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Mini-Workshop: Free Boundary Problems Arising in Fluid Mechanics

Organized by
Diego Alonso Orán, La Laguna
Claudia García, Granada
Juan J. L. Velázquez, Bonn

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Abstract. Fluid mechanics is one of the classical areas in the study of partial differential equations and has been a vast subject of research in the last centuries. A relevant class of problems are those in which the evolution of fluids of different nature and their interaction is described through the dynamics of their common boundary. Such problems are called free-boundary problems. The key topic of this workshop deals with recent advances on the analysis of free-boundary problems which open up a whole new area of research activity. More precisely, we will address problems as the vortex-patch, the study of water waves, interface flows in porous media and Hele-Shaw cells as well as atmospheric front dynamics where the formation of finite time singularities is a fundamental open question.

Mathematics Subject Classification (2020): 35Q35, 35R35, 76B03.

Introduction by the Organizers

The mini-workshop Free Boundary Problems arising in Fluid Mechanics organised by Diego Alonso-Orán (La Laguna), Claudia García (Granada) and Juan J. L. Velázquez (Bonn) was attended by 17 participants including the organizers from 12 institutions. This workshop was a nice blend of researchers working in free boundary problems with various backgrounds and with the participation of promising young researchers as well as known senior experts in the field. The successfully achieved main goal of the mini-workshop was to promote scientific exchange of ideas and enhance collaborations and explore possible future directions of research of this area.

The format of the mini-workshop followed the classical spirit and tradition of Oberwolfach which consisted in two to four one-hour talks per day, leaving enough time for discussion and interaction between the participants. The talks were usually followed by several questions and mathematical exchange of ideas which created spontaneous discussions around the lectured topics. We strongly believe that these interactions where particularly fruitful and helpful for young researchers. Incidentally we also underline a significant number of women participants as well as researchers coming from different nationalities such as Germany, Spain, United Arab Emirates and USA.

The mini-workshop treated three main research directions: the vortex patch problem and existence of rotating solutions, the study of viscous fluids in free boundary problems and the more classical problem of steady and unsteady water waves.

Concerning the vortex patch problem, the first talk was given by Joan Mateu who presented new results regarding global in time solutions to the 3D quasi-geostrophic model. The main technique towards the result being the use of the bifurcation Crandall-Rabinowitz theorem. Similar ideas, but for corotating vortex pairs for the Euler equation where exposed by Susanna V. Haziot which explained the use of global bifurcation arguments to study the problem. Closely, Zineb Hassainia offered a very interesting and innovative talk about the use of KAM theory to construct quasi-periodic doubly-connected vortex patch solutions for the Euler equation. Although being a very technical but deep result, several questions and intriguing discussions emerge after Zineb's presentation. The PhD student Bernhard Kepka closed this topic by presenting his work on rotating solutions for the Euler-Poisson equation with external particles. Several possible ideas to extend Kepka's work where suggested and discussed by the attending participants.

A majority of the contributions were devoted to the theme of free boundary problems of viscous fluids. In particular, it was analysed in the talks of Helmut Abels, Bogdan Matioc, Francisco Gancedo, Eduardo García Juárez and Xian Liao. Helmut Abels studied the diffusive interface for a two-phase incompressible viscous problem. More precisely, he studied several asymptotic models and the rigorous convergence from the diffusive interface to the sharp interface. After the talk, the audience had an animated discussion regarding the so-called contact line problem and the correct use of boundary value conditions. Bogdan Matioc started with a review of the known results regarding the free boundary problem for the Stokes equation and also presented his new result on local existence of smooth solutions. The use of potential theory and the abstracts results for evolution equation of Lunardy were strongly used in his approach. Francisco Gancedo presented two different problems: the one-phase Muskat problem and the interface problem by two fluids of different densities evolving by the linear Stokes law. He gave an overall view of the before-mentioned problems and later an analysis regarding global in time results in critical spaces. A similar problem known as the Peskin problem describing the evolution of a two-dimensional elastic membrane immersed in a three-dimensional steady Stokes flow was treated in the talk of Eduardo García-Juárez. To conclude this theme, the board-talk by Xian Liao studied the existence of strong solutions and enhance dissipation phenomena for the inhomogeneous Navier-Stokes equation. Moreover, she discussed about the existence of stationary solutions and constructed some explicit examples.

The more classical water wave problem was addressed by: Erik Wáhlen, who presented a double periodic steady water wave problem and constructed solutions via an approach first derived by Lortz in the context of magnetohydrodynamics; Sijue Wu, who introduced the classical water wave problem and showed the existence of solutions with crests angles, this is, solutions with non-C^1 interfaces and Nastasia Grubic, who complemented Sijue Wu's talk by providing the existence of solutions with crests whose angle changes in time.

Finally, we also had the talks by Martina Magliocca who presented some asymptotic free boundary models for tumor growth and well-posedness results and by Christian Zillinger, who discussed about stratification and non-linear resonances for the 2D Boussinesq equations. Furthermore, C. Zillinger also mentioned some perspectives concerning echoes for the magnetohydrodynamics equations.

Mini-Workshop: Free Boundary Problems Arising in Fluid Mechanics

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Abstracts

Approximation of Incompressible Two-Phase Flows by Diffuse Interface Models

Helmut Abels
(joint work with Mingwen Fei)

We consider the singular limit $\varepsilon \to 0$ of the following system:

$$\partial_t \mathbf{v}_\varepsilon + \mathbf{v}_\varepsilon \cdot \nabla \mathbf{v}_\varepsilon - \operatorname{div}(2\nu(c_\varepsilon) D\mathbf{v}_\varepsilon) + \nabla p_\varepsilon = -\varepsilon \operatorname{div}(\nabla c_\varepsilon \otimes \nabla c_\varepsilon), \tag{1}$$

$$\operatorname{div} \mathbf{v}_\varepsilon = 0, \tag{2}$$

$$\partial_t c_\varepsilon + \mathbf{v}_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - \tfrac{1}{\varepsilon^2} f'(c_\varepsilon), \tag{3}$$

in $\Omega \times (0, T_0)$ together with suitable boundary and initial data. Here $\mathbf{v}_\varepsilon, p_\varepsilon$ are the velocity and the pressure of the fluid mixture, c_ε is the order parameter, which is related to the concentration difference of the fluids, $\nu(c_\varepsilon)$ describes the viscosity in dependence on c_ε, and f is a suitable smooth double well potential, e.g., $f(c) = \frac{1}{8}(c^2-1)^2$. Moreover, $D\mathbf{v}_\varepsilon = \frac{1}{2}(\nabla \mathbf{v}_\varepsilon + (\nabla \mathbf{v}_\varepsilon)^T)$ and $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with smooth boundary. We prove the convergence of (1)-(3) to the following sharp interface limit system:

$$\partial_t \mathbf{v}_0^\pm + \mathbf{v}_0^\pm \cdot \nabla \mathbf{v}_0^\pm - \nu^\pm \Delta \mathbf{v}_0^\pm + \nabla p_0^\pm = 0 \qquad \text{in } \Omega^\pm(t), t \in (0, T_0), \tag{4}$$

$$\operatorname{div} \mathbf{v}_0^\pm = 0 \qquad \text{in } \Omega^\pm(t), t \in (0, T_0), \tag{5}$$

$$[\![2\nu^\pm D\mathbf{v}_0^\pm - p_0^\pm \mathbf{I}]\!] \mathbf{n}_{\Gamma_t} = -\sigma H_{\Gamma_t} \mathbf{n}_{\Gamma_t} \qquad \text{on } \Gamma_t, t \in (0, T_0), \tag{6}$$

$$[\![\mathbf{v}_0^\pm]\!] = 0 \qquad \text{on } \Gamma_t, t \in (0, T_0), \tag{7}$$

$$V_{\Gamma_t} - \mathbf{n}_{\Gamma_t} \cdot \mathbf{v}_0^\pm = H_{\Gamma_t} \qquad \text{on } \Gamma_t, t \in (0, T_0), \tag{8}$$

where $\nu^\pm = \nu(\pm 1)$, Ω is the disjoint union of $\Omega^+(t), \Omega^-(t)$, and Γ_t for every $t \in [0, T_0]$, $\Omega^\pm(t)$ are smooth domains, $\Gamma_t = \partial \Omega^+(t)$, and $\mathbf{n}_{\Gamma_t}$ is the interior normal of Γ_t with respect to $\Omega^+(t)$. Moreover,

$$[\![u]\!](p, t) = \lim_{h \to 0+} [u(p + \mathbf{n}_{\Gamma_t}(p)h) - u(p - \mathbf{n}_{\Gamma_t}(p)h)]$$

is the jump of a function $u \colon \Omega \times [0, T_0] \to \mathbb{R}^2$ at Γ_t in direction of $\mathbf{n}_{\Gamma_t}$, H_{Γ_t} and V_{Γ_t} are the curvature and the normal velocity of Γ_t, both with respect to $\mathbf{n}_{\Gamma_t}$. Furthermore, $\sigma = \int_{\mathbb{R}} \theta_0'(\rho)^2 \, d\rho$, where θ_0 is the so-called optimal profile that is the unique solution of

$$-\theta_0''(\rho) + f'(\theta_0(\rho)) = 0 \quad \text{for all } \rho \in \mathbb{R}, \qquad \lim_{\rho \to \pm\infty} \theta_0(\rho) = \pm 1, \quad \theta_0(0) = 0.$$

