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Mini-Workshop: Zero-Range and Point-Like Singular Perturbations: For a Spillover to Analysis, PDE and Differential Geometry

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Abstract. The field of contact interactions and perturbations of differential operators supported on subsets with non-trivial co-dimension is an increasingly active mainstream of mathematical physics (in particular, operator and spectral theory and quantum mechanics), with intimately related applications and mathematical challenges in partial differential equations, and neighbouring sectors of analysis, PDEs, and differential geometry. This Mini-Workshop fostered intense and prolific discussions on recent advances and trends in the field.

Mathematics Subject Classification (2020): 35B40, 35J75, 35Q55, 47N20, 53B21, 81Q35.

Introduction by the Organizers

The Oberwolfach Mini-Workshop 2240a, *Zero-Range and Point-Like Singular Perturbations: For a Spillover to Analysis, PDE and Differential Geometry*, organised by Vladimir Georgiev (Pisa) and Alessandro Michelangeli (Bonn) took place on 2-8 October 2022 in hybrid on-side and online form, and was structured in seminar presentations and an amount of round table discussions. The 13 speakers, of which 11 on-site and 2 online, came from universities from France, Germany, Italy, Portugal, Serbia, and Spain.

This meeting was aimed at promoting and facilitating the transfer of techniques, tools, and open problems from the field of **contact interactions** and **perturbations of differential operators supported on subsets with non-trivial co-dimension**, an increasingly active mainstream of mathematical physics (in particular, operator and spectral theory and quantum mechanics), to intimately related applications and mathematical challenges in the area of partial differential equations, and neighbouring sectors of analysis, PDEs, and differential geometry. The latter areas are naturally involved when one investigates the heat or Schrödinger evolution flow generated by singular-perturbed Hamiltonian of relevance, the associated standing waves, or when one transposes the problem in the language of stochastic processes generated by such operators, or again when point-like singular-perturbed PDE's are placed on relevant manifold (typically, almost-Riemannian structures).

Inter-particle interactions of virtually zero range ('contact interactions') became a natural tool when in the 1930's quantum mechanics began to be applied to the newly observed nuclear phenomena, as the decrease by a factor 10^{-5} from the atomic to the nuclear scale made it plausible to model the interaction among nucleons as a delta-like interaction. In 1932 Wigner [77] calculated that the nuclear forces must be of very short range and very strong magnitude, leading first Bethe and Peierls [21, 22], then Thomas [76], Fermi, [42], and Breit [28] to describe the neutron-proton scattering, and then some 20 years later, in 1955, Ter-Martirosyan and Skornyakov [74] to describe the three-body problem with zero-range interaction, by means of the Schrödinger equation in the approximation of a two-body potential of very short range, virtually null. Whereas nuclear physics obviously developed into a much finer corpus of knowledge, the 'delta-like' idealisation remained for some decades an efficient tool for a formal first-order perturbation theory in application to atomic physics [40]. Modern advances in the manipulation of cold atoms have today conferred the zero-range modelling a renewed topicality, owing to the experimental possibility of tuning the interaction's effective scattering length to very large values and the interaction's effective range to zero, by means of a magnetically induced Feshbach resonance [71, Section 5.4.2]. Ultra-cold gases in the so-called 'unitary regime' (infinite scattering length and zero-range) are nowadays well preparable and intensively studied (see [30, 27, 69] and references therein).

An intense mathematical activity, at the crossroads of functional analysis, operator theory, and spectral theory paralleled the above physical developments, often being triggered by the latter, until the present days. This includes: the rigorous constructions of self-adjoint Hamiltonians for two-, three-, and N-body quantum systems with zero-range interactions [18, 65, 75, 39, 32, 61, 62, 63, 33, 60, 16, 58, 59, 67, 68, 15, 56], the study of their spectral properties [64, 8, 66, 62, 63, 79, 17, 56], the approximation by means of ordinary Schrödinger operators with potentials supported at very short scales [8, 9, 13, 50, 14], among other themes, with tools and techniques from spectral and scattering theory, self-adjoint extension schemes, and the like.

Notably, each of the above subjects has witnessed and is witnessing both fabulous, technically ingenious breakthroughs, and almost desperate, deep open problems. And most importantly, for the purposes of the Mini-Workshop, many mathematical questions that are central in the math-phys field sketched above naturally involves specific techniques from, and naturally give rise independent problems in the realm of neighbouring subjects such as the theory of partial differential equations, differential geometry, and stochastic processes, where several studies have been already initiated.

Let us survey two main representative lines around which the workshop's discussions unfolded.

