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Morphisms in Low Dimensions

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Abstract. This workshop brought together experts on interrelated topics in low-dimensional topology, centred around the common theme of 'morphisms'. Our goal was to improve community understanding of recent developments in the field and to promote new advances in the study of global properties of 4-manifolds.

Mathematics Subject Classification (2020): Primary: 57M99; Secondary: 57K99.

Introduction by the Organizers

In many areas of pure mathematics, it is a mantra that in order to understand an object, one tries to understand its automorphism group. This point of view has taken a while to percolate through to low-dimensional topology – typically most work has been directed towards classifying the *objects* of the category of interest (for example smooth or topological 4-manifolds, 3-manifolds, knots, or embedded surfaces). Furthermore, in the case that a low-dimensional invariant is functorial (Floer homologies and Khovanov-type homologies for example), most applications of the invariant have tended to make use of the values taken by the invariant on objects and not on morphisms. Nevertheless, in very recent years, there has been a change. To pick two notable examples: Watanabe's work on embedding calculus has shown us how diffeomorphisms of the 4-sphere may be studied, and Zemke's work has used the functoriality of Floer homology to derive powerful new obstructions to ribbon concordance. A common theme in these works is the emphasis on *functoriality* of the invariants, rather than simply computation, and a renewed interest in understanding diffeomorphism groups of a manifold rather than single isotopy groups.

'Morphisms' served as the central organising theme of the workshop, variously interpreted by the speakers as referring to homeomorphism or diffeomorphism groups, mapping class groups of 4-dimensional manifolds, morphisms in functorial knot invariants, and functions on the knot concordance group.

With the goal of facilitating discussions among the participants, we ended each day of the workshop with a lightning talk session, featuring 5-minute talks by six presenters, including both junior and senior researchers. Each day also featured three 45-minute talks (except only one on Wednesday). To highlight the theme of the conference, consider the following talks that took place over the week.

- In the topological category: "Stable homeomorphism and homotopy equivalence," by Anthony Conway (MIT),
- Regarding Floer cobordism maps: "Torsion in the knot concordance group and cabling," by Sungkyung Kang (IBS),
- On diffeomorphism groups of 4-manifolds: "Showing implanted barbell diffeomorphisms are non-trivial," by Ryan Budney (Victoria).

We selected these titles to illustrate how participants interpreted the theme broadly and considered a range of related topics. The light schedule left plenty of time for lively discussions among the participants; we heard from several participants that this week was productive and sparked interesting conversations on avenues for future research.

The workshop was well attended with 45 participants present at Oberwolfach, and a further 15 participants online. Of the 45 in-person participants, 10 were current PhD students and another 11 were postdoctoral scholars.

Workshop: Morphisms in Low Dimensions

Table of Contents

Abstracts

A survey of mapping class groups in dimension 2, 4 and 6 through the eyes of Dehn twists

Matthias Kreck

We denote the mapping class group, the group of isotopy classes of self-diffeomorphisms of a manifold M, by $\pi_0(\mathrm{Diff}\,(M))$. All information in this talk about mapping class groups of surfaces is from [1].

In this talk I started out with the Dehn–Nielsen–Baer theorem saying that the mapping class group of a closed oriented surface is isomorphic to the outer automorphism group of the fundamental group. So in a way it is done. Unfortunately the outer automorphism group is hard to work with. So one is looking for another approach.

This approach works identically up to the last step for a certain interesting class of 6-manifolds, complex 3-dimensional complete intersections. So we presented this approach in parallel for surfaces and complex 3-dimensional complete intersections. Whereas the case of surfaces has a long tradition, the work for complete intersections is new (it is joint work with Su Yang [3]).

Here are the results. Let X stand either for a closed oriented surface of genus g or a 3-dimensional complex complete intersection with 3^{rd} Betti number $2g$.

Theorem.

(1) The action on the integral middle homology with its intersection form, which is a skew-symmetric hyperbolic form, gives (after choosing a symplectic basis) a surjective homomorphism.

$$\pi_0(\mathrm{Diff}\,(M)) \to \mathrm{Sp}(2g, \mathbb{Z}).$$

The surjectivity can be obtained from considering **Dehn twists**, which in the 6-dimensional case are given by embeddings $S^3 \times D^3$ into X and choosing $\alpha \in \pi_3(SO(4))$ represented by a map $\alpha(D^3, S^3) \to (SO(4), 1)$ and mapping $(x, y) \in S^3 \times D^3$ to $(\alpha(y)x, y)$. The kernel of this map is called the **Torelli group** $T(X)$.

(2) There is an explicitly known abelian group A_X and a surjective homomorphism

$$J : T(X) \to A_X.$$

For surfaces this was constructed by Johnson and in the other case by Kreck and Su Yang. In spirit these homomorphisms are similar.

(3) There is an explicitly known abelian group B_X and a surjective homomorphism

$$K : \mathrm{Ker}(J) \to B_X.$$

For surfaces this was constructed by Birman, Craiggs and Johnson, and in the other case by Kreck and Su Yang. Again in spirit they are similar.

(4) Whereas for surfaces the kernel of K is widely unknown the main result of Kreck and Su Yang is that K is an isomorphism.

There are a lot of other similarities between the two cases, like that Dehn twists generate the mapping class groups and that the groups are finitely presentable. For more details we refer to the literature.

We didn't say much in dimension 4 where very little is known. Here is a striking result which indicates the difficulty in dimension 4:

Theorem. (Kronheimer and Mrowka, [4]) The Dehn twist along the separating 3-sphere in the connected sum $K_3\sharp K_3$ of two Kummer surfaces is not isotopic to the identity.

Whereas for surfaces and complex 3-dimensional intersections isotopy is equivalent to pseudo-isotopy the result of Kronheimer and Mrowka is a very simple example of a diffeomorphism which is pseudo-isotopic to the identity (by an old result of the speaker [2]) but not isotopic. Earlier examples of this type were found by Rubermann.

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Mapping class groups of 4-manifolds

Mark Powell

(joint work with Patrick Orson)

I will report on joint work with Patrick Orson on the mapping class groups of compact, simply-connected 4-manifolds.

Given an oriented, topological manifold X, with (possibly empty) boundary, we consider $\mathrm{Homeo}^+(X, \partial X)$, the topological group of orientation preserving self-homeomorphisms that restrict to the identity on the boundary ∂X, with the compact-open topology. The set of connected components $\pi_0 \mathrm{Homeo}^+(X, \partial X)$ is the *topological mapping class group* of X, the group of isotopy classes of orientation preserving self-homeomorphisms that fix the boundary pointwise. We study topological mapping class groups for X a compact, oriented, simply connected 4-manifold.

Let $\lambda_X \colon H_2(X) \times H_2(X) \to \mathbb{Z}$ be the intersection pairing of X. When $\partial X = \emptyset$, it was shown by Perron and Quinn [Qui86, Per86] (cf. Kreck [Kre79, Theorem 1]), that if two orientation preserving self-homeomorphisms of X induce the same isometry of the intersection form then they are isotopic. Freedman [Fre82, Theorem 1.5, Addendum] showed that every automorphism of the intersection form is induced by a homeomorphism. Therefore the results of Perron, Quinn and Freedman combine to compute the mapping class group of every closed, simply connected 4-manifold, in the sense of reducing the problem to algebra:

$$\pi_0 \operatorname{Homeo}^+(X) \xrightarrow{\cong} \operatorname{Aut}(H_2(X), \lambda_X);\ F \mapsto F_*.$$

When X has nonempty boundary, we need to consider a refinement of the automorphism group $\operatorname{Aut}(H_2(X), \lambda_X)$ to capture the algebraic data of a homeomorphism. A map $F \in \operatorname{Homeo}^+(X, \partial X)$ determines a homomorphism

$$\Delta_F \colon H_2(X, \partial X) \to H_2(X)$$

called a *variation* [Lam75, DK75, Kau74], defined by $[x] \mapsto [x - F(x)]$. Using that X has Poincaré-Lefschetz duality, Saeki [Sae06] showed that Δ_F satisfies an additional condition, making it what we call a *Poincaré variation*. There is a binary operation on the set of Poincaré variations, together with which they form a group $\mathcal{V}(H_2(X), \lambda_X)$. The map $F \mapsto F_*$ factors through this group via homomorphisms:

$$\pi_0 \operatorname{Homeo}^+(X, \partial X) \xrightarrow{F \mapsto \Delta_F} \mathcal{V}(H_2(X), \lambda_X) \xrightarrow{\Delta \mapsto \operatorname{Id} - \Delta \circ j} \operatorname{Aut}(H_2(X), \lambda_X),$$

where $j \colon H_2(X) \to H_2(X, \partial X)$ is the quotient map. In general Δ_F contains more information than F_*. Saeki [Sae06] used $\mathcal{V}(H_2(X), \lambda_X)$ to describe the smooth stable mapping class group for simply connected 4-manifolds with nonempty, connected boundary.

When ∂X has more than one connected component and X admits a spin structure, there is a further invariant that does not appear in the closed case nor when the boundary is connected. For $F \in \operatorname{Homeo}^+(X, \partial X)$ we may compare a (topological) spin structure $\mathfrak{s}$ on X with the induced spin structure $F^*\mathfrak{s}$. The two agree on ∂X because F fixes the boundary pointwise. There is a free, transitive action of $H^1(X, \partial X; \mathbb{Z}/2)$ on the set of isomorphism classes of spin structures on X that agree on ∂X, and we denote by $\Theta(F) \in H^1(X, \partial X; \mathbb{Z}/2)$ the class representing the difference between $\mathfrak{s}$ and $F^*\mathfrak{s}$.

Our main result shows that these invariants describe the entire topological mapping class group.

Theorem (Orson-Powell)**.** *Let $(X, \partial X)$ be a compact, simply connected, oriented, topological 4-manifold.*

(1) When X is spin, the map $F \mapsto (\Theta(F), \Delta_F)$ induces a group isomorphism

$$\pi_0 \operatorname{Homeo}^+(X, \partial X) \xrightarrow{\cong} H^1(X, \partial X; \mathbb{Z}/2) \times \mathcal{V}(H_2(X), \lambda_X).$$

(2) When X is not spin, the map $F \mapsto \Delta_F$ induces a group isomorphism

$$\pi_0 \operatorname{Homeo}^+(X, \partial X) \xrightarrow{\cong} \mathcal{V}(H_2(X), \lambda_X).$$

Our key contribution is injectivity of the maps in the theorem. Let us outline the proof strategy. First recall that a *topological pseudo-isotopy* is a homeomorphism $F \colon X \times I \to X \times I$ such that $F|_{\partial X \times I} = \operatorname{Id}_{\partial X \times I}$. The restrictions $F_0 = F|_{X \times \{0\}}$ and $F_1 := F|_{X \times \{1\}}$ are said to be *topologically pseudo-isotopic*. We classify homeomorphisms of simply connected 4-manifolds with boundary, up to topological pseudo-isotopy. The strategy builds on that of [Kre79, Proposition 2]. In broad strokes, if we can find a 6-manifold with boundary the (capped off) mapping torus of F, such that the 6-manifold is a rel. boundary h-cobordism from $X \times [0,1]$ to itself, then it follows that F is pseudo-isotopic to the identity. Our proof consists of an analysis of the obstructions to finding such an h-cobordism, and uses Kreck's modified surgery theory [Kre99] as the main technical tool in its construction. With the pseudo-isotopy classification in hand, the proof that the maps in the theorem are injective concludes by appealing to Quinn's result [Qui86, Theorem 1.4] that topological pseudo-isotopy implies topological isotopy for homeomorphisms of simply connected, compact 4-manifolds.

Of course, injectivity can be applied to diffeomorphisms of smooth 4-manifolds, yielding a topological isotopy. This is a important step in the hunt for exotic diffeomorphisms, which is currently a topic of considerable interest. For example the main theorem was applied in this way recently by Iida-Konno-Mukherjee-Taniguchi [IKMT22].