Theorem. *Let $N \geq 3$, $N \in \mathbb{N}$, $(\mathbf{v}_0^\pm, \Gamma)$ be a smooth solution of (4)-(8) together with $\mathbf{v}_0^\pm|_{\partial\Omega} = 0$ for some $T_0 \in (0, \infty)$. Then there are smooth $c_{A,0} \colon \Omega \to \mathbb{R}$ and $\mathbf{v}_{A,0} \colon \Omega \to \mathbb{R}^2$, depending on $\varepsilon \in (0,1)$, such that the following is true: Let $(\mathbf{v}_\varepsilon, c_\varepsilon)$*

be strong solutions of (1)-(3) *with initial values* $c_{0,\varepsilon}\colon \Omega \to [-1,1]$, $\mathbf{v}_{0,\varepsilon}\colon \Omega \to \mathbb{R}^2$, $0 < \varepsilon \leq 1$ *and* $(\mathbf{v}_\varepsilon, c_\varepsilon)|_{\partial\Omega} = (0,-1)$, *satisfying*

$$\|c_{0,\varepsilon} - c_{A,0}\|_{L^2(\Omega)} + \varepsilon^2 \|\nabla(c_{0,\varepsilon} - c_{A,0})\|_{L^2(\Omega)} + \|\mathbf{v}_{0,\varepsilon} - \mathbf{v}_{A,0}\|_{L^2(\Omega)} \leq C\varepsilon^{N+\frac{1}{2}} \tag{9}$$

for all $\varepsilon \in (0,1]$ *and some* $C > 0$. *Then there are some* $\varepsilon_0 \in (0,1]$, $R > 0$, *and* $c_A\colon \Omega \times [0,T_0] \to \mathbb{R}$, $\mathbf{v}_A\colon \Omega \times [0,T_0] \to \mathbb{R}^2$ *(depending on* ε*) such that*

$$\sup_{0\leq t\leq T_0} \|c_\varepsilon(t) - c_A(t)\|_{L^2(\Omega)} + \|\nabla(c_\varepsilon - c_A)\|_{L^2(\Omega\times(0,T_0)\setminus\Gamma(\delta))} \leq R\varepsilon^{N+\frac{1}{2}},$$

$$\|\nabla_{\boldsymbol{\tau}}(c_\varepsilon - c_A)\|_{L^2(\Omega\times(0,T_0)\cap\Gamma(2\delta))} + \varepsilon\|\partial_{\mathbf{n}}(c_\varepsilon - c_A)\|_{L^2(\Omega\times(0,T_0)\cap\Gamma(2\delta))} \leq R\varepsilon^{N+\frac{1}{2}},$$

$$\|\nabla(c_\varepsilon - c_A)\|_{L^\infty(0,T_0;L^2(\Omega))} + \|\nabla^2(c_\varepsilon - c_A)\|_{L^2(\Omega\times(0,T_0))} \leq R\varepsilon^{N-\frac{3}{2}}$$

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_A\|_{L^\infty(0,T_0;L^2(\Omega))} + \|\mathbf{v}_\varepsilon - \mathbf{v}_A\|_{L^2(0,T_0;H^1(\Omega))} \leq C(R)\varepsilon^{N+\frac{1}{2}}$$

hold true for all $\varepsilon \in (0,\varepsilon_0]$ *and some* $C(R) > 0$. *Here* $\Gamma(\delta)$, $\Gamma(2\delta)$ *are* δ-, 2δ-*neighborhoods of* Γ, *respectively. Moreover,*

$$\lim_{\varepsilon\to 0} c_A = \pm 1 \quad \text{uniformly on compact subsets of } \Omega^\pm = \bigcup_{t\in[0,T_0]} \Omega^\pm(t) \times \{t\},$$

$$\mathbf{v}_A = \mathbf{v}_0^\pm + O(\varepsilon) \qquad \text{in } L^\infty(\Omega \times (0,T_0)) \text{ as } \varepsilon \to 0.$$

For the proof an approximate solution $(\mathbf{v}_A, p_A, c_A)$ is constructed using finitely many terms from formally matched asymptotic calculations and a novel ansatz for a highest order term. Then the error $c_A - c_\varepsilon$ is estimated with the aid of a refined spectral estimate for the linearized Allen-Cahn operator. We refer to [2] for the details.

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Steady three-dimensional water waves with vorticity

Erik Wahlén

(joint work with Douglas Svensson Seth, Kristoffer Varholm and Jörg Weber)

While the two-dimensional steady water wave problem with vorticity has received a lot of attention, the corresponding three-dimensional problem is almost completely open. One reason is that there is no general analogue of the stream function formulation in 3D and the free boundary problem is generally not elliptic. The irrotational problem can, on the other hand, be formulated as an elliptic problem in terms of the velocity potential. In the recent work [2], we construct solutions which are symmetric and have small nonzero vorticity. Inspired by an approach by Lortz [1] for magnetohydrostatic equilibria we obtain small-amplitude doubly periodic solutions bifurcating from uniform flows.

We consider gravity-capillary waves travelling at constant speed in the x-direction. In a moving frame of reference, the problem is given by the stationary incompressible Euler equations with kinematic and dynamic boundary conditions,

$$\begin{aligned} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &= -\nabla(p+gz) && \text{in } \Omega^\eta, \\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega^\eta, \\ \boldsymbol{u}\cdot\boldsymbol{n} &= 0 && \text{on } \partial\Omega^\eta, \\ p &= -\sigma\mathcal{H} && \text{on } z=\eta, \end{aligned}$$

where $\Omega^\eta = \{\boldsymbol{x} = (x,y,z) \in \mathbb{R}^3 : -d < z < \eta(x,y)\}$ is the unknown fluid domain, $g > 0$ the gravitational constant of acceleration, $\sigma > 0$ the coefficient of surface tension, $\mathcal{H} = \nabla\cdot(\nabla\eta/\sqrt{1+|\nabla\eta|^2})$ the mean curvature and $\boldsymbol{n}$ a normal vector. The unknowns are the surface η and the velocity field $\boldsymbol{u}$, from which one can recover the pressure p. We seek solutions which are λ_1-periodic in x and λ_2-periodic in y, and satisfy the symmetry conditions $\eta(x,y) = \eta(-x,y) = \eta(x,-y)$, $(u_1,u_2,u_3)(-x,y,z) = (u_1,-u_2,-u_3)(x,y,z)$, $(u_1,u_2,u_3)(x,-y,z) = (u_1,-u_2,u_3)(x,y,z)$. Under the additional assumption that $u_1 > 0$ throughout Ω^η, this has the consequence that the streamlines are also periodic. A trivial family of solutions is given by uniform flows $\boldsymbol{u} = (c,0,0)$ with a flat surface $\eta = 0$. In fact, any shear flow $(U(y,z),0,0)$ with U periodic and even in y is a solution. This means that if we linearise the above problem directly without imposing further restrictions, we face the challenge of an infinite-dimensional kernel. In order to suitably restrict the solution space, we follow Lortz' approach and use a Clebsch type representation of the vorticity $\boldsymbol{\omega} := \nabla\times\boldsymbol{u}$ of the form $\boldsymbol{\omega} = \nabla H \times \nabla\tau$, where $H = \frac{1}{2}|\boldsymbol{u}|^2 + gz + p$ is the *Bernoulli function* and $\tau(\boldsymbol{x})$ the travel time along a streamline from the symmetry plane $\{x = 0\}$ to the point $\boldsymbol{x}$. The Bernoulli function is constant along streamlines. So is $q(\boldsymbol{x})$, the total travel time along the streamline passing through $\boldsymbol{x}$ form the symmetry plane $\{x = 0\}$ to the shifted plane $\{x = \lambda_1\}$. It is therefore natural to impose a functional relationship $H = h(q)$ between the two. This has the effect of making the kernel finite-dimensional.

The problem is now reformulated as

$$\begin{aligned} \nabla\times\boldsymbol{u} &= h'(q)\nabla q\times\nabla\tau && \text{in } \Omega^\eta, \\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega^\eta, \\ \boldsymbol{u}\cdot\boldsymbol{n} &= 0 && \text{on } \partial\Omega^\eta, \\ \tfrac{1}{2}|\boldsymbol{u}|^2 + g\eta - \sigma\mathcal{H} &= h(q) + Q && \text{on } z=\eta, \end{aligned}$$

where Q is the Bernoulli constant, and $\tau = \tau[\boldsymbol{u},\eta]$ is defined by $\boldsymbol{u}\cdot\nabla\tau = 1$, $\tau|_{x=0} = 0$ while $q = \tau(\cdot + \lambda_1\boldsymbol{e}_x) - \tau$. Note that the uniform flows are solutions to this problem for any h (since q is constant), whereas non-uniform shear flows require the specific choice $h(q) = \lambda_1^2/(2q^2)$. Due to the hyperbolic nature of the equations defining τ and q, they are *not* Fréchet differentiable in the classical sense with respect to $\boldsymbol{u}$ and η, but only if one allows some loss of spatial regularity. This precludes the use of the Crandall–Rabinowitz local bifurcation theorem. However, the problem is sufficiently 'tame' that an adaptation which allows for some loss of regularity works. The issue that $\boldsymbol{u}$ depends implicitly on η through its domain of definition can be handled by the standard trick of 'flattening' the domain.

Theorem ([2]). *Assume that $h \in C^{k,\gamma}(\mathbb{R})$, $k > 4$, $\gamma \in (0,1)$ with $\|h\|_{k,\gamma} \ll 1$ and that $\boldsymbol{k} = 2\pi(\lambda_1^{-1}, \lambda_2^{-1})$ satisfies the dispersion relation $g + \sigma|\boldsymbol{k}|^2 - \frac{c_*^2 k_1^2}{|\boldsymbol{k}|}\coth(|\boldsymbol{k}|d) = 0$, and that there are no other solutions $\boldsymbol{k} \in (2\pi/\lambda_1)\mathbb{Z} \times (2\pi/\lambda_2)\mathbb{Z}$ to the dispersion relation except for reflections of $\boldsymbol{k}$. Then there is a family of periodic and symmetric solutions $(\boldsymbol{u}(t), \eta(t), c(t)) \in C^{k,\gamma}(\overline{\Omega^{\eta(t)}}) \times C^{k+1,\gamma}(\mathbb{R}) \times \mathbb{R}$, $|t| < \varepsilon$, bifurcating from a uniform flow at $t = 0$, and satisfying*

$$\eta(t)(x,y) = t\cos\left(\frac{2\pi}{\lambda_1}x\right)\cos\left(\frac{2\pi}{\lambda_2}y\right) + o(t),$$

in $C^{k,\gamma}$. Under the additional hypothesis $h'(\lambda_1/c_*) \neq 0$, $\boldsymbol{\omega}(t) \neq 0$ for $0 < |t| \ll 1$.