Subject line I – Linear and non-linear Schrödinger and heat equations with point-like Laplace perturbation. This is a natural and active continuation, in the field of PDE, of the quantum mechanical problem of the *time evolution* of particle systems subject to mutual zero-range interaction. The prototypical object of interest is the singular-perturbed Laplacian $-\Delta_\alpha$ on $L^2(\mathbb{R}^d)$, $d \in \{1,2,3\}$, which is a self-adjoint extension of $-\Delta|_{C_0^\infty(\mathbb{R}^d\setminus\{0\})}$, namely of the free Laplacian initially restricted to smooth functions that vanish in the vicinity of the origin. The parameter $\alpha \in \mathbb{R} \cup \{\infty\}$ measures, through a suitable expression, the scattering length of the quantum particle that feels the non-trivial interaction supported at $x = 0$ (or, re-interpreting the x-variable as a relative variable, the scattering length in the two-body interaction of zero range). For example, in $d = 3$ dimensions,

$$\begin{aligned} \mathrm{dom}(-\Delta_\alpha) \;&=\; \Big\{ u \in L^2(\mathbb{R}^3) \,\Big|\, u = f + \frac{f(0)}{4\pi\alpha + 1}\,\frac{e^{-|x|}}{|x|} \text{ with } f \in H^2(\mathbb{R}^3) \Big\}\,, \\ (-\Delta_\alpha + 1)u \;&=\; (-\Delta + 1)f \end{aligned} \tag{1}$$

(dom ≡ domain). The realisation with $\alpha = \infty$ is nothing but the self-adjoint Laplacian on the H^2-domain. This naturally leads to considering linear and non-linear Schrödinger equations

$$\mathrm{i}\partial_t u \;=\; -\Delta_\alpha u\,, \tag{2}$$

$$\mathrm{i}\partial_t u \;=\; -\Delta_\alpha u + g(u)\,, \tag{3}$$

for non-linearities of physical relevance such as

$$g(u) \;=\; \lambda |u|^p u\,, \qquad g(u) \;=\; (w * |u|^2)u \tag{4}$$

(for suitable $\lambda, p > 0$ and $w : \mathbb{R}^3 \to \mathbb{R}$). For the linear problem, in all dimensions, the explicit propagator is known [7, 36], and dispersive and Strichartz estimates [41, 51, 38], as well as the completeness of the wave operators [38, 31, 78] have been recently proved. In dimension $d = 1$ the point-like nature of the singularity has the explicit structure $-\Delta + \delta(x)$ (in the sense of sum of quadratic form, where $\delta(x)$ is the Dirac distribution), thereby allowing for an accurate analysis of (2)-(3) in terms of local and global well-posedness, blow-up, scattering, asymptotic stability, solitons, standing waves, ground state [5, 37, 2, 4, 3, 10, 12, 53, 52, 34, 11, 54, 55, 35]. The same questions in the case of *dimension two or three* encounter a much more uncharted territory, with very recent characterisations [49, 29] of the adapted (i.e., singular-perturbed) fractional Sobolev spaces

$$H^s_\alpha(\mathbb{R}^3) \;:=\; \mathrm{dom}\big((-\Delta_\alpha + \lambda\mathbb{1})^{\frac{s}{2}}\big)\,, \qquad s > 0\,, \tag{5}$$

for $\lambda > 0$ large enough, the proof of local and global well-posedness for certain three-dimensional [5, 29] and two-dimensional [1, 44] singular-perturbed NLS. This leaves an ample room open for improvements and unanswered questions concerning solution theory, standing waves, scattering, stability and instability, etc., when $d = 2, 3$. Part of the workshop focussed on this, as well as on the counterpart issues for the linear and non-linear heat equation with point-like perturbation.

Subject line II – Dispersive and scattering properties of the Schrödinger and heat flow on degenerate Riemannian manifold of Grushin type. This is another natural and active spin-off, in the field of Riemannian geometry and analysis of PDEs on manifold, of the problem of a quantum particle subject to the zero-range interaction supported in a region of non-trivial co-dimension, which is in fact a region where the metric blows up. The study of a quantum particle on degenerate Riemannian manifolds, and the problem of the purely geometric confinement away from the singularity locus of the metric, as opposite to the dynamical transmission across the singularity, has recently attracted a considerable amount of attention in relation to Grushin structures and to the induced confining effective potentials on cylinder, cone, and plane (as in the works [70, 23, 26, 73, 25, 43, 46, 24, 72, 47, 20, 45, 48]), as well as, more generally, on two-step two-dimensional almost-Riemannian structures [23, 20, 19], or also generalisations to almost-Riemannian structures in any dimension and of any step, and even to sub-Riemannian geometries, provided that certain geometrical assumptions on the singular set are taken [43, 73]. On a related note, a satisfactory interpretation of the heat-confinement in the Grushin cylinder is known in terms of Brownian motions [24] and random walks [6]. The prototypical playground is the Grushin-type manifold $M_\alpha \equiv (M, g_\alpha)$, for given $\alpha \in \mathbb{R}$, with