When X has nonempty, connected boundary, surjectivity of the variation map $\pi_0 \operatorname{Homeo}^+(X, \partial X) \to \mathcal{V}(H_2(X), \lambda_X)$ was already known, and is a consequence of Boyer's classification of simply connected compact 4-manifolds with connected boundary, and a subsequent result of Saeki [Boy86, Boy93, Sae06]. To show that the map (1) of the theorem is surjective, in particular to realise the Θ invariants topologically, requires a novel geometric construction, again in combination with Boyer and Saeki's results [Boy86, Boy93, Sae06].

Dehn twists. An important type of self-homeomorphism of 4-manifolds is the *Dehn twist*, which arises as follows. Let $\phi_t \in \pi_1(\mathrm{SO}(4))$ be a generator based at the identity matrix, represented by a smooth map $S^1 \to \mathrm{SO}(4)$ that is constant near the basepoint. This induces a smooth loop of self-diffeomorphisms of S^3, which generates $\pi_1(\operatorname{Diffeo}^+(S^3)) \cong \mathbb{Z}/2$, and thence a self-diffeomorphism

$$\Phi\colon S^3 \times I \xrightarrow{\cong} S^3 \times I; \qquad (x,t) \mapsto (\phi_t(x), t).$$

Given an embedding of $S^3 \times I$ into a 4-manifold, one can extend the map Φ by the identity to obtain a self-homeomorphism of the entire 4-manifold, and we call any self-homeomorphism obtained this way a *Dehn twist.* If X is smooth to begin with, and $S^3 \times I$ is smoothly embedded, then the Dehn twist is a self-diffeomorphism.

Now let X be a closed, simply connected 4-manifold and decompose $X \setminus \mathring{D}^4$ as the union $N \cup_{S^3 \times \{1\}} S^3 \times I$ of a collar neighbourhood of $\partial(X \setminus \mathring{D}^4)$ and the closure of its complement. The diffeomorphism Φ induces a Dehn twist homeomorphism

$$t_X\colon X \setminus \mathring{D}^4 \to X \setminus \mathring{D}^4; \qquad y \mapsto \begin{cases} \Phi(x,t) & y = (x,t) \in S^3 \times I, \\ y & y \in N. \end{cases}$$

Corollary. *For every closed, simply connected, topological manifold X, the Dehn twist t_X is topologically isotopic rel. boundary to $\mathrm{Id}_{X \setminus \mathring{D}^4}$.*

An explicit geometric argument of Giansiracusa shows that $t_{\mathbb{CP}^2}$ is smoothly isotopic to the identity [Gia08]. This result can be extended to show that t_X

is smoothly isotopic to the identity for any non-spin, smooth, simply connected, closed 4-manifold X; this argument was communicated to us by Auckly, Kronheimer, and Ruberman. On the other hand, it was shown independently by Baraglia-Konno [BK22] and Kronheimer-Mrowka [KM20] that t_{K3} is not smoothly isotopic to the identity. This prompts the obvious question.

Question. For which closed, spin, simply connected, smooth manifolds X is t_X smoothly isotopic to the identity?

Homeomorphisms not restricting to the identity on the boundary. We consider the implications of our results when we relax the assumption that homeomorphisms must fix the boundary pointwise. Let X be a compact, oriented, simply connected 4-manifold. There is a fibre sequence

$$\mathrm{Homeo}^+(X, \partial X) \to \mathrm{Homeo}^+(X) \to \mathrm{Homeo}^+(\partial X).$$

Consequently there is an exact sequence in homotopy groups, extending to the left,

(1)

$$\pi_1 \mathrm{Homeo}^+(\partial X) \to \pi_0 \mathrm{Homeo}^+(X, \partial X) \to \pi_0 \mathrm{Homeo}^+(X) \to \pi_0 \mathrm{Homeo}^+(\partial X).$$

Here, the first arrow can be defined by inserting the loop of diffeomorphisms of ∂X (based at $\mathrm{Id}_{\partial X}$) into a collar of the boundary, and extending by the identity. Taking the basepoint of each group of homeomorphisms to be the respective identity map, the sequence (1) is an exact sequence of groups. Here the π_0 terms are also groups because they are connected components of topological groups. The sequence suggests that the problem of whether two homeomorphisms F_1, $F_2 \colon (X, \partial X) \to (X, \partial X)$ are isotopic in $\mathrm{Homeo}^+(X)$ can be decomposed into two stages, as follows.

The first-stage question is purely about 3-manifolds: are $F_1|_{\partial X}$ and $F_2|_{\partial X}$ isotopic? This is a highly nontrivial question in general, but thanks to the modern spectacular understanding of 3-manifolds, we have a good chance of being able to decide. Self-homeomorphisms of ∂X must respect the prime decomposition [Kne29, Mil62] and the JSJ decomposition [JS79, Joh79]; see also [Hat07]. Restricting to geometric pieces it often suffices to understand the isometry groups (in the sense of Riemannian geometry), by [Gab01, HKMR12, BK21, BK17] and the references therein. For simple 3-manifolds their mapping class groups were known earlier. For lens spaces the mapping class groups were computed by Bonahon [Bon83], while for Seifert fibred spaces in general see e.g. [BO91]. For Haken 3-manifolds, Hatcher and Ivanov [Hat76, Iva79] showed that the mapping class group equals the group of homotopy self-equivalences. So with enough work, the first-stage question can in principle be answered with our current knowledge.

If there is no isotopy between $F_1|_{\partial X}$ and $F_2|_{\partial X}$, then certainly F_1 and F_2 are not isotopic. So let us assume that the 3-manifold question has been solved affirmatively. Then, after an isotopy of F_1 supported in a collar of ∂X we can assume that $F_1|_{\partial X} = F_2|_{\partial X}$. We may ask the second-stage question: is $G := F_2 \circ F_1^{-1} \in \mathrm{Homeo}^+(X, \partial X)$ in the image of $\pi_1 \mathrm{Homeo}^+(\partial X)$?

In some cases, $\pi_1 \operatorname{Homeo}^+(\partial X) = 0$ and so it causes no additional complications. A general condition for this, using work of Gabai, Hatcher, Ivanov, and Waldhausen [Gab01, Hat76, Iva79, Wal67], is as follows.

Proposition. *Let X be a compact, simply connected, oriented, topological 4-manifold and suppose that every connected component of ∂X is irreducible but not Seifert fibred. Then $\pi_1 \operatorname{Homeo}^+(\partial X) = 0$ and so there is exact sequence of groups*

$$0 \to \pi_0 \operatorname{Homeo}^+(X, \partial X) \to \pi_0 \operatorname{Homeo}^+(X) \to \pi_0 \operatorname{Homeo}^+(\partial X).$$

Our main theorem describes the left group. The image of the right hand map was described precisely by Boyer [Boy86, Boy93], for all 3-manifolds. So in the case that every connected component of ∂X is irreducible but not Seifert fibred, the combination of our work with Boyer's results can be employed to complete the two-stage process discussed above.

We considered Seifert fibred 3-manifold boundary components, and studied the problem of realising the invariants in the theorem using loops of diffeomorphisms in a boundary collar. For S^3, lens spaces, and $S^1 \times S^2$ we found some success, showing that for X spin and ∂X a disjoint union of 3-manifolds of Heegaard genus at most one, every element of $0 \times H^1(X, \partial X; \mathbb{Z}/2)$ can be obtained by collar insertion. In addition, if the dimension of $H_1(\partial X; \mathbb{Q})$ is at most one, then we can identify $\mathcal{V}(H_2(X), \lambda_X)$ with a subgroup $\operatorname{Aut}^{\mathrm{fix}}_\partial(H_2(X), \lambda_X)$ of $\operatorname{Aut}(H_2(X), \lambda_X)$. We obtain the following corollary.

Corollary. *Let X be a compact, simply connected, orientable, topological 4-manifold. Suppose that every connected component of ∂X has Heegaard genus at most 1, and at most one of the connected components is $S^1 \times S^2$. Then there is an exact sequence of groups*

$$0 \to \operatorname{Aut}^{\mathrm{fix}}_\partial(H_2(X), \lambda_X) \to \pi_0 \operatorname{Homeo}^+(X) \to \pi_0 \operatorname{Homeo}^+(\partial X).$$

Note that this statement is independent of whether or not X admits a spin structure. Let me end by setting the following challenge.

Challenge. Compute the collar insertion map $\pi_1 \operatorname{Homeo}^+(\partial X) \to \pi_0 \operatorname{Homeo}^+(X, \partial X)$ when ∂X consists of more general Seifert fibred spaces.

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Quasimorphisms, knot invariants, and slice-Bennequin inequalities

Peter Feller

An approach to studying a group is to investigate (group homo/quasi)morphisms from it to some other group. The study of group representations (group homomorphisms to matrix groups) is a famous instance of this approach. Here we take the group of interest to be Artin's braid group B_n [Art25] for a fixed integer $n \geq 3$ and the target is the group of real numbers $\mathbb{R}$.

While, up to scaling, the only group homomorphism to $\mathbb{R}$ is given by the writhe $\mathrm{wr}\colon B_n \to \mathbb{Z} \subset \mathbb{R}$ (since the Abelianization of B_n is infinite cyclic), B_n is known to have many homogeneous quasimorphisms[1], written *hqm* below. The latter is implied by the fact that the mapping class group of a surface (which the braid group can be seen as) acts interestingly on a hyperbolic space (the curve complex) [BF02]. We make 'many' precise: the $\mathbb{R}$-vector space of hqms on B_n is known to be uncountably infinite-dimensional.

Meta question: do hqms on B_n (which we know exist plentifully) relate to knot concordance in the same way group homomorphisms do? We make this precise (see the conjecture and the question below) after explaining how group homomorphisms (i.e. the writhe) relate to knot concordance.

The slice-Bennequin inequality

For a link L—a non-empty oriented closed smooth 1-submanifold of the 3-sphere S^3—denote by $\chi_4(L)$ the largest integer among the Euler characteristics of smooth oriented surfaces in the 4-ball B^4 without closed components and oriented boundary $L \subset \partial B^4 = S^3$. In particular, for a knot K—a connected link—one has $2g_4(K) = 1 - \chi_4(K)$, where g_4 denotes the slice genus. Denoting by $\widehat{\beta}$ the link obtained as the closure of β, we state a celebrated affine linear relation between the writhe and χ_4.

Slice-Bennequin inequality ([Rud93, KM94])**.** $|\mathrm{wr}(\beta)| \leq n - \chi_4\left(\widehat{\beta}\right)\ \forall \beta \in B_n$.

One may wonder whether the writhe is special or whether in fact a similar inequality holds for all hqms.

Conjecture. For every hqm $f\colon B_n \to \mathbb{R}$, there exist constants $A, C \in \mathbb{R}$ such that $|f(\beta)| \leq A\chi_4\left(\widehat{\beta}\right) + C$ for all $\beta \in B_n$.

We answer the conjecture for *the fractional Dehn twist coeeficient* $\omega\colon B_n \to \mathbb{R}$, which is a hqm with many interesting relations to low-dimensional topology (knots, 3-manifolds, contact topology, ...) [GO89, Mal04, HKM07, HKM08, FH19].

Theorem ([Fel22, Theorem 3])**.** $|\omega(\beta)| \leq n - \chi_4\left(\widehat{\beta}\right)$ for all $\beta \in B_n$.

[1] Recall that a *quasimorphism* on a group G is a function $f\colon G \to \mathbb{R}$ such that $\sup_{a,b\in G} |f(ab) - f(a) - f(b)| < \infty$, where $\sup_{a,b\in G} |f(ab) - f(a) - f(b)|$ is called the *defect* of f and is denoted by D_f. A function $f\colon G \to \mathbb{R}$ is said to be *homogeneous* if $f\left(g^k\right) = kf(g)$ for all $g \in G$ and integers k.