The fact that all non-uniform shear flows have the same h indicates that the above approach does not in that case. Going back to the original formulation of the steady water wave problem we show in a work in progress that there are no C^2 curves of symmetric solutions bifurcating from non-uniform flows. In fact, the infinite-dimensional kernel of the linearisation implies that the nonlinearities have to satisfy infinitely many compatibility conditions, and one can show that these are not satisfied at quadratic order unless the underlying flow is uniform.

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Time periodic solutions for the 3D Quasigeostrophic model

Joan Mateu

(joint work with C. García and T. Hmidi)

This note is related with the construction of time periodic solutions for the inviscid 3D quasi-geostrophic model. In the papers [2] and [3] is proved the existence of non-trivial rotating patches, obtained by suitable perturbations of stationary solutions. These stationary solutions are given by generic revolutions shapes around the vertical axis. The construction of these special solutions is done using some ideas introduced by Burbea in last century [1]. More precisely, using bifurcation theory through the Crandall-Rabinowitz Theorem, rotating patches are obtained for the simply connected case and for the doubly connected case.

The quasi–geostrophic system is described by the potential vorticity q which is merely advected by the fluid. In fact, we consider a transport equation given by

$$(1) \qquad \begin{cases} \partial_t q + u\partial_1 q + v\partial_2 q = 0, & (t,x) \in [0,+\infty) \times \mathbb{R}^3, \\ \Delta\psi = q, \\ u = -\partial_2\psi, \ v = \partial_1\psi, \\ q(0,x) = q_0(x). \end{cases}$$

This system is a model commonly used in the ocean and atmosphere circulations to describe the vortices and to track the emergence of long–lived structures.

The main question to investigate is the existence of non trivial relative equilibria close to the stationary revolution shapes r_0.

We look for smooth domains D with the following parametrization,

$$D = \left\{(re^{i\theta}, \cos(\phi)) \,:\, 0 \le r \le r(\phi,\theta), 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\right\},$$

where the shape is sufficiently close to a revolution shape domain, meaning that

$$r(\phi,\theta) = r_0(\phi) + f(\phi,\theta),$$

with small perturbation f (in a given function space).

The result obtained for the simply connected case follows, if some conditions are required for the initial profile r_0, and denoted throughout this note by **(H)**. These conditions are

(H1) $r_0 \in C^2([0,\pi])$, with $r_0(0) = r_0(\pi) = 0$ and $r_0(\phi) > 0$ for $\phi \in (0,\pi)$.

(H2) There exists $C > 0$ such that

$$\forall\, \phi \in [0,\pi], \quad C^{-1}\sin\phi \le r_0(\phi) \le C\sin(\phi).$$

(H3) r_0 is symmetric with respect to $\phi = \frac{\pi}{2}$, i.e., $r_0\left(\frac{\pi}{2} - \phi\right) = r_0\left(\frac{\pi}{2} + \phi\right)$, for any $\phi \in [0, \frac{\pi}{2}]$.

Under these condition the main result reads as follows.

Theorem. *Assume that r_0 satisfies the assumptions* **(H)**. *Then for any $m \ge 2$, there exists a curve of non trivial rotating solutions with m-fold symmetry to the equation* (1) *bifurcating from the trivial revolution shape associated to r_0 at some angular velocity Ω_m.*

For the doubly connected case we also can prove the existence of a family of rotating domains which has been obtained bifurcating form an stationary doubly connected revolution shape.

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The capillarity driven Stokes flow as a small viscosity limit

Bogdan-Vasile Matioc
(joint work with Georg Prokert)

The two-dimensional quasistationary Stokes flow of a fluid layer $\Omega(t)$ of infinite depth driven by surface tension at the free boundary $\Gamma(t) = \partial\Omega(t)$ is governed by the following system of equations

$$\left.\begin{array}{rcll} \mu\Delta v - \nabla p &=& 0 & \text{in } \Omega(t), \\ \operatorname{div} v &=& 0 & \text{in } \Omega(t), \\ T_\mu(v,p)\nu &=& \sigma\kappa\nu & \text{on } \Gamma(t), \\ (v,p)(x) &\to& 0 & \text{for } |x| \to \infty, \\ V_n &=& v\cdot\nu & \text{on } \Gamma(t) \end{array}\right\} \tag{1a}$$

for $t > 0$, where the interface $\Gamma(t)$ is given as the graph of a map $f(t,\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$, that is

$$\Omega(t) := \{x = (x_1,x_2) \in \mathbb{R}^2 \,:\, x_2 < f(t,x_1)\},$$

$$\Gamma(t) := \partial\Omega(t) := \{(\xi, f(t,\xi)) \,:\, \xi \in \mathbb{R}\}.$$

Additionally, the interface $\Gamma(t)$ is assumed to be known initially

$$f(0,\cdot) = f^{(0)}. \tag{1b}$$

In (1a), $v = v(t) : \Omega(t) \longrightarrow \mathbb{R}^2$ and $p = p(t) : \Omega(t) \longrightarrow \mathbb{R}$ are the velocity and the pressure of the Newtonian fluid, ν is the unit exterior normal to $\partial\Omega$, κ denotes the curvature of the interface, and $T_\mu(v,p) = (T_{\mu,ij}(v,p))_{1\le i,\,j\le 2}$ is the stress tensor which is given by

$$T_\mu(v,p) := -pE_2 + \mu\big[\nabla v + (\nabla v)^\top\big], \quad (\nabla v)_{ij} := \partial_j v_i.$$

Moreover, V_n is the normal velocity of the interface $\Gamma(t)$, $a\cdot b$ denotes the Euclidean scalar product of two vectors a, $b \in \mathbb{R}^2$, $E_2 \in \mathbb{R}^{2\times 2}$ is the identity matrix, and the positive constants μ and σ are the dynamic viscosity of the fluid and the surface tension coefficient at the interface $\Gamma(t)$, respectively.

In the recent reference [1] we have shown that this problem is well-posed and that it can be rigorously identified as the singular limit of the corresponding two-phase Stokes flow

$$\left.\begin{array}{rcll} \mu^\pm\Delta w^\pm - \nabla q^\pm &=& 0 & \text{in } \Omega^\pm(t), \\ \operatorname{div} w^\pm &=& 0 & \text{in } \Omega^\pm(t), \\ [w] &=& 0 & \text{on } \Gamma(t), \\ [T_\mu(w,q)]\nu &=& -\sigma\kappa\nu & \text{on } \Gamma(t), \\ (w^\pm,q^\pm)(x) &\to& 0 & \text{for } |x| \to \infty, \\ V_n &=& w^\pm\cdot\nu & \text{on } \Gamma(t) \end{array}\right\} \tag{2a}$$

for $t > 0$ and

$$f(0,\cdot) = f^{(0)}, \tag{2b}$$

with $\mu^- = \mu$ fixed, when the viscosity coefficient μ^+ vanishes. In (2a) it is again assumed that $\Gamma(t)$ is the graph of a function $f(t,\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$,

$$\Omega^\pm(t) := \{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 \gtrless f(t, x_1)\},$$

and ν is the unit exterior normal to $\partial\Omega^-(t)$. Moreover, $w^\pm(t)$ and $q^\pm(t)$ represent the velocity and pressure fields in $\Omega^\pm(t)$, respectively, and $[w]$ (respectively $[T_\mu(w,q)]$) is the jump of the velocity (respectively stress tensor) across the moving interface.

We point out that the limit $\mu^+ \to 0$ in (2) is singular because the ellipticity of the boundary value problem is lost in this limit.

It is shown in [2, Theorem 1.1] that, given $f^{(0)} \in H^s(\mathbb{R})$, $s \in (3/2, 2)$ there exists a unique maximal solution $(f_{\mu^+}, w^\pm_{\mu^+}, q^\pm_{\mu^+})$ to (2) such that

- $f_{\mu^+} = f_{\mu^+}(\cdot, f^{(0)}) \in \mathrm{C}([0, T_{+,\mu^+}), H^s(\mathbb{R})) \cap \mathrm{C}^1([0, T_{+,\mu^+}), H^{s-1}(\mathbb{R}))$,
- $w^\pm_{\mu^+}(t) \in \mathrm{C}^2(\Omega^\pm(t)) \cap \mathrm{C}^1(\overline{\Omega^\pm(t)})$, $q^\pm_{\mu^+}(t) \in \mathrm{C}^1(\Omega^\pm(t)) \cap \mathrm{C}(\overline{\Omega^\pm(t)})$ for all $t \in (0, T_{+,\mu^+})$,
- $w^\pm_{\mu^+}(t)|_{\Gamma(t)} \circ \Xi_{f(t)} \in H^2(\mathbb{R})^2$ for all $t \in (0, T_{+,\mu^+})$,

where $T_{+,\mu^+} = T_{+,\mu^+}(f^{(0)})$ is the maximal existence time and $\Xi_{f(t)}(\xi) := (\xi, f(t,\xi))$ for $\xi \in \mathbb{R}$. As $H^{3/2}(\mathbb{R})$ is a critical spaces for both problems (1) and (2), this well-posedness result covers all subcritical L_2-Sobolev spaces.

In [1, Theorem 1.1] it is shown that also the one-phase problem is well-posed in the same setting.