$$
\begin{aligned}
M &= \mathbb{R}_x^- \times \mathbb{R}_y + \mathbb{R}_x^+ \times \mathbb{R}_y \qquad \text{(Grushin plane)}\,, \\
M &= \mathbb{R}_x^- \times \mathbb{S}_y^1 + \mathbb{R}_x^+ \times \mathbb{S}_y^1 \qquad \text{(Grushin cylinder)}
\end{aligned} \tag{6}
$$

and with metric

$$
g_\alpha = \mathrm{d}x \otimes \mathrm{d}x + |x|^{-2\alpha}\, \mathrm{d}y \otimes \mathrm{d}y\,, \tag{7}
$$

thus with associated Laplace-Beltrami operator

$$
\Delta_\alpha = \frac{\partial^2}{\partial x^2} + |x|^{2\alpha} \frac{\partial^2}{\partial y^2} - \frac{\alpha}{|x|} \frac{\partial}{\partial x}\,. \tag{8}
$$

For such models, the *geometric quantum confinement* in each half-cylinder corresponds to the essential self-adjointness of $-\Delta_\alpha$ in $L^2(M, |x|^{-\alpha}\, \mathrm{d}x \wedge \mathrm{d}y)$ on its minimal domain of smooth functions supported away from the singularity; the *quantum transmission* between the two halves of M corresponds instead to the lack of essential self-adjointness, in which case the type of transmission is governed by the Schrödinger equation induced by a self-adjoint extension of $-\Delta_\alpha$. We underline the fundamental novelty of the quantum setting, where depending on the boundary conditions for Δ_α the particle may or may not trespass the singularity locus, whereas instead classically M_α is always geodesically incomplete (the classical particle reaches the boundary in finite time). Next to the recently established identification of the α-regimes of essential self-adjointness or lack-of, the classification of the self-adjoint extensions of $-\Delta_\alpha$, and the study of their spectral and scattering properties [46, 72, 47, 45], the relevant problem arises of the dispersive and scattering properties of the Schrödinger and heat flow on M_α. For concreteness in the case of the Grushin cylinder (compact y-variable), one can see [45, 48] that in each Fourier mode conjugate to y one is led to the study of PDEs of the form

$$(9)\qquad \begin{cases} \mathrm{i}\partial_t u \;=\; -\partial_x^2 u + \frac{\alpha(\alpha+2)}{4x^2}\, u\,, \\ u_0(t)^- = u_0(t)^+ \quad \text{with } u_0^\pm(t) := \lim\limits_{x\to 0^\pm} |x|^{\frac{\alpha}{2}} u(t,x)\,, \\ u_1(t)^- = -u_1(t)^+ \text{with } u_1^\pm(t) := \lim\limits_{x\to 0^\pm} |x|^{-(1+\frac{\alpha}{2})}(u(t,x) - |x|^{-\frac{\alpha}{2}} u_0^\pm(t))\,, \\ u(0,x) = \varphi(x) \qquad \text{with } \varphi \in L^2(\mathbb{R}) \end{cases}$$

in the unknown $u \equiv u(t,x)$, with $t \geq 0$ and $x \in \mathbb{R} \setminus \{0\}$: a free Schrödinger equation except for the non-trivial *boundary conditions of contact interaction* at $x = 0$ (a signature of a non-trivial left↔right transmission protocol). Similar considerations hold for the associated heat equation. Along a completely open programme recently proposed and advertised in [48], the next crucial challenges are the study of dispersive, smoothing, and Strichartz estimates for linear problems of the type (9), as well as the well-posedness, ground states, scattering, stability/instability of the associated non-linear equations.

The Mini-Workshop aimed at gathering a small number of mathematicians of various degrees of expertise, with an amount of direct or indirect scientific links already transversely present among them, and who are carriers of the mathematical-physical, differential-geometric, functional-analytic, and differential equation theoretic backgrounds, with the general goal of discussing the setup and the perspectives of ongoing and future investigations in each of the above fields, all stemming from, or intimately motivated by the grand picture of zero-range and contact interactions sketched in the previous Section. At the end of a week of mutual and profitable exchanges, this little group of already mildly connected researchers was meant to import into their own network of collaborations the corpus of general motivations, cross-disciplinary stimuli, technical tools, and bibliographic references discussed in the workshop. In this respect, the event set the basis of future larger meetings among neighbouring mathematical communities.