This theorem in particular answers [HKK$^+$20, Question 1.6]. It turns out that both the slice-Bennequin inequality and the theorem can be proven by realizing the involved hqms (wr and ω, respectively) as the homogenization of a certain type of knot invariant.

Homogenization of knot invariants

A real-valued knot invariant $I\colon \mathfrak{Knots} \to \mathbb{R}$ is called *1-Lipschitz concordance homomorphism*, if $I(K\#J) = I(K)+I(J)$ and $|I(K)| \leq g_4(K)$ for all K, J in $\mathfrak{Knots}$, where $\mathfrak{Knots}$ denotes the set of isotopy classes of knots. For each 1-Lipschitz concordance homomorphism I, the map

$$\widetilde{I} \to \mathbb{R}, \quad \beta \mapsto \widetilde{I}(\beta) := \lim_{k\to\infty} \frac{I\left(\widehat{\beta^{nk}\delta}\right)(t)}{nk},$$

where δ is any element in B_n with $\widehat{\delta}$ a knot, is a hqm with defect $D_{\widetilde{I}} \leq \frac{n-1}{2}$ [FH19, Lemma A.1], and the following holds.

Key Observation ([Fel22, Appendix A])**.** $\widetilde{I}(\beta) \leq \frac{n-\chi_4(\widehat{\beta})}{2}$ for all $\beta \in B_n$.

In light of the key observation, one approach towards proving above the conjecture would be to answer the following question in the positive.

Question. Let $f\colon B_n \to \mathbb{R}$ be a hqm. Does there exist a 1-Lipschitz concordance homomorphism I and $r \in \mathbb{R}$ such that $f = r\widetilde{I}$?

Example 1. We consider the case when I is a *slice torus invariant*—a 1-Lipschitz concordance homomorphism I with $I(T_{p,p+1}) = g_4(T_{p,p+1}) = (p-1)p/2$ for positive integers p. Slice torus invariants include Ozsváth-Szabó's τ [OS03] and Rasumussen's s [Ras10]. In this case we have $\widetilde{I} = \mathrm{wr}/2$; see e.g. [FH19, Lemma A.3]. Hence, for such I, the key observation recovers the slice-Bennequin inequality.[2]

Example 2. If $I(K) := \frac{\Upsilon_K(\frac{2}{n-1})}{n-1} + \frac{\tau(K)}{2}$ (where Υ is as defined in [OSS17]), then $\widetilde{I} = \omega/2$ [FH19, Theorem 1.3]. Hence, the key observation yields the above theorem.

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[2] We find this to be philosophically pleasing for the following reason. Observe that slice torus invariants are by definition the 1-Lipschitz concordance homomorphisms that are strong enough to reprove the local Thom conjecture (since the latter can be phrased as $g_4(T_{p,p+1}) = (p-1)p/2$ for all positive integers p [KM94]). In fact, the existence of slice torus invariants can be seen to be equivalent to the statement of the local Thom conjecture without making use of any explicit construction of such an invariant [FLL22, Remark 4]. It is then fitting that, via homogenization, slice torus invariants recover the slice Bennequin inequality, which Rudolph derived by elementary means using only the local Thom conjecture as an input.

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Floer homology and non-fibered knot detection

Steven Sivek

(joint work with John A. Baldwin)

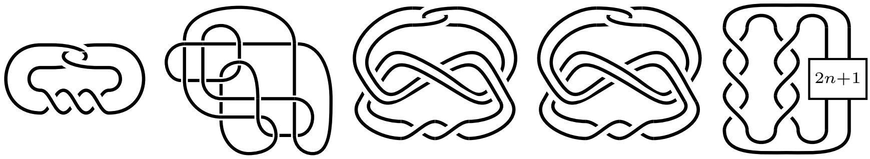
Knot Floer homology assigns to any knot $K \subset S^3$ a bigraded abelian group

$$\widehat{HFK}(K) = \bigoplus_{a,m \in \mathbb{Z}} \widehat{HFK}_m(K,a),$$

and the Seifert genus $g(K)$ is the maximal a such that $\widehat{HFK}_*(K,a)$ is nonzero [OS04a]. Moreover, K is a fibered knot if and only if $\widehat{HFK}(K, g(K))$ has rank 1 [Ghi08, Ni07]. These facts imply that $\widehat{HFK}$ detects the unknot, meaning that $\widehat{HFK}(K) \cong \widehat{HFK}(U)$ as bigraded groups if and only if $K = U$, and likewise the trefoils and figure eight, because these are the only fibered knots of genus ≤ 1. It is also known to detect the cinquefoils [FRW22], which are fibered of genus 2.

This talk focused on recent work with John Baldwin [BS22a], where we proved for the first time that $\widehat{HFK}$ can detect knots which are *not* fibered. The main result is a classification of the "nearly fibered" knots of genus 1.

Theorem 1. Let $K \subset S^3$ be a knot of Seifert genus 1. Then $\dim \widehat{HFK}(K, 1; \mathbb{Q}) = 2$ if and only if K or its mirror is one of the following:



5_2 $15n_{43522}$ $\mathrm{Wh}^-(T_{2,3}, 2)$ $\mathrm{Wh}^+(T_{2,3}, 2)$ $P(-3, 3, 2n+1)$

Among these knots, we note that $\widehat{HFK}$ uniquely detects 5_2 and $\mathrm{Wh}^+(T_{2,3}, 2)$; it cannot distinguish $15n_{43522}$ from $\mathrm{Wh}^-(T_{2,3}, 2)$, or any of the pretzel knots $P(-3, 3, 2n + 1)$ from each other. With a little extra work, we can then use other knot homologies to tell the pretzels apart:

Theorem 2. Reduced Khovanov homology detects 5_2, and reduced HOMFLY homology detects each of the pretzel knots $P(-3, 3, 2n + 1)$.

Remark 3. We expect that reduced Khovanov homology should be enough to detect each of the pretzels $P(-3, 3, 2n + 1)$, but we were unable to prove it.

Theorem 1 also lets us draw some purely topological conclusions. We say $r \in \mathbb{Q}$ is a *characterizing slope* for $K \subset S^3$ if $S^3_r(K) \cong S^3_r(J)$ implies that $K = J$.

Theorem 4 ([BS22b, BS22c])**.** Every $r \in \mathbb{Q} \setminus \mathbb{Z}_{>0}$ is characterizing for 5_2. If K is any of the knots of Theorem 1, then 0 is characterizing for K.

The first step in the proof of Theorem 1 is to classify the possible complements of genus-minimizing Seifert surfaces. If F is a Seifert surface for K, then the sutured Floer homology of

$$S^3(F) = (S^3 \setminus N(F), \lambda_K)$$

can be identified with $\widehat{HFK}(K, g(F))$. When $\dim SFH(S^3(F)) = 1$, properties of SFH tell us that

$$S^3(F) \cong (F \times [-1, 1], \partial F \times \{0\}),$$

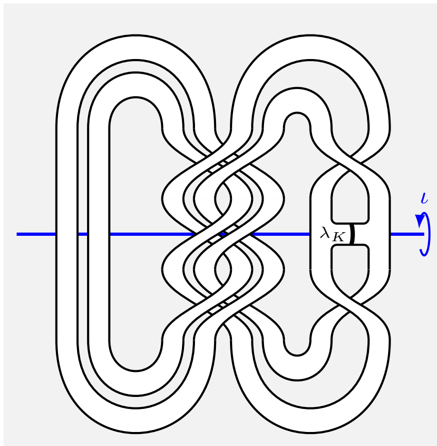
so this recovers the fact that K must be fibered. We are instead concerned with the 2-dimensional case, so $S^3(F)$ is no longer a product sutured manifold; however, work of Juhász [Juh10] tells us that since $\dim SFH(S^3(F))$ is sufficiently small, there must be an essential product annulus in $S^3(F)$. We decompose $S^3(F)$ along this annulus and repeat, and eventually we have simplified the topology enough that only two possibilities remain:

Proposition 5. Let F be a genus-1 Seifert surface for K, and suppose that $\dim SFH(S^3(F)) = 2$. Then $S^3(F)$ is the complement of the union of

- the $(2, 4)$-cable of either the unknot or the right-handed trefoil, and
- a properly embedded, non-separating arc in the cabling annulus,

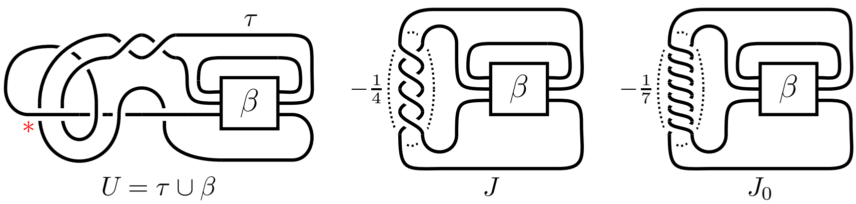
up to orientation reversal. Its suture is a meridian of that arc.

Once we know $S^3(F)$, viewed as the complement of a product $F \times [-1,1]$, it remains to be seen how we can glue $F \times \{1\}$ to $F \times \{-1\}$ to recover the complement of K. The key observation is that in either case, $S^3(F)$ admits an involution ι which restricts to $F \times \{\pm 1\}$ as a hyperelliptic involution. Since $g(F) = 1$, this involution is central in the mapping class group of F, and this allows us to extend ι across $F \times [-1,1]$ to the whole of S^3. Here we illustrate $(S^3(F), \iota)$ in case where $S^3(F)$ is built from a cable of a trefoil:



Taking the quotient by ι, we realize $S^3(F)$ as the branched double cover of a fixed tangle τ in the 3-ball, and $F \times [-1,1]$ as the branched double cover of some 3-braid β in $D^2 \times [-1,1]$. Then $\tau \cup \beta$ must be unknotted, since its branched cover is S^3, so it remains to determine all such β and produce the corresponding K.

We can only give a hint here of how to enumerate the possible braids β in the trefoil case. After some simplification, we are led to the unknot diagram at left:



Changing the indicated crossing turns U into a knot of the form $T_{-2,3} \# J$. The Montesinos trick tells us that its branched double cover $L(3,2)\#\Sigma_2(J)$ arises as some $\frac{2n+1}{2}$-surgery on a knot c in $\Sigma_2(U) \cong S^3$. But half-integer surgeries must be irreducible [GL87], so $L(3,2)\#\Sigma_2(J) \cong L(3,2)$, and then c and J are unknotted and $\frac{2n+1}{2} = \frac{3}{2}$. Now we instead take the 0-resolution of that crossing of U to get J_0; its branched double cover is $S^3_n(c) \cong S^3_1(U) \cong S^3$, so J_0 is unknotted as well.

Both J and J_0 are unknots differing in a single rational tangle, so we can replace it with another rational tangle of slope $\frac{p}{q}$ to get a 2-bridge link with fraction $\frac{p}{q}$.

In the cases 0 () or ∞ () we see that the braid closure $\hat{\beta}$ is a 2-component unlink, and that a certain 2-bridge plat closure involving β is unknotted. The 3-braids with $\hat{\beta} = U \sqcup U$ are known up to conjugacy, and from there we can pin down the actual braids β, which end up giving rise to $K = \mathrm{Wh}^{\pm}(T_{2,3}, 2)$.

The remaining knots in Theorem 1 arise when $S^3(F)$ comes from a $(2,4)$-cable of the unknot, and that case is harder but based on similar ideas. These arguments could plausibly generalize to knots K for which $S^3(F)$ comes from a $(2,2n)$-cable of the unknot or of $T_{2,3}$, at least for small values of n, and this would be useful in enumerating genus-1 knots with $\dim \widehat{HFK}(K,1) = n > 2$. The problem is that at present we do not know how to prove the analogue of Proposition 5 that would classify all possible $S^3(F)$, even for $n = 3$.