Theorem ([1, Theorem 1.1]). *Let $s \in (3/2, 2)$ be given. Then, the following statements hold true:*

(i) (Well-posedness) *Given $f^{(0)} \in H^s(\mathbb{R})$, there exists a unique maximal solution (f, v, p) to (1) such that*
  - $f = f(\cdot; f^{(0)}) \in \mathrm{C}([0, T_+), H^s(\mathbb{R})) \cap \mathrm{C}^1([0, T_+), H^{s-1}(\mathbb{R}))$,
  - $v(t) \in \mathrm{C}^2(\Omega(t)) \cap \mathrm{C}^1(\overline{\Omega(t)})$, $p(t) \in \mathrm{C}^1(\Omega(t)) \cap \mathrm{C}(\overline{\Omega(t)})$ *for all* $t \in (0, T_+)$,
  - $v(t)|_{\Gamma(t)} \circ \Xi_{f(t)} \in H^2(\mathbb{R})^2$ *for all* $t \in (0, T_+)$,

  where $T_+ = T_+(f^{(0)}) \in (0, \infty]$ is the maximal existence time. Moreover, the set

$$\mathcal{M} := \{(t, f^{(0)}) \,:\, f^{(0)} \in H^s(\mathbb{R}),\, 0 < t < T_+(f^{(0)})\}$$

  is open in $(0,\infty) \times H^s(\mathbb{R})$, and $[(t, f^{(0)}) \longmapsto f(t; f^{(0)})]$ is a semiflow on $H^s(\mathbb{R})$ which is smooth in $\mathcal{M}$.

(ii) (Parabolic smoothing)
  (iia) *The map $[(t,\xi) \longmapsto f(t,\xi)] : (0, T_+) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a C^∞-function.*
  (iib) *For any $k \in \mathbb{N}$, we have $f \in \mathrm{C}^\infty((0, T_+), H^k(\mathbb{R}))$.*

(iii) (Global existence) *If*

$$\sup_{[0,T]\cap[0,T_+(f^{(0)}))} \|f(t)\|_{H^s} < \infty$$

  for each $T > 0$, then $T_+(f^{(0)}) = \infty$.

Moreover, the one-phase problem is rigorously identified as the limit of the two-phase problem when $\mu^+ \to 0$.

Theorem ([1, Theorem 1.2])**.** *Let $s \in (3/2, 2)$ and $f^{(0)} \in H^s(\mathbb{R})$ be given. Let $(f(\cdot; f^{(0)}), v, p)$ denote the maximal solution to* (1) *identified in Theorem and choose $T < T_+(f^{(0)})$. Then, there exist constants $\varepsilon > 0$ and $M > 0$ such that for all $\mu^+ \in (0, \varepsilon]$, we have $T < T_{+,\mu^+}(f^{(0)})$ and*

$$\left\| f(\cdot; f^{(0)}) - f_{\mu^+}(\cdot; f^{(0)}) \right\|_{C([0,T], H^s(\mathbb{R}))} + \left\| \frac{d}{dt}\big(f(\cdot; f^{(0)}) - f_{\mu^+}(\cdot; f^{(0)})\big) \right\|_{C([0,T], H^{s-1}(\mathbb{R}))} \leq M\mu^+.$$

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Global bifurcation for co-rotating vortex patches

Susanna V. Haziot
(joint work with Claudia García)

We will consider *steady* (*V-states*) rotating vortex patch solutions to the two-dimensional Euler equations: the patches are rotating at constant angular velocity Ω without changing shape. We consider rotating configurations of two point vortices. When perturbatively desingularized, one obtains pairs of symmetric patches with same circulations (see [3]). Most of the work done on V-state vortex patches is of perturbative nature: the solutions obtained are very close to the explicit solutions they are bifurcating from. However, numerics suggest that interesting and beautiful solutions form as one moves further away from the trivial solutions.

The first global bifurcation result for the simple patch bifurcating from the disk is due to Hassainia, Masmoudi, and Wheeler in [2]. They obtain a global curve which limits to a vanishing of the angular fluid velocity. We construct a global curve of solutions for the corotating vortex pairs. Since in our case the explicit solutions to the problem are points rather than patches, the formulation of the problem contains a singularity, the main major difficulty of this problem.

The two-dimensional incompressible Euler equations expressed in the vorticity form are given by

$$\partial_t \omega + (u \cdot \nabla)\omega = 0, \qquad u = \nabla^\perp \psi, \qquad \Delta\psi = \omega. \tag{1}$$

Here u denotes the velocity field, ψ the stream function and ω the vorticity. We identify $(x, y) \in \mathbb{R}^2$ with $z = x + iy \in \mathbb{C}$. We seek solution of (1) which satisfy the initial data

$$\omega_0(z) := \omega(0,z) = \frac{1}{\varepsilon^2\pi}(\chi_{D_1(0)}(z) + \chi_{D_2(0)}(z)), \tag{2}$$

where the $D_m(t)$ are disjoint simply connected regions. If this solution takes the form $\omega(t,z) = \omega_0(e^{-it\Omega}z)$, we get a rotating vortex pair (D_1, D_2) about the origin $(0,0)$ with angular velocity Ω. We choose the center of mass of D_1 to be $(l,0)$ for some $l \in \mathbb{R}$ and we set $D_2 := -D_1$. By moving to a frame of reference rotating at this same speed Ω, the regions appear to be stationary. By expressing (1) in terms of the relative stream function $\Psi = \psi_0 - \frac{1}{2}\Omega|z|^2$ we for $\Psi \in C^1(\mathbb{C})$ then get

$$\Delta\Psi = \frac{1}{\varepsilon^2\pi}\chi_{D_1} + \frac{1}{\varepsilon^2\pi}\chi_{D_2} - 2\Omega, \tag{3a}$$

$$\nabla(\Psi + \tfrac{1}{2}\Omega|z|^2) \to 0, \quad \text{as } |z| \mapsto \infty, \qquad \text{and} \qquad \Psi = c_m, \quad \text{on } \partial D_m, \tag{3b}$$

for some constants c_m, $m = 1,2$. Since both the D_m and the function Ψ are unknowns, this is a free boundary problem. We obtain the following result.

Theorem. *There exists a continuous curve $\mathcal{C}$ of corotating vortex patch solutions to* (3)*, parameterized by $s \in (0,\infty)$. The following properties hold along $\mathcal{C}$:*

- (i) (Bifurcation from point vortex) *The solution at $s = 0$ is a pair of points z_1, z_2 lying on the horizontal axis at a distance l from each other.*
- (ii) (Limiting configurations) *As $s \to \infty$*

$$\min\left\{ \min_{z\in\partial D_1} \varepsilon\nabla\Psi(z)\cdot\left(\frac{z-l}{|z-l|}\right), \min_{z_m\in\partial D_m} |z_1 - z_2| \right\} \to 0 \tag{4}$$

- (iii) (ε bounded away from 0) *The value of the parameter $\varepsilon(s)$ is bounded away from 0 for all s away from the local curve.*
- (iv) (Analyticity) *For each $s > 0$, the boundary ∂D_m is analytic.*
- (v) (Graphical boundary) *For each $s > 0$, the boundary of the patch can be expressed as a polar graph.*

The first term in (4) indicates that there are points on the boundaries of the patches for which the angular fluid velocity becomes arbitrarily small. The factor of ε is necessary in order to catch the domains in (3), where the vorticity (for the purpose of the desingularization of point vortices) has been normalized to $1/(\pi\varepsilon^2)$. The slightly complicated formulation of angular fluid velocity comes from the fact that the patches are not centered at the origin. The formation of a corner or of a cusp would require that $\varepsilon\nabla\Psi = 0$ at a given point on the boundary of the patches and numerical evidence indicates that this does in fact happen. The second term in (4) vanishes if and only if the boundaries ∂D_m of the two patches intersect at some point z. Numerical work [4] suggests that the limiting scenario consists of the two patches intersecting at a corner with a 90° angle. This conjecture, also known as the *Overman conjecture*, would imply that the two terms in (4) would occur simultaneously. The core of the proof relies on the following rigidity theorem.

Theorem. *If $\omega_0 = (\pi\varepsilon^2)^{-1}(\chi_{D_1} + \chi_{D_2})$ is a solution to the vortex pair problem then $\Omega \in (0, 1/(2\pi\varepsilon^2))$. Moreover, if $\varepsilon \leq l/10$ then $|\Omega| \lesssim |l|^{-2}$, where l is the distance of the center of the patch D_1 to the y-axis.*

The first part of the theorem is an adaptation of the result in [1] to the two-patch setting. The second part is new with the striking property that it provides uniform bounds on Ω regardless of where along the global curve the solution lies.

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On the role of viscosity stratification

Xian Liao

(joint work with Zihui He, Christian Zillinger)

We investigate the role played by the viscosity stratification in the theory of existence, regularity and stability for the two-dimensional (inhomogeneous) incompressible fluids. The global-in-time wellposedness for the two-dimensional inhomogeneous incompressible Navier-Stokes equations in the presence of density-dependent viscosity coefficient with large variation is still open. Instead, we propose to consider the stationary model, and also an evolutionary model with constant density but variable viscosity coefficient.