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Abstracts

Two-dimensional systems with non-linear point interactions

Riccardo Adami

Non-linear point interactions were introduced two decades ago in order to describe localised phenomena like trapping, beatings, concentration of wave packets, and so on. Seminal studies focussed on one and three-dimensional systems, leaving untouched the more technical two-dimensional case up to two years ago. We review the results and comment on the specific features exhibited by this case, together with open problems.

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Self-adjoint extensions for the curvature Laplacian on Grushin manifolds

Ivan Beschastnyi

I discuss some results from an ongoing project with H. Quan (University of Washington), in which we study the curvature Laplacian on curved high-dimensional analogues of the α-Grushin planes. We are able to prove that all self-adjoint properties of the Laplacian are encoded in a second order polynomial, which can be easily read off from the equation itself. In particular, we can determine whether the operator is essentially self-adjoint by computing its discriminant and construct all of its self-adjoint extensions from its roots.

NLS ground states with singularities

Filippo Boni

We investigate the existence of ground states at fixed mass of the L^2-subcritical non-linear Schrödinger equation with a point interaction. First, the problem is considered in dimension two and three. We prove that ground states exist for every value of the mass and, up to a multiplication by a phase factor, they are positive, radially symmetric, decreasing along the radial direction and present a singularity where the interaction is placed. In order to obtain qualitative features of the ground states, we refine a classical result on rearrangements and move to equivalent variational formulations of the problem. Then, we present some future developments, among them the possibility of generalising similar models on hybrid structures.

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Geometric confinement of the curvature Laplacian on almost Riemannian manifolds

Ugo Boscain

Two-dimensional almost-Riemannian structures of step 2 are natural generalisations of the Grushin plane. They are generalised Riemannian structures for which the vectors of a local orthonormal frame can become parallel. Under the 2-step assumption the singular set Z, where the structure is non-Riemannian, is a one-dimensional embedded submanifold. While approaching the singular set, all Riemannian quantities diverge. A remarkable property of these structures is that the geodesics can cross the singular set without singularities, but the heat and the solution to the Schrödinger equation (with the Laplace-Beltrami operator Δ cannot. This is due to the fact that (under a natural compactness hypothesis), the Laplace-Beltrami operator is essentially self-adjoint on a connected component of the manifold without the singular set. In the literature such a counter-intuitive phenomenon is called geometric confinement.

For the heat equation an intuitive explanation of this fact can be given in terms of random walks. For the Schrödinger equation an intuitive explanation is more subtle, as the evolution of a quantum particle on a manifold can be done in several non-equivalent way.

In this talk I describe the evolution (and the confinement) of a quantum particle described by the curvature Laplacian $-\Delta + cK$ (here K is the Gaussian curvature and $c > 0$ a constant) which originates in coordinate-free quantisation procedures (as, for instance, in path-integral or covariant Weyl quantisation).

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About Schrödinger and Dirac operators with scaling critical potentials

Luca Fanelli

(joint work with Luz Roncal, and Nico Michele Schiavone)

Lower-order perturbations of the free Hamiltonians usually appear in Quantum Mechanics, ad models describing the interaction of a free particle with an external field. In some cases, the perturbation lies at the same level as the free Hamiltonian, and the resulting conflict can generate interesting phenomena. We will introduce the Inverse Square and Coulomb potentials as toy models, and describe the main features of the complete Hamiltonians from the point of view of Fourier Analysis, Spectral Theory, and dispersive evolutions. In this talk, the potentials will usually belong to $\mathbb{C}$, which is responsible of a lack of symmetry, and we hence are in the case of non-self-adjoint Hamiltonians. In the recent years, there is a growing interest devoted to this kind of operators, and it is quite surprising that it started around 25 years ago, after almost one century since the advent of Quantum Mechanics. Motivated by needs of Nuclear Physics, Scholtz-Geyer-Hahne [12] suggested an interesting model in which observables are represented by operators which are not necessarily self-adjoint, but merely quasi-self-adjoint" (roughly speaking, similar to self-adjoint operators). In this case, it is sufficient to change the inner product in the base Hilbert space, with the help of a metric operator related to the similarity, to end back in the usual Hermitian case. Later on, Bender-Boettcher [2] noticed that a large class of non-self-adjoint operators still possesses a real spectrum, and renewed the interest to the topic.