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Khovanov homology and the Involutive Heegaard Floer homology of branched double covers

Melissa Zhang

(joint work with Akram Alishahi, Linh Truong)

Since the turn of the millennium, Khovanov and Heegaard Floer homologies have been prominent tools in the study of knots and their relationship with 3- and 4-manifolds. Khovanov's original link homology theory [6], *Kh*, is an invariant of isotopy classes of smooth links in S^3, and is computed from a flat diagram of the link. Ozsváth and Szabó's original Heegaard Floer homology [12, 13] is an invariant of 3-manifolds, and is computed from a Heegaard diagram of the 3-manifold.

The first direct relationship between *Kh* and *HF* was Ozsváth and Szabó's *link surgeries spectral sequence* [14]. Given a knot $K \subset S^3$, they define a spectral sequence $E^{\bullet}$ such that

$$E^2 \cong \widetilde{Kh}(\bar{K}),$$

where $\bar{K}$ is the mirror of the knot K (necessary due to convention choices), and $\widetilde{Kh}$ is *reduced Khovanov homology*, a version of Khovanov homology for based links. This spectral sequence abuts to

$$E^\infty \cong \widehat{HF}(\Sigma(K)),$$

where $\Sigma(K)$ is the branched double cover of S^3 along K, and $\widehat{HF}$ is the simplest version of Heegaard Floer homology, a vector space over the field $\mathbb{F}_2$. In particular, this spectral sequence is shown to collapse immediately (i.e. $E^2 \cong E^\infty$) for *quasi-alternating knots*, which are a class of knots extending the class of alternating knots, defined by a recursive condition on their knot determinants.

In the past decade, Hendricks and Manolescu's *involutive Heegaard Floer homology* [5], HFI, has proven to be immensely powerful in numerous applications in low-dimensional topology. These homology theories incorporate the action of ι, a homotopy involution on the Heegaard Floer complex obtained by modifying Heegaard diagrams for the 3-manifold. The hat flavor, $\widehat{HFI}$, assigns to each 3-manifold an $\mathbb{F}_2[Q]/(Q^2)$-module.

Following Hendricks and Manolescu's construction, Lin defined *involutive monopole Floer homology* [9], $\widetilde{HMI}$, an involutive version of a gauge-theoretic 3-manifold invariant, *monopole Floer homology*, $\widetilde{HM}$ [7]. Lin also constructed a spectral sequence of $\mathbb{F}_2[Q]/(Q^2)$ modules, relating a version of Bar-Natan homology (a perturbation of Khovanov homology [2]), $E^2 \cong \widetilde{Kh}(\bar{K})$, to $E^\infty \cong \widetilde{HMI}(\Sigma(K))$. This is an involutive version of Bloom's spectral sequence relating $\widetilde{Kh}(\bar{K})$ to $\widetilde{HM}(\Sigma(K))$, an analogue to Ozsváth and Szabó's link surgeries spectral sequence, for monopole Floer homology.

The historical account above begs the following question: "Is there a link surgeries spectral sequence relating some version of Khovanov homology to involutive Heegaard Floer homology?" The purpose of this talk is to announce upcoming work with Alishahi and Truong, where we construct an involutive version of Ozsváth and Szabó's spectral sequence, with

$$E^2 \cong \widetilde{Kh}(\bar{K}) \qquad \text{and} \qquad E^\infty \cong \widehat{HFI}(\Sigma(K)).$$

This is a spectral sequence of $\mathbb{F}_2[Q]/(Q^2)$-modules, and is analogous to Lin's spectral sequence.

This result is possible due to the availability of the following tools. First, Lipshitz, Ozsváth, and Thurston defined a modular version of HF called *bordered Floer homology* [11], and re-constructed Ozsváth and Szabó's link surgeries spectral sequence in this context [10]. Second, Hendricks and Lipshitz defined a surgery exact triangle for *bordered involutive Heegaard Floer homology* [4], allowing us to make explicit computations of the cobordism maps needed in constructing the multicone used in the construction of our spectral sequence.

We know that our spectral sequence collapses immediately for quasi-alternating knots. We also know of examples where the spectral sequence does not collapse immediately; we obtain these by computing $\widetilde{BN}$ for these knots using Lewark's `khoca` program [8].

We expect that our spectral sequence is *weakly functorial*, in the sense of [1]. We also have the following questions:

- Is there a (describable) larger class of knots for which the spectral sequence collapses immediately?
- What is the significance of the page of the spectral sequence on which it collapses?
- In what situations can this spectral sequence be used to deduce information about the involutive Heegaard Floer homology of a branched double cover?
- Does $\widetilde{BN}$ come from a homotopy involution on (some chain homotopy representative of) the Khovanov complex?

We conclude by speculating wildly about the last question. Consider knot diagram D, drawn on the xy-plane. Now reflect the diagram across the y-axis, and reverse all the crossings. This diagram, which we denote by $\bar{D}$, is another diagram for the same knot. There is a sequence of Reidmeister moves that transforms $\bar{D}$ back into D. What is the induced endomorphism on the Khovanov chain complex for D? Is this is a non-trivial homotopy involution ι? If so, we could define *involutive Khovanov homology*; one would hope this would be isomorphic to $\widetilde{BN}$!

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The joy of not being a PID

Lukas Lewark

This talk, based on work in progress, is about Khovanov homology giving rise to a homomorphism from the smooth concordance group $\mathcal{C}$ to some abelian group $\mathcal{G}$, which unifies and generalizes the Rasmussen invariants defined over $\mathbb{Q}$ [8], over the prime fields $\mathbb{F}_p$ [6], and over the integers $\mathbb{Z}$ [9].

First, let us describe how to treat Khovanov homology as a black box satisfying certain properties. Equip the polynomial ring $\mathbb{Z}[x]$ with a so-called *quantum grading* denoted by qdeg, by setting $\operatorname{qdeg} x = -2$. The *Khovanov chain complex* $[\![K]\!]$ of a knot K is constructed as a graded chain complex over $\mathbb{Z}[x]$, well-defined up to graded chain homotopy equivalence $\simeq$. For a $\mathbb{Z}[x]$-module M, *Khovanov homology with coefficients in* M, denoted by $\mathrm{Kh}(K; M)$, is defined as the homology of $[\![K]\!] \otimes M$ (all tensor products are over $\mathbb{Z}[x]$). If M is a graded module, then so is $\mathrm{Kh}(K; M)$. Note that the isomorphism type of $\mathrm{Kh}(K; M)$ is a knot invariant. The following properties hold for all knots K and J:

(1) $[\![K]\!]$ consists of grading shifted free modules, and is of finite total rank.
(2) $[\![\text{unknot}]\!] \simeq \mathbb{Z}[x]$.
(3) $[\![K\#J]\!] \simeq [\![K]\!] \otimes [\![J]\!]$.
(4) $[\![-K]\!] \simeq [\![K]\!]^*$, where $-K$ denotes the mirror image of K with reversed orientation (so that the concordance classes of K and $-K$ are inverse), and C^* denotes the graded dual of C, i.e. C_i^* is the $\mathbb{Z}[x]$-module of graded $\mathbb{Z}[x]$-homomorphisms from C_i to $\mathbb{Z}[x]$.
(5) For the ungraded $\mathbb{Z}[x]$-module $Z \coloneqq \mathbb{Z}[x]/(x-1)$, the homology module $\mathrm{Kh}_i(K; Z)$ is isomorphic to Z for $i = 0$, and trivial for $i \neq 0$.
(6) If there exists a smooth connected cobordism of genus g from K to J, then there exists a graded chain map $[\![K]\!] \to [\![J]\!]$ of quantum degree $-2g$ that induces an isomorphism on $\mathrm{Kh}(\,\cdot\,; Z)$.

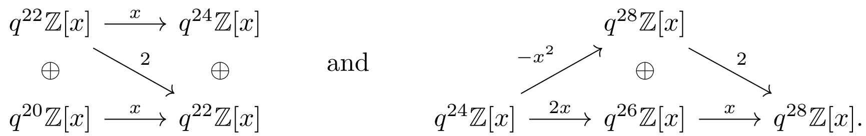
Note that these properties are not sufficient to determine $[\![\cdot]\!]$. For example, setting $[\![K]\!] \coloneqq \mathbb{Z}[x]$ for all knots K would satisfy (1)–(6). For experts in the subject, let us remark that the chain complex $[\![\cdot]\!]$ we are using here is the one coming from the Frobenius algebra $\mathbb{Z}[x][y]/(y^2 + yx)$ over $\mathbb{Z}[x]$, or equivalently from Bar-Natan homology of the 2-ended tangle obtained by cutting K open [4, 1, 7, 3]. For example, one has (where q^k signifies a quantum grading shift by k)

$$[\![(3,4)\text{-torus knot}]\!] \simeq q^6\mathbb{Z}[x] \longrightarrow 0 \longrightarrow q^{10}\mathbb{Z}[x] \xrightarrow{x} q^{12}\mathbb{Z}[x] \xrightarrow{0} q^{12}\mathbb{Z}[x] \xrightarrow{x^2} q^{16}\mathbb{Z}[x].$$

For a field $\mathbb{F}$, the chain complex $[\![K]\!] \otimes \mathbb{F}[x]$ is a graded chain complex over a PID. Thus, up to $\simeq$ it decomposes essentially uniquely into summands $\mathbb{F}[x]$ and summands $\mathbb{F}[x] \to \mathbb{F}[x]$ with differential x^n for some $n \geq 1$. From property (5), it follows that there is a unique summand $\mathbb{F}[x]$, which has homological degree 0. A *Rasmussen invariant* may now be defined as the quantum grading of a generator of that summand. The characteristic c of $\mathbb{F}$ determines this Rasmussen invariant, so one may denote it by s_c. From the properties (1)–(6) one deduces quickly that

$s_c/2$ gives a homomorphism $\mathcal{C} \to \mathbb{Z}$, and that $|s_c(K)/2|$ is a lower bound for the smooth slice genus of K.

How can one extract a concordance homomorphism from $[\![\,\cdot\,]\!]$ without tensoring with a simpler module first? Over the non-PID $\mathbb{Z}[x]$, indecomposable graded chain complexes may be quite complicated (in particular, they are not classified). For example, $[\![(4,5)\text{-torus knot}]\!]$ admits the indecomposable summands



Moreover, the decomposition of chain complexes as sum of indecomposables is generally not unique [3]. So, instead of focusing on a special summand, let us pursue a different strategy. Let $\mathcal{R}$ be the set of graded chain complexes over $\mathbb{Z}[x]$ consisting of grading shifted free modules, of finite total rank, modulo the equivalence relation $\sim$. Here, $C \sim D$ if there are graded chain maps $f\colon C \to D$ and $g\colon D \to C$ that induce isomorphisms $f\colon H_*(C \otimes Z) \to H_*(D \otimes Z)$ and $g\colon H_*(D \otimes Z) \to H_*(C \otimes Z)$. Then $\mathcal{R}$ is a commutative semiring with addition $\oplus$, zero-element the trivial complex, multiplication $\otimes$, and one-element $\mathbb{Z}[x]$. Let $\mathcal{G} \subset \mathcal{R}$ consist of those C with $H_i(C \otimes Z)$ isomorphic to Z for $i = 0$, and trivial for $i \neq 0$. For $C \in \mathcal{G}$, one can show that $C \otimes C^* \sim \mathbb{Z}[x]$. It follows that $\mathcal{G}$ is an abelian group. From the properties (1)–(6), one may quickly deduce the following.

Theorem 1. Sending a knot K to $[\![K]\!]/\!\sim$ induces a group homomorphism $\mathcal{C} \to \mathcal{G}$.