The stationary model. We are first concerned with the two-dimensional stationary inhomogeneous incompressible Navier–Stokes equations

$$\begin{cases} \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu S u) + \nabla \Pi = 0, \\ \operatorname{div} u = 0,\ \operatorname{div}(\rho u) = 0. \end{cases} \tag{1}$$

The unknown density function $\rho \geq 0$, the unknown velocity vector field $u \in \mathbb{R}^2$ and the unknown pressure $\Pi \in \mathbb{R}$ depend on the spatial variable $(x, y) \in \mathbb{R}^2$. In the viscosity term $-\operatorname{div}(\mu S u)$, $Su = \nabla u + (\nabla u)^T$ denotes the (symmetric) deformation strain tensor, and the variable viscosity coefficient depends continuously on the *unknown* density function

$$\mu = b(\rho) = b(\rho(x, y)) \in [\mu_*, \mu^*], \quad 0 < \mu_* \leq \mu^*.$$

Observe that

- The pair of the following (Frolov) form solves $(1)_2$ automatically:

$$(\rho, u) = (\eta(\Phi), \nabla^\perp \Phi), \quad \eta \in L^\infty(\mathbb{R}; [0, \infty)) \tag{2}$$

- Applying $\nabla^{\perp}\cdot$ to $(1)_1$ gives the following equation for stream function Φ:

$$L_{\mu}\Phi = \nabla^{\perp}\cdot \operatorname{div}(\rho\nabla^{\perp}\Phi\otimes\nabla^{\perp}\Phi), \tag{3}$$

where L_μ denotes the fourth-order elliptic operator with $\mu = (b\circ\eta)(\Phi)$:

$$L_{\mu} = (\partial_{x_2x_2}-\partial_{x_1x_1})\mu(\partial_{x_2x_2}-\partial_{x_1x_1}) + (2\partial_{x_1x_2})\mu(2\partial_{x_1x_2}).$$

Theorem ([1])**.** *Let $\eta\in L^{\infty}(\mathbb{R};[0,\infty))$, $b\in C(\mathbb{R};[\mu_*,\mu^*])$, $0<\mu_*\le\mu^*$ be given. Let $\Omega\subset\mathbb{R}^2$ be a bounded simply connected $C^{1,1}$ domain, and $u_0\in H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^2)$ with $\int_{\partial\Omega}u_0\cdot n=0$ be the boundary value condition. Then*

- *There exists a weak solution (ρ,u) of form (2), with $u\in H^1(\Omega;\mathbb{R}^2)$, of the boundary value problem for (1).*
- *If (ρ,u) is a weak solution of (1) with $u_0\in W^{1,\infty}(\partial\Omega)$, and the viscosity coefficient μ is e.g. pieceweise-constant, then the weak solution satisfies*

$$\nabla u\in L^p(\Omega) \text{ and } \mathbb{P}div(\mu Su)\in L^p(\Omega),\quad \forall p\in[2,\infty),$$

where $\mathbb{P}$ denotes the Leray-Helmholtz projector on div-free vectors.
- *There exist explicit solutions of form (2) for (1), e.g. shear flows: $\rho=\rho(y)$, $u=\begin{pmatrix}U(y)\\0\end{pmatrix}$, which satisfies*

$$\partial_y(\mu(y)\partial_yU)=const. \tag{4}$$

Remark. *One observes that the assumption with positive lower and upper bounds on μ is enough for existence results $\nabla u\in L^2$, while one needs some further regularity assumption on μ to arrive at $\nabla u\in L^p$, $\forall p\in(2,\infty)$: Otherwise, $\exists\mu:\Omega\to\{K,\frac{1}{K}\}$, $K>1$, s.t. $L_\mu\Phi=0$ has a solution with $\nabla u\notin L^p$, $\forall p\ge\frac{2K}{K-1}$. The solution (4) implies explicitly the inheritance of the irregularity from μ to ∇u.*

The evolutionary model. We consider the two-dimensional evolutionary Navier-Stokes equations with constant density but variable viscosity coefficient:

$$\begin{cases}\partial_t u + u\cdot\nabla u - \operatorname{div}(\mu Su) + \nabla\Pi = 0,\\ \operatorname{div} v = 0,\end{cases} \tag{5}$$

where $\mu=\mu(y)\in C^2(\mathbb{R};\mathbb{R}^+)$ is a *given* stratified viscosity profile. Let $u=\begin{pmatrix}U(y)\\0\end{pmatrix}$, $(x,y)\in\mathbb{T}\times\mathbb{R}$ be a shear flow of (5) such that

$$\mu\partial_yU=\text{const.} \tag{6}$$

Notice that (6) is a special case of (4).

Theorem ([2])**.** *Suppose also that μ only varies gradually, in the sense that*

$$\|(\ln\mu)'\|_{W^{1,\infty}(\mathbb{R})}<0.0001.$$

The linearized equations around the steady flow (in vorticity formulation)

$$\partial_t\omega + U\partial_x\omega = U''u_2 + div(\mu\nabla\omega) - div(\mu'\nabla u_1) - \mu''\partial_xu_2,\ \begin{pmatrix}u_1\\u_2\end{pmatrix} = -\nabla^{\perp}(-\Delta)^{-1}\omega,$$

are stable and exhibit enhanced dissipation: There exists a time-dependent family of operators $A(t)$ with

$$0.1\|\omega(t)\|^2_{L^2(\mathbb{T}\times\mathbb{R},U'dxdy)} \leq \|A(t)\omega(t)\|^2_{L^2(\mathbb{T}\times\mathbb{R},U'dxdy)} \leq \|\omega(t)\|^2_{L^2(\mathbb{T}\times\mathbb{R},U'dxdy)},$$

such that, if the x-average of the initial vorticity vanishes: $\int_{\mathbb{T}} \omega_0 dx = 0$, then for all times $t > 0$ it holds that

$$\frac{d}{dt}\|A(t)\omega(t)\|^2_{L^2(\mathbb{T}\times\mathbb{R},U'dxdy)} \leq -0.0001\|(\mu(U')^2))^{1/6}A(t)\omega\|^2_{L^2(\mathbb{T}\times\mathbb{R},U'dxdy)}.$$

Remark. *Let $\mu U' = \sigma > 0$ be a constant, then the effective dissipation rate*

$$\left(\mu(U')^2\right)^{\frac{1}{3}} = \frac{\sigma^{\frac{2}{3}}}{\mu^{\frac{1}{3}}}$$

increases as μ decreases while decreases as μ increases.

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On Resonance Chains in the Boussinesq Equations

Christian Zillinger

We consider the evolution of a two-dimensional, incompressible, heat conducting viscous fluid in a periodic channel as modeled by the Boussinesq equations:

$$\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla p &= \nu \Delta v + \theta e_2, \\
\partial_t \theta + v \cdot \nabla \theta &= \kappa \Delta \theta, \\
\mathrm{div}(v) &= 0, \\
(t, x, y) &\in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}.
\end{aligned}$$

These equations are a system of the Navier-Stokes or Euler equations and an advection diffusion equation and are coupled by a buoyancy term θe_2, which causes hotter fluid to rise above colder fluid.

This system of equations includes several interacting (de)stabilizing effects:

- Layers of hotter fluid below colder fluid give rise to *Rayleigh-Bénard instability.*
- Shear flows in the fluid, chosen here as Couette flow $v = (y, 0)$, can lead to *mixing* and the appearance of very fine scales in the dynamics.
- The interaction of mixing and viscous effects leads to dissipation of kinetic energy on time scales much faster than without mixing, a phenomenon known as *enhanced dissipation.*

As a first main result we show that in the setting of purely viscous dissipation ($\kappa = 0$, $\nu > 0$) mixing enhanced dissipation is sufficiently strong to suppress Rayleigh-Bénard instability for the linearized problem and establish stability in Sobolev regularity [3, 2]. However, in the nonlinear problem even in the stably stratified case new instabilities appear, which limit regularity to a suitable Gevrey class (i.e. between C^∞ and analytic) [5]. We show that this instability, which is commonly considered a purely non-linear effect, can already be captured in the linearized problem around non-trivial low frequency solutions, which we call traveling waves. While viscosity suppresses (chains of) so-called *fluid echoes*, as in the Euler equations, the system structure causes thermal fluctuations to induce resonances in the velocity: the equations exhibit *thermal echoes* [1].

Finally, we consider the inviscid Boussinesq equations in the stably stratified regime, where stratification and shear flow give rise to algebraic instabilities. The interaction between these instabilities and fluid echoes as captured in terms of traveling waves is shown to exhibit highly frequency-dependent norm inflation. In particular, we match the upper bounds of nonlinear estimates [6] for a specific choice of frequencies and establish improved bounds for both lower and higher frequencies [4].

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Asymptotic models for tumors growth

Martina Magliocca

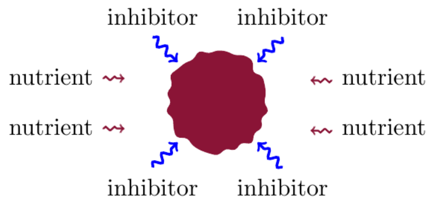
(joint work with Rafael Granero-Belinchón)

The aim of this talk is deriving asymptotic models describing the growth of vascular tumors under the action of inhibitors, i.e. tumors that form from cells that make blood vessels or lymph vessels. Note that vascular tumors grow faster than avascular ones.

The mathematical point of view we assume is the free boundary one, see [1, 2].

These results are part of an ongoing project of R. Granero-Belinchón (Universidad de Cantabria) and M. Magliocca.

The tumor model we consider can be seen as a region $\Omega(t)$ with free boundary $\Gamma(t)$ and surrounded by the vacuum, whose growth is regulated by an externally supplied nutrient $\sigma = \sigma(t,x)$ (oxygen, glucose). We assume that only proliferating cancer cells constitute $\Omega(t)$, and that a certain inhibitor $\beta = \beta(t,x)$ "fights against" the tumor propagation. Both the nutrient σ and the inhibitor β verify reaction-diffusion equations.



We assume that tumor receives constant nutrient supply from the tumor surface and the pressure on the tumor surface is proportional to the mean curvature to maintain the cell-to-cell adhesiveness of the tumor. This assumption is called surface tension effect.
Cells move according to pressure gradients p created by the birth and death of cells, and the velocity vector of the tumor u is assumed to follow Darcy's law.