In the recent papers [4, 6, 7], we proved by purely real analytic methods some uniform resolvent estimates for the free Hamiltonian, inspired to the pioneer results by Kato and Yajima in [11]. Those are uniform version over the complex plane of some weighted variant of the so called *Hardy-Rellich* inequality, which has been recently investigated in [3] with a particular interest on the sharp constants. In this talk, we will present the recent results obtained in [8], concerned with the analogous questions in the sub-Riemannian setting of the Heisenberg Group. Some interesting open problems related with the spectrum of mass-less Dirac operators and the sharp constants of the uniform resolvent estimates will also be introduced.

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Pointed sub-Laplacians in three dimensions and Hardy inequalities

Valentina Franceschi

The aim of this seminar is to present recent results on the essential self-adjointness of pointed sub-Laplacians in three dimensions. We show that, unlike the Euclidean case, pointed sub-Laplacians (associated with smooth measures) are essentially self-adjoint in dimension 3. To this purpose, we focus on the case of the 3D Heisenberg sub-Laplacian, and we show its essential self-adjointness by exploiting non-commutative Fourier transform techniques. We then generalise the result to a class of three-dimensional sub-Riemannian manifolds. In connection with the main result, we present a discussion on Hardy inequalities in the Heisenberg group: contrary to the Euclidean case, a radial Hardy inequality, i.e., a Hardy inequality taking into account only the directional derivative with respect to the sub-Riemannian distance, does not hold in this context for any dimension. This underlines again a difference with respect to the Euclidean case, where essential self-adjointness of pointed Laplacians can be derived from Hardy inequalities.

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The Laplace-Beltrami operator on the Grushin Cylinder

Matteo Gallone

(joint work with Alessandro Michelangeli, Eugenio Pozzoli)

When a quantum particle is constrained on an orientable Riemannian manifold, one challenging problem that arises naturally is the question of the so-called 'geometric quantum confinement'. This is the possibility that a particle whose initial wave-function is supported inside some portion of space may remain confined in such a region for all times when evolving according to the unitary group generated by the free Hamiltonian. This occurrence is related to the presence of singularities in the metric and to the (essential) self-adjointness of the Laplace-Beltrami operator. The prototypical example of space exhibiting this phenomenon is the 'Grushin-like cylinder', that is, roughly speaking, a cylinder with metric $\mathrm{d}s^2 = \mathrm{d}x^2 + |x|^{-2\alpha}\mathrm{d}y^2$. In this talk I consider this manifold and present the classification of a physically interesting sub-family of self-adjoint realisations of the Laplace-Beltrami operator in the regime where it is not essentially self-adjoint. I discuss the advantages of the usage of the Kreĭn-Višik-Birman self-adjoint extension theory.

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Boundary conditions, product formulae and classical limit

Marilena Ligabò
(joint work with Paolo Facchi, Fabio Deelan Cunden, Giancarlo Garnero)

In this seminar some connection between (classical and quantum) boundary condition and product formulae will be illustrated. The starting point of this presentation is a result concerning a parametrisation of all the possible self-adjoint extensions of the Laplace operator on a bounded domain in terms of unitary operators at the boundary [1]. Then, using the Trotter product formula, a composition law for quantum boundary conditions will be defined [2]. Furthermore, the link between the occurrence of boundary conditions and frequent measurements on a quantum system will be discussed. This phenomenon is due to the existence of a one-parameter unitary group obtained as the limit of the Zeno product formula (defined by intertwining the time evolution group with an orthogonal projection) [3, 4]. The limiting dynamics is called Zeno dynamics and the corresponding generator is called Zeno Hamiltonian. Finally, it will be shown that the classical limit of a Zeno dynamics obtained in a cavity quantum electrodynamics experiment [5, 6] induces a change of topology in the classical phase space [7].

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The NLS equation with a point interaction in two and three dimensions

Diego Noja
(joint work with Claudio Cacciapuoti and Domenico Finco)

The NLS equation with a power non-linearity and point interaction is described by the Cauchy problem

$$i\partial_t \psi = H_\alpha \psi + g|\psi|^{p-1}\psi \qquad p > 1, \quad g = \pm 1$$

where H_α is the abstract Schrödinger-like singular perturbation of the Laplacian known as *point interaction* or *delta potential*, sometimes improperly written as $H_\alpha = -\Delta + \alpha\delta$. The model is well studied in dimension one but only recently the two and three dimensional cases have been considered by several authors. In the talk, after a review of the definition, construction and properties of point interactions especially relevant to the applications in the nonlinear setting, the well-posedness of the Cauchy problem will be treated, as regards the known facts and reviewing also some still open problems. Moreover, some asymptotic properties of the model will be discussed, and in particular as regards the blow-up for strong non-linearities in dimension two. Namely, it will be shown that in two dimensions and in the super-critical regime $p > 3$ large sets of initial data undergo blow-up and that the standing waves are strongly unstable, in the sense that arbitrarily close (in the energy norm) to any standing wave there exist blowing-up initial data.

