being somewhere with certain probability. Following the evolution of particle 1 only, it is then possible to prove that its distribution is driven by a linear Boltzmann equation for all times (actually as shown in [2], up to diffusive times). In both cases mentioned above, the densities are bounded a priori in strong, uniform (L^∞) norms, and Lanford's perturbative method can be implemented (with some extra care) on time intervals of arbitrary length. The study of the fluctuation field $(\xi_t^\varepsilon)_{t\in\mathbb{R}^+}$ is more subtle: in spite of the proximity to

the invariant measure, there is no simple a priori mechanism telling us that the moment densities stay well bounded in a uniform norm, globally in time.

We state our result next. Following the general fluctuation theory, as reported in [14], we introduce the linearized collision operator (around $f = M(1+g)$)

$$\mathcal{L}(g)(x,v_1) = -v_1\cdot\nabla_x g + \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} [\omega\cdot(v_1-v_2)]_+ M(v_2)\ \Delta g(x,v_1,v_2,\omega)\,d\omega\,dv_2$$

where $\Delta g(x,v_1,v_2,\omega) := g(x,v_1') + g(x,v_2') - g(x,v_1) - g(x,v_2)$. Furthermore we define the Gaussian noise $d\eta_t$, with mean zero and covariance

$$\begin{aligned}&\mathbb{E}\Big[\int dt\ dz_1\ \varphi(z_1)\eta_t(z_1)\int ds\ dz_2\ \psi(z_2)\eta_s(z_2)\Big]\\ &= \frac{1}{2}\int dt\ dx\,dv_1\,dv_2\,d\omega\,[\omega\cdot(v_1-v_2)]_+\ M(v_1)M(v_2)\ \Delta\varphi\ \Delta\psi\ .\end{aligned}$$

As explained in [13], the (space-time) white noise is due to memory effects in the collision process, such as recollisions. Moreover in the quoted references, a central limit theorem is conjectured to hold. The conjecture is confirmed by the following

Theorem 1 ([6]). *In the Boltzmann-Grad limit $\varepsilon\to 0$, the fluctuation field $(\xi_t^\varepsilon)_{t\in\mathbb{R}^+}$ converges in law for all times to the Ornstein-Uhlenbeck process solving the fluctuating Boltzmann equation: $d\xi_t = \mathcal{L}(\xi_t)\,dt + d\eta_t$.*

In the context of a purely stochastic dynamics, fluctuation fields have been studied in [12] where the same equation is obtained in an appropriate kinetic limit.

Our proof is based on a weak convergence method focusing on the moments of ξ_t^ε. We rely on L^2 bounds on dynamical observables coding the perturbation of the invariant measure (replacing the standard L^∞ bounds on correlation functions of Lanford's approach) and on a suitable coupling of the deterministic and the stochastic limiting dynamics.

A similar result for nonequilibrium states has been proved for short times in [4], while (local-in-time) convergence of the equilibrium covariance goes back to [1], recently extended to longer times in [5]. For hard disks in two dimensions, the global-in-time convergence of the covariance has been also obtained in [3] with a different proof, by using the uniform boundedness of the partition function (which is a special feature of hard disks in the canonical ensemble).

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High moments of the 2D polymer partition function

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(joint work with Ofer Zeitouni)

Let $W_N(\beta, x) = \mathrm{E}_x\left[e^{\sum_{n=1}^N \beta\omega(n,S_n)-N\beta^2/2}\right]$ be the partition function of a two dimensional directed polymer in a random environment, where $\omega(i,x), i \in \mathbb{N}, x \in \mathbb{Z}^2$ are i.i.d. standard normal and $\{S_n\}$ is the path of a random walk. With $\beta = \beta_N = \hat{\beta}\sqrt{\pi/\log N}$ and $\hat{\beta} \in (0,1)$ (the subcritical window), $\log W_N(\beta_N, 0)$ has been shown by Caravenna, Sun and Zygouras [1] to converge in distribution to a Gaussian law of mean $-\lambda^2/2$ and variance λ^2, with $\lambda^2 = \log(1/(1-\hat{\beta}^2))$.

In a follow-up work [2], the same authors have proven that for all $\hat{\beta} < 1$,

$$\sqrt{R_N}\left(\log W_N(\beta_N, x\sqrt{N}) - \mathbb{E}\log W_N(\beta_N, x\sqrt{N})\right) \xrightarrow{(d)} \sqrt{\frac{\hat{\beta}^2}{1-\hat{\beta}^2}}G(x),$$

where $G(x)$ a log-correlated Gaussian field on $\mathbb{R}^2$. Log-correlated fields, and in particular their extremes and large values, have played an important role in various problems ranging from branching walks, random matrices, Liouville quantum, turbulence and more. In the context of polymers, the above convergence opens the door to the study of extremal values of the polymer partition function.