We study the case in which the boundary $\Gamma(t)$ is a graph. In particular,we assume that

$$\Omega(t) = \left\{x \in \mathbb{R}^2, x_1 \in L\mathrm{T},\ -\infty < x_2 < h(x_1,t)\right\},$$
$$\Gamma(t) = \left\{x \in \mathbb{R}^2, x_1 \in L\mathrm{T},\ x_2 = h(x_1,t)\right\},$$

being $\mathrm{T} = [-\pi, \pi]$ the one-dimensional torus with periodic boundary conditions.

Once the model has been constructed, we first pass to the dimensionless formulation of the system. We later apply a time dependent diffeomorphism to this dimensionless problem in order to obtain an equivalent formulation over a fixed boundary domain Ω.
Then, our main goal consists in finding the equation verified by

$$\partial_t h_0 + \varepsilon \partial_t h_1,$$

being h_0 and εh_1 the first two terms of the expansion $h = \sum_{n\geq 0} \varepsilon^n h_n$.

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The Peskin Problem: Immersed Elastic Interfaces

Eduardo García-Juárez
(joint work with Po-Chun Kuo, Yoichiro Mori, Robert Strain)

The 2D Peskin problem models the dynamics of a closed elastic filament immersed in a two-dimensional incompressible fluid. Since the recent breakthrough works [2] and [3], which provided the strong solution theory together with the global behavior, the problem has attracted a lot of attention. We introduce its three-dimensional counterpart: consider a three-dimensional incompressible Stokes fluid that interacts with an elastic closed two-dimensional membrane in $\mathbb{R}^3$, [1].

An important feature of the Peskin problem is that it admits a Boundary Integral formulation. The problem can be written as an evolution equation for $\boldsymbol{X}$, the deformation map from the sphere to the evolving membrane:

$$\begin{aligned}\partial_t \boldsymbol{X}(\widehat{\boldsymbol{x}}) &= \int_{\mathbb{S}^2} G(\boldsymbol{X}(\widehat{\boldsymbol{x}}) - \boldsymbol{X}(\widehat{\boldsymbol{y}}))\nabla_{\mathbb{S}^2}\cdot\Big(\mathcal{T}(|\nabla_{\mathbb{S}^2}\boldsymbol{X}(\widehat{\boldsymbol{y}})|)\frac{\nabla_{\mathbb{S}^2}\boldsymbol{X}(\widehat{\boldsymbol{y}})}{|\nabla_{\mathbb{S}^2}\boldsymbol{X}(\widehat{\boldsymbol{y}})|}\Big)d\mu_{\mathbb{S}^2}(\widehat{\boldsymbol{y}}),\\ \boldsymbol{X}(\widehat{\boldsymbol{x}})|_{t=0} &= \boldsymbol{X}_0(\widehat{\boldsymbol{x}}),\end{aligned} \tag{1}$$

where $G(\boldsymbol{x})$ is the Stokeslet tensor in $\mathbb{R}^3$:

$$G(\boldsymbol{x}) = \frac{1}{8\pi}\Big(\frac{1}{|\boldsymbol{x}|}I_3 + \frac{\boldsymbol{x}\otimes\boldsymbol{x}}{|\boldsymbol{x}|^3}\Big).$$

We have suppressed the dependence of $\boldsymbol{X}$ on t to avoid cluttered notation. It will be sometimes convenient in the analysis to work with coordinates. Let $\boldsymbol{\theta} = (\theta_1, \theta_2)$ be a (local) coordinate system on $\mathbb{S}^2$ and let $\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{X}}(\boldsymbol{\theta}) \in \mathbb{S}^2 \subset \mathbb{R}^3$ be the point on $\mathbb{S}^2$ corresponding to $\boldsymbol{\theta}$. Let $\boldsymbol{X}(\boldsymbol{\theta}) = \boldsymbol{X}(\widehat{\boldsymbol{X}}(\boldsymbol{\theta})) \in \Gamma \subset \mathbb{R}^3$ be the position on Γ corresponding to the coordinate point $\boldsymbol{\theta}$ (see figure below). If $\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{X}}(\boldsymbol{\theta})$, we will write $\boldsymbol{X}(\widehat{\boldsymbol{x}})$ and $\boldsymbol{X}(\boldsymbol{\theta})$ in an abuse of notation. Then, after integration by parts and choosing an isothermal coordinate system, equation (1) becomes

$$\partial_t \boldsymbol{X}(\boldsymbol{\theta}) = -\text{p.v.}\int_{\mathbb{R}^2} \partial_{\eta_i} G(\boldsymbol{X}(\boldsymbol{\theta}) - \boldsymbol{X}(\boldsymbol{\eta}))\tilde{\boldsymbol{F}}_{\text{el},i}(\boldsymbol{X})(\boldsymbol{\eta})d\eta_1 d\eta_2, \tag{2}$$

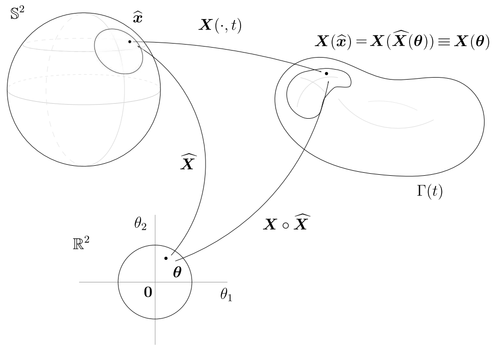
where we denote

$$\tilde{\boldsymbol{F}}_{\text{el},i}(\boldsymbol{X})(\boldsymbol{\eta}) = \frac{\mathcal{T}(\lambda(\boldsymbol{\eta}))}{\lambda(\boldsymbol{\eta})}\partial_{\eta_i}\boldsymbol{X}(\boldsymbol{\eta}), \quad \lambda(\boldsymbol{\eta}) = \sqrt{\text{tr}(\widehat{g}^{-1}(\boldsymbol{\eta})g(\boldsymbol{\eta}))}.$$

We will show that the problem is well-posed. We will first show the existence and uniqueness of strong solutions with initial data in little Hölder spaces, $h^{1,\gamma}(\mathbb{S}^2)$, $\gamma \in (0,1)$, defined as the completion of the set of smooth functions in $C^{1,\gamma}(\mathbb{S}^2)$.

Theorem. *Consider the 3D Peskin problem* (1) *with initial data satisfying* $\boldsymbol{X}_0 \in h^{1,\gamma}(\mathbb{S}^2)$, *and the arc-chord condition:*

$$|\boldsymbol{X}_0|_* = \inf_{\substack{\widehat{\boldsymbol{x}}\neq\widehat{\boldsymbol{y}}\\ \widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}}\in\mathbb{S}^2}} \frac{|\boldsymbol{X}(\widehat{\boldsymbol{x}}) - \boldsymbol{X}(\widehat{\boldsymbol{y}})|}{|\widehat{\boldsymbol{x}} - \widehat{\boldsymbol{y}}|} > 0,$$



and $\mathcal{T} \in C^3$ such that $\mathcal{T} > 0$, $d\mathcal{T}/d\lambda \geq 0$. Then, there exists some time $T > 0$ such that (1) *has a unique strong solution $\boldsymbol{X}$,*

$$\boldsymbol{X} \in C([0,T]; h^{1,\gamma}(\mathbb{S}^2)) \cap C^1([0,T]; h^{\gamma}(\mathbb{S}^2)).$$

Next, we will show that the solutions become smooth instantly in time, and hence are classical solutions.

Theorem. *Let $\boldsymbol{X}$ be the solution to the Peskin problem with initial data $\boldsymbol{X}_0 \in h^{1,\gamma}(\mathbb{S}^2)$ constructed in the previous Theorem . Then, for any $\alpha \in (0,1)$, it holds that $\boldsymbol{X} \in C^1((0,T]; C^{3,\alpha}(\mathbb{S}^2))$. Moreover, for any $3 \leq n \in \mathbb{N}$ and $\alpha \in (0,1)$, assuming that $\mathcal{T} \in C^{n,\alpha}$, it holds that $\boldsymbol{X} \in C^1((0,T]; C^{n+1,\beta}(\mathbb{S}^2))$, for any $\beta < \alpha$.*

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Global-in-time dynamics for Muskat and two-phase Stokes gravity waves

Francisco Gancedo
(joint work with H. Dong, R. Granero-Belinchón, H.Q. Nguyen and E. Salguero)

We consider the evolution of an interface evolving by an incompressible flow. On the one hand, we study the one-phase Muskat problem, where the fluid is filtered in a porous medium. In the gravity-stable case, we show that initial Lipschitz graphs of arbitrary size provide global-in-time well-posedness. On the other hand, we study the interface dynamics given by two fluids of different densities evolving by the linear Stokes law. We show stability to the flat stable case and exponential growth in the unstable regime.

1. Muskat. The Muskat problem models the evolution of the interface of a two-diemnsional incompressible fluid

$$\nabla \cdot u(x,y,t) = 0, \quad (x,y) \in \mathbb{R}^2, \quad t \geq 0,$$

evolving by Darcy's law Darcy's law (1856)

$$\mu(x,y,t)u(x,y,t) = -\nabla p(x,y,t) - \rho(x,y,t)(0,1), \tag{1}$$

with μ viscosity, p pressure, and ρ. The gravity constant g and permeability of the medium κ are taken equal to one for the sake of simplicity. The fluid occupy the domain

$$D(t) = \{(x,y) \in \mathbb{R}^2, \quad y < f(x,t); \quad f : \mathbb{T} \times [0,T] \to \mathbb{R}\}$$

having constant density and viscosity as follows

$$(\mu,\rho)(x,y,t) = \begin{cases} (\mu,\rho), & (x,y) \in D(t), \\ (0,0), & (x,y) \in \mathbb{R}^2 \setminus \overline{D(t)}, \end{cases}$$

with horizontally periodic moving free boundary $f(x,t)$.