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On the square of Laplacian with inverse square potential

Mario Rastrelli
(joint work with Vladimir S. Georgiev)

The aim of the talk is to show an explicit characterisation of the domain of the scaling-invariant perturbation of the Laplacian, the operator $A = -\Delta + \frac{\beta}{|x|^2}$, and of its square. For the domain of A, we require $\beta > \frac{n(n-4)}{4}$ that is the necessary condition for essentially self-adjointness of the operator [8, 9]. For A^2 it is still not clear what happens when $\beta = 8 + \frac{n(n-4)}{4}$. It is also more difficult to give an expression of the domain in even dimensions, when $n < 8$, due to the problems that arise with weighted Rellich inequalities. The description of $D(A)$ gives to us a Rellich type inequality that can be used to study Schrödinger operators with singular potential of the inverse-square type [5], that hold even if $n = 3, 4$. The characterisation of $D(A^2)$ improve this last result, giving to us a weighted version of the inequality.

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Non-linear Schrödinger equations with point interactions: results and perspectives

Raffaele Scandone
(joint work with Vladimir Georgiev, Alessandro Michelangeli)

In this talk I review recent advances in the study of non-linear Schrödinger equations with delta-like potentials. After introducing a suitable functional framework, the notion of sub-critical non-linearity, and a class of dispersive-type estimates, I discuss some results concerning the global well-posedness and the existence (and symmetries) of standing waves. The last part of the talk is devoted to open questions and future research perspectives.

Generalised solutions to non–linear Schrödinger equations with singularities

Ivana Vojnović

(joint work with Nevena Dugandžija, Alessandro Michelangeli)

We consider Schrödinger equations of Hartree and cubic type in three spatial dimensions and its approximations of singular, point-like perturbations.

As approximants to the Hartree equation, we analyse the equation of the form

$$i\partial_t u_\varepsilon \;=\; -\Delta u_\varepsilon + V_\varepsilon u_\varepsilon + (w * |u_\varepsilon|^2)u_\varepsilon \,, \tag{1}$$

for $\varepsilon \in (0,1]$. Here V_ε is a real-valued potential and is meant to represent a singular, delta-like profile centred at $x = 0$.

We assume that

$$V_\varepsilon(x) \;:=\; \frac{1}{\varepsilon^\sigma} V\Big(\frac{x}{\varepsilon}\Big)\,, \tag{2}$$

for a given measurable function $V : \mathbb{R}^3 \to \mathbb{R}$ and a given $\sigma \geq 0$.

The corresponding nets of approximate solutions represent generalised solutions for the singular-perturbed Schrödinger equation. The behaviour of such nets is investigated for $\sigma \in [0,3]$.

We also study a generalised solution in the Colombeau algebra $\mathcal{G}_{C^1,H^2}$ for cubic and Hartree equation with delta potential, which correspond to the case $\sigma = 3$ in (2). In the case of the Hartree equation with delta potential compatibility between the Colombeau solution and the solution of the classical Hartree equation is established. More precisely, we prove that

$$\lim_{\varepsilon\downarrow 0} \|u_\varepsilon - u\|_{L^\infty([0,\mathsf{T}],L^2(\mathbb{R}^3))} \;=\; 0\,, \tag{3}$$

where Colombeau solution is represented by net (u_ε) and u is the unique solution in $C(\mathbb{R}, L^2(\mathbb{R}^3))$ to the Cauchy problem

$$i\partial_t u \;=\; -\Delta u + (w * |u|^2)u$$

with initial datum $a \in L^2(\mathbb{R}^3)$.