There are similar constructions of concordance homomorphisms coming from other knot homologies, such as $CFK^\infty(K)/\!\sim$ from knot Floer homology [2] with coefficients $\mathbb{F}_2[U,V]$, and $S_n(K)$ from $\mathfrak{sl}_n$ homology [5] with coefficients $\mathbb{C}[x, a_1, \ldots, a_{n-1}]$. In all three cases, a coefficient ring that is not a PID appears as crucial ingredient for a rich invariant.

What can one say about the isomorphism type of $\mathcal{G}$? For a field $\mathbb{F}$ and $C \in \mathcal{G}$, there is a unique $\mathbb{F}[x]$ summand of $C \otimes \mathbb{F}[x]$. Denote by $s_c(C)$ the quantum degree of a generator of this summand, where $c = \operatorname{char} \mathbb{F}$. If $C \sim [\![K]\!]$ for some knot K, then $s_c(C) = s_c(K)$. In particular $[\![K]\!]/\!\sim$ determines the Rasmussen invariants s_c. The following will be proved in an upcoming paper.

Theorem 2. There is a surjective group homomorphism $\mathcal{G} \to \mathbb{Z}^\infty$ given by $1/2(s_0(C), s_2(C) - s_0(C), s_3(C) - s_0(C), s_5(C) - s_0(C), \ldots)$. The kernel of this homomorphism is infinitely generated.

Recently, Schütz [9] introduced an integral version of the Rasmussen invariant, denoted by $s_\mathbb{Z}$, which is a concordance invariant, but not a homomorphism. The invariant $s_\mathbb{Z}$ is defined in terms of the spectral sequence coming from the filtration induced on $[\![K]\!] \otimes Z$ by the quantum grading. In fact, $s_\mathbb{Z}$ encodes precisely the isomorphism type of the filtered abelian group $\mathrm{Kh}_0(K; Z) \cong \mathbb{Z}$. It follows that $s_\mathbb{Z}(K)$ is also determined by $[\![K]\!]/\!\sim$. Schütz's astonishing discovery that there are knots K for which $s_c(K) = 0$ holds for all c, but $s_{\mathbb{Z}}(K) \neq s_{\mathbb{Z}}(U)$, thus implies the non-triviality of the kernel of the homomorphism $\mathcal{G} \to \mathbb{Z}^\infty$ mentioned in Theorem 2.

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Stable homeomorphism and homotopy equivalence

Anthony Conway
(joint work with Diarmuid Crowley, Lisa Piccirillo, Mark Powell, Joerg Sixt)

In what follows, manifolds are assumed to be compact, connected and oriented. We work in the topological category unless stated otherwise.

Context. How can one try to decide if two 4-manifolds M and N are homeomorphic? Classical surgery theory suggests one should first find a homotopy equivalence $M \simeq N$ and then "improve" it to a homeomorphism [Wal99]. Modified surgery theory proposes to first find a homeomorphism $M \#_r S^2 \times S^2 \cong N \#_r S^2 \times S^2$ (from now on we write $\cong_s$ for stable homeomorphism) and then attempt to cancel off the $S^2 \times S^2$ summands [Kre99].

The idea behind modified surgery theory is that the stable classification is less difficult to approach than the homotopy classification. To get a sense of why this might be true, note for simply-connected, spin, closed 4-manifolds M and N, it is known that $M \cong_s N$ if and only if the signatures of M and N agree, whereas $M \simeq N$ if and only if the intersection forms of M and N agree. More formally, Kreck [Kre99] has reduced the stable classification to a (difficult) bordism problem that can be approached using spectral sequence arguments; see e.g. [Tei92, Spa03, Dav05, KLPT17, HH19, KPT20, KPT21]

What about the cancellation aspect of the modified surgery approach? A striking result in this direction is due to Hambleton and Kreck [HK93] who showed that every closed 4-manifold that is stably homeomorphic to $M := M_0 \# S^2 \times S^2$ with $\pi_1(M_0)$ finite is homeomorphic to M; see [CS11, Kha17] for related results. Naturally, such cancellation results cannot hold in general as can be seen by using Freedman's results [Fre82] to realise distinct, stably isometric, nonsingular symmetric bilinear forms by closed simply-connected 4-manifolds.

Since the (equivariant) intersection form gets in the way of cancellation, it is natural to wonder whether stably homeomorphic 4-manifolds with isometric equivariant intersection forms are homeomorphic. Here the *equivariant intersection form* refers to the Hermitian form

$$\lambda_M \colon H_2(\widetilde{M}) \times H_2(\widetilde{M}) \to \mathbb{Z}[\pi_1(M)]$$

on the universal cover $\widetilde{M}$ of M, defined by $\lambda_M([A],[B]) := \sum_{g \in \pi} (A \cdot gB)\ g^{-1}$, where $A \cdot gB$ is the algebraic intersection between A and the g-translate of B.

This "cancellation with fixed form" holds when $\pi := \pi_1(M) \cong \pi_1(N)$ is trivial [Fre82], infinite cyclic [FQ90], finite cyclic [HK88] and a Baumslag-Solitar group of the form $BS(1,n) \cong \langle a, b, \mid aba^{-1} = b^n \rangle$ [HKT09] but does not hold in general: for every integer $k \geq 1$, Kreck and Schafer [KS84] found examples of closed, smooth $4k$-manifolds M, N with $\pi \cong \mathbb{Z}_5^3$ that are stably diffeomorphic and have hyperbolic equivariant intersection form but are not homotopy equivalent.

Reformulating (but stating the results in the topological category for brevity), for every integer $k \geq 1$, Kreck and Schafer found a closed $4k$-manifold M for which the following set (which we call the *fixed-form homotopy stable class*) has cardinality at least 2:

$$\mathcal{S}_\lambda^{st,h}(M) = \{N \mid N \cong_s M, \quad \lambda_M \cong \lambda_N\} / \text{ homotopy equivalence}.$$

The Kreck-Schafer $4k$-manifolds are constructed by taking the boundary of thickened $2k$-complexes; they are distinguished using their k-invariants. Some questions now arise: can one systematise the Kreck-Schafer construction to produce manifolds with larger $\mathcal{S}_\lambda^{st,h}(M)$? Can one get a sense of the fundamental groups for which $\mathcal{S}_\lambda^{st,h}(M)$ is trivial? finite? infinite?

Statement of results. In high dimensions we are able to produce families of smooth manifolds with large fixed-form homotopy stable class. In dimension 4, we obtain a similar result when the boundary is nonempty.

Theorem 1 ([CCPS21a, CCPS21b, CCP22])**.**

- For every $k \geq 2$ and $n > 0$, there is a smooth, closed, simply-connected $4k$-manifold M with hyperbolic intersection form and $|\mathcal{S}_\lambda^{st,h}(M)| > n$.
- For every $k \geq 2$ and $n > 0$, there is a smooth, closed $4k$-manifold M with $\pi_1(M) \cong \mathbb{Z}$, hyperbolic equivariant intersection form and $|\mathcal{S}_\lambda^{st,h}(M)| = \infty$.
- For every odd prime q, there is a 4-manifold M_q with $\pi_1(M_q) = \mathbb{Z}$, equivariant intersection form $(2q)$, boundary $L(2q,1)$ and $|\mathcal{S}_\lambda^{st,h}(M_q)| = \infty$.

Summarising, our contribution is to produce manifolds that have both large $\mathcal{S}^{st,h}_{\lambda}(M)$ and simple fundamental group. At the time of writing, the closed 4-dimensional case remains challenging.

Remark 2. Here are some further comments on Theorem 1.

- In the closed higher dimensional cases:
  - The tangent bundles of the manifolds are stably trivial; this procludes them from being distinguished by their characteristic classes.
  - The manifolds are stably diffeomorphic; we stated the results in the topological category to avoid introducing more notation.
  - The manifolds are distinguished by the cohomology ring (of their universal cover) but in an indirect way: we do not calculate the ring.
- In the 4-dimensional case with boundary:
  - In the definition of $\mathcal{S}^{st,h}_{\lambda}(M)$, we mod out by homotopy equivalences that restrict to homotopy equivalences on the boundary. If we require our homotopy equivalences to be homeomorphisms on the boundary, we obtain stronger results.
  - The invariant is, roughly speaking, the map $H_2(\widetilde{M}, \partial\widetilde{M}) \to H_1(\partial\widetilde{M})$. In practice it takes the form of an isometry of the Blanchfield form

    $$\mathrm{Bl}_{\partial M} \colon H_1(\partial\widetilde{M}) \times H_1(\partial\widetilde{M}) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}].$$

    This invariant takes values in a set $\mathrm{Aut}(\mathrm{Bl}_{\partial M})/\,\mathrm{hAut}^+(\partial M) \times \mathrm{Aut}(\lambda_M)$ whose definition we omit.

Proof ideas.

Proof idea of the 4*-dimensional result.* We consider the case $q = 3$ for concreteness, but the proof is identical in general. Consider $M := (S^1 \times D^3) \natural X_6(U)$, where $X_6(U)$ denotes the 6-trace on the unknot. The first step of the proof consists of showing that our invariant gives rise to a surjection $\mathcal{S}^{st,h}_{\lambda}(M) \twoheadrightarrow \mathrm{Aut}(\mathrm{Bl}_{\partial M})/\,\mathrm{hAut}^+(\partial M) \times \mathrm{Aut}(\lambda_M)$, whereas the second step shows that the target is infinite. The second step is tedious but explicit: it involves understanding the Blanchfield form of ∂M and its isometries; this is the main technical work of [CCP22]. We therefore describe the idea behind the first step, i.e. how to construct a 4-manifold with boundary realising the algebra; this construction comes from [CPP22].

As we mentioned above, for the purposes of this note we think of our obstruction as a $\mathbb{Z}[t^{\pm 1}]$-linear map $f \colon \mathbb{Z}[t^{\pm 1}] = H_2(\widetilde{M}, \partial\widetilde{M}) \to H_1(\partial\widetilde{M}) = \mathbb{Z}[t^{\pm 1}]/(6)$. Add 2-handles to the top of $\partial\widetilde{M} \times [0,1]$ along all the $\mathbb{Z}$-translates of a knot representing $f(1)$ with "equivariant framing" 6. Use surgery theory to show that the top of the resulting cobordism W_f bounds a 4-manifold B that has the homotopy type of S^1. The manifold we are after is $M_f := W_f \cup_\partial B$.

The manifold M_f is spin, has boundary ∂M, fundamental group $\pi_1(M_f) = \mathbb{Z}$, equivariant intersection form (6), and realises f. One can then use [Kre99] (with an argument from [CP20]) to show that since our manifolds are spin, have $\pi_1 = \mathbb{Z}$, $\lambda \cong (6)$, and the same boundary, they are stably homeomorphic. □

Proof idea of the higher-dimensional results. The obstruction, despite being extracted from the cohomology ring, is challenging to describe and so we focus on the construction of our manifolds. To this effect, we first recall (odd-dimensional) *Wall realisation* from classical surgery theory. Given a closed smooth $2q$-manifold M and $x \in L_{2q+1}(\mathbb{Z}[\pi_1(M)])$, Wall realisation outputs a cobordism (W_x, M_x, M) based on M and a degree one normal map

$$(F, f, \mathrm{id}_M)\colon (W_x, M_x, M) \to (M \times [0,1], M, M)$$

whose surgery obstruction is x. The cobordism W_x is obtained from $M \times [0,1]$ by first attaching trivial q-handles and then adding an equal number of $(q+1)$-handles in a way that is "dictated" by x. The manifolds M and M_x are stably diffeomorphic, as well as homotopy equivalent; thus Wall realisation cannot be used to prove Theorem 1.