Using potential theory, it is possible to find the evolution equation for f given by

$$\partial_t f = -\frac{\rho}{\mu} G(f) f,$$

with the operator $G(f)g$ (Dirichlet-Neumann) given by

$$G(f)g(x) = \frac{1}{4\pi} p.v. \int_{\mathbb{T}} \frac{\sin(x-x') + \sinh(f(x)-f(x'))\partial_x f(x)}{\cosh(f(x)-f(x')) - \cos(x-x')} \theta(x')dx',$$

and $\theta : \mathbb{T} \to \mathbb{R}$ satisfies

$$\frac{1}{2}\theta(x) + \frac{1}{2\pi} p.v. \int_{\mathbb{T}} \frac{\sinh(f(x)-f(x')) - \sin(x-x')\partial_x f(x)}{\cosh(f(x)-f(x')) - \cos(x-x')} \theta(x')dx' = \partial_x g(x).$$

The main result is the following

Theorem (H. Dong, **G.**, H.Q. Nguyen-21)**.** *For all $f_0 \in W^{1,\infty}(\mathbb{T})$, there exists*

$$f \in C(\mathbb{T} \times [0,\infty)) \cap L^\infty([0,\infty); W^{1,\infty}(\mathbb{T})), \quad \partial_t f \in L^\infty([0,\infty); L^2(\mathbb{T}))$$

such that $f|_{t=0} = f_0$, f satisfies One-Fluid-Muskat in $L^\infty_t L^2_x$, and

$$\|f(t)\|_{W^{1,\infty}(\mathbb{T})} \leq \|f_0\|_{W^{1,\infty}(\mathbb{T})} \quad a.e.\ t > 0.$$

Moreover, f is unique in its class (viscosity solution).

It provides the first construction of unique global strong solutions for the Muskat problem with initial data of arbitrary size in a critical space.

2. Two-phase Stokes gravity waves. In this interface problem we replace (1) by the Stokes law

$$-\Delta u(x,y,t) = -\nabla p(x,y,t) - \rho(x,y,t)(0,1),$$

considering different density and equal viscosity fluids as follows

$$(\mu,\rho)(x,y,t)=\begin{cases}(1,\rho^1), & (x,y)\in D^1(t),\\ (1,\rho^2), & (x,y)\in D^2(t)=\mathbb{R}^2\setminus\overline{D^1(t)}.\end{cases}$$

Using the x periodic Stokeslet

$$8\pi\mathcal{S}(x,y)=\log\left(2(\cosh(x)-\cos(y))\right)I-\frac{y}{\cosh(y)-\cos(x)}\begin{pmatrix}-\sinh(y) & \sin(x)\\ \sin(x) & \sinh(y)\end{pmatrix},$$

we obtain the evolution equation for the interface

$$z_t(\alpha,t)=(\rho^2-\rho^1)\int_{\mathbb{T}}\mathcal{S}(z(\alpha,t)-z(\beta,t))\cdot\partial_\beta z^\perp(\beta,t)z_2(\beta,t)d\beta.$$

For interfaces giving by a graph, $z(x,t)=(x,h(x,t))$, we have stability in the stable case (denser fluid below):

Theorem (**G.**, R. Granero-Belinchón, E. Salguero-22). *If $\|h_0\|_{H^3}<\delta$, for $\delta>0$ small enough, there exists a unique global classical solution such that*

$$h\in C([0,\infty);H^3(\mathbb{T})),$$

and

$$(1+t)^s\|h\|_{L^2}(t)+\|\partial_x^3 h\|_{L^2}(t)\le C\|h_0\|_{H^3},$$

for $3/2<s<2$.

On the other hand, in the unstable case (denser fluid above) we show instability. We use Wiener spaces. For $\nu>0$ we define

$$A_\nu=\Big\{h\in L^1\,:\,\|h\|_{A^s_\nu}=\sum_{k\in\mathbb{Z}}e^{\nu|k|}|\hat{h}(k)|<\infty\Big\}.$$

Theorem (**G.**, R. Granero-Belinchón, E. Salguero-23). *Let $T>0$ arbitrary. There exists a family of smooth initial data $g_0\in A_{\nu*}$ such that*

$$g\in C([0,T];A_{\nu*}),$$

is a solution in the unstable regime, $\rho^2-\rho^1<0$, and

$$\|g\|_{A_{\nu^*}}(\tau)\ge c(g_0)\exp\Big(\sqrt{(\rho^1-\rho^2)\nu^*\tau}\Big),\quad \tau\in[0,T].$$

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Invariant KAM tori around annular vortex patches for the planar Euler equations

Zineb Hassainia
(joint work with Taoufik Hmidi, Emeric Roulley)

The motion of a two-dimensional ideal homogeneous incompressible fluid follows the the classical planar incompressible Euler equations that can be reformulated in the vorticity/velocity form as follows

$$\partial_t \omega + (\nabla^\perp \psi) \cdot \nabla \omega = 0 , \qquad \Delta \psi := \omega , \tag{1}$$

where $\nabla^\perp \psi := (-\partial_y \psi, \partial_x \psi)$ is the fluid velocity. Such Hamiltonian system manifests diverse interesting dynamical behaviors, which are at the center of intensive studies with a wide range of applications in natural sciences and engineering.

Global existence and uniqueness of weak solutions of (1) for bounded and integrable initial vortices follows from Yudovich's theory [16]. This allows to deal with discontinuous vortices of the patch form, where the vorticity is uniformly distributed in a bounded domain, that is $\omega(0) = \mathbf{1}_{D_0}$. Due to the transportation of the vorticity by the flow, such structure is preserved in time and the boundary evolves according to a suitable contour dynamics equation. In particular, for any parametrization $z(t) : \mathbb{T} \to \partial D_t$ of the patch boundary one has

$$\partial_t z(t,\theta) \cdot \mathbf{n}\big(t, z(t,\theta)\big) = \partial_\theta \Big[\psi\big(t, z(t,\theta)\big) \Big], \tag{2}$$

where $\mathbf{n}\big(t, z(t,\theta)\big) \triangleq \mathrm{i}\partial_\theta z(t,\theta)$ is a normal vector to the boundary at the point $z(t,\theta)$.

In 1858, Rankine observed that any radial initial domain D_0 (disc, annulus, etc...) generates a stationary solution to (2). Thus, it is quite natural from a dynamical system point of view to explore whether time periodic solutions may exist around these equilibrium states. The first result in this direction is due to Kirchhoff [14], where he proved that any vorticity uniformly distributed inside an elliptic region performs uniform rotation about its center with a constant angular velocity related to its aspect ratio. In addition to the ellipses, several structures undergoing a rigid rotation of fixed shape, called V-states, were found by using bifurcation theory [2, 3, 4, 5, 11, 10, 12, 8]. However, very few results are known in the non-rigid case. In the present work, we investigate the emergence of time quasi-periodic solutions in the vortex patches setting. Quasi-periodic functions generalize periodic ones to several mutually irrational frequencies of oscillations and naturally appear as invariant structures in Hamiltonian dynamical systems.

The construction of quasi-periodic vortex patches to (1) or to various active scalar equations (generalized surface quasi-geostrophic equations, quasi-geostrophic shallow-water equations and Euler-α equations) has been explored in the recent papers [1, 7, 9, 13, 15, 6]. All of them deal with simply-connected quasi-periodic patches vortices provided that the suitable external parameter is selected in a massive Cantor set. Our main gaol here is to construct quasi-periodic vortex patch solutions with one hole for (1) near the annulus

$$A_b \triangleq \big\{z \in \mathbb{C} \quad \text{s.t.} \quad b < |z| < 1\big\}. \tag{3}$$

The motivation behind that is the existence of time periodic patches around the annulus as stated in [11] and one may get time quasi-periodic solutions at the linear level by mixing a finite number of frequencies. One of the main difficulties in the construction of quasi-periodic solutions at the nonlinear level stems from the vectorial structure of the problem because we are dealing with two coupled interfaces. This leads to more time-space resonances coming in part from the interaction between the transport equations advected by two different speeds. Informally stated, our main result is the following;

Theorem. *Consider a compact interval of moduli* $[b_*, b^*] \subset (0,1)$. *Then, for any* $d \in \mathbb{N}$, *there exists a set* $\mathcal{C} \subset [b_*, b^*]$ *with asymptotically full Lebesgue measure such that, for any* $b \in \mathcal{C}$ *there exists a time quasi-periodic vortex patch solution* $\omega(t) = \mathbf{1}_{D_t}$ *of the Euler equations* (1) *with a diophantine frequency vector* $\Omega \in \mathbb{R}^d$, *where* D_t *is doubly connected domain,* m*-fold symmetric, with* m *large enough, and close to the annulus* A_b, *described in* (3).

The proof of this theorem is based on a KAM reducibility scheme and a Nash-Moser iterative scheme. We use the modulus b of the annulus A_b to verify all the Melnikov non-resonance conditions along the KAM iteration. Moreover, we take advantage of the m-fold symmetry structures in order to eliminate the degeneracy of the mode 2.

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On sharp crested water waves whose angles change with time

Nastasia Grubic
(joint work with D. Cordoba and A. Enciso)

We consider the motion of an inviscid incompressible irrotational fluid in the plane with a free boundary. A time-dependent interface

$$\Gamma(t) := \{z(\alpha,t) \,|\, \alpha \in \mathbb{R}\}$$

separates the plane into two open sets: the water region, denoted by $\Omega(t)$, and the vacuum region, $\mathbb{R}^2 \backslash \overline{\Omega(t)}$. The evolution of the fluid is described by the Euler equations,

$$\partial_t v + (v \cdot \nabla) v = -\nabla P - g e_2 \quad \text{in} \quad \Omega(t), \tag{1a}$$

$$\nabla \cdot v = 0 \quad \text{and} \quad \nabla^{\perp} \cdot v = 0 \quad \text{in} \quad \Omega(t), \tag{1b}$$

$$(\partial_t z - v) \cdot \vec{n} = 0 \quad \text{on} \quad \Gamma(t), \tag{1c}$$

$$p = 0 \quad \text{on} \quad \Gamma(t), \tag{1d}$$

where v and P are the water velocity and pressure in $\Omega(t)$, e_2 is the second vector of a Cartesian basis, $g > 0$ is the gravity constant and $\vec{n}$ is the unit normal vector.