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Operators on Groups and Non-Commutative Fourier Transforms

Jens Wirth
(joint work with Jonas Brinker, Michael Ruzhansky)

The study of differential and more general pseudo-differential operators is usually based on Euclidean Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi \mathrm{i} x\cdot\xi} f(x)\,\mathrm{d}x, \qquad f \in \mathcal{S}(\mathbb{R}^n),$$

which allows to treat differential operators $p(\mathrm{D}) = \sum_{|\alpha|\le m} a_\alpha \mathrm{D}^\alpha$, $\mathrm{D} = \frac{\mathrm{i}}{2\pi}\partial$ in terms of their symbols $p(\xi)$. More generally, for $\sigma_A(x,\xi) \in \mathrm{C}_b^\infty(\mathbb{R}^n)\widehat{\otimes}\mathcal{O}_M(\mathbb{R}^n)$ one considers operators of the form

$$Af(x) = \int_{\mathbb{R}^n} \mathrm{e}^{2\pi\mathrm{i}x\cdot\xi}\sigma_A(x,\xi)\widehat{f}(\xi)\,\mathrm{d}\xi.$$

For details on pseudo-differential operators and their calculus see [1] and [2]. Although, (almost) any operator can be treated in such a way, the translation invariance of the constant coefficient differential operators is built into the calculus.

Sometimes it is advantageous to treat operators based on different symmetries and therefore use non-commutative analogues of the classical Fourier transform.

The non-commutative Fourier transform. For a locally compact type-I group G we denote by $\widehat{G}$ the set of equivalence classes of irreducible unitary representations $\boldsymbol{\xi} : G \to \mathcal{L}(H_{\boldsymbol{\xi}})$ and for each $[\boldsymbol{\xi}] \in \widehat{G}$ define

$$\widehat{f}(\boldsymbol{\xi}) = \int_G f(x)\boldsymbol{\xi}(x)^*\,\mathrm{d}x$$

as non-commutative Fourier transform. For sufficiently nice functions $\widehat{f}(\boldsymbol{\xi})$ is of trace class and there exists a unique measure on $\widehat{G}$, the Plancherel measure μ, such that the inversion formula

$$f(x) = \int_{\widehat{G}} \mathrm{trace}(\widehat{f}(\boldsymbol{\xi})\boldsymbol{\xi}(x))\,\mathrm{d}\mu(\boldsymbol{\xi})$$

holds true. Again this can be used as starting point to develop a pseudo-differential calculus, for details see e.g. [6]. It is worth looking at some examples to see how such a calculus will look like and what benefit it brings.

Compact groups. If G is compact, irreducible representations $[\boldsymbol{\xi}] \in \widehat{G}$ are finite-dimensional, i.e. $H_{\boldsymbol{\xi}} \simeq \mathbb{C}^{d_{\boldsymbol{\xi}}}$, and the inversion of non-commutative Fourier transform reduces to Peter–Weyl theorem

$$L^2(G) = \bigoplus_{[\boldsymbol{\xi}]\in\widehat{G}} \operatorname{span}\{\xi_{ij} \mid \boldsymbol{\xi}(x) = (\xi_{ij}(x))_{1\le i,j\le d_{\boldsymbol{\xi}}}\}$$

in the form of series representations

$$f(x) = \sum_{[\boldsymbol{\xi}]\in\widehat{G}} d_{\boldsymbol{\xi}} \operatorname{trace}(\widehat{f}(\boldsymbol{\xi})\boldsymbol{\xi}(x)).$$

For G a compact Lie group any linear continuous operator $A : \mathrm{C}^\infty(G) \to \mathcal{D}'(G)$ can be represented in terms of a matrix-valued symbol

$$\sigma_A(x,\boldsymbol{\xi}) = \boldsymbol{\xi}(x)^*(A\boldsymbol{\xi})(x),$$

where A is applied to the matrix entries ξ_{ij}, as series

$$Af(x) = \sum_{[\boldsymbol{\xi}]\in\widehat{G}} d_{\boldsymbol{\xi}} \operatorname{trace}(\widehat{f}(\boldsymbol{\xi})\sigma_A(x,\boldsymbol{\xi})\boldsymbol{\xi}(x)).$$

Again symbols encode properties of the operator and allow for a full symbolic calculus under appropriate conditions, see [4] and [5]. To mention a typical example, Hörmander class pseudo-differential operators defined in local coordinates allow a global characterisation in terms of such global symbols. The operator A is pseudo-differential of order m and type (ρ,δ) if

$$\sup_{x\in G} \|\partial_x^\alpha \triangle_\xi^\beta \sigma_A(x,\boldsymbol{\xi})\|_{\mathrm{op}} \le C_{\alpha,\beta}\langle\boldsymbol{\xi}\rangle^{m-\rho|\beta|+\delta|\alpha|}$$

holds true with $\langle\boldsymbol{\xi}\rangle = \sqrt{1+\lambda_{\boldsymbol{\xi}}}$ given in terms of the Laplace eigenvalue $\lambda_{\boldsymbol{\xi}}$ on the eigenfunctions ξ_{ij}, an admissible selection of left-invariant vector fields ∂_x and an admissible selection of first order difference operators $\triangle_\xi$. First order difference operators by itself are defined in terms of the non-commutative Fourier transform,

$$\triangle\widehat{f} = \widehat{qf},$$

using differentiable functions $q : G \to \mathbb{C}$ vanishing to first order in the identity element $1 \in G$ and admissibility for a selection of such difference operators essentially means that their differentials in the identity span the full cotangent space.