As a consequence we prove a "modified surgery analogue of Wall realisation": we show how to realise subsets of the ℓ-monoid $\ell_{2q+1}(\mathbb{Z}[\pi_1(M)])$ by cobordisms based on M [CCPS21b]. The construction is similar to that of Wall realisation (in particular M and M_x are stably diffeomorphic) but for $q \geq 4$ even, we show that M_x need not be homotopy equivalent to M (extracting a suitable homotopy invariant from the cohomology ring takes some effort). Some additional work is then needed to ensure that the equivariant intersection forms of M and M_x are isometric (in general, the realisation process only ensures stable isometry). □

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Diagrams for contractible spaces of 4–manifolds

David Gay

We begin this talk by considering the following thought experiment: I draw a Kirby diagram for a closed manifold, on a piece of paper, and pass it to audience member A, who copies it down on their own piece of paper. Person A passes their copy to person B, who makes their own copy, and the process repeats until everyone in the room has their own copy. We are all careful topologists so we have all correctly copied the diagram, in other words all of our diagrams are equivalent modulo planar isotopies. Now we each use our diagram to build a closed 4–manifold; person A's 4–manifold is called X_A, person B's 4–manifold is X_B, and so on. One of the basic teachings of the smooth 4–manifold establishment is that all of these 4–manifolds are diffeomorphic. In other words, none of the choices made after my initial drawing of the diagram "matter", in the sense that the given diagram uniquely determines a 4–manifold up to diffeomorphism. So far so good.

Now, however, we remember that this is a workshop on *morphisms* in low-dimensional topology, not on *objects*. If the (isomorphism classes of) objects can be described by diagrams, can the morphisms be described by diagrams? One might imagine that if a Kirby diagram D describes 4–manifold $X(D)$ and another diagram D' describes 4–manifold $X(D')$, then a sequence of moves on Kirby diagrams connecting D to D' should describe a diffeomorphism from $X(D)$ to $X(D')$. However, just as $X(D)$ and $X(D')$ are only specified *up to diffeomorphism*, one should at most expect the sequence of diagrams to describe a diffeomorphism up to isotopy. In general, this should be possible but the purpose of this talk is to think about potential pitfalls and solutions along the way, and what we should be thinking about if we want to improve this idea still further to think about higher homotopy groups (not just π_0) of spaces of diffeomorphisms between 4–manifolds. In particular, we have started with Kirby diagrams since they are the most standard diagrams for 4–manifolds, but we will see that in fact *trisection diagrams* (or at least some kind of enhanced version of trisection diagrams) have some advantages over Kirby diagrams for at least the 0'th level in this program.

Returning to our thought experiment, persons A and B could discover a diffeomorphism $\phi_{AB} : X_A \to X_B$, persons B and C could discover a diffeomorphism $\phi_{BC} : X_B \to X_C$ and persons C and A could discover a diffeomorphism $\phi_{CA} : X_C \to X_A$. After all, all three people were using the same diagram! Then we could ask about $\phi_{CA}\phi_{BC}\phi_{AB} : X_A \to X_A$. Surely, since we all started with the same diagram, we might hope that this diffeomorphism is isotopic to the identity. However, there is really no reason from the general theory of Kirby diagrams to expect this, and the real problem is that the spaces of choices one makes along the way from a diagram to a manifold might itself have nontrivial topology.

Thus the 0'th level referred to above is to ensure that our diagrams determine 4–manifolds in such a way that we can honestly speak about the *identity diffeomorphism* from a diagram-determined 4–manifold back to itself. Since it is not reasonable to ask that a diagram uniquely determine one single manifold (i.e. an actual set of points with an actual atlas of charts), the next best thing is to ask that all the manifolds determined by a single diagram fit together into a larger space as fibers of a bundle over a contractible base space. There is a natural base space in this case, namely the space of *auxiliary choices* one needs to make to go from a diagram to a manifold. These spaces of choices would generally be embedding spaces, or bundles of embedding spaces over embedding spaces, and so on. For example, for a Kirby diagram decribing a 4–manifold with two 2–handles, a 3–handle and a 4–handle, these choices include the space of embeddings of two solid tori into S^3 such that projections of their cores are isotopic to the given diagram, together with the space of embeddings of $S^2 \times I$ into the result of surgery along the embedded solid tori (a bundle of embedding spaces over the space of embeddings of the solid tori), and finally the 4–handle attaching map, a space of diffeomorphisms of S^3 with the result of surgery along all the 2– and 3–handles.

We now define a class of diagrams for 4–manifolds which we claim fit the bill, i.e. which are diagrams for contractible spaces of 4–manifolds.

Definition 1. A *framed point* on an oriented surface Σ is a point $p \in \Sigma$ together with an oriented frame for $T_p\Sigma$, i.e. a positively oriented pair of linearly independent tangent vectors (U_p, V_p) at p.

Definition 2. A *pre-marked* surface is an oriented surface Σ together with a finite number of framed points $\{p_1, \dots, p_n\}$ with frames $\{(U_{p_1}, V_{p_1}), \dots, (U_{p_n}, V_{p_n})\}$. Thus the full data is the tuple $\mathbf{\Sigma} = (\Sigma, \{p_1, \dots, p_n\}, \{(U_{p_1}, V_{p_1}), \dots, (U_{p_n}, V_{p_n})\})$.

Once and for all, at the beginning of time, for each genus g and each $n \in \mathbb{N}$, fix a "standard" pre-marked genus g surface $\mathbf{\Sigma_{g,n}}$ with n framed marked points. This is important so that our spaces of choices below do not include the choice of surface, marked points and frames.

Definition 3. A *framed, marked, generalized cut system* on a pre-marked surface $\mathbf{\Sigma}$ is a collection of disjoint oriented simple closed curves $\mathbf{C} = \{C_1, \dots, C_k\}$ in Σ and a preferred subset $P_{\mathbf{C}} \subset \{p_1, \dots, p_n\}$ of the pre-marked framed points satisfying the following conditions:

- The simple closed curves cut Σ into a collection of genus 0 components. (In particular, k is greater than or equal to the genus of Σ.)
- Each C_i passes through exactly one marked point p_j, with the first vector U_j in the frame at p_j being positively tangent to C_i.
- Each component of $\Sigma \setminus (C_1 \cup \ldots \cup C_k)$ contains exactly one of the preferred marked points $P_{\mathbf{C}}$.

An *isotopy* of a framed, marked, generalized cut system is an isotopy of the curves $C_1, \dots, C_k$ preserving all the above properties. In particular, the pre-marked points do not move, the curves do not move at the marked points, and the curves do not cross any of the extra pre-marked points during their isotopies.

Now we are ready to describe our diagrams for 4–manifolds.

Definition 4. A *framed, marked, generalized trisection diagram $\mathcal{D}$* consists of the following data

- A standard pre-marked surface $\mathbf{\Sigma}_{g,n}$ for some g and n.
- A cyclically ordered list $(\mathbf{C}^1, \dots, \mathbf{C}^r)$ of *isotopy classes* of framed, marked, generalized cut systems on $\mathbf{\Sigma}_{g,n}$.

such that each $\mathbf{C}^i$ is related to $\mathbf{C}^{i+1}$ in one of these two ways:

- $\mathbf{C}^{i+1}$ contains all the curves of $\mathbf{C}^i$ together with one more curve, and all the preferred marked points of $\mathbf{C}^i$ together with one more preferred point, or vice versa.
- One curve C of $\mathbf{C}^i$ is replaced by one curve C_* of $\mathbf{C}^{i+1}$, all other curves remain the same, and C and C_* meet transversely at one point.

The following theorem is a statement of a preliminary result to the effect that these diagrams in fact do describe contractible spaces of 4–manifolds. Having given all the gory details for the definition of the kinds of diagrams we are considering, we skip some definitions needed to make completely rigorous sense of the statement, but we hope that the essential idea is clear.

Theorem 5. Given any framed, marked, generalized trisection diagram $\mathcal{D}$, there is a contractible space $A(\mathcal{D})$ of *auxiliary choices*, and a bundle $X(\mathcal{D})$ over $A(\mathcal{D})$

the fibers of which are smooth 4–manifolds. Furthermore, For every smooth 4–manifold X, there is a diagram $\mathcal{D}$ such that X is diffeomorphic to a fiber of $X(\mathcal{D})$. If $\mathcal{D}$ and $\mathcal{D}'$ are two diagrams such that the fibers of $X(\mathcal{D})$ and $X(\mathcal{D}')$ are diffeomorphic then $\mathcal{D}$ and $\mathcal{D}'$ are related by certain moves.

These diagrams are described as generalizations of trisection diagrams because they can be produced somewhat systematically from trisection diagrams, and a trisection diagram is like one of these diagrams with only three collections of curves, except that in a trisection diagrams, the successive collections of curves are not as simply related as those in our generalized trisection diagrams.

The next step in this program is to show that any diffomorphism between 4–manifolds can be described, up to isotopy, by a sequence of diagrams. This itself is a baby step towards a grand goal of constructing a cell complex in which the vertices are these diagrams, one-cells are these moves, and higher cells are described diagrammatically, so that each component of this cell complex corresponds to a diffeomorphism class of 4–manifolds and each component is homotopy equivalent to $\mathrm{BDiff}(X)$ for a 4–manifold X in its diffeomorphism class.

A pleasant outcome of giving this talk was that I learned from audience members that the space discussed in the preceding paragraph (a copy of $\mathrm{BDiff}(X)$ for each diffeomorphism class of 4–manifolds) is known as the *moduli space of* 4*–manifolds*, and that my goal is to constract a diagrammatic cell complex approximation of this space.

Computing invariants of barbell diffeomorphisms

Ryan Budney
(joint work with David Gabai)

Given a compact hyperbolic manifold of dimension ≥ 2 its isometry group is known to be finite. In dimensions ≥ 3, Mostow Rigidity tells us the space of homotopy self-equivalences of the manifold has the isometry group as a deformation-retract. Due to work of Hatcher, Waldhausen and Gabai, we know the group of homeomorphisms, PL-automorphisms and diffeomorphisms also has the isometry group as a deformation-retract. In dimensions ≥ 11, Farrell and Jones [5] used the machinery of Higher Simple Homotopy Theory (K-theory, pseudoisotopy, etc) [9] to show that smooth, PL and topological automorphism groups of compact hyperbolic manifolds do not have the homotopy-type of the isometry group, moreover they do not have the homotopy-type of finite CW-complexes. We extend these results to dimension ≥ 4 by avoiding the use of Higher Simple Homotopy Theory, and constructing diffeomorphisms explicitly using 'barbells' [1]. The core of our argument involves extending our theorem about the non-finite generation of $\pi_{n-4}\mathrm{Diff}(S^1 \times D^{n-1})$ to the non-finite generation of $\pi_{n-4}\,\mathrm{Homeo}(S^1 \times D^{n-1})$ for all $n \geq 4$. To do this, we construct an invariant of topological embedding spaces, modelled on the Embedding Calculus. Specifically, we consider a 'scanning' map $\mathrm{Homeo}(S^1 \times D^{n-1}) \to \Omega^{n-2}\mathrm{Emb}(I, S^1 \times D^{n-1})$, where $\mathrm{Emb}(I, S^1 \times D^{n-1})$ is a space of topological embeddings of the interval in $S^1 \times D^{n-1}$ with fixed boundary conditions. Our invariant of $\pi_{2n-6}\mathrm{Emb}(I, S^1 \times D^{n-1})$ takes place in a quotient of the rational $(2n-3)$-rd homotopy group of a space analogous to an 'orbit configuration space' of points in the universal cover of $S^1 \times D^{n-1}$ [4].

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The Dehn surgery characterisation of satellite knots

Laura Wakelin

The Dehn surgery characterisation problem asks: for which non-trivial slopes $p/q \in \mathbb{Q}$ does the manifold $\mathbb{S}^3_K(p/q)$ obtained by performing Dehn surgery of slope p/q on a knot $K \subset \mathbb{S}^3$ uniquely determine K? We combine JSJ decompositions and hyperbolic techniques to investigate this question in the case where K is a certain type of satellite knot, with a particular focus on Whitehead doubles.