The local well-posedness for the corresponding Cauchy problem in Sobolev spaces has extensive literature. We only mention that the first general result was established by Wu [5] and that (at present) lowest allowed interface regularity is $\mathcal{C}^{3/2}$, cf [3]. These results assume Rayleigh–Taylor sign condition, that is,

$$-\partial_n P \geq c > 0 \quad \text{on} \;\; \Gamma(t). \tag{2}$$

From the point of view of the energy estimates, $\partial_n P$ appears directly in the definition of the energy, and its sign is directly related to the positivity of the energy. Note that (2) is automatically satisfied as long as $\Gamma(t)$ is of class $\mathcal{C}^{1,\lambda}$.

Here, we presented the results of a recent preprint [2]. We were interested in the existence of solutions to (1) with interface of class $\mathcal{C}^{1,\lambda}$ everywhere except at one point $z_*(t) \in \Gamma(t)$ (or more generally finitely many points) by jump discontinuities in the tangent vector which thus correspond to corners in the fluid domain. In the vicinity of such points, the strictly positive lower bound in (2) cannot be assured and we are led to consider weighted Sobolev spaces with weight given in terms of the distance to the corner tip. Two classes of solutions allowing (short-time) propagation of angled crests were known previously. The rigid angle class constructed in [4, 6], and solutions constructed by the authors in [1] which do allow time-dependent angles, but are highly symmetric and thus do not allow gravity.

In [2], we were able to remove all restrictions on the symmetry of the domain and prove local well-posedness in a suitable scale of weighted Sobolev spaces that allow for interfaces with corners of time-dependent angle in the range $\left(0, \frac{\pi}{2}\right)$, under the degenerate Rayleigh-Taylor condition

$$-\partial_n P(z,t) \sim |z - z_*|.$$

Morever, the corresponding fluid velocity is of class $C^{1,\lambda}$ up to the boundary inside the water region, which implies these are classical solutions of the Euler equations. In addition, by choosing weighted Sobolev spaces of a sufficiently high order, we can ensure that the interface and velocity in the water region are arbitrarily smooth away from the corner tip.

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Rotating solutions to the incompressible Euler-Poisson equation with external particle

Bernhard Kepka
(joint work with Diego Alonso-Orán and Juan J. L. Velázquez)

We consider a two-dimensional, self-interacting, incompressible fluid body $E \subset \mathbb{R}^2$ which is perturbed by an external particle X with small mass $m > 0$. The shape of the fluid body is assumed to be closed to the unit disk $\mathbb{D}$ and is deformed due to the interaction with the particle. Furthermore, both the fluid body and the particle are assumed to rotate around their center of mass at angular speed Ω_0. The center of mass can be chosen to be (w.l.o.g.) in the origin. We construct solutions which are steady states in a rotating frame of reference. In addition, differently from the results on self-gravitating, ellipsoidal figures reviewed in [1] (excluding the shapes studied by Riemann), we consider solutions which contain a non-trivial internal motion $v \neq 0$ in any coordinate system.

The equations under considerations read

$$(1)\qquad \begin{cases} (v\cdot\nabla)v + 2\Omega_0 Jv - \Omega_0^2 x = -\nabla p - \nabla U_E - m\nabla U_X & \text{in } E \\ \nabla\cdot v = 0 & \text{in } E \\ n\cdot v = 0 & \text{on } \partial E \\ p = 0 & \text{on } \partial E \\ \Omega_0^2 X = \nabla U_E(X) \\ |E| = \pi \\ \int_E x\,dx + mX = 0. \end{cases}$$

Here, J is defined by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Furthermore, we consider various interaction potentials, in particular

$$(2)\qquad U_X(x) = \ln|x-X|, \quad U_E(x) = \int_E \ln|x-y|\,dy.$$

Let us mention that the equation $\Omega_0^2 X = \nabla U_E(X)$ ensures that the centrifugal force acting on X balances with the force due to the interaction with the fluid. By invariance of (1) w.r.t. rotations we seek for $X = (a,0)$ on the x_1-axis.

In order to solve the system (1) we use conformal mappings and the Grad-Shafranov method [2, 3]. More precisely, $E_h = f_h(\mathbb{D})$ with $f_h(z) = z + h(z)$. The velocity is given by the stream function ψ, i.e. $v = \nabla^\perp\psi$, and

$$(3)\qquad \begin{cases} \Delta\psi = G(\psi) & \text{in } E_h, \\ \psi = 0 & \text{on } \partial E_h, \end{cases}$$

for some given function $G : \mathbb{R} \to \mathbb{R}$. Using the fact that the Bernoulli head is constant on the free-boundary ∂E_h one can reduce the system (1) to

$$(4)\qquad \begin{cases} \frac{1}{2}|\nabla\psi_h|^2 - \frac{\Omega_0^2}{2}|x|^2 + U_{E_h} + mU_X = \lambda & \text{on } \partial E_h, \\ \Omega_0^2 a = \partial_{x_1} U_{E_h}(a,0), \\ |E_h| = \pi. \end{cases}$$

Here, (h, a, λ) are the unknowns. Note that the second equation is only the first component of the Newton equation for the particle. The other component is satisfied automatically, since E_h is symmetric w.r.t. the x_1-axis. The unperturbed solution ($m = 0$) is given by $(h, a, \lambda) = (0, a_0, \lambda_0)$ solving (4). In particular, $a_0\Omega_0^2 = U'_{\mathbb{D}}(a_0)$ relates a_0 to Ω_0. We assume that $a_0 \geq 2$, say, so that the particle does not intersect the fluid.

We apply the implicit function theorem and to this end, study the linearized operator. This operator acts on h via some Fourier multipliers $\omega_n = \omega_n(\Omega_0)$. The main result reads as follows.

Theorem. *Let $k \in \mathbb{N}_0$, $\alpha \in (0,1)$ and choose $a_0(\Omega_0) \geq 2$ with $a_0\,\Omega_0^2 = U'_0(a_0)$. In addition, assume that*

(1) $G \in C^{k+3}(\mathbb{R};\mathbb{R})$ *is non-decreasing;*
(2) $\psi_0'(1) \neq 0$*;*
(3) $\omega_n(\Omega_0) \neq 0$ *for all* $n \neq 0$.

Then, for any sufficiently small $m \geq 0$ *there is a unique solution* $(h, a, \lambda) \in C^{k+2,\alpha}(\overline{\mathbb{D}}) \times \mathbb{R}^2$ *to* (4) *close to* $(0, a_0, \lambda_0)$*. Finally, the domain* $E_h = f_h(\mathbb{D})$ *is symmetric w.r.t. the* x_1*-axis and the corresponding velocity field* $v = \nabla^{\perp}\psi_h$ *together with the position of the particle* $X = (a, 0)$ *yield a solution to* (1).

Remark. *Assumption (3) ensure that the linearized operator is invertible. An analysis shows* $\omega_n = \frac{1}{2}\psi_0'(1)^2|n| + \mathcal{O}(1)$ *as* $|n| \to \infty$*. In particular, (3) holds automatically for large* $|n|$*.*

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Reporter: Bernhard Kepka

Participants

Prof. Dr. Helmut Abels
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
GERMANY

Dr. Diego Alonso-Orán
Dept. of Mathematical Analysis
Facultad de Matematicas
Universidad de la Laguna
38271 La Laguna Tenerife
SPAIN

Prof. Dr. Francisco Gancedo
Departamento de Análisis Matemático
Universidad de Sevilla
Campus Reina Mercedes
C/Tarfia, s/n
41012 Sevilla
SPAIN

Dr. Claudia García
Departamento de Matematica Aplicada
Universidad de Granada
18071 Granada
SPAIN

Dr. Eduardo Garcia-Juarez
Departamento de Análisis Matemático
Universidad de Sevilla
Campus Reina Mercedes
C/ Tarfia s/n
41012 Sevilla
SPAIN

Dr. Nastasia Grubić
Departamento de Matematicas
Universidad Autonoma de Madrid
Ciudad Universitaria de Cantoblanco
28049 Madrid
SPAIN

Prof. Dr. Zineb Hassainia
New York University Abu Dhabi
Division of Science and Mathematics
P.O. Box 198-A
129188 Abu Dhabi
UNITED ARAB EMIRATES

Dr. Susanna Haziot
Department of Mathematics
Brown University
Box 1917
Providence, RI 02912
UNITED STATES

Bernhard Kepka
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Xian Liao
Institut für Analysis
Karlsruher Institut für Technologie
(KIT)
Englerstraße 2
76131 Karlsruhe
GERMANY

Prof. Dr. Juan José López Velázquez
Hausdorff Center for Mathematics
Institute for Applied Mathematics
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Martina Magliocca
Universidad de Sevilla
Campus Reina Mercedes
C/ Tarfia s/n
41012 Sevilla
SPAIN

Prof. Dr. Joan Mateu
Departament de Matematiques
Universitat Autonoma de Barcelona
Campus Universitari
08193 Bellaterra, Barcelona
SPAIN

Dr. Bogdan Matioc
Fakultät für Mathematik
Universität Regensburg
Universitätsstr. 31
93053 Regensburg
GERMANY

Prof. Dr. Erik Wahlén
Centre for Mathematical Sciences
Lund University
P.O. Box 118
22100 Lund
SWEDEN

Prof. Dr. Sijue Wu
Department of Mathematics
University of Michigan
2074 East Hall
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Dr. Christian Zillinger
Institut für Analysis
Karlsruher Institut für Technologie
(KIT)
Englerstraße 2
76131 Karlsruhe
GERMANY