The Heisenberg group. On the Heisenberg group $\mathbb{H}_n = \mathbb{R}^{2n+1}$ with group law

$$(x,\xi,\tau) \bullet (y,\eta,\sigma) = (x+y, \xi+\eta, \tau+\sigma+\frac{1}{2}(y\cdot\eta - x\cdot\xi))$$

irreducible representations are characterised by Stone–von Neumann theorem. Of interest for us are only the Schrödinger representations

$$\boldsymbol{\rho}_\lambda(x,\xi,\eta) = \mathrm{e}^{2\pi\mathrm{i}\tau+\pi\mathrm{i}\lambda x\cdot\xi} M_\xi T_{\lambda x}$$

parametrised by $\lambda \in \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ and given in terms of translations T_x and modulations M_ξ on the Hilbert space $L^2(\mathbb{R}^n)$. The Haar measure of the Heisenberg group is given by Lebesgue measure and the Fourier inversion formula is given by

$$f(x) = \int_{\mathbb{R}^\times} \operatorname{trace}(\widehat{f}(\lambda)\boldsymbol{\rho}_\lambda(x,\xi,\tau))|\lambda|^n \, \mathrm{d}\lambda, \qquad \widehat{f}(\lambda) = \widehat{f}(\boldsymbol{\rho}_\lambda).$$

For more details, in particular on the resulting pseudo-differential calculus, see [7]. The main difference to the case of compact groups is that symbols σ_A of pseudo-differential operators A are by itself unbounded operators on the representation space $L^2(\mathbb{R}^n)$ and should better by itself be seen as Weyl quantised pseudo-differential operators. The following table collects examples for this:

operator on $\mathbb{H}_n$	symbol as operator on $\mathbb{R}^n$	Weyl symbol
$\partial_{x_j} - \frac{1}{2}\xi_j\partial_\tau$	$\sqrt{\lvert\lambda\rvert}\partial_{y_j}$	$\sqrt{\lvert\lambda\rvert}\eta_j$
$\partial_{\xi_j} + \frac{1}{2}x_j\partial_\tau$	$\mathrm{i}\sqrt{\lambda}y_j$	$\mathrm{i}\sqrt{\lambda}y_j$
∂_τ	$\mathrm{i}\lambda$	$\mathrm{i}\lambda$
$\mathcal{L}_{sub}$	$\lvert\lambda\rvert^2\sum_{j=1}^n(\partial_{u_j}^2 - u_j^2)$	$-\lvert\lambda\rvert^2\sum_{j=1}^n(y_j^2+\eta_j^2)$

Symbols of pseudo-differential operators are again characterised in terms of smoothness and difference conditions. Now difference operators are in fact also differential operators acting on Weyl symbols

function on $\mathbb{H}_n$	difference operator on $\mathbb{R}^n$	difference operators on symbols
x_j	$\lvert\lambda\rvert^{-1/2}\,\mathrm{ad}\,y_j$	$\lvert\lambda\rvert^{-1/2}\partial_{\eta_j}$
ξ_j	$-\mathrm{i}\lambda^{-1/2}\,\mathrm{ad}\,\partial_{y_j}$	$-\mathrm{i}\lambda^{-1/2}\partial_{y_j}$
τ	$\cdots$	$\partial_\lambda - \frac{1}{2\lambda}\sum_{j=1}^n(y_j\partial_{y_j} + \eta_j\partial_{\eta_j})$

and in essence from these operators it can be seen that symbols of Hörmander class pseudo-differential operators on $\mathbb{R}^n$ are in fact itself parameter-dependent pseudo-differential operators with symbols from Shubin classes.

Smooth orbit characterisations. Properties of operators A on a Hilbert space H can be encoded into smoothness properties of the orbit

$$x \mapsto \boldsymbol{\pi}(x)^* A \boldsymbol{\pi}(x)$$

with respect to unitary representations $\boldsymbol{\pi} : G \to \mathcal{L}(H)$. It is well-known that C^∞-smoothness of such orbits with respect to Schrödinger representations characterise pseudo-differential operators with symbols from $S^0_{0,0}$, see [3]. In [8], [9] this is generalised to ultra-differential smoothness for actions of compact and also for homogeneous groups.

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