The first step to tackle this question is to study the moments $\mathbb{E}[W_N(\beta_N)^q]$ in the subcritical window, for $q = O(\sqrt{\log N})$. Our result is the following.

Theorem 1 ([3]). *There exists $\hat{\beta}_0 \le 1$ so that if $\hat{\beta} < \hat{\beta}_0$ and*

$$\limsup_{N\to\infty} \frac{3\hat{\beta}^2}{\left(1-\hat{\beta}^2\right)} \frac{1}{\log N} \binom{q}{2} < 1,$$

then,

$$\mathbb{E}[W_N(\beta_N)^q] \le e^{\binom{q}{2}\lambda^2(1+\varepsilon_N)},$$

where $\varepsilon_N = \varepsilon(N,\hat{\beta}) \searrow 0$ as $N \to \infty$.

Our starting point is the moment formula:

$$\mathbb{E}[W_N^q] = \mathrm{E}^{\otimes q}\left[e^{\beta_N^2 \sum_{1\le i<j\le q}\sum_{n=1}^N \mathbf{1}_{S_n^i=S_n^j}}\right],$$

where $S^1,\dots,S^q$ are q independent copies of the simple random walk. The analysis is based on ruling out triple (or more) intersections and estimating the contribution of diagrams of successive 2-particle interactions.

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A soft proof of delocalization for the SOS model on the square lattice

Sébastien Ott

(joint work with P. Lammers)

The Solid-On-Solid (SOS) model is a random integer valued height function. In two dimensions, the model is known to be *delocalized* at high temperatures (i.e.: the variance of the height difference between far away points diverges). I discussed a recent soft argument developed in [1] to prove this fact. The proof is a combination of a planar percolation argument originating from S. Sheffield's work [2] and a "loss of memory" property of the SOS model conditioned to be larger than a constant which follows from the Lipschitz nature of the interaction potential. The method works at any slope (macroscopic tilt of the model), and for any planar graph invariant under a $\mathbb{Z}^2$ action.

This type of delocalization proof was also used by P. Lammers (see [3]) on a restricted class of planar graphs (with maximal degree 3) to a wide class of interaction potentials (without the Lipschitz constraint), but limited to the flat interface (zero slope). This latter result was then used in [4, 5] to prove the existence of a massless phase for the dual models (plane rotor with suitable interaction on the dual graph) at low temperature.

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Queues, stationarity, and stabilisation of last passage percolation

Márton Balázs
(joint work with Ofer Busani, Timo Seppäläinen)

Place i.i.d. Exponential weights on the vertices of the 2-dimensional integer lattice. Take a point a on this lattice and another one y North-East from the first. The *last passage time* between the two points is the maximal sum of these weights which can be collected by a path that takes North and East steps, and the *geodesic* between a and y is the a.s. unique path that realises this. The process of these weights as the endpoint(s) varies is a difficult one.

It has been known that as a is sent to infinity along any of the South-West directions, the geodesic up to a fixed distance from y eventually settles and doesn't move anymore [5]. We call this *stabilisation*, and it gives rise to *Busemann functions*, differences of last passage times from a, to two fixed locations y and y' in this limit. When $||y-y'|| = 1$, Busemann functions take particularly nice properties in the Exponential model, which had also been known under the name *stationarity* [2].

The speed of convergence with stabilisation was unknown. We show that as a gets N far from y, the geodesics to points in a $\mathcal{O}(N^{2/3})$-neighbourhood will jointly stabilise. Corollaries are coalescence of point to point geodesics on the same scale, furthermore an approximation result of the Airy_2 function by Brownian motion in total variation distance.

The proof is based on *joint* stationarity recently discovered in [3] for two different directions. In this setup the structure of joint nearest-neighbour Busemann functions becomes the same as inter-departure and service times of customers in M/M/1 queues. (Such ideas had been around since [4] for a similar situation in exclusion.) In particular when the directions are close to each other the queue becomes heavy-traffic, and inter-departure times become identical to service times for long periods of time. Translating back to the language of stationary last passage this exactly means that stationary paths from these two directions to anywhere in a small box coincide within that box with high probability. We can now include one or multiple finite points a (and b) between two such directions to include coalescence of point-to-point paths as well and complete our results [1].

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