Introduction

Throughout, we denote by K a knot in a 3-manifold M (usually either $\mathbb{S}^3$ or $\mathbb{S}^1 \times \mathbb{D}^2$) and write its complement as $M_K := M \setminus \nu(K)$. The manifold obtained by performing Dehn surgery of slope p/q on the knot $K \subset M$ is expressed as $M_K(p/q)$. We use similar notation for links; any ambiguity should be resolved by the context.

Definition 1. A *satellite* $K = P(C)$ with *pattern* $P = U \cup Q$ and *companion* C is the image of Q in the oriented homeomorphism $\mathbb{S}^3_U \to \nu(C)$ preserving longitudes.

We denote the winding number of Q inside $\mathbb{S}^3_U \cong \mathbb{S}^1 \times \mathbb{D}^2$ by w.

Definition 2. A slope $p/q \in \mathbb{Q}$ is called *characterising* for a knot $K \subset \mathbb{S}^3$ if the existence of an orientation-preserving homeomorphism $\mathbb{S}^3_K(p/q) \cong \mathbb{S}^3_{K'}(p/q)$ for a knot $K' \subset \mathbb{S}^3$ implies that $K = K'$, and *non-characterising* otherwise.

Question 3. Given a knot $K \subset \mathbb{S}^3$, can we classify all of its characterising and non-characterising slopes?

Non-characterising slopes. There are many knots with non-characterising slopes. Brakes suggested a method of constructing these using the idea of gluing together a pair of knot complements [3]; this is realised in the following example [17].

Example 4. For any $m, n, q \in \mathbb{Z} \setminus \{0\}$ with $m \neq n$, there is a pair of distinct multiclasped doubles of double twist knots, $K = W^n(T_q^m)$ and $K' = W^m(T_q^n)$, and an orientation-preserving homeomorphism $\mathbb{S}^3_K(1/q) \cong \mathbb{S}^3_{K'}(1/q)$.

Baker and Motegi showed that there are also knots for which infinitely many slopes are non-characterising [2]. The simplest of their examples is the hyperbolic knot 8_6, for which no integer slopes are characterising.

Characterising slopes. Every slope is characterising for the unknot [7, 8], as well as for the trefoils and figure eight knot [15]. Moreover, all but finitely many non-integer slopes are characterising for torus knots [14, 11]. Lackenby showed that for any knot $K \subset \mathbb{S}^3$, there exists a constant $C(K)$ for which every slope $p/q \in [-1, 1]$ is characterising [9]; Sorya recently refined this in [16].

Inspired by the existence of $C(K)$, the motivation for this work is to explicitly construct such a lower bound in some special cases. The purpose of this talk is to sketch the proofs of the following theorems from [17].

Theorem 5. Let $K = P(C)$ be a satellite knot by a pattern P in whose complement the length of the shortest geodesic is at least 0.136. Then every slope p/q with $\gcd(p, w) \neq 1$, $|q| \geq 36$ is characterising for K.

Theorem 6. Let $K = W(C)$ be a Whitehead double and assume that the SnapPea census of the first n orientable, 2-cusped hyperbolic 3-manifolds ordered by volume is complete. Then every slope p/q with $|p| \neq 1$, $|q| \geq \phi(n)$ is characterising for K.

n	4	6	8	10	16	18	24	49	289
$\phi(n)$	44	33	31	29	28	27	26	25	24

Table 1. Choices of n and the corresponding values of $\phi(n)$.

JSJ decompositions

The JSJ decomposition theorem is a key part of our argument: we use the classification of JSJ pieces in a satellite knot complement to ensure that a homeomorphism $\mathbb{S}^3_K(p/q) \cong \mathbb{S}^3_{K'}(p/q)$ between two fillings identifies the filled JSJ pieces.

Satellite knot complements. The JSJ pieces in a satellite knot complement can take precisely 4 different forms [4]. In particular, by considering each possibility for the outermost JSJ piece in turn, we gain crucial information about the filling.

Homeomorphic fillings. Let $K = P(C)$ be a satellite by a hyperbolic pattern and suppose that $\mathbb{S}^3_K(p/q) \cong \mathbb{S}^3_{K'}(p/q)$. We would like to show that $K' = K$.

Question 7. Must we have another satellite knot, $K' = P'(C')$?

The answer is yes, provided we assume $|q| \geq 9$; then we can avoid exceptional surgeries on hyperbolic knots by [10] and eliminate torus knots by [12].

Question 8. Must the homeomorphism identify the filled pieces?

The answer is yes, provided we assume $\gcd(p, w) \neq 1$; this prevents the filled pattern piece from being a knot complement.

Proposition 9. Let $K = P(C)$ be a satellite by a hyperbolic pattern and suppose that $\mathbb{S}^3_K(p/q) \cong \mathbb{S}^3_{K'}(p/q)$, where $\gcd(p, w) \neq 1$ and $|q| \geq 9$. Then $K' = P'(C)$ is a satellite of the same companion C by a pattern P', and one of the following holds:

(i) $P' = U \cup Q'$ is hyperbolic and $\mathbb{S}^3_P(p/q) \cong \mathbb{S}^3_{P'}(p/q)$;

(ii) $P' = J_{r,s}(\hat{P})$ is a cabled pattern whose cabling slope satisfies $|p - qrs| = 1$, $\hat{P} = U \cup \hat{Q}$ is hyperbolic and $\mathbb{S}^3_P(p/q) \cong \mathbb{S}^3_{\hat{P}}(p/qs^2)$.

Hyperbolic techniques

By Proposition 9, we have reduced our problem to showing that if P and $\tilde{P}$ are hyperbolic patterns such that $\mathbb{S}^3_P(p/q) \cong \mathbb{S}^3_{\tilde{P}}(p/qt^2)$ for some integer $t \geq 1$, then $P = \tilde{P}$. (We can subsequently rule out case (ii) by noting that the existence of a cosmetic surgery $\mathbb{S}^3_K(p/q) \cong \mathbb{S}^3_K(p/qt^2)$ with $t > 1$ would contradict [13]; this leaves us with case (i), which immediately implies that $K = K'$.) There are two distinct hyperbolic techniques which we can use to address this.

Minimal geodesics. Let P be a hyperbolic pattern such that the shortest geodesic in its complement has length at least 0.136. By assuming that $|q| \geq 36$ and invoking quantitative bounds from [6], we show that the shortest geodesic in both filled manifolds can only be the core curve, thus implying that the homeomorphism $\mathbb{S}^3_P(p/q) \cong \mathbb{S}^3_{\tilde{P}}(p/qt^2)$ takes one to the other; hence $\tilde{P} = P$. This completes the proof of Theorem 5.

Minimal volumes. The Whitehead link complement and its sister jointly achieve the smallest volume of all orientable 2-cusped hyperbolic 3-manifolds [1]. By assuming that $|q| \geq 24$ and utilising hyperbolic volume inequalities from [5], the conjectural completeness of the SnapPea census up to a certain point allows us to limit the possibilities for such a manifold M satisfying $\mathbb{S}^3_W(p/q) \cong M(p/qt^2)$ to a finite list. We then obstruct the existence of a homeomorphism $M \cong \mathbb{S}^3_{\tilde{P}}$ to a suitable link complement unless $\tilde{P} = W$. This completes the proof of Theorem 6.

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Multisections

Delphine Moussard

The goal of this talk was to present a generalization of Heegaard splittings of 3–manifolds and trisections of 4–manifolds to higher-dimensional manifolds. It is a joint work in progress with Fathi Ben Aribi, Sylvain Courte and Marco Golla.

All manifolds are compact, connected and oriented. By *handlebody* we mean a 1–handlebody.

A *Heegaard splitting* of a closed 3–manifold M is a decomposition $M = H_1 \cup H_2$ where H_1 and H_2 are handlebodies with common boundary a closed surface Σ. A *trisection* of a smooth closed 4–manifold X is a decomposition $X = X_1 \cup X_2 \cup X_3$ where the X_i are 4–dimensional handlebodies, the $X_{ij} = X_i \cap X_j$ are 3–dimensional handlebodies and $\Sigma = X_1 \cap X_2 \cap X_3$ is a closed surface.

From a Heegaard splitting or a trisection, one can produce a new one *via* a stabilization move. This move is defined by removing a neighborhood of a boundary-parallel arc properly embedded in one handlebody of the splitting and adding it to the other handlebody. Similarly, from a trisection $X = X_1 \cup X_2 \cup X_3$, take a boundary-parallel arc in one of the X_{ij} and add a tubular neighborhood of it to the opposite X_k.

Theorem 1 (Heegaard, Reidemeister, Singer, Gay–Kirby)**.** Any smooth closed 3/4–manifold admits a Heegaard splitting/trisection, unique up to stabilization.

These decompositions can be represented by diagrams. First define a *cut system* for a handlebody as a family of disjoint curves on its boundary that bound disjoint disks, such that cutting the handlebody along these disks gives a 3–ball.

A *Heegaard diagram* for a given Heegaard splitting is given by two families of curves on the central surface that are cut systems for the two handlebodies of the splitting, see Figure 1.

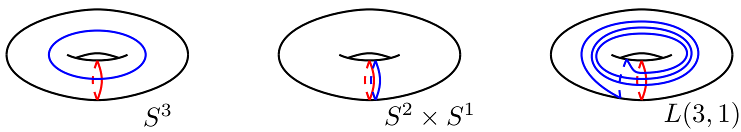

Figure 1. Heegaard diagrams

A *trisection diagram* for a given trisection consists of three cut systems on the central surface Σ for the three 3–dimensional handlebodies X_{ij} of the trisection.

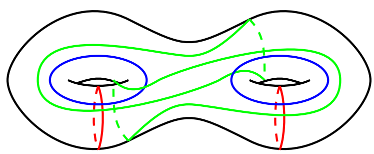

Figure 2. A trisection diagram for $S^2 \times S^2$

Abstractly, a Heegaard diagram (resp. trisection diagram) is a triple $(\Sigma; \alpha, \beta)$ (resp. a quadruple $(\Sigma; \alpha, \beta, \gamma)$) where Σ is a closed genus–g surface and α, β, γ are families of disjoint simple closed curves on Σ; for a trisection diagram, we add the condition that each pair of families defines a Heegaard splitting of a connected sum of copies of $(S^1 \times S^2)$, that is the boundary of a 4–dimensional handlebody. A Heegaard diagram (resp. a trisection diagram) determines a unique smooth 3/4–manifold.

Beyond Heegaard splittings of 3–manifolds and trisections of smooth 4–manifolds, it is natural to ask whether a similar decomposition can be provided in higher dimensions. We call *n–section* or *multisection* of a closed smooth $(n+1)$–manifold W a decomposition $W = \cup_{i=1}^{n} W_i$ such that:

- $W_I = \cap_{i\in I} W_i$ is an $(n+2-|I|)$–dimensional handlebody for any proper non-empty subset I of $\{1,\dots,n\}$,
- $\cap_{1\le i\le n} W_i$ is a closed surface.

It should be noted that the W_I cannot be all simultaneously diffeomorphic to a handlebody. We deal here with manifolds with corners, requiring that their canonical smoothing is diffeomorphic to a handlebody.

An interesting feature of these multisections is the following inductive property: for each I, the manifold ∂W_I has a natural $(n-|I|)$–section given by the W_J for $J = I \cup \{j\}$, $j \notin I$.

Note that the definition of multisections extends to the PL setting by replacing diffeomorphisms by PL homeomorphisms everywhere.

Given an n–section $\mathcal{M}$ of a smooth $(n+1)$–manifold W as above, denote $\Sigma = \cap_{1\le i\le n} W_i$ the central surface and choose for all $i \in \{1,\dots,n\}$ a cut system $(\alpha^i_j)_{1\le j\le g}$ for the 3–dimensional handlebody $\cap_{k\ne i} W_k$. Then $(\Sigma;\alpha^1,\dots,\alpha^n)$ is an *n–section diagram* for $(W,\mathcal{M})$. This is not unique, but it is well-known that each system of curves $(\alpha^i_1,\dots,\alpha^i_g)$ is unique up to handleslides. Hence the n–section diagram associated to a multisected manifold is unique up to handleslides (performed independently in each family α^i).

An *abstract n–section diagram* is a genus–g closed surface Σ with n families of g disjoint simple closed curves, such that any subcollection of $k \in \{2,\dots,n\}$ of these families is a k–section diagram for a multisection of a connected sum of copies of $S^1 \times S^k$.

Theorem 2 (Ben Aribi, Courte, Golla, M.)**.** Let $n \ge 2$.

- For $n \le 6$, any abstract n–section diagram is the diagram of some smooth multisected $(n+1)$–manifold, which is unique up to multisection preserving diffeomorphism if $n \le 5$.
- For arbitrary n, any abstract n–section diagram is the diagram of some PL multisected $(n+1)$–manifold which is unique up to multisection preserving PL homeomorphism.

The proof relies on results of Laudenbach–Poénaru [LP72], Montesinos [Mon79] and Cavicchioli–Hegenbarth [CH93]: for $n \ge 4$, any PL homeomorphism of the boundary of an n–dimensional handlebody extends to the whole handlebody; this holds for diffeomorphisms if $n \le 6$.

The uniqueness part of the result is optimum. Indeed, for $n \ge 6$, exotic $(n+1)$–spheres are known to be twisted spheres, so that they admit genus–0 n–sections; hence they all admit the 2–sphere as an n–section diagram.

From dimension 8, it is known that there exist PL manifolds that do not admit any smooth structure. If such manifolds can be multisected, then their multisection diagrams cannot be realized by a smooth manifold.

There is obvioulsy a single genus–0 n–section diagram. For $n \geq 4$, all genus–1 diagrams are given by two groups of parallel curves, pairwise dual (see Figure 3 for $n = 4$). When all curves are parallel, the diagram represents $S^1 \times S^n$. Otherwise, it represents S^{n+1}. The stabilization moves for $n \geq 4$ correspond to connected sums with the corresponding multisections of S^{n+1}. These moves can be described by cutting and pasting of embedded arcs or higher dimensional disks.

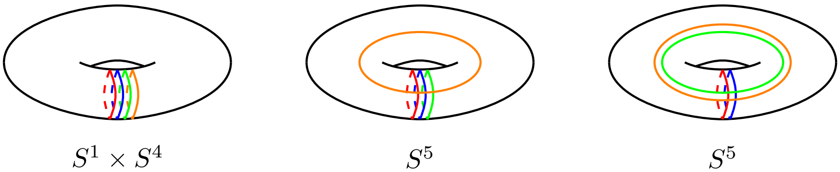

Figure 3. Genus–1 quadrisection diagrams

The main question concerning multisections is that of existence and uniqueness up to stabilizations.

Theorem 3 (Ben Aribi, Courte, Golla, M.)**.** Any closed smooth 5–manifold admits a quadrisection.

The proof relies on morse functions on smooth manifolds, adapting in dimension 5 the approach of Lambert-Cole and Miller [LCM21] of the initial proof of existence of trisections by Gay and Kirby [GK16].

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Involutive knot Floer homology and the Arf invariant

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(joint work with JungHwan Park)

Given a knot K and relatively prime integers p, q, one can consider the cabled knot $K_{p,q}$. When K is slice and $q = 1$, then it is clear that $K_{p,q}$ is also slice. Miyazaki asked if the converse of this fact also holds. In fact, we go one step further and focus on the following stronger conjecture.

Conjecture 1. The (p,q)-cable of K is torsion in the knot concordance group $\mathcal{C}$ if and only if K is slice and $|q|=1$.

It can be easily seen that if $|q|>1$, then $K_{p,q}$ has infinite order in $\mathcal{C}$, so the conjecture reduces to the case when $q=1$. There are many nonslice knots where their $(p,1)$-cables have infinite order in $\mathcal{C}$, but most of those arguments only work when the given knot K also has infinite order in $\mathcal{C}$ as well. The only known result when K is torsion in $\mathcal{C}$ appeared recently in a work of the author with Hom, Park, and Stoffregen, where it is shown that the $(p,1)$-cable of the figure-eight knot has infinite order in $\mathcal{C}$ whenever p is odd and not equal to ± 1. In our work, we expand this result to a much larger family. Roughly, we show that "half" of the torsion knots have infinite order once cabled. Let $K_{p_1,q_1;p_2,q_2;\dots;p_m,q_m}$ denote the iterated cable of K.

Theorem 2. Denote by $\mathcal{C}_T$ the torsion subgroup of the knot concordance group $\mathcal{C}$. Then there is a nontrivial group homomorphism

$$\mathfrak{A}\colon \mathcal{C}_T \to \mathbb{Z}/2\mathbb{Z}.$$

Moreover, if K is torsion in $\mathcal{C}$ with $\mathfrak{A}(K)=1$, then for any sequence of positive integers $n_1,n_2,\dots,n_m$ the iterated cable $K_{2n_1+1,1;2n_2+1,1;\dots;2n_m+1,1}$ has infinite order in $\mathcal{C}$. In particular, for any nonzero integer n the cable $K_{2n+1,1}$ has infinite order in $\mathcal{C}$.

I was also able to prove linear independence of (odd,1)-cables of knots which are torsion in $\mathcal{C}$ and has nonzero $\mathfrak{A}$-invaraint.

Theorem 3. If K is torsion in $\mathcal{C}$ with $\mathfrak{A}(K)=1$, then the set of cables $\{K_{2n+1,1}\}_{n>0}$ contains an infinite subset which is linearly independent in $\mathcal{C}$.

It is a very interesting question whether the invariant $\mathfrak{A}$ takes the same value as the Arf invariant, so we leave this as a conjecture.

Question 4. If K is torsion in $\mathcal{C}$, then is $\mathfrak{A}(K)=\mathrm{Arf}(K)$?

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Bordered perspectives on the link surgery formulas

Ian Zemke

In this talk, I discussed work (much of which is in progress) on the surgery formulas in Heegaard Floer theory. In the early 2000s, Ozsváth and Szabó introduced a powerful set of 3-manifold invariants called *Heegaard Floer homology.* Some of the most important developments in Heegaard Floer theory are the result of its computability relative to other Floer theoretic invariants. Some of the most important and useful computational results are the surgery formulas of Manolsecu-Ozsváth-Szabó [5, 6].

For a knot $K \subset S^3$ (or more generally a null-homologous knot in a 3-manifold), the surgery formulas give an isomorphism

$$HF^-(Y_n(K)) \cong Cone(v + h_n \colon \mathbb{A}(K) \to \mathbb{B}(K)),$$

for some complexes $\mathbb{A}(K)$, $\mathbb{B}(K)$ and maps v and h_n.

Fundamentally, Dehn surgery is an operation involving 3-manifolds with torus boundaries, i.e. we cut out a neighborhood of K and glue in a solid torus. In Heegaard Floer theory, there is a parallel theory for 3-manifolds with torus boundary, called *Bordered Heegaard Floer homology*, due to Lipshitz, Ozsváth and Thurston [4].

In this talk, I described some ways in which the Manolescu-Ozsváth-Szabó surgery formulas could be interpreted as a bordered theory [8]. This interpretation is through an algebra due to the author, denoted $\mathcal{K}$. This algebra is over an idempotent ring of 2 elements, denoted $I_0 \oplus I_1$. We set

$$I_0 \cdot \mathcal{K} \cdot I_0 = \mathbb{F}[\mathcal{U}, \mathcal{V}], \quad I_0 \cdot \mathcal{K} \cdot I_1 = 0, \quad I_1 \cdot \mathcal{K} \cdot I_1 = \mathbb{F}[U, T, T^{-1}].$$

We declare $I_1 \cdot \mathcal{K} \cdot I_0$ to be

$$\mathbb{F}[U, T, T^{-1}] \otimes_{\mathbb{F}} \langle \sigma, \tau \rangle.$$

The relations are

$$UT^{-1}\sigma = \sigma,\mathcal{U} \quad T\sigma = \sigma\mathcal{V}, \quad T^{-1}\tau = \tau\mathcal{U}, \quad UT\tau = \tau\mathcal{V}.$$

To a 3-manifold M with torus boundary, the author describes a type-D module $\mathcal{X}(M)^{\mathcal{K}}$ (i.e. a projective module), as well as a type-A module ${}_{\mathcal{K}}\mathcal{X}(M)$ (i.e. an A_∞-module). For 3-manifolds with n torus boundary components, one can construct a type-D module over the algebra $\mathcal{K} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{K}$.

The type-A module for a knot complement is easy to describe. Namely, the underlying vector space is $\mathbb{A}(K) \oplus \mathbb{B}(K)$. The action of σ is by the map v, and the action of τ is by h_n.

The author has proven gluing formulas for gluing along torus boundary components. These amount to connected sum formulas for the link surgery formula. Topologically, these exploit the fact that

$$S^3_{n_1+n_2}(K_1 \# K_2) \cong \big(S^3 \setminus \nu(K_1)\big) \cup_\phi \big(S^3 \setminus \nu(K_2)\big)$$

where ϕ is the map which sends the meridian of K_1 to minus the meridian of K_2, and sends the n_1-framed longitude of K_1 to the n_2-framed longitude of K_2.

Many developments in the Heegaard Floer surgery formulas have interpretations in this theory. As a concrete example, Hedden-Levine and Eftekhary have computed the knot Floer homology of the “dual” knot of a surgery [3, 7]. In the theory I propose, their formula corresponds to the DA-bimodule for the Hopf link. Note that the complement of the Hopf link is $\mathbb{T}^2 \times [0, 1]$, so the complement of the Hopf link may naturally be viewed as the mapping cylinder of a simple diffeomorphism of the torus. This fits nicely in parallel with the bordered theory of Lipshitz-Ozsváth-Thurston.

The theory has proven useful in many contexts. One example is that I proved a conjecture of Némethi about the equivalence of Heegaard Floer and lattice homology [9]. In another work [2] with Liu and Borodzik, we describe a combinatorial model of the link Floer homologies of all algebraic links in S^3.

In the talk, I discussed new work on the surgery formulas. Precisely, I discussed a new proof of the surgery formulas of Manolescu-Ozsváth-Szabó. This proof simplifies many technical and annoying aspects of the original proofs, which are obstacles to extending their formulas. (To the experts, the obstacles lie mainly in truncations and gradings). The proof I am working on develops a new exact triangle in the Fukaya category of the torus. Namely, we suppose that β_λ denotes a Lagrangian in the torus, and β_0 is a curve intersecting β_λ in a single point. We let β_1 be obtained by winding β_0 exactly once in the direction of β_λ. We decorate β_0 and β_1 with certain notions of local systems, and we denote $\beta_0^{V_0}$ and $\beta_1^{V_1}$ for these Lagrangians equipped with these local systems. There are two canonical morphisms $\theta^\sigma, \theta^\tau \colon \beta_0^{V_0} \to \beta^{V_1}$. In forthcoming work, I prove that there is an isomorphism in the Fukaya category of the torus

$$\beta_\lambda \simeq Cone(\theta^\sigma + \theta^\tau \colon \beta_0^{V_0} \to \beta^{V_1}).$$

A fairly routine argument shows that this implies the standard surgery formulas. (In fact, it extends them outside of the context of null-homologous knots and links).

Finally, we also consider the endomorphism A_∞-algebra $End(\beta_0^{V_0} \oplus \beta_1^{V_1})$. In forthcoming work, I show that there is an equivalence of (A_∞)-algebras

$$End(\beta_0^{V_0} \oplus \beta_1^{V_1}) \simeq \mathcal{K}.$$

This is reminiscent of work of Auroux [1] in the ordinary setting of bordered Heegaard Floer homology